Towards Computing Distances between Programs via Scott Domains*

Michael A. Bukatin[1] and Joshua S. Scott[2]

[1] Department of Computer Science, Brandeis University, Waltham, MA 02254, USA; **bukatin@cs.brandeis.edu**; *http://www.cs.brandeis.edu/~bukatin/papers.html*; Parametric Technology Corp., 128 Technology Drive, Waltham, MA 02154, USA

[2] Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA; **josh@neu.edu**

Abstract. This paper introduces an approach to defining and computing distances between programs via *continuous generalized distance functions* $\rho : A \times A \to D$, where A and D are directed complete partial orders with the induced Scott topology, A is a semantic domain, and D is a domain representing distances (usually, some version of interval numbers). A continuous distance function ρ can define a T_0 topology on a nontrivial domain A only if the axiom $\exists 0 \in D. \forall x \in A. \rho(x,x) = 0$ does not hold. Hence, the notion of *relaxed metric* is introduced for domains — the axiom $\rho(x,x) = 0$ is eliminated, but the axiom $\rho(x,y) = \rho(y,x)$ and a version of the triangle inequality tailored for the domain D remain.

The paper constructs continuous relaxed metrics yielding the Scott topology for all continuous Scott domains with countable bases. This construction is closely related to partial metrics of Matthews and valuation spaces of O'Neill, but it describes a wider class of domains in a more intuitive way from the computational point of view.

1 Introduction

In this paper we presume that the methods of denotational semantics allow us to obtain adequate descriptions of program behavior (e.g., see [10]). The term *domain* in this paper denotes a directed complete partial order (*dcpo*) equipped with the Scott topology.

The traditional paradigm of denotational semantics states that all data types should be represented by domains and all computable functions should be represented by Scott continuous functions between domains. For the purposes of this paper all *continuous* functions are Scott continuous.

Consider the typical setting in denotational semantics — a syntactic domain of programs, P, a semantic domain of meanings, A, and a continuous semantic function, $[\![\ ]\!] : P \to A$. The syntactic domain P (called a syntactic lattice in [10]) represents a data type of program parse trees, but we say colloquially that programs belong to P.

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Assume that we have a domain representing distances, D, and a continuous generalized distance function, $\rho : A \times A \to D$. Assume that we can construct a *generalized metric topology*, $\mathcal{T}[\rho]$, on A via ρ. It would be reasonable to say that ρ reflects computational properties of A, if $\mathcal{T}[\rho]$ is the Scott topology on A.

Then $\rho([\![p_1]\!], [\![p_2]\!])$ would yield a computationally meaningful distance between programs p_1 and p_2. The continuous function ρ cannot possess all properties of ordinary metrics because we want $\mathcal{T}[\rho]$ to be non-Hausdorff.

1.1 Axiom $\rho(x, x) = 0$ Cannot Hold

Recall that a non-empty partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. A poset, $(A, \sqsubseteq)$, is a *dcpo* if it has the least element, $\perp$, and for any directed $S \subseteq A$, A contains $\sqcup S$ — the least upper bound of S. A set $U \subseteq A$ is *Scott open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$ and for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$. The Scott open subsets of a dcpo form the *Scott topology*.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. $f : A \to B$ is (Scott) continuous iff it is monotonic ($x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$) and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

Assume that there is an element $0 \in D$ representing the ordinary numerical 0. Let us show that $\forall x.\ \rho(x, x) = 0$ cannot be true under reasonable assumptions. We will see later that all other properties of ordinary metrics can be preserved at least for continuous Scott domains with countable bases (Sect. 5).

It seems reasonable to assume that any reasonable construction of $\mathcal{T}[\rho]$ for any generalized distance function $\rho : A \times A \to D$, should satisfy the following axiom, regardless of whether the distance space D is a domain, or whether ρ is continuous:

Axiom 1. *For all $x, y \in A$, $\rho(x, y) = \rho(y, x) = 0$ implies that x and y share the same system of open sets, i.e. for all open sets $U \in \mathcal{T}[\rho]$, $x \in U$ iff $y \in U$.*

We assume this axiom for the rest of the paper. A topology is called T_0, if different elements do not share the systems of open set.

Corollary 2. *If there are $x, y \in A$, such that $x \neq y$ and $\rho(x, y) = \rho(y, x) = 0$, then $\mathcal{T}[\rho]$ is not a T_0 topology.*

Let us return to our main case, where D is a domain and ρ is a continuous function.

Lemma 3. *Assume that there are at least two elements $x, y \in A$, such that $x \sqsubset_A y$. Assume that $\rho : A \times A \to D$ is a continuous function. If $\rho(x, x) = \rho(y, y) = d \in D$, then $\rho(x, y) = \rho(y, x) = d$*

Proof. The continuity of ρ implies its monotonicity with respect to the both of its arguments. Then $x \sqsubset_A y$ implies $d = \rho(x, x) \sqsubseteq_D \rho(x, y) \sqsubseteq_D \rho(y, y) = d$. This yields $\rho(x, y) = d$, and, similarly, $\rho(y, x) = d$. $\square$

Then we can obtain the following simple, but important result.

Theorem 4. *Assume that there are at least two elements $x, y \in A$, such that $x \sqsubset_A y$. Assume that $\rho : A \times A \to D$ is a continuous generalized distance function and $\mathcal{T}[\rho]$ is a T_0 topology. Then the double equality $\rho(x,x) = \rho(y,y) = 0$ does not hold.*

Proof. By Lemma 3, $\rho(x,x) = \rho(y,y) = 0$ would imply $\rho(x,y) = \rho(y,x) = 0$. Then, by Corollary 2, $\mathcal{T}[\rho]$ would not be T_0, contradicting our assumptions. □

The topologies used in domain theory are usually T_0; in particular, the Scott topology is T_0. This justifies studying continuous generalized metrics ρ, such that $\rho(x,x) = 0$ is false for some x, more closely.

1.2 Intuition behind $\rho(x,x) \neq 0$

There are compelling intuitive reasons not to expect $\rho(x,x) = 0$, when x is not a maximal element of A. The computational intuition behind $\rho(x,y)$ is that the elements in question are actually x' and y', $x \sqsubseteq_A x', y \sqsubseteq_A y'$, but not all information is usually known about them. The correctness condition $\rho(x,y) \sqsubseteq_D \rho(x',y')$ is provided by the monotonicity of ρ.

In particular, even if $x = y$, this only means that we know the same information about x' and y', but this does not mean that $x' = y'$. Consider $x' \neq y'$, such that $x \sqsubset_A x'$ and $x \sqsubset_A y'$. Then $\rho(x,x) \sqsubseteq_D \rho(x',y')$ and $\rho(x,x) \sqsubseteq_D \rho(y',x')$, and at least one of $\rho(x',y')$ and $\rho(y',x')$ is non-zero, if we want ρ to yield a T_0 topology (we do not assume symmetry yet).

Example 1. Here is an important example — a continuous generalized distance on the domain of interval numbers R^I — $\rho : R^I \times R^I \to R^I$ (See Sect. 2 for the definition of R^I). Consider intervals $[a,b]$ and $[c,d]$ and set $S = \{|x' - y'| \mid a \leq x' \leq b,\ c \leq y' \leq d\}$. Define $\rho([a,b],[c,d]) = [\min S, \max S]$. In particular, $\rho([a,b],[a,b]) = [0, b-a] \neq [0,0]$, and $\rho([a,a],[b,b]) = [|a-b|, |a-b|]$.

1.3 Related Work: Quasi-Metrics

Quasi-metrics [9] and Kopperman-Flagg generalized distances [4] are asymmetric generalized distances. They satisfy axiom $\rho(x,x) = 0$ and yield Scott topology for various classes of domains via a construction satisfying Axiom 1.

Theorem 4 means that if one wishes to represent the distance spaces via domains, these asymmetric distances cannot be made continuous unless their nature is changed substantially.

The practice of representing all computable functions via continuous functions between domains suggests that quasi-metrics cannot, in general, be computed (see Sect. 7).

1.4 Related Work: Partial Metrics

Historically, partial metrics are the first generalized distances on domains for which axiom $\rho(x,x)=0$ does not hold. They were introduced by Matthews [7, 6] and further investigated by Vickers [11] and O'Neill [8].

Partial metrics satisfy a number of additional axioms in lieu of $\rho(x,x)=0$ (see Sect. 4). Matthews and Vickers state that $\rho(x,x)\neq 0$ is caused by the fact, that x expresses a partially defined object. The most essential component of the central construction in our paper is a partial metric (Sect. 5).

1.5 Our Contribution

We build partial metrics yielding Scott topologies for a wider class of domains that was known before (Sect. 5, Theorem 7). This class — all continuous Scott domains with countable bases — is sufficiently big to solve interesting domain equations and to define denotational semantics of at least sequential deterministic programming languages [10].

We introduce the notion of *relaxed metric* (Sect. 2), which maintains the intuitively clear requirement to reject axiom $\rho(x,x)=0$, but does not impose the specific axioms of partial metrics. We believe that the applicability of these specific axioms is more limited (see Sect. 8.1).

We introduce the idea that a space of distances should be thought of as a data type in the context of denotational semantics and thus, should be represented by a domain. We also introduce the requirement that distance functions should be computable and thus, Scott continuous (the use of *continuous valuations* in [8] should be considered as a step in this direction).

These considerations lead to an understanding that relaxed metrics should map pairs of partial elements to *upper estimates* of some "ideal" distances, where the *distance domain of upper estimates*, R^-, is equipped with a dual informational order: $\sqsubseteq_{R^-}=\geq$. We also consider lower estimates of "ideal" distances, thus, introducing the *distance domain of interval numbers*, R^I. Continuous lower estimates are useful during actual computations of distances (Sect. 7) and for defining and computing an induced metric structure on the space of total elements (Theorem 8).

There is a comparison in [7] between partial metrics and alternative generalized distance structures such as quasi-metrics and weighted metrics [5]. We provide what we believe to be the strongest argument in favor of partial metrics so far — among all those alternatives only partial metrics can be thought of as Scott continuous, computable functions.

2 Relaxed Metrics

Consider distance domains in greater detail. It is conventional to think about distances as non-negative real numbers. When it comes to considering approximate information about reals, it is conventional to use some kind of *interval numbers*.

We follow both conventions in this text. The distance domain consists of pairs $< a, b >$ (also denoted as $[a, b]$) of non-negative reals ($+\infty$ included), such that $a \leq b$. We denote this domain as R^I. $[a, b] \sqsubseteq_{R^I} [c, d]$ iff $a \leq c$ and $d \leq b$.

We can also think about R^I as a subset of $R^+ \times R^-$, where $\sqsubseteq_{R^+} = \leq$, $\sqsubseteq_{R^-} = \geq$, and both R^+ and R^- consist of non-negative reals and $+\infty$. We call R^+ a *domain of lower bounds*, and R^- — a *domain of upper bounds*. Thus a distance function $\rho : A \times A \to R^I$ can be thought of as a pair of distance functions $< l, u >$, $l : A \times A \to R^+$, $u : A \times A \to R^-$.

We think about $l(x, y)$ and $u(x, y)$ as, respectively, lower and upper bounds of some "ideal" distance $\sigma(x, y)$. We do not try to formalize the "ideal" distances, but we refer to them to motivate our axioms. There are good reasons to impose the triangle inequality, $u(x, z) \leq u(x, y) + u(y, z)$. Assume that for our "ideal" distance, the triangle inequality, $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$, holds. If $u(x, z) > u(x, y) + u(y, z)$, then $u(x, y) + u(y, z)$ gives a better upper estimate for $\sigma(x, z)$ than $u(x, z)$. This means that unless $u(x, z) \leq u(x, y) + u(y, z)$, u could be easily improved and, hence, would be very imperfect.

This kind of reasoning is not valid for $l(x, z)$. In fact, there are reasonable situations, when $l(x, z) \neq 0$, but $l(x, y) = l(y, z) = 0$. E.g., consider Example 1 and take $x = [2, 2], y = [2, 3], z = [3, 3]$.

Also only u plays a role in the subsequent definition of the relaxed metric topology, and the most important results remain true even if we take $l(x, y) = 0$. In the last case we sometimes take $D = R^-$ instead of $D = R^I$ making the distance domain look more like ordinary numbers (it is important to remember, that $\sqsubseteq_{R^-} = \geq$ and, hence, 0 is the largest element of R^-).

We also impose the symmetry axiom on the function ρ. The motivation here is that we presume our "ideal" distance to be symmetric, hence, we should be able to make symmetric upper and lower estimates.

We state a definition summarizing the discourse above:

Definition 5. A symmetric function $u : A \times A \to R^-$ is called a *relaxed metric* when it satisfies the triangle inequality. A symmetric function $\rho : A \times A \to R^I$ is called a *relaxed metric* when its upper part u is a relaxed metric.

3 Relaxed Metric Topology

An *open ball* with a center $x \in A$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x, y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those points y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

Definition 6. A subset U of A is *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > \rho(x, x)$ such that $B_{x,\epsilon} \subseteq U$.

It is easy to show that for a continuous relaxed metric on a dcpo all relaxed metric open sets are Scott open and form a topology.

4 Partial Metrics

The distances p with $p(x,x) \neq 0$ were first introduced by Matthews [7, 6]. They are known as *partial metrics* and obey the following axioms:

1. $x = y$ iff $p(x,x) = p(x,y) = p(y,y)$.
2. $p(x,x) \leq p(x,y)$.
3. $p(x,y) = p(y,x)$.
4. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The last axiom (due to Vickers [11]) implies the ordinary triangle inequality, since the distances are non-negative. O'Neill found it useful to introduce negative distances in [8], but this is avoided in the present paper.

Whenever partial metrics are used to describe a partially ordered domain, a stronger form of the first two axioms is used: If $x \sqsubseteq y$ then $\rho(x,x) = \rho(x,y)$, otherwise $\rho(x,x) < \rho(x,y)$. We include the stronger form in the definition of partial metrics for the purposes of this paper.

5 Central Construction

Here we construct continuous relaxed metrics yielding the Scott topology for all continuous Scott domains with countable bases. Our construction closely resembles one by O'Neill [8]. We also use valuations, but we consider continuous valuations on the powerset of the basis instead of continuous valuations on the domain itself. This allows us to handle a wider class of domains.

We define continuous Scott domains in the spirit of [3]. Consider a dcpo $(A, \sqsubseteq)$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$. An element x, such that $x \ll x$, is called *compact*. We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B \in A$.

Consider a set $K \subseteq A$. Assume that $\perp_A \in K$. We say that a bounded complete dcpo A is a *continuous Scott domain* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements.

Enumerate elements of $K \setminus \{\perp_A\}$: $k_1, ..., k_i, ...$. Associate weights with all basic elements: $w(\perp_A) = 0$, and let $w(k_i)$ form a converging sequence of strictly positive weights. For convenience we agree that the sum of weights of all basic elements equals 1. For example, one might wish to consider $w(k_i) = 2^{-i}$ or $w(k_i) = \epsilon(1+\epsilon)^{-i}, \epsilon > 0$. Then for any $K_0 \subseteq K$, the weight of set K_0, $W(K_0) = \sum_{k \in K_0} w(k_0)$, is well defined and belongs to $[0,1]$.

We have several versions of function ρ. For most purposes it is enough to consider $u(x,y) = 1 - W(K_x \cap K_y)$ and $l(x,y) = 0$. Sometimes it is useful to consider a better lower bound function $l(x,y) = W(I_x \cup I_y)$, where $I_x = \{k \in K \mid k \sqcup x \text{ does not exist}\}$ for the computational purposes (see Sect. 7).

Theorem 7. *The function u is a partial metric. The function ρ is a continuous relaxed metric. The relaxed metric topology coincides with the Scott topology.*

If, in addition, we would like the next theorem to hold, we have to consider a different version of ρ with $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y)$ and $l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x)$. The previous theorem still holds.

We introduce the notion of a *regular basis.* A set of maximal elements in A is denoted as $Total(A)$. We say that the basis K is *regular* if $\forall k \in K, x \in Total(A).\ k \sqsubseteq x \Rightarrow k \ll x$. In particular, if K consists of compact elements, thus making A an algebraic Scott domain, K is regular.

Theorem 8. *Let K be a regular basis in A. Then for all x and y from $Total(A)$, $l(x,y) = u(x,y)$. Consider $\mu : Total(A) \times Total(A) \to \mathbb{R}$, $\mu(x,y) = l(x,y) = u(x,y)$. Then $(Total(A), \mu)$ is a metric space, and its metric topology is the subspace topology induced by the Scott topology on A.*

6 Proofs of Theorems 7 and 8

Here we prove relatively difficult parts of these theorems for the case when $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y)$ and $l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x)$. Lemma 10 is needed for Theorem 8, and other lemmas are needed for Theorem 7.

Lemma 9 (Correctness of lower bounds). $l(x,y) \leq u(x,y)$.

Proof. Using $K_x \cap I_x = \emptyset$ we can rewrite u and l. $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) = W(\overline{U})$, where $U = (K_x \cap K_y) \cup (I_x \cap I_y)$. $l(x,y) = W(V)$, where $V = (K_x \cap I_y) \cup (K_y \cap I_x)$.

We want to show that $V \subseteq \overline{U}$, for which it is enough to show that $V \cap U = \emptyset$. We show that $(K_x \cap I_y) \cap U = \emptyset$. Then by symmetry the same will be true for $K_y \cap I_x$, and hence for V.

$(K_x \cap I_y) \cap U = (K_x \cap I_y \cap K_x \cap K_y) \cup (K_x \cap I_y \cap I_x \cap I_y)$. But $K_x \cap I_y \cap K_x \cap K_y \subseteq I_y \cap K_y = \emptyset$. Similarly, $K_x \cap I_y \cap I_x \cap I_y = \emptyset$. □

Lemma 10. *If K is a regular basis and $x, y \in Total(A)$, then $l(x,y) = u(x,y)$.*

Proof. Using the notations of the previous proof we want to show that $\overline{U} \subseteq V$.

Let us show first that $K_x \cup I_x = K_y \cup I_y = K$. Consider $k \in K$. Since $x \in Total(A)$, if $k \notin I_x$, then $k \sqsubseteq x$. Now from the regularity of K we obtain $k \ll x$ and $k \in K_x$. Same for y.

Now, if $k \notin U$, then $k \notin K_x$ or $k \notin K_y$. Because of the symmetry it is enough to consider $k \notin K_x$. Then $k \in I_x$. Then, using $k \notin U$ once again, $k \notin I_y$. Then $k \in K_y$. So we have $k \in K_y \cap I_x \subseteq V$. □

Lemma 11 (Vickers-Matthews triangle inequality for upper bounds). $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$.

Proof. We want to show $1 - W(K_x \cap K_z) - W(I_x \cap I_z) \leq 1 - W(K_x \cap K_y) - W(I_x \cap I_y) + 1 - W(K_y \cap K_z) - W(I_y \cap I_z) - 1 + W(K_y) + W(I_y)$. This is equivalent to $W(K_x \cap K_y) + W(K_y \cap K_z) + W(I_x \cap I_y) + W(I_y \cap I_z) \leq W(K_y) + W(I_y) + W(K_x \cap K_z) + W(I_x \cap I_z)$.

Notice that $W(K_x \cap K_y) + W(K_y \cap K_z) = W(K_x \cap K_y \cap K_z) + W(K_y \cap (K_x \cup K_z))$, and the similar formula holds for I's.

Then the result follows from the following simple facts: $W(K_x \cap K_y \cap K_z) \leq W(K_x \cap K_z)$, $W(K_y \cap (K_x \cup K_z)) \leq W(K_y)$, and the similar inequalities for I's. □

Lemma 12. *Function $\rho : A \times A \to R^I$ is continuous.*

Proof. Monotonicity of ρ is trivial.

Consider a directed set $B \subseteq A$ and some $z \in A$. We have to show that $\rho(z, \sqcup B) = \sqcup_{R^I}\{\rho(z,x) \mid x \in B\}$, which is equivalent to $u(z, \sqcup B) = \inf\{u(z,x) \mid x \in B\}$ and $l(z, \sqcup B) = \sup\{l(z,x) \mid x \in B\}$.

Rewriting this, we want to show that $W(K_z \cap K_{\sqcup B}) + W(I_z \cap I_{\sqcup B}) = \sup\{W(K_z \cap K_x) + W(I_z \cap I_x) \mid x \in B\}$ and $W(K_z \cap I_{\sqcup B}) + W(I_z \cap K_{\sqcup B}) = \sup\{W(K_z \cap I_x) + W(I_z \cap K_x) | \ x \in B\}$. Monotonicity considerations trivially yield both "$\geq$" inequalities, so it is enough to show "$\leq$" inequalities. In fact, we will show that for any sets $C \subseteq A$ and $D \subseteq A$, $W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B}) \leq \sup\{W(C \cap K_x) + W(D \cap I_x) \mid x \in B\}$ holds.

It is easy to show that $K_{\sqcup B} = \cup\{K_x \mid x \in B\}$ by showing first that the set $\cup\{K_x \mid x \in B\}$ is directed and $\sqcup B = \sqcup(\cup\{K_x \mid x \in B\})$. Let us prove that $I_{\sqcup B} = \cup\{I_x \mid x \in B\}$. "$\supseteq$" is trivial. Let us prove "$\subseteq$". Assume that $k \notin \cup\{I_x \mid x \in B\}$, i.e. $\forall x \in B \; k \sqcup x$ exists. It is easy to see that because B is a directed set, $\{k \sqcup x \mid x \in B\}$ is also directed. Then $k \sqsubseteq \sqcup\{k \sqcup x \mid x \in B\} \sqsupseteq \sqcup B$, implying existence of $k \sqcup (\sqcup B)$ and, hence, $k \notin I_{\sqcup B}$.

Now consider enumerations of the countable or finite sets $K_{\sqcup B}$ and $I_{\sqcup B}$: $k_1, ..., k_n, ...$ and $k'_1, ..., k'_n, ...$, respectively. Define tail sums $S_n = w(k_n) + w(k_{n+1}) + ...$ and $S'_n = w(k'_n) + w(k'_{n+1}) + ...$ Observe that (S_n) and (S'_n) converge to 0.

Pick for every k_n some $x_n \in B$ such that $k_n \in K_{x_n}$. Pick for every k'_n some $x'_n \in B$ such that $k'_n \in I_{x'_n}$. Then using the directness of B, we can for any n pick such $y_n \in B$, that $x_1, ..., x_n, x'_1, ..., x'_n \sqsubseteq y_n$. Then $k_1, ..., k_n \in K_{y_n}$ and $k'_1, ..., k'_n \in I_{y_n}$. It is easy to see that $(W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B})) - (W(C \cap K_{y_n}) + W(D \cap I_{y_n})) = W(C \cap K_{\sqcup B}) - W(C \cap K_{y_n}) + W(D \cap I_{\sqcup B}) - W(D \cap I_{y_n}) < S_n + S'_n$, which can be made as small as we like. □

Lemma 13. *If $B \subseteq A$ is Scott open then it is relaxed metric open.*

Proof. Consider $x \in B$. Because B is Scott open there is a basic element $k \in B$ such that $k \ll x$. We must find $\epsilon > 0$ such that $x \in B_{x,\epsilon} \subseteq B$. Let $\epsilon = u(x,x) + w(k)/2$. Clearly $x \in B_{x,\epsilon}$. We claim that $B_{x,\epsilon} \subseteq \{y | y \gg k\} \subseteq B$. Assume, by contradiction, that $y \gg k$ is false. Then $k \notin K_y$ and $k \in K_x$. Then $W(K_x \cap K_y) + W(I_x \cap I_y) + w(k) \leq W(K_x) + W(I_x)$. Then $u(x,y) - w(k) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) - w(k) \geq 1 - W(K_x) - W(I_x) = u(x,x)$. Therefore $u(x,y) \geq u(x,x) + w(k)$ and thus $y \notin B_{x,\epsilon}$. □

7 Computability and Continuity

We do not have space for a proper discussion on effective structures on domains (see [1] for some of it). We have to ask the reader to take on faith that whenever one expects element x to be computable, it is reasonable to impose the requirement that K_x and I_x must be recursively enumerable, but one almost never should expect them to be recursive.

The actual computation of $\rho(x, y)$ goes as follows. Start with $[0, 1]$ and go along the recursive enumerations of K_x, K_y, I_x, and I_y. Whenever we discover that some k occurs in both K_x and K_y, or in both I_x and I_y, subtract $w(k)$ from the upper boundary. Whenever we discover that some k occurs in both K_x and I_y, or in both I_x and K_y, add $w(k)$ to the lower boundary. If this process continues long enough, $[l(x, y), u(x, y)]$ is approximated as well as desired.

However, there is no general way to compute a better lower estimate for $u(x, y)$ than $l(x, y)$, or to compute a better upper estimate for $l(x, y)$ than $u(x, y)$. Consequently, there is no general way to determine how close is the convergence process to the actual values of $l(x, y)$ and $u(x, y)$, except that we know that $u(x, y)$ is not less than currently computed lower bound, and $l(x, y)$ is not greater than currently computed upper bound. Of course, for large x and y this knowledge might provide a lot of information, and if the basis of our domain is regular, for total elements x and y this knowledge provides us with precise estimates — i.e. if the basis is regular, then the resulting classical metric on $Total(A)$ can be nicely computed.

The computational situation is very different with regard to quasi-metrics. Consider $u(x, y) = 1 - W(K_x \cap K_y)$ and $d(x, y) = u(x, y) - u(x, x) = W(K_x \setminus K_y)$. This is a quasi-metric in the style of [9, 4], and it yields a Scott topology [1]. However, as discussed in [1], typically $K_x \setminus K_y$ is not recursive. Moreover, one should not expect $K_x \setminus K_y$ or its complement to be recursively enumerable. This precludes us from building a generally applicable method computing $d(x, y)$ and illustrates that it is computationally incorrect to subtract one upper bound from another.

8 Some Open Issues in the Theory of Relaxed Metrics

8.1 Should the Axioms of Partial Metrics Hold?

Consider relaxed metric ρ and its upper part u. Should we expect function u to satisfy the axioms of partial metrics? Example 1 describes a natural relaxed metric on interval numbers, where u is not a partial metric. Function u gives a better upper estimate of the "ideal" distance between interval numbers, than the partial metric $p([a, b], [c, d]) = \max(b, d) - \min(a, c)$ described in [7]. For example, $u([0, 2], [1, 1]) = 1$, which is what one expects — if we know that one of the numbers is somewhere between 0 and 2, and another number equals 1, then we know that the distance between them is no greater than 1. However, $p([0, 2], [1, 1]) = p([0, 2], [0, 2]) = 2$.

Now we describe two situations when the axioms of partial metrics are justified. Consider again function $d(x,y) = u(x,y) - u(x,x)$ from Sect. 7. Vickers notes in [11] that the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ is equivalent to $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$, and $d(x,y) \geq 0$ is equivalent to $u(x,x) \leq u(x,y)$. This means that function u is a partial metric if and only if function d is a quasi-metric.

Another justification comes from the consideration of the proof of Lemma 11. Whenever the upper part $u(x,y)$ of a relaxed metric is based on *common information* shared by x and y yielding a *negative contribution* to the distance (we subtract the weight of common information from the universal distance 1 in this paper), both $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$ and $u(x,x) \leq u(x,y)$ should hold.

However, to specify function u in Example 1, we use information about x and y, which cannot be thought of as common information shared by x and y. In such case we still expect a relaxed metric ρ, but its upper part u does not have to satisfy the axioms of partial metrics.

8.2 Potential Relationship with Measure Theory

Important applications of domains to measure theory have emerged recently [2]. The construction developed in this paper suggests that fruitful applications in the opposite direction are possible. For example, in order to compute upper bound $u(x,y)$ we measure common information shared by x and y and subtract it from the universal distance 1. Our way to measure this information is fairly primitive — it is based on assigning a converging system of weights to a countable set of basic points and adding weights of basic points belonging to the sets of interest. This is similar to measuring areas on the plane via assigning a converging system of finite weights to points with rational coordinates and adding weights of such points belonging to the sets of interest. We should be able to do better, at least for domains without many compact elements.

9 Conclusion

Let us briefly state where we stand with regard to the applications to programs. We are able to introduce relaxed metrics on a class of domains sufficiently large for practical applications in the spirit of [10].

For example, consider $X = [\![\texttt{while B do S}]\!]$, and the sequence of programs, P_1 = `loop`;...;P_N = `if B then S`;P_{N-1}`else skip endif`;.... Define $X_N = [\![P_N]\!]$. Typically $X_{N-1} \sqsubseteq X_N$ and $X = \sqcup X_N$ hold. We agreed that the distances between programs will be distances between their meanings. Assume that $M \leq N$. Then regardless of specific weights, $u(P_M, P_M) = u(P_M, P_N) = u(P_M, P)$, also $u(P_M, P_M) \geq u(P_N, P_N) \geq u(P, P)$, and $u(P, P) = \inf u(P_N, P_N)$. Of course, none of these distances has to be zero.

However, we do not yet know how to build relaxed distances so that not only nice convergence properties are true, but also that distances between particular pairs of programs "look right" — a notion, which is more difficult to formalize, than convergence. Also, we compute these distances via recursive enumeration now, and a more efficient scheme is needed.

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