Physics 111B: [Non-Linear Dynamics and Chaos]

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In the field of nonlinear dynamics (NLD) and chaos, we present a structure for studying intricate systems whose sensitives to initial conditions and characteristic unpredictability govern their complex behavior. This lab applies a variety of LabView programs that allow us to experiment and visualize the certain parameters that drastically affect the outcome of a system. The two real-time continuous NLD systems used in this lab are the PN-Junction and Bouncing Ball circuits. The time-series, power spectra, return maps, and bifurcation diagrams of these circuits are analyzed to understand inherit features of chaotic systems holistically. Since a lot of physical interactions in real-life are non-linear, our observations provide us with an important understanding of how to analyze and model these convoluted systems.

I. INTRODUCTION

A. Non-Linear Dynamics

Linear systems are governed by a general rule that some change in the input (position, voltage, etc..) is proportional to a change in the output (force, current, etc..). One example of of a linear system is a mass on a spring with a driving force:

$$m\ddot{x} = -kx + F(t) \tag{1}$$

where x is position, m is mass, k is the spring, and F(t) is the time-dependent driving force.

In contrast, a change in the input of a non-linear system is not proportional to a change in output. A simple example of a non-linear, conservative system is an oscillating pendulum, as represented by Figure 1. The equation of motion for this system can be written by the following:

$$mL^2\ddot{\theta} = -mgLsin(\theta) \tag{2}$$

where m is mass, L is the constant length of the pendulum, θ is the angular displacement, and g is the gravitational acceleration. We observe that the equation of motion has a dependence on a trigonometric term $\sin(\theta)$; this is a characteristic feature of non-linear systems. For small θ, we can approximate a simple pendulum as linear. But as these dynamic systems evolve over time, their parameters change non-linearly.

B. Chaos

1. Initial Conditions

While chaotic systems depend on a non-linear equation of motion, non-linearity does not imply that the system

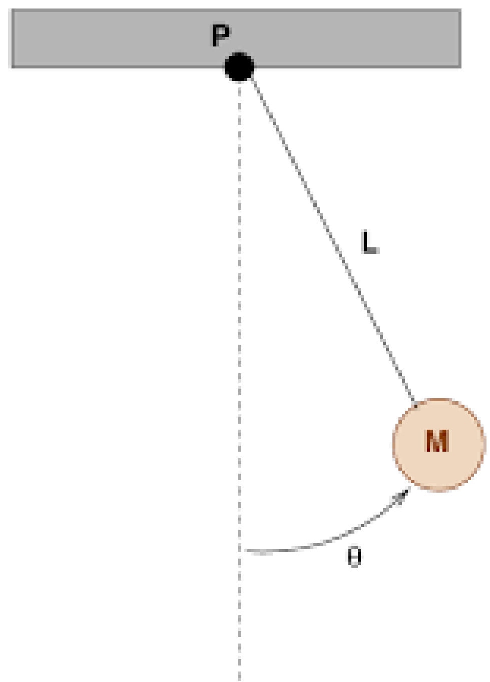

FIG. 1. A diagram of a simple oscillating pendulum with mass m, length L, and anglular displacement θ.

is chaotic. The simple pendulum in the last section is not-chaotic. However if were to apply a damping and driving force to the pendulum, then the pendulum is not longer "simple". The equation of motion for this damped, driven pendulum (DDP) would be in the form:

$$mL^2\ddot{\theta} = -mgLsin(\theta) - bL^2\dot{\theta} + LF(t) \tag{3}$$

where b is the damping coefficient (dependent on the angular velocity of the mass) and F(t) is the driving amplitude. A DDP is an example of a non-conservative (i.e.

dissipative) chaotic system because it is highly dependent on the initial conditions (position, damping, and driving amplitude). Any small change in any of the initial could result in an entirely different outcome. We will see later the PN Junction and Bouncing Ball experiment are examples of DDPs.

2. Liapunov Exponents and Error

Another method for determining whether is a state is chaotic is by measuring the Liapunov exponent λ. Let's assume that a system is represented by the function f(t, ζ). If we set the displacement after some time increment δ equal to an exponential function, that is:

$$|f(t,\zeta) - f(t,\zeta+\delta)| = \delta e^{\lambda t} \tag{4}$$

the system is chaotic if λ is positive. In other words, a system is chaotic if the normalized displacement of a system exponentially increases.

Sensitivity to initial conditions is an important characteristic of chaotic systems. Since our prediction of a system's state becomes harder as it temporally evolves, we can measure how long we can accurately measure the system's evolution to some tolerance. This is represented by the equation:

$$t_{horizon} \approx \frac{1}{\lambda} ln \frac{a}{|\delta_0|} \tag{5}$$

where a is the tolerance and δ_0 is the uncertainty in our initial conditions. If the uncertainty in the initial conditions is minimized, then we can accurately predict the evolution of the system (i.e. $t_{horizon}$) for a longer time. It also makes sense that $t_{horizon}$ and the Liapunov exponent have an inverse relationship because the more chaotic a system is, the less predictable it will become.

3. Phase Space and Return Maps

If a system is allowed to evolve with time, we can map out the various outcomes (i.e. trajectories) the system takes. The space that the trajectories confine themselves is called the phase space. This can be represented by 2-D position plots, velocity-position state spaces, or current-voltage characteristics. The set of states in which a system tends to evolve to is called the attractor.

A Return Map is a space that is 2-dimensionally mapped out by two discrete states x_n and x_{n+1}. A general function that is used to understand return maps is an inverse parabola, written as:

$$f(x) = Rx(1-x) \tag{6}$$

where R is the chaotic parameter. We will see in the LabView section why return maps are important to understanding chaotic systems.

4. PN Junction and Bifurcations

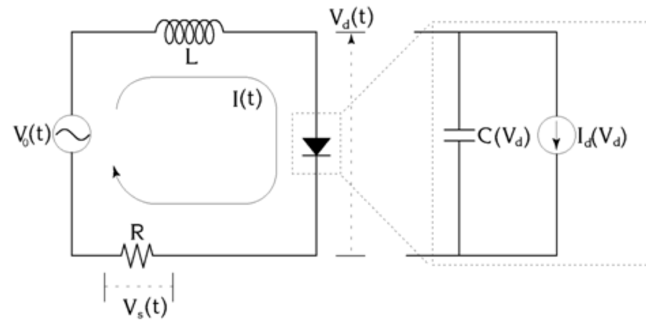

FIG. 2. A circuit diagram for the PN-Junction DDP with a schematic of the diode.

The circuit for the PN Junction is presented in Figure 2 and the equations of motion are the following:

$$\dot{I} = \frac{V_0 - RI - V_d}{L} \tag{7}$$

$$\dot{V_d} = \frac{I - I_D(V_d)}{C(V_d)} \tag{8}$$

where I the circuit's current, I_d is the current across the diode, V_d is the voltage across the diode, V_0 is the driving voltage, R is the resistance of the resistor, L is the inductance of the inductor, C(V_d) is the voltage-dependent capacitance of the capacitor.

V_0 is a non-linear driving voltage can be further expressed:

$$V_0 = V_{os} cos(w_0 t) \tag{9}$$

where V_{os} is the driving amplitude, and $w_0 \approx \frac{1}{\sqrt{LC_0}}$.

The voltage-dependent diode current is further expressed as:

$$I_d(V_d) = I_0[e^{\frac{eV_d}{kT}} - 1] \tag{10}$$

where e is the elementary charge, k is the Boltzmann constant, and T is the temperature of the diode.

For V_d¿0, the voltage-dependent capacitance is also expressed as:

$$C(V_d) = C_0 e^{\frac{eV_d}{kT}} \tag{11}$$

From the non-linearity in the equations of motions of themselves we can observe how the PN Junction represents a chaotic system.

For small values of V_{os} (i.e. driving amplitude, chaotic parameter), the phase space of the PN Junction is presented in the top left diagram of Figure 3. We observe

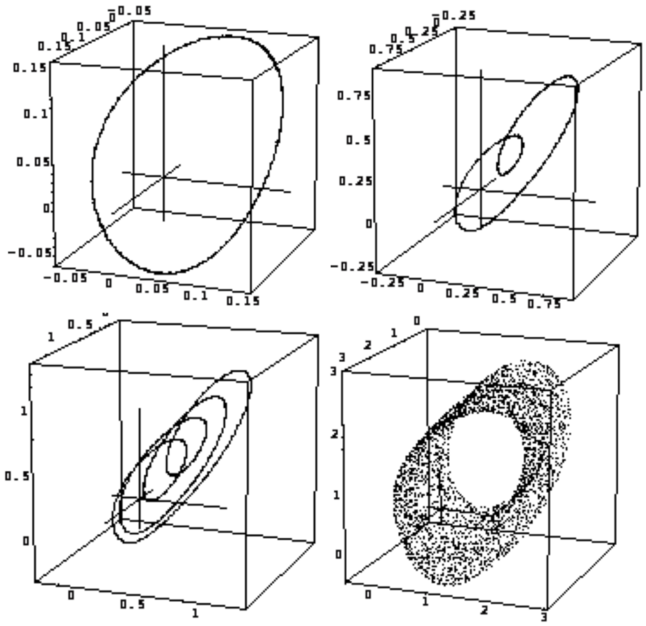

FIG. 3. Phase spaces of PN Junction where period doubling bifurcation (top right), period quadrupling bifurcation (bottom left), and chaos (bottom right) are observed.

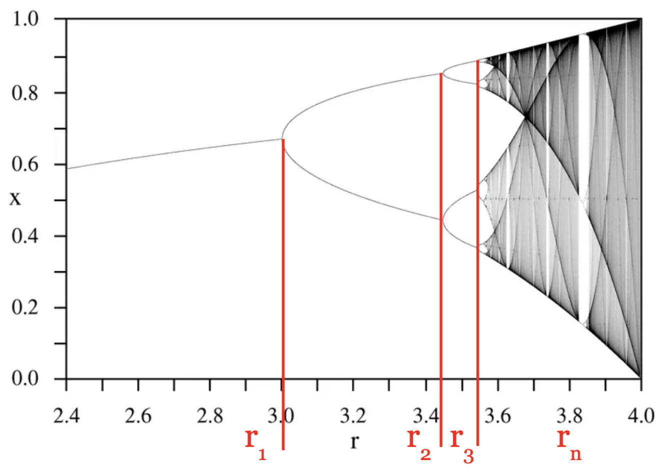

FIG. 4. Example of a bifurcation diagram. The labelled values of r_n represent the critical chaotic parameter values in which the system's period doubles (i.e. bifurcates).

the I-V phase space to oscillate periodically with time. As the chaotic parameter is increased, we see the period of the PN Junction doubles (top right of Figure 3) and quadruples (bottom left). When the chaotic parameters reaches some value, the system is in a state of chaos where all states are simultaneously occupied along the system's attractor. This process of period evolution is known as bifurcation.

A bifurcation diagram is represented in Figure 4. We observe that the displacement between each successive bifurcation decreases. The ratio of displacements between each successive bifurcation (i.e. r_n) is given by the equation:

$$\delta_F = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669 \tag{12}$$

where δ_F is known as the Feigenbaum constant.

5. *Time Domain and Fourier Transforms*

The equations of motion for simple, non-linear systems (e.g. simple harmonic oscillator) obey periodic functions (i.e. cos(x)). For periodic systems in the time-domain space, we can Fourier transform the system to uncover a characteristic frequency. This allows us to plot the system from time-domain to frequency domain.

For more complicated non-linear systems (i.e. small amplitude DDP's), we can observe the system in periodicity in several spaces. The characteristic frequencies of each of these phases can be mapped on a frequency spectrum. A Fourier Transform of a chaotic (i.e. large amplitude DDP's) system will have a frequency spectrum with peaks at several different frequencies.

The systems mentioned in this section are mapped in position-time space. However the PN Junction and Bouncing Ball system used in this experiment are mapped in current-voltage space, where a Fourier Transform would produce a power spectrum, which is very similar to a frequency spectrum.

C. LabView

All of the computational programs used in this experiment were ran through LabView, a visually programming language where an input an be manipulated to a produced a desired output. The experimented consisted of a mix of pre-made LabView programs and programs that my lab partner and I had to create and implement ourselves.

1. *Cobweb Analyzer*

The first major program was the Cobweb Analyzer. This program takes in two function, a chaotic parameter, and an initial guess to try to find a solution to the two equations. In Figure 5, the two equations are:

$$f(x) = Rx(1 - x) \tag{13}$$

$$g(x) = x \tag{14}$$

where f(x) is a parabola and R is the chaotic parameter. We see that given some initial x guess, the analyzer is able to converge to a solution of f(x)=g(x).

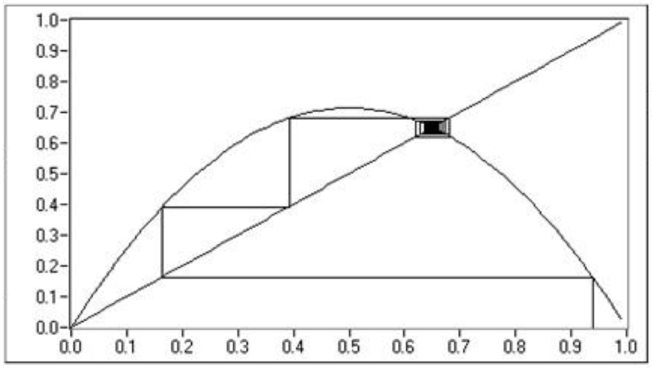

FIG. 5. LabView Cobweb Analyzer program that takes in an initial guess and tries to find a solution to system. In this specific case, the analyzer converges to the solution and the system well-defined.

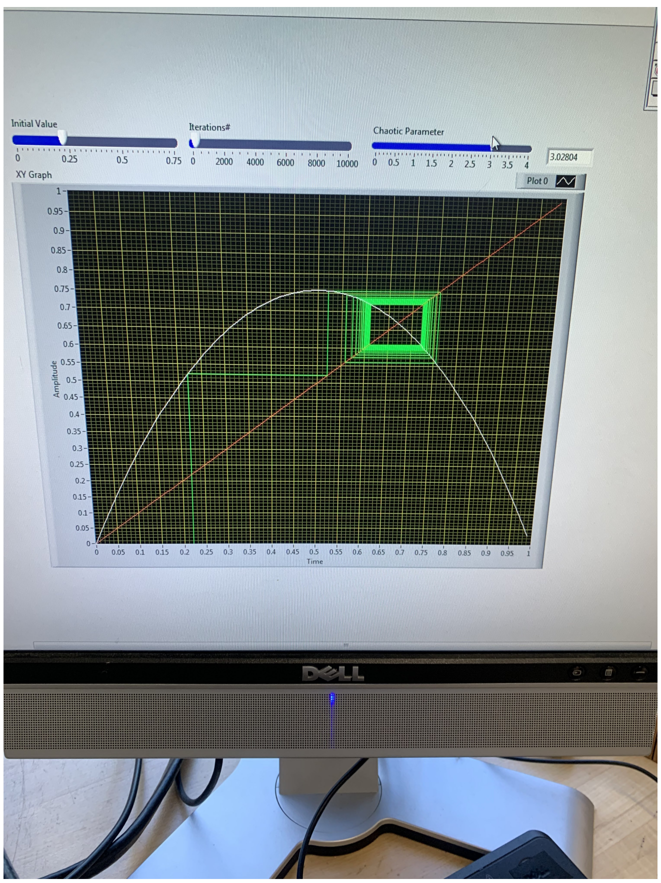

FIG. 6. LabView Cobweb Analyzer program where f(x) = Rsin(x). We see that at some given initial guess x, the program is not able to find a solution.

In trying to understand the LabView apparatus, my partner and I varied f(x) to equal Rsin(x). Figure 6 showed the program trying to find a solution for Rsin(x) = x given some inital guess; the system isn't able to find a solution and diverges.

2. *NLDGEN*

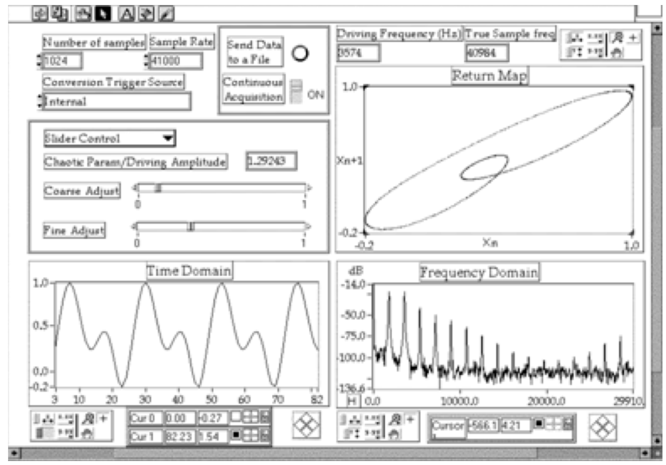

FIG. 7. LabView "nldgen" program that outputs a return map, frequency-domain, and time-domain diagram for a system.

3. *NLDBIFUR*

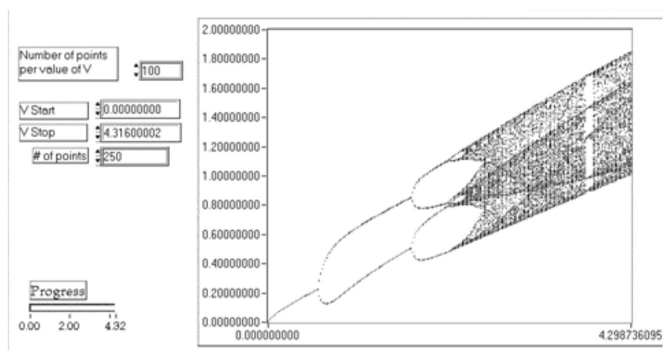

FIG. 8. LabView "nldbifur" program that samples a range of voltages for a system and outputs a bifurcation diagram.

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II. EXPERIMENTAL PROCEDURES

The instruments used in this experiment is the SRS DS 345 Signal Function Generator, +15V/-15V/+5V Power Supply, DAQ box, and Bouncing Ball/Lorentz Attractor NLD Apparatus. The wired set up for the PN Junction is presented in Figure 9 and the set up for the Bouncing Ball is in Figure 10.

In general, the procedure was to simply make sure all the connections were wired up properly and the function generator had the correct frequency and amplitude so that the computer reads out the correct results. We are also allowed to adjust the chaotic parameter/driving amplitude on the NLDGEN program window and the gain on the NLD apparatus.

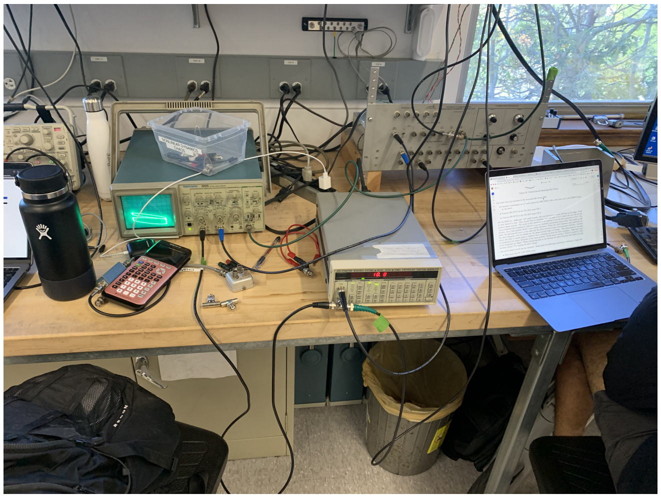

FIG. 9. The laboratory apparatus for the PN Junction experiment.

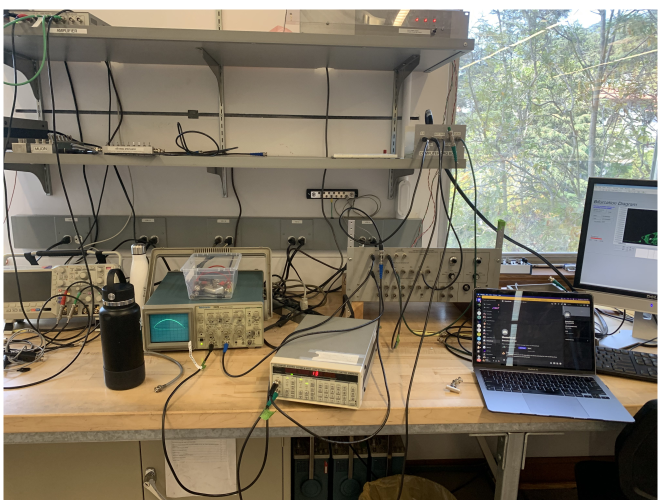

FIG. 10. The laboratory apparatus for the Bouncing Ball experiment.

III. RESULTS & ANALYSIS

This section will go over the analysis of each experimental system and various measurements of the Feigenbaum ratio. The chaotic parameters in which each bifurcation was made was measured both with the return map and bifurcation diagram.

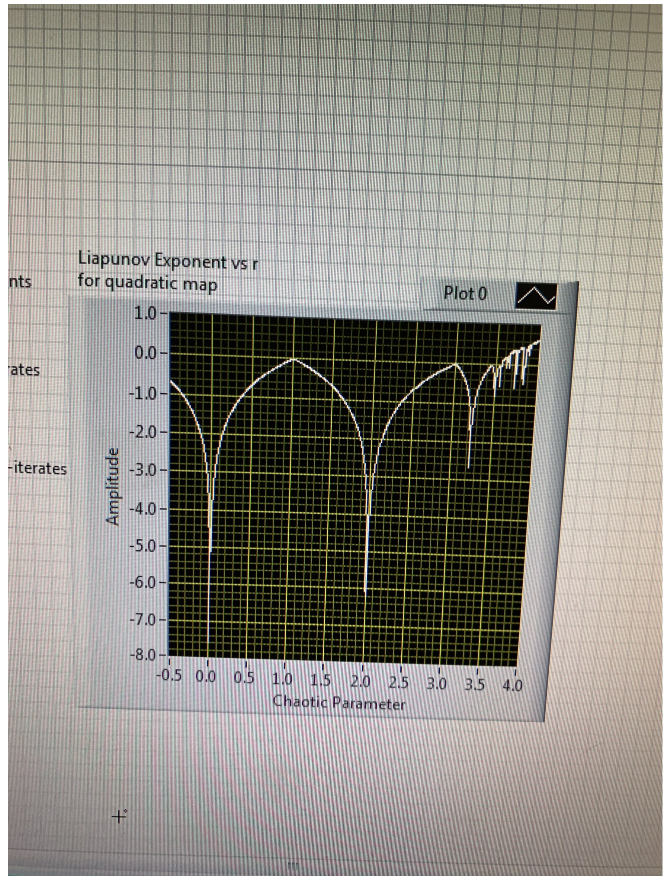

FIG. 11. A plot of the Lyapunov exponent vs. the chaotic parameter.

A. Lyapunov

Figure 11 is a plot of the Lyapunov exponent versus the chaotic paramter/driving amplitude. Recalling the introductory section, a system exhibits chaotic behavior when the Lyapunov exponent λ is positive. The values of chaotic parameters where the Lyapunov exponent equals zero is where the system bifurcates. As we continue increasing the chaotic parameter, the graph shifts above $\lambda > 0$, representing how large driving amplitudes result in chaos.

The Feigenbaum ratio for Figure 11 was measured to be about $\delta \approx 4.65$. This was done by measuring where the system bifurcates and using the Feigenbaum ratio formula.

B. PN Junction

The first section of this paper goes over the derivation of the PN Junction and how varying the driving amplitude (i.e. chaotic parameters) results in a chaotic system. The PN Junction is an example of a dissipative (i.e. non-conservative) system because the total change energy of

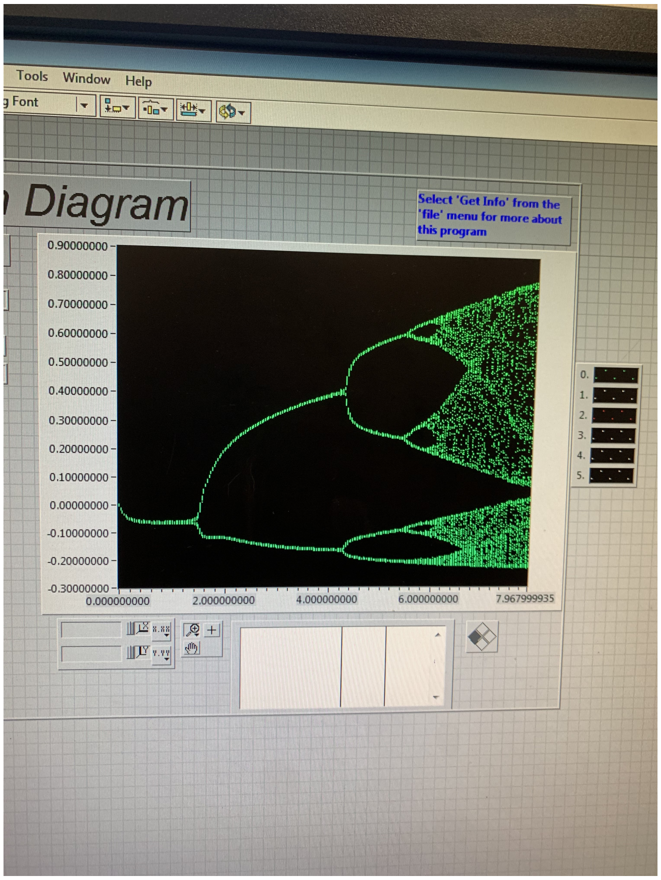

FIG. 12. The bifurcation diagram of the PN Junction.

the system is non-zero.

The bifurcation diagram of the PN Junction is presented in Figure 12. From this diagram, we measured $\delta \approx 3.19$ and $\alpha \approx 1.54$ (width of the bifurcations).

Since δ should be about 4.67, we suspect our errors comes from the lack of resolution to higher order bifurcations; if we were able to see the more fine bifurcations we theoretically could've calculated a more accurate Feigenbaum ratio.

C. Bouncing Ball

D. PN Junction

The Bouncing Ball experiment is an electronic representation of a ball bouncing on a vertically oscillating platform, where frequency is how many times the table oscillates per second and the driving amplitude is how hard the table hits the ball. This is also another example of a non-conservative system because energy is being applied to the ball.

The current-voltage characteristic in Figure 14 is a digital analog to a phase space diagram, where we can replace the axes with velocity-position. At 90Hz, we did

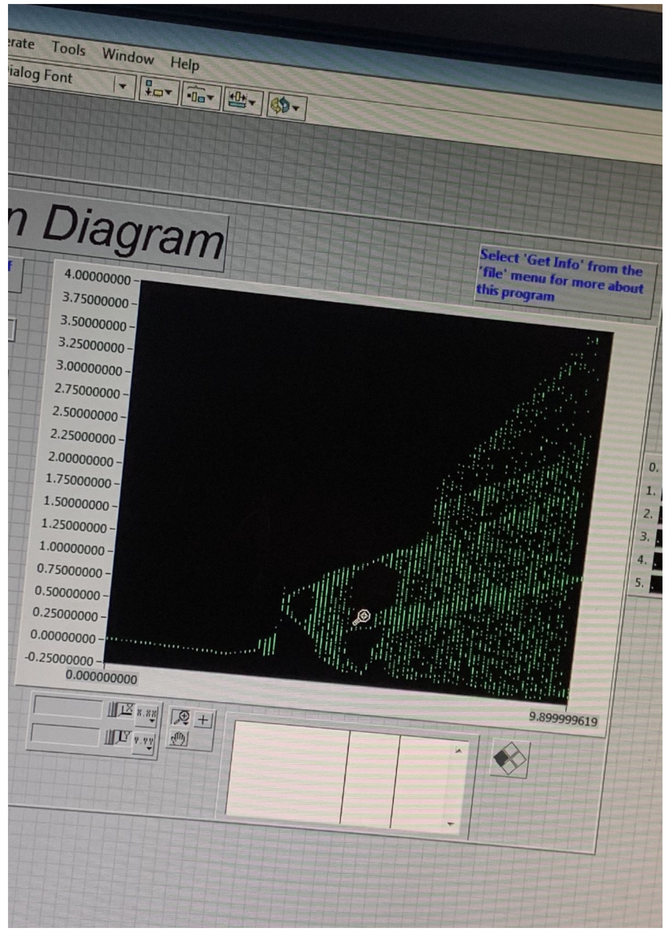

FIG. 13. The bifurcation diagram of the Bouncing Ball at 130Hz.

not observe any bifurcations as the driving amplitude was varied. However, when increasing the frequency to 130Hz, we observe asymmetric bifurcations where the system didn't bifurcate evenly into 2^n periods. For large driving amplitudes, all of the phase space was occupied.

Figure 14 presents the assymmetric nature of the Bouncing Ball. This also made measurements of δ and α impractical.

IV. CONCLUSION

We were able to calculate the Feigenbaum ratio for the with Lyapunov LabView Program and the PN Junction by measuring where the system bifurcates. Our analysis of chaotic systems and their sensitive dependence to initial conditions and non-linearity was well-demonstrated in the Cobweb, NLDGEN, and NLDBIF programs. Since non-linear dynamics and chaos have real-life applications (turbulent flow, plasma physics, stellar clusters), this experiment gave us much insight into modelling reality.

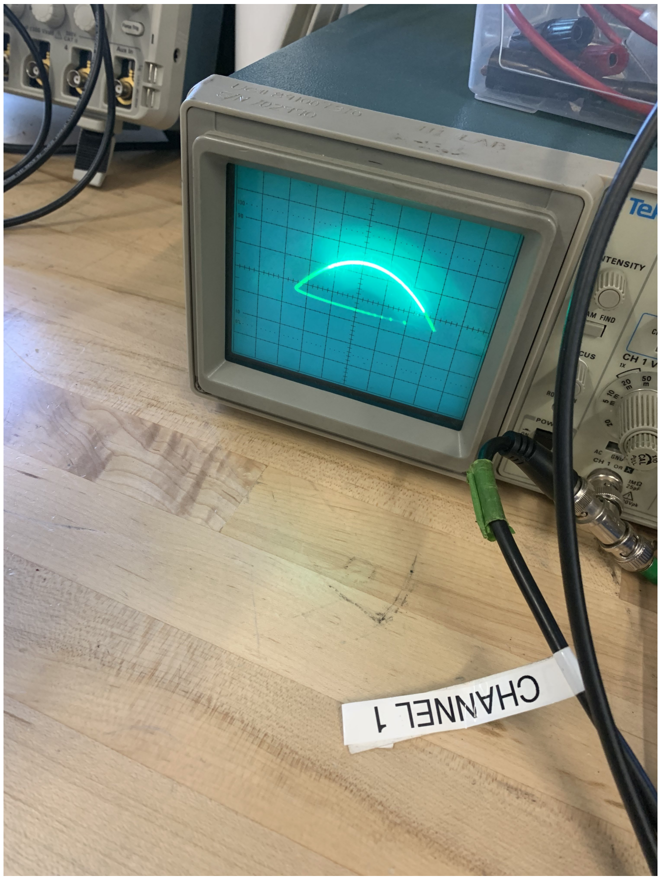

FIG. 14. The I-V characteristic of the Bouncing Ball experiment.