Derivation of Local Dynamical Indices

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Suppose the state of interest in the Lorenz system is $\varsigma_{\mathbf{x}} = \mathbf{x}(t = t_\varsigma)$. To analyse growth rates of trajectories, $\mathbf{x}(t)$, within a neighborhood of $\varsigma_{\mathbf{x}}$, I can calculate the negative logarithmic returns as:

$$g(\varsigma_{\mathbf{x}}, \mathbf{x}(t)) = -\log\left[\mathrm{dist}(\varsigma_{\mathbf{x}}, \mathbf{x}(t))\right]$$

The trajectory, $\mathbf{x}(t)$, falls within a neighborhood of $\varsigma_{\mathbf{x}}$ when the time series $g(\varsigma_{\mathbf{x}}, \mathbf{x}(t))$ is above a given threshold $s(q, \varsigma_{\mathbf{x}})$, which is the qth quantile $g(\varsigma_{\mathbf{x}}, \mathbf{x}(t))$.The exceedances of the neighbouring states of $\varsigma_{\mathbf{x}}$ is given as:

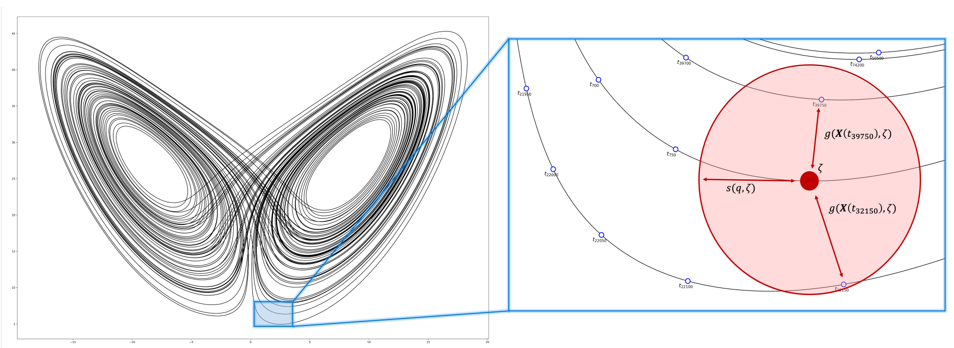
$$U(\varsigma_{\mathbf{x}}) = g(\varsigma_{\mathbf{x}}, \mathbf{x}(t)) - s(q, \varsigma_{\mathbf{x}}), \forall g(\varsigma_{\mathbf{x}}, \mathbf{x}(t)) > s(q, \varsigma_{\mathbf{x}})$$

The interpretation of the above quantities for the idealised Lorenz '63 attractor is illustrated graphically below:

Assuming the independence of the exceedances, the cumulative probability distribution, $F(U, \varsigma_{\mathbf{x}})$, converges to the exponential member of the generalised Pareto distribution, i.e.:

$$F(U, \varsigma_{\mathbf{x}}) \sim \exp\left[-\theta \frac{U(\varsigma_{\mathbf{x}})}{\sigma(\varsigma_{\mathbf{x}})}\right]$$

U and σ are parameters of the distribution and both depend on the state $\varsigma_{\mathbf{x}}$. The **Extremal Index** is given by θ, while the **Local Dimension** is given by:



$$d(\varsigma_{\mathbf{x}}) = \frac{1}{\sigma(\varsigma_{\mathbf{x}})}$$

Finally, we define the co-recurrence ratio by considering two trajectories $\mathbf{x}(t)$, and $\mathbf{y}(t)$ and a corresponding joint state of interest, $\varsigma = (\varsigma_{\mathbf{x}}, \varsigma_{\mathbf{y}})$. The **Predictability** is then given by:

$$\alpha(\varsigma) = \frac{\nu[g(\mathbf{x}(t)) > s_{\mathbf{x}}(q) | g(\mathbf{y}(t)) > s_{\mathbf{y}}(q)]}{\nu[g(\mathbf{x}(t)) > s_{\mathbf{x}}(q)]}$$

ν[...] is the number of events satisfying the condition [...]

This derivation is written with the help of the following article which is available online:

Characterising and comparing different palaeoclimates with dynamical systems theory D. Faranda and G. Messori *European Geosciences Union*, 17(1): 545–563, 2021 `https://cp.copernicus.org/articles/17/545/2021`