Matrix Analysis - Review

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0 Review and miscellanea

0.1 Vector spaces

- If S_1, S_2 are two subspaces of V, then

$$\dim(S_1 \cap S_2) + \dim(S_1 + S_2) = \dim S_1 + \dim S_2$$

$$\dim(S_1 \cap S_2) \geq \dim S_1 + \dim S_2 - \dim V$$

0.2 Matrices

- $\dim(\text{range } A) + \dim(\text{nullspace } A) = \text{rank } A + \text{nullity } A = n$
- $(AB)^* = B^*A^*, \quad (AB)^T = A^TB^T, \quad \overline{AB} = \overline{A}.\overline{B}$
- $(y^*x)^* = \overline{y^*x} = x^*y = y^T\overline{x}$
- Some definitions:

symmetric	$A^T = A$	*skew symmetric*	$A^T = -A$
orthogonal	$A^TA = I$	*skew Hermitian*	$A^* = -A$
Hermitian	$A^* = A$	*essentially Hermitian*	$\exists\theta \in \mathbb{R}$:$e^{i\theta}A$ is Hermitian
unitary	$A^*A = I$	*normal*	$A^*A = AA^*$

- Each $A \in M_n(\mathbb{C})$ can be written in exactly one way as $A = H(A) + iK(A)$, in which $H(A), K(A)$ are Hermitian.
- $\text{tr}AA^* = \text{tr}A^*A = \sum_{i,j} |a_{ij}|^2, A \in M_n(\mathbb{C})$
- range $A +$ range $B =$ range $[A \quad B]$
- nullspace $A \ \cap$ nullspace $B =$ nullspace $\begin{bmatrix} A \\ B \end{bmatrix}$

0.3 Determinants

- **Laplace expansion by minors along a row or column**

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik} = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj}$$

0.4 Rank

0.4.1 Rank inequalities

- If $A \in M_{m,n}(\mathbb{F})$ then: rank $A \leq \min\{m, n\}$
- If $A \in M_{m,k}(\mathbb{F}), B \in M_{k,n}(\mathbb{F})$ then:
$$(\text{rank } A + \text{rank } B) - k \leq \text{rank } AB \leq \min\{\text{rank } A, \text{rank } B\}$$
- If $A, B \in M_{m,n}(\mathbb{F})$ then:
$$|\text{rank } A - \text{rank } B| \leq \text{rank } (A + B) \leq \text{rank } A + \text{rank } B$$
- If rank $B = 1$ then (*changing one entry of a matrix can change its rank by at most 1*): $|\text{rank } (A + B) - \text{rank } A| \leq 1$
- (Frobenius inequality): If $A \in M_{m,k}(\mathbb{F}), B \in M_{k,p}(\mathbb{F}), C \in M_{p,n}(\mathbb{F})$ then:
$$\text{rank } AB + \text{rank } BC \leq \text{rank } B + \text{rank } ABC$$
with equality if and only if there are matrices X, Y such that $B = BCX + YAB$

0.4.2 Rank equalities

- If $A \in M_{m,n}(\mathbb{C})$, then : rank $A^* =$ rank $A^T =$ rank $\overline{A} =$ rank A
- If $A \in M_m(\mathbb{F})$ and $C \in M_n(\mathbb{F})$ are nonsingular and $B \in M_{m,n}(\mathbb{F})$, then: rank $AB =$ rank $B =$ rank $BC =$ rank ABC
- If $A, B \in M_{m,n}(\mathbb{F})$, then rank $A =$ rank $B \Leftrightarrow$ there exist a nonsingular $X \in M_m(\mathbb{F})$ and a nonsingular $Y \in M_n(\mathbb{F})$ such that $B = XAY$
- If $A \in M_{m,n}(\mathbb{C})$, then: rank $A^*A =$ rank A

- If $A \in M_{m,n}(\mathbb{F})$, then rank $A = k \Leftrightarrow A = XY^T$ for some $X \in M_{m,k}(\mathbb{F})$ and $Y \in M_{n,k)}(\mathbb{F})$ that each have independent columns

- rank $A = k \iff \exists$ nonsingular matrices $S \in M_n(\mathbb{F})$ and $T \in M_n(\mathbb{F})$ such that $A = S \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T$

- Let $A \in M_{m,n}(\mathbb{F})$. If $X \in M_{n,k}(\mathbb{F})$ and $Y \in M_{m,k}(\mathbb{F})$ and if $W = Y^T AX$ is nonsingular, then:

$$\text{rank } (A - AXW^{-1}Y^T A) = \text{rank } A - \text{rank } AXW^{-1}Y^T A$$

When $k = 1$ (*Wedderburn's rank-one redution formula*): If $x \in \mathbb{F}^n$ and $y \in \mathbb{F}^m$, and if $\omega = y^T Ax \neq 0$, then:

$$\text{rank } (A - \omega^{-1} Axy^T A) = \text{rank } A - 1$$

Conversely, if $\sigma \in \mathbb{F}, u \in \mathbb{F}^n, v \in \mathbb{F}^m$, and rank $(A - \sigma uv^T) <$ rank A, then: rank $(A - \sigma uv^T) =$ rank $A - 1$ and there are $x \in \mathbb{F}^n, y \in \mathbb{F}^m$ such that $u = Ax, v = A^T y, y^T Ax \neq 0$, and $\sigma = (y^T Ax)^{-1}$

0.5 Nonsingularity

- $(A^{-1})^T = (A^T)^{-1}$

0.6 The Euclidean inner product and norm

- $\langle x, y \rangle = y^* x, \qquad \|x\|_2 = \langle x, x \rangle = (x * x)^{1/2}$

- $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$, $\langle x, \alpha y_1 + \beta y_2 \rangle = \overline{\alpha} \langle x, y_1 \rangle + \overline{\beta} \langle x, y_2 \rangle$

0.7 Partitioned sets and matrices

0.7.1 The inverse of a partitioned matrix

- $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B - D)^{-1} \\ D^{-1}C(BD^{-1}C - A)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & (BD^{-1}C - A)^{-1}BD^{-}1 \\ (CA^{-1}B - D)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

assume that the relevent inverses exist.

0.7.2 The Sherman-Morrion-Woddbury formula

Let $A \in M_n(\mathbb{F})$ be nonsingular and $B = A + XRY, X \in M_{n,r}(\mathbb{F}), Y \in M_{r,n}(\mathbb{F}), R \in M_{r,r}(\mathbb{F})$. If B and $R^{-1} + YA^{-1}X$ are nonsingular, then:

$$B^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}$$

If $r \ll n$, then R and $R^{-1} + YA^{-1}X$ may be much easier to invert than B. If $x, y \in \mathbb{F}^n$ are nonzero vectors, $X = x, Y = y^T y^T A^{-1}x \neq 0$, and $R = [1]$ then:

$$(A + xy^T)^{-1} = A^{-1} - (1 + y^T A^{-1}x)^{-1} A^{-1}xy^T A^{-1}$$

In particular, if $B = I + xy^T$ for $x, y \in \mathbb{F}^n$ and $y^T x \neq -1$, then

$$B^{-1} = I - (1 + y^T x)^{-1} xy^T$$

0.7.3 Complementary nullities

Let $A \in M_n(\mathbb{F})$ is nonsingular. The *law of complementary nullities* is:

$$\text{nullity}(A[\alpha, \beta]) = \text{nullity}(A^{-1}[\beta^c, \alpha^c])$$

which is equivalent to the rank identity:

$$\text{rank } (A[\alpha, \beta]) = \text{rank } (A^{-1}[\beta^c, \alpha^c]) + r - s - n, \quad r = |\alpha|, s = |\beta|$$

0.7.4 Rank in a partitioned matrix and rank-principal matrices

- $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, A_{11} \in M_r(\mathbb{F}), A_{22} \in M_{n-r}(\mathbb{F})$. If A_{11} is nonsingular, then

$$\text{rank } \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} = \text{rank } \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = r$$

- The converse is true:
if rank $A_{11} =$ rank $\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} =$ rank $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$, then A_{11} is nonsingular.

0.8 Determinants again

0.8.1 The adjugate and the inverse

- If $A \in M_n(\mathbb{F}), n \geq 2$, the *adjugate* of A is: adj $A = [(-1)^{i+j} \det A[\{j\}^c, \{i\}^c]]$
- (adj $A)A = A($adj $A) = (\det A)I$, and $\det($adj $A) = (\det A)^{n-1}$
- adj $(A^{-1}) = A/ \det A = ($adj $A)^{-1}$
- If rank $A \leq n - 2$, then adj $A = 0$
- If rank $A = n - 1$ then rank adj $A = 1$. Suppose adj $A = \alpha xy^T$ for some $\alpha \in \mathbb{F}$ and nonzero $x, y \in \mathbb{F}^n$. From:

$$(Ax)y^T = A(\text{adj } A) = 0 = (\text{adj } A)A = x(y^T A)$$

we conclude that: $Ax = 0$ and $y^T A = 0$

- adj $(AB) = (\text{adj } A)(\text{adj } B)$ for all $A, B \in M_n$
- If A is nonsingular, then:

$$\begin{array}{rclcl} \text{adj }(\text{adj } A) & = & \text{adj }((\det A)A^{-1}) & = & (\det A)^{n-1}\text{adj } A^{-1} \\ & = & (\det A)^{n-1}(A/\det A) & = & (\det A)^{n-2}A \end{array}$$

- If $A+B$ is nonsingular, then : $A(A+B)^{-1}B = B(A+B)^{-1}A$, so continuity ensures that:

$$A\text{adj }(A+B)B = B\text{adj }(A+B)A$$

- $(\text{adj } A)B = B(\text{adj } A)$ whenever $AB = BA$, even if A is singular.
- $(\text{adj } A) = \left[\dfrac{\partial}{\partial a_{ij}} \det A\right]^T$

0.8.2 Minors of the inverse

$\det A^{-1}[\alpha^c, \beta^c] = (-1)^{p(\alpha,\beta)} \dfrac{\det A[\beta, \alpha]}{\det A}$, in which $p(\alpha, \beta) = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$.

In particular: $\det A^{-1}[\alpha^c] = \dfrac{\det A[\alpha]}{\det A}$

0.8.3 Schur complements and determinantal formulae

- Definition: The *Schur complement of* $A[\alpha]$ *in* A:

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c]$$

- $\det A = \det A[\alpha] \det \left(A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c]\right)$
- When α^c consists of a single element. Then:

$$\begin{array}{rcl} \det A & = & \det A[\alpha]\left(A[\alpha^c] - A[\alpha^c, \alpha](A[\alpha]^{-1}A[\alpha, \alpha^c]\right) \\ & = & A[\alpha^c] \det A - A[\alpha^c, \alpha](\text{adj } A[\alpha])A[\alpha, \alpha^c] \end{array}$$

- *Cauchy's formula for the determinant of a rank-one perturbation*

$$\begin{array}{rcl} \det \begin{bmatrix} \hat{A} & x \\ y^T & a \end{bmatrix} & = & a \det(\hat{A} - a^{-1}xy^T) \\ \det \begin{bmatrix} \hat{A} & x \\ y^T & a \end{bmatrix} & = & a \det \hat{A} - y^T(\text{adj } \hat{A})x \end{array}$$

$$\Rightarrow a \det(\hat{A} - a^{-1}xy^T) = a \det \hat{A} - y^T(\text{adj } \hat{A})x$$

$$a = -1 \Rightarrow \det(\hat{A} + xy^T) = \det \hat{A} + y^T(\text{adj } \hat{A})x$$

0.8.4 Determinantal identities of Sylvester and Kronecker

0.8.10 Derivative of the determinant

- $\dfrac{d}{dt} \det A(t) = \text{tr}(\text{adj } A(t))A'(t)$
- $\dfrac{d}{dt} \det(tI - A) = \text{tr adj }(tI - A)$

0.9 Special types of matrices

0.9.1 Bock diagonal matrices and direct sums

- A matric $A \in M_n(\mathbb{F})$ of the form: $A = \begin{bmatrix} A_{11} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & A_{kk} \end{bmatrix}$ in which $A_{ii} \in M_{n_i}(\mathbb{F}), i = 1, \dots, k, \sum_{i=1}^k n_i = n$ is called *block diagonal*. We also write:

$$A = A_{11} \oplus A_{22} \oplus \cdots \oplus A_{kk} = \bigoplus_{i=1}^k A_{ii}$$

- $\det \left(\bigoplus_{i=1}^k A_{ii}\right) = \prod_{i=1}^k \det A_{ii}$
- $\text{rank } \left(\bigoplus_{i=1}^k A_{ii}\right) = \sum_{i=1}^k \text{rank } A_{ii}$
- If $A \in M_n$ and $B \in M_m$ are nonsingular, then:
  1. $(A \oplus B)^{-1} = A^{-1} \oplus B^{-1}$
  2. $(\det(A \oplus B))(A \oplus B)^{-1} = (\det A)(\det B)(A^{-1} \oplus B^{-1}) = ((\det B)(\det A)A^{-1} \oplus (\det A)(\det B)B^{-1})$
- a continuity argument ensures that:

$$\text{adj }(A \oplus B) = (\det B)\text{adj } A \oplus (\det A)\text{adj } B$$

0.9.2 Triangular matrices

- If $T \in M_n$ is triangular, has distinct diagonal entries, and commutes with $B \in M_n$, then B must be triangular of the same type as T (upper, stricly upper, lower, stricly lower).
- If a square triangular matrix is nonsingular, its inverse is a triangular matrix of the same type.
- A product of square triangular matrices of the same size and type is a triangular matrix of the same type; each i, i diagonal entry of such a matrix product is the product of the i, i entries of the factors.

0.9.3 Permutation matrices

- A squate matric P is a *permutation matric* if exactly one entry in each row and column is equal to 1 and all other entries are 0.
- $P^T = P^{-1}$ and $\det P = \pm 1$
- The product of two permuation matrices is again a permuation matrix.
- A matrix $A \in M_n$ such that PAP^T is triangular for some permutation matric P is called *essentially triangular*
- The n-by-n *reversal matrix* is the permutation matrix:

$$K_n = \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}$$

0.9.4 Ciculant matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix}$$

0.9.5 Toplits matrices

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_n \\ a_{-1} & a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{-n} & a_{-n+1} & a_{-n+2} & a_{-n+3} & \dots & a_0 \end{bmatrix}$$

0.9.6 Hankel matrices

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n+1} \\ a_2 & a_3 & a_4 & \dots & a_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n} \end{bmatrix}$$

0.9.7 Hessenberg matrices

- A matric $A = [a_{ij}] \in M_n(\mathbb{F})$ is said to be in *upper Hessenberg form* or to be an *upper Hessenberg matric* if $a_{ij} = 0$ for all $i > j + 1$:

$$A = \begin{bmatrix} a_{11} & & & & * \\ a_{21} & a_{22} & & & \\ & a_{32} & \ddots & & \\ & & & \ddots & \ddots \\ 0 & & & a_{n,n-1} & a_{nn} \end{bmatrix}$$

- An upper Hessenberg matrix A is *unreduced* if $a_{i+1,i} \neq 0$ for all $i = 1, \dots, n-1$; the rank of such a matrix is atl least $n-1$ since its first $n-1$ colums are independent.

0.9.8 Tridiagonal, bidiagonal, and other structured matrices

- A matrix that is both upper and lower Hessenberg is called tridiagonal, that is, A is tridiagonal if $a_{ij} = 0, \forall\ |i-j| > 1$
- A *Jacobi matrix* is a real symmetric tridiagonal matrix with positive subdiagonal entries.
- A matrix $A = [a_{ij}] \in M_n(\mathbb{F})$ is *persymmetric* if $a_{ij} = a_{n+1-j,n+1-i}$ for all $i, j = 1, \dots, n-1$
- A is persymmetric if $K_n A = A^T K_n$
- If A is persymmetric and invertible, then A^{-1} is also persymmetric since $K_n A^{-1} = (AK_n)^{-1} = (K_n A^T)^{-1} = (A^{-1})^T K_n$
- A is *skew persymmetric* if $K_n A = -A^T K_n$. The inverse of a nonsingular skew-persymmetric matrix is skew persymmetric.
- $A \in M_n$ is *perhermitian* if $K_n A = A^* K_n$, is *skew perhermitian* if $K_n A = -A^* K_n$
- A matrix $A = [a_{ij}] \in M_n(\mathbb{F})$ is *centrosymmetric* if $a_{ij} = a_{n+1-i,n+1-j}$ for all $i, j = 1, \dots, n$. A is centrosymmetric if $K_n A = AK_n$.
- if A and B are centrosymmetric, then AB is centrosymmetric. If A and B are skew centrosymmetric, then AB is centrosymmetric.

0.9.9 Vandermonde matrices and Lagrange interpolation

- A *Vandermonde matrix* $A \in M_n(\mathbb{F})$ has the form

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

- $\det A = \prod_{1 \le i < j \le n}^{n} (x_i - x_j)$

0.9.10 Cauchy matrices

- A *Cauchy matric* $A \in M_n(\mathbb{F})$ is matric of the form $A = [(a_i + b_j)^{-1}]_{i,j=1}^n$, in which $a_1, \dots, a_n, b_1, \dots, b_n$ are scalars such that $a_i + b_j \neq 0$ for all $i, j = 1, \dots, n$.
- $\det A = \dfrac{\prod_{1 \le i < j \le n}(a_j - a_i)(b_j - b_i)}{\prod_{1 \le i < j \le n}(a_i + b_j)}$
- A *Hilbert matrix* $H_n = [i + j - 1]^{-1}]_{i,j=1}^n$ is a Cauchy matrix that is also a Hankel matrix.

$$\det H_n = \frac{(1!2!\dots(n-1)!)^4}{1!2!\dots(2n-1)!}$$

- So a Hilbert matrix is always nonsingular. The entries of its inverse $H_n^{-1} = [h_{ij}]_{i,j=1}^n$ are:

$$h_{ij} = \frac{(-1)^{i+j}(n+i-1)!(n+j-1)!}{\Big((i-1)!(j-1)!\Big)^2 (n-i)!(n-j)!(i+j+1)!}$$

0.9.11 Involution, nilpotent, projection, coninvolution

A matrix $A \in M_n(\mathbb{F})$ is

- an *involution* if $A^2 = I$
- *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}*$; the least such k is the *index of nulpotence* of A.
- a *projection/idempotent* if $A^2 = A$

Suppose that $\mathbb{F} = \mathbb{C}$. A matrix $A \in M_n$ is:

- a *Hermitian projection/ orthogonal projection* if $A^* = A$ and $A^2 = A$.
- a *coninvolution/coninvolutory* if $A\overline{A} = I$

0.10 Change of basis

0.11 Equivalence relations

Equivalence Relation $\sim$	$A \sim B$
congruence	$A = SBS^T$
unitary congruence	$A = UBU^T$
*congruence	$A = SBS^*$
consimilarity	$A = SB\overline{S}^{-1}$
equivalence	$A = SBT$
unitary equivalence	$A = UBV$
diagonal equivalence	$A = S_1BD_2$
similarity	$A = SBS^{-1}$
unitary similarity	$A = UBU^*$
triangular equivalence	$A = LBR$

in which:

- D_1, D_2, S, T, L and R are square and nonsingular.
- U and V are unitary
- L is lower triangular
- R is upper triangular
- D_1 and D_2 are diagonal
- A and B need not be square for equivalence, unitary equivalence, triangular equivalence, or diagonal equivalence.

1 Eigenvalues, Eigenvectors and Similarity

1.1 The eigenvalue-eigenvector equation

- $A \in M_n$, $p()$ is a given polynomial. Then if $Ax = \lambda x \Rightarrow p(A)x = p(\lambda)x$
- **Theorem 1.1.6** *$A \in M_n$, $p()$ is a given polynomial. Then if $Ax = \lambda x \Rightarrow p(A)x = p(\lambda)x$. Conversely, if $k \ge 1$ and if μ is an eigenvalue of $p(A)$, then $\exists \lambda \in \sigma(A) | \mu = p(\lambda)$*
- **Observation 1.1.8** *Let $A \in M_n$ and $\lambda, \mu \in \mathbb{C}$ be given. Then $\lambda \in \sigma(A) \iff \lambda + \mu \in \sigma(A + \mu I)$*
- **Theorem 1.1.9** *Let $A \in M_n$ be given. Thus, for each $(y \neq 0) \in \mathbb{C}^n$, $\exists$ a polynomial $g(t)$ of degree at most $n-1$ such that $g(A)y$ is an eigenvector of A.*

1.2 The characteristic polynomial and algebraic multilicity

- **Theorem 1.2.8 (Brauer's theorem)** . *Let $x, y \in C^n, x \neq 0$ and $A \in M_n$. Suppose that $Ax = \lambda x$ and let the eigenvalues of A be $\lambda, \lambda_2, \dots, \lambda_n$. Then, the eigenvalues of $A + xy^*$ are $\lambda + y^*x, \lambda_2, \dots, \lambda_n$. In other words,*

$$(t - \lambda)p_{A+xy^*}(t) = (t - (\lambda + y^*x))p_A(t)$$

- a_k is the coefficient of t^k in $p_A(t)$. $E_k(A)$ is the sum of all k-by-k principal minors of A. Then

$$a_k = \frac{1}{k!}p_A^{(k)}(0) = (-1)^{n-k}E_{n-k}(A)$$

- $E_k(A) = S_k(\lambda_1, \dots, \lambda_n) = \displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \lambda_{i_j},\ \lambda_i \in \sigma(A)$

$$p_A(t) = t^n - E_1(A)t^{n-1} + \cdots + (-1)^{n-1}E_{n-1}t + (-1)^n E_n(A)$$

- **Theorem 1.2.17** *Let $A \in M_n$. There is some $\delta > 0$ such that $A + \varepsilon I$ is nonsingular whenever $\varepsilon \in \mathbb{C}$ and $0 < |\varepsilon| < \delta$ (See Observation 1.1.8)*

- α is a zero of $p(t)$ of multiplicity k iff $\begin{cases} p'(\alpha) = \cdots = p^{(k-1)}(\alpha) = 0 \\ p^{(k)}(\alpha) \neq 0 \end{cases}$

- **Theorem 1.2.18** *Let $A \in M_n$ and suppose that $\lambda \in \sigma(A)$ has algebraic multiplicity k. Then* rank $(A - \lambda I) \ge n - k$ *with equality for $k = 1$.*

- Let $A \in M_n$ and $x, y \in \mathbb{C}^n$ be given. Let $f(t) = \det(A + txy^T)$. Then, for any $t_1 \neq t_2$

$$\det A = \frac{t_2 f(t_1) - t_1 f(t_2)}{t_2 - t_1}$$

1.3 Similarity

- **Theorem 1.3.7** *Let $A \in M_n$ be given. Then*

  1. *A is similar to a bloock matrix of the form*

$$\begin{bmatrix} \Lambda & C \\ 0 & D \end{bmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_k), D \in M_{n-k}, 1 \le k < n$$

     iff there are k linearly independent vectors in C^n, each of which is an eigenvector of A.

  2. *The matrix A is diagonalizable iff it has n linearly independent eigenvectors.*

  3. *If $x^{(1)}, \dots, x^{(n)}$ are linearly independent vectors of A and if $S = [x^{(1)}, \dots, x^{(n)}]$, then $S^{-1}AS$ is a diagonal matrix*

  4. *If A is similar to a matrix of the above form, then the diagonal entries of Λ are eigenvalues of A. If A is similar to a diagonal matrix Λ, then the diagonal entries of Λ are all of the eigenvalues of A*

- **Lemma 1.3.8** *Let $\lambda_1, \dots, \lambda_k, (k \ge 2)$ be distinct eigenvalues of $A \in M_n$ and suppose that $x^{(i)}$ is an eigenvector associated with λ_i for each $i = 1, \dots, k$. Then the vectors $x^{(1)}, \dots, x^{(k)}$ are linearly independent.*

- **Theorem 1.3.9** *If $A \in M_n$ has n distinct eigenvalues, then A is diagonalizable.*

- **Lemma 1.3.10** *Let $B_1 \in M_{n1}, \dots, B_d \in M_{nd}$ be gien and let B be the direct sum*

$$B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_d \end{bmatrix} = B_1 \oplus \cdots \oplus B_d$$

  Then B is diagonalizable iff each of $B_1, \dots, B_d$ is diagonalizable.

- **Theorem 1.3.12** *Let $A, B \in M_n$ be diagonalizable. Then $AB = BA$ iff they are simultaneously diagonalizable.*

- $\mathcal{F} \subseteq M_n$ is a commuting family $\Rightarrow \exists x \in \mathbb{C}^n$ that is an eigenvector of every $A \in \mathcal{F}$

- $\mathcal{F}$ is a family of diagonalizable matrices. Then $\mathcal{F}$ is a commuting family $\Leftrightarrow$ it is a simultaneously diagonalizable family.

- $A \in M_{m,n}, B \in M_{n,m}, m \le n$. Then $p_{BA}(t) = t^{n-m}p_{AB}(t)$

- **Theorem 1.3.28** *Let $S \in M_n$ be nonsingular and let $S = C + iD$, in which $C, D \in M_n(\mathbb{R})$. There is a real number τ such that $T = C + \tau D$ is nonsingular.*

- **Theorem 1.3.29** *Two real matrices that are similar over $\mathbb{C}$ are similar over $\mathbb{R}$.*

- **Theorem 1.3.31 (Misky)** *Let an interger $n \ge 2$ and complex scalars $\lambda_1, \dots, \lambda_n$ and $d_1, \dots, d_n$ be given. There is an $A \in M_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and main diagonal entries $d_1, \dots, d_n$ iff $\displaystyle\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i$.*
  If $\lambda_1, \dots, \lambda_n$ and $d_1, \dots, d_n$ are all real and have the same sums, there is an $A \in M_n(\mathbb{R})$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and main diagonal entries $d_1, \dots, d_n$.

2 Unitary equivalence and normal

2.1 Unitary matrices

- *Orthogonal*: The vectors $x_1, \dots, x_k \in \mathbf{C}^n$ form an orthogonal set if $x_i^* x_j = 0, \forall$ pairs$1 \leq i < j \leq k$.
- *Orthonormal*: If an orthogonal set has vectors normalized, $x_i^* x_i = 1, \forall i = 1, \dots, k$, then the set is call orthonormal. An orthonormal set of cectors is linearly independent.
- A matrix $U \in M_n$ is said to be *unitary* if $U^*U = I$. If, in addition, $U \in M_n(\mathbb{R})$, U is said to be *real orthogonal.*
- **Theorem**: For all $x \in \mathbb{C}^n$ and matrix U is unitary, the Euclidean length of $y = Ux$ is the same as that of x; that is, $y * y = x * x$.
- **Theorem**: Let $A \in M_n$ be a nonsingular matrix. Then A^{-1} is similar to A^* iff there is a nonsingular matrix $B \in M_n$ such that $A = B^{-1}B^*$

2.2 Unitary equivalence

- **Def**: A matrix $B \in M_n$ is said to be *unitarily equivalent to* $A \in M_n$ if there is a unitary matrx $U \in M_n$ such that $B = U^*AU$. If U may be taken to be real, then B is said to be *(real) orthogonally equivalent* to A.
- **Thr**: If $A = [a_{ij}]$ and $B = [b_{ij}] \in M_n$ are unitary equivalent, then:

$$\sum_{i,j=1}^{n} |b_{ij}|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$$

- **Householder transformations**: Let $w \in C^n$ be a nonzero vector and define $U_w \in M_n$ by $U_w = I - tww^*$ in which $t = 2(w^*w)^{-1}$. Then,
  - $U_w x = x$ if $x \perp w$ and $W_w w = -w$
  - U_w is both unitary and Hermitian
- **Specht's Thr**: Two given matrices $A, B \in M_n$ are unitarily equivalent iff: tr $W(A, A^*) =$ tr $W(B, B^*)$ for every word $W(s, t)$ in two noncommuting variables.

$$W(A, A^*) = A^{m_1}(A^*)^{n_1} A^{m_2}(A^*)^{n_2} \dots A^{m_k}(A^*)^{n_k}$$

- **Pearcy's Thr**: Two given matrices $A, B \in M_n$ are unitarily equivalent iff tr $W(A, A^*) =$ tr $W(B, B^*)$ for every word $W(s, t)$ of degre at most $2n^2$.

2.3 Schur's unitary triangularization theorem

- **Schur's Thr**: Given $A \in M_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ in any prescribed order, there is a unitary matrix $U \in M_n$ such that:

$$U * AU = T = [t_{ij}]$$

  is upper triangular, with diagonal entries $t_{ii} = \lambda_i, i = 1, \dots, n$. That is, every square matrix A is unitarily equivalent to a triangular matrix whose diagonal entries are the eigenvalues of A in a prescribed order.
- **Thr**: Let $\mathcal{F} \subseteq M_n$ be a commuting family. There is a unitary matrix $U \in M_n$ such that U^*AU is upper triangular for every $A \in \mathcal{F}$
- **Thr**: if $A \in M_n(\mathbb{R})$, there is a real orthogonal matrix $Q \in M_n(\mathbb{R})$ such that:

$$Q^T AQ = \begin{bmatrix} A_1 & & * \\ & A_2 & \\ & & \ddots & \\ & 0 & & A_k \end{bmatrix} \in M_n(\mathbb{R}), \quad 1 \leq k \leq n \tag{13}$$

  where each A_i is a real 1-by-1 matrix, or a real 2-by-2 matrix with a non-real pair of complex conjugate eigenvalues. The diagonal blocks A_i may be arranged in any prescribed order.
- **Thr**: Let $\mathcal{F} \subseteq M_n(\mathbb{R})$ be a commuting family. There is a real orthogonal matrix $Q \in M_n(\mathbb{R})$ such that $Q^T AQ$ is of the form 13 for every $A \in \mathcal{F}$.

2.4 Some implications of Schur's theorem

- **Lem**: Supose that $R = [r_{ij}], T = [t_{ij}] \in M_n$ are upper triangular and that $r_{ij} = 0, 1 \leq i, j \leq k < n$, and $t_{k+1,k+1} = 0$. Let $T' = [t'_{ij}]RT$ then $t'_{ij} = 0, 1 \leq i, j \leq k + 1$.
- **Cayley-Hamilton Thr**: Let $P_A(t)$ be the characteristic polynomial of $A \in M_n$. Then: $P_A(A) = 0$.
- **Thr**: Let $A = [a_{ij}] \in M_n$. For every $\epsilon > 0$, there exists a matrix $A(\epsilon) = [a_{ij}(\epsilon)] \in M_n$ that has n distinct eigenvalues (and therefore diagonalizabal) and is such that:

$$\sum_{i,j=1}^{n} |a - [ij] - a_{ij}(\epsilon)|^2$$

- **Thr**: Let $A \in M_n$. For every $\epsilon > 0$, there exists a nonsingular matrix $S_\epsilon \in M_n$ such that

$$S_\epsilon^{-1} AS_\epsilon = T_\epsilon = [t_{ij}(\epsilon)]$$

  is upper triangular and $|T_{ij}(\epsilon)| < \epsilon, \forall 1 \leq i <, > j \leq n$

- **Thr**: Suppose that $A \in M_n$ has eigenvalues $\lambda + i$ with multiplicity $n_i, i = 1, \dots, k$ and that $\lambda_1, \dots, \lambda_k$ are distint. Then A is similar to a matrix of the form
$$\begin{bmatrix} T_1 & & 0 & \\ & T_2 & & \\ & & \ddots & \\ & 0 & & T_k \end{bmatrix}$$
where $T_i \in M_{n_i}$ is upper triangular with all diagonal entries equal to $\lambda_i, i = 1, \dots, k$. If $A \in M_n(\mathbb{R}))$ and if all the eigenvalues of A are real, then the same result holds, and the similarity matrix may be taken to be real.

- **Thr**: Let $A, B \in M_n$ have eigencalues $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ respectively. If A and B commute, there is a permutation $i_1, \dots, i_n$ of the indices $1, \dots, n$ such that the eigenvalues of $A + B$ are $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$. In particular, $\sigma(A + B) \subseteq \sigma(A) + \sigma(B)$ if A and B are commute.

2.5 Normal matrices

- **Def**: A matrix $A \in M_n$ is said to be *normal* if $A^*A = AA^*$.

- **Thm**: if $A = [a_{ij}] \in M_n$ has eigenvalues $\lambda_1, \dots, \lambda_n$, the following statements are equivalents:
  1. A is normal;
  2. A is unitary diagonalizable;
  3. $\sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$; and
  4. There is an orthonormal set of n eigenvectors of A.

- A normal matrix is nondefective (the geometric's and algebraic's multiplicity are the same)

- If $A \in M_n$ is normal, $x \in \mathbb{C}^n$ then $Ax = \lambda x \Leftrightarrow x^*A = \lambda x^*$

- **Thm**: If $\mathcal{N} \subseteq M_n$ is a commuting family of normal matrices, then $\mathcal{N}$ is simultaneously unitary diagonalizable.

- **Lem**: If $A \in M_n$ is Hermitian and $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$, then all the eigenvalues of A are nonnegative. If, in addition, tr $A = 0$ then A = 0.

4 Hermitian and symmetric matrices

4.1 Definitions, properties of Hermitian matrices

- **Thm**: Let $A = [a_{ij}] \in M_n$ be given. Then A is Hermitian iff at least one of the follwong holds:
  1. x^*Ax is real for all $x \in \mathbb{C}^n$;
  2. A is normal and all the eigenvalues of A is are real; or
  3. S^*AS is Hermitian for all $S \in M_n$

- **Thm**: Let $A \in M_n$ be gien. Then A is Hermitian iff there is a nunitarry matrix $U \in M_n$ and a real diagonal matrix $\Lambda \in M_n$ s.t. $A = U\Lambda U^*$

4.2 Variational characterizations of eigenvalues of Hermitian matrices

- **Thm**: Let $A \in M_n$ be a Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \lambda_n$, and let k be the given integer with $1 \leq k \leq n$. Then
$$\min_{w_1, w_2, \dots, w_{n-k} \in \mathbb{C}^n} \max_{x \neq 0, x \perp w_1, \dots, w_{n-k}} \frac{x^*Ax}{x^*x} = \lambda_k$$
$$\max_{w_1, w_2, \dots, w_{k-1} \in \mathbb{C}^n} \min_{x \neq 0, x \perp w_1, \dots, w_{k-1}} \frac{x^*Ax}{x^*x} = \lambda_k$$

4.3 Some apps of the variational characterizations

- **Thm:** Let $A, B \in M_n$ be Hermitian. Let the respective eigenvalues A, B, and $A + B$ be $\{\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)\}$, $\{\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(B)\}$, and $\{\lambda_1(A + B) \leq \lambda_2(A + B) \leq \cdots \leq \lambda_n(A + B)\}$. Then:
$$\lambda_i(A + B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad i = 1, 2, \dots, n; j = 0, 1, \dots, n - 1$$

- **Thm**: Let $A, B \in M_n$ be Hermitian and let the eigenvalues $\lambda_i(A), \lambda_i(B)$, and $\lambda_i(A + B)$ be arranged in increasing order. For each $k = 1, 2, \dots, n$, we have:
$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$$

- **Corollary**: Let $A, B \in M_bn$ be Hermitian. Assume that B is positive semidefinite and that the eigenvalues of A and $A + B$ are arranged in increasing order. Then: $\lambda_k(A) \leq \lambda_k(A + B)$ for all $k = 1, 2, \dots, n$

- **Thm 4.3.4**: Let $A \in M_n$ be Hermitian and let $z \in \mathbb{C}^n$ be a given vector. If the eigenvalues of A and $A \pm zz^*$ are arranged in increasing order, we have:
  1. $\lambda_k(A \pm zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+2}(A \pm zz^*), k = 1, 2, \dots, n - 2$
  2. $\lambda_k(A) \leq \lambda_{k+1}(A \pm zz^*) \leq \lambda_{k+2}(A)$

- Let $A, B \in M_n$ be Hermitian and suppose that B has rank at most r. Then:
  1. $\lambda_k(A + B) \leq \lambda_{k+r}(A) \leq_{k+2r} (A), k = 1, 2, \dots, n - 2r$
  2. $\lambda_k(A) \leq \lambda_{k+r}(A + B) \leq \lambda_{k+2r}(A), k = 1, 2, ..., n - 2r$

- **C**: A, B Hermitian.

1. $j+k \geq n+1$, then: $\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B)$.
2. $j+k \leq n+1$, then $\lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A+B)$

- **Thm**: A Hermitian. $\hat{A} = \begin{bmatrix} A & y \\ y^* & a \end{bmatrix}$. Then:

$$\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \leq \cdots \leq \hat{\lambda}_n \leq \lambda_n \leq \hat{\lambda}_{n+1}$$

- **Thm 4.3.10** If: $\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \leq \cdots \leq \hat{\lambda}_n \leq \lambda_n \leq \hat{\lambda}_{n+1}$. Let $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_n$. Then $\exists a, y \in \mathbb{R}^n$ s.t. $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\}$ is the set of the real symmetric matrix: $\begin{bmatrix} A & y \\ y^* & a \end{bmatrix}$

- **Thm**: A is Hermitian, A_r is r-by-r principal submatrix of A.

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A), \quad 1 \leq k \leq r$$

- **Def**: The vector β is said to *majorize* the vector α if sum of k smallest elements of $\beta \geq$ that of α, for $k = 1, \ldots, n$ and *equality* for $k = n$

- **Thm**: A is Hermitian. Then the vector of diagonal entries of A majorizes the vector of eigenvalues of A.

- **Thm**: $a_1 \leq \cdots \leq a_n; \lambda_1 \leq \cdots \leq \lambda_n$ and vector $a = [a_i]$ majorizes the vector $\lambda = [\lambda_i]$, then $\exists A = [a_{ij}]$ is real symmetric s.t. $a_{ii} = a_i$ and $\{\lambda_i\}$ is the set of eigenvalues of A

- **Weyl's Thm**: Let $A, B \in M_n$ be Hermitian. Let the respective eigenvalues of A, B, and $A+B$ be $\{\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)\}$, $\{\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(B)\}$, and $\{\lambda_1(A+B) \leq \lambda_2(A+B) \leq \cdots \leq \lambda_n(A+B)\}$. Then:

$$\lambda_{i-k+1}(A) + \lambda_k(B) \leq \lambda_i(A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B),$$
$$i = \overline{1,n}; j = \overline{0, n-1}; k = \overline{1,i}$$

  - If B has exactly π positive eigenvalues, μ negative eignenvalues:

$$\lambda_i(A+B) \leq \lambda_{i+\pi}(A); i = \overline{1, n-\pi}, \tag{14}$$
$$\lambda_{i-\mu}(A) \leq \lambda_i(A+B); i = \overline{\mu+1, n} \tag{15}$$

  - If rank $(B) = r < n$:

$$\lambda_i(A+B) \leq \lambda_{i+r}(A); i = \overline{1, n-r} \tag{16}$$
$$\lambda_{i-r} \leq \lambda_i(A+B); i = \overline{r+1, n} \tag{17}$$

  - $0 \neq z \in \mathrm{C}^n$:

$$\lambda_i(A) \leq \lambda_i(A + zz^*) \leq \lambda(A); i = \overline{1, n-1} \tag{18}$$
$$\lambda_n(A) \leq \lambda_n A + zz^* \tag{19}$$

  - $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B); k = \overline{1,n}$

- **Cauchy's Thm:** Let $B \in M_n$ be Hermitian, let $y \in \mathrm{C}^n$ and $a \in \mathrm{R}$ be a given, and let $A = \begin{bmatrix} B & y \\ y^* & a \end{bmatrix} \in M_{n+1}$. Then:

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

- **Thm 4.3.10** If: $\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \leq \cdots \leq \hat{\lambda}_n \leq \lambda_n \leq \hat{\lambda}_{n+1}$. Let $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_n)$. Then $\exists a, y \in \mathbb{R}^n$ s.t. $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\}$ is the set of the real symmetric matrix: $\begin{bmatrix} A & y \\ y^* & a \end{bmatrix}$

- If $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_n \leq \mu_n$. Let $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_2)$. There exists $z \in \mathrm{R}^n$ such that: $\sigma(\Lambda + zz^T) = \{\mu_1, \ldots, \mu_n\}$

5 Norms for vectors and matrices

5.4 Analytic properties of vector norms

- Let $B = \{x^{(1)}, \ldots, x^{(n)}\}$ be a basix for V. A fuction $f : V \to \mathrm{R}$ is said to be **pre-norm** if it satisfies:

  a) Positive: $f(x) \geq 0,\ \forall x; f(x) = 0 \Leftrightarrow x = 0$
  b) Homogeneous: $f(\alpha x) = \alpha f(x)$
  c) Continunous: $f(x(z))$ is continuous on $(\mathbb{F})^n$, where $z = [z_1, \ldots, z_n]^T \in (\mathbb{F})^n$ and $x(x) = z_1 x^{(1)} + \cdots + z_n x^{(n)}$

  A norm is always a pre-norm.

- Let $f()$ be a pre-norm on $V = \mathrm{R}^n$ or C^n. The function:

$$f^D(y) = \max_{f(x)=1} \mathrm{Re} y^* x$$

  is called the **dual norm** of f . Also: $f^D(y) = \max_{f(x)=1} |y^* x|$. The dual norm of a pre-norm is always a norm.

  - $|y^* x| \leq f(x) f^D(y)$
  - $(\|\cdot\|_1)^D = \|\cdot\|_\infty; \quad (\|\cdot\|_\infty)^D = \|\cdot\|_1;$
  - $(\|\cdot\|_2)^D = \|\cdot\|_2$

- $((\|\cdot\|)^D)^D = \|\cdot\|$

- $x \in \mathrm{C}^n$ be a given vector and let $\|\cdot\|$ be a given vector norm on C^n. The set

$$\{y \in \mathrm{C}^n : \|y\|^D \|x\| = y^* x = 1\}$$

is said to be **the dual of x with respec to** $\|\cdot\|$

5.5 Matrix norms

- We call a function $\|\cdot\| : M_n \to \mathrm{R}$ a **matrix norm** if for all $A, B \in M_n$, it astisfies the following five axioms:

(1)	$\|A\| \geq 0$	Nonegative
(1a)	$\|A\| = 0$ if and only if $A = 0$	Positive
(2)	$\|cA\| = \|c\|\|A\|,\ \forall c \in C$	Homogeneous
(3)	$\|A+B\| \leq \|A\| + \|B\|$	Triangle inequality
(4)	$\|AB\| \leq \|A\|\|B\|$	Submutiplicative

- Let $\|\cdot\|$ be a vector norm on C^n. Define $\|\cdot\|$ on M_n by:

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Then, $\|\cdot\|$ is a matrix norm. $\|\cdot\|$ is the matrix norm **induced** by the vector norm $\|\cdot\|$.

- The **maximum column sum matrix norm** $\|\cdot\|_1$:

$$\|A\|_1 = \max_{a \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

- The **maximum row sum matrix norm** $\|\cdot\|_\infty$:

$$\|A\|_\infty = \max_{a \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

- **Spectral norm** $\|\cdot\|_2$ is defined on M_n by:

$$\|A\|_2 = \max\{\sqrt{\lambda} \text{ is an eigenvalues of } A^*A\}$$

- **Spectrul radius** $\rho(A)$ of a matrix $A \in M_n$ is:

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigen value of } A\}$$

- If $\|\cdot\|$ is any matrix norm, then $\rho(A) \leq \|A\|$.

- Let $A \in M_n$ and $\epsilon > 0$ be given. There is a matrix norm $\|\cdot\|$ such that: $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$

- If $A \in M_n$, then th series $\sum_{k=0}^{\infty} a_k A^k$ converges if there is a matrix norm $\|\cdot\|$ on M_n such that the numerical series $\sum_{k=1}^{\infty} |a_k| \|A\|^k$ converges, or even if the partial sums of this series are bounded.

- Define $\|A\|^* = \|A^*\|$

- A matrix norm $\|\cdot\|$ on M_n is a **minimal matrix norm** if the only matrix norm $N(\cdot)$ on M_n such that $N(A) \leq \|A\|$ for all $A \in M_n$ is $N(\cdot) = \|\cdot\|$

-

- **Thm:** Let $\|\cdot\|$ be a given matrix norm on M_n. Then:

  a) $\|\cdot\|^*$ is an induced norm if and only if $\|\cdot\|$ is an induced norm.

  b) If the matrix norm $\|\cdot\|$ is induced by the vector norm $\|\cdot\|$, then $\|\cdot\|^*$ is induced by the dual norm $\|\cdot\|^D$

  c) The spectral norm $\|\cdot\|_2$ is the only matrix norm on M_n that is both induced and self-adjoint.

5.7 Vector norms on matrices

- If f is a pre-norm on M_n, then $\lim_{k\to\infty}[f(A^k)]^{1/k}$ exists for all $A \in M_n$ and:

$$\lim_{k\to\infty} [f(A^k)]^{1/k} = \rho(A)$$

- For each vectr norm $G(\cdot)$ on M_n, there is a finite positive constant $c(G)$ such that $c(G)G(\cdot)$ is a matrix norm on M_n. If $\|\cdot\|$ is a matrix norm on M_n, and if:

$$C_m\|A\| \leq G(A) \leq C_M\|A\| \text{ for all } A \in M_n$$

then $c(G) \leq \dfrac{C_M}{C_m^2}$

- The vector norm $\|\cdot\|$ on C^n is said to be compatible with the vector norm $G(\cdot)$ on M_n if:

$$\|Ax\| \leq G(A)\|x\|,\ \forall x \in \mathrm{C}^n, A \in M_n$$

- If $\|\cdot\|$ is a matrix norm on M_n, then there is some vectr norm on C^n that is compatible with it.