Binomial Model

In this lecture. . .

- a simple model for an asset price random walk
- delta hedging
- no arbitrage
- the basics of the binomial method for valuing options
- risk neutrality

By the end of this lecture you will be able to

- understand how hedging is used to eliminate risk

- use the binomial method to price simple options

- explain the concept of risk neutrality

Introduction

The most ‘accessible’ approach to option pricing is the **binomial model**. This requires only basic arithmetic and no complicated stochastic calculus.

In this model we will see the ideas of hedging and no arbitrage used.

The end result is a simple algorithm for determining the correct value for an option.

The framework

We are going to examine a very simple model for the behavior of a stock, and based on this model see how to value options.

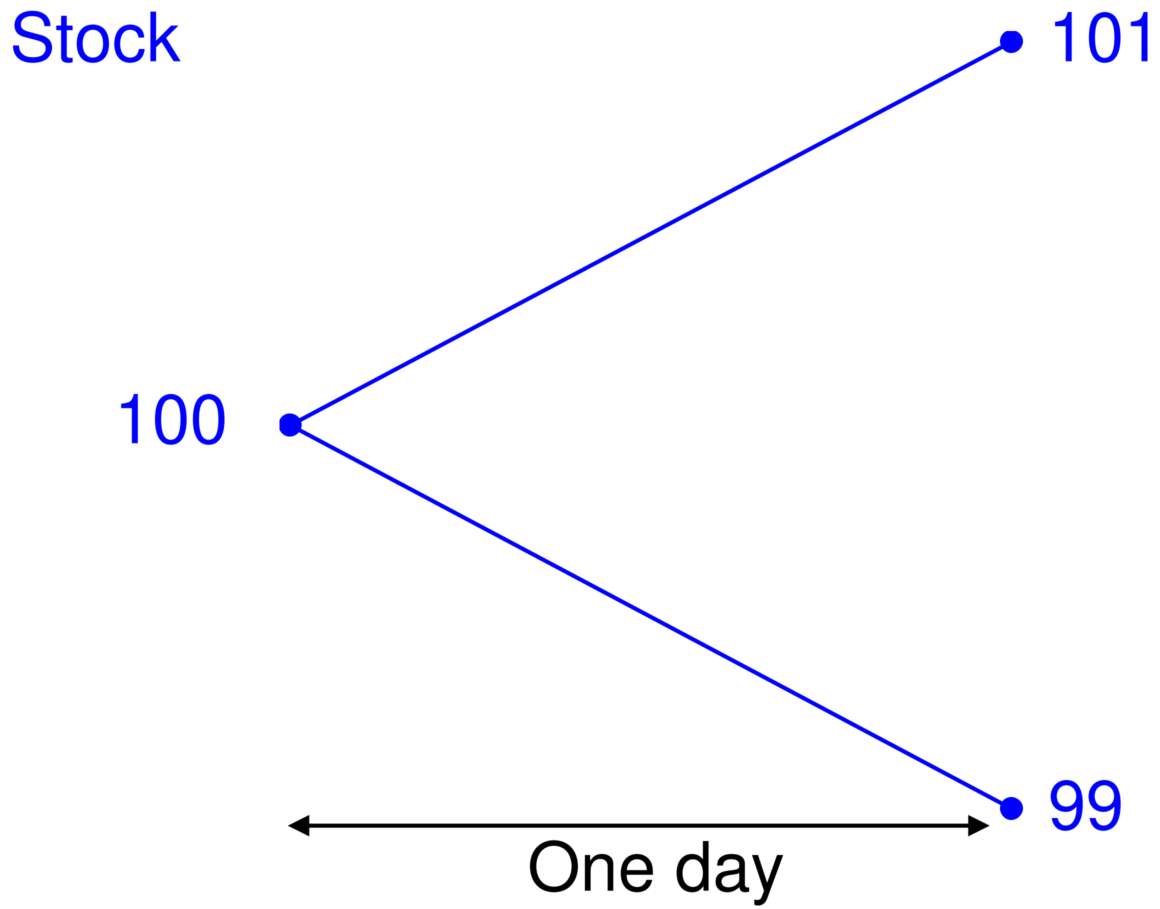
We will have

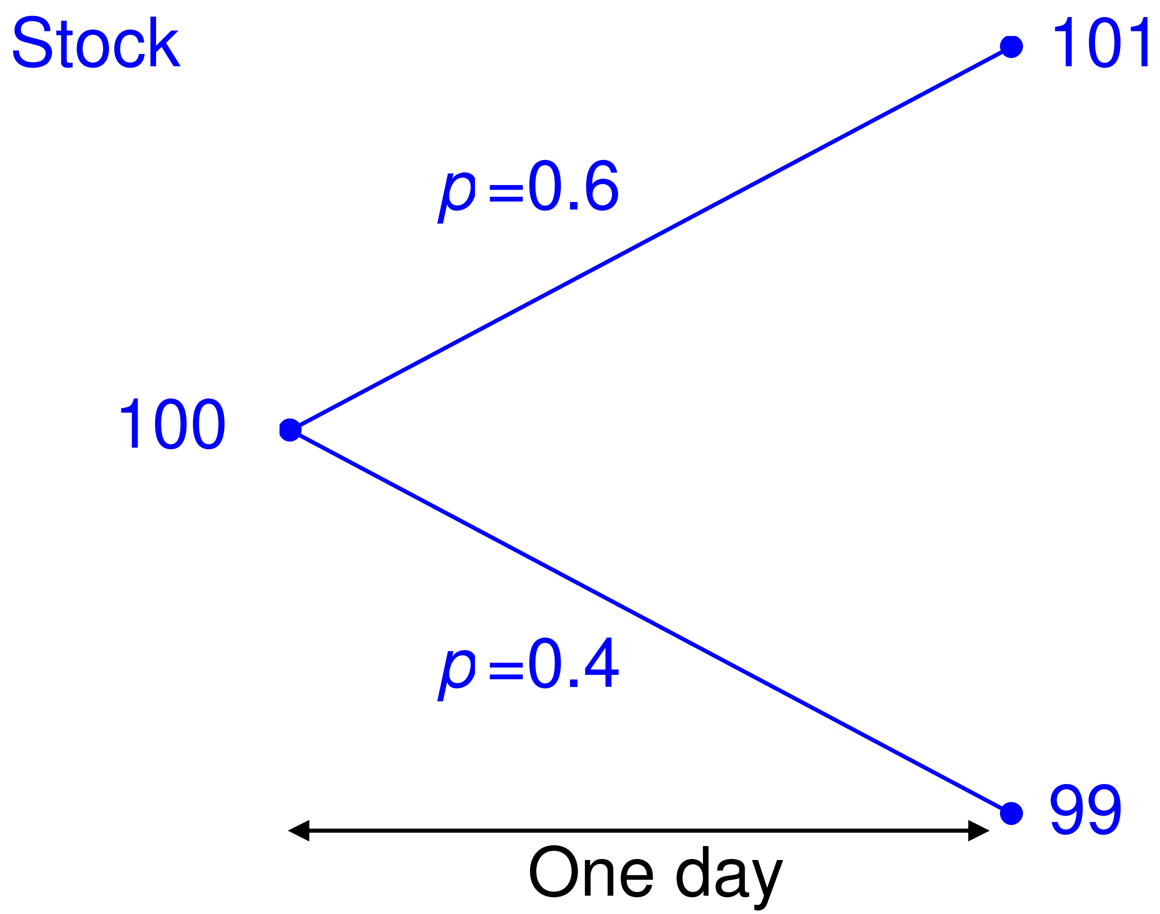
- a stock

- a call option on that stock expiring tomorrow

The stock can either rise or fall by a known amount between today and tomorrow.

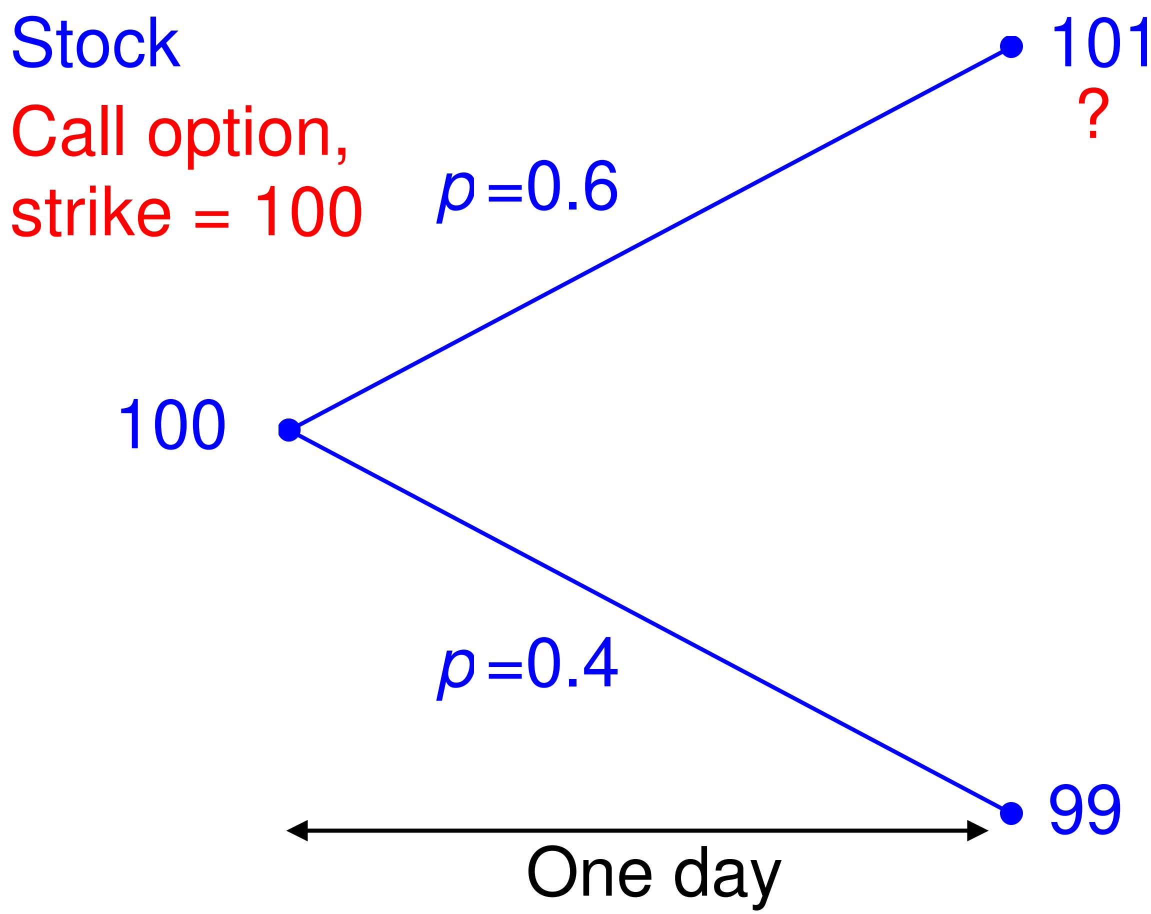
- Interest rates are zero

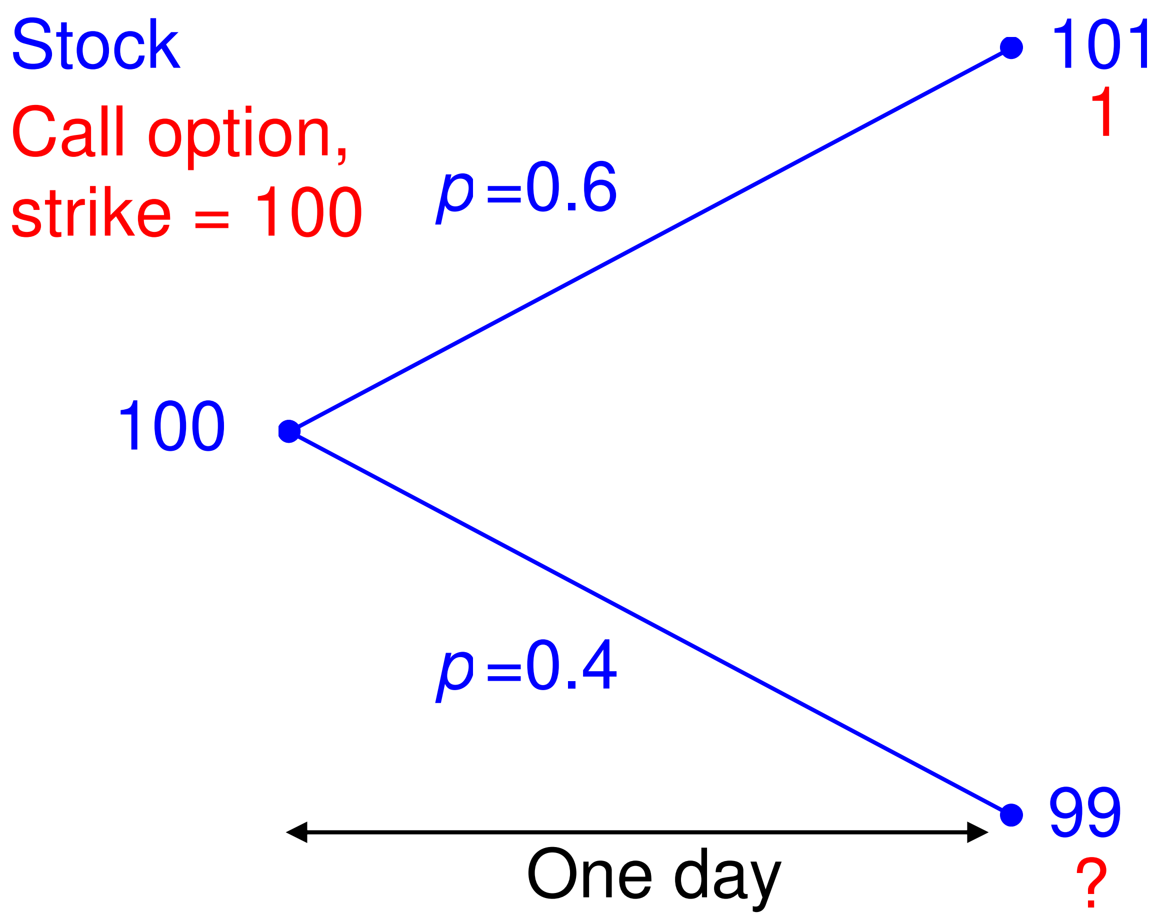

Stock
101
100
99
One day

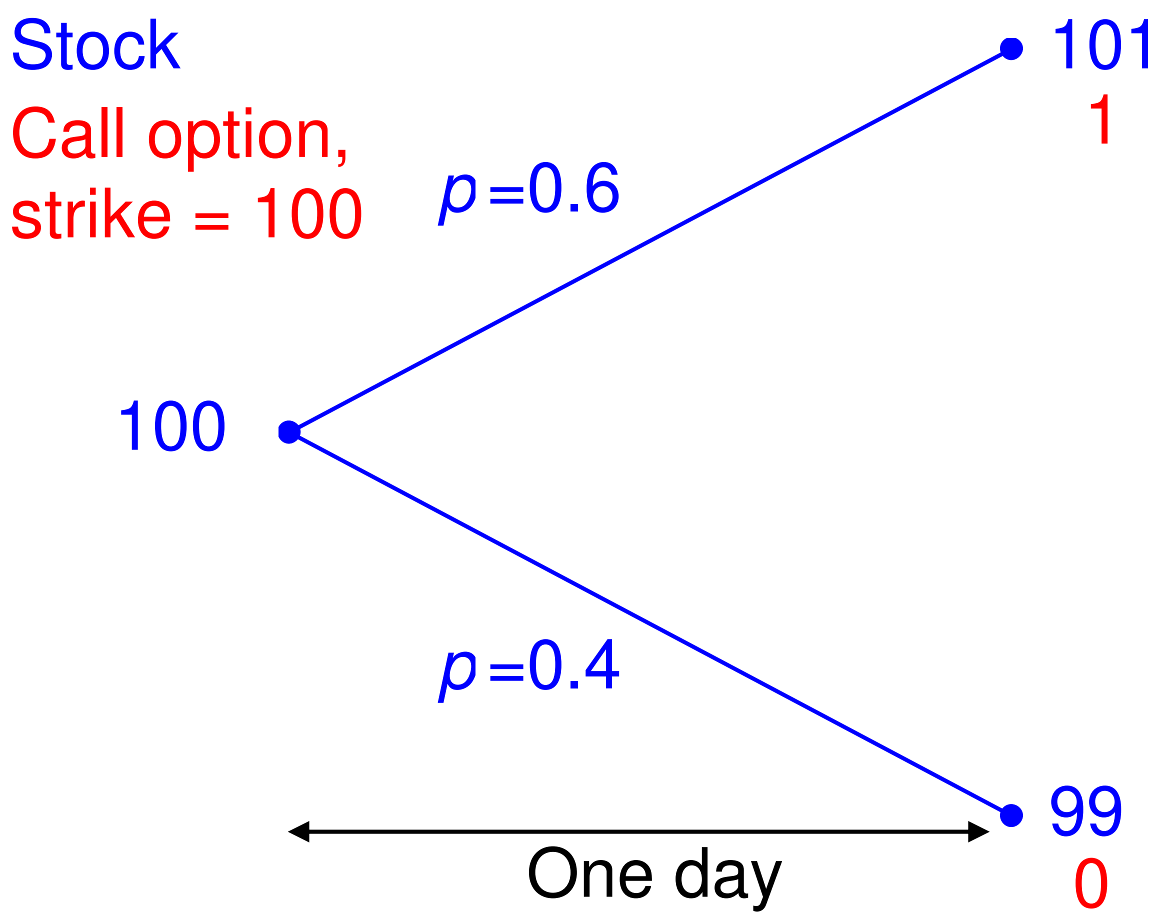

Stock
101
p =0.6
100
p =0.4
99
One day

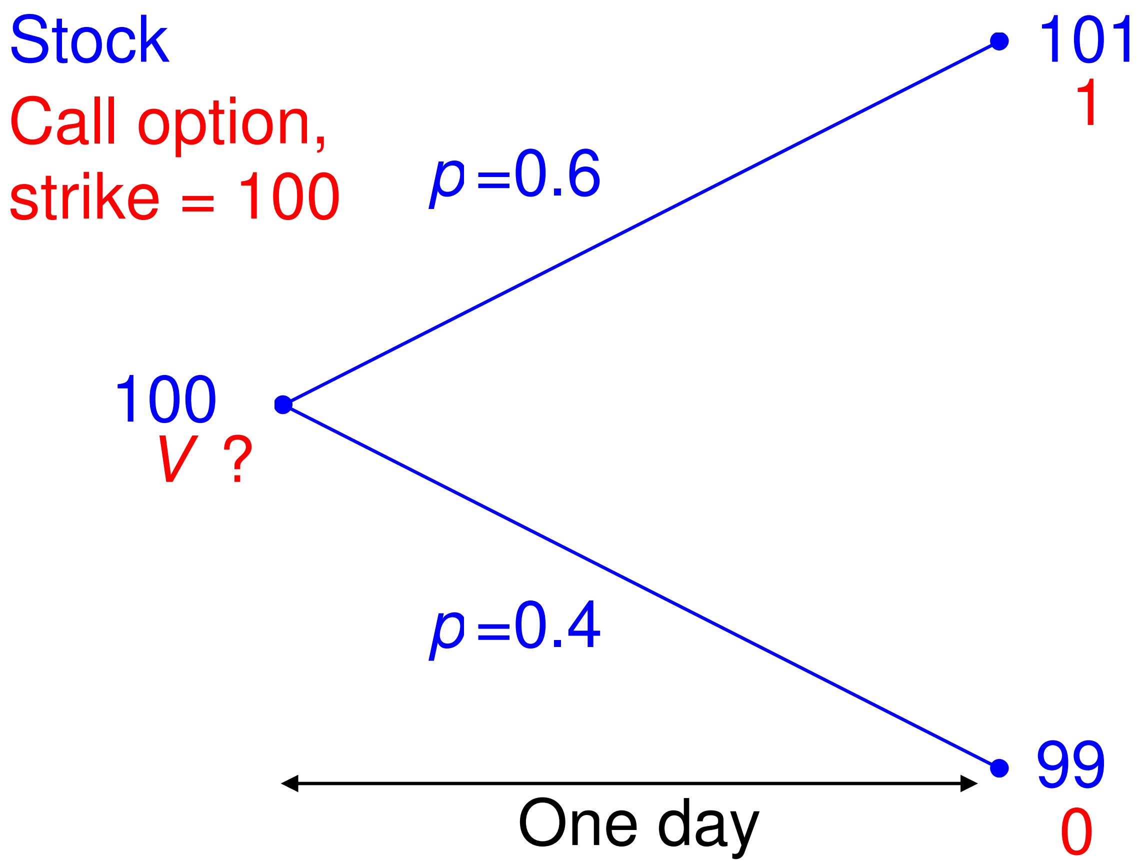
Where do those probabilities come from?

Soon I shall be choosing them so that this binomial random walk has the same growth and volatility as the continuous-time SDE random walk that we've already seen.


Stock
Call option,
strike = 100
101
?
p=0.6
100
p=0.4
99
One day


Stock
Call option,
strike = 100
p =0.6
101
1
100
p =0.4
99
?
One day

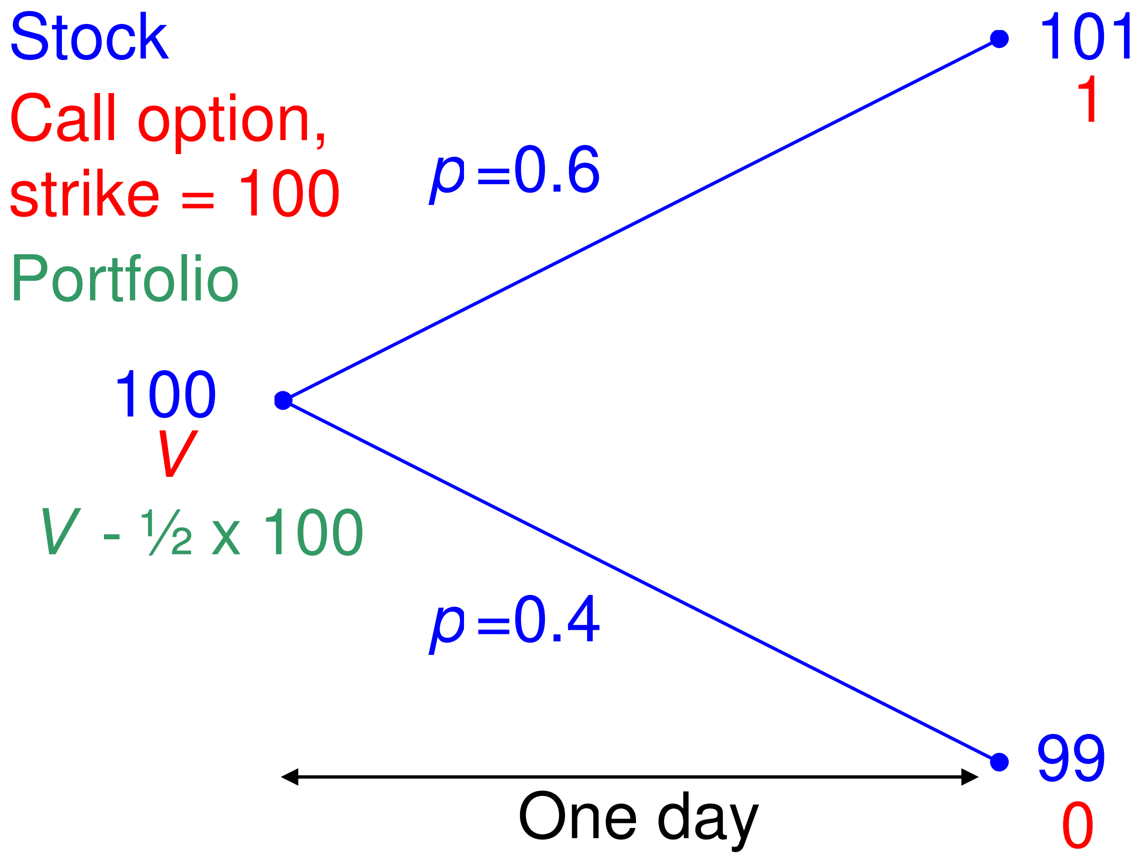

Stock
Call option,
strike = 100
p =0.6
101
1
100
p =0.4
99
0
One day

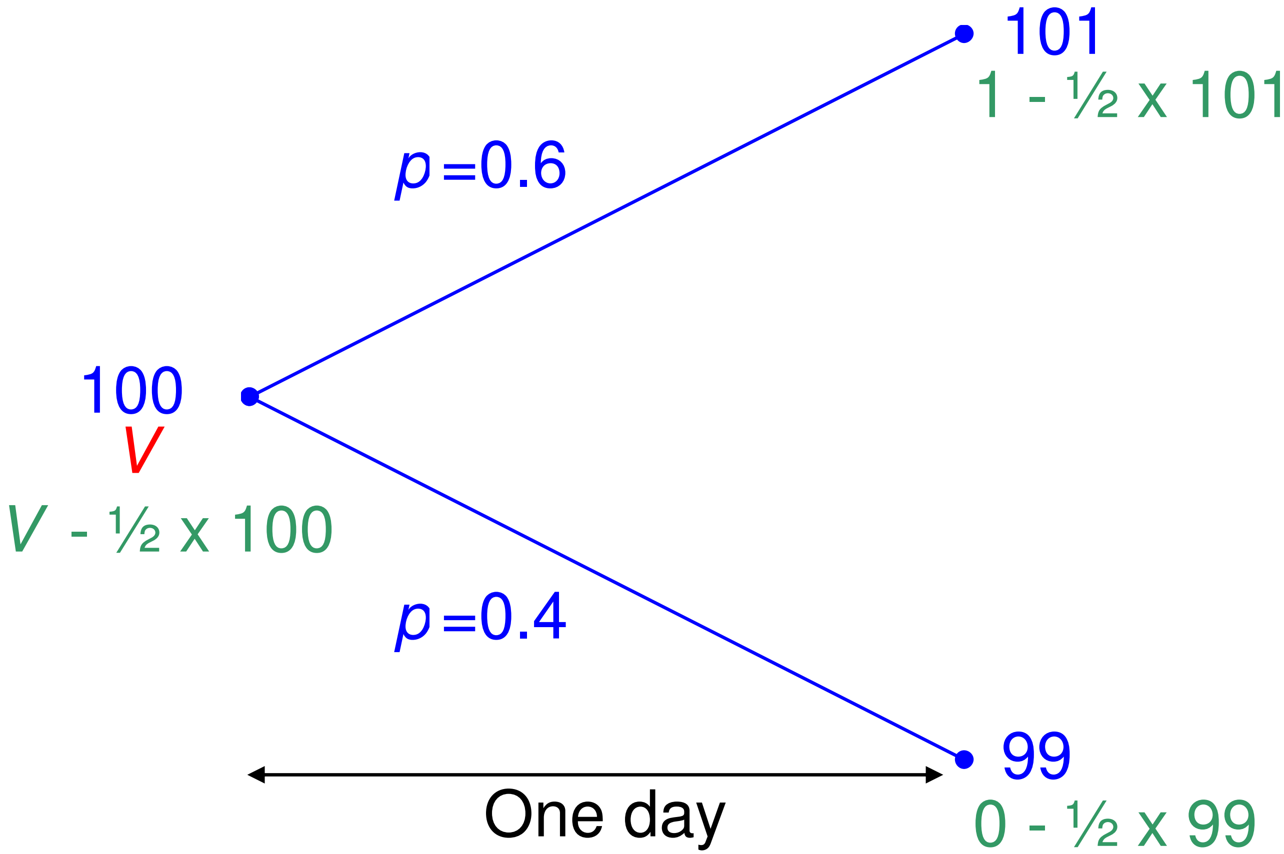

Stock
Call option,
strike = 100
p =0.6
101
1
100
V ?
p =0.4
99
0
One day

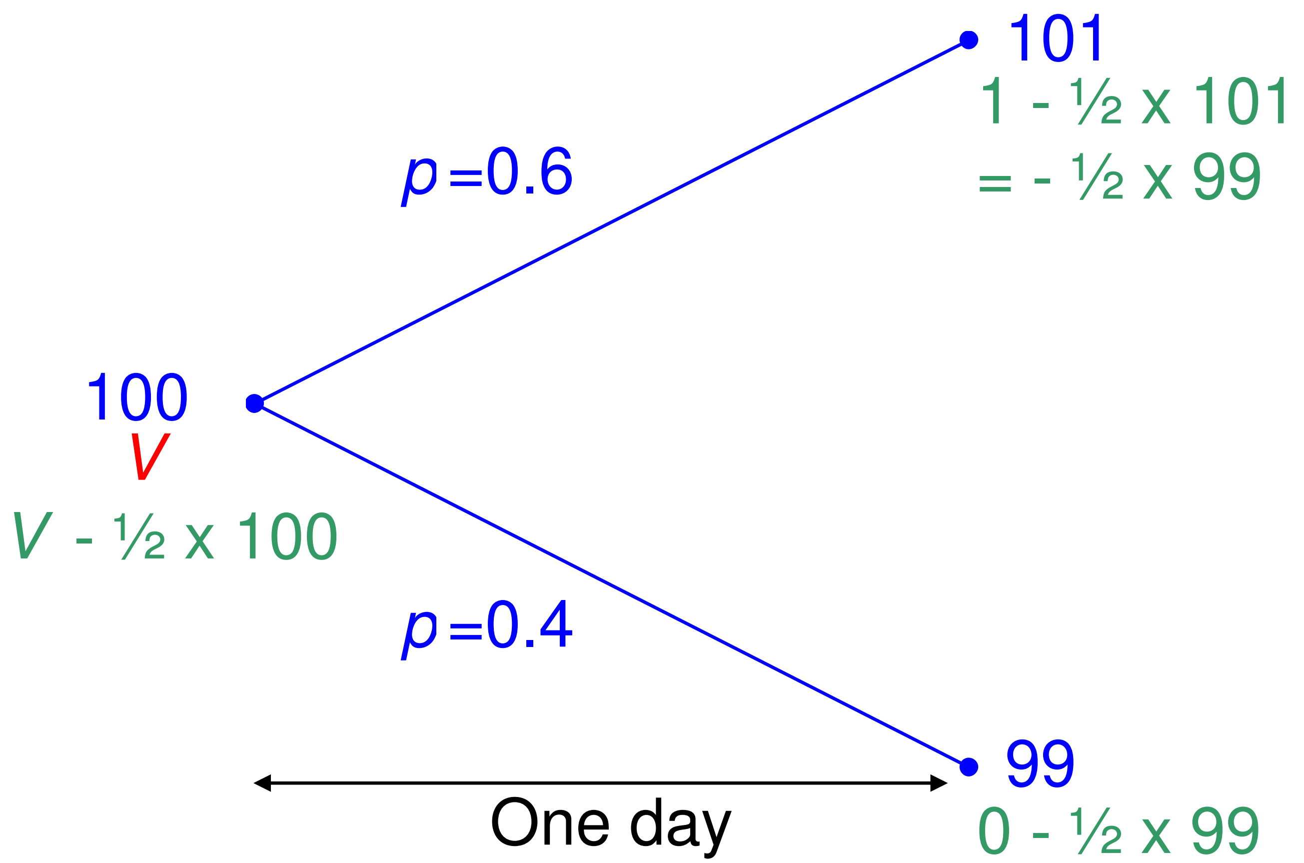
The answer is. . .

Why?

To see how this can be the only correct answer we must first construct a *portfolio* consisting of one option and short $\frac{1}{2}$ of the underlying stock.


Stock
Call option,
strike = 100
Portfolio
p =0.6
101
1
100
V
V - ½ x 100
p =0.4
99
0
One day


101
1 - ½ x 101
p =0.6
100
V
V - ½ x 100
p =0.4
99
0 - ½ x 99
One day


101
1 - ½ x 101
= - ½ x 99
p =0.6
100
V
V - ½ x 100
p =0.4
99
One day
0 - ½ x 99

Tomorrow, at expiration, the portfolio takes the value

$$-\frac{99}{2}$$

regardless of whether the stock rises or falls.

We have constructed a perfectly risk-free portfolio.

If the portfolio is worth $-99/2$ tomorrow, and interest rates are zero, how much is this portfolio worth today?

It must also be worth $-99/2$ today.

This is an example of **no arbitrage**:

There are two ways to ensure that we have $-99/2$ tomorrow.

1. Buy one option and sell one half of the stock
2. Put the money under the mattress

Both of these 'portfolios' must be worth the same today.

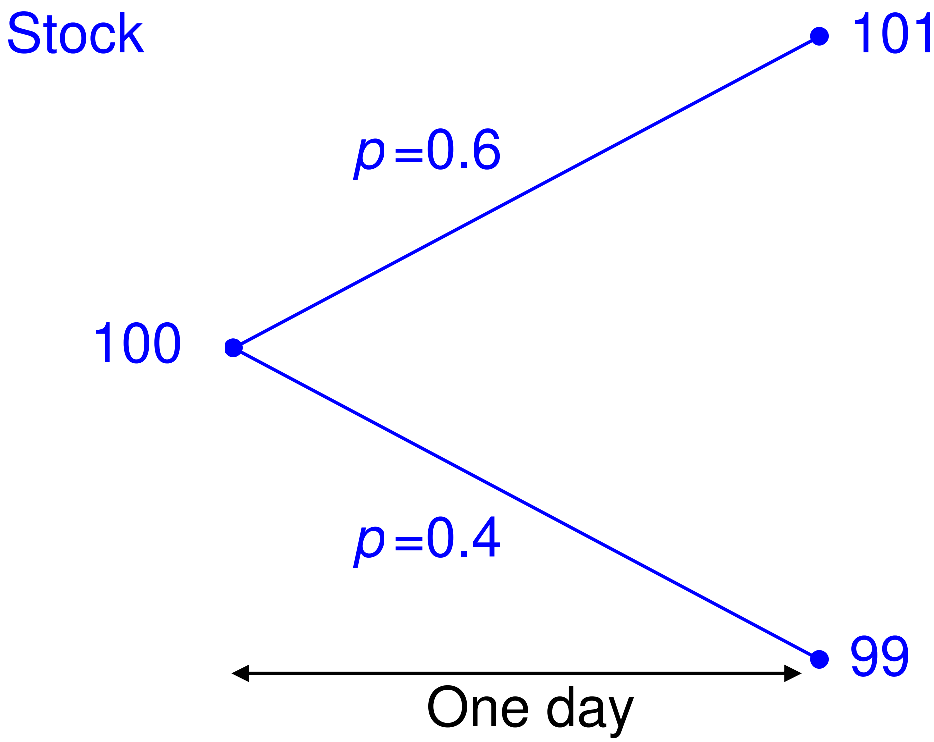
Therefore

$$V - \tfrac{1}{2} \times 100 = -\tfrac{1}{2} \times 99$$

and so

$$V = \text{the option value} = \frac{1}{2}.$$

Which part of our 'model' didn't we need?



The value of an option does not depend on the probability of the stock rising or falling.

- **This is equivalent to saying that the stock growth rate is irrelevant for option pricing**

This is because we have **hedged** the option with the stock.

We do not care whether the stock rises or falls.

We *do* care about the stock price range, however.

- **The stock volatility is very important in the valuation of options**

Three questions:

- Why should this ‘theoretical price’ be the ‘market price’?

- How did I know to sell $\frac{1}{2}$ of the stock for hedging?

- How does this change if interest rates are non zero?

Why should this 'theoretical price' be the 'market price'?

Because if it's not, then there is risk-free money to be made!

If the option costs less than 0.5 simply buy it and hedge to make a profit.

If it is worth more than 0.5 in the market then sell it and hedge, and make a guaranteed profit.

Supply and demand should act to make the option price converge to the 0.5.

The role of probability and expectations

The expected payoff is 0.6.

How much would someone buy or sell for?

Buying. . .

- **Would anyone pay 0.6 or more for the option?** No, unless they were **risk seeking**

- **Would anyone pay from 0.5 to 0.6?** Possibly. You would have a positive expected return if you bought the option for speculation

- **Would anyone pay less than 0.5?** Yes, definitely! Either speculate with the option. Or even hedge it and lock in a profit!

Selling. . . Sellers tend to hedge options. Let's assume they are doing that.

- **Would anyone sell the option for 0.5 or more?** Yes, definitely! Lock in the profit

- **Would anyone sell the option for less than 0.5?** No!

Note that if you are hedging then the 0.6 expectation is unimportant. (Except it will affect how many options you sell to speculators!)

How did I know to sell $\frac{1}{2}$ of the stock for hedging?

Introduce a symbol!

Use Δ to denote the quantity of stock that must be sold for hedging. We start off with one option, $-\Delta$ of the stock, giving a portfolio value of

$$V - \Delta \times 100.$$

Tomorrow the portfolio is worth

$$1 - \Delta \times 101$$

if the stock rises, or

$$0 - \Delta \times 99$$

if it falls.

Make these two equal to each other:

$$1 - \Delta \times 101 = 0 - \Delta \times 99.$$

Therefore

$$\Delta(101 - 99) = 1 - 0$$

$$\Delta = \frac{1 - 0}{101 - 99} = 0.5.$$

(This is a different 0.5 from the option value!)

The general formula for Δ

Delta hedging means choosing Δ such that the portfolio value does not depend on the direction of the stock.

NB when we generalize this (using symbols instead of numbers) we will find that

$$\Delta = \frac{\text{Range of option payoffs}}{\text{Range of stock prices}}.$$

(What does this become when we go to continuous time?)

We can think of Δ as the sensitivity of the option to changes in the stock.

Another example:

Stock price is 100, can rise to 103 or fall to 98.

Value a call option with a strike price of 100.

Interest rates are zero.

Again use Δ to denote the quantity of stock that must be sold for hedging.

The portfolio value is

$$V - \Delta \times 100.$$

Tomorrow the portfolio is worth either

$$3 - \Delta 103$$

or

$$0 - \Delta 98.$$

So we must make

$$3 - \Delta 103 = 0 - \Delta 98.$$

That is,

$$\Delta = \frac{3-0}{103-98} = \frac{3}{5} = 0.6.$$

The portfolio value tomorrow is then

$$-0.6 \times 98.$$

With zero interest rate, the portfolio value today must equal the risk-free portfolio value tomorrow:

$$V - 0.6 \times 100 = -0.6 \times 98.$$

Therefore the option value is 1.2.

How does this change if interest rates are non zero?

Simple.

We delta hedge as before to construct a risk-free portfolio. (Exactly the same delta.)

Then we present value that back in time, by multiplying by a discount factor.

Example: First example, but now $r = 0.1$.

The discount factor for going back one day is

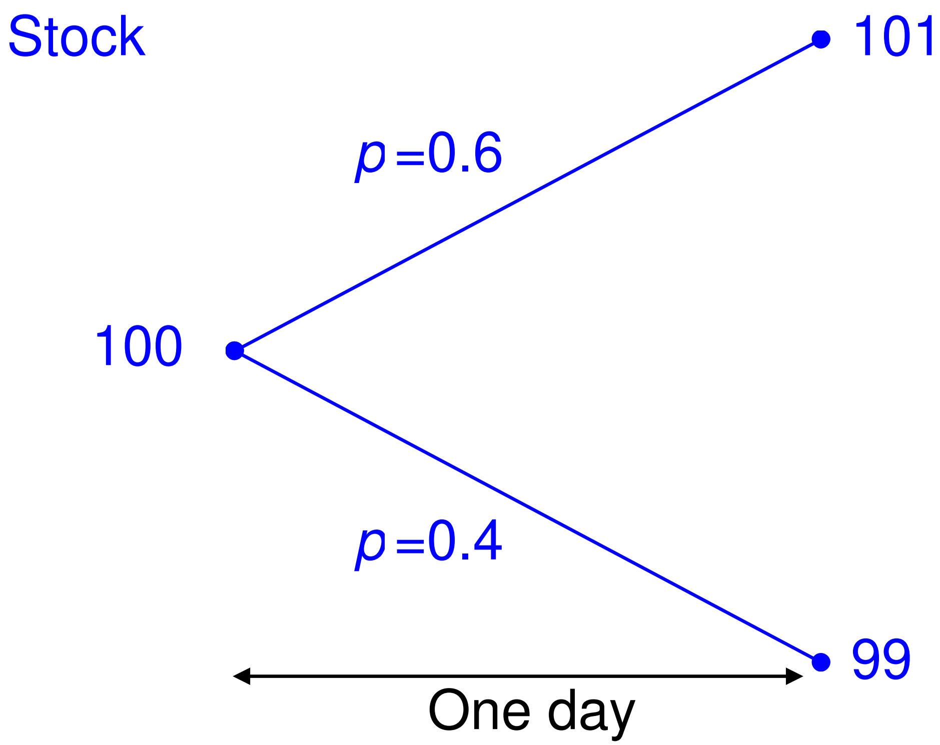
$$\frac{1}{1 + 0.1/252} = 0.9996.$$

The portfolio value today must be the *present value* of the portfolio value tomorrow

$$V - 0.5 \times 100 = -0.5 \times 99 \times 0.9996.$$

$$V = 0.51963.$$

Is the stock itself 'correctly' priced?



The expected stock value tomorrow is

$$0.6 \times 101 + 0.4 \times 99 = 100.2.$$

In an expectation's sense, the stock itself seems incorrectly priced.

Shouldn't it be valued at 100.2 today?

No.

We pay less than the future expected value because the stock is risky.

We want a positive expected return to compensate for the risk.

Complete markets

Options are redundant in this ‘world.’

Option payoffs can be **replicated** by stocks.

(Or cash can be replicated by the stock and the option. . . or the stock can be replicated by cash and the option.)

The real and risk-neutral worlds

In our world, the **real world**, we have used our statistical skills to estimate the future possible stock prices (99 and 101) and the probabilities of reaching them (0.4 and 0.6).

Some properties of the real world:

- We know all about delta hedging and risk elimination
- We are very sensitive to risk, and expect greater return for taking risk
- We don't value options by taking expectations

People often refer to the **risk-neutral world** in which people don't care about risk.

The risk-neutral world has the following characteristics:

- We don't care about risk, and don't expect any extra return for taking unnecessary risk

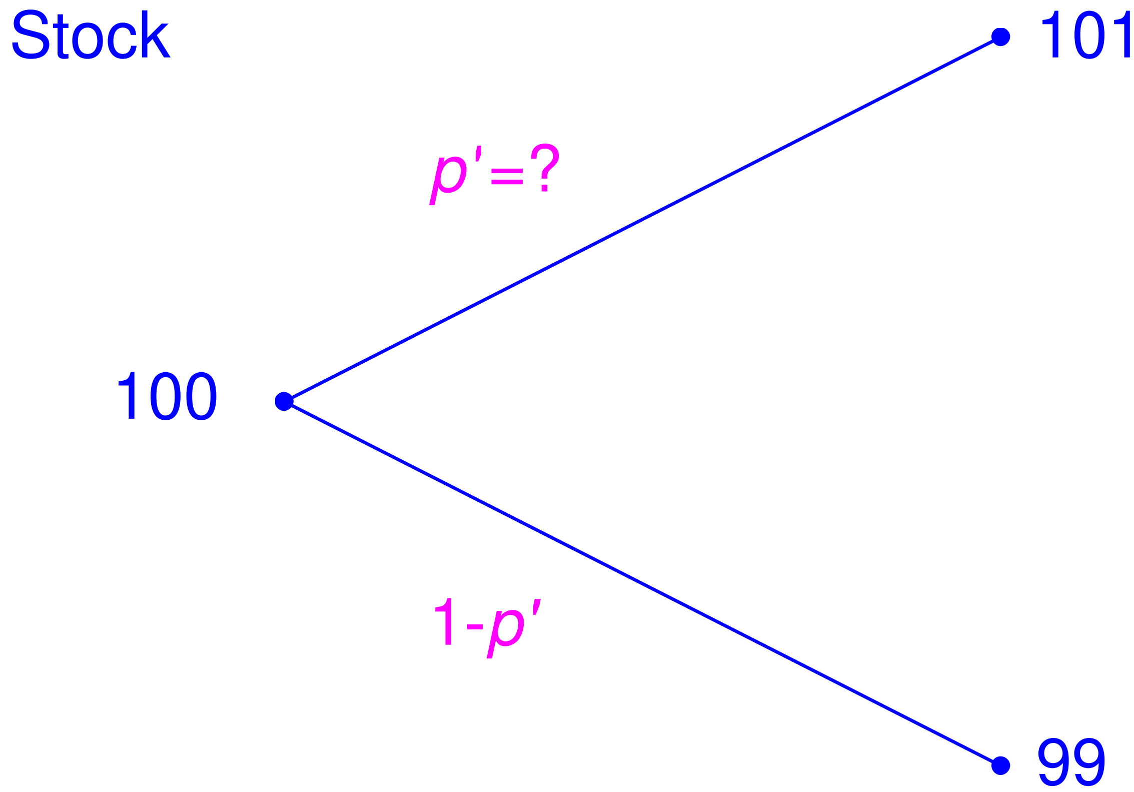
- We don't ever need to estimate probabilities of events happening

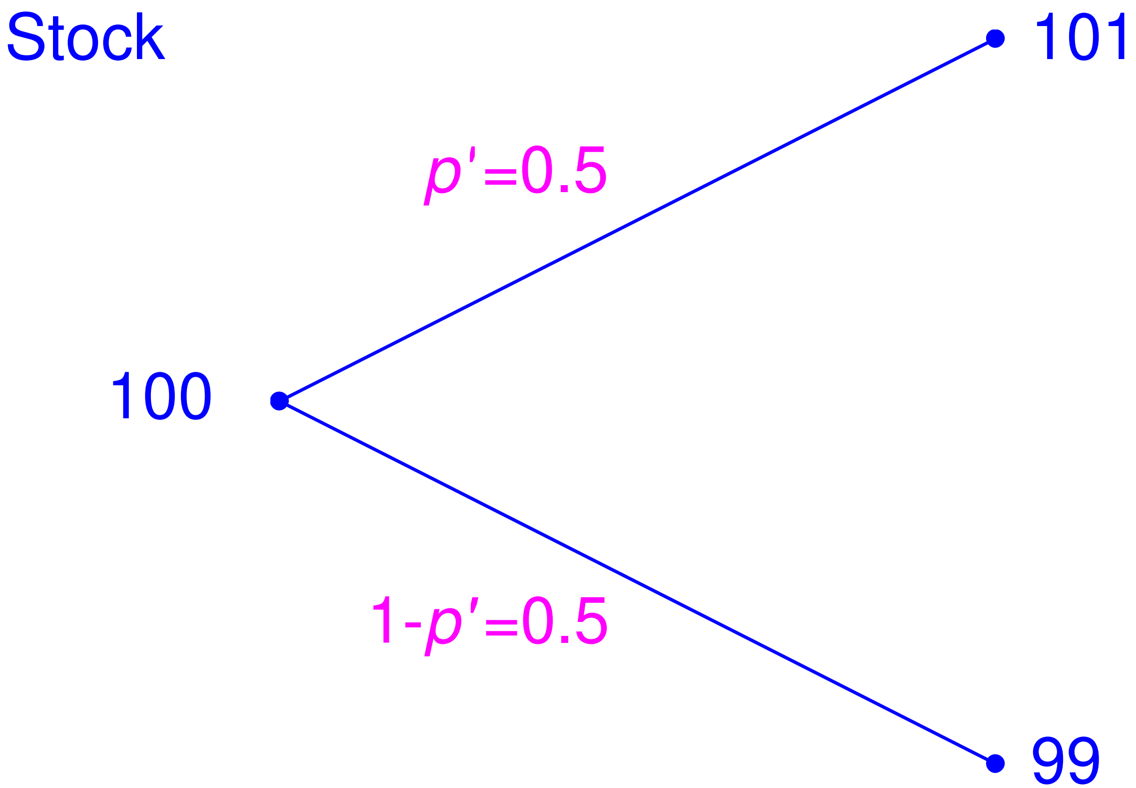
- We believe that everything is priced using simple expectations

Imagine yourself in the risk-neutral world. . .

. . . let's look at the stock price model.

If the stock is correctly priced today, using simple expectations, what would you deduce to be the probabilities of the stock price rising or falling?


Stock
101
p'=?
100
1-p'
99

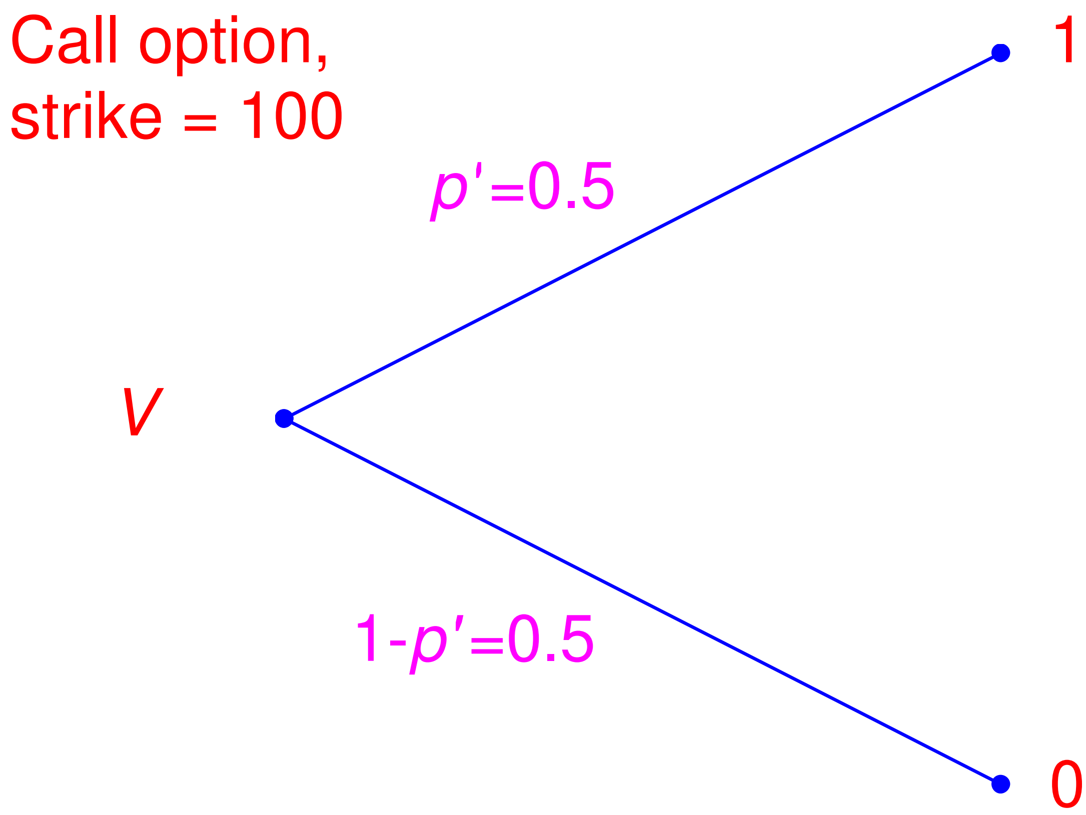

Stock
101
p'=0.5
100
1-p'=0.5
99

On the risk-neutral planet they calculate **risk-neutral probabilities** p' from the equation

$$p' \times 101 + (1 - p') \times 99 = 100.$$

From which $p' = 0.5$.

How would they then value the call option?


Call option,
strike = 100
1
p'=0.5
V
1-p'=0.5
0

Just take the **risk-neutral expectation**.

The answer is

$$\frac{1}{2}.$$

In the risk-neutral world they have exactly the same price for the option (but for different reasons!).

Non-zero interest rates

When interest rates are non zero we must perform exactly the same operations, but whenever we equate values at different times we must allow for present valuing.

With $r = 0.1$ we calculate the risk-neutral probabilities from

$$0.9996 \times \left(p' \times 101 + (1 - p') \times 99 \right) = 100.$$

So

$$p' = 0.51984.$$

The expected option payoff is now

$$0.51984 \times 1 + (1 - 0.51984) \times 0 = 0.51984.$$

And the present value of this is

$$0.9996 \times 0.51984 = 0.51963.$$

And this must be the option value. (It is the same as we derived the 'other' way!)

Symbols

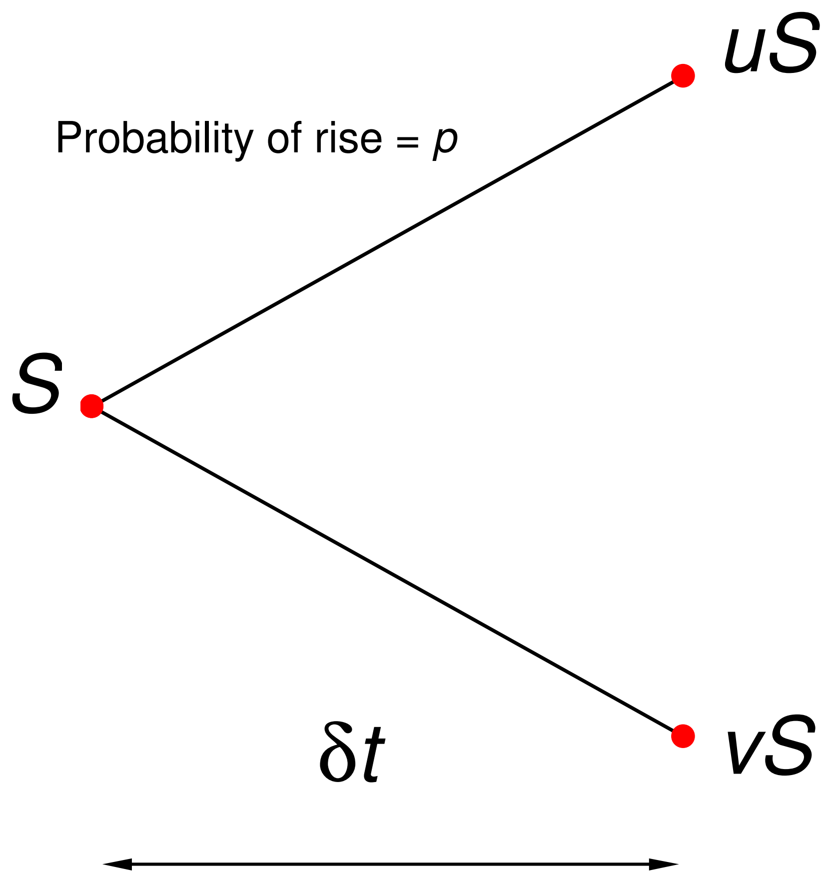
In the binomial model we assume that the asset, which initially has the value S, can, during a time step δt, either

- rise to a value $u \times S$ or

- fall to a value $v \times S$,

with $0 < v < 1 < u$.

- The probability of a rise is p and so the probability of a fall is $1 - p$.


uS
Probability of rise = p
S
vS
δt

Note: By *multiplying* the asset price by constants rather than *adding* constants, we will later be able to build up a whole tree of prices.

An equation for the value of an option

Suppose that we know the value of the option at the time $t+\delta t$. For example, this time may be the expiration of the option, say.

Now construct a portfolio at time t consisting of one option and a short position in a quantity Δ of the underlying. At time t this portfolio has value

$$\Pi = V - \Delta S,$$

where the option value V is for the moment unknown.

You'll recognize this as exactly what we did before, but now we're using symbols instead of numbers.

At time $t + \delta t$ the option takes one of two values, depending on whether the asset rises or falls

$$V^+ \text{ or } V^-.$$

At the same time the portfolio becomes either

$$V^+ - \Delta u S \text{ or } V^- - \Delta v S.$$

Since we know V^+, V^-, u, v and S the values of both of these expressions are just linear functions of Δ.

Hedging

Having the freedom to choose Δ, we can make the value of this portfolio the same whether the asset rises or falls. This is ensured if we make

$$V^+ - \Delta u S = V^- - \Delta v S.$$

This means that we should choose

$$\Delta = \frac{V^+ - V^-}{(u - v)S} \tag{1}$$

for hedging.

The portfolio value is then

$$V^+ - \Delta u S = V^+ - \frac{u(V^+ - V^-)}{(u - v)}$$

if the stock rises or

$$V^- - \Delta v S = V^- - \frac{v(V^+ - V^-)}{(u - v)}$$

if it falls.

And, of course these two expressions are the same.

Present valuing

Since the value of the portfolio ‘tomorrow’ has been guaranteed, we can say that its value ‘today’ must simply be tomorrow’s vale discounted. (This is the no-arbitrage argument.)

So the portfolio value today must be

$$\frac{1}{1 + r\,\delta t}\left(V^+ - \frac{u(V^+ - V^-)}{(u - v)}\right)$$

$$= V - \Delta S = V - \frac{V^+ - V^-}{(u - v)S}S.$$

Which is just an equation for V.

Rearranging as an equation for V we get

$$V = \frac{V^+ - V^-}{u - v} + \frac{1}{1 + r\,\delta t}\frac{(uV^- - vV^+)}{(u - v)}.$$

This is an equation for V given V^+, and V^-, the option values at the next time step, and the parameters u and v describing the random walk of the asset.

But it can be written more elegantly than this.

This equation can also be written as

$$V = \frac{1}{1 + r\,\delta t}\left(p' V^{+} + (1 - p') V^{-}\right), \tag{2}$$

where

$$p' = \frac{1 - v + r\,\delta t}{u - v} \tag{3}$$

The left-hand side of Equation (2) is today's option value.

The right-hand side of Equation (2) is just the present value of something like an expectation; it's the sum of probabilities multiplied by events.

If only the expression contained p, the real probability of a stock rise, then this expression would be the expected value at the next time step.

We see that the probability of a rise or fall is irrelevant as far as option pricing is concerned. But what if we interpret p' as a probability? Then we could 'say' that the option price is the present value of an expectation. But not the real expectation.

We are back with risk-neutral expectations again!

Now let's bring together the stats of our previous (stochastic differential equation) model and the present option-valuation model!

How should we choose u, v and p?

Let's choose them so that they have the same mean and standard deviation as our earlier lognormal random walk!

u, v and p will therefore be related to μ and σ.

So we are going to make our binomial be like a discrete-time version of the continuous-time lognormal.

(Why only match mean and standard deviation? CLT!)

Problem. . . **three unknowns (u, v and p) only two equations (mean and standard deviation).**

Not really a problem, there are just an infinite number of solutions!

The mean change in the asset price for the lognormal random walk over a time step δt is

$$\mu S \, \delta t.$$

For the binomial the mean change is

$$puS + (1-p)vS - S.$$

Equate these two, that's the first equation.

The variance of the change in the asset price for the lognormal random walk over a time step δt is

$$\sigma^2 S^2 \, \delta t.$$

And for the binomial the variance is

$$S^2 \Big(p(u - 1 - (pu + (1-p)v - 1))^2 \\ + (1-p)(v - 1 - (pu + (1-p)v - 1))^2 \Big)$$

Equate these two, and that's the second equation!

So which of the infinite number of solutions should we choose?

Note: Often people throw in the third equation that $uv = 1$. That is so an up (down) followed by a down (up) gets us back to where we started. (And I use this in my later numerical example.)

But I'm going to use the (approximate) solution that makes the mathematics prettier (i.e. more compact!)

Here's one solution

$$u = 1 + \sigma\sqrt{\delta t},$$

$$v = 1 - \sigma\sqrt{\delta t}$$

and

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}.$$

Of the infinite number of ways of matching mean and standard deviation, this is the neatest (and therefore the best for teaching purposes!).

Exercise!

Real versus risk neutral!

Let's compare the expression for p' with the expression for the actual probability p. Remembering that

$$p' = \frac{1 - v + r\,\delta t}{u - v}$$

we have

$$p' = \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}$$

but

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}.$$

The two expressions differ in that where one has the interest rate r the other has the drift μ, but are otherwise the same. Strange.

- We call p' the **risk-neutral probability**. It's like the real probability, but the real probability if the drift rate were r instead of μ.

Observe that the risk-free interest plays two roles in option valuation. It's used once for discounting to give present value, and it's used as the drift rate in the asset price random walk.

Where did the probability p go?

What happened to the probability p and the drift rate μ?

Interpreting p' as a probability, (2) is the statement that

the option value at any time is the present value of the risk-neutral expected value at any later time.

In reading books or research papers on mathematical finance you will often encounter the expression 'risk-neutral' this or that, including the expression risk-neutral probability.

You can think of an option value as being the present value of an expectation, only it's not the real expectation.

Don't worry we'll come back to this several more times until you get the hang of it.

Counterintuitive?

Two stocks A and B.

Both have same value, same volatility and are denominated in the same currency.

Both have call options with the same strike and expiration.

Stock A is doubling in value every year, stock B is halving.

Both call options have the same value.

But which would you buy?

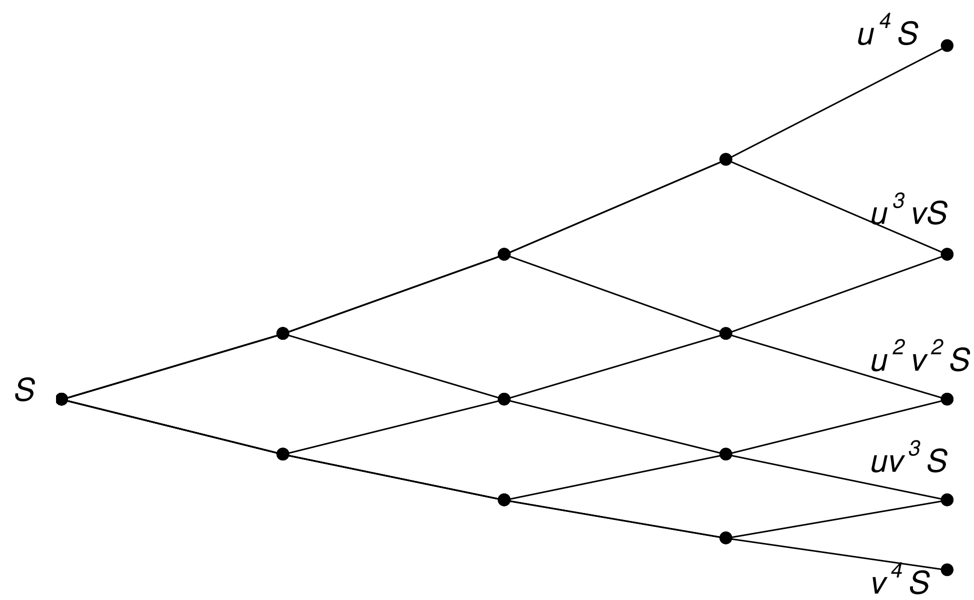
The binomial tree

The binomial model allows the stock to move up or down a prescribed amount over the next time step. If the stock starts out with value S then it will take either the value uS or vS after the next time step.

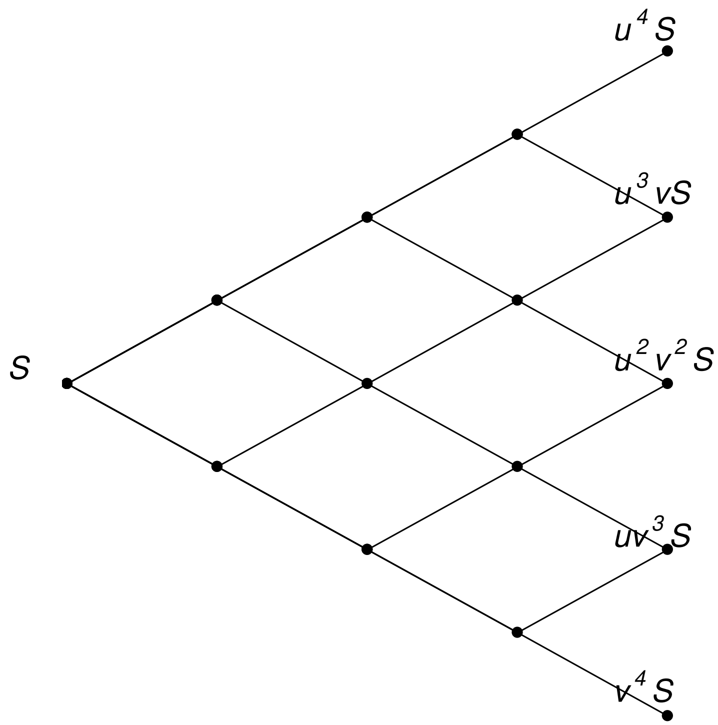
We can extend the random walk to the next time step. After two time steps the asset will be at either u^2S, if there were two up moves, uvS, if an up was followed by a down or vice versa, or v^2S, if there were two consecutive down moves.

After three time steps the asset can be at u^3S, u^2vS, etc.

Imagine extending this random walk out all the way until expiry. The result is the **binomial tree**.



Observe how the tree bends due to the geometric nature of the asset growth.



Often this tree is drawn as here because it is easier to draw, but this doesn't quite capture the correct structure.

Valuing back down the tree

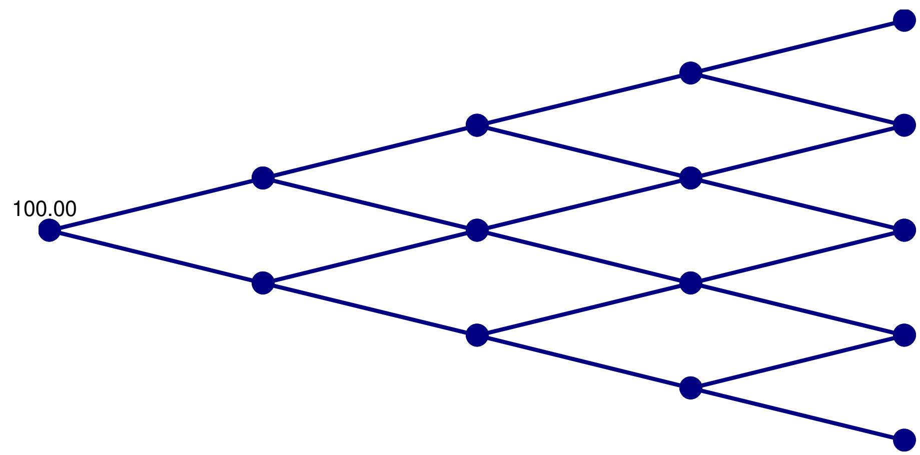
We certainly know V^+ and V^- at expiry, time T, because we know the option value as a function of the asset then, this is the payoff function.

If we know the value of the option at expiry we can find the option value at the time $T - \delta t$ for all values of S on the tree. But knowing these values means that we can find the option values one step further back in time.

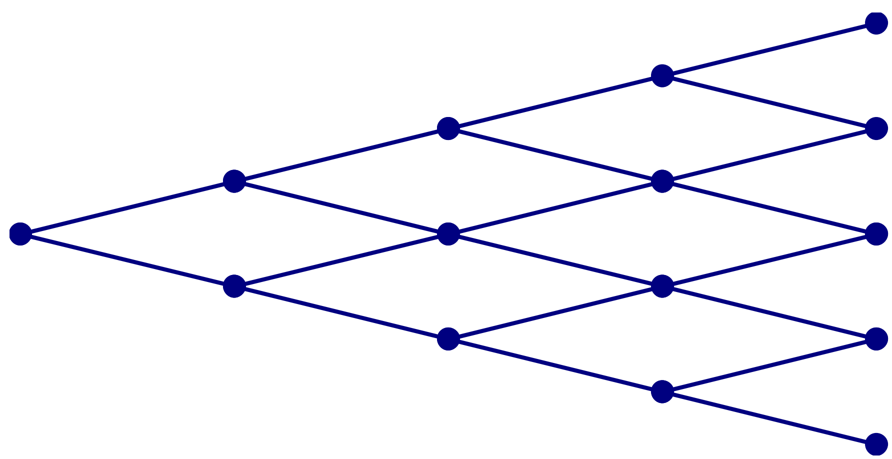
- Thus we work our way back down the tree until we get to the root.

This root is the current time and asset value, and thus we find the option value today.

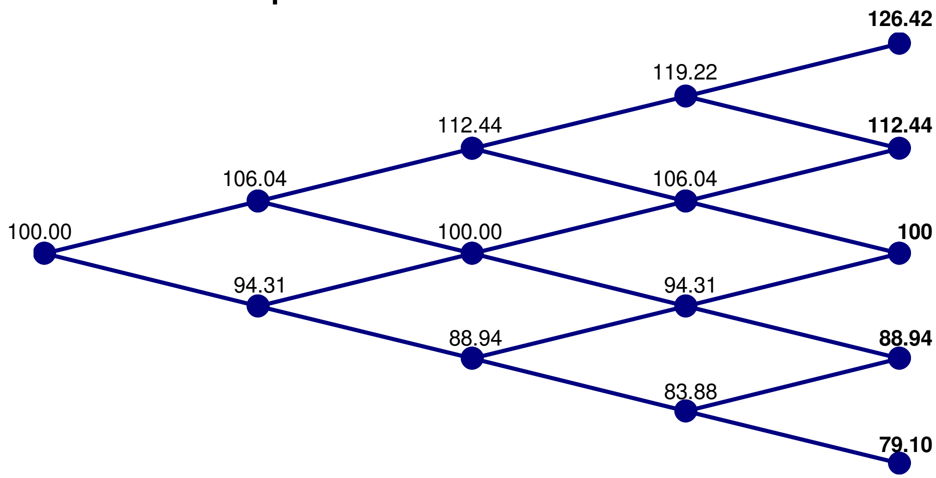
Tree of asset prices



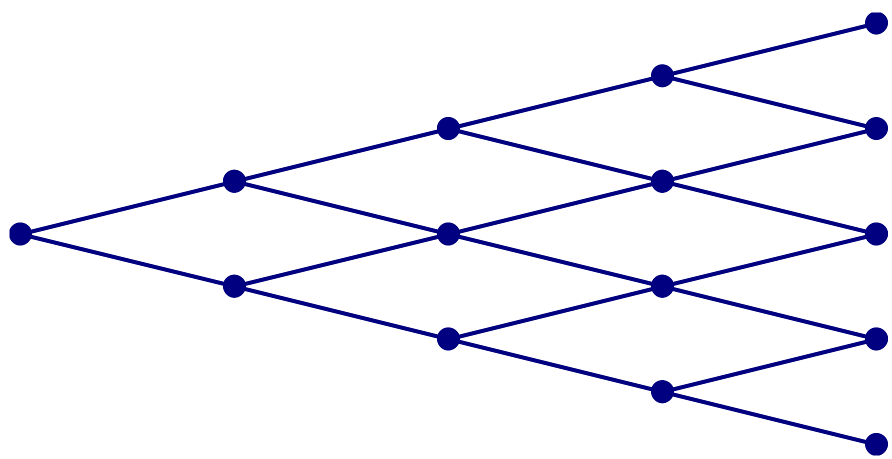
Tree of option prices



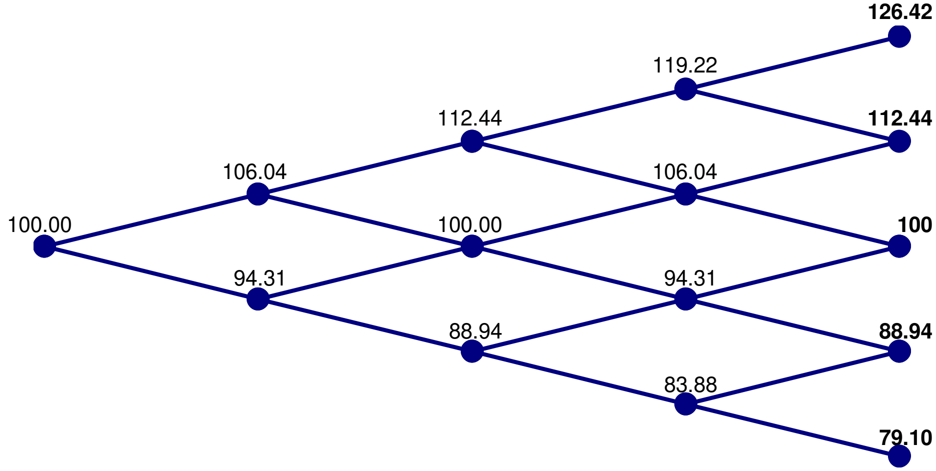
Tree of asset prices



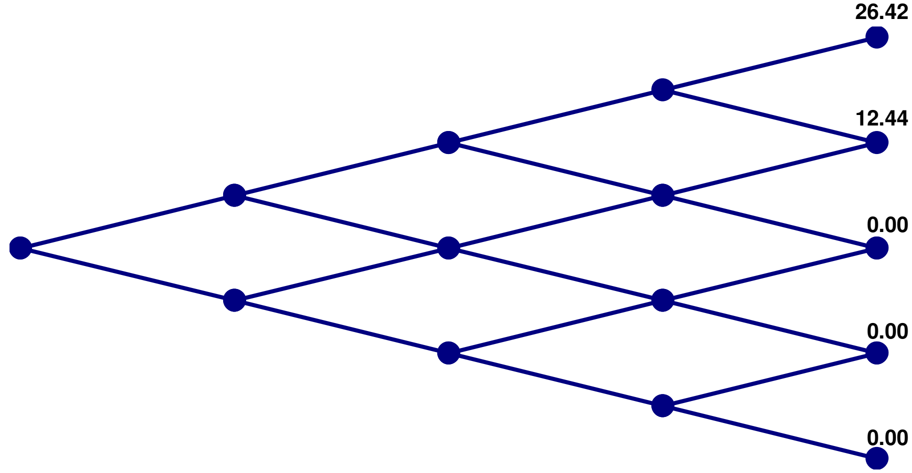
Tree of option prices



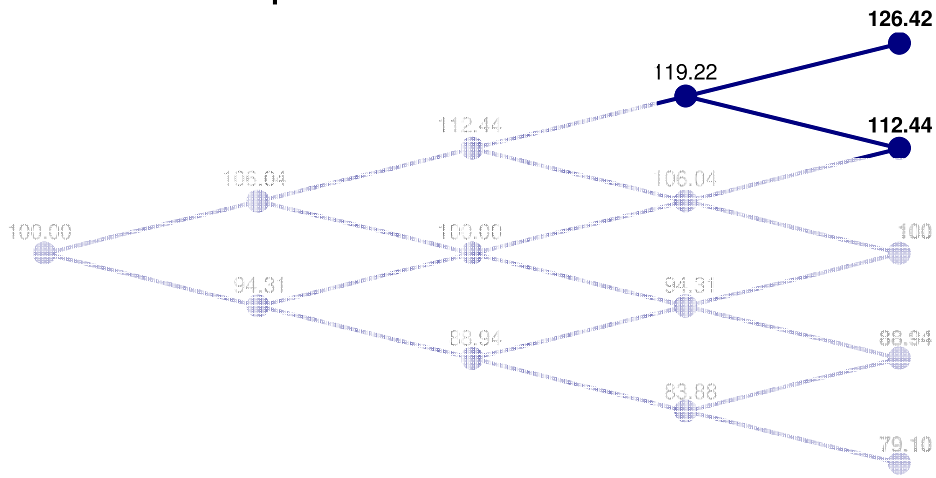
Tree of asset prices



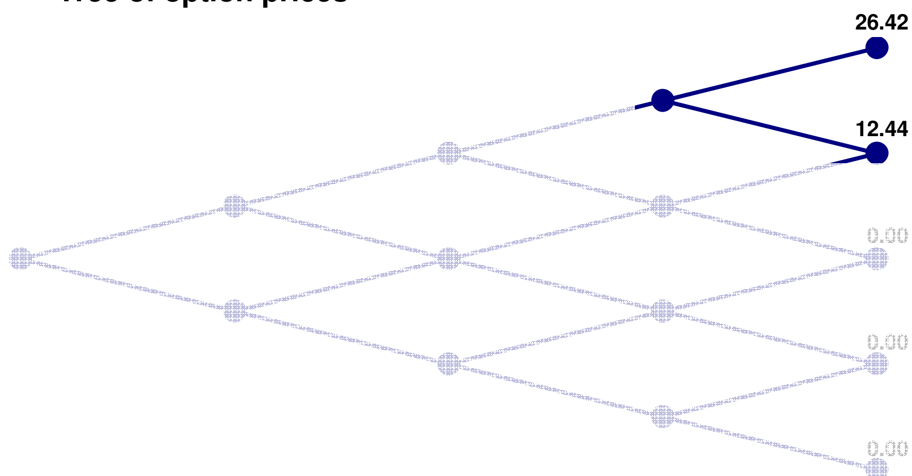
Tree of option prices



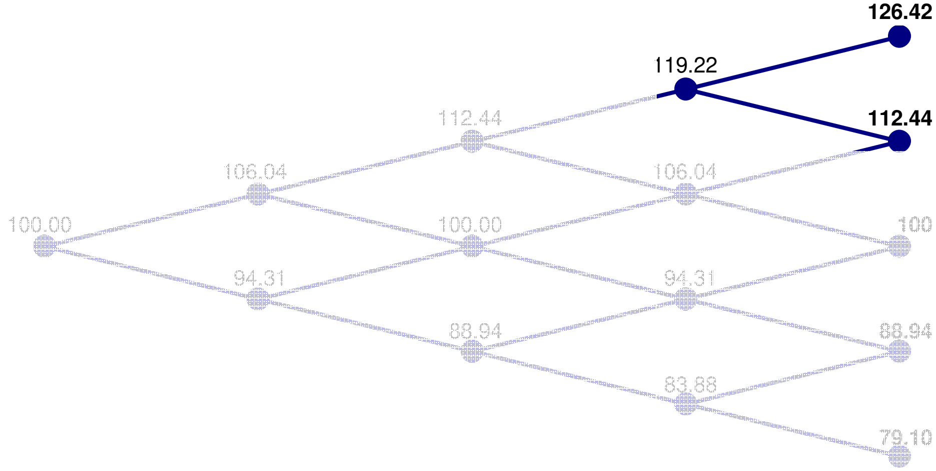
Tree of asset prices



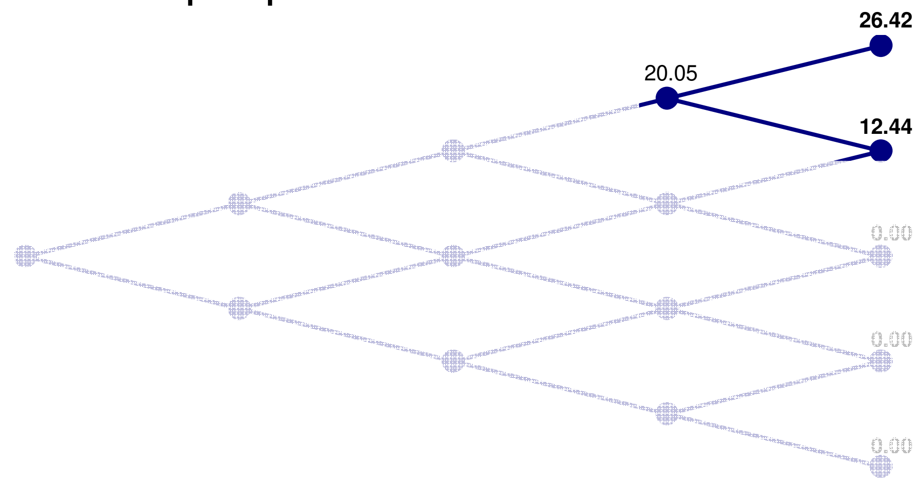
Tree of option prices



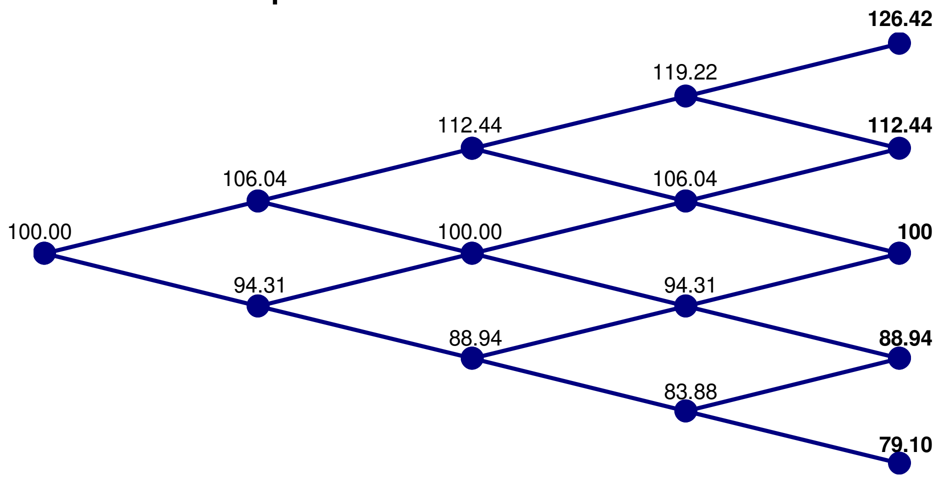
Tree of asset prices



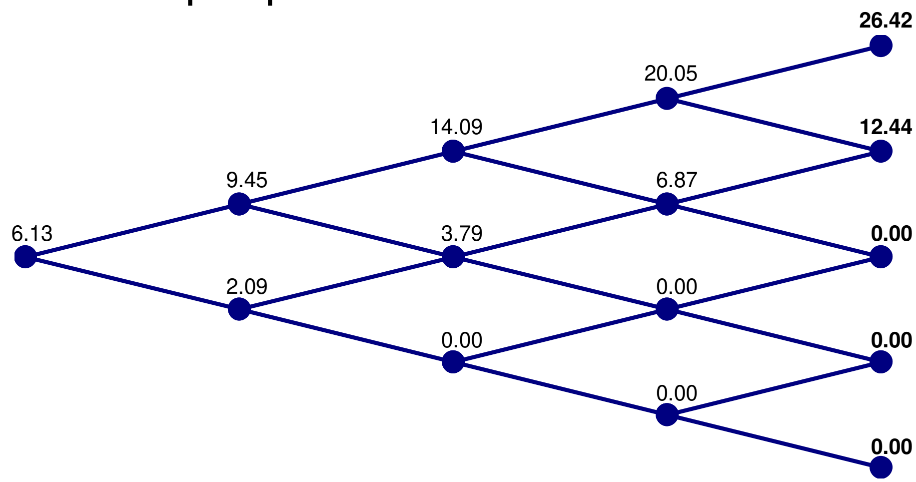
Tree of option prices



Tree of asset prices



Tree of option prices



The continuous-time limit

Let's examine (2) as $\delta t \to 0$.

We'll end up with what's known as a **partial differential equation**, the famous **Black–Scholes equation**.

Recall that

$$u = 1 + \sigma\sqrt{\delta t}$$

and

$$v = 1 - \sigma\sqrt{\delta t}.$$

Now let's represent the option prices, not by a collection of numbers associated with places in a tree but by a function of stock price S and time t.

Call that function $V(S,t)$.

Think of $V(S,t)$ being the option value at the root of a single branch.

In this notation we can write V^+ in terms of V according to

$$V^+ = V(uS, t+\delta t) = V\left((1+\sigma\sqrt{\delta t})S, t+\delta t\right).$$

And similarly

$$V^- = V(vS, t+\delta t) = V\left((1-\sigma\sqrt{\delta t})S, t+\delta t\right).$$

And, of course, we also have

$$\Delta = \frac{V^+ - V^-}{(u-v)S}$$

$$= \frac{V\left((1+\sigma\sqrt{\delta t})S, t+\delta t\right) - V\left((1-\sigma\sqrt{\delta t})S, t+\delta t\right)}{2\sigma\sqrt{\delta t}S}$$

Now expand for small δt to get

$$V\left((1+\sigma\sqrt{\delta t})S, t+\delta t\right) \approx$$

$$V(S,t) + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\sigma\sqrt{\delta t}\,S + \tfrac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\delta t\,S^2 + o(\delta t).$$

And a similar expression for $V\left((1-\sigma\sqrt{\delta t})S, t+\delta t\right)$.

Substituting these expressions into that for Δ we get we find that

$$\Delta \sim \frac{\partial V}{\partial S} \quad \text{as} \quad \delta t \to 0.$$

Similarly, the 'pricing equation' becomes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

This is the Black–Scholes equation.

Summary

Please take away the following important ideas

- Delta hedging can be used for the elimination of risk
- The binomial method is a simple way to value options
- The concept of risk neutrality, its meaning and use
- The continuous-time limit of the binomial model is the Black–Scholes equation