

Solutions to CQF Module 1 Assignment Summer 2020

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W} dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(t), t)$ where $dW(t)$ is an increment of a Brownian motion.

If $W(0) = 0$ evaluate

$$\begin{aligned} & \int_0^t \tau^2 \sin W dW(\tau). \\ \downarrow \frac{\partial F}{\partial W} &= t^2 \sin W \longrightarrow F = -t^2 \cos W \downarrow \\ \frac{\partial^2 F}{\partial W^2} &= t^2 \cos W \quad \frac{\partial F}{\partial t} = -2t \cos W \end{aligned}$$

and substitute into the integral formula

$$\int_0^t \tau^2 \sin W dW(\tau) = -t^2 \cos W - \int_0^t \left(-2\tau \cos W + \frac{1}{2} \tau^2 \cos W \right) d\tau$$

- b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Obtain a SDE $df(S, t)$ for each of the following functions. Here we use Itô IV

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial f}{\partial S_t} dW_t.$$

- i $f(S, t) = \alpha^t + \beta t S^n$ α, β are constants

$$\begin{aligned} \frac{\partial f}{\partial t} &= \alpha^t \log \alpha + \beta S^n; \quad \frac{\partial f}{\partial S} = n\beta t S^{n-1}; \quad \frac{\partial^2 f}{\partial S^2} = n(n-1)\beta t S^{n-2} \\ df &= \left(\alpha^t \log \alpha + \beta S^n + n\mu\beta t S^n + \frac{1}{2} n(n-1)\beta t \sigma^2 S^n \right) dt + \sigma n\beta t S^n dW_t \end{aligned}$$

- ii $f(S, t) = \log tS + \cos tS$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{t} - S \sin tS; \quad \frac{\partial f}{\partial S} = \frac{1}{S} - t \sin tS; \quad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} - t^2 \cos tS \\ df &= \left(\frac{1}{t} - S \sin tS + \mu S \left(\frac{1}{S} - t \sin tS \right) + \frac{1}{2} \sigma^2 S^2 \left(-\frac{1}{S^2} - t^2 \cos tS \right) \right) dt + \sigma S \left(\frac{1}{S} - t \sin tS \right) dW_t \end{aligned}$$

You can simplify, but it just creates more terms.

2. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\ dr_t &= \gamma(m - r_t) dt + c dW_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E} \left[dW_t^{(1)} dW_t^{(2)} \right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

Begin by writing a 3D Taylor expansion for $V(t, S_t, r_t)$

$$\begin{aligned} &V(t + dt, S_t + dS, r_t + dr) - V(t, S_t, r_t) \\ &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial r \partial S} dr dS \end{aligned}$$

Since $dW_i^2 \rightarrow dt$ in the mean square limit for $i = 1, 2$, we see that

$$\begin{aligned} dS_t^2 &\rightarrow \sigma^2 S_t^2 dt, \\ dr_t^2 &\rightarrow c^2 dt, \end{aligned}$$

Also, since $dW_t^{(1)} dW_t^{(2)} = \rho dt$, we see that

$$dS_t dr_t \rightarrow \rho c \sigma S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for V is given by

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma(m - r_t) \frac{\partial V}{\partial r} \right. \\ &\quad \left. + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_t \frac{\partial^2 V}{\partial r \partial S} \right) dt \\ &\quad + \sigma S_t \frac{\partial V}{\partial S} dW_t^{(1)} + c \frac{\partial V}{\partial r} dW_t^{(2)} \end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned} V(t, S_t, r_t) &= \underbrace{V(0, S_0, r_0)}_{=v} + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} + \gamma(m - r_\tau) \frac{\partial V}{\partial r} \right. \\ &\quad \left. + \frac{1}{2} \sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_\tau \frac{\partial^2 V}{\partial r \partial S} \right) d\tau \\ &\quad + \int_0^t \sigma S_\tau \frac{\partial V}{\partial S} dW_\tau^{(1)} + \int_0^t c \frac{\partial V}{\partial r} dW_\tau^{(2)} \end{aligned}$$

3. An equity price S evolves according to Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are constants. We know that an explicit solution is

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

where S_0 is S_t at time $t = 0$.

By working through all the integration steps, deduce that the expected value of S_t at time $t > 0$, given S_0 , is

$$\mathbb{E}[S_t | S_0] = S_0 e^{\mu t}.$$

You are required to present all your integration steps to obtain the expectation.

We wish to calculate the expected terminal value of this risky asset. So

$$\begin{aligned} \mathbb{E}[S_t] &= S_0 \mathbb{E} \left[\exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \right] \\ &= S_0 e^{(\mu - \frac{1}{2} \sigma^2) t} \mathbb{E}[e^{\sigma W_t}] \end{aligned}$$

$$\mathbb{E}[S(t) | S_0] = \mathbb{E} \left[S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t} \right] = S_0 e^{(\mu - \sigma^2/2)t} \mathbb{E}[e^{\sigma W_t}]$$

Recall $W_t \sim N(0, t)$. To calculate the shortened expectation $\mathbb{E}[e^{\sigma W_t}] =$

$$\begin{aligned} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x} e^{-x^2/2t} dx &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x - x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x - \sigma t)^2/2t} e^{\sigma^2 t/2} dx \\ &= e^{\sigma^2 t/2} \underbrace{\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x - \sigma t)^2/2t} dx}_{=1} \end{aligned}$$

$$\text{So } \mathbb{E}[S(t) | S_0] = S_0 e^{(\mu - \sigma^2/2)t} e^{\sigma^2 t/2} = S_0 e^{\mu t}$$

4. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}\sigma^2 \frac{d^2 p_\infty}{du^2} + \theta \frac{d}{du} (up_\infty) = 0$$

$p_\infty = p_\infty(u)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation: Integrate wrt u

$$\frac{1}{2}\sigma^2 \frac{dp}{du} + \theta (up) = k$$

where k is a constant of integration and can be calculated from the conditons, that as

$$u \rightarrow \infty : \begin{cases} \frac{dp}{du} \rightarrow 0 \\ p \rightarrow 0 \end{cases} \Rightarrow k = 0$$

which gives

$$\frac{1}{2}\sigma^2 \frac{dp}{du} = -\theta (up),$$

a first order variable separable equation. So

$$\begin{aligned} \frac{1}{2}\sigma^2 \int \frac{dp}{p} &= -\theta \int u du \rightarrow \\ \frac{1}{2}\sigma^2 \ln p &= -\theta \left(\frac{u^2}{2} \right) + C, \quad C \text{ is arbitrary.} \\ \ln p &= -\frac{\theta}{\sigma^2} u^2 + c \end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p = \exp \left(-\frac{\theta}{\sigma^2} u^2 + c \right) = A \exp \left(-\frac{\theta}{\sigma^2} u^2 \right)$$

and we know as p_∞ is a PDF

$$\int_{-\infty}^{\infty} p_\infty du' = 1 \rightarrow A \int_{-\infty}^{\infty} e^{-(\frac{\theta}{\sigma^2} u^2)} du = 1$$

A few (related) ways to calculate A . Now use the error function, i.e. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. So put

$$x = \sqrt{\frac{\theta}{\sigma^2}} u \rightarrow dx = \sqrt{\frac{\theta}{\sigma^2}} du$$

which transforms the integral above

$$\frac{A\sigma}{\sqrt{\theta}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1 \rightarrow A\sigma \sqrt{\frac{\pi}{\theta}} = 1 \therefore A = \frac{1}{\sigma} \sqrt{\frac{\theta}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty = \frac{1}{\sigma} \sqrt{\frac{\theta}{\pi}} \exp \left(-\frac{\theta}{\sigma^2} u'^2 \right).$$

If we compare this to

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{1}{2\sigma^2} (x - \mu)^2 \right)$$

we find u' is normally distributed with mean 0. For the variance write $-\frac{\theta}{\sigma^2} u'^2$ as

$$-\frac{1}{2} \cdot 2 \cdot \frac{\theta}{\sigma^2} u'^2 \equiv -\frac{1}{2} \frac{1}{\sigma^2/2\theta} u'^2$$

to give the variance as $\sigma^2/2\theta$ and standard deviation is $\sqrt{\sigma^2/2\theta}$

5. The steady state equation for our model becomes

$$\frac{1}{2}\nu^2 \frac{d^2}{dr^2} (r^{2\beta} p_\infty(r)) - \frac{d}{dr} (u(r) p_\infty(r)) = 0$$

This can be simply integrated once to give

$$\begin{aligned} \frac{1}{2}\nu^2 \frac{d}{dr} (r^{2\beta} p_\infty(r)) - (u(r) p_\infty(r)) &= \text{const} \\ \frac{1}{2}\nu^2 \left(r^{2\beta} \frac{dp_\infty}{dr} \right) + \nu^2 \beta r^{2\beta-1} p_\infty(r) - (u(r) p_\infty(r)) &= \text{const} \end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left. \begin{aligned} p_\infty(r) \\ \frac{dp_\infty}{dr} \end{aligned} \right\} \longrightarrow 0$$

$$\begin{aligned} u(r) p_\infty(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{dp_\infty}{dr} + \nu^2 \beta r^{2\beta-1} p_\infty(r) \\ u(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{1}{p_\infty(r)} \frac{dp_\infty}{dr} + \nu^2 \beta r^{2\beta-1} \end{aligned}$$

We can write $\frac{1}{p_\infty} \frac{dp_\infty}{dr}$ as $\frac{d}{dr} (\log p_\infty)$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr} (\log p_\infty) + \nu^2 \beta r^{2\beta-1}$$

a. Let X_t, Y_t be two one-dimensional stochastic processes, where

$$\begin{aligned} dX_t &= a(t, X_t) dt + b(t, X_t) dW_t^{(1)}, \\ dY_t &= c(t, Y_t) dt + d(t, Y_t) dW_t^{(2)}. \end{aligned} \tag{2}$$

The Wiener processes are correlated such that $\mathbb{E} [W_t^{(1)} W_t^{(2)}] = \rho t$. By applying the two-dimensional form of Itô's lemma with $f(X, Y) = XY$

i.

$$\begin{aligned} df &= \frac{\partial f}{\partial X} dX + \frac{\partial f}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} dX^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y^2} dY^2 + \frac{\partial^2 f}{\partial X \partial Y} dX dY \\ \frac{\partial f}{\partial X} &= Y \quad \frac{\partial f}{\partial Y} = X \\ \frac{\partial^2 f}{\partial X^2} &= 0 \quad \frac{\partial^2 f}{\partial Y^2} = 0 \quad \frac{\partial^2 f}{\partial Y \partial X} = 1 = \frac{\partial^2 f}{\partial X \partial Y} \end{aligned}$$

which gives

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

ii. Rearranging the above product rule

$$X_t dY_t = d(X_t Y_t) - Y_t dX_t - dX_t dY_t$$

and integrating over $[0, t]$

$$\begin{aligned} \int_0^t X_s dY_s &= \int_0^t d(X_s Y_s) - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s \\ &= X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s. \end{aligned}$$

ii.

$$\begin{aligned}\frac{\partial f}{\partial X} &= 1/Y & \frac{\partial f}{\partial Y} &= -X/Y^2 & \frac{\partial^2 f}{\partial X^2} &= 0 \\ \frac{\partial^2 f}{\partial Y^2} &= 2X/Y^3 & \frac{\partial^2 f}{\partial X \partial Y} &= -1/Y^2 = \frac{\partial^2 f}{\partial Y \partial X}\end{aligned}$$

which gives

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y} \left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dXdY}{XY} + \left(\frac{dY}{Y}\right)^2 \right)$$

- b. Consider a function $V(t, X_t, Y_t)$, where X_t, Y_t are defined by (1.1). Using Itô and suitable integration over $[0, T]$ obtain an expression for $V(T, X_T, Y_T)$.

$$\begin{aligned}dV &= \left(\frac{\partial V}{\partial t} + a \frac{\partial V}{\partial X} + c \frac{\partial V}{\partial Y} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} d^2 \frac{\partial^2 V}{\partial Y^2} + bd \frac{\partial^2 V}{\partial X \partial Y} \right) dt \\ &\quad + b \frac{\partial V}{\partial X} dW_t^{(1)} + d \frac{\partial V}{\partial Y} dW_t^{(2)}\end{aligned}$$

Integrating and rearranging then gives $V(T, X_T, Y_T) =$

$$\begin{aligned}V(0, X_0, Y_0) &+ \int_0^T \left(\frac{\partial V}{\partial t} + a \frac{\partial V}{\partial X} + c \frac{\partial V}{\partial Y} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2} d^2 \frac{\partial^2 V}{\partial Y^2} + \rho bd \frac{\partial^2 V}{\partial X \partial Y} \right) dt + \\ &\int_0^T b \frac{\partial V}{\partial X} dW_t^{(1)} + \int_0^T d \frac{\partial V}{\partial Y} dW_t^{(2)}.\end{aligned}$$

c.

$$\begin{aligned}X_{i+1} &= X_i + a\delta t + b\phi_1\sqrt{\delta t} \\ Y_{i+1} &= Y_i + c\delta t + d\phi_2\sqrt{\delta t}\end{aligned}$$

$\phi_1, \phi_2 \sim N(0, 1)$ and $\mathbb{E}[\phi_1\phi_2] = \rho$. The pair $\epsilon_1, \epsilon_2 \sim N(0, 1)$ with $\mathbb{E}[\epsilon_1\epsilon_2] = 0$. Construct $\phi_1 = \epsilon_1$; $\phi_2 = \alpha\epsilon_1 + \beta\epsilon_2$,

$$\begin{aligned}\mathbb{E}[\phi_1\phi_2] &= \rho = \mathbb{E}[\epsilon_1(\alpha\epsilon_1 + \beta\epsilon_2)] \\ \mathbb{E}[\epsilon_1(\alpha\epsilon_1 + \beta\epsilon_2)] &= \rho \\ \alpha\mathbb{E}[\epsilon_1^2] + \beta\mathbb{E}[\epsilon_1\epsilon_2] &= \rho \rightarrow \alpha = \rho\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\phi_2^2] &= 1 = \mathbb{E}[(\alpha\epsilon_1 + \beta\epsilon_2)^2] \\ &= \mathbb{E}[\alpha^2\epsilon_1^2 + \beta^2\epsilon_2^2 + 2\alpha\beta\epsilon_1\epsilon_2] \\ &= \alpha^2\mathbb{E}[\epsilon_1^2] + \beta^2\mathbb{E}[\epsilon_2^2] + 2\alpha\beta\mathbb{E}[\epsilon_1\epsilon_2] = 1 \\ \rho^2 + \beta^2 &= 1 \rightarrow \beta = \sqrt{1 - \rho^2}\end{aligned}$$

The correlated SDEs become

$$\begin{aligned}X_{i+1} &= X_i + a\delta t + b\epsilon_1\sqrt{\delta t} \\ Y_{i+1} &= Y_i + c\delta t + d\left(\rho\epsilon_1 + \sqrt{1 - \rho^2}\epsilon_2\right)\sqrt{\delta t}\end{aligned}$$

- a. For which values of k is the process

$$Y_t = W_t^4 - 6tW_t^2 + kt^2, \quad t \geq 0,$$

a martingale? The problem is asking you to calculate the value of k such that Y_t has zero drift. Using Itô

$$\begin{aligned}dY_t &= \left(\frac{\partial Y_t}{\partial t} + \frac{1}{2} \frac{\partial^2 Y_t}{\partial W^2} \right) dt + \frac{\partial Y_t}{\partial W} dW \\ \frac{\partial Y_t}{\partial t} &= -6W_t^2 + 2kt; \quad \frac{\partial Y_t}{\partial W} = 4W_t^3 - 12tW_t; \quad \frac{\partial^2 Y_t}{\partial W^2} = 12W_t^2 - 12t\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_t}{\partial t} + \frac{1}{2} \frac{\partial^2 Y_t}{\partial W^2} &= 0 \rightarrow -6W_t^2 + 2kt + 6W_t^2 - 6t = 0 \\ k &= 3.\end{aligned}$$

b. Show that $X_t = \cosh(\theta W_t) e^{-\theta^2 t/2}$; $\theta \in \mathbb{R}$, is a martingale.

$$F(W_t, t) = \cosh(\theta W_t) e^{-\theta^2 t/2}$$

Using Itô

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) dt + \frac{\partial F}{\partial W} dW$$

So checking that $\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} = 0$, i.e. a driftless process.

$$\begin{aligned}\frac{\partial F}{\partial t} &= \cosh(\theta W_t) e^{-\theta^2 t/2} = -\frac{\theta^2}{2} \cosh(\theta W_t) e^{-\theta^2 t/2} \\ \frac{\partial F}{\partial W} &= \theta \sinh(\theta W_t) e^{-\theta^2 t/2}; \quad \frac{\partial^2 F}{\partial W^2} = \theta^2 \cosh(\theta W_t) e^{-\theta^2 t/2}\end{aligned}$$

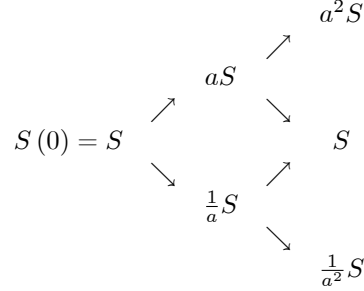
$$\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} = -\frac{\theta^2}{2} \cosh(\theta W_t) e^{-\theta^2 t/2} + \frac{1}{2} \left(\theta^2 \cosh(\theta W_t) e^{-\theta^2 t/2} \right)$$

Hence a Martingale.

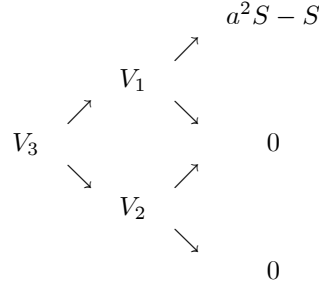
6. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant. Asset:



Option:



- a. Find all the one period risk-neutral probabilities and the corresponding probability measure \mathbb{Q} on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price. These are when $r = 0$,

$$\begin{aligned} q(\text{up}) &= \frac{s - s_d}{s_u - s_d} \\ q(\text{down}) &= \frac{s_u - s}{s_u - s_d} \end{aligned}$$

For each time-step we have the probabilities:

$$\begin{aligned} q(\text{up}) &= \frac{S - \frac{1}{a}S}{aS - \frac{1}{a}S} = \frac{1}{a+1}, \\ q(\text{down}) &= \frac{aS - S}{aS - \frac{1}{a}S} = \frac{a}{a+1}. \end{aligned}$$

\therefore

$$\begin{aligned} \omega_1 &= \frac{1}{(a+1)^2} \\ \omega_2 &= \frac{a}{(a+1)^2} \\ \omega_3 &= \frac{a}{(a+1)^2} \\ \omega_4 &= \frac{a^2}{(a+1)^2}. \end{aligned}$$

So the expected value is:

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} p(\omega) X(\omega) = p(\omega_1)(a^2S - S) + 0 + 0 + 0 = \frac{a-1}{a+1}S,$$

as before!

- b. Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.

This is a special case of the above model when $a = 2$. Substituting in $a = 2$ into the option gives

$$\frac{2-1}{2+1}S = \frac{1}{3}S.$$