Solutions - Transition Density Functions

Consider a **symmetric** random walk which starts with a marker placed at a point x at time s; written (x, s). Suppose at a later time $t > s$ the marker is at y; the future state denoted (y, t). The marker can move in step sizes of δy in a time step of δt. At the previous step the marker must have been at one of $(y - \delta y, t - \delta t)$ or $(y + \delta y, t - \delta t)$. The transition probability density function of the position y of the diffusion at a later time t, is written $p(x, s; y, t)$. Derive the Forward Equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}. \tag{1.1}$$

You may omit the dependence on (x, s) in your working as they will not change.
Assume a solution of (1.1) exists and takes the following form

$$p(y, t) = t^{-1/2} f(\eta);\ \eta = \frac{y}{t^{1/2}}.$$

Solve (1.1) to show that a particular solution of this is

$$p(x, s; y, t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right).$$

You may use the result $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$, in your working.

$$p(y', t') = \tfrac{1}{2} p(y' + \delta y, t' - \delta t) + \tfrac{1}{2} p(y' - \delta y, t' - \delta t)$$

Taylor series expansion gives

$$\begin{aligned} p(y' + \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + ... \\ p(y' - \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + ... \end{aligned}$$

Substituting into the above

$$\begin{aligned} p(y', t') &= \tfrac{1}{2}\left(p(y', t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) \\ &\quad + \tfrac{1}{2}\left(p(y', t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) \end{aligned}$$

$$0 = -\frac{\partial p}{\partial t'}\delta t + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y, \delta t \longrightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}$$

To solve, write

$$p(y, t) = t^{-1/2} f(\eta)$$

therefore

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial \eta}\frac{\partial \eta}{\partial y} = t^{-1/2} f'(\eta) \times t^{-1/2} = t^{-1} f'(\eta)$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{-1} f'(\eta)\right) = t^{-3/2} f''(\eta)$$

$$\begin{aligned} \frac{\partial p}{\partial t} &= t^{-1/2}\frac{\partial}{\partial t} f(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= t^{-1/2}\left(-\frac{1}{2} y t^{-3/2}\atop 1\right) f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= -\frac{1}{2}\eta t^{-3/2} f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta), \end{aligned}$$

and then substituting

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{1}{2}t^{-3/2}\left(\eta f'(\eta)+f(\eta)\right)\\ \frac{\partial^2 p}{\partial y^2} &= t^{-3/2}f''(\eta)\end{aligned}$$

gives

$$-\frac{1}{2}t^{-3/2}\left(\eta f'(\eta)+f(\eta)\right)=\frac{1}{2}t^{-3/2}f''(\eta)$$

simplifying to the ODE

$$-\left(f+\eta f'\right)=f''.$$

We have an exact derivative on the lhs, i.e. $\frac{d}{d\eta}(\eta f)=f+\eta f'$, hence

$$-\frac{d}{d\eta}(\eta f)=f''$$

and we can integrate once to get

$$-\eta f=f'+K.$$

We set $K=0$ (see class notes for justification) in order to get the correct solution, i.e.

$$-\eta f=f'$$

which can be solved as a simple first order variable separable equation:

$$f(\eta)=A\exp\left(-\tfrac{1}{2}\eta^2\right)$$

A is a normalizing constant, so write

$$A\int_{\mathbb{R}}\exp\left(-\tfrac{1}{2}\eta^2\right)d\eta=1$$

put $x=\eta/\sqrt{2}\rightarrow\sqrt{2}dx=d\eta$

$$\sqrt{2}A\underbrace{\int_{\mathbb{R}}\exp\left(-x^2\right)d\eta}_{=\sqrt{\pi}}=1\rightarrow A=\frac{1}{\sqrt{2\pi}}$$

$$u(y,t)=t^{-1/2}f(\eta)\ \text{ becomes }u(y,t)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right).$$

If the random variable y has value x at time s then we can generalize to

$$p(x,s;y,t)=\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$

To show

$$\int_{\mathbb{R}}p(x,s;y,t)\,dy=1.$$

$$\tfrac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}}\exp\left(-\tfrac{(y-x)^2}{2(t-s)}\right)dy=1.$$

Integration by substitution

$$\begin{aligned}u &= \frac{y-x}{\sqrt{2(t-s)}}\\ \sqrt{2(t-s)}du &= dy\end{aligned}$$

$$\tfrac{\sqrt{2(t-s)}}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}}\exp\left(-u^2\right)du$$

$$\tfrac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\exp\left(-u^2\right)du$$

we know $\int_{\mathbb{R}}\exp\left(-u^2\right)du=\sqrt{\pi}$, therefore $\frac{1}{\sqrt{\pi}}\times\sqrt{\pi}$ gives the desired result.