Detailed Mathematical Workings

Applied Stochastic Calculus

The diagram above shows the flucatuations in the price of apple stock over a 35 year period. This uncertainty that is quite conspicuous is the most important feature of financial modelling. Because there is so much randomness, the most successful mathematical models of financial assets have a probabilistic foundation.

The evolution of financial assets is random and depends on time. They are examples of *stochastic processes* which are random variables indexed (parameterized) with time. If the movement of an asset is discrete it is called a *random walk*. A continuous movement is called a *diffusion process*. We will consider the asset price dynamics to exhibit continuous behaviour and each random path traced out is called a *realization*. If we used a to denote the price of apple stock at time t, we could express this as a_t. This would be an example of a stochastic process.

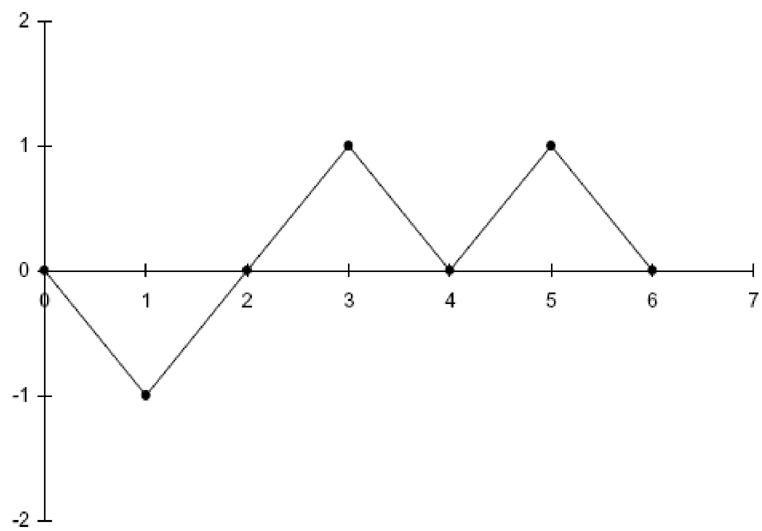
Hence the need for defining a robust set of properties for the randomness observed in an asset price realization, which is **Brownian Motion.** This was named after the Scottish Botanist who in the summer of 1827, while examining grains of pollen of the plant Clarkia pulchella suspended in water under a microscope, observed minute particles, ejected from the pollen grains, executing a continuous highly irregular fidgety motion. Further study showed that finer particles moved more rapidly, and that the motion was stimulated by heat and by a decrease in the viscosity of the liquid. The findings were published in *A Brief Account of Microscopical Observations Made in the Months of June, July and August 1827* There is little doubt that as well as the most well known (and famous) stochastic process, Brownian motion is the most widely used.

The origins of quantitative finance can be traced back to the start of the twentieth century. Louis Jean-Baptiste Alphonse Bachelier (March 11, 1870 - April 28, 1946) is credited with being the first person to derive the price of an option where the share price movement was modelled by Brownian motion, as part of his PhD at the Sorbonne, entitled Théorie de la Spéculation (published 1900).

Thus, Bachelier may be considered a pioneer in the study of financial mathematics and one of the earliest exponents of Brownian Motion. Five years later Einstein used Brownian motion to study diffusions; which described the microscopic transport of material and heat. In 1920 Norbert Wiener, a mathematician at MIT provided a mathematical construction of Brownian motion together with numerous results about the properties of Brownian motion - in fact he was the first to show that Brownian motion exists and is a well-defined entity. Hence Wiener process is also used as a name for this, and denoted W, $W(t)$ or W_t.

Construction of Brownian Motion

Brownian Motion can be constructed as a carefully scaled limit of a symmetric random walk, in the context of a simple gambling game. Consider the coin tossing experiment



where we define the random variable

$$R_i = \begin{cases} +1 & \text{if } H \\ -1 & \text{if } T \end{cases}$$

and examine the statistical properties of R_i.

Firstly the mean

$$\mathbb{E}[R_i] = (+1)\frac{1}{2} + (-1)\frac{1}{2} = 0$$

and secondly the variance

$$\begin{aligned} \mathbb{V}[R_i] &= \mathbb{E}\left[R_i^2\right] - \underbrace{\mathbb{E}^2[R_i]}_{=0} \\ &= \mathbb{E}\left[R_i^2\right] = 1 \end{aligned}$$

Suppose we now wish to keep a score of our winnings after the n^{th} toss - we introduce a new random variable

$$W_n = \sum_{i=1}^{n} R_i$$

This allows us to keep a track of our total winnings. This represents the position of a marker that starts off at the origin (no winnings). So starting with no money means

$$W_0 = 0$$

Now we can calculate expectations of W_n

$$\mathbb{E}[W_n] = \mathbb{E}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathbb{E}[R_i] = 0$$

$\mathbb{E}[X_n^2] =$

$$\begin{aligned}
\mathbb{E}\left[X_n^2\right] &= \mathbb{E}\left[R_1^2 + R_2^2 + ...R_n^2 + 2R_1R_2 + ... + 2R_{n-1}R_n\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} R_i^2\right] + \mathbb{E}\left[\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} R_iR_j\right] = \sum_{i=1}^{n} \mathbb{E}\left[R_i^2\right] + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbb{E}[R_i]\,\mathbb{E}[R_j] \\
&= n \times 1 + 0 \times 0 = n
\end{aligned}$$

A Note on Variations

Consider a function f_t, where $t_i = i\frac{t}{n}$, we can define different measures of how much f_t varies over time as

$$V^N = \sum_{i=1}^{n} \left|f_{t_i} - f_{t_{i-1}}\right|^N$$

The cases $N = 1, 2$ are important.

$$V = \sum_{i=1}^{n} \left|f_{t_i} - f_{t_{i-1}}\right| \text{ \textit{total variation} of trajectory - sum of absolute changes}$$

$$V^2 = \sum_{i=1}^{n} \left|f_{t_i} - f_{t_{i-1}}\right|^2 \text{ \textit{quadratic variation} - sum of squared changes}$$

Now look at the *quadratic variation* of the random walk. After each toss, we have won or lost \$1. That is

$$W_n - W_{n-1} = \pm 1 \Longrightarrow |W_n - W_{n-1}| = 1$$

Hence

$$\sum_{i=1}^{n} \underbrace{(W_i - W_{i-1})^2}_{=1} = n$$

Let's now extend this by introducing time dependence. Perform six tosses of a coin in a time t. So each toss must be performed in time $t/6$, and a bet size of $\sqrt{t/6}$ (and not \$1), i.e. we win or lose $\sqrt{t/6}$ depending on the outcome.

Let's examine the quadratic variation for this experiment

$$\begin{aligned}
&\sum_{i=1}^{6} (W_i - W_{i-1})^2 \\
&= \sum_{i=1}^{6} \left(\pm\sqrt{t/6}\right)^2 \\
&= 6 \times \frac{t}{6} = t
\end{aligned}$$

Now speed up the game. So we perform n tosses within time t with each bet being $\sqrt{t/n}$. Time for each toss is t/n.

$$W_i - W_{i-1} = \pm\sqrt{t/n}$$

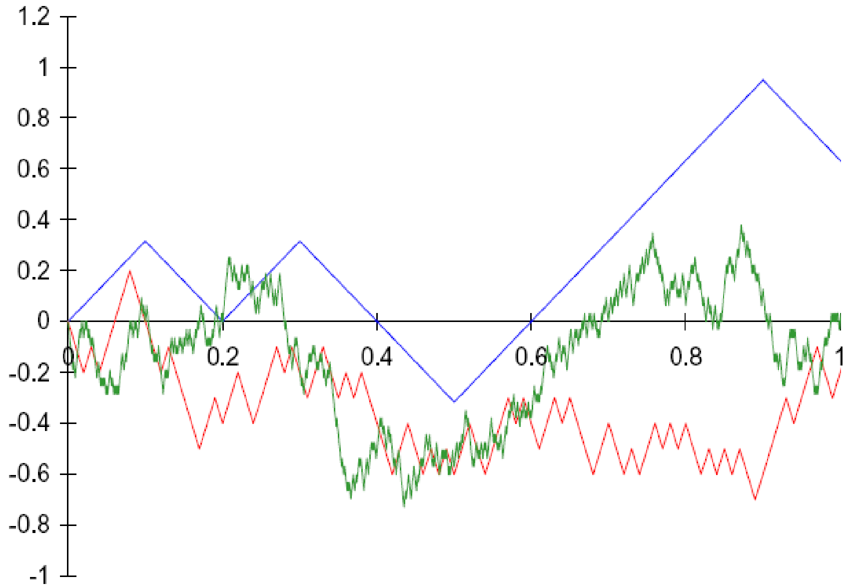
The quadratic variation is

$$\begin{aligned}\sum_{i=1}^{n}(W_i - W_{i-1})^2 &= n \times \left(\pm\sqrt{t/n}\right)^2 \\ &= t\end{aligned}$$

As n becomes larger and larger, time between subsequent tosses decreases and the bet sizes become smaller. The time and bet size decrease in turn like

$$\begin{aligned}\text{time decrease} &\sim O\left(\frac{1}{n}\right) \\ \text{bet size} &\sim O\left(\frac{1}{\sqrt{n}}\right)\end{aligned}$$

The diagram below shows a series of coin tossing experiments.



The scaling we have used has been chosen carefully to both keep the random walk finite and also not becoming zero. i.e. In the limit $n \longrightarrow \infty$, the random walk stays finite. It has an expectation conditional on a starting value of zero, of

$$\begin{aligned}\mathbb{E}[W_t] &= \mathbb{E}\left[\lim_{n\to\infty}\sum_{i=1}^{n} R_i\right] \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}[R_i] = n\cdot 0 \\ \text{Mean of } W_t &= 0\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[W_t^2\right] &= \mathbb{E}\left[\lim_{n\to\infty}\sum_{i=1}^{n} R_i^2\right] \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}\left[R_i^2\right] = \lim_{n\to\infty} n \cdot \left(\sqrt{t/n}\right)^2 \\
\mathbb{V}\left[W_t\right] &= \mathbb{E}\left[W_t^2\right] = t
\end{aligned}
$$

This limiting process as dt tends to zero is called Brownian Motion and denoted W_t.

Alternative notation for Brownian motion/Wiener process is X_t or B_t.

Another construction of a Brownian motion

Here we provide an alternative construction. Consider the coin tossing experiment with payoff

$$
Z = \begin{cases} +1 & \text{if } H \\ -1 & \text{if } T \end{cases}
$$

Let

$$
X_n = \sum_{i=1}^{n} Z_i
$$

this is the position of a random walker starting at the origin and after each toss moves one unit up or down with equal probability of a half. We know

$$
\mathbb{E}\left[X_n\right] = \sum_{i=1}^{n}\mathbb{E}\left[Z_i\right] = 0
$$

$$
\mathbb{V}\left[X_n\right] = \sum_{i=1}^{n}\mathbb{V}\left[Z_i\right] = n
$$

Now introduce time t; steps N. That is, perform N tosses in time t. Can we find a continuous time limit? Each step takes $\delta t = t/N$. If we let $N \to \infty$, steps of size ± 1 would become infinite (or may possibly collapse to zero). We require a finite random walk. So look for a suitable scaling of the time-step (away from ± 1), keeping in mind the Central Limit Theorem. Let

$$
Y = \alpha_N Z
$$

for some suitable α_N and let

$$
\{X_n : n = 0, 1, ..., N\}
$$

be the path of the random walk with steps of size α_N.

Thus

$$
\mathbb{E}\left[X_N\right] = 0 \; \forall N
$$

$$
\begin{aligned}
\mathbb{V}\left[X_N\right] &= \mathbb{E}\left[(X_N)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{N} Y_i\right)^2\right] = \mathbb{E}\left[\alpha_N^2 \sum_{i=1}^{N} Z_i^2\right] \\
&= \alpha_N^2 N = \left(\frac{t}{\delta t}\right)\alpha_N^2
\end{aligned}
$$

We need $\alpha_N^2/\delta t \sim O(1)$. Choose $\alpha_N^2/\delta t = 1$. Therefore

$$\mathbb{E}\left[(X_N)^2\right] = \mathbb{V}[X_N] = t.$$

As $N \to \infty$, the random walk

$$\{X_t : t \in [0, \infty)\}$$

converges to Brownian Motion.

Quadratic Variation

Consider a function f_t, which has at most a finite number of jumps or discontinuities. Define the Quadratic Variation

$$Q[f] = \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| f_{t_{i+1}} - f_{t_i} \right|^2$$

Take the time period $[0, T]$ with N partitions so $dt = T/N$; $t_i = idt$. For Brownian Motion W_t on the interval $[0, T]$ we have

$$\begin{aligned} Q[W_t] &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| W_{t_{i+1}} - W_{t_i} \right|^2 \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| \pm\sqrt{dt} \right|^2 = \lim_{N\to\infty} N dt \\ &= T \end{aligned}$$

So the Quadratic Variation $Q[W_t] = T$.

The Variation given

$$\begin{aligned} V[W_t] &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| W_{t_{i+1}} - W_{t_i} \right| \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| \pm\sqrt{dt} \right| \\ &= \lim_{N\to\infty} N\sqrt{dt} \to \infty \end{aligned}$$

Properties of a Wiener Process/Brownian motion

A stochastic process $\{W_t : t \in \mathbb{R}_+\}$ is defined to be Brownian motion (or a Wiener process) if

- Brownian motion starts at zero, i.e. $W_0 = 0$ (with probability one).
- Continuity - paths of W_t are continuous (no jumps) a.s. (differentiable nowhere)
- Brownian motion has independent Gaussian increments, with zero mean and variance equal to the temporal extension of the increment. That is for each $t > 0$ and $s > 0$, $W_t - W_s$ is normal with mean 0 and variance $|t-s|$,

  i.e.
  $$W_t - W_s \sim N(0, |t-s|).$$
  Coin tosses are Binomial, but due to a large number and the Central Limit Theorem we have a distribution that is normal. $W_t - W_s$ $(= x)$ has a pdf given by
  $$p(x) = \frac{1}{\sqrt{2\pi |t-s|}} \exp\left(-\frac{x^2}{2|t-s|}\right)$$
- More specifically $W_{t+s} - W_t$ is independent of W_t. This means if
  $$0 \le t_0 \le t_1 \le t_2 \le .....$$
  $$\begin{aligned} dW_1 &= W_1 - W_0 \text{ is independent of } dW_2 = W_2 - W_1 \\ dW_3 &= W_3 - W_2 \text{ is independent of } dW_4 = W_4 - W_3 \\ &\quad \text{and so on} \end{aligned}$$

Also called *standard Brownian motion* if the above properties hold. More importantly is the result (in stochastic differential equations)
$$dW = W_{t+dt} - W_t \sim N(0, dt)$$

Mean Square Convergence

Consider a function $F(X)$. If

$$\mathbb{E}\left[(F(X)-l)^2\right] \longrightarrow 0$$

then we say that $F(X)=l$ in the *mean square limit*, also called *mean square convergence*. We present a full derivation of the mean square limit. Starting with the quantity:

$$\mathbb{E}\left[\left(\sum_{j=1}^{n}\left(W(t_j)-W(t_{j-1})\right)^2-t\right)^2\right]$$

where $t_j = \dfrac{jt}{n} = j\Delta t$.

Hence we are saying that *up to mean square convergence,*

$$dW^2 = dt.$$

This is the symbolic way of writing this property of a Wiener process, as the partitions Δt become smaller and smaller.

Developing the terms inside the expectation

First, we will simplify the notation in order to deal more easily with the outer (right most) squaring. Let $Y(t_j) = \left(W(t_j)-W(t_{j-1})\right)^2$, then we can rewrite the expectation as:

$$\mathbb{E}\left[\left(\sum_{j=1}^{n} Y(t_j)-t\right)^2\right]$$

Expanding we have:

$$\mathbb{E}\left[\left(Y(t_1)+Y(t_2)+\ldots+Y(t_n)-t\right)\times\left(Y(t_1)+Y(t_2)+\ldots+Y(t_n)-t\right)\right]$$

The term inside the Expectation is equal to

$$\begin{aligned}
&Y(t_1)^2+Y(t_1)Y(t_2)+\ldots+Y(t_1)Y(t_n)-Y(t_1)t\\
&+Y(t_2)^2+Y(t_2)Y(t_1)+\ldots+Y(t_2)Y(t_n)-Y(t_2)t\\
&\vdots\\
&+Y(t_n)^2+Y(t_n)Y(t_1)+\ldots+Y(t_n)Y(t_{n-1})-Y(t_n)t\\
&-tY(t_1)-tY(t_2)-\ldots-tY(t_n)+t^2
\end{aligned}$$

Rearranging

$$\begin{aligned}
&Y(t_1)^2+Y(t_2)^2+\ldots+Y(t_n)^2\\
&2Y(t_1)Y(t_2)+2Y(t_1)Y(t_3)+\ldots+2Y(t_{n-1})Y(t_n)\\
&-2Y(t_1)t-2Y(t_2)t-\ldots-2Y(t_n)t\\
&+t^2
\end{aligned}$$

We can now factorize to get

$$\sum_{j=1}^{n} Y(t_j)^2 + 2\sum_{i=1}^{n}\sum_{j<i} Y(t_i)Y(t_j) - 2t\sum_{j=1}^{n} Y(t_j) + t^2$$

Substituting back $Y(t_j) = (W(t_j) - W(t_{j-1}))^2$ and taking the expectation, we arrive at:

$$\begin{aligned}
&\mathbb{E}\,[\sum_{j=1}^{n} (W(t_j) - W(t_{j-1}))^4 \\
&+2\sum_{i=1}^{n}\sum_{j<i} (W(t_i) - W(t_{i-1}))^2 (W(t_j) - W(t_{j-1}))^2 \\
&-2t\sum_{j=1}^{n} (W(t_j) - W(t_{j-1}))^2 \\
&+t^2\,]
\end{aligned}$$

Computing the expectation

By linearity of the expectation operator, we can write the previous expression as:

$$\begin{aligned}
&\sum_{j=1}^{n} \mathbb{E}\left[(W(t_j) - W(t_{j-1}))^4\right] \\
&+2\sum_{i=1}^{n}\sum_{j<i} \mathbb{E}\left[(W(t_i) - W(t_{i-1}))^2 (W(t_j) - W(t_{j-1}))^2\right] \\
&-2t\sum_{j=1}^{n} \mathbb{E}\left[(W(t_j) - W(t_{j-1}))^2\right] \\
&+t^2
\end{aligned}$$

Now, since $Z(t_j) = W(t_j) - W(t_{j-1})$ follows a Normal distribution with mean 0 and variance $\frac{t}{n}\,(= dt)$, it follows (standard result) that its fourth moment is equal to $3\frac{t^2}{n^2}$. We will show this shortly.

Firstly we know that $Z(t_j) \sim N\left(0, \frac{t}{n}\right)$, i.e.

$$\mathbb{E}\left[Z(t_j)\right] = 0, \quad \mathbb{V}\left[Z(t_j)\right] = \frac{t}{n}$$

therefore we can construct its PDF. For any random variable $\psi \sim N(\mu, \sigma^2)$ its probability density is given by

$$p(\psi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\frac{(\psi-\mu)^2}{\sigma^2}\right)$$

hence for $Z(t_j)$ the PDF is

$$p(z) = \frac{1}{\sqrt{t/n}\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\frac{z^2}{t/n}\right)$$

$$\begin{aligned}
\mathbb{E}\left[(W(t_j) - W(t_{j-1}))^4\right] &= \mathbb{E}\left[Z^4\right] \\
&= 3\frac{t^2}{n^2} \qquad \text{for } j = 1, \ldots, n
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{E}\left[Z^4\right] &= \int_{\mathbb{R}} Z^4 p(z)\,dz \\
&= \sqrt{\frac{n}{2t\pi}} \int_{\mathbb{R}} Z^4 \exp\left(-\tfrac{1}{2}\frac{z^2}{t/n}\right) dz
\end{aligned}$$

now put

$$u = \frac{z}{\sqrt{t/n}} \longrightarrow du = \sqrt{n/t}dz$$

Our integral becomes

$$\sqrt{\frac{n}{2t\pi}} \int_{\mathbb{R}} \left(\sqrt{\frac{t}{n}}u\right)^4 \exp\left(-\tfrac{1}{2}u^2\right) \sqrt{\frac{t}{n}}du$$

$$\begin{aligned}
&= \sqrt{\frac{1}{2\pi}\frac{t^2}{n^2}} \int_{\mathbb{R}} u^4 \exp\left(-\tfrac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2} \cdot \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} u^4 \exp\left(-\tfrac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2} \cdot \mathbb{E}\left[u^4\right].
\end{aligned}$$

So the problem reduces to finding the fourth moment of a standard normal random variable. Here we do not have to explicitly calculate any integral. Two ways to do this.

Either use the MGF as we did earlier and obtained the fourth moment to be three.

Or the other method is to make use of the fact that the kurtosis of the standardised normal distribution is 3.

That is

$$\mathbb{E}\left[\frac{(\phi-\mu)^4}{\sigma^4}\right] = \mathbb{E}\left[\frac{(\phi-0)^4}{1^4}\right] = 3.$$

Hence $\mathbb{E}\left[u^4\right] = 3$ and we can finally write $3\frac{t^2}{n^2}$.

and

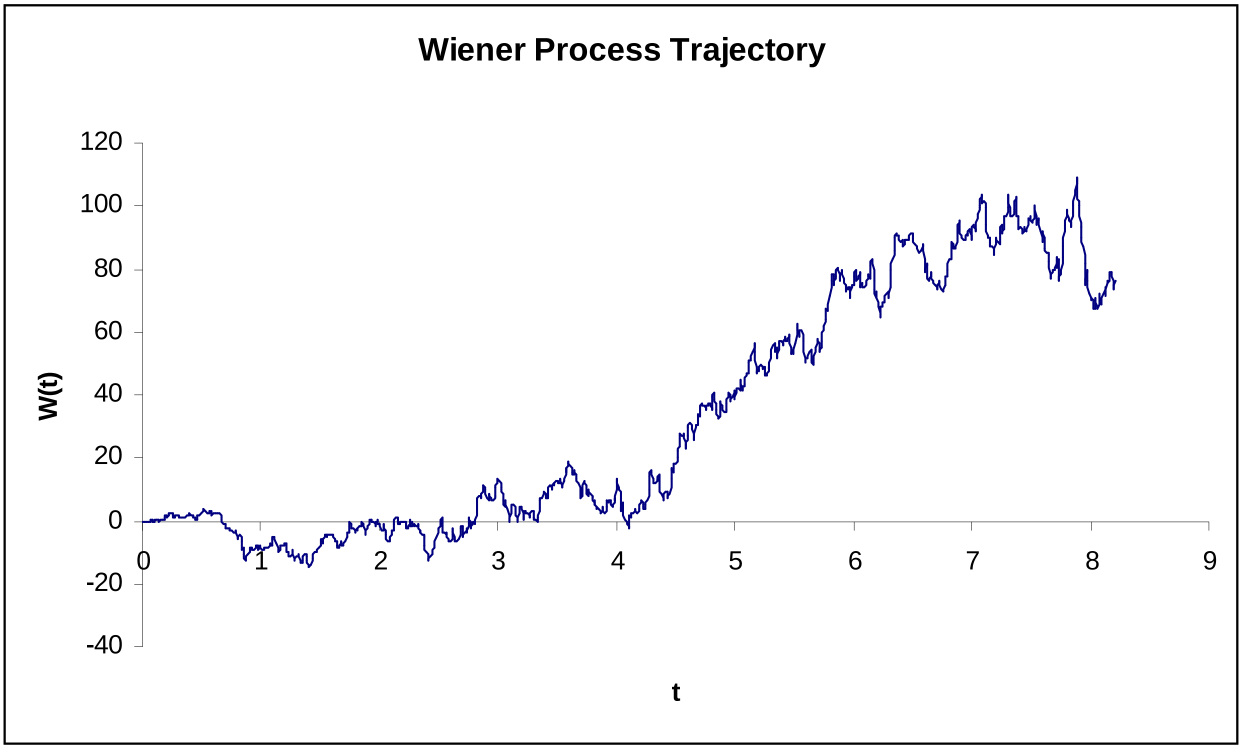
$$\mathbb{E}\left[(W(t_j) - W(t_{j-1}))^2\right] = \frac{t}{n} \qquad \text{for } j = 1, \ldots, n$$

Because of the single summation, the fourth moment and the variance multiplied by t actually recur n times. Because of the double summation, the product of variances occurs $\frac{n(n-1)}{2}$ times.

We can now conclude that the expectation is equal to:

$$\begin{aligned}
& 3n\frac{t^2}{n^2} + n(n-1)\frac{t^2}{n^2} - 2tn\frac{t}{n} + t^2 \\
&= 3\frac{t^2}{n} + t^2 - \frac{t^2}{n} - 2t^2 + t^2 = 2\frac{t^2}{n} \\
&= O(\frac{1}{n})
\end{aligned}$$

So, as our partition becomes finer and finer and n tends to infinity, the quadratic variation will tend to t *in the mean square limit.*


Wiener Process Trajectory
W(t)
120
100
80
60
40
20
0
-20
-40
0
1
2
3
4
5
6
7
8
9
t

Taylor Series and Itô

If we were to do a naive Taylor series expansion of $F(W_t)$, completely disregarding the nature of W_t, and treating dW_t as a small increment in W_t, we would get

$$F(W_t + dW_t) = F(W_t) + \frac{dF}{dW_t} dW_t + \frac{1}{2}\frac{d^2F}{dW_t^2} dW_t^2,$$

ignoring higher-order terms. We could argue that $F(W_t + dW_t) - F(W_t)$ was just the 'change in' F and so

$$dF = \frac{dF}{dW_t} dW_t + \frac{1}{2}\frac{d^2F}{dW_t^2} dW_t^2.$$

This is *almost* correct.
Because of the way that we have defined Brownian motion, and have seen how the quadratic variation behaves, it turns out that the dW_t^2 term isn't really random at all. The dW_t^2 term becomes (as all time steps become smaller and smaller) the same as its average value, dt. Taylor series and the 'proper' Itô are very similar. The only difference being that the correct Itô's lemma has a dt instead of a dW_t^2.

You can, with little risk of error, use Taylor series with the 'rule of thumb'

$$dW_t^2 = dt.$$

and in practice you will get the right result.

We can now answer the question, "If $F = W_t^2$ what is dF?" In this example

$$\frac{dF}{dW_t} = 2W_t \text{ and } \frac{d^2F}{dW_t^2} = 2.$$

Therefore Itô's lemma tells us that

$$dF = dt + 2W_t dW_t.$$

This is an example of a **stochastic differential equation (SDE),** written more generally as

$$dF = A(W_t)\,dt + B(W_t)\,dW_t.$$

There are two parts. The part before the plus, $A(W_t)\,dt$, is the deterministic bit. The random component follows $B(W_t)\,dW_t$. More importantly $A(W_t)$ is the drift; $B(W_t)$ is the diffusion.

Now consider a slight extension. A function of a Wiener Process $f = f(t, W_t)$, so we can allow both t and W_t to change, i.e.

$$\begin{aligned} t &\longrightarrow t + dt \\ W_t &\longrightarrow W_t + dW_t. \end{aligned}$$

Using Taylor as before

$$\begin{aligned} f(t + dt, W_t + dW_t) &= f(t, W_t) + \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W_t} dW_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial W_t^2} dW_t^2 + .. \\ df &= f(t + dt, W_t + dW_t) - f(t, W_t) \end{aligned}$$

This gives another form of Itô:

$$df = \left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right) dt + \frac{\partial f}{\partial W_t} dW_t. \qquad (*)$$

This is also a SDE.

Examples:

1. Obtain a SDE for $f = te^{W_t}$. We need $\frac{\partial f}{\partial t} = e^{W_t}$; $\frac{\partial f}{\partial W_t} = te^{W_t} = \frac{\partial^2 f}{\partial W_t^2}$, then substituting in $(*)$

$$df = \left(e^{W_t} + \tfrac{1}{2}te^{W_t}\right) dt + te^{W_t} dW_t.$$

   We can factor out te^{W_t} and rewrite the above as

$$\frac{df}{f} = \left(\tfrac{1}{t} + \tfrac{1}{2}\right) dt + dW_t.$$

2. Consider the function of a stochastic variable $f = t^2 W_t^n$

$$\frac{\partial f}{\partial t} = 2tW_t^n; \ \frac{\partial f}{\partial W_t} = nt^2 W_t^{n-1}; \ \frac{\partial^2 f}{\partial W_t^2} = n(n-1)t^2 W_t^{n-2},$$

   in $(*)$ gives

$$df = \left(2tW_t^n + \tfrac{1}{2}n(n-1)t^2 W_t^{n-2}\right) dt + nt^2 W_t^{n-1} dW_t.$$

Itô multiplication table:

$\times$	dt	dW_t
dt	$dt^2 = 0$	$dtdW_t = 0$
dW_t	$dW_t dt = 0$	$dW_t^2 = dt$

A Formula for Stochastic Integration

If we take the 2D form of Itô given by $(*)$, rearrange and integrate over $[0, t]$, we obtain a very nice formula for integrating functions of the form $f(t, W(t))$:

$$\int_0^t \tfrac{\partial f}{\partial W} dW = f(t, W(t)) - f(0, W(0)) - \int_0^t \left(\tfrac{\partial f}{\partial \tau} + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial W^2}\right) d\tau$$

This was derived in class during lecture 1.4 (30 January 2020).

Example: Show that

$$\int_0^t \left(t + e^W\right) dW = tW + e^W - 1 - \int_0^t \left(W_\tau + \tfrac{1}{2}e^{W_\tau}\right) d\tau.$$

Comparing this to the stochastic integral formula above, we see that $\frac{\partial f}{dW} \equiv t + e^W \implies f = tW + e^W$. Also

$$\tfrac{\partial^2 f}{\partial W^2} = e^{W_t}, \quad \tfrac{\partial f}{\partial t} = W_t.$$

Substituting all these terms in to the formula and noting that $f(0, W(0)) = 1$ verifies the result.

Naturally if $f = f(W(t))$ then the integral formula simply collapses to

$$\int_0^t \tfrac{df}{dW} dW = f(W(t)) - f(W(0)) - \tfrac{1}{2}\int_0^t \tfrac{d^2 f}{dW^2} d\tau$$

Example

Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W}\, dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(\tau), \tau)$ where $dW(\tau)$ is an increment of a Brownian motion.

If $W(0) = 0$ evaluate

$$\int_0^t \tau^2 \sin W\, dW(\tau).$$

$$\begin{array}{ll} \downarrow \dfrac{\partial F}{\partial W} = t^2 \sin W \longrightarrow & F = -t^2 \cos W \downarrow \\ \dfrac{\partial^2 F}{\partial W^2} = t^2 \cos W & \dfrac{\partial F}{\partial t} = -2t \cos W \end{array}$$

and substitute into the integral formula

$$\int_0^t \tau^2 \sin W\, dW(\tau) = -t^2 \cos W - \int_0^t \left(-2\tau \cos W + \frac{1}{2} \tau^2 \cos W \right) d\tau$$