MI14 extra notes are on Moodle.

To do: to upload extra notes on Binomial Model (Replication)

Differential Equations

Problem 1

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad V = V(S,t)$$

B.S. PDE.

Method of sol^n: $\exists$ a sol^n of form $V(S,t) = f(t)g(S)$

Notation $\quad \bullet \equiv \frac{d}{dt}$, $\quad ' \equiv \frac{d}{dS}$

$$\boxed{\frac{\partial V}{\partial t} = \dot{f}g; \quad \frac{\partial V}{\partial S} = fg'; \quad \frac{\partial^2 V}{\partial S^2} = fg''}$$

Method of separation of variables.

Subst in B.S.E above

$$\dot{f}g + \frac{1}{2}\sigma^2 S^2 f g'' + rSfg' - rfg = 0$$

$$-\dot{f}g = \frac{1}{2}\sigma^2 S^2 f g'' + rS fg' - rfg \qquad \div \text{ thro' by } fg$$

$$-\frac{\dot{f}}{f} = \frac{\frac{1}{2}\sigma^2 s^2 \cancel{f} g'' + rs\cancel{f} g' - r\cancel{f} g}{g}$$

$$-\frac{\dot{f}}{f} = \frac{\frac{1}{2}\sigma^2 s^2 g'' + rsg' - rg}{g}$$

depends on t — depends on s

Indep. of s — Indep of t

Consider LHS, vary the t — other side doesn't change

Consider RHS, vary the s — " " " "

$\therefore$ Each side must equal a function not dep. on (t, s)

i.e a constant

$$\therefore \quad \frac{\dot{f}}{f} = \frac{-\frac{1}{2}\sigma^2 s^2 g'' - rsg' + rs}{s} = \text{constant } c$$

$\rightarrow$ 2 ODEs (1) $\frac{df}{dt} = fc$

(2) $\frac{1}{2}\sigma^2 S^2 \frac{d^2 g}{dS^2} + rS\frac{dg}{dS} - (r-c)g = 0$

Sep. of var.s now gives a 1st order de and 2nd order Cauchy–Euler eq^{n}

To solve (1) : $\int \frac{df}{f} = c\int dt \rightarrow \log f(t) = ct + D$

$f(t) = \alpha e^{ct}$ $\alpha \in \mathbb{R}$

To solve (2) : $\exists$ a sol^{n} $g(s) = S^{\lambda}$

A.E : $\frac{1}{2}\sigma^2\lambda^2 + (r - \frac{1}{2}\sigma^2)\lambda - (r-c) = 0$

$\lambda^2 + \left(\frac{2r}{\sigma^2} - 1\right)\lambda - \frac{2}{\sigma^2}(r-c) = 0$

This quadratic has one of three solutions

rearrange

$ax^2y'' + bxy' + cy = 0$ $y = x^{\lambda}$

$y' = \lambda x^{\lambda-1}$; $y'' = \lambda(\lambda-1)x^{\lambda-2}$

$ax^2(\lambda^2-\lambda)x^{\lambda-2} + bx\lambda x^{\lambda-1} + cx^{\lambda} = 0$

$ax^{\lambda}(\lambda^2-\lambda) + b\lambda x^{\lambda} + cx^{\lambda} = 0$

$x^{\lambda}\underbrace{\left[a\lambda^2 + (b-a)\lambda + c\right]}_{} = 0$

$$\lambda_{\pm} = -\frac{1}{2}\left(\frac{2r}{\sigma^2}-1\right) \pm \frac{1}{2}\sqrt{\left(\frac{2r}{\sigma^2}-1\right)^2 - \frac{8}{\sigma^2}(c-r)}$$

(1) $\left(\frac{2r}{\sigma^2}-1\right)^2 - \frac{8}{\sigma^2}(c-r) > 0 \implies \lambda_+ \neq \lambda_- \in \mathbb{R}$

$\therefore \quad g(s) = A s^{\lambda_+} + B s^{\lambda_-}$

G.S $\quad V(s,t) = f(t)\, g(s) = e^{-ct}\left[\beta s^{\lambda_+} + \gamma s^{\lambda_-}\right] \quad \beta, \gamma \in \mathbb{R}$

(2) $\left(\frac{2r}{\sigma^2}-1\right)^2 - \frac{8}{\sigma^2}(c-r) = 0 \qquad \lambda_+ = \lambda_- = \lambda \in \mathbb{R} \qquad \lambda$ - 2 fold root

If s^{λ} is one sol$^{\underline{n}}$, second sol$^{\underline{n}}$ is $s^{\lambda} \log s$

$\therefore \quad g(s) = s^{\lambda}(\varepsilon + \delta \log s) \qquad$ G.S $V = e^{-ct} s^{\lambda}(\varepsilon + \delta \log s)$

$\varepsilon, \delta \in \mathbb{R}$

(3) $\left(\frac{2r}{\sigma^2}-1\right)^2 - \frac{8}{\sigma^2}(c-r) < 0$

$\Rightarrow \quad \lambda_{\pm} = p \pm iq \qquad i = \sqrt{-1}$

$$g(S) = S^p\left[A\cos(q\log S) + B\sin(q\log S)\right]$$

To get G.S $\quad V(S,t) = e^{-ct}S^p\left[\bar{A}\cos(q\log S) + \bar{B}\sin(q\log S)\right]$

$\bar{A}, \bar{B} \in \mathbb{R}$

Fourier series

Example 2: A popular problem in option pricing is the following B.V.P.

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0 \qquad V = V(S)$$

$V(0) = 0$ ①

$V(S^*) = S^* - E$ ②

S^* - special value of S

E - strike price

$S^*, E \in \mathbb{R}$

Again this 2^{nd} order ODE is an Euler eq^n: $\exists$ sol^n $V(S) = S^\lambda$

Resulting A.E: $\frac{1}{2}\sigma^2\lambda^2 + \left(r - \frac{1}{2}\sigma^2\right)\lambda - r = 0$

$$\lambda^2 + \left(\frac{2r}{\sigma^2} - 1\right)\lambda - \frac{2r}{\sigma^2} = (\lambda - 1)\left(\lambda + \frac{2r}{\sigma^2}\right) = 0 \Rightarrow \lambda_+ = 1 \ ; \ \lambda_- = -\frac{2r}{\sigma^2}$$

G.S. $V(S) = AS^{\lambda_+} + BS^{\lambda_-} = AS + BS^{-2r/\sigma^2}$

Applying 1 $S=0$ $V=0$

Gs $V(S) = AS + BS^{-2r/\sigma^2} = AS + \dfrac{B}{S^{2r/\sigma^2}}$

$V(0)=0$ means V is finite $\therefore$ in order to guarantee this $\Longrightarrow$ $B=0$

Hence $V(S) = AS$

Applying 2 ie at $S=S^*$ $V(S) = S^* - E$

$S^* - E = AS^*$ $\therefore$ $A = \dfrac{S^* - E}{S^*}$

$\longrightarrow$ $V(S) = (S^* - E)\left(\dfrac{S}{S^*}\right)$

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