✓ Martingales

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⊛ Martingales

⊛ Simulations

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game. We encounter Martingales through three distinct, but closely connected ideas:

1. *Martingales* as a class of stochastic process;   MIL4

2. *Exponential martingales*, which are a specific and extremely useful example of a martingale;   Visit this concept → Mod. J

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

Discrete Time Martingales

$\mathbb{T} = \{0, 1, 2, 3, \dots\}$

already seen

A discrete time stochastic process $\{M_t : t = 0, \dots, T\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \{0, \dots, T\}$ is a **martingale** if $\mathbb{E}|M_t| < \infty$ and
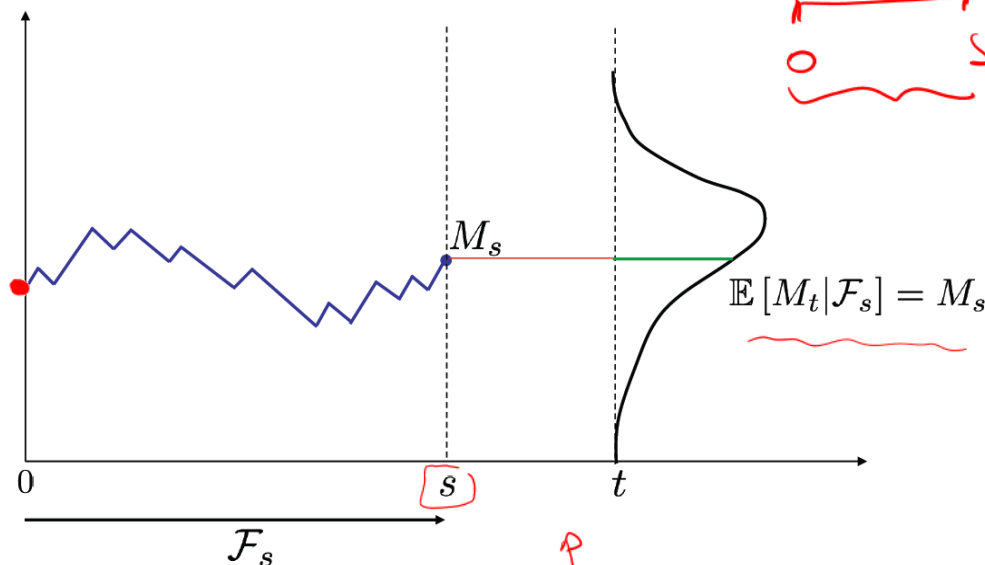
$$\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t \tag{1}$$

$\mathbb{E}[\cdot]$

improper integral

$\int_{\mathbb{R}} = \int_{-\infty}^{\infty}$

$\longrightarrow$ integrability condition

bounded

M_{t+1}

t

M_{t+1}

0 s t

M_s

$\mathbb{E}[M_t|\mathcal{F}_s] = M_s$

0 s t

$\mathcal{F}_s$

M_s $\mathcal{F}_s$

$t > s$

diagramatically

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The first equation represents a standard integrability condition.

Constant mean process   driftless process

The second equation tells you that the expected value of M at time $t+1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**.

If we take expectation on both sides of eqn. 1, then

$$\mathbb{E}\left[M_{t+1}\right] = \mathbb{E}\left[M_t\right]$$

$= M_t$ $\mathbb{E}\left[M_o\right] \Rightarrow$ *original mean*

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

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In addition, Doob and Meyer have developed a powerful theory centred around martingales.

$$\{M_t : t = 0, \ldots, n\}$$

✳ Continuous Time Martingales ✳

Next, we generalize our definitions to continuous time: A continuous time stochastic process

time cts.

$$\left\{ M_t : t \in \mathbb{R}^+ \right\}$$

or
adapted

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

① $$\mathbb{E}|M_t| < \infty \qquad > s$$

and

special

② $$\boxed{\mathbb{E}_s[M_t|\mathcal{F}_s] = M_s, \quad 0 \le s \le t.}$$

Mean of
our process
i) some finite
∴ expectation, are improper integral
$n > 1$

s   t

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Lévy's Martingale Characterisation: Let X_t, $t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. X_t is a Brownian motion iff the following conditions are satisfied:

1. $X_0 = 0$ a.s.;

   Standard BM

2. the sample paths $t \mapsto X_t$ are continuous a.s.;

3. X_t is a martingale with respect to the filtration $\mathcal{F}_t$;

   $\mathbb{E}_s[X_t | \mathcal{F}_s] = X_s$

   $s < t$

   t

Quadratic variation:

4. $|X_t|^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

   $X_t^2 - t$ $\boxed{\mathbb{E}(X_t^2) = t} \longleftarrow$ $\mathbb{E}[dX^2] = dt$

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process X_t satisfying:

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$$X_t \text{ is a Martingale} \implies \mathbb{E}[X_t | F_s] = X_s \qquad s < t$$

$$\underbrace{N(0, |t-s|)}_{} \qquad \text{Variance} = |t-s|$$

$$\mathbb{E}_s[X_t - X_s + X_s | F_s] = \mathbb{E}_s[\overbrace{X_t - X_s}^{N(0, |t-s|)} | F_s]$$

$$+ \underbrace{\mathbb{E}_s[X_s | F_s]}_{}$$

$$= 0 + \mathbb{E}_s[X_s]$$

$$= X_s$$

$$N(\text{mean, variance})$$

$$X_t - X_s \sim N(0,1)$$

Standard B.M

1. $X_0 = 0$ a.s.;

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$, $X_{t_2} - X_{t_1}$ are independent;

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t - s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

T

$dy = \dfrac{dy}{dx} dx + \dfrac{1}{2} \dfrac{d^2y}{dx^2} dt$

$\dfrac{dy}{dx} = 2x$ $\qquad \dfrac{d^2y}{dx^2} = 2$

$\displaystyle\int_0^t d(x_t^2)$

Itô Integrals and Martingales

$\displaystyle\int d(x^2) = \int_5 2x\,dx + \int_0^T \dfrac{1}{2}\cdot 2\,dt$

$\left. X^2 \right|_0^T$

Next, we explore the link between Itô integration and martingales.

Write $d(x_t^2)$

Consider the stochastic process $\boxed{Y(t) = X^2(t).}$ By Itô, we have

Itô I

$$X^2(T) = T + \int_0^T 2X(t)\,dX(t)$$

the $\displaystyle\int_0^T$

Taking the expectation, we get

Q.V.

property of
~ B.m.

$$\boxed{\mathbb{E}[X^2(T)] = T} + \mathbb{E}\left[\int_0^T 2X(t)\,dX(t)\right]$$

Itô Integral

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[X^2(T)] = T$$

else this is
contradicted

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)\,dX(t)\right] = 0$$

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Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

Itô integrals are martingales

is a martingale.

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\mathbb{E}\left[\int_0^T g(t, X_t)dX_t\right] = 0 \Rightarrow \text{Martingale}$$

is a martingale.

So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

$\mathbb{E}\left[\int \left| f_s \right| \right] = 0$

Module 3

Martingale Representation Theorem: If M_t is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$M_T = M_0 + \int_0^T g(t, X_t)dX_t$$

No information

Example Using only Itô and the fact that Itô integrals are martingales, we will show that

$$\mathbb{E}\left[X^2(T)\right] = T.$$

New exercise

Consider the function $F(t, X_t) = X_t^2$, then by Itô's lemma,

$$X_T^2 = X_0^2 + \frac{1}{2}\int_0^T 2\,dt + \int_0^T 2X_t\,dX_t$$

$$X_T^2 = \int_0^T dt + 2\int_0^T X_t\,dX_t$$

since $X_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\int_0^T X_t\,dX_t\right]$$

Now,

$$\int_0^T X_t\,dX_t$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t\,dX_t\right] = 0$

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Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}[T] = T$$

We can conclude that

$$\mathbb{E}\left[X^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that
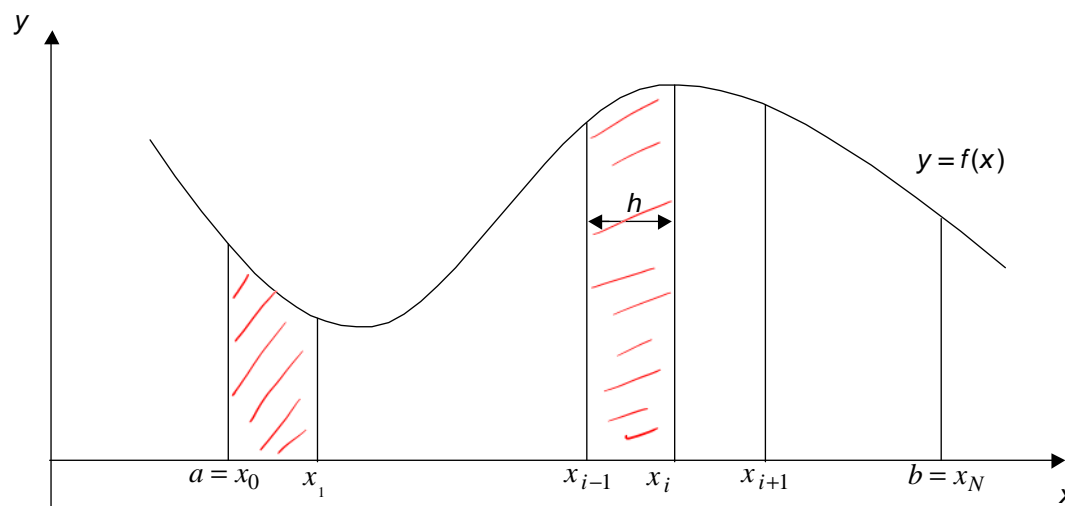
$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}[f(X_t)]\,dt$$

This is due to an analysis result known as **Fubini's Theorem**.

Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x)\, dx$$

which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against x.

Assuming f is a "well behaved" function on $[a, b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a, b]$ into N intervals with end points $x_0 = a < x_1 < x_2 < \ldots\ldots < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are N intervals and $N + 1$ points x_i.

i here is a index

$dx = \dfrac{b-a}{N}$

$N \to \infty$ $dx \to 0$

$\therefore \; L \to 0$
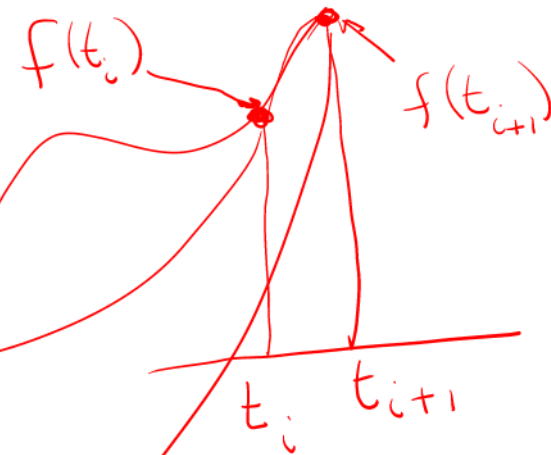
Discretising x gives

$$x_i = a + idx$$

Now consider the definite integral

$$\int_0^T f(t)\, dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;

$$\int_0^T f\left(t\right)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_i\right)\left(t_{i+1} - t_i\right)$$

2. right hand rectangle rule;

$$\int_0^T f\left(t\right)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+1}\right)\left(t_{i+1} - t_i\right)$$

3. trapezium rule;

$$\int_0^T f\left(t\right)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}\left(f\left(t_i\right) + f\left(t_{i+1}\right)\right)\left(t_{i+1} - t_i\right)$$

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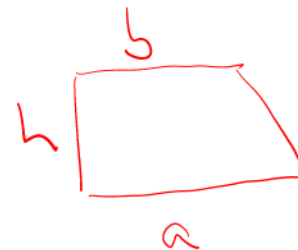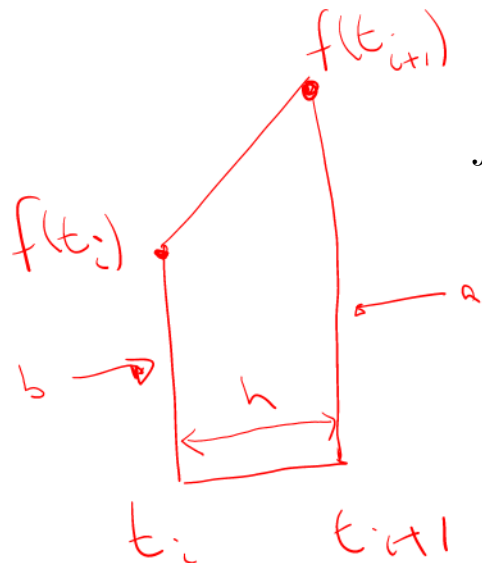
4. midpoint rule

$$\int_0^T f(t)\, dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

$$\longrightarrow \int_0^T f(t, X)\, dX = \int_0^T f(t, X(t))\, dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

left hand rule no. 1

slide 15

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where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

Similar to 2. right hand rule.

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

Very Important: In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

anticipate this

$$I = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$
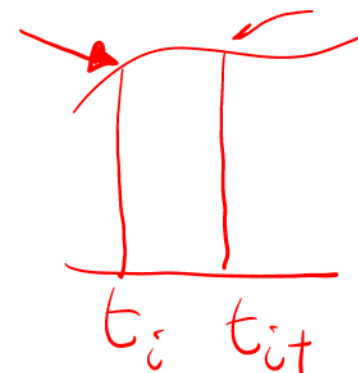
At t_i

$X_i \equiv X(t_i)$

is special. This definition results in the **Itô Integral**, and is important.

It is special because it is **non-anticipatory**; given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $X_{i+1} - X_i$ term.

Compare this to a definition such as

$$\lim_{N \to \infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ — there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f(t_{i+1}, X_{i+1})$.

The main thing to note about Itô integrals is that I is a random variable (unlike the deterministic case). Additionally, since I is essentially the limit of a sum of normal random variables, then by the CLT I is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\, dt.$$

Show that the result also can be found by writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} X_i{}^2 (X_{i+1} - X_i)$$

Hint: use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$.

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\, dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i^2 (X_{i+1} - X_i)$$

Now note the hint:

$$3b^2 (a - b) = a^3 - b^3 - 3b (a - b)^2 - (a - b)^3$$

hence

$$
\begin{aligned}
&\equiv\ 3X_i^2 (X_{i+1} - X_i) \\
&=\ X_{i+1}^3 - X_i^3 - 3X_i (X_{i+1} - X_i)^2 - (X_{i+1} - X_i)^3,
\end{aligned}
$$

so that

$$\sum_{i=0}^{N-1} 3X_i^2 (X_{i+1} - X_i) =$$

$$
\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 - \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2
$$
$$
- \sum_{i=0}^{N-1} (X_{i+1} - X_i)^3
$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$

$$= X(T)^3 - X(0)^3 .$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N \to \infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = (X_{i+1} - X_i)^2 \cdot (X_{i+1} - X_i)$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0.$

product

Hence putting together gives

Itô integration

$$X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt$$

from Itô principles.

which is consistent with Itô's lemma.

$$\mathbb{E}\left[f(t_{i+1}, X_{i+1}) \right]$$

The other important property that the Itô integral has is that it is a martingale. We know that

Monotone
Convergence $\lim^{n'}$

zero drift

$$\mathbb{E}\left[X_{i+1} - X_i \right] = 0$$

$N(0, dt)$

is a martingale; i.e. in the context

given : $\mathbb{E}[X_{i+1} - X_i] = 0.$

$$\int_0^T f\, dX = \lim_{N \to \infty} \sum_0^{N-1} f(t_i, X_i)$$

are here.

$(X_{i+1} - X_i)$

Since

Starting point: $\lim_{N \to \infty}$ $\mathbb{E}\left[\sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i) \right] =$
at time t_i

t_i t_{i+1}

$$\sum_{i=0}^{N-1} f(t_i, X_i)\, \underbrace{\mathbb{E}[X_{i+1} - X_i]}_{0} = 0$$

$\therefore X_{i+1} - X_i = dX \sim N(0, dt)$

Thus

$$\mathbb{E}\left[\int_0^T f(t, X(t))\, dX(t) \right] = 0.$$

t_i

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

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Exercise We know from Itô's lemma that

$$4 \int_0^T X^3(t) \, dX(t) = X^4(T) - X^4(0) - 6 \int_0^T X^2(t) \, dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4 \int_0^T X^3 \, dX = \lim_{N \to \infty} 4 \sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)$$

Hint: use $4b^3(a-b) = a^4 - b^4 - 4b(a-b)^3 - 6b^2(a-b)^2 - (a-b)^4$.

For you to try.

$$f(t_i, X_i)$$

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Proving that a Continuous Time Stochastic Process is a Martingale

$$dY = f\,dt + g\,dx$$

$$f = 0$$

$$dY = g\,dx$$

Consider a stochastic process $Y(t)$ solving the following SDE:

$$dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0$$

How can we tell whether $Y(t)$ is a martingale?

driftless process

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE between s and t to get an exact form for $Y(t)$:

$$\int_s^t$$

Itô integral

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

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$$dy = \left(f \, dt\right) + g \, dX$$

Taking the expectation conditional on the filtration at time s, we get

$$\mathbb{E}\left[Y_t | \mathcal{F}_s\right] = \mathbb{E}\left[Y(s) + \int_s^t f(Y_u, u) du + \int_s^t g(Y_u, u) dX(u) | \mathcal{F}_s\right]$$

$$= Y(s) + \mathbb{E}\left[\int_s^t f(Y_u, u) du | \mathcal{F}_s\right]$$

$$\mathbb{E}[\ ] = 0$$

Itô integral.

where the last line follows from the fact that a Itô integral is a martingale, $\therefore$

$$\mathbb{E}\left[\int_s^t g(Y_u, u) dX(u) | \mathcal{F}_s\right] = \int_s^s g(Y_u, u) dX(u) = 0.$$

So, $Y(t)$ is a martingale iff

$$\mathbb{E}\left[\int_s^t f(u) du | \mathcal{F}_s\right] = 0$$

f must be zero ele

$$\mathbb{E}[Y_t | \mathcal{F}_s] \neq Y_s$$

This condition is satisfied only if $f(Y_t, t) = 0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$dY(t) = g(Y_t, t) dX(t), \qquad Y(0) = Y_0$$

SDE has

no drift $\implies$ Y_t is a Martingale elte Y_t is not a Martingale.

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Exponential Martingales

Let's start with a motivating example. (special)

Consider the stochastic process $Y(t)$ satisfying the SDE

SDE | differential:

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \qquad (2)$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Z_t take Y_t a)

Define a new process $Z(t) = e^{Y(t)}$.

an exp. func^

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

dZ ?

Itô III

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

$$\frac{dt}{dY} = e^{Y} = Z = \frac{dZ}{dY}$$

$$
\begin{aligned}
dZ(t) &= \frac{dZ}{dY}dY(t) + \frac{1}{2}\frac{d^2Z}{dY^2}dY^2(t) \\
&= \frac{dZ}{dY}(f(t)dt + g(t)dX(t)) + \frac{1}{2}\frac{d^2Z}{dY^2}g^2(t)dt \\
&= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dX(t) \\
dZ &= Z(t)\left[\underbrace{\left(f(t) + \frac{1}{2}g^2(t)\right)}dt + g(t)dX(t)\right]
\end{aligned}
$$

only drift is

iff $\left(\quad\right)dt = 0$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

drift $Z(f + \frac{1}{2}g^2)$

$$f(t) + \frac{1}{2}g^2(t) = 0 \qquad \text{diffusion } Z\,g$$

This is only possible if

$dZ = Z g dX$

$$\boxed{f(t) = -\frac{1}{2}g^2(t)}$$

update dY.

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$$E\left[|M_t|^2\right] < \infty \quad f = -\frac{1}{2}g^2_t$$

Going back to the process $Y(t)$, we must have

$$\text{S.D.E} \quad dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

implying that   Integrate   over $[0, T]$ growth.

$$y_T$$
$$e = t_T$$

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of $Z(t)$:

$$dG = f dt + g dx$$

$$dZ(t) = Z(t)g(t)dX(t).$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

$$E[G] = f$$

$$Z_t = e^{y_t}$$

$$y_0 ----$$

Let's simplify this $Z(T) = e Z_0 e$

$$Z_0 = e^{y_0}$$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

$$Z_0 = e^{y_T}$$

to give

$$Z_T = e^{y_T}$$

$$Z(T) = Z_0\exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

exponential

Mart.

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$$\int_0^{X_t} \frac{\partial F}{\partial x} dx = F(t, X_t) - F(0, X_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right) ds$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

$$\int F(t, X_t) \, dt$$

$\underline{Vasicek}$

$$dr = (\eta - \gamma r) \, dt + \sigma \, dx$$

$\gamma -$ speed of reversion

$\sigma -$ vol.

$$r_{i+1} = r_i + (\eta - \gamma r_i) \, \delta t + \sigma \phi \sqrt{\delta t}$$

$$\bar{r} = \eta$$
$$\frac{1}{\gamma}$$

$$\eta = \gamma \bar{r}$$

$$dr = (\eta - \gamma r) \, dt$$

$$ln(r - \bar{r}) = -\gamma t + c$$
$$r = \bar{r} + A e^{-\gamma t}$$

$$dr = -\gamma (r - \bar{r}) \, dt$$

$$\int \frac{dr}{r - \bar{r}} = -\gamma \int dt$$

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Key Condition (Novikov Condition)

definition

A trading strategy $(\phi, \psi) = \{(\phi_t, \psi_t) ; t \in [0, ..., T]\}$ is a previsible process in that $\phi_t \in \mathcal{F}_{t-}$.

γ_t

$\mathcal{F}_{t-}$ $\mathcal{F}_t$

£1

$dB_t = r B_t \, dt$

A ~~stochastic~~ process γ_t satisfies the *Novikov condition* if

Integrability condition.

$$\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^T \gamma_s^2 ds \right) \right] < \infty$$

where is a γ_t previsible process.

option

Cash

hedged portfolio

$$\Pi = V - \Delta S$$

amount of risky asset

Replication no. of shares

£1 cts compounded → e^r

$$\boxed{\Pi = \phi s + \psi}$$ e

no. of bonds

Shreve

1. discrete time finance
2. cts " "

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Key Fact

$$Change \ of \ measure$$
$$from \ \mathbb{P} \longrightarrow \mathbb{Q}$$

Given a process γ_t satisfying the Novikov condition, then the process M_t^γ defined as we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$M_t^\gamma = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.

In our earlier example $\gamma_t = -g(t); \ M_t^\gamma = Z(t)$.

Key Fact (Girsanov's Theorem)

Given a process θ_t γ satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X_t^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} + \int_0^t \gamma_s dX_s$$

as is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

$\mathbb{P} \sim \mathbb{Q}$