Problem Sheet

1. Consider the Forward Kolmogorov equation (FKE), given by
$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2} \tag{1.1}$$
for the transition density function $p(y, t\ ; y', t')$; $c^2 \in \mathbb{R}^+$. The states (y, t) are past and are **fixed** while (y', t') refers to future ones and are variables. By simple substitution show that
$$p(y, t; y', t') = \frac{1}{2c\sqrt{\pi(t'-t)}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right), \tag{1.2}$$
satisfies the FKE. **You may drop the (y, t) from your working as they won't change.** Show that (1.2) satisfies
$$\int_{\mathbb{R}} p(y, t; y', t') dy' = 1.$$

2. Consider a **symmetric** random walk which starts with a marker placed at a point x at time s; written (x, s). Suppose at a later time $t > s$ the marker is at y; the future state denoted (y, t). The marker can move in step sizes of δy in a time step of δt. At the previous step the marker must have been at one of $(y - \delta y, t - \delta t)$ or $(y + \delta y, t - \delta t)$. The transition probability density function of the position y of the diffusion at a later time t, is written $p(x, s; y, t)$. Derive the Forward Equation
$$\frac{\partial p}{\partial t} = \tfrac{1}{2} \frac{\partial^2 p}{\partial y^2}.$$
You may omit the dependence on (x, s) in your working as they will not change.

3. A FKE of the following form is given
$$\frac{\partial p}{\partial t} = \tfrac{1}{2} \frac{\partial^2 p}{\partial y^2}, \tag{3.1}$$
for the transition probability density function $p(y, t)$. At time t, the diffusion has position y. Assume a solution of (3.1) exists and takes the following form
$$p(y, t) = t^{-1/2} f(\eta);\ \eta = \frac{y}{t^{1/2}}.$$
Solve (3.1) to show that a particular solution of this is
$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$