Solutions to CQF Module 1 Assignment

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W}\, dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(t), t)$ where $dW(t)$ is an increment of a Brownian motion.
If $W(0) = 0$ evaluate

$$\int_0^t \tau^2 \sin W \, dW(\tau).$$

$$\downarrow \frac{\partial F}{\partial W} = t^2 \sin W \longrightarrow \quad F = -t^2 \cos W \downarrow$$
$$\frac{\partial^2 F}{\partial X^2} = t^2 \cos W \qquad \frac{\partial F}{\partial t} = -2t \cos W$$

and substitute into the integral formula

$$\int_0^t \tau^2 \sin W \, dW(\tau) = -t^2 \cos W - \int_0^t \left(-2\tau \cos W + \frac{1}{2}\tau^2 \cos W \right) d\tau$$

b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Obtain a SDE $df(S,t)$ for each of the following functions. Here we use Itô IV

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial f}{\partial S_t} dW_t.$$

i $f(S,t) = \alpha^t + \beta t S^n \qquad \alpha$, β are constants

$$\begin{aligned} \frac{\partial f}{\partial t} &= \alpha^t \log \alpha + \beta S^n; \ \frac{\partial f}{\partial S} = n\beta t S^{n-1}; \ \frac{\partial^2 f}{\partial S^2} = n(n-1)\beta t S^{n-2} \\ df &= \left(\alpha^t \log \alpha + +\beta S^n + n\mu\beta t S^n + \frac{1}{2} n(n-1)\beta t \sigma^2 S^n \right) dt + \sigma n \beta t S^n dW_t \end{aligned}$$

ii $f(S,t) = \log tS + \cos tS$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{t} - S \sin tS; \ \frac{\partial f}{\partial S} = \frac{1}{S} - t \sin tS; \ \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} - t^2 \cos tS \\ df &= \left(\frac{1}{t} - S \sin tS + \mu S \left(\frac{1}{S} - t \sin tS \right) + \frac{1}{2}\sigma^2 S^2 \left(-\frac{1}{S^2} - t^2 \cos tS \right) \right) dt + \sigma S \left(\frac{1}{S} - t \sin tS \right) dW_t \end{aligned}$$

You can simplify, but it just creates more terms.

2. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\
dr_t &= \gamma (m - r_t) dt + c dW_t^{(2)},
\end{aligned}
$$

in turn. and where $\mathbb{E}\left[dW_t^{(1)} dW_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

Begin by writing a 3D Taylor expansion for $V(t, S_t, r_t)$

$$
\begin{aligned}
& V(t + dt, S_t + dS, r_t + dv) - V(t, S_t, r_t) \\
= \; & \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial r \partial S} dr dS
\end{aligned}
$$

Since $dW_i^2 \to dt$ in the mean square limit for $i = 1, 2$, we see that

$$dS_t^2 \to \sigma^2 S_t^2 dt,$$

$$dr_t^2 \to c^2 dt,$$

Also, since $dW_t^{(1)} dW_t^{(2)} = \rho dt$, we see that

$$dS_t dr_t \to \rho c \sigma S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for V is given by

$$
\begin{aligned}
dV = & \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma (m - r_t) \frac{\partial V}{\partial r} \right. \\
& \left. + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_t \frac{\partial^2 V}{\partial r \partial S} \right) dt \\
& + \sigma S_t \frac{\partial V}{\partial S} dW_t^{(1)} + c \frac{\partial V}{\partial r} dW_t^{(2)}
\end{aligned}
$$

Integrating over $[0, t]$, we get

$$
\begin{aligned}
V(t, S_t, r_t) = & \underbrace{V(0, S_0, r_0)}_{=v} + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} + \gamma (m - r_\tau) \frac{\partial V}{\partial r} \right. \\
& \left. + \frac{1}{2} \sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_\tau \frac{\partial^2 V}{\partial r \partial S} \right) d\tau \\
& + \int_0^t \sigma S_\tau \frac{\partial V}{\partial S} dW_\tau^{(1)} + \int_0^t c \frac{\partial V}{\partial r_\tau} dW_\tau^{(2)}
\end{aligned}
$$

3. An equity price S evolves according to Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are constants. We know that an explicit solution is

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}$$

where S_0 is S_t at time $t = 0$.
By working through all the integration steps, deduce that the expected value of S_t at time $t > 0$, given S_0, is

$$\mathbb{E}\left[\left. S_t \right| S_0\right] = S_0 e^{\mu t}.$$

You are required to present all your integration steps to obtain the expectation.
We wish to calculate the expected terminal value of this risky asset. So

$$\begin{aligned}
\mathbb{E}\left[S_t\right] &= S_0 \mathbb{E}\left[\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\right] \\
&= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} \mathbb{E}\left[e^{\sigma W_t}\right]
\end{aligned}$$

$$\mathbb{E}\left[\left. S(t) \right| S_0\right] = \mathbb{E}\left[S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}\right] = S_0 e^{\left(\mu - \sigma^2/2\right)t} \mathbb{E}\left[e^{\sigma W_t}\right]$$

Recall $W_t \sim N(0, t)$. To calculate the shortened expectation $\mathbb{E}\left[e^{\sigma W_t}\right] =$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x} e^{-x^2/2t} dx &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x - x^2/2t} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x - \sigma t)^2/2t} e^{\sigma^2 t/2} dx \\
&= e^{\sigma^2 t/2} \underbrace{\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x - \sigma t)^2/2t} dx}_{=1}
\end{aligned}$$

So $\mathbb{E}\left[\left. S(t) \right| S_0\right] = S_0 e^{\left(\mu - \sigma^2/2\right)t} e^{\sigma^2 t/2} = S_0 e^{\mu t}$

4. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}\sigma^2\frac{d^2p_\infty}{du^2}+\theta\frac{d}{du}\left(up_\infty\right)=0$$

$p_\infty = p_\infty\left(u\right)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation: Integrate wrt u

$$\frac{1}{2}\sigma^2\frac{dp}{du}+\theta\left(up\right)=k$$

where k is a constant of integration and can be calculated from the condtions, that as

$$u\rightarrow\infty:\left\{\begin{array}{l}\frac{dp}{du}\rightarrow 0\\ p\rightarrow 0\end{array}\right.\Rightarrow k=0$$

which gives

$$\frac{1}{2}\sigma^2\frac{dp}{du}=-\theta\left(up\right),$$

a first order variable separable equation. So

$$\begin{aligned}\frac{1}{2}\sigma^2\int\frac{dp}{p} &= -\theta\int udu\rightarrow\\ \frac{1}{2}\sigma^2\ln p &= -\theta\left(\frac{u^2}{2}\right)+C\ , \qquad C\ \text{ is arbitrary.}\\ \ln p &= -\frac{\theta}{\sigma^2}u^2+c\end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p=\exp\left(-\frac{\theta}{\sigma^2}u^2+c\right)=A\exp\left(-\frac{\theta}{\sigma^2}u^2\right)$$

and we know as p_∞ is a PDF

$$\int_{-\infty}^{\infty}p_\infty\ du'=1\rightarrow A\int_{-\infty}^{\infty}e^{-\left(\frac{\theta}{\sigma^2}u^2\right)}du=1$$

A few (related) ways to calculate A . Now use the error function, i.e. $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$. So put

$$x=\sqrt{\frac{\theta}{\sigma^2}}u\rightarrow dx=\sqrt{\frac{\theta}{\sigma^2}}du$$

which transforms the integral above

$$\frac{A\sigma}{\sqrt{\theta}}\int_{-\infty}^{\infty}e^{-x^2}\ dx=1\rightarrow A\sigma\sqrt{\frac{\pi}{\theta}}=1\therefore A=\frac{1}{\sigma}\sqrt{\frac{\theta}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty=\frac{1}{\sigma}\sqrt{\frac{\theta}{\pi}}\exp\left(-\frac{\theta}{\sigma^2}u'^{\ 2}\right).$$

If we compare this to

$$p\left(x\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right)$$

we find u' is normally distributed with mean 0. For the variance write $-\frac{\theta}{\sigma^2}u'^{\ 2}$ as

$$-\frac{1}{2}.2.\frac{\theta}{\sigma^2}u'^{\ 2}\equiv-\frac{1}{2}\frac{1}{\sigma^2/2\theta}u'^{\ 2}$$

to give the variance as $\sigma^2/2\theta$ and standard deviation is $\sqrt{\sigma/2\theta}$

5. The steady state equation for our model becomes

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}\left(r^{2\beta}p_\infty(r)\right)-\frac{d}{dr}\left(u(r)\,p_\infty(r)\right)=0$$

This can be simply integrated once to give

$$\begin{aligned}\frac{1}{2}\nu^2\frac{d}{dr}\left(r^{2\beta}p_\infty(r)\right)-\left(u(r)\,p_\infty(r)\right) &= \text{const}\\ \frac{1}{2}\nu^2\left(r^{2\beta}\frac{dp_\infty}{dr}\right)+\nu^2\beta r^{2\beta-1}p_\infty(r)-\left(u(r)\,p_\infty(r)\right) &= \text{const}\end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left.\begin{array}{c}p_\infty(r)\\ \frac{dp_\infty}{dr}\end{array}\right\}\longrightarrow 0$$

$$\begin{aligned}u(r)\,p_\infty(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{dp_\infty}{dr}+\nu^2\beta r^{2\beta-1}p_\infty(r)\\ u(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{1}{p_\infty(r)}\frac{dp_\infty}{dr}+\nu^2\beta r^{2\beta-1}\end{aligned}$$

We can write $\dfrac{1}{p_\infty}\dfrac{dp_\infty}{dr}$ as $\dfrac{d}{dr}\left(\log p_\infty\right)$

$$u(r)=\frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}\left(\log p_\infty\right)+\nu^2\beta r^{2\beta-1}$$

a. Let X_t, Y_t be two one-dimensional stochastic processes, where

$$\begin{aligned}dX_t &= a(t,X_t)\,dt+b(t,X_t)\,dW_t^{(1)},\\ dY_t &= c(t,Y_t)\,dt+d(t,Y_t)\,dW_t^{(2)}.\end{aligned}\qquad(2)$$

The Wiener processes are correlated such that $\mathbb{E}\left[W_t^{(1)}W_t^{(2)}\right]=\rho t$.By applying the two-dimensional form of Itô's lemma with $f(X,Y)=XY$

i.

$$df=\frac{\partial f}{\partial X}dX+\frac{\partial f}{\partial Y}dY+\tfrac{1}{2}\frac{\partial^2 f}{\partial X^2}dX^2+\tfrac{1}{2}\frac{\partial^2 f}{\partial Y^2}dY^2+\frac{\partial^2 f}{\partial X\partial Y}dX\,dY$$

$$\begin{array}{lll}\frac{\partial f}{\partial X}=Y & \frac{\partial f}{\partial Y}=X & \\ \frac{\partial^2 f}{\partial X^2}=0 & \frac{\partial^2 f}{\partial Y^2}=0 & \frac{\partial^2 f}{\partial Y\partial X}=1=\frac{\partial^2 f}{\partial X\partial Y}\end{array}$$

which gives

$$d(X_tY_t)=X_tdY_t+Y_tdX_t+dX_tdY_t.$$

ii. Rearranging the above product rule

$$X_tdY_t=d(X_tY_t)-Y_tdX_t-dX_tdY_t$$

and integrating over $[0,t]$

$$\begin{aligned}\int_0^t X_sdY_s &= \int_0^t d(X_sY_s)-\int_0^t Y_sdX_s-\int_0^t dX_sdY_s\\ &= X_tY_t-X_0Y_0-\int_0^t Y_sdX_s-\int_0^t dX_sdY_s.\end{aligned}$$

ii.

$$\frac{\partial f}{\partial X} = 1/Y \qquad \frac{\partial f}{\partial Y} = -X/Y^2 \qquad \frac{\partial^2 f}{\partial X^2} = 0$$
$$\frac{\partial^2 f}{\partial Y^2} = 2X/Y^3 \qquad \frac{\partial^2 f}{\partial X \partial Y} = -1/Y^2 = \frac{\partial^2 f}{\partial Y \partial X}$$

which gives

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dXdY}{XY} + \left(\frac{dY}{Y}\right)^2\right)$$

b. Consider a function $V(t, X_t, Y_t)$, where X_t, Y_t are defined by (1.1). Using Itô and suitable integration over $[0, T]$ obtain an expression for $V(T, X_T, Y_T)$.

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial X} + c\frac{\partial V}{\partial Y} + \tfrac{1}{2}b^2\frac{\partial^2 V}{\partial X^2} + \tfrac{1}{2}d^2\frac{\partial^2 V}{\partial Y^2} + bd\frac{\partial^2 V}{\partial X \partial Y}\right)dt \\ &\quad + b\frac{\partial V}{\partial X}dW_t^{(1)} + d\frac{\partial V}{\partial Y}dW_t^{(2)} \end{aligned}$$

Integrating and rearranging then gives $V(T, X_T, Y_T) =$

$$V(0, X_0, Y_0) + \int_0^T \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial X} + c\frac{\partial V}{\partial Y} + \tfrac{1}{2}b^2\frac{\partial^2 V}{\partial X^2} + \tfrac{1}{2}d^2\frac{\partial^2 V}{\partial Y^2} + \rho bd\frac{\partial^2 V}{\partial X \partial Y}\right)dt +$$
$$\int_0^T b\frac{\partial V}{\partial X}dW_t^{(1)} + \int_0^T d\frac{\partial V}{\partial Y}dW_t^{(2)}.$$

c.

$$\begin{aligned} X_{i+1} &= X_i + a\delta t + b\phi_1\sqrt{\delta t} \\ Y_{i+1} &= Y_i + c\delta t + d\phi_2\sqrt{\delta t} \end{aligned}$$

$\phi_1, \phi_2 \sim N(0,1)$ and $\mathbb{E}[\phi_1\phi_2] = \rho$. The pair $\epsilon_1,\ \epsilon_2 \sim N(0,1)$ with $\mathbb{E}[\epsilon_1\epsilon_2] = 0$. Construct $\phi_1 = \epsilon_1;\ \phi_2 = \alpha\epsilon_1 + \beta\epsilon_2$,

$$\begin{aligned} \mathbb{E}[\phi_1\phi_2] &= \rho = \mathbb{E}[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)] \\ \mathbb{E}[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)] &= \rho \\ \alpha\mathbb{E}\left[\varepsilon_1^2\right] + \beta\mathbb{E}[\varepsilon_1\varepsilon_2] &= \rho \to \alpha = \rho \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left[\phi_2^2\right] &= 1 = \mathbb{E}\left[(\alpha\varepsilon_1 + \beta\varepsilon_2)^2\right] \\ &= \mathbb{E}\left[\alpha^2\varepsilon_1^2 + \beta^2\varepsilon_2^2 + 2\alpha\beta\varepsilon_1\varepsilon_2\right] \\ &= \alpha^2\mathbb{E}\left[\varepsilon_1^2\right] + \beta^2\mathbb{E}\left[\varepsilon_2^2\right] + 2\alpha\beta\mathbb{E}[\varepsilon_1\varepsilon_2] = 1 \\ \rho^2 + \beta^2 &= 1 \to \beta = \sqrt{1-\rho^2} \end{aligned}$$

The correlated SDEs become

$$\begin{aligned} X_{i+1} &= X_i + a\delta t + b\epsilon_1\sqrt{\delta t} \\ Y_{i+1} &= Y_i + c\delta t + +d\left(\rho\epsilon_1 + \sqrt{1-\rho^2}\epsilon_2\right)\sqrt{\delta t} \end{aligned}$$

a. For which values of k is the process

$$Y_t = W_t^4 - 6tW_t^2 + kt^2,\ t \geq 0,$$

a martingale? The problem is asking you to calculate the value of k such that Y_t has zero drift. Using Itô

$$dY_t = \left(\frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial W^2}\right)dt + \frac{\partial Y_t}{\partial W}dW$$

$$\frac{\partial Y_t}{\partial t} = -6W_t^2 + 2kt;\ \frac{\partial Y_t}{\partial W} = 4W_t^3 - 12tW_t;\ \frac{\partial^2 Y_t}{\partial W^2} = 12W_t^2 - 12t$$

$$
\begin{aligned}
\frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial W^2} &= 0 \rightarrow -6W_t^2 + 2kt + 6W_t^2 - 6t = 0 \\
k &= 3.
\end{aligned}
$$

b. Show that $X_t = \cosh(\theta W_t)\, e^{-\theta^2 t/2};\ \theta \in \mathbb{R}$, is a martingale.

$$F(W_t, t) = \cosh(\theta W_t)\, e^{-\theta^2 t/2}$$

Using Itô

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) dt + \frac{\partial F}{\partial W} dW$$

So checking that $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} = 0$, i.e. a driftless process.

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= \cosh(\theta W_t)\, e^{-\theta^2 t/2} = -\frac{\theta^2}{2}\cosh(\theta W_t)\, e^{-\theta^2 t/2} \\
\frac{\partial F}{\partial W} &= \theta \sinh(\theta W_t)\, e^{-\theta^2 t/2}; \frac{\partial^2 F}{\partial W^2} = \theta^2 \cosh(\theta W_t)\, e^{-\theta^2 t/2}
\end{aligned}
$$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} = -\frac{\theta^2}{2}\cosh(\theta W_t)\, e^{-\theta^2 t/2} + \frac{1}{2}\left(\theta^2 \cosh(\theta W_t)\, e^{-\theta^2 t/2}\right)$$

Hence a Martingale.

6. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant. Asset:

$$\begin{array}{ccccc} & & & & a^2S \\ & & & \nearrow & \\ & & aS & & \\ & \nearrow & & \searrow & \\ S(0) = S & & & & S \\ & \searrow & & \nearrow & \\ & & \frac{1}{a}S & & \\ & & & \searrow & \\ & & & & \frac{1}{a^2}S \end{array}$$

Option:

$$\begin{array}{ccccc} & & & & a^2S - S \\ & & & \nearrow & \\ & & V_1 & & \\ & \nearrow & & \searrow & \\ V_3 & & & & 0 \\ & \searrow & & \nearrow & \\ & & V_2 & & \\ & & & \searrow & \\ & & & & 0 \end{array}$$

a. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price.
These are when $r = 0$,

$$\begin{aligned} q(\text{up}) &= \frac{s - s_d}{s_u - s_d} \\ q(\text{down}) &= \frac{s_u - s}{s_u - s_d} \end{aligned}$$

For each time-step we have the probabilities:

$$\begin{aligned} q(\text{up}) &= \frac{S - \frac{1}{a}S}{aS - \frac{1}{a}S} = \frac{1}{a+1}, \\ q(\text{down}) &= \frac{aS - S}{aS - \frac{1}{a}S} = \frac{a}{a+1}. \end{aligned}$$

$\therefore$

$$\begin{aligned} \omega_1 &= \frac{1}{(a+1)^2} \\ \omega_2 &= \frac{a}{(a+1)^2} \\ \omega_3 &= \frac{a}{(a+1)^2} \\ \omega_3 &= \frac{a^2}{(a+1)^2}. \end{aligned}$$

So the expected value is:

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} p(\omega) X(\omega) = p(\omega_1)\left(a^2 S - S\right) + 0 + 0 + 0 = \frac{a-1}{a+1} S,$$

as before!

b. Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.
This is a special case of the above model when $a = 2$. Substituting in $a = 2$ into the option gives

$$\frac{2-1}{2+1} S = \frac{1}{3} S.$$