Further Stochastic Differential Equations and Stochastic Integration

W_t is a Brownian Motion (Wiener Process) and dW_t or $dW(t)$ is its increment. $W_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\,dW_t + B(S,t)\,dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following SDE

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dW_t.$$

Can (non-zero) A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.
A function $g(S)$ will satisfy the shorter SDE

$$dg = \left(B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2g}{dS^2}\right)dt + A\frac{dg}{dS}dW_t.$$

For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition

$$B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2g}{dS^2} = 0.$$

i.e.

$$\frac{dg}{dS} + \frac{1}{2}\frac{A^2}{B}\frac{d^2g}{dS^2} = 0$$

We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

2. Show that $F(W_t) = \arcsin(2aW_t + \sin F_0)$ is a solution of the SDE

$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dW_t,$$

where F_0 and a is a constant. The following standard result may be used

$$\frac{d}{dx}\sin^{-1}ax = \frac{a}{\sqrt{1-a^2x^2}}$$

$F = \arcsin(2aW_t + \sin F_0)$ implies $\sin F = 2aW_t + \sin F_0$ hence

$$\frac{dF}{dW_t} = \frac{2a}{\sqrt{1-(2aW_t+\sin F_0)^2}} = 2a\left\{1-(2aW_t+\sin F_0)^2\right\}^{-1/2}$$

$$\frac{d^2F}{dW_t^2} = \frac{(2a)^2(2aW_t+\sin F_0)}{\left\{1-(2aW_t+\sin F_0)^2\right\}^{3/2}}$$

So Itô gives

$$dF = \frac{2a}{\sqrt{1-(2aW_t+\sin F_0)^2}}dW + \frac{1}{2}\frac{(2a)^2(2aW_t+\sin F_0)}{\left\{1-(2aW_t+\sin F_0)^2\right\}^{3/2}}dt$$

We know $\cos^2 F + \sin^2 F = 1 \Longrightarrow \cos F = \sqrt{1-\sin^2 F} = \sqrt{1-(2aW_t+\sin F_0)^2}$. Some more trig.

$$\sec F=\frac{1}{\cos F}=\frac{1}{\sqrt{1-(2aW_t+\sin F_0)^2}}$$

and

$$(\tan F)\left(\sec^2 F\right)=\frac{\sin F}{\cos F}\frac{1}{\cos^2 F}=\frac{\sin F}{\cos^3 F}=\frac{2aW_t+\sin F_0}{\left\{1-(2aW_t+\sin F_0)^2\right\}^{3/2}}$$

which gives

$$dF=2a^2(\tan F)\left(\sec^2 F\right)dt+2a(\sec F)\,dW_t.$$

3. Show that

$$\int_0^t W_\tau\left(1-e^{-W_\tau^2}\right)dW_\tau=\overline{F}(W_t)+\int_0^t G(W_\tau)\,d\tau.$$

where the functions $\overline{F}$ and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t W(\tau)\left(1-e^{-W^2(\tau)}\right)dW(\tau)=\overline{F}(W(t))+\int_0^t G(W(t))\,d\tau$$

with

$$\int_0^t \frac{\partial F}{\partial W}dW(\tau)=F(W(t),t)-F(W(0),0)+\int_0^t-\left(\frac{\partial F}{\partial \tau}+\tfrac{1}{2}\frac{\partial^2 F}{\partial W^2}\right)d\tau$$

suggests that

$$\frac{\partial F}{\partial W}=W(\tau)\left(1-e^{-W^2(\tau)}\right)$$

so integrating over $[0,t]$ gives $\overline{F}(W(t),t)$, which we will do by substitution, i.e. put $u=W^2$ which gives

$$F(W(t),t)-F(W(0),0)=\tfrac{1}{2}W^2(t)+\tfrac{1}{2}e^{-W^2(t)}-\tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial W}$ allows us to easily obtain $\frac{\partial^2 F}{\partial W^2}=2W^2(t)\,e^{-W^2(t)}-e^{-W^2(t)}+1$. Hence

$$G(W(t))=-\tfrac{1}{2}\frac{\partial^2 F}{\partial W^2}=-\tfrac{1}{2}\left(1-e^{-W^2(t)}\right)-W^2(t)\,e^{-W^2(t)}$$

and we have shown

$$\int_0^t W(\tau)\left(1-e^{-W^2(\tau)}\right)dW(\tau)=\overline{F}(W(t))+\int_0^t G(W(t))\,d\tau$$

where

$$\begin{aligned}\overline{F}(W(t),t)&=\tfrac{1}{2}W^2(t)+\tfrac{1}{2}e^{-W^2(t)}-\tfrac{1}{2}\\ G(W(t))&=-\tfrac{1}{2}\left(1-e^{-W^2(t)}\right)-W^2(t)\,e^{-W^2(t)}.\end{aligned}$$

4. Consider the process

$$d(\log y)=(\alpha-\beta\log y)\,dt+\delta dW_t.$$

The parameters $\alpha,\ \beta,\ \delta$ are constant. Show that y satisfies

$$\frac{dy}{y}=\left(\alpha-\beta\log y+\frac{1}{2}\delta^2\right)dt+\delta dW_t.$$

By Ito's lemma if $dZ = a(Z,t)dt + b(Z,t)dW_t$ and $Y = f(Z)$ then

$$dY = \left(a\frac{\partial Y}{\partial Z} + \frac{1}{2}b^2\frac{\partial^2 Y}{\partial Z^2} \right) dt + b\frac{\partial Y}{\partial Z}dW_t$$

here $Z \equiv \log y_t,\ a \equiv (\alpha - \beta Z),\ b \equiv \delta,\ ,\ Y = e^Z = y,\ \frac{\partial Y}{\partial Z} = e^Z = \frac{\partial^2 f}{\partial Z^2}$, putting all these in Ito's lemma we have

$$dY \equiv dy_t = \left((\alpha - \beta \log y_t)\, y_t + \frac{1}{2}\delta^2 y_t \right) dt + \delta y_t dW_t$$

hence

$$\frac{dy_t}{y_t} = \left(\alpha - \beta \log y_t + \frac{1}{2}\delta^2 \right) dt + \delta dW_t$$

5. Show that

$$G = e^{t + ae^{W_t}}$$

is a solution of the stochastic differential equation

$$dG_t = G_t \left(1 + \tfrac{1}{2}(\ln G_t - t) + \tfrac{1}{2}(\ln G_t - t)^2\right) dt + G_t (\ln G_t - t)\, dW_t,$$

where a is a constant.

$$\frac{\partial G_t}{\partial t} = G_t, \quad \frac{\partial G_t}{\partial W_t} = aG_t e^{W_t}, \quad \frac{\partial^2 G_t}{\partial W_t^2} = ae^{W_t}G_t + ae^{W_t}\frac{\partial G_t}{\partial W_t} = ae^{W_t}G_t + a^2e^{2W_t}G_t$$

In Itô, i.e.

$$\begin{aligned} dG_t &= \left(\frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial W_t^2} \right) dt + \frac{\partial G_t}{\partial W_t} dW_t \\ &= \left(G_t + \tfrac{1}{2}ae^{W_t}G_t + \tfrac{1}{2}a^2e^{2W_t}G_t\right) dt + ae^{W_t}G_t dW_t \end{aligned}$$

From $G_t = e^{t+ae^{W_t}}$ we have

$$ae^{W_t} + t = \ln G_t \Longrightarrow ae^{W_t} = \ln G_t - t$$

so we can write the SDE in terms of the process G_t

$$dG_t = G_t\left(1 + \tfrac{1}{2}ae^{W_t} + \tfrac{1}{2}a^2e^{2W_t}\right) dt + ae^{W_t}G_t dW_t$$

So

$$dG_t = G_t \left(1 + \tfrac{1}{2}(\ln G_t - t) + \tfrac{1}{2}(\ln G_t - t)^2\right) dt + G_t (\ln G_t - t)\, dW_t.$$

6. A spot rate r_t, evolves according to the popular form

$$dr_t = u(r_t)\, dt + \nu r_t^{\beta} dW_t, \qquad (*)$$

where ν and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies the drift structure of $(*)$ is given by

$$u(r_t) = \nu^2 \beta r_t^{2\beta - 1} + \frac{1}{2}\nu^2 r_t^{2\beta}\frac{d}{dr}(\log p_\infty).$$

The forward F.P equation for $dr = u(r,t)\,dt + w(r,t)\,dW_t$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(w^2(r,t)\,p(r,t)\right) - \frac{\partial}{\partial r}\left(u(r,t)\,p(r,t)\right)$$

for the probability density $p(r,t)$. The steady state equation for our model becomes

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}\left(r^{2\beta}p_\infty(r)\right) - \frac{d}{dr}\left(u(r)\,p_\infty(r)\right) = 0$$

This can be simply integrated once to give

$$\begin{aligned}
\frac{1}{2}\nu^2\frac{d}{dr}\left(r^{2\beta}p_\infty(r)\right) - \left(u(r)\,p_\infty(r)\right) &= \text{const} \\
\frac{1}{2}\nu^2\left(r^{2\beta}\frac{dp_\infty}{dr}\right) + \nu^2\beta r^{2\beta-1}p_\infty(r) - \left(u(r)\,p_\infty(r)\right) &= \text{const}
\end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left.\begin{matrix} p_\infty(r) \\ \frac{dp_\infty}{dr} \end{matrix}\right\} \longrightarrow 0$$

$$\begin{aligned}
u(r)\,p_\infty(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1}p_\infty(r) \\
u(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{1}{p_\infty(r)}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1}
\end{aligned}$$

We can write $\dfrac{1}{p_\infty}\dfrac{dp_\infty}{dr}$ as $\dfrac{d}{dr}(\log p_\infty)$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}(\log p_\infty) + \nu^2\beta r^{2\beta-1}$$