CQF FitchLearning

# TUTORIAL

# Statistical Essentials for VaR and ES
# VaR Backtesting and Breaches
# EVT Results

Dr Richard Diamond
CQF ARPM

March 2021

Knowledge | Skills | Conduct

# Learning outcomes

$Pr(x < 0.95) \equiv \phi(\,.\,.\,)$

- understand the first principles: expectations algebra, percentile, analytical vs empirical work with distributions
- be able to read probability notation maths for VaR and ES
  conditional
- EXCEL walkthrough via VaR Backtesting (simple scheme)
- EXCEL example of EVT calculation

These slides will be released AFTER tutorial.

During the session, we refer to Market Risk Lecture SOLUTIONS – please download from CQF Portal and open.

# Percentile VaR

3. Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* to derive analytical expression for VaR calculation.

Random Variable

$\int f_X(x) = F_X$ — $f_X(x)$: PDF, $F_X$: CDF

**Solution:**

specific value of R.V. X

The probability of loss $x < 0$ being worse than VaR $< 0$ is

95%

$$\Pr(x \le \mathrm{VaR}(X)) = 1 - c = 1 - 0.99 = 0.01$$

CDF

$$\mathrm{VaR}_c(X) = \inf\{x \mid \Pr(X > x) \le 1 - c\} = \inf\{x \mid F_X(x) \ge c\}$$

max bound

for or 99% confidence, the probability that $X$ above loss $x$ is less than $(1 - 0.99) = 0.01$.

If P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

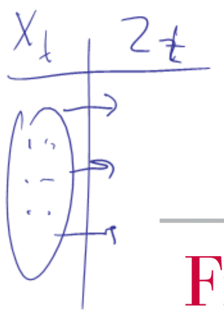
standardise

$$Z_t = \frac{X_t - \hat{\mu}}{\sigma}$$

$$\Phi^{-1}\left[\Pr\left(\phi \le \frac{\mathrm{VaR}(X) - \mu}{\sigma}\right)\right] = \Phi^{-1}[1 - c] \implies$$

$$\boxed{\mathrm{VaR}(X) = \mu + \Phi^{-1}(1 - c) \times \sigma}$$

$\phi^{-1}(\Pr)$ NORMSINV()




**Inverse CDF for a probability distribution is known as 'percentile function'.**

# Analytical VaR / Parametric

$$\cancel{\phi^{-1}}\left[\cancel{\phi}\left(\frac{VaR(x)-\mu}{\sigma}\right)\right] = \phi^{-1}[1-c]$$

$F_X(x) = \phi(\;)$

inverse of CDF

$$VaR(x) - \mu = \phi^{-1}[1-c] \times \sigma$$

$$VaR(x) = \mu + \underbrace{\phi^{-1}[1-c] \times \sigma}$$

Factor

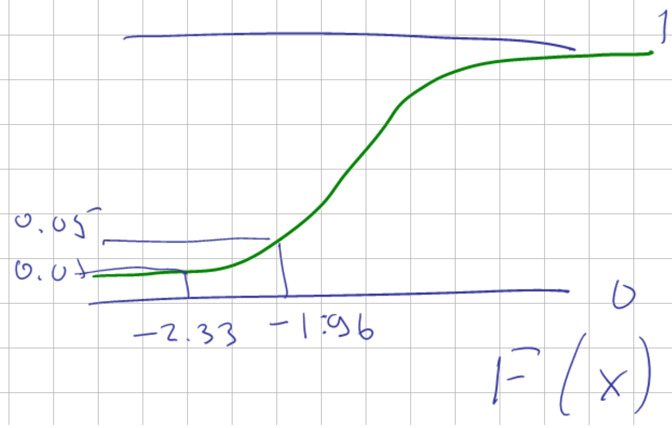
99% VaR

$$\phi^{-1}[1-0.99] = \phi^{-1}[0.01] = -2,32635$$

NORMSINV(0.01)

Percentile function

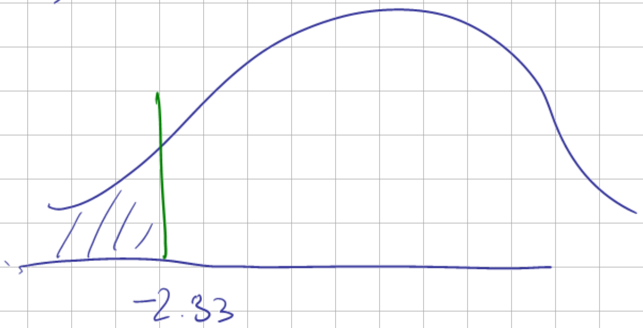
Standardised Percentile



# Empirical VaR / Historical Simulation

VaR

$\text{P\&L}_{\text{returns}} \sim N(\mu, \sigma^2)$




| Standardised residuals, Z | Returns (sorted) X |
| --- | --- |
| -2.7 | -0.17 |
| -2.5 | -0.155 |
| -2.33 | -0.14 |
| -2.18 | -0.12 |
| -1.98 | -0.10 |
| | ... |

1% of $N_{obs} = 300$

inf { }

VaR

$Z_t = \dfrac{X_t - \mu}{\sigma}$

PERCENTILE(0,01)

$VaR = \mu - 2.32635 \times \sigma$

↳ Analytical VaR

-2.33

↳ Analytical ES

-2.67

(-2.6652)

$ES = \mu - 2.6652 \times \sigma$

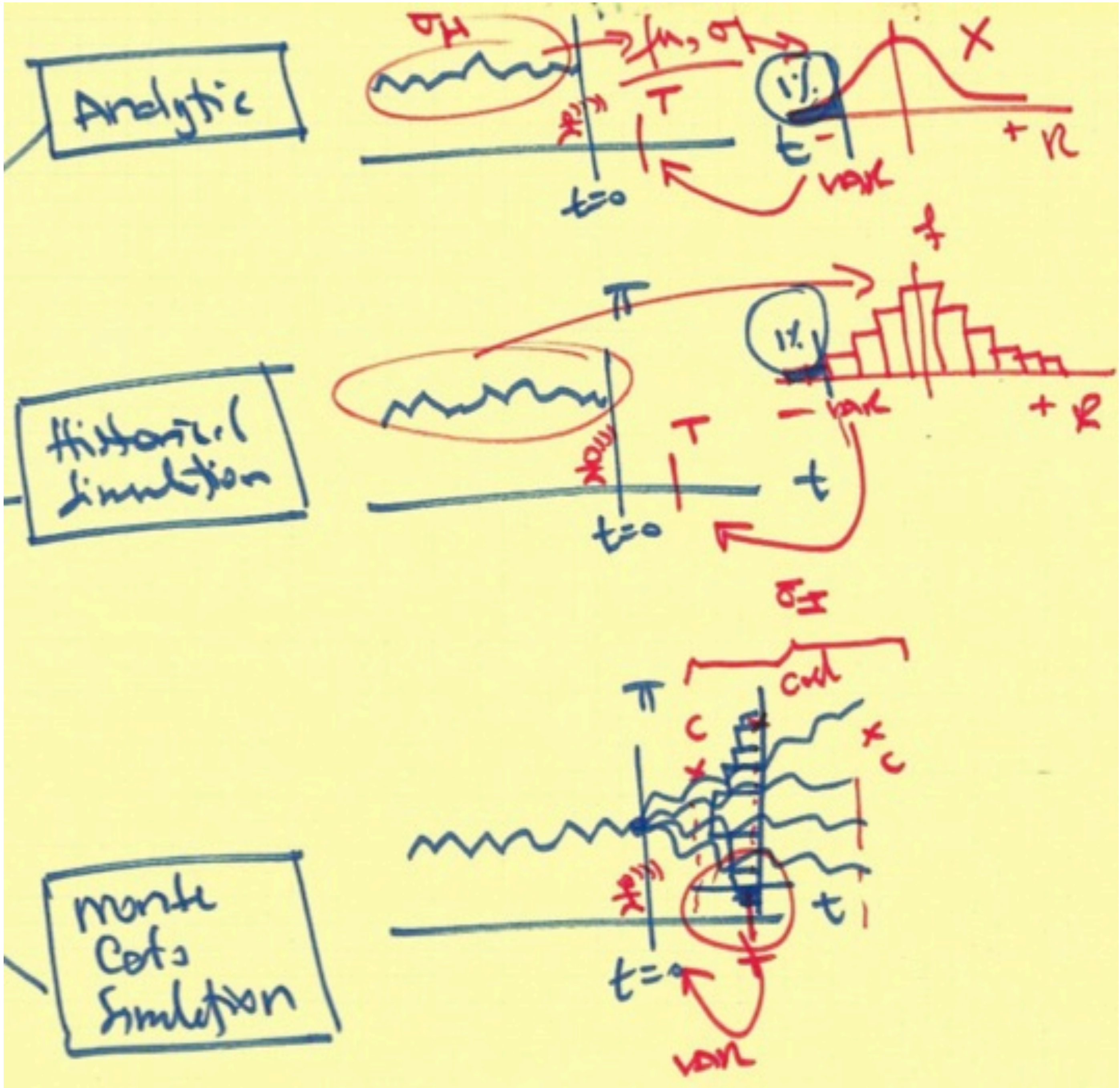
$VaR = -0.14$

$ES_{99\%} = -\dfrac{(0.17 + 0.155 + 0.14)}{3}$

↳ Historic Simulation $X \to Z$

↳ Monte Carlo Simulation

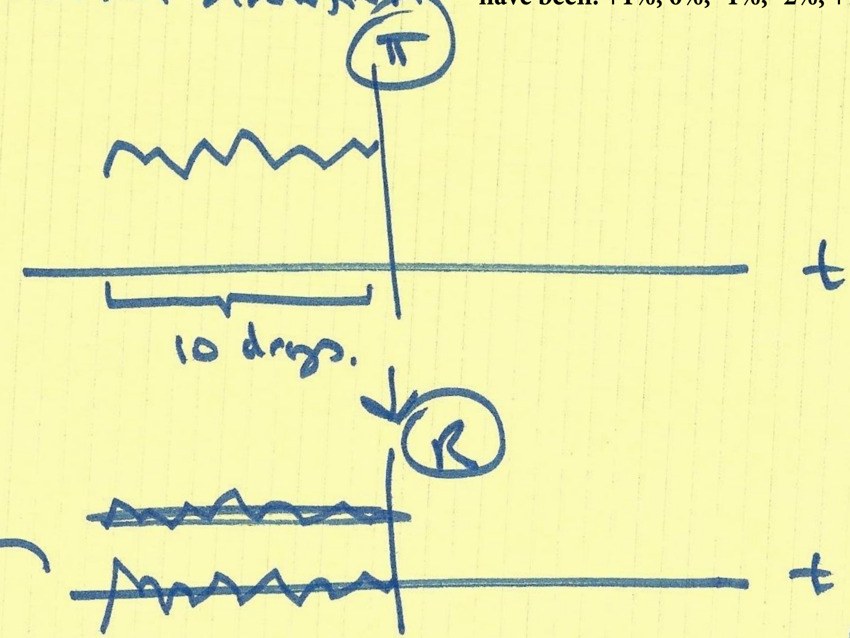
$Z_{sim} \to X$


Analytic
Historical Simulation
Monte Carlo Simulation
t=0
T
t
VaR
1%
+R
−
X

**Problem 1: Historical Simulation Compute the 1-day VAR at 90% confidence (both in percent and monetary terms) for a portfolio of £3 million whose recent daily returns have been: +1%, 0%, -1%, -2%, +1%, +3%, -1%, 0%, -3%, 0%**

① VaR Historical Simulation

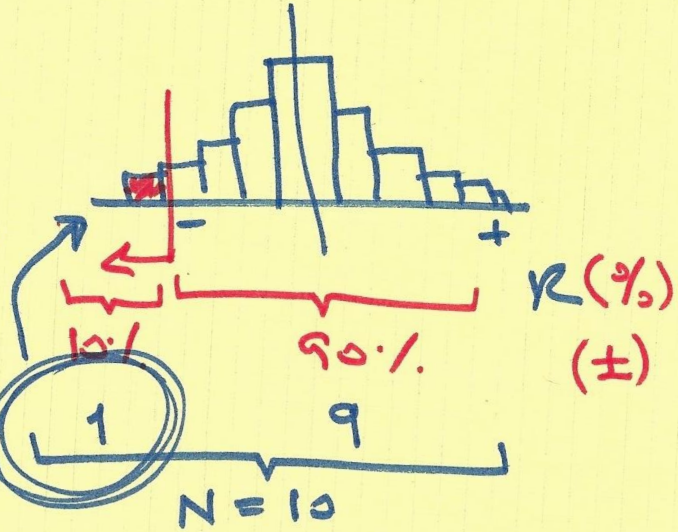
Π : $\boxed{3m £}$




VaR(1d, 90%)

VaR(1d, X)

T




| R | sorted R |
|---|---|
| +1% | −3% |
| 0% | −2% |
| −1% | −1% |
| −2% | −1% |
| +1% | 0% |
| +3% | 0% |
| −1% | 0% |
| 0% | +1% |
| −3% | +1% |
| 0% | +3% |

N=10

1 | 9

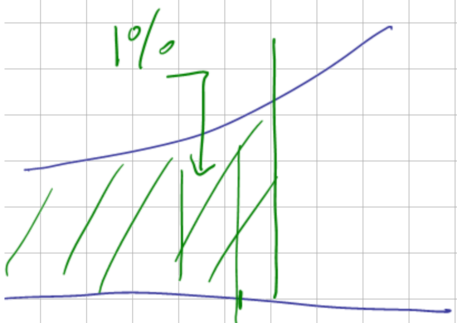
N = 10

→ VaR(1d, 90%) = +3% (%)

→ VaR(1d, 90%) = (−3%)(3m£) = + 90,000 £ (£)

# Expected Shortfall

ES




$VaR_{0,008}$ $VaR_{0,009}$ $VaR_{0.01}$

$$ES(X) = E[X \mid X \le VaR_c(X)]$$

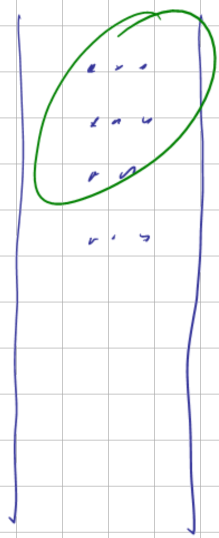
mean E[X] but here

mean of the tail

$$ES(X) = \frac{1}{1-c}\int_0^{1-c} VaR_u \, du$$

$$\underbrace{\frac{\sum VaR_u}{0.01}}_{\text{averaging}}$$

Return



Average is ES

$du = 0.001$ (10 obs)

or

$0.0001$ (100 obs)

$VaR_{99\%} \approx -2,33$

$VaR_{99,1\%} \approx -2.33$

$VaR_{99.2\%} \approx -2,34$

ES is

average worst outcome

mean of the tail values

# VaR Backtesting DEMO

Imagine that each morning you calculate 99%/10day VaR from available prior data only. Once ten days pass you compare that VaR number to the realised return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE 100 index levels, continue in Excel.

- Calculate Value at Risk for each day $t$ (starting on Day 21) as follows:

$$\mathrm{VaR} = \mu_{10D} + \mathrm{Factor} \times \sigma_{10D} \quad \dagger$$

where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have a final column with $\mathrm{VaR}_t$ as a percentage since calculation is done on returns.

# Backtesting: where to improve?

**C.1** Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation of log-returns, as follows:

- The rolling standard deviation for a sample of 21 is computed for days 1-21, 2-22, ..., there must be 21 observations in the sample. So, you have a time series of $\sigma_t$.
- Scale standard deviation to reflect a ten days move $\sigma_{10D} = \sqrt{10 \times \sigma^2}$ (we can add variances) and scale an average daily return as $\mu_{10D} = \mu \times 10$ where $\mu$ is a mean return of all data given.

- Sample window – regulators' approach is 200 days.
- Volatility estimation – is it responsive enough? Why ARCH-filtered volatility might make the breach count **worse**?
- Is Normal Percentile a good predictor for P&L? We should aim to at least choose t distribution Percentile – requires d.f. parameter.

***Exercise: Build Q-Q plots vs. Normal and OTHER DISTRIBUTIONS.***

# RiskMetrics (EWMA)

$$\boldsymbol{\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)u_{t-1}^2}$$

**<u>Low $\lambda$</u>**

Low $\lambda$ leads to more weight for the $(1-\lambda)u_{n-1}^2$ term, so the model is very responsive to the previous day's returns, i.e. news from the market.

**<u>High $\lambda$</u>**

High $\lambda$ leads to a slow response to new information.

Example: The RiskMetrics database made available by by JP Morgan in 1994 uses the EWMA model with $\lambda = 0.94$ for updating daily estimation of variance across a range of markets.

# Volatility Filtering (Risk Metrics, ARCH)

EWMA

~~0.94~~ 0.72

$$\sigma_{t+1}^2 = \alpha\,\sigma_t^2 + (1-\alpha)\,r_t^2$$

forecast

past vol$^2$ (all / nearly all obs avail.)

past ret$^2$

"Moving Averges"

$r_t \equiv u_t$ or $r_t \equiv \varepsilon_t$

GARCH

$$\sigma_{t+1}^2 = \beta\,\sigma_t^2 + \alpha\,r_t^2 + \omega$$

$$\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$$

long term average variance (const)

# EVT Result and DEMO

- tails of wide range of distributions share the common properties.

| | Exceedance |
|---|---|
| -0.17 | 3% |
| -0.155 | 1.5% |
| -0.14 | 0 |

$$\Pr\left[(\text{Loss} - \underset{\uparrow\,\text{VaR}}{u_0}) \leq y \,|\, \text{Loss} > u_0\right] = \text{GPD}_{\xi,\beta}(y)$$

- Given that value does exceed *u*, the probability for each outcome – the amount by which *v* **exceeds** *u*

$$F_u(y) = \frac{\overset{\Phi[-3]}{F(u+y)} - \overset{\Phi[-2.33]}{F(u)}}{\underbrace{1 - F(u)}_{0.01}} \qquad G_{\xi,\beta}(y) = 1 - \left[1 + \xi\frac{y}{\beta}\right]^{-1/\xi}$$

cdist

is given by Generalised Pareto Distribution with tail parameter xi. ξ

parameter β like σ for Normal

***Exercise: differentiate over G(y) to obtain PDF and construct an MLE function "sum of log-likelihoods". See 13.7 Hull textbook for answer.***

# Basel II and legacy backtesting

- Independence of breaches in VaR: Christoffersen’s 1998 Exceedance Independence Test
- Measures the dependence between consecutive days only

REF Christoffersen’s 1998 Exceedance Independence Test

https://www.value-at-risk.net/backtesting-independence-tests/

REF Statistical tests for VaR backtesting

https://www.mathworks.com/help/risk/overview-of-var-backtesting.html