条件波动率模型的估计与检验

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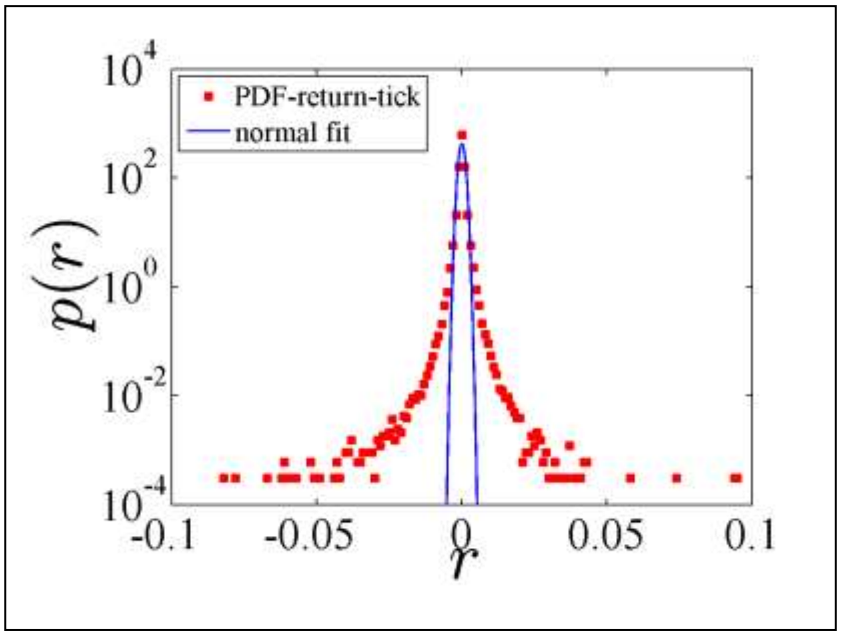
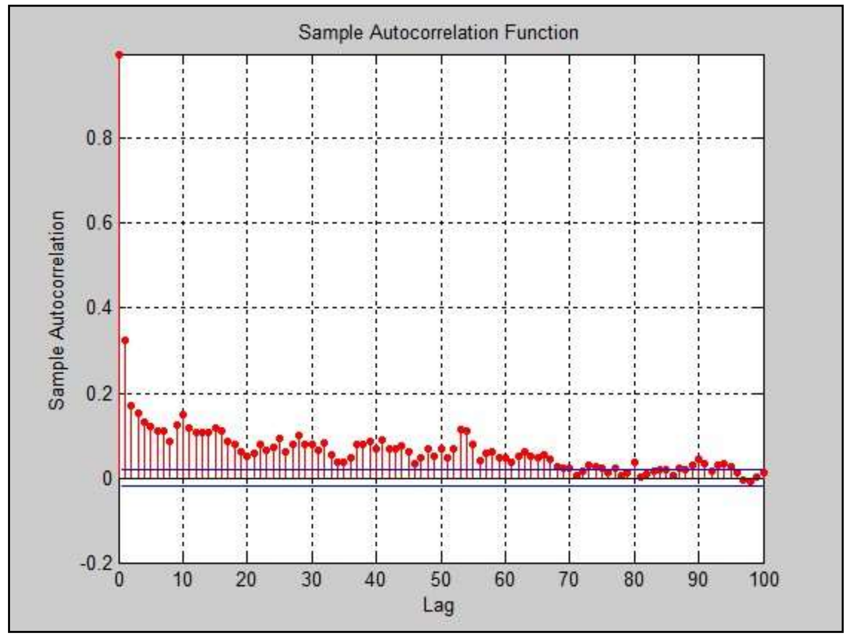
金融时间序列特征

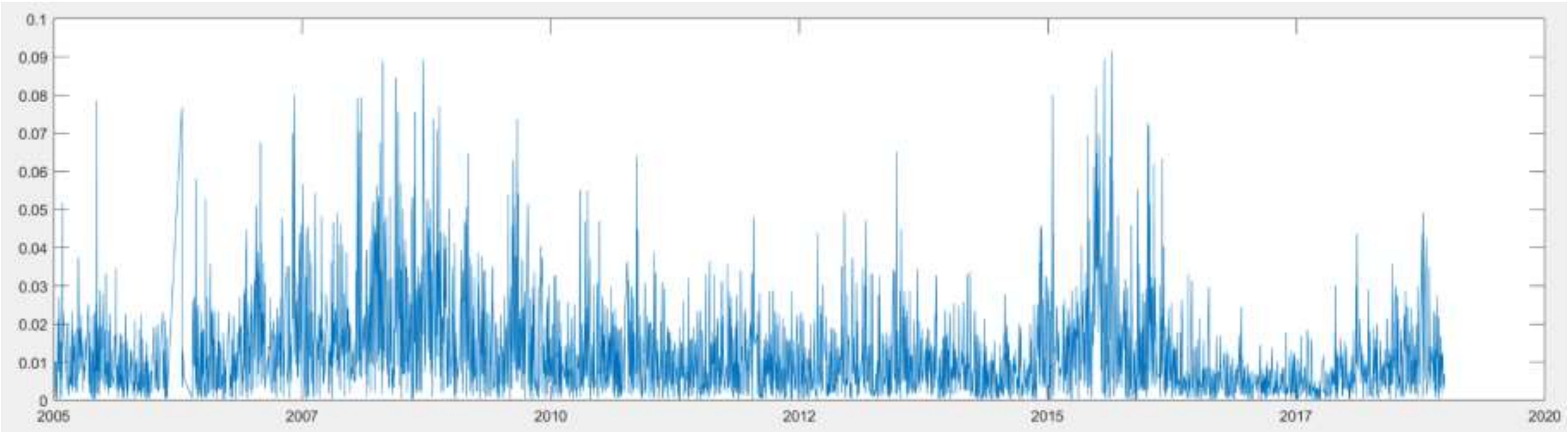
- 收益率尖峰厚尾
- 波动率集聚（自相关）
- 正负冲击的非对称性





传统线性回归模型无效

波动率



Volatility clustering

- "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes."
—Mandelbrot (1963)

波动率

收益率 → 波动率

- 反应市场的不确定性程度
- 信息流的度量：高波动率，大信息冲击
- 特点：时变、集聚
- 简单估计：收益率的绝对值

金融时序胖尾特征 → 建模改进

- 条件波动：ARCH和GARCH
- 其他分布形式：*t* 分布、GED分布等

条件波动率和收益率模型

条件波动率 $\sigma_t = \sqrt{\mathrm{Var}[r_t | \mathbf{I}_{t-1}]}$

- $\mathbf{I}_{t\text{-}1}$ 投资者在 t-1 时刻所拥有的信息
- Var[$\cdot$|$\mathbf{I}_{t\text{-}1}$] 基于信息集$\mathbf{I}_{t\text{-}1}$的条件方差

条件波动率、非条件波动率

条件方差、非条件方差

收益率观察值的样本方差是非条件方差的一个估计值

条件波动率和收益率模型

收益率模型 $\begin{cases} r_t = \mu_t + \epsilon_t \\ \epsilon_t \equiv \sigma_t z_t, \end{cases}$

- $\mu_t = E(r_t \mid \mathbf{I}_{t\text{-}1})$ 收益率条件期望
- σ_t 条件波动率，信息集$\mathbf{I}_{t\text{-}1}$的非负函数
- $\mathbf{I}_{t\text{-}1}$表示时刻 t-1 的信息集
- ε_t 是一个白噪声过程
- Z_t 满足标准正态分布，IID

条件波动率和收益率模型

收益率模型 $\begin{cases} r_t = \mu_t + \epsilon_t \\ \epsilon_t \equiv \sigma_t z_t, \end{cases}$

μ_t 的设定：

- **常数**
- **ARMA过程**

$$r_t = \phi_0 + \sum_{p=1}^{P} \phi_p r_{t-p} + \sum_{q=1}^{Q} \theta_q \epsilon_{t-q} + \epsilon_t$$

有： $\text{Var}[r_t|\mathbf{I}_{t-1}] = \sigma_t^2$

条件波动率和收益率模型

收益率模型

$$\begin{cases} r_t = \mu_t + \epsilon_t \\ \epsilon_t \equiv \sigma_t z_t, \end{cases}$$

σ_t^2的设定：

- **ARCH**
- **GARCH**
- **EGARCH**
- **ARCH-M**
- **……**

条件波动率和收益率模型

收益率模型

$$\begin{cases} r_t = \mu_t + \epsilon_t \\ \epsilon_t \equiv \sigma_t z_t, \end{cases}$$

z_t的设定：

- **标准正态分布**
- **标准*t*分布**
- **标准的广义误差分布**
- **混合正态分布**
- ……

ARCH

- **ARCH(1)**

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2, \quad \alpha_0 > 0, 0 < \alpha_1 < 1.$$

- **ARCH(q)**

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 \equiv \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

- ✓ **保证 $\sigma_t^2 > 0$，要求 $\alpha_i > 0$ $(i = 0, 1, \ldots, q)$**
- ✓ **保证平稳，要求** $\sum_{i=1}^{q} \alpha_i < 1$

ARCH

- **ARCH(1)的性质**

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2, \quad \alpha_0 > 0, 0 < \alpha_1 < 1.$$

- 均值 $E[\epsilon_t] = 0$
- 方差 $\mathrm{Var}[\epsilon_t] = \dfrac{\alpha_0}{1-\alpha_1}$
- 峰度 $\mathrm{Kurt}[\epsilon_t] = 3\dfrac{1-\alpha_1^2}{1-3\alpha_1^2} > 3, \quad 0 < \alpha_1 < 1/\sqrt{3}$

尖峰胖尾

ARCH

基于ARCH(1)的收益率模拟

1.设置模型参数

$$\mu = 0.033\%, \alpha_0 = 1.52 \times 10^{-4}, \alpha_1 = 0.4735, \sigma_0^2 = 2.89 \times 10^{-4}$$

2.利用标准正态分布产生随机数

$$Z_0, Z_1, Z_2, \cdots, Z_T$$

3.迭代计算下式

$$\epsilon_t = \sigma_t z_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2,$$

$$r_t = \mu + \epsilon_t.$$

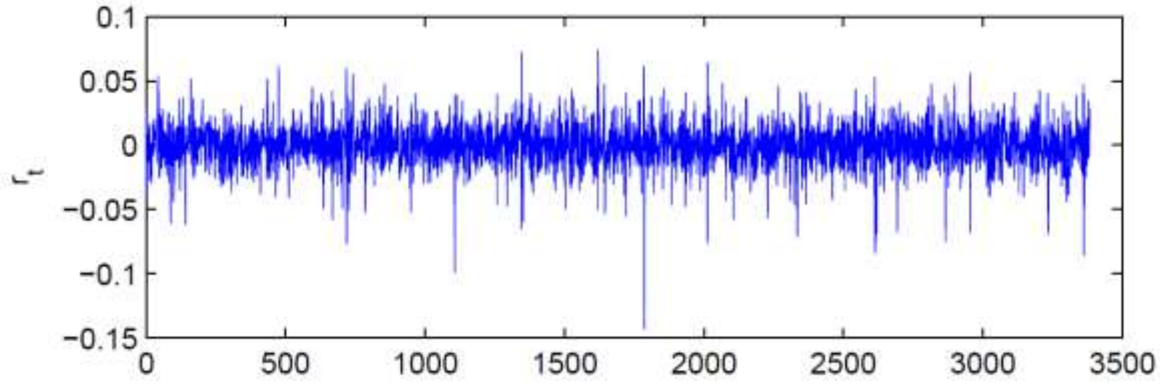
ARCH

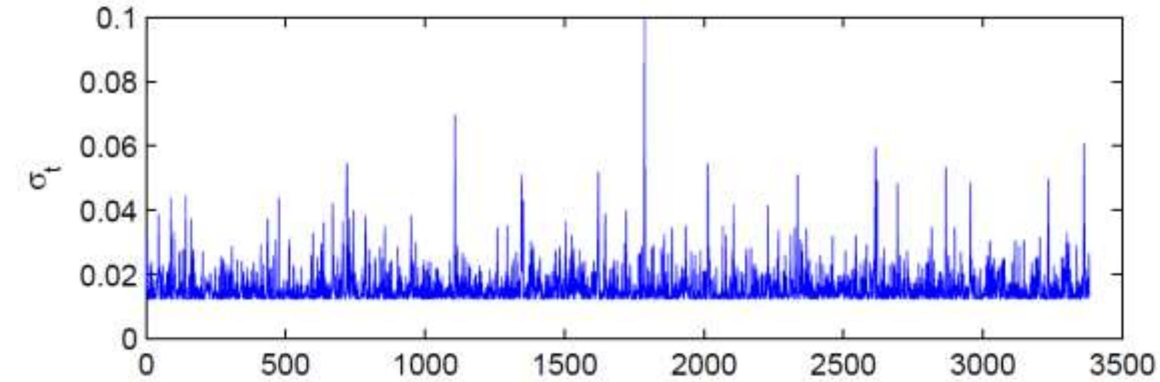
ARCH(1)的模拟

$$\epsilon_t = \sigma_t z_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2,$$

$$r_t = \mu + \epsilon_t.$$





ARCH

ARCH效应检验（条件异方差？）

Engle(1982)'s ARCH test

- $\boxed{\hat{\epsilon}_t \text{ as the demeaned return series,}}$ and run regression:

$$\hat{\epsilon}_t^2 = \alpha_0 + \alpha_1 \hat{\epsilon}_{t-1}^2 + \alpha_2 \hat{\epsilon}_{t-2}^2 + \cdots + +\alpha_m \hat{\epsilon}_{t-m}^2 + u_t.$$

- Record the R^2 of the regression, then under the null hypothesis of no ARCH effect

$$T \cdot R^2 \overset{a}{\sim} \chi^2(m).$$

GARCH

- **GARCH(p, q)**

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

- ✓ **保证 $\sigma_t^2 > 0$，要求 α_i $\beta_i > 0$ $(i = 0,1,\ldots,q)$**
- ✓ **保证平稳，要求**

$$\sum_{i=1}^{p} \beta_i + \sum_{i=1}^{q} \alpha_i < 1$$

GARCH

- **GARCH(1, 1)的性质**

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2$$

$$\omega > 0, \beta_1 > 0, \alpha_1 > 0, \beta_1 + \alpha_1 < 1$$

- 均值 $E[\epsilon_t] = 0$
- 方差 $\mathrm{Var}(\epsilon_t) = \dfrac{\omega}{1-(\beta_1+\alpha_1)}$
- 峰度 $\mathrm{Kurt}[\epsilon_t] = 3\dfrac{1-(\alpha_1+\beta_1)^2}{1-(\alpha_1+\beta_1)^2-2\alpha_1^2} > 3$

尖峰胖尾

GARCH

基于GARCH(1, 1)的收益率模拟

1.设置模型参数

$$\mu = 0.0245\%, \omega = 5.18 \times 10^{-6}, \beta_1 = 0.8869, \alpha_1 = 0..1010, \sigma_0^2 = 0.0207^2$$

2.利用标准正态分布产生随机数

$$Z_0, Z_1, Z_2, \cdots, Z_T$$

3.迭代计算下式

$$\epsilon_t = \sigma_t Z_t$$

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2$$

$$r_t = \mu_t + \epsilon_t$$

EGARCH

- **EGARCH(*p*, *q*)**

$$\log \sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \log \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i (|z_{t-i}| - E|z_{t-i}|) + \sum_{i=1}^{q} \theta_i z_{t-i}$$

✓ **杠杆效应** Leverage effect: $\theta_i < 0$

ARCH-M

- **ARCH-M**

$$
\begin{cases}
r_t = \mu_t + \epsilon_t \\
\mu_t \equiv \mu + \delta\sigma_t, \quad \delta > 0 \\
\epsilon_t = \sigma_t z_t.
\end{cases}
$$

- **风险溢价** Risk premium: $\delta\sigma_t$

其他模型

- Nonlinear GARCH (NGARCH) model: Engle and Ng (1993)

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha(\epsilon_{t-1} - \theta\sigma_{t-1})^2,$$

- Quadratic GARCH (QGARCH) model: Sentana (1995)

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\epsilon_{t-1}^2 + \theta\epsilon_{t-1};$$

- QJR-GARCH model or threshold GARCH (TGARCH) model: Glosten, Jagannathan, and Runkle (1993)

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\epsilon_{t-1}^2 + \theta\epsilon_{t-1}^2 \cdot 1_{\{\epsilon_{t-1}\geq 0\}}.$$

- Similar to EGARCH, the NGARCH, QGARCH, and TGARCH models also can describe leverage effect

Z_t设定

- 标准正态分布

$$f(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z_t^2}{2}}$$

- 标准 t 分布

$$g(x) = \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)} \left(1 + \frac{x^2}{v}\right)^{-(v+1)/2},$$

$$f(z_t) = \sigma g(\sigma z_t), \quad \sigma \equiv \sqrt{v/(v-2)}.$$

$$v(v > 2)$$

Z_t设定

- **标准的广义误差分布**

$$f(z_t) = \frac{\nu \exp[-(1/2)|z_t/\lambda|^{\nu}]}{\lambda 2^{(1+1/\nu)}\Gamma(1/\nu)}, \quad 0 < \nu \leq \infty,$$

$$\lambda \equiv \left[\frac{2^{-2/\nu}\Gamma(1/\nu)}{\Gamma(3/\nu)}\right]^{1/2}$$

- **混合正态分布**

$$z_t \sim \text{mixed normal}(\lambda, \eta) = \begin{cases} N(0, \sigma^2) & \text{with probability}(1-\eta) \\ N(0, \lambda\sigma^2) & \text{with probability}\eta \end{cases}$$

where $\sigma^2 \equiv 1/(1-\eta+\lambda\eta)$ to make the variance of z_t equal to 1.

ARCH(1)参数估计

- ARCH(1) model:

$$\begin{cases} r_t = \mu + \epsilon_t \\ \epsilon_t = \sigma_t z_t, \quad z_t \sim \mathcal{N}(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2. \end{cases}$$

- Conditional density:

$$f(r_t|r_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{(r_t-\mu)^2}{2\sigma_t^2}\right].$$

- Log likelihood function:

$$L(\theta) \equiv \ln f(r_T, \cdots, r_t|r_0) = \sum_{t=1}^{T} \ln f(r_t|r_{t-1})$$

$$= -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T} \ln \sigma_t - \frac{1}{2}\sum_{t=1}^{T} \frac{(r_t-\mu)^2}{\sigma_t^2}$$

GARCH(1, 1)参数估计

- GARCH(1,1) model:

$$\begin{cases} r_t = \mu + \epsilon_t \\ \epsilon_t = \sigma_t z_t, \quad z_t \sim \mathcal{N}(0,1) \\ \sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2, \end{cases}$$

- Log likelihood function:

$$L(\boldsymbol{\theta}) \equiv -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T} \ln \sigma_t - \frac{1}{2}\sum_{t=1}^{T} \frac{(r_t - \mu)^2}{\sigma_t^2},$$

where $\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2$.

- Initial value σ_0^2: Bollerslev proposes using sample variance:

$$\hat{\sigma}^2 \equiv \frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2$$

EARCH(1, 1)参数估计

- EGARCH(1,1) model:

$$\begin{cases} r_t = \mu + \epsilon_t \\ \epsilon_t = \sigma_t z_t, \quad z_t \sim \mathcal{N}(0,1) \\ \log \sigma_t^2 = \omega + \beta_1 \log \sigma_{t-1}^2 + \alpha_1(|z_{t-1}| - E|z_{t-1}|) + \theta z_{t-1}. \end{cases}$$

- Log likelihood function:

$$L(\boldsymbol{\theta}) \equiv -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T} \ln \sigma_t - \frac{1}{2}\sum_{t=1}^{T} \frac{(r_t - \mu)^2}{\sigma_t^2},$$

where

$$\sigma_t^2 = \exp\left[\omega + \beta_1 \log \sigma_{t-1}^2 + \alpha_1(|z_{t-1}| - E|z_{t-1}|) + \theta z_{t-1}\right].$$

- Initial value σ_0^2:

$$\hat{\sigma}^2 \equiv \frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2$$

检验冲击 Z_t 的ARCH效应

$$
\begin{cases}
r_t = \mu + \epsilon_t \\
\epsilon_t = \sigma_t z_t, \quad z_t \sim \mathcal{N}(0,1) \\
\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \epsilon_{t-1}^2,
\end{cases}
$$

1. 利用估计参数计算ε_t
2. 设σ_0=0，计算σ_t
3. 计算冲击$Z_t = \varepsilon_t / \sigma_t$
4. 利用archtest检验Z_t