Bayesian Convolutional Neural Networks with Variational Inference

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Abstract

1. Introduction

We introduce Bayesian convolutional neural networks with variational inference, a variant of convolutional neural networks (CNNs), in which the intractable posterior probability distributions over weights are inferred by Bayes by Backprop. We demonstrate how this reliable variational inference method can serve as a fundamental construct for various network architectures. On multiple datasets in supervised learning settings (MNIST, CIFAR-10, CIFAR-100), our proposed variational inference method achieves performances equivalent to frequentist inference in identical architectures, while the two desiderata, a measure for uncertainty and regularization are incorporated naturally. We examine in detail how this measure for uncertainty, namely the predictive variance, can be decomposed into aleatoric and epistemic uncertainties. In the past, Bayes by Backprop has been successfully implemented in feedforward and recurrent neural networks, but not in convolutional ones. This work represents the extension of the group of Bayesian neural networks with variational inference which encompasses now all three aforementioned types of network architectures.

Convolutional neural networks (CNNs) excel at tasks in the realm of image classification (e.g. [10, 28, 18]). However, from a probability theory perspective, it is unjustifiable to use single point-estimates as weights to base any classification on. CNNs with frequentist inference require substantial amounts of data examples to train on and are prone to overfitting on datasets with few examples per class.

In this work, we apply Bayesian methods to CNNs in order to add a measure for uncertainty and regularization in their predictions, respectively their training. This approach allows the network to express uncertainty via its parameters in form of probability distributions (see Figure 1). At the same time, by using a prior probability distribution to integrate out the parameters, we compute the average across many models during training, which gives a regularization effect to the network, thus preventing overfitting.

We build our Bayesian CNN upon Bayes by Backprop [8, 1], and approximate the intractable true posterior probability distributions $p(w|\mathcal{D})$ with variational probability distributions $q_{\theta}(w|\mathcal{D})$, which comprise the properties of Gaussian distributions $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^d$, denoted $\mathcal{N}(\theta|\mu,\sigma^2)$, where d is the total number of parameters defining a probability distribution. The shape of these Gaussian variational posterior probability distributions, determined by their variance σ^2 , expresses an uncertainty estimation of every model parameter. The main contributions of our work are as fol-

lows:

- 1. We present how *Bayes by Backprop* can be efficiently applied to CNNs. We therefore introduce the idea of applying two convolutional operations, one for the mean and one for the variance.
- We empirically show that our proposed generic and reliable variational inference method for Bayesian CNNs can be applied to various CNN architectures without any limitations on their performances.
- 3. We examine how to estimate the aleatoric and epistemic uncertainties and add regularization in CNNs by means of applying Bayesian methods to their training. We compare the performances of these Bayesian CNNs to CNNs which use single point-estimates as weights, i.e. which are trained by frequentist inference.

This work builds on the foundations laid out by Blundell et al. [1], who introduced *Bayes by Backprop* for feedforward neural networks. Together with the extension to recurrent neural networks, introduced by Fortunato et al. [5], *Bayes by Backprop* is now applicable on the three most frequently used types of neural networks, i.e., feedforward, recurrent, and convolutional neural networks.

The paper is structured as subsequently outlined: after we have first introduced our work, we secondly review briefly related work in this field; third, we explain the *Bayes by Backprop* method on which our work is built; fourth, we explain our proposed Bayesian CNN with variational inference in details; fifth, we introduce aleatoric and epistemic uncertainties; sixth, we present our results and findings through experimental evaluation of the proposed method on various architectures and datasets before we finally conclude our work.

2. Related Work

Applying Bayesian methods to neural networks has been studied in the past with various approximation methods for the intractable true posterior probability distribution $p(w|\mathcal{D})$. Buntine and Weigend [2] started to propose various maximum-a-posteriori (MAP) schemes for neural networks. They were also the first who suggested second order derivatives in the prior probability distribution p(w) to encourage smoothness of the resulting approximate posterior probability distribution. In subsequent work by Hinton and Van Camp [11], the first variational methods were proposed which naturally served as a regularizer in neural networks. Hochreiter and Schmidhuber [12] suggest taking an information theory perspective into account and utilising a minimum description length (MDL) loss. This penalises nonrobust weights by means of an approximate penalty based upon perturbations of the weights on the outputs. Denker and LeCun [3], and MacKay [23] investigated the posterior probability distributions of neural networks by using Laplace approximations. As a response to the limitations of Laplace approximations, Neal [25] investigated the use of hybrid Monte Carlo for training neural networks, although it has so far been difficult to apply these to the large sizes of neural networks built in modern applications. More recently, Graves [8] derived a variational inference scheme for neural networks and Blundell et al. [1] extended this with an update for the variance that is unbiased and simpler to compute. Graves [9] derives a similar algorithm in the case of a mixture posterior probability distribution.

Several authors have claimed that Dropout [29] and Gaussian Dropout [30] can be viewed as approximate variational inference schemes [7, 16]. We compare our results to Gal's & Ghahramani's [7] and discuss the methodological differences in detail.

3. Bayes by Backprop

Bayes by Backprop [8, 1] is a variational inference method to learn the posterior distribution on the weights $w \sim q_{\theta}(w|\mathcal{D})$ of a neural network from which weights w can be sampled in backpropagation. Since the true posterior is typically intractable, an approximate distribution $q_{\theta}(w|\mathcal{D})$ is defined that is aimed to be as similar as possible to the true posterior $p(w|\mathcal{D})$, measured by the Kullback-Leibler (KL) divergence [19]. Hence, we define the optimal parameters θ^{opt} as

$$\theta^{opt} = \underset{\theta}{\arg \min} \text{ KL } [q_{\theta}(w|\mathcal{D}) || p(w|\mathcal{D})]$$

$$= \underset{\theta}{\arg \min} \text{ KL } [q_{\theta}(w|\mathcal{D}) || p(w)]$$

$$- \mathbb{E}_{q(w|\theta)} [\log p(\mathcal{D}|w)] + \log p(\mathcal{D})$$
(1)

where

$$KL\left[q_{\theta}(w|\mathcal{D})\|p(w)\right] = \int q_{\theta}(w|\mathcal{D})\log\frac{q_{\theta}(w|\mathcal{D})}{p(w)}dw. \quad (2)$$

This derivation forms an optimisation problem with a resulting cost function widely known as *variational free energy* [26, 31, 6] which is built upon two terms: the former, KL $[q_{\theta}(w|\mathcal{D})||p(w)]$, is dependent on the definition of the prior p(w), thus called complexity cost, whereas the latter, $\mathbb{E}_{q(w|\theta)}[\log p(\mathcal{D}|w)]$, is dependent on the data $p(\mathcal{D}|w)$, thus called likelihood cost. The term $\log p(\mathcal{D})$ can be omitted in the optimisation because it is constant.

Since the KL-divergence is also intractable to compute exactly, we follow a stochastic variational method [8, 1]. We sample the weights w from the variational distribution $q_{\theta}(w|\mathcal{D})$ since it is much more probable to draw samples which are appropriate for numerical methods from the variational posterior $q_{\theta}(w|\mathcal{D})$ than from the true posterior

 $p(w|\mathcal{D})$. Consequently, we arrive at the tractable cost function (3) which is aimed to be optimized, i.e. minimised w.r.t. θ , during training:

$$\mathcal{F}(\mathcal{D}, \theta) \approx \sum_{i=1}^{n} \log q_{\theta}(w^{(i)}|\mathcal{D}) - \log p(w^{(i)}) - \log p(\mathcal{D}|w^{(i)})$$
(3)

where n is the number of draws.

We sample $w^{(i)}$ from $q_{\theta}(w|\mathcal{D})$. The uncertainty afforded by *Bayes by Backprop* trained neural networks has been used successfully for training feedforward neural networks in both supervised and reinforcement learning environments [1, 22, 13], for training recurrent neural networks [5], but has not been applied to convolutional neural networks to-date.

4. Bayesian convolutional neural networks with variational inference

In this section, we explain our algorithm of building a CNN with probability distributions over its weights in each filter, as seen in Figure 1, and apply variational inference, i.e. *Bayes by Backprop*, to compute the intractable true posterior probability distribution, as described in the previous section. Notably, a fully Bayesian perspective on a CNN is for most CNN architectures not accomplished by merely placing probability distributions over weights in convolutional layers; it also requires probability distributions over weights in fully-connected layers (see Figure 2).

4.1. Local reparameterization trick for convolutional layers

We utilise the local reparameterization trick [16] and apply it to CNNs. Following [16, 27], we do not sample the weights w, but we sample instead layer activations b due to its consequent computational acceleration. The variational posterior probability distribution $q_{\theta}(w_{ijhw}|\mathcal{D}) = \mathcal{N}(\mu_{ijhw}, \alpha_{ijhw}\mu_{ijhw}^2)$ (where i and j are the input, respectively output layers, h and w the height, respectively width of any given filter) allows to implement the local reparamerization trick in convolutional layers. This results in the subsequent equation for convolutional layer activations b:

$$b_j = A_i * \mu_i + \epsilon_j \odot \sqrt{A_i^2 * (\alpha_i \odot \mu_i^2)}$$
 (4)

where $\epsilon_j \sim \mathcal{N}(0,1)$, A_i is the receptive field, * signalises the convolutional operation, and \odot the component-wise multiplication.

4.2. Applying two sequential convolutional operations (mean and variance)

The crux of equipping a CNN with probability distributions over weights instead of single point-estimates and be-

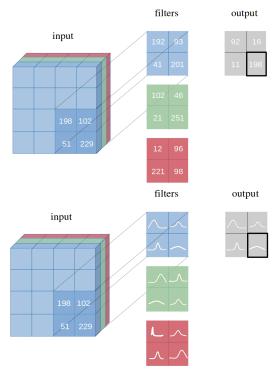


Figure 1. Input image with exemplary pixel values, filters, and corresponding output with point-estimates (top) and probability distributions (bottom) over weights.

ing able to update the variational posterior probability distribution $q_{\theta}(w|\mathcal{D})$ by backpropagation lies in applying *two* convolutional operations whereas filters with single pointestimates apply *one*. As explained in the previous section, we deploy the local reparametrization trick and sample from the output b. Since the output b is a function of mean μ_{ijwh} and variance $\alpha_{ijhw}\mu_{ijhw}^2$ among others, we are then able to compute the two variables determining a Gaussian

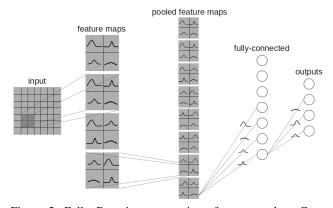


Figure 2. Fully Bayesian perspective of an exemplary CNN. Weights in filters of convolutional layers, and weights in fully-connected layers have the form of a probability distribution.

probability distribution, namely mean μ_{ijhw} and variance $\alpha_{ijhw}\mu_{ijhw}^2$, separately.

We do this in two convolutional operations: in the first, we treat the output b as an output of a CNN updated by frequentist inference. We optimize with Adam [15] towards a single point-estimate which makes the validation accuracy of classifications increasing. We interpret this single point-estimate as the mean μ_{ijwh} of the variational posterior probability distributions $q_{\theta}(w|\mathcal{D})$. In the second convolutional operation, we learn the variance $\alpha_{ijhw}\mu_{ijhw}^2$. As this formulation of the variance includes the mean μ_{ijwh} , only α_{ijhw} needs to be learned in the second convolutional operation [24]. In this way, we ensure that only one parameter is updated per convolutional operation, exactly how it would have been with a CNN updated by frequentist inference.

In other words, while we learn in the first convolutional operation the MAP of the variational posterior probability distribution $q_{\theta}(w|\mathcal{D})$, we observe in the second convolutional operation how much values for weights w deviate from this MAP. This procedure is repeated in the fully-connected layers. In addition, to accelerate computation, to ensure a positive non-zero variance $\alpha_{ijhw}\mu_{ijhw}^2$, and to enhance accuracy, we learn $\log \alpha_{ijhw}$ and use the *Softplus* activation function as further described in the Experiments section.

5. Uncertainty estimation in CNNs

In classification tasks, we are interested in the predictive distribution $p_{\mathcal{D}}(y^*|x^*)$, where x^* is an unseen data example and y^* its predicted class. For a Bayesian neural network, this quantity is given by:

$$p_{\mathcal{D}}(y^*|x^*) = \int p_w(y^*|x^*) p_{\mathcal{D}}(w) dw$$
 (5)

In Bayes by Backprop, Gaussian distributions $q_{\theta}(w|\mathcal{D}) \sim \mathcal{N}(w|\mu,\sigma^2)$, where $\theta=\{\mu,\sigma\}$ are learned with some dataset $\mathcal{D}=\{x_i,y_i\}_{i=1}^n$ as we explained previously. Due to the discrete and finite nature of most classification tasks, the predictive distribution is commonly assumed to be a categorical. Incorporating this aspect into the predictive distribution gives us

$$p_{\mathcal{D}}(y^*|x^*) = \int \operatorname{Cat}(y^*|f_w(x^*)) \mathcal{N}(w|\mu, \sigma^2) \, dw \qquad (6)$$

$$= \int \prod_{c=1}^{C} f(x_c^*|w)^{y_c^*} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(w-\mu)^2}{2\sigma^2}} dw \quad (7)$$

where C is the total number of classes and $\sum_{c} f(\boldsymbol{x}_{c}^{*}|\boldsymbol{w}) = 1.$

As there is no closed-form solution due to the lack of conjugacy between categorical and Gaussian distributions, we cannot recover this distribution. However, we can construct

an unbiased estimator of the expectation by sampling from $q_{\theta}(w|\mathcal{D})$:

$$\mathbb{E}_q[p_{\mathcal{D}}(y^*|x^*)] = \int q_{\theta}(w|\mathcal{D}) \ p_w(y|x) \ dw \qquad (8)$$

$$\approx \frac{1}{T} \sum_{t=1}^{T} p_{w_t}(y^*|x^*)$$
 (9)

where T is the pre-defined number of samples. This estimator allows us to evaluate the uncertainty of our predictions by the definition of variance, hence called *predictive variance* and denoted as Var_q :

$$\operatorname{Var}_q(p(y^*|x^*)) = \mathbb{E}_q[yy^T] - \mathbb{E}_q[y]\mathbb{E}_q[y]^T$$
 (10)

This quantity can be decomposed into the aleatoric and epistemic uncertainty [14, 20].

$$\operatorname{Var}_{q}(p(y^{*}|x^{*})) = \underbrace{\frac{1}{T} \sum_{t=1}^{T} \operatorname{diag}(\hat{p}_{t}) - \hat{p}_{t} \ \hat{p}_{t}^{T}}_{\text{aleatoric}} + \underbrace{\frac{1}{T} \sum_{t=1}^{T} (\hat{p}_{t} - \bar{p})(\hat{p}_{t} - \bar{p})^{T}}_{\text{enistemic}}$$

$$(11)$$

where $\bar{p} = \frac{1}{T} \sum_{t=1}^{T} \hat{p}_t$ and $\hat{p}_t = \text{Softmax}(f_{w_t}(x^*))$. It is of paramount importance that uncertainty is sp

It is of paramount importance that uncertainty is split into aleatoric and epistemic quantities since it allows the modeler to evaluate the room for improvements: while aleatoric uncertainty (also known as statistical uncertainty) is merely a measure for the variation of ("noisy") data, epistemic uncertainty is caused by the model. Hence, a modeler can see whether the quality of the data is low (i.e. high aleatoric uncertainty), or the model itself is the cause for poor performances (i.e. high epistemic uncertainty). The former can be improved by gathering more data, whereas the latter requests to refine the model [4].

6. Experiments

For all conducted experiments, we implement the foregoing description of Bayesian CNNs with variational inference in LeNet-5 [21] and AlexNet [18]. The exact architecture specifications can be found in the Appendix and in our GitHub repository*. We train the networks with the MNIST dataset of handwritten digits [21], and with the CIFAR-10 and CIFAR-100 datasets [17] since these datasets serve widely as benchmarks for CNNs' performances. The originally chosen activation functions in all architectures are *ReLU*, but we must introduce another, called *Softplus*, see (12), because of our method to apply two convolutional

^{*}https://github.com/felix-laumann/Bayesian_CNN

or fully-connected operations. As aforementioned, one of these is determining the mean μ , and the other the variance $\alpha\mu^2$. Specifically, we apply the *Softplus* function because we want to ensure that the variance $\alpha\mu^2$ never becomes zero. This would be equivalent to merely calculating the MAP, which can be interpreted as equivalent to a maximum likelihood estimation (MLE), which is further equivalent to utilising single point-estimates, hence frequentist inference. The *Softplus* activation function is a smooth approximation of *ReLU*. Although it is practically not influential, it has the subtle and analytically important advantage that it never becomes zero for $x \to -\infty$, whereas *ReLU* becomes zero for $x \to -\infty$.

Softplus
$$(x) = \frac{1}{\beta} \cdot \log (1 + \exp(\beta \cdot x))$$
 (12)

where β is by default set to 1.

All experiments are performed with the same hyperparameters settings as stated in the Appendix.

6.1. Datasets

As aforementioned, we train various architectures on multiple datasets, namely MNIST, CIFAR-10, and CIFAR-100.

Classification on MNIST. The MNIST dataset of handwritten digits consists of 60,000 training and 10,000 validation images of 28 by 28 pixels. Each image is labelled with its corresponding number (between zero and nine, inclusive).

Classification on CIFAR-10. The CIFAR-10 dataset consists of 60,000 and the consists o

sists of 60,000 colour images in 10 classes, with 6,000 images per class, each image 32 by 32 pixels large. Each of the classes has 5,000 training images and 1,000 validation images.

Classification on CIFAR-100. This dataset is similar to the CIFAR-10, except it has 100 classes containing 600 images each. There are 500 training images and 100 validation images per class. The resolution of the images is as in CIFAR-10 32 by 32 pixels.

6.2. Results

First, we evaluate the performance of our proposed method, Bayesian CNNs with variational inference. Table 1 shows a comparison of validation accuracies (in percentage) for architectures trained by two disparate Bayesian approaches, namely variational inference, i.e. *Bayes by Backprop* and Dropout as proposed by Gal and Ghahramani [7], plus frequentist inference for all three datasets. Bayesian CNNs trained by variational inference achieve validation accuracies comparable to their counter-architectures trained by frequentist inference. On MNIST, validation accuracies of the two disparate Bayesian approaches are comparable, but a Bayesian LeNet-5 with Dropout achieves a considerable higher validation accuracy on CIFAR-10, although we

were not able to reproduce these reported results.

In Figure 3, we show how Bayesian networks incorporate naturally effects of regularization, exemplified on AlexNet. While an AlexNet trained by frequentist inference without any regularization overfits greatly on CIFAR-100, an AlexNet trained by Bayesian inference on CIFAR-100 does not. It performs equivalently to an AlexNet trained by frequentist inference with three layers of Dropout after the first, fourth, and sixth layers in the architecture. In initial epochs, Bayesian CNNs trained by variational inference start with a low validation accuracy compared to architectures trained by frequentist inference. This must deduce from the initialization of the variational posterior probability distributions $q_{\theta}(w|\mathcal{D})$ as uniform distributions, while initial point-estimates in architectures trained by frequentist inference are randomly drawn from a standard Gaussian distribution. The latter initialization method ensures the initialized weights are neither too small nor too large. In other words, the motivation of the latter initialization is to start with weights such that the activation functions do not let them begin in saturated or dead regions. This is not true in case of uniform distributions and hence, Bayesian CNNs' starting validation accuracies can be comparably low.

Figure 4 displays the convergence of the standard deviation σ of the variational posterior probability distribution $q_{\theta}(w|\mathcal{D})$ of a random model parameter over epochs. As aforementioned, all prior probability distributions p(w) are initialized as uniform distributions. The variational posterior probability distributions $q_{\theta}(w|\mathcal{D})$ are approximated as Gaussian distributions which become more confident as more data is processed - observable by the decreasing standard deviation over epochs in Figure 4. Although the validation accuracy for MNIST on Bayesian LeNet-5 has already reached 99%, we can still see a fairly steep decrease in the parameter's standard deviation. In Figure 5, we plot the actual Gaussian variational posterior probability distributions $q_{\theta}(w|\mathcal{D})$ of a random parameter of LeNet-5 trained on CIFAR-10 at some epochs.

Finally, Table 2 compares the means of aleatoric and epis-

	MNIST	CIFAR-10	CIFAR-100
Bayesian AlexNet (with VI)	99	73	36
Frequentist AlexNet	99	73	38
Bayesian LeNet-5 (with VI)	98	69	31
Frequentist LeNet-5	98	68	33
Bayesian LeNet-5 (with Dropout)	99.5	83	

Table 1. Comparison of validation accuracies (in percentage) for different architectures with variational inference (VI), frequentist inference and Dropout as a Bayesian approximation as proposed by Gal and Ghahramani [7] for MNIST, CIFAR-10, and CIFAR-100

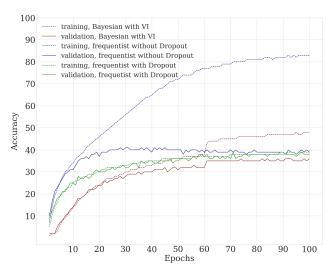


Figure 3. AlexNet trained on CIFAR-100 by Bayesian and frequentist inference. The frequentist AlexNet without Dropout overfits while the Bayesian AlexNet naturally incorporates an effect of regularization, comparable to a frequentist AlexNet with three Dropout layers.

temic uncertainties for a Bayesian LeNet-5 with variational inference on MNIST and CIFAR-10. The aleatoric uncertainty of CIFAR-10 is about twenty times as large as that of MNIST. Considering that the aleatoric uncertainty measures the irreducible variability and depends on the predicted values, a larger aleatoric uncertainty for CIFAR-10 can be directly deduced from its lower validation accuracy and may be further due to the smaller number of training examples. The epistemic uncertainty of CIFAR-10 is about

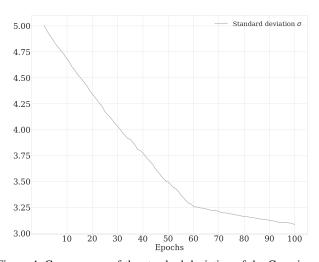


Figure 4. Convergence of the standard deviation of the Gaussian variational posterior probability distribution $q_{\theta}(w|\mathcal{D})$ of a random model parameter at epochs 1, 5, 20, 50, and 100. MNIST is trained on Bayesian LeNet-5.

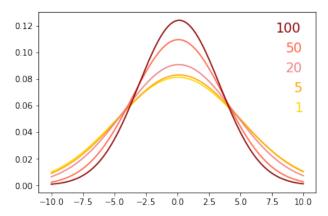


Figure 5. Convergence of the Gaussian variational posterior probability distribution $q_{\theta}(w|\mathcal{D})$ of a random model parameter at epochs 1, 5, 20, 50, and 100. CIFAR-10 is trained on Bayesian LeNet-5.

	Aleatoric uncertainty	Epistemic uncertainty
Bayesian LeNet-5 (MNIST)	0.0096	0.0026
Bayesian LeNet-5 (CIFAR-10)	0.1920	0.0404

Table 2. Aleatoric and epistemic uncertainty for Bayesian LeNet-5 calculated for MNIST and CIFAR-10, computed as proposed by Kwon et al. [20].

fifteen times larger than that of MNIST, which we anticipated, since epistemic uncertainty decreases proportional to validation accuracy.

7. Conclusion

We propose Bayesian CNNs utilizing *Bayes by Back-prop* as a reliable, variational inference method for CNNs which has not been studied to-date, and estimate the models' aleatoric and epistemic uncertainties.

There has been previous work by Gal and Ghahramani [7] who utilized the various outputs of a Dropout function to define a distribution, and concluded that one can then speak of a Bayesian CNN. This approach finds, perhaps also due its ease, a large confirming audience. However, we argue against this approach, and claim deficiencies. Specifically, in Gal's and Ghahramani's [7] approach, no prior probability distributions p(w) are placed on the CNN's parameters. But, these are a substantial part of a Bayesian interpretation for the simple reason that Bayes' theorem includes them. Thus we argue, starting with prior probability distributions p(w) is essential in Bayesian methods. In comparison, we place prior probability distributions over all model parameters, and update them according to Bayes' theorem with variational inference, precisely Bayes by Backprop. We show that these neural networks achieve state-of-the-art results as those achieved by the same network architectures trained by frequentist inference. Furthermore, we examine how aleatoric and epistemic uncertainties can be computed

for our proposed method and show the natural regularization effect of Bayesian methods.

As an add-on method to further enhance the stability of the optimization, posterior sharpening [5] could be applied to Bayesian CNNs in future work. There, the variational posterior distribution $q_{\theta}(w|\mathcal{D})$ is conditioned on the training data of a batch $\mathcal{D}^{(i)}$. We can see $q_{\theta}(w|\mathcal{D}^{(i)})$ as a proposal distribution, or hyper-prior when we rethink it as a hierarchical model, to improve the gradient estimates of the intractable likelihood function $p(\mathcal{D}|w)$.

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8. Appendix

8.1. Experiment specifications

variable	value
learning rate	0.001
epochs	100
batch size	128
sample size	10
$(lpha\mu^2)_{init}$ of approximate posterior $q_{oldsymbol{ heta}}(w \mathcal{D})$	-10
optimizer	Adam [15]
λ in ℓ -2 normalisation	0.0005
eta_i	$\frac{2^{M-i}}{2^{M-1}} [1]$

8.2. Model architectures

8.3. LeNet-5

layer type	width	stride	padding	input shape	nonlinearity
convolution (5 × 5)	6	1	0	$M\times1\times32\times32$	Softplus
Mmax-pooling (2 \times 2)		2	0	$M\times 6\times 28\times 28$	
convolution (5 \times 5)	16	1	0	$M\times1\times14\times14$	Softplus
max-pooling (2 \times 2)		2	0	$M\times16\times10\times10$	
fully-connected	120			$M \times 400$	Softplus
fully-connected	84			$M \times 120$	Softplus
fully-connected	10			$M \times 84$	

8.4. AlexNet

layer type	width	stride	padding	input shape	nonlinearity
convolution (11 × 11)	64	4	5	$M\times3\times32\times32$	Softplus
max-pooling (2 $ imes$ 2)		2	0	$M\times64\times32\times32$	
convolution (5 \times 5)	192	1	2	$M\times64\times15\times15$	Softplus
max-pooling (2 $ imes$ 2)		2	0	$M\times192\times15\times15$	
convolution (3×3)	384	1	1	$M\times192\times7\times7$	Softplus
convolution (3×3)	256	1	1	$M\times384\times7\times7$	Softplus
convolution (3×3)	128	1	1	$M\times256\times7\times7$	Softplus
max-pooling (2 \times 2)		2	0	$M\times128\times7\times7$	
fully-connected	128			$M\times128$	