

Introduction to the Curry-Howard correspondence

The logic of types in functional programming languages

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Type constructions in functional programming

The common ground between OCaml, Haskell, Scala, Rust, and other languages

Five type constructions are common in FP languages:

- Tuple (“product”) type: $\text{Int} \times \text{String}$
- Function type: $\text{Int} \Rightarrow \text{String}$
- Disjunction (“sum”) type: $\text{Int} + \text{String}$
- Unit type (“empty tuple”): 1
- Type parameters: List^T

Up to differences in syntax, the FP languages share all these features

Type constructions: Scala syntax

- Tuple type: `(Int, String)`
 - ▶ Create: `val pair: (Int, String) = (123, "abc")`
 - ▶ Use: `val y: String = pair._2`
- Function type: `Int ⇒ String`
 - ▶ Create: `val f: (Int ⇒ String) = x ⇒ "Value is " + x.toString`
 - ▶ Use: `val y: String = f(123)`
- Disjunction type: `Either[Int, String]` defined in standard library
 - ▶ Create:
`val x: Either[Int, String] = Left(123)`
`val y: Either[Int, String] = Right("abc")`
 - ▶ Use: `val z: Boolean = x match {`
 `case Left(i) ⇒ i > 0`
 `case Right(_) ⇒ false`
 `}`
- Unit type: `Unit`
 - ▶ Create: `val x: Unit = ()`

Type constructions: OCaml syntax

- Tuple type: `int * string`
 - ▶ Create: `let pair: int * string = (123, "abc")`
 - ▶ Use: `let y: string = snd pair`
- Function type: `Int \Rightarrow String`
 - ▶ Create: `let f: int -> string =
 fun x -> Printf.sprintf "Value is %d" x`
 - ▶ Use: `let y: string = f 123`
- Disjunction type: `type e = Left of int | Right of string`
 - ▶ Create:
 `let x: e = Left 123`
 `let y: e = Right "abc"`
 - ▶ Use: `let z: bool = match x with
 Left i -> i > 0
 Right _ -> false`
- Unit type: `unit`
 - ▶ Create: `let x: unit = ()`

Type constructions: Haskell syntax

- Tuple type: `(Int, String)`
 - ▶ Create: `pair = (123, "abc")`
 - ▶ Use: `(_, y) = pair`
- Function type: `Int ⇒ String`
 - ▶ Create: `f = \x -> "Value is " ++ show x`
 - ▶ Use: `y = f 123`
- Disjunction type: `data E = Left Int | Right String`
 - ▶ Create:
`x = Left 123`
`y = Right "abc"`
 - ▶ Use: `z = case x of`
`Left i -> i > 0`
`Right _ -> false`
- Unit type: `Unit`
 - ▶ Create: `x = ()`

From types to propositions

The code `val x: T = ...` shows that *we can compute a value of type T* as part of our program expression

- Let's denote this *proposition* by $\mathcal{CH}(T)$ – “Code \mathcal{H} has a value of type T ”
- Correspondence between types and propositions, for a given program:

Type	Proposition	Short notation
T	$\mathcal{CH}(T)$	T
(A, B)	$\mathcal{CH}(A)$ and $\mathcal{CH}(B)$	$A \wedge B; A \times B$
<code>Either[A, B]</code>	$\mathcal{CH}(A)$ or $\mathcal{CH}(B)$	$A \vee B; A + B$
$A \Rightarrow B$	$\mathcal{CH}(A)$ implies $\mathcal{CH}(B)$	$A \Rightarrow B$
<code>Unit</code>	<i>True</i>	1

- Type parameter `[T]` in a function type means $\forall T$
- Example: `def dupl[A]: A \Rightarrow (A, A)`. The type of this function, $A \Rightarrow A \times A$, corresponds to the (valid) theorem $\forall A: A \Rightarrow A \wedge A$

Translating language constructions into the logic I

What are the derivation rules for the logic of types?

Code expressions create *logical relationships* between propositions $\mathcal{CH}(\dots)$

- Expressions (program code) are represented by **sequents**
 - ▶ $A, B \vdash C$ represents an *expression* of type C that uses $x: A$ and $y: B$
 - ★ Sequents only describe the *types* of expressions and their parts
 - ▶ In $A, B, \dots \vdash C$ the **premises** are A, B, \dots and the **goal** is C
- Some sequents are immediate, others follow from previous ones
 - ▶ Tuple type: $A \times B$
 - ★ Create: $A, B \vdash A \times B$
 - ★ Use: $A \times B \vdash A$ and also $A \times B \vdash B$
 - ▶ Function type: $A \Rightarrow B$
 - ★ Create: if we have $A \vdash B$ then we will have $\emptyset \vdash A \Rightarrow B$
 - ★ Use: $A \Rightarrow B, A \vdash B$
 - ▶ Disjunction type: $A + B$
 - ★ Create: $A \vdash A + B$ and also $B \vdash A + B$
 - ★ Use: $A + B, A \Rightarrow C, B \Rightarrow C \vdash C$
 - ▶ Unit type: 1
 - ★ Create: $\emptyset \vdash 1$

Translating language constructions into the logic II

Additional rules for the logic of types

In addition to constructions that use types, we have “trivial” constructions:

- a single, unmodified value of type A is a valid expression of type A
 - ▶ For any A we have the sequent $A \vdash A$
- if a value can be computed using some given data, it can also be computed if given *additional* data
 - ▶ If we have $A, \dots, C \vdash G$ then also $A, \dots, C, D \vdash G$ for any D
 - ▶ For brevity, we denote by Γ a sequence of arbitrary premises
- the order in which data is given does not matter, we can still compute all the same things given the same premises in different order
 - ▶ If we have $\Gamma, A, B \vdash G$ then we also have $\Gamma, B, A \vdash G$

Syntax conventions:

- the implication operation associates *to the right*
 - ▶ $A \Rightarrow B \Rightarrow C$ means $A \Rightarrow (B \Rightarrow C)$
- precedence order: implication, disjunction, conjunction
 - ▶ $A + B \times C \Rightarrow D$ means $(A + (B \times C)) \Rightarrow D$

Quantifiers: implicitly, all our type variables are universally quantified

- When we write $A \Rightarrow B \Rightarrow A$, we mean $\forall A : \forall B : A \Rightarrow B \Rightarrow A$

The logic of types I

Now we have all the axioms and the derivation rules of the logic of types.

- What theorems can we derive in this logic?
- Example: $A \Rightarrow B \Rightarrow A$
 - ▶ Start with an axiom $A \vdash A$; add an unused extra premise B : $A, B \vdash A$
 - ▶ Use the “create function” rule with B and A , get $A \vdash B \Rightarrow A$
 - ▶ Use the “create function” rule with A and $B \Rightarrow A$, get the final sequent $\emptyset \vdash A \Rightarrow (B \Rightarrow A)$ showing that $A \Rightarrow B \Rightarrow A$ is a **theorem** since it is derived from no premises
- What code does this describe?
 - ▶ The axiom $A \vdash A$ represents the expression x where x is of type A
 - ▶ The unused premise B corresponds to unused variable y of type B
 - ▶ The “create function” rule gives the function $y \Rightarrow x$
 - ▶ The second “create function” rule gives $x \Rightarrow (y \Rightarrow x)$
 - ▶ Scala code: `def f[A, B]: A \Rightarrow B \Rightarrow A = (x: A) \Rightarrow (y: B) \Rightarrow x`
- Any code expression's type can be translated into a sequent
- A proof of a theorem directly guides us in writing code for that type

Correspondence between programs and proofs

- By construction, any theorem can be implemented in code

Proposition	Code
$\forall A : A \Rightarrow A$	<code>def identity[A](x: A): A = x</code>
$\forall A : A \Rightarrow 1$	<code>def toUnit[A](x: A): Unit = ()</code>
$\forall A \forall B : A \Rightarrow A + B$	<code>def inLeft[A,B](x:A): Either[A,B] = Left(x)</code>
$\forall A \forall B : A \times B \Rightarrow A$	<code>def first[A,B](p: (A,B)): A = p._1</code>
$\forall A \forall B : A \Rightarrow (B \Rightarrow A)$	<code>def const[A,B](x: A): B \Rightarrow A = (y:B) \Rightarrow x</code>

- Also, non-theorems *cannot be implemented* in code
 - ▶ Examples of non-theorems:
 $\forall A : 1 \Rightarrow A$; $\forall A \forall B : A + B \Rightarrow A$;
 $\forall A \forall B : A \Rightarrow A \times B$; $\forall A \forall B : (A \Rightarrow B) \Rightarrow A$
- Given a type's formula, can we implement it in code? Not obvious.
 - ▶ Example: $\forall A \forall B : (((A \Rightarrow B) \Rightarrow A) \Rightarrow A) \Rightarrow B \Rightarrow B$
 - ★ Can we write a function with this type?

The logic of types II

What kind of logic is this?

This is called “intuitionistic propositional logic”, IPL (also “constructive”)

- Disjunction works very differently from classical (Boolean) logic
 - ▶ Example: $A \Rightarrow B + C \vdash (A \Rightarrow B) + (A \Rightarrow C)$ does not hold in IPL
 - ▶ This is counter-intuitive!
 - ▶ We cannot implement a function with this type:

```
def q[A,B,C](f: A  $\Rightarrow$  Either[B, C]): Either[A  $\Rightarrow$  B, A  $\Rightarrow$  C]
```

- ▶ Disjunction is “constructive”: need to supply one of the parts
 - ★ But `Either[A \Rightarrow B, A \Rightarrow C]` is not a function of `A`
- Implication works somewhat differently
 - ▶ Example: $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ holds in Boolean logic but not in IPL
 - ▶ Cannot compute an `x: A` because of insufficient data
- Conjunction works the same as in Boolean logic
 - ▶ Example:

$$A \Rightarrow B \times C \vdash (A \Rightarrow B) \times (A \Rightarrow C)$$

The logic of types III

How to determine whether a given IPL formula is a theorem?

- The IPL cannot have a truth table with a fixed number of truth values
 - ▶ This was shown by Gödel in 1932 (see [Wikipedia page](#))
- The IPL has a decision procedure (algorithm) that either finds a proof for a given IPL formula, or determines that there is no proof
- There may be several inequivalent proofs of an IPL theorem
- Each proof can be *automatically translated* into code
 - ▶ The [curryhoward](#) library implements an IPL prover as a Scala macro, and generates Scala code from types
 - ▶ The [djinn-ghc](#) compiler plugin and the [JustDolt plugin](#) implement an IPL prover in Haskell, and generate Haskell code from types
- All these IPL provers use the same basic algorithm called LJT
 - ▶ and all cite the same paper [\[Dyckhoff 1992\]](#)
 - ▶ because most other papers on this subject are incomprehensible to engineers or describe algorithms that are too complicated

Proof search I: looking for an algorithm

Why our initial presentation of IPL does not give a proof search algorithm

The FP type constructions give nine axioms and three derivation rules:

$$\bullet \Gamma, A, B \vdash A \times B$$

$$\bullet \Gamma, A \times B \vdash A$$

$$\bullet \Gamma, A \times B \vdash B$$

$$\bullet \Gamma, A \Rightarrow B, A \vdash B$$

$$\bullet \Gamma, A \vdash A + B$$

$$\bullet \Gamma, B \vdash A + B$$

$$\bullet \Gamma, A + B, A \Rightarrow C, B \Rightarrow C \vdash C$$

$$\bullet \Gamma \vdash 1$$

$$\bullet \Gamma, A \vdash A$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash G}{\Gamma, D \vdash G}$$

$$\frac{\Gamma, A, B \vdash G}{\Gamma, B, A \vdash G}$$

Can we use these rules to obtain a finite and complete search tree? No.

- Try proving $A, B + C \vdash A \times B + C$: cannot find matching rules
- Need a better formulation of the logic

Proof search II: Gentzen's calculus LJ (1935)

- A “complete and sound calculus” is a set of axioms and derivation rules that will yield all (and only!) theorems of the logic

$$\begin{array}{c}
 (X \text{ is atomic}) \frac{}{\Gamma, X \vdash X} Id \\
 \frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C} L \Rightarrow \\
 \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A + B \vdash C} L+ \\
 \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \times A_2 \vdash C} L \times_i \\
 \frac{}{\Gamma \vdash \top} \top \\
 \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} R \Rightarrow \\
 \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} R+_i \\
 \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B} R \times
 \end{array}$$

- Two axioms and eight derivation rules
- Each rule says: The sequent at bottom will be proved if proofs are given for sequent(s) at top
- Use these rules “bottom-up” to perform a proof search
 - Sequents are nodes and proofs are edges in the proof search tree

Proof search example I

Example: to prove $\emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q$

- Root sequent $S_0 : \emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q$
- S_0 with rule $R \Rightarrow$ yields $S_1 : (R \Rightarrow R) \Rightarrow Q \vdash Q$
- S_1 with rule $L \Rightarrow$ yields $S_2 : (R \Rightarrow R) \Rightarrow Q \vdash R \Rightarrow R$ and $S_3 : Q \vdash Q$
- Sequent S_3 follows from the *Id* axiom; it remains to prove S_2
- S_2 with rule $L \Rightarrow$ yields $S_4 : (R \Rightarrow R) \Rightarrow Q \vdash R \Rightarrow R$ and $S_5 : Q \vdash R \Rightarrow R$
 - ▶ We are stuck here because $S_4 = S_2$ (we are in a loop)
 - ▶ We can prove S_5 , but that will not help
 - ▶ So we backtrack (erase S_4, S_5) and apply another rule to S_2
- S_2 with rule $R \Rightarrow$ yields $S_6 : (R \Rightarrow R) \Rightarrow Q; R \vdash R$
- Sequent S_6 follows from the *Id* axiom

Therefore we have proved S_0 .

Q.E.D.

Proof search III: The calculus LJT

Vorobieff-Hudelmaier-Dyckhoff, 1950-1990

- The Gentzen calculus LS will loop if rule $L \Rightarrow$ is applied ≥ 2 times
- The calculus LJT keeps all rules of LJ except rule $L \Rightarrow$
- Replace rule $L \Rightarrow$ by pattern-matching on A in the premise $A \Rightarrow B$:

$$\begin{array}{c} (X \text{ is atomic}) \frac{\Gamma, X, B \vdash D}{\Gamma, X, X \Rightarrow B \vdash D} L \Rightarrow_1 \\ \frac{\Gamma, A \Rightarrow (B \Rightarrow C) \vdash D}{\Gamma, (A \times B) \Rightarrow C \vdash D} L \Rightarrow_2 \\ \frac{\Gamma, A \Rightarrow C, B \Rightarrow C \vdash D}{\Gamma, (A + B) \Rightarrow C \vdash D} L \Rightarrow_3 \\ \frac{\Gamma, B \Rightarrow C \vdash A \Rightarrow B \quad \Gamma, C \vdash D}{\Gamma, (A \Rightarrow B) \Rightarrow C \vdash D} L \Rightarrow_4 \end{array}$$

- When using LJT rules, the proof tree has no loops and terminates
 - ▶ See [this paper](#) for an explicit decreasing measure on the proof tree

Proof search IV: The calculus LJT

“It is obvious that it is obvious” – a mathematician after thinking for a half-hour

- Rule $L \Rightarrow_4$ is based on the key theorem:

$$((A \Rightarrow B) \Rightarrow C) \Rightarrow (A \Rightarrow B) \iff (B \Rightarrow C) \Rightarrow (A \Rightarrow B)$$

- The key theorem for rule $L \Rightarrow_4$ is attributed to Vorobieff (1958):

be extracted from Lemma 7 in [22]. One could also go further and make subproofs sensible.

LEMMA 2. $\vdash_{LJ} \Gamma, (C \supset D) \supset B \Rightarrow C \supset D$ iff $\vdash_{LJ} \Gamma, D \supset B \Rightarrow C \supset D$.
PROOF. Trivial [34].

THEOREM 1. *The systems LJ and LJT are equivalent.*

PROOF. As noted earlier, it is routine to show that any sequent provable

[R. Dyckhoff, *Contraction-Free Sequent Calculi for Intuitionistic Logic*, 1992]

- A stepping stone to this theorem:

$$((A \Rightarrow B) \Rightarrow C) \Rightarrow B \Rightarrow C$$

Proof (*obviously* trivial): $f^{(A \Rightarrow B) \Rightarrow C} \Rightarrow b^B \Rightarrow f(x^A \Rightarrow b)$

► *Details are left as exercise for the reader*

Proof search V: From deduction rules to code

- The new rules are equivalent to the old rules, therefore...
 - ▶ Proof of a sequent $A, B, C \vdash G \Leftrightarrow$ code/expression $t(a, b, c) : G$
 - ▶ Also can be seen as a function t from A, B, C to G
- Sequent in a proof follow from an axiom or from a transforming rule
 - ▶ The two axioms are fixed expressions, $x^A \Rightarrow x$ and 1
 - ▶ Each rule has a *proof transformer* function: $PT_{R \Rightarrow}$, PT_{L+} , etc.
- Examples of proof transformer functions:

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A + B \vdash C} L+$$

$$PT_{L+}(t_1^{A \Rightarrow C}, t_2^{B \Rightarrow C})(x^{A+B}) = x \text{ match } \begin{cases} a \Rightarrow & t_1(a) \\ b \Rightarrow & t_2(b) \end{cases}$$

$$\frac{\Gamma, A \Rightarrow (B \Rightarrow C) \vdash D}{\Gamma, (A \times B) \Rightarrow C \vdash D} L \Rightarrow_2$$

$$PT_{L \Rightarrow_2}(f^{(A \Rightarrow B \Rightarrow C) \Rightarrow D})(g^{A \times B \Rightarrow C}) = f(x^A \Rightarrow y^B \Rightarrow g(x, y))$$

- Verify that we can indeed produce PTs for every rule of LJ

Proof search VI: Example deduction

Once a proof tree is found, start from leaves and apply PTs

- Example: to prove S_0 , start from S_6 backwards:

$$S_6 : (R \Rightarrow R) \Rightarrow Q; R \vdash R \quad (\text{axiom } Id) \quad t_6(rrq, r) : R = r$$

$$S_2 : (R \Rightarrow R) \Rightarrow Q \vdash (R \Rightarrow R) \quad PT_{R \Rightarrow}(t_6) \quad t_2(rrq) : (R \Rightarrow R) = (r \Rightarrow t_6(rrq, r))$$

$$S_3 : Q \vdash Q \quad (\text{axiom } Id) \quad t_3(q) : Q = q$$

$$S_1 : (R \Rightarrow R) \Rightarrow Q \vdash Q \quad PT_{L \Rightarrow}(t_2, t_3) \quad t_1(rrq) : Q = t_3(rrq(t_2(rrq)))$$

$$S_0 : \emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q \quad PT_{R \Rightarrow}(t_1) \quad t_0 = (rrq \Rightarrow t_1(rrq))$$

- The expression for the proof of S_0 is

$$\begin{aligned} t_0 &= rrq \Rightarrow t_3(rrq(t_2(rrq))) = rrq \Rightarrow rrq(r \Rightarrow t_6(rrq, r)) \\ &= rrq \Rightarrow rrq(r \Rightarrow r) \end{aligned}$$

Simplified final code (proof term) having the required type:

$$t_0 : ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q = (rrq \Rightarrow rrq(r \Rightarrow r))$$

Type isomorphisms I: identities

Using known properties of propositional logic and arithmetic

Are $A + B$, $A \times B$ more like logic ($A \vee B$, $A \wedge B$) or like arithmetic?

- Some standard identities in logic ($\forall A \forall B \forall C$ is assumed):

$$A \times 1 = A; \quad A \times B = B \times A$$

$$A \vee 1 = 1; \quad A \vee B = B \vee A$$

$$(A \times B) \times C = A \times (B \times C); \quad A \vee (B \times C) = (A \vee B) \times (A \vee C)$$

$$(A \vee B) \vee C = A \vee (B \vee C); \quad A \times (B \vee C) = (A \times B) \vee (A \times C)$$

$$(A \times B) \Rightarrow C = A \Rightarrow (B \Rightarrow C)$$

$$A \Rightarrow (B \times C) = (A \Rightarrow B) \times (A \Rightarrow C)$$

$$(A \vee B) \Rightarrow C = (A \Rightarrow C) \times (B \Rightarrow C)$$

- Each identity means 2 function types: $X = Y$ is $X \Rightarrow Y$ and $Y \Rightarrow X$
 - ▶ Do these functions convert values between the types X and Y ?

Type isomorphisms II

- Types A and B are isomorphic, $A \equiv B$, if there is a 1-to-1 correspondence between the sets of values of these types
 - ▶ Need to find two functions $f : A \Rightarrow B$ and $g : B \Rightarrow A$ such that $f \circ g = id$ and $g \circ f = id$

Example 1: Is $\forall A : A \times 1 \equiv A$? Types in Scala: `(A, Unit)` and `A`

- Two functions with types $\forall A : A \times 1 \Rightarrow A$ and $\forall A : A \Rightarrow A \times 1$:

```
def f1[A]: ((A, Unit)) => A = { case (a, ()) => a }  
def f2[A]: A => (A, Unit) = a => (a, ())
```

- Verify that their compositions equal `identity`

Example 2: Is $\forall A : 1 + A \equiv 1$? (The formula $\forall A : A \vee 1 = 1$ is a theorem!)

- Types in Scala: `Option[A]` and `Unit`
 - ▶ These types are obviously *not* equivalent

Some logic identities yield isomorphisms of types

- Which ones *do not* yield isomorphisms, and why?

Type isomorphisms III

Verifying type equivalence by implementing isomorphisms

- Need to verify that $f_1 \circ f_2 = id$ and $f_2 \circ f_1 = id$

Example 3: $\forall A \forall B \forall C : (A \times B) \times C \equiv A \times (B \times C)$

```
def f1[A,B,C]: (((A, B), C))  $\Rightarrow$  (A, (B, C)) = ???  
def f2[A,B,C]: ((A, (B, C)))  $\Rightarrow$  ((A, B), C) = ???
```

Example 4: $\forall A \forall B \forall C : (A + B) \times C \equiv A \times C + B \times C$

```
def f1[A,B,C]: ((Either[A,B], C))  $\Rightarrow$  Either[(A,C), (B,C)] = ???  
def f2[A,B,C]: Either[(A,C), (B,C)]  $\Rightarrow$  (Either[A, B], C) = ???
```

Example 5: $\forall A \forall B \forall C : (A + B) \Rightarrow C \equiv (A \Rightarrow C) \times (B \Rightarrow C)$

```
def f1[A,B,C]: (Either[A, B]  $\Rightarrow$  C)  $\Rightarrow$  (A  $\Rightarrow$  C, B  $\Rightarrow$  C) = ???  
def f2[A,B,C]: ((A  $\Rightarrow$  C, B  $\Rightarrow$  C))  $\Rightarrow$  Either[A, B]  $\Rightarrow$  C = ???
```

Example 6: $\forall A \forall B \forall C : A + B \times C \not\equiv (A + B) \times (A + C)$ – “information loss”

```
def f1[A,B,C]: Either[A, (B,C)]  $\Rightarrow$  (Either[A,B], Either[A,C]) = ???  
def f2[A,B,C]: ((Either[A,B], Either[A,C]))  $\Rightarrow$  Either[A, (B,C)] = ???
```

Type isomorphisms IV

Logical CH vs. arithmetical CH

- WLOG consider types A, B, \dots that have *finite* sets of possible values
 - ▶ Sum type $A + B$ (size $|A| + |B|$) provides a disjoint union of sets
 - ▶ Product type $A \times B$ (size $|A| \cdot |B|$) provides a Cartesian product of sets
 - ★ Have identities $(a + b) + c = a + (b + c)$, $(a \times b) \times c = a \times (b \times c)$,
 $1 \times a = a$, $(a + b) \times c = a \times c + b \times c$, ... – as in “school-level” algebra
 - ▶ Function type $A \Rightarrow B$ provides the set of all maps between sets
 - ★ The size of $A \Rightarrow B$ is $|B|^{|A|}$
 - ★ Have identities $a^c \times b^c = (a \times b)^c$, $a^{b+c} = a^b \times a^c$, $a^{b \times c} = (a^b)^c$
- If the set size (**cardinality**) differs, A and B cannot be equivalent

The meaning of the type/logic/arithmetic correspondence:

- Arithmetical identities signify type equivalence (isomorphism)
- Logic identities only signify *equal implementability* of types

Reasoning about types is *school-level algebra* with polynomials and powers

- **Exp-polynomial** expressions: constants, sums, products, exponentials
 - ▶ exp-poly types: primitive types, disjunctions, tuples, functions
 - ▶ polynomial types are commonly called “algebraic types”

Making practical use of the CH correspondence I

Implications for actually writing code

What can we do now?

- Given a fully parametric type, decide whether it can be implemented in code (“type is inhabited”); if so, *generate* the code
- Let the computer fill in the code when it is “trivial” to do so
 - ▶ This is often (not always) the case for fully type-parametric functions
- Decide type isomorphisms using the “arithmetical CH”
- Isomorphically transform types using school-level algebra

What problems cannot be solved with these tools?

- Automatically generate code satisfying properties (e.g. isomorphism)
- Express complicated conditions via types (e.g. “array is sorted”)
- Generate code using type constructors with properties (e.g. `map`)
 - ▶ Scala type signature: `(x: List[A]).map[B](f: A \Rightarrow B): List[B]`
 - ▶ This formula has a quantifier *inside*: $\text{List}^A \Rightarrow (\forall B : f^{A \Rightarrow B} \Rightarrow \text{List}^B)$
 - ▶ This requires **first-order logic**, which is generally *undecidable* (no algorithm can guarantee finding a proof or showing its absence)

Some caveats

- The CH correspondence becomes informative only with parameterized types. For concrete types, e.g. `Int`, we can always produce *some* value even with no previous data, so $\mathcal{CH}(\text{Int})$ is always true.
- Functions such as `(x: Int) \Rightarrow x + 1` have type `Int \Rightarrow Int`, and the type signature is insufficient to specify the code. Only for fully type-parametric functions the type signature can be, in some cases, informative enough for deriving the code automatically.
- Having an arithmetic identity does not *guarantee* that we have a type equivalence via CH (it is a necessary but not a sufficient condition); but it does yield a type equivalence in all cases I looked at so far.
- Scala's type `Nothing` and Haskell's type `Void` correspond to the logical constant *False*; but the practical uses of *False* are extremely limited.
- We did not talk about the logical negation because it is defined as $\neg A \equiv A \Rightarrow \text{False}$ and its practical use is as limited as that of *False*.

Making practical use of the CH correspondence II

Implications for designing new programming languages

- The CH correspondence maps the type system of each programming language into a certain system of logical propositions
- Scala, Haskell, OCaml, F#, Swift, Rust, etc. are mapped into the full constructive logic (all logical operations are available)
 - ▶ C, C++, Java, C#, etc. are mapped to *incomplete logics* – without “or” and without “true” / “false”
 - ▶ Python, JavaScript, Ruby, Clojure, etc. have only one type (“any value”) and are mapped to logics with only one proposition
- The CH correspondence is a principle for designing type systems:
 - ▶ Choose a complete logic, free of inconsistency
 - ★ Mathematicians have studied all kinds of logics and determined which ones are interesting, and found the minimal sets of axioms for them
 - ★ Modal logic, temporal logic, linear logic, etc.
 - ▶ Provide a type construction for each basic operation (e.g. “or”, “and”)
 - ★ There should be a type for every logical operation and vice versa