Chapter 7: Computations lifted to a functor context II. Monads

Part 2: Laws and structure of semimonads

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Semimonad laws I: The intuitions

What properties of functor block programs do we expect to have?

- In `x ← c`, the value of `x` will *go over items* held in container `c`
- Manipulating items in container is followed by a generator:

```
x ← cont1
y = f(x)
z ← cont2(y)
```

```
y ← cont1
      .map(x ⇒ f(x))
z ← cont2(y)
```

`cont1.flatMap(x ⇒ cont2(f(x))) = cont1.map(f).flatMap(y ⇒ cont2(y))`

- Manipulating items in container is preceded by a generator:

```
x ← cont1
y ← cont2(x)
z = f(y)
```

```
x ← cont1
z ← cont2(x)
      .map(f)
```

`cont1.flatMap(cont2).map(f) = cont1.flatMap(x ⇒ cont2(x).map(f))`

- After `x ← cont`, further computations will use *all those* `x`

```
x ← cont
y ← p(x)
z ← cont2(y)
```

```
y ← for { x ← cont
          yy ← p(x) } yield yy
z ← cont2(y)
```

`cont.flatMap(x ⇒ p(x).flatMap(cont2)) = cont.flatMap(p).flatMap(cont2)`

Semimonad laws II: The laws for `flatMap`

To get a more concise notation, use flm instead of `flatMap`

A **semimonad** S^A has $\text{flm}^{[S,A,B]} : (A \Rightarrow S^B) \Rightarrow S^A \Rightarrow S^B$ with 3 laws:

1. $\text{flm}\,(f^{A\Rightarrow B} \circ g^{B\Rightarrow S^C}) = \text{fmap}\,f \circ \text{flm}\,g$ (naturality in A)

$$\begin{tikzcd} & S^B \arrow[dr, "\text{flm}\,g^{B\Rightarrow S^C}"] & \\ S^A \arrow[ur, "\text{fmap}\,f^{A\Rightarrow B}"] \arrow[rr, "\text{flm}\,(f^{A\Rightarrow B}\circ g^{B\Rightarrow S^C})"'] & & S^C \end{tikzcd}$$

2. $\text{flm}\left(f^{A\Rightarrow S^B} \circ \text{fmap}\,g^{B\Rightarrow C}\right) = \text{flm}\,f \circ \text{fmap}\,g$ (naturality in B)

$$\begin{tikzcd} & S^B \arrow[dr, "\text{fmap}\,g^{B\Rightarrow C}"] & \\ S^A \arrow[ur, "\text{flm}\,f^{A\Rightarrow S^B}"] \arrow[rr, "\text{flm}\,(f^{A\Rightarrow S^B}\circ \text{fmap}\,g^{B\Rightarrow C})"'] & & S^C \end{tikzcd}$$

3. $\text{flm}\left(f^{A\Rightarrow S^B} \circ \text{flm}\,g^{B\Rightarrow S^C}\right) = \text{flm}\,f \circ \text{flm}\,g$ (associativity)

$$\begin{tikzcd} & S^B \arrow[dr, "\text{flm}\,g^{B\Rightarrow S^C}"] & \\ S^A \arrow[ur, "\text{flm}\,f^{A\Rightarrow S^B}"] \arrow[rr, "\text{flm}\left(f^{A\Rightarrow S^B}\circ \text{flm}\,g^{B\Rightarrow S^C}\right)"'] & & S^C \end{tikzcd}$$

Is there a shorter formulation of the laws?

Semimonad laws III: The laws for `flatten`

The methods `flatten` (denoted by ftn) and `flatMap` are equivalent:

$$\text{ftn}^{[S,A]} : S^{S^A} \Rightarrow S^A = \text{flm}^{[S,S^A,A]}(m^{S^A} \Rightarrow m)$$

$$\text{flm}\left(f^{A\Rightarrow S^B}\right) = \text{fmap}\, f \circ \text{ftn}$$

$$\begin{tikzcd}
 & S^{S^B} \arrow[dr, "\text{ftn}"] & \\
S^A \arrow[ur, "\text{fmap}\, f^{A\Rightarrow S^B}"] \arrow[rr, "\text{flm}\left(f^{A\Rightarrow S^B}\right)"'] & & S^B
\end{tikzcd}$$

It turns out that `flatten` has only 2 laws:

1. $\text{fmap}\left(\text{fmap}\, f^{A\Rightarrow B}\right) \circ \text{ftn}^{[S,B]} = \text{ftn}^{[S,A]} \circ \text{fmap}\, f$ (naturality)

$$\begin{tikzcd}
 & S^{S^B} \arrow[dr, "\text{ftn}^{[S,B]}"] & \\
S^{S^A} \arrow[ur, "\text{fmap}\left(\text{fmap}\, f^{A\Rightarrow B}\right)"] \arrow[dr, "\text{ftn}^{[S,A]}"'] & & S^B \\
 & S^A \arrow[ur, "\text{fmap}\, f^{A\Rightarrow B}"'] &
\end{tikzcd}$$

2. $\text{fmap}\,(\text{ftn}^{[S,A]}) \circ \text{ftn}^{[S,A]} = \text{ftn}^{[S,S^A]} \circ \text{ftn}^{[S,A]}$ (associativity)

$$\begin{tikzcd}
 & S^{S^A} \arrow[dr, "\text{ftn}^{[S,A]}"] & \\
S^{S^{S^A}} \arrow[ur, "\text{fmap}\,(\text{ftn}^{[S,A]})"] \arrow[dr, "\text{ftn}^{[S,S^A]}"'] & & S^A \\
 & S^{S^A} \arrow[ur, "\text{ftn}^{[S,A]}"'] &
\end{tikzcd}$$

Equivalence of a natural transformation and a "lifting"

- Equivalence of flm and ftn: $\text{ftn} = \text{flm}\,(\text{id})$; $\text{flm}\,f = \text{fmap}\,f \circ \text{ftn}$
- We saw this before: $\text{deflate} = \text{fmapOpt}\,(\text{id})$; $\text{fmapOpt}\,f = \text{fmap}\,f \circ \text{deflate}$
  - Is there a general pattern where two such functions are equivalent?
- Let $\text{tr} : F^{G^A} \Rightarrow F^A$ be a natural transformation (F and G are functors)
- Define $\text{ftr} : (A \Rightarrow G^B) \Rightarrow F^A \Rightarrow F^B$ by $\text{ftr}\,f = \text{fmap}\,f \circ \text{tr}$
- It follows that $\text{tr} = \text{ftr}\,(\text{id})$, and we have equivalence between tr and ftr:

$$\text{tr} : F^{G^A} \Rightarrow F^A = \text{ftr}(m^{G^A} \Rightarrow m)$$

$$\text{ftr}\left(f^{A\Rightarrow G^B}\right) = \text{fmap}\,f \circ \text{tr}$$

$$\begin{tikzcd} & F^{G^B} \arrow[dr, "\text{tr}"] & \\ F^A \arrow[ur, "\text{fmap}\,f^{A\Rightarrow G^B}"] \arrow[rr, "\text{ftr}\left(f^{A\Rightarrow G^B}\right)"'] & & F^B \end{tikzcd}$$

- An automatic law for ftr ("naturality in A") follows from the definition: $\text{fmap}\,g \circ \text{ftr}\,f = \text{fmap}\,g \circ \text{fmap}\,f \circ \text{tr} = \text{fmap}\,(g \circ f) \circ \text{tr} = \text{ftr}\,(g \circ f)$
  - This is why tr has *one law fewer* than ftr
- To demonstrate equivalence in the direction $\text{ftr} \rightarrow \text{tr}$: start with an arbitrary ftr satisfying "naturality in A", then obtain $\text{tr} = \text{ftr}\,(\text{id})$ from it, then verify $\text{ftr}\,f = \text{fmap}\,f \circ \text{tr}$ with that tr: $\text{fmap}\,f \circ \text{ftr}\,(\text{id}) = \text{ftr}\,(f \circ id) = \text{ftr}\,f$

Semimonad laws IV: Deriving the laws for `flatten`

Denote for brevity $q^{\uparrow} \equiv \text{fmap}^{[S]}\, q$ for any function q

Express $\text{flm}\, f = f^{\uparrow} \circ \text{ftn}$ and substitute that into flm's 3 laws:

1. $\text{flm}\,(f \circ g) = f^{\uparrow} \circ \text{flm}\, g$ gives $(f \circ g)^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ g^{\uparrow} \circ \text{ftn}$ – this law holds automatically due to functor composition law
2. $\text{flm}\left(f \circ g^{\uparrow}\right) = \text{flm}\, f \circ g^{\uparrow}$ gives $(f \circ h)^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ \text{ftn} \circ h$; using the functor composition law, we reduce this to $h^{\uparrow} \circ \text{ftn} = \text{ftn} \circ h$ – this is the naturality law
3. $\text{flm}\,(f \circ \text{flm}\, g) = \text{flm}\, f \circ \text{flm}\, g$ with functor composition law gives $f^{\uparrow} \circ g^{\uparrow\uparrow} \circ \text{ftn}^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ \text{ftn} \circ g^{\uparrow} \circ \text{ftn}$; using ftn's naturality and omitting the common factor $f^{\uparrow} \circ g^{\uparrow\uparrow}$, we get $\text{ftn}^{\uparrow} \circ \text{ftn} = \text{ftn} \circ \text{ftn}$ – associativity law

- `flatten` has the simplest type signature *and* the fewest laws
- It is usually easy to check naturality!
  - **Parametricity theorem**: Any *pure, fully parametric* code for a function of type $F^A \Rightarrow G^A$ will implement a natural transformation
- Checking `flatten`'s associativity needs *a lot* more work!

The `cats` library has a `FlatMap` type class, defining `flatten` via `flatMap`

Checking the associativity law for standard monads

- Implement `flatten` for these functors and check the laws (see code):
  - `Option` monad: $F^A \equiv 1 + A$; $\text{ftn} : 1 + (1 + A) \Rightarrow 1 + A$
  - `Either` monad: $F^A \equiv Z + A$; $\text{ftn} : Z + (Z + A) \Rightarrow Z + A$
  - `List` monad: $F^A \equiv \text{List}^A$; $\text{ftn} : \text{List}^{\text{List}^A} \Rightarrow \text{List}^A$
  - Writer monad: $F^A \equiv A \times W$; $\text{ftn} : (A \times W) \times W \Rightarrow A \times W$
  - Reader monad: $F^A \equiv R \Rightarrow A$; $\text{ftn} : (R \Rightarrow (R \Rightarrow A)) \Rightarrow R \Rightarrow A$
  - State: $F^A \equiv S \Rightarrow A \times S$; $\text{ftn} : (S \Rightarrow (S \Rightarrow A \times S) \times S) \Rightarrow S \Rightarrow A \times S$
  - Continuation monad: $F^A \equiv (A \Rightarrow R) \Rightarrow R$;
    $\text{ftn} : ((((A \Rightarrow R) \Rightarrow R) \Rightarrow R) \Rightarrow R) \Rightarrow (A \Rightarrow R) \Rightarrow R$
- Code implementing these `flatten` functions is *fully parametric* in A
  - Naturality of these functions follows from parametricity theorem
  - Associativity needs to be checked for each monad!
- Example of a useful semimonad that is *not* a full monad:
  - $F^A \equiv A \times V \times W$; $\text{ftn}\,((a \times v_1 \times w_1) \times v_2 \times w_2) = a \times v_1 \times w_2$
- Examples of *non-associative* (i.e. wrong) implementations of `flatten`:
  - $F^A \equiv A \times W \times W$; $\text{ftn}\,((a \times v_1 \times v_2) \times w_1 \times w_2) = a \times w_2 \times w_1$
  - $F^A \equiv \text{List}^A$, but `flatten` concatenates the nested lists in reverse order

Motivation for monads

- Monads represent values with a "special computational context"
- Specific monads will have methods to create various contexts
- Monadic composition will "combine" the contexts associatively
- It is generally useful to have an "empty context" available

$$\text{pure} : A \Rightarrow M^A$$

- Combining empty context with another context works as a no-op
- Empty context is followed by a generator:

```
y ← pure(x)
z ← cont(y)
```

```
y = x
z ← cont(y)
```

`pure(x).flatMap(y ⇒ cont(y)) = cont(x)`

$\text{pure} \circ \text{flm}\, f = f$ – left identity

- Empty context is preceded by a generator:

```
x ← cont
y ← pure(x)
```

```
x ← cont
y = x
```

`cont.flatMap(x ⇒ pure(x)) = cont`

$\text{flm}\,(\text{pure}) = \text{id}$ – right identity

The monad laws formulated in terms of `pure` and `flatten`

- Naturality law for `pure`: $f \circ \mathrm{pure} = \mathrm{pure} \circ f^{\uparrow}$

$$\begin{tikzcd}
 & B \arrow[dr, "\mathrm{pure}^{[S,B]}"] & \\
A \arrow[ur, "f^{A \Rightarrow B}"] \arrow[dr, "\mathrm{pure}^{[S,A]}"'] & & S^B \\
 & S^A \arrow[ur, "\mathrm{fmap}\, f^{A \Rightarrow B}"'] &
\end{tikzcd}$$

- Left identity: $\mathrm{pure} \circ \mathrm{flm}\, f = \mathrm{pure} \circ f^{\uparrow} \circ \mathrm{ftn} = f \circ \mathrm{pure} \circ \mathrm{ftn} = f$ requires that $\mathrm{pure} \circ \mathrm{ftn} = \mathrm{id}$ (both sides applied to S^A)

$$\begin{tikzcd}
 & S^{S^A} \arrow[dr, "\mathrm{ftn}"] & \\
S^A \arrow[ur, "\mathrm{pure}^{[S,S^A]}"] \arrow[rr, "\mathrm{id}"'] & & S^A
\end{tikzcd}$$

- Right identity: $\mathrm{flm}\,(\mathrm{pure}) = \mathrm{pure}^{\uparrow} \circ \mathrm{ftn} = \mathrm{id}$

$$\begin{tikzcd}
 & S^{S^A} \arrow[dr, "\mathrm{ftn}"] & \\
S^A \arrow[ur, "\mathrm{fmap}(\mathrm{pure}^{[S,A]})"] \arrow[rr, "\mathrm{id}"'] & & S^A
\end{tikzcd}$$

Formulating laws via Kleisli functions

- Recall: we formulated the laws of filterables via `fmapOpt`
  - $\text{fmapOpt} : (A \Rightarrow 1 + B) \Rightarrow S^A \Rightarrow S^B$
- And then we had to compose functions of types $A \Rightarrow 1 + B$ with $\diamond_{\text{Opt}}$
- Here we have $\text{flm} : \left(A \Rightarrow S^B\right) \Rightarrow S^A \Rightarrow S^B$ instead of `fmapOpt`
- Can we compose **Kleisli functions** with "twisted" types $A \Rightarrow S^B$?
- Use flm to define **Kleisli composition**: $f^{A \Rightarrow S^B} \diamond g^{B \Rightarrow S^C} \equiv f \circ \text{flm}\, g$
- Define **Kleisli identity** $\text{id}_\diamond$ of type $A \Rightarrow S^A$ as $\text{id}_\diamond \equiv \text{pure}$
- Composition law: $\text{flm}\,(f \diamond g) = \text{flm}\, f \circ \text{flm}\, g$ (same as for `fmapOpt`)
  - Shows that `flatMap` is a "lifting" of $A \Rightarrow S^B$ to $S^A \Rightarrow S^B$
- These laws are similar to functor "lifting" laws...
  - except that $\diamond$ is used for composing Kleisli functions
- What are the properties of $\diamond$?
  - Exactly similar to the properties of function composition $f \circ g$

Reformulate flm's laws in terms of the $\diamond$ operation:

- flm's left and right identity laws: $\text{pure} \diamond f = f$ and $f \diamond \text{pure} = f$
- Associativity law: $(f \diamond g) \diamond h = f \diamond (g \diamond h)$
  - Follows from the flm law: $f \circ \text{flm}\,(g \circ \text{flm} h) = f \circ \text{flm}\, g \circ \text{flm}\, h$

From Kleisli back to `flatMap`

Compare different "liftings" seen so far:

Category	Function type	Identity	Composition
plain functions	$A \Rightarrow B$	$\mathrm{id} : A \Rightarrow A$	$f^{A \Rightarrow B} \circ g^{B \Rightarrow C}$
lifted to F	$F^A \Rightarrow F^B$	$\mathrm{id} : F^A \Rightarrow F^A$	$f^{F^A \Rightarrow F^B} \circ g^{F^B \Rightarrow F^C}$
Kleisli over F	$A \Rightarrow F^B$	$\mathrm{pure} : A \Rightarrow F^A$	$f^{A \Rightarrow F^B} \diamond g^{B \Rightarrow F^C}$

Category axioms: identity and associativity for composition

General **functor**: a "lifting" maps functions from one category to another

- Functor laws: "lifting" must preserve identity and composition

Reformulate `map` and `flatMap` in terms of the $\diamond$ operation:

- Define `flatMap` through Kleisli composition: $\mathrm{flm}\, f^{A \Rightarrow S^B} \equiv \mathrm{id}^{S^A \Rightarrow S^A} \diamond f$
- Define `flatten` through Kleisli: $\mathrm{ftn} \equiv \mathrm{id}^{S^{S^A} \Rightarrow S^{S^A}} \diamond \mathrm{id}^{S^A \Rightarrow S^A}$
- Express `fmap` through Kleisli: $\mathrm{fmap}\, f \equiv (\mathrm{fmap}\, \mathrm{id}) \diamond (f \circ \mathrm{pure})$
- Need additional laws to connect $\diamond$ and $\circ$:
    - Left naturality: $f^{A \Rightarrow B} \circ g^{B \Rightarrow S^C} = (f \circ \mathrm{pure}) \diamond g$
    - Right naturality: $f^{A \Rightarrow B} \circ \mathrm{fmap}\, g^{B \Rightarrow S^C} = f \diamond (g \circ \mathrm{pure})$
        - ★ With these laws, monad laws follow from category axioms for Kleisli

Structure of semigroups and monoids

- Semimonad contexts are combined associatively, as in a semigroup
- A full monad includes an "empty" context, i.e. the identity element
- Semigroup with an identity element is a monoid

Some constructions of semigroups and monoids:

1. Any type Z is a semigroup with operation $z_1 \circledast z_2 = z_1$ (or z_2)
2. $1 + S$ is a monoid if S is (at least) a semigroup
3. List^A is a monoid (for any type A), also Seq^A etc.
4. The function type $A \Rightarrow A$ is a monoid (for any type A)
   - The operation $f \circledast g$ is either $f \circ g$ or $g \circ f$
5. Any totally ordered type is a monoid, with $\circledast$ defined as max or min
6. $S_1 \times S_2$ is a semigroup (monoid) if S_1, S_2 are semigroups (monoids)
7. $S \times P$ is a semigroup (monoid) if S is a semigroup (monoid) such that S acts on P. ("Twisted product.") Example: $(A \Rightarrow A) \times A$
   - The "action" is $a : S \Rightarrow P \Rightarrow P$ such that $a(s_1) \circ a(s_2) = a(s_1 \circledast s_2)$.
8. $Z \Rightarrow S$ is a semigroup/monoid, for any Z, if S is a semigroup/monoid

- There are other examples: Int, String, Set^A, Akka routes, ...
- Non-examples: trees; $S_1 + S_2$ where $S_{1,2}$ are different monoids

Structure of (semi)monads

How to recognize a (semi)monad by its type? Open question!

Intuition from `flatten`: reshuffle data in F^{F^A} to fit into F^A

Some constructions of exponential-polynomial semimonads:

1. $F^A \equiv Z$ (constant functor) for a fixed type Z
   - For a full monad, need to choose $Z = 1$
2. $F^A \equiv A \times G^A$ for any functor G^A (a full monad only if $G^A \equiv 1$)
3. $F^A \equiv Z + A \times W$ for a fixed type Z and a semigroup W
   - For a full monad, need W to be a monoid
4. $F^A \equiv G^{Z+A\times W}$ if $Z + A \times W$ is a (semi)monad
5. $F^A \equiv G^A \times H^A$ for any (semi)monads G^A and H^A
6. $F^A \equiv A + G^A$ for any semimonad G^A
7. $F^A \equiv A + G^{F^A}$ (recursive) for any functor G^A (**free monad** over G)
8. $F^A \equiv G^A + G^{F^A}$ (recursive) for any functor G^A (semimonad only!)
9. $F^A \equiv R \rightarrow G^A$ for any (semi)monad G^A
10. $F^A \equiv H^A \Rightarrow A \times G^A$ for any contrafunctor H^A and functor G^A
    - For a full monad, need to set $G^A \equiv 1$

* Worked examples II: Constructions of filterable functors I

(2) The `fmapOpt` laws hold for $F^A \times G^A$ if they hold for F^A and G^A

- For $f^{A \Rightarrow 1+B}$, get $\text{fmapOpt}_F(f) : F^A \Rightarrow F^B$ and $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\text{fmapOpt}_{F \times G} f \equiv p^{F^A} \times q^{G^A} \Rightarrow \text{fmapOpt}_F(f)(p) \times \text{fmapOpt}_G(f)(q)$
- Identity law: $f = \text{id}_{\diamond_{\text{Opt}}}$, so $\text{fmapOpt}_F f = \text{id}$ and $\text{fmapOpt}_G f = \text{id}$
  - Hence we get $\text{fmapOpt}_{F+G}(f)(p \times q) = \text{id}(p) \times \text{id}(q) = p \times q$
- Composition law:

$$
\begin{aligned}
& (\text{fmapOpt}_{F \times G}\, f_1 \circ \text{fmapOpt}_{F+G}\, f_2)(p \times q) \\
&= \text{fmapOpt}_{F \times G}(f_2)\,(\text{fmapOpt}_F(f_1)(p) \times \text{fmapOpt}_G(f_1)(q)) \\
&= (\text{fmapOpt}_F\, f_1 \circ \text{fmapOpt}_F\, f_2)(p) \times (\text{fmapOpt}_G\, f_1 \circ \text{fmapOpt}_G\, f_2)\,(q) \\
&= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(p) \times \text{fmapOpt}_G(f_1 \diamond f_2)(q) \\
&= \text{fmapOpt}_{F \times G}(f_1 \diamond_{\text{Opt}} f_2)(p \times q)
\end{aligned}
$$

- Exactly the same proof as that for functor property for $F^A \times G^A$
  - this is because `fmapOpt` corresponds to a generalized functor
- New proofs are necessary only when using non-filterable functors
  - these are used in constructions 4 – 6

* Worked examples II: Constructions of filterable functors II

(5) The `fmapOpt` laws hold for $F^A \equiv 1 + A \times G^A$ if they hold for G^A

- For $f^{A \Rightarrow 1+B}$, get $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\text{fmapOpt}_F(f)(1 + a^A \times q^{G^A})$ by returning $0 + b \times \text{fmapOpt}_G(f)(q)$ if the argument is $0 + a \times q$ and $f(a) = 0 + b$, and returning $1 + 0$ otherwise
- Identity law: $f = \text{id}_{\diamond_{\text{Opt}}}$, so $f(a) = 0 + a$ and $\text{fmapOpt}_G f = \text{id}$
  - Hence we get $\text{fmapOpt}_F(\text{id}_{\diamond_{\text{Opt}}})(1 + a \times q) = 1 + a \times q$
- Composition law: need only to check for arguments $0 + a \times q$, and only when $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, in which case $(f_1 \diamond_{\text{Opt}} f_2)(a) = 0 + c$; then

$$
\begin{aligned}
& (\text{fmapOpt}_F\, f_1 \circ \text{fmapOpt}_F\, f_2)(0 + a \times q) \\
&= \text{fmapOpt}_F(f_2)\left(\text{fmapOpt}_F(f_1)(0 + a \times q)\right) \\
&= \text{fmapOpt}_F(f_2)\left(0 + b \times \text{fmapOpt}_G(f_1)(q)\right) \\
&= 0 + c \times (\text{fmapOpt}_G\, f_1 \circ \text{fmapOpt}_G\, f_2)(q) \\
&= 0 + c \times \text{fmapOpt}_G(f_1 \diamond_{\text{Opt}} f_2)(q) \\
&= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(0 + a \times q)
\end{aligned}
$$

This is a “greedy filter”: if $f(a)$ is empty, will delete all data in G^A

* Worked examples II: Constructions of filterable functors III

(6) The `fmapOpt` laws hold for $F^A \equiv G^A + A \times F^A$ if they hold for G^A

- For $f^{A \Rightarrow 1+B}$, we have $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$ and $\text{fmapOpt}'_F(f) : F^A \Rightarrow F^B$ (for use in recursive arguments as the inductive assumption)
- Define $\text{fmapOpt}_F(f)(q^{G^A} + a^A \times p^{F^A})$ by returning $0 + \text{fmapOpt}'_F(f)(p)$ if $f(a) = 1 + 0$, and $\text{fmapOpt}_G(f)(q) + b \times \text{fmapOpt}'_F(f)(p)$ otherwise
- Identity law: $\text{id}_{\diamond_{\textbf{Opt}}}(x) \neq 1 + 0$, so $\text{fmapOpt}_F(\text{id}_{\diamond_{\textbf{Opt}}})(q + a \times p) = q + a \times p$
- Composition law:
  $(\text{fmapOpt}_F(f_1) \circ \text{fmapOpt}_F(f_2))(q + a \times p) = \text{fmapOpt}_F(f_1 \diamond_{\textbf{Opt}} f_2)(q + a \times p)$
- For arguments $q + 0$, the laws for G^A hold; so assume arguments $0 + a \times p$. When $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, the proof of the previous example will go through. So we need to consider the two cases $f_1(a) = 1 + 0$ and $f_1(a) = 0 + b$, $f_2(b) = 1 + 0$
- If $f_1(a) = 1 + 0$ then $(f_1 \diamond_{\textbf{Opt}} f_2)(a) = 1 + 0$; to show $\text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\textbf{Opt}} f_2)(p)$, use the inductive assumption about $\text{fmapOpt}'_F$ on p
- If $f_1(a) = 0 + b$ and $f_2(b) = 1 + 0$ then $(f_1 \diamond_{\textbf{Opt}} f_2)(a) = 1 + 0$; to show $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\textbf{Opt}} f_2)(p)$, rewrite $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p))$ and again use the inductive assumption about $\text{fmapOpt}'_F$ on p

This is a "list-like filter": if $f(a)$ is empty, will recurse into nested F^A data

Worked examples II: Constructions of filterable functors IV

Use known filterable constructions to show that

$F^A \equiv (\text{Int} \times \text{String}) \Rightarrow (1 + \text{Int} \times A + A \times (1 + A) + (\text{Int} \Rightarrow 1 + A + A \times A \times \text{String}))$

is a filterable functor

- Instead of implementing `Filterable` and verifying laws by hand, we analyze the structure of this data type and use known constructions
- Define some auxiliary functors that are parts of the structure of F^A,
  - $R_1^A = (\text{Int} \times \text{String}) \Rightarrow A$ and $R_2^A = \text{Int} \Rightarrow A$
  - $G^A = 1 + \text{Int} \times A + A \times (1 + A)$ and $H^A = 1 + A + A \times A \times \text{String}$
- Now we can rewrite $F^A = R_1\left[G^A + R_2\left[H^A\right]\right]$
  - G^A is filterable by construction 5 because it is of the form $G^A = 1 + A \times K^A$ with filterable functor $K^A = 1 + \text{Int} + A$
  - K^A is of the form $1 + A + X$ with constant type X, so it is filterable by constructions 1 and 3 with the `Option` functor $1 + A$
  - H^A is filterable by construction 5 with $H^A = 1 + A \times (1 + A \times \text{String})$, while $1 + A \times \text{String}$ is filterable by constructions 5 and 1
- Constructions 3 and 4 show that $R_1\left[G^A + R_2\left[H^A\right]\right]$ is filterable

Note that there are more than one way of implementing `Filterable` here

Exercises II

1. For an arbitrary monad M^A, show that the functor $F^A \equiv \text{Boolean} \times M^A$ can be defined as a semimonad but not a monad.
2. If W and R are arbitrary fixed types, which of the functors can be made into a semimonad: $F^A \equiv W \times (R \Rightarrow A)$, $G^A = R \Rightarrow (W \times A)$?
3. Suppose a functor F^A has a natural transformation $\text{ex}^{[A]} : F^A \Rightarrow A$ that "extracts the value" from F^A. Would F^A be a semimonad if we defined `flatten` as $\text{ftn} = \text{ex}^{[F^A]}$ or $\text{ftn} = \text{fmap}\,\text{ex}$?
4. A programmer implemented the `fmap` method for the type constructor $F^A \equiv A \times (A \Rightarrow Z)$ as

   ```
   def fmap[A,B](f: A⇒B): ((A, A⇒Z)) ⇒ (B, B⇒Z) =
     { case(a, az) ⇒ (f(a), (_: B) ⇒ az(a)) }
   ```

   Show that this implementation fails to satisfy the functor laws.
5. Implement the `flatten` and `pure` methods for the type constructor $F^A \equiv 1 + A \times A$ (`type F[A] = Option[(A, A)]`) in at least two different ways, and show that the monad laws always fail.
6. Implement the monad methods for $F^A \equiv (Z \Rightarrow 1 + A) \times \text{List}^A$ using the known monad constructions.

(continued from the previous slide)

7. For an arbitrary monad M^A, show that the functor $F^A \equiv \text{Boolean} \times M^A$ can be defined as a semimonad but not a monad.
8. If W and R are arbitrary fixed types, which of the functors can be made into a semimonad: $F^A \equiv W \times (R \Rightarrow A)$, $G^A = R \Rightarrow (W \times A)$?
9. Suppose a functor F^A has a natural transformation $\text{ex}^{[A]} : F^A \Rightarrow A$ that "extracts the value" from F^A. Would it be correct to define F^A as a semimonad if we defined `flatten` as $\text{ftn} = \text{ex}^{[F^A]}$ or $\text{ftn} = \text{fmap}\,\text{ex}$?
10. A programmer implemented the `fmap` method for the type constructor $F^A \equiv A \times (A \Rightarrow Z)$ as

```
def fmap[A,B](f: A⇒B): ((A, A⇒Z)) ⇒ (B, B⇒Z) =
  { case(a, az) ⇒ (f(a), (_: B) ⇒ a2z(a)) }
```

Show that this implementation fails to satisfy the functor laws.

11. A programmer implemented the `flatMap` method for the type constructor $F^A \equiv 1 + A \times A$ as

```
def flatMap[A,B](f: A⇒F[B]): ((A,A⇒Z)) ⇒ (B,B⇒Z) =
{ case (a, az) ⇒ (f(a), (_: B) ⇒ az(a)) }
```