

Chapter 4: Functors. Part 1: Functor laws

How to recognize a functor

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“Container-like” type constructors

- Visualize `Seq[T]` as a container with some items of type `T`
 - ▶ How to formalize this idea as a property of `Seq`?
- Another example of a container: `Future[T]`
 - ▶ a value of type `T` will be available later, or may fail to arrive

Let us separate the “bare container” functionality from other functionality

- A “bare container” will allow us to:
 - ▶ manipulate items held within the container
 - ★ In FP, to “manipulate items” means to *apply functions to values*
- “Container holds items” = we can apply a function to the items
 - ▶ but the new items *remain* within the same container!
 - ▶ need `map`: $\text{Container}[A] \Rightarrow (A \Rightarrow B) \Rightarrow \text{Container}[B]$
- A “bare container” will *not* allow us to:
 - ▶ make a new container out of a given set of items
 - ▶ read values out of the container
 - ▶ add more items into container, delete items from container
 - ▶ wait until items are available in container, etc.

Option[T] as a container I

- In the short notation: $\text{Option}^A = 1 + A$
- The `map` function is required to have the type

$$\text{map}^{A,B} : 1 + A \Rightarrow (A \Rightarrow B) \Rightarrow 1 + B$$

Main questions:

- How to avoid “information loss” in this function?
- Does this `map` allow us to “manipulate values within the container”?

Option[T] as a container II

Avoiding “information loss” means:

- `map[A,A](opt)(x⇒x) == opt` – “identity law” for `map`
- We have two implementations of the type:

$$\text{map}^{[A,B]} = (1 + a^A) \Rightarrow (f^{A \Rightarrow B}) = 1 + f(a)$$

and

$$\text{map}^{[A,B]} = (1 + a^A) \Rightarrow (f^{A \Rightarrow B}) = 1 + 0^B$$

The second implementation has “information loss”!

- Short notation for code (type annotations are optional):

Short notation	Scala code
a^A	<code>val a: A</code>
$f^{[A]} B \Rightarrow C$	<code>def f[A]: B ⇒ C ...</code>
$a^A + b^B$	<code>x: Either[A, B] match {...}</code>
$a^A + 0^B$	<code>Left(a): Either[A, B]</code>
1	<code>()</code>

Option[T] as a container III

What it means to “be able to manipulate values in a container”

- Flip the two curried arguments in the type signature of `map`:

$$\text{fmap}^{[A,B]} : (A \Rightarrow B) \Rightarrow \text{Option}^A \Rightarrow \text{Option}^B$$

- A function is “**lifted**” from $A \Rightarrow B$ to $\text{Option}^A \Rightarrow \text{Option}^B$ by `fmap`:

$$\text{fmap}(f^{A \Rightarrow B}) : \text{Option}^A \Rightarrow \text{Option}^B$$

- Being able to manipulate values means that functions *behave normally* when lifted, i.e. when applied within the container
- The standard properties of function composition are

$$f^{A \Rightarrow B} \circ id^{B \Rightarrow B} = f^{A \Rightarrow B}$$

$$id^{A \Rightarrow A} \circ f^{A \Rightarrow B} = f^{A \Rightarrow B}$$

$$f^{A \Rightarrow B} \circ (g^{B \Rightarrow C} \circ h^{C \Rightarrow D}) = (f^{A \Rightarrow B} \circ g^{B \Rightarrow C}) \circ h^{C \Rightarrow D}$$

and should hold for the “lifted” functions as well!

- The “identity law” already requires that $\text{fmap}(id^{A \Rightarrow A}) = id^{\text{Option}^A \Rightarrow \text{Option}^A}$
- It remains to require that `fmap` should preserve function composition:

$$\text{fmap}(f^{A \Rightarrow B} \circ g^{B \Rightarrow C}) = \text{fmap}(f^{A \Rightarrow B}) \circ \text{fmap}(g^{B \Rightarrow C})$$

Functor: the definition

An abstraction for “bare container” functionality

A **functor** is:

- a type constructor with a type parameter, e.g. `MyType[T]`
- such that a function `map` or, equivalently, `fmap` is available:

$$\text{map}^{[A,B]} : \text{MyType}^A \Rightarrow (A \Rightarrow B) \Rightarrow \text{MyType}^B$$

$$\text{fmap}^{[A,B]} : (A \Rightarrow B) \Rightarrow \text{MyType}^A \Rightarrow \text{MyType}^B$$

- such that the identity law and the composition law hold for any type `T`
 - ▶ The laws are easier to formulate in terms of `fmap`:

$$\text{fmap}^{A,A}(\text{id}^A) = \text{id}^{F^A}$$

$$\text{fmap}(f \circ g) = \text{fmap}(f) \circ \text{fmap}(g)$$

- Verify the laws for `Option[A]`: see test code

```
def fmap[A,B]: (A => B) => Option[A] => Option[B] = f => {  
  case Some(a) => Some(f(a))  
  case None => None  
}
```

Examples of functors I

(Almost) everything that has a “map” is a functor

- Specific functors will have methods for creating them, reading values out of them, adding / removing items, waiting for items to arrive, etc.
 - ▶ Common to all functors is the `map` function
- Need to verify the laws!

Examples of functors in the Scala standard library:

- `Option[T]`
- `Either[L, R]` with respect to `R`
- `Seq[T]` and `Iterator[T]`
- the many subtypes of `Seq` (`Range`, `List`, `Vector`, `IndexedSeq`, etc.)
- `Future[T]`
- `Try[T]`
- `Map[K, V]` with respect to `V` (using `mapValues`)

Examples of non-functors that have a `map`:

- `Set[T]` – it works only when `T` has a well-behaved “==” operation
- `Map[K, V]` with respect to both `K` and `V`, because it is a `Set` w.r.t. `K`

See test code and <https://gist.github.com/tpolecat/7401433>

Examples of functors II

Polynomial type constructors as functors

- ❶ Short notation: $\text{QueryResult}^A = \text{String} \times \text{Int} \times A$

```
case class QueryResult[A](name: String, time: Int, data: A)
```

- ❷ Short notation: $\text{Vec3}^A = A \times A \times A$

```
case class Vec3[A](x: A, y: A, z: A)
```

- ❸ Short notation: $\text{QueryResult}^A = \text{String} + \text{String} \times \text{Int} \times A$

```
sealed trait QueryResult[A]  
case class Error[A](message: String) extends QueryResult[A]  
case class Success[A](name: String, time: Int, data: A)  
    extends QueryResult[A]
```

See test code

Examples of functors III: non-functors I

Type constructors that cannot have “map”

- 1 Type constructors that *consume* a value of the parameter type

$$\text{NotContainer}^A = (A \Rightarrow \text{Int}) \times A$$

```
case class NotContainer[A](x: A ⇒ Int, y: A)
```

- 2 Disjunction type constructors with non-parametric type values

```
sealed trait ServerAction[Res]  
case class GetResult[Res](r: Long ⇒ Res) extends ServerAction[Res]  
case class StoreId(x: Long, y: String) extends ServerAction[Long]  
case class StoreName(name: String) extends ServerAction[String]
```

- The type constructor `ServerAction[Res]` is called a GADT (“generalized algebraic data type”)
 - ▶ Not sure what the short notation should be for GADT types!

Examples of functors III: non-functors II

They could be functors, except for incorrect implementations of “fmap”

We need:

$$\text{fmap}(f^{A \Rightarrow B}) : \text{Container}^A \Rightarrow \text{Container}^B$$

- ① `fmap(f)` ignores `f` – e.g. always returns `None` for `Option[B]`
- ② `fmap(f)` reorders data items in a container:

$$\text{Container}^A \equiv A \times A; \quad \text{fmap}^{A,B}(f^{A \Rightarrow B})(x^A, y^A) = (f(y), f(x))$$

or swap some elements in $A \times A \times A$:

```
def fmap(f: A=>B): Vec3[A] => Vec3[B] =  
  { case Vec3(x, y, z) => Vec3(f(y), f(x), f(z)) }
```

- ③ Does a special computation if types are equal: if `A` and `B` are the same type, do `fmap[A, A](f) = identity`, otherwise `f(x)` is applied
- ④ Does a special computation if type is equal to a specific type, e.g. if `A = B = Int` then do `f(f(x))` else `f(x)`
- ⑤ Does a special computation if `f` is equal to some `f0`, otherwise use `f(x)`

See test code

Recursive polynomial types as functors

- Example: List of pairs defined as a recursive type,

$$LP^A \equiv 1 + A \times A \times LP^A$$

```
sealed trait LP[A]
final case class LPEmpty[A]() extends LP[A]
final case class LPpair[A](x: A, y: A, tail: LP[A]) extends LP[A]
```

- We can implement `map` as a recursive function:

```
def map[A, B](lp: LP[A], f: A ⇒ B): LP[B] = lp match {
  case LPEmpty() ⇒ LPEmpty[B]()
  case LPpair(x, y, tail) ⇒ LPpair[B](f(x), f(y), map(tail, f))
}
```

- This is the only way to implement `map` that satisfies the functor laws!

See test code for checking the functor laws

Contrafunctors

- The type constructor $C^A \equiv A \Rightarrow \text{Int}$ is not a functor (impossible to implement `map`), but we can implement `contrafmap`:

$$\text{contrafmap}^{A,B} : (B \Rightarrow A) \Rightarrow C^A \Rightarrow C^B$$

- The contrafunctor laws are analogous to functor laws:

$$\text{contrafmap}^{A,A}(\text{id}^A) = \text{id}^{C^A}$$

$$\text{contrafmap}(g \circ f) = \text{contrafmap}(f) \circ \text{contrafmap}(g)$$

The “contra-” reverses the arrow between A and B

- The type parameter A is to the left of the function arrow (“consumed”)
- “Functors contain data; contrafunctors consume data”

Example of non-contrafunctor:

- The type $\text{NotContainer}^A = (A \Rightarrow \text{Int}) \times A$ is neither a functor nor a contrafunctor

Covariance and contravariance

Functors, contrafunctors, and subtyping

Example of subtyping:

```
sealed trait AtMostTwo
final case class Zero() extends AtMostTwo
final case class One(x: Int) extends AtMostTwo
final case class Two(x: Int, y: Int) extends AtMostTwo
```

- Here `Zero`, `One`, and `Two` are **subtypes** of `AtMostTwo`
- We can pass `Two(10, 20)` to a function that takes an `AtMostTwo`
- This is equivalent to an automatic type conversion `Two \Rightarrow AtMostTwo`
- A container `C[A]` is **covariant** if `C[Two]` is a subtype of `C[AtMostTwo]`
 - ▶ And then a type conversion function `C[Two] \Rightarrow C[AtMostTwo]` exists
- More generally, when `x` is a subtype of `y` then we have `x \Rightarrow y` and we need `C[x] \Rightarrow C[y]`, which is guaranteed if we have a function of type

$$(A \Rightarrow B) \Rightarrow (C^A \Rightarrow C^B)$$

Functors are covariant, contrafunctors are contravariant

Worked examples

- Decide if a type constructor is a functor, a contrafunctor, or neither
 - Implement a `map` or a `contramap` function that satisfies the laws
- 1 Define case classes for these type constructors, and implement `map`:
 - 1 $\text{Data}^A \equiv \text{String} + A \times \text{Int} + A \times A \times A$
 - 2 $\text{Data}^A \equiv 1 + A \times (\text{Int} \times \text{String} + A)$
 - 3 $\text{Data}^A \equiv (\text{String} \Rightarrow \text{Int} \Rightarrow A) \times A + (\text{Boolean} \Rightarrow \text{Double} \Rightarrow A) \times A$
 - 2 Decide which of these type constructors are functors or contrafunctors, and implement `fmap` or `contrafmap` respectively:
 - 1 $\text{Data}^A \equiv (A \Rightarrow \text{Int}) + (A \Rightarrow A \Rightarrow \text{String})$
 - 2 $\text{Data}^{A,B} \equiv (A + B) \times ((A \Rightarrow \text{Int}) \Rightarrow B)$
 - 3 Rewrite this code in the short notation; identify covariant and contravariant type parameters; verify that with Scala:

```
sealed trait Coi[A, B]
case class Pa[A, B](b: (A, B), c: B⇒Int) extends Coi[A, B]
case class Re[A, B](d: A, e: B, c: Int) extends Coi[A, B]
case class Ci[A, B](f: String⇒A, g: B⇒A) extends Coi[A, B]
```

Exercises

Define case classes for these type constructors, decide if they are covariant or contravariant, and implement `map` or `contramap` as needed:

- 1 $\text{Data}^A \equiv (1 + A) \times (1 + A) \times \text{String}$
- 2 $\text{Data}^A \equiv (A \Rightarrow \text{String}) \Rightarrow (A \times (\text{Int} + A))$
- 3 $\text{Data}^{A,B} \equiv (A \Rightarrow \text{String}) \times ((A + B) \Rightarrow \text{Int})$
- 4 $\text{Data}^A \equiv (1 + (A \Rightarrow \text{String})) \Rightarrow (1 + (A \Rightarrow \text{Int})) \Rightarrow \text{Int}$
- 5 $\text{Data}^B \equiv (B + (\text{Int} \Rightarrow B)) \times (B + (\text{String} \Rightarrow B))$
- 6 Rewrite this code in the short notation; identify covariant and contravariant type parameters; verify that with Scala:

```
sealed trait Result[A]
case class P[A](a: A, b: String, c: Int) extends Result[A]
case class Q[A](d: Int⇒A, e: Int⇒String) extends Result[A]
case class R[A](f: A⇒A, g: A⇒String) extends Result[A]
```