

# Chapter 6: Computations lifted to a functor context I

## Filterable functors, their laws and structure

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# Computations within a functor context

- Example:

$$\sum_{x \in \mathbb{Z}; 0 \leq x \leq 100; \cos x > 0} \sqrt{\cos x} \approx 38.71$$

Scala code:

```
(0 to 100).map(math.cos(_)).filter(_ > 0).map(math.sqrt).sum
```

- Using Scala's `for`/`yield` syntax (“functor block”, “`for` comprehension”)

```
(for { x ← 0 to 100
      y = math.cos(x)
      if y > 0
    } yield math.sqrt(y)
).sum
```

```
(0 to 100).map { x ⇒
  math.cos(x) }.filter { y ⇒
  y > 0 }.map { y ⇒
  math.sqrt(y)
}.sum
```

- ▶ “Functor block” is a syntax for manipulating data within a container
  - ★ Container must be a functor (has `map` such that the laws hold)
  - ★ Data changes but remains within the same container
- A **filterable functor** is a functor that has a `withFilter` method
- Can use “`if`” when `withFilter(p: A⇒Boolean): F[A] ⇒ F[A]` is defined
  - ▶ What are the required laws for `withFilter`?
  - ▶ What data types are filterable functors?

# Filterable functors: Intuitions I

Intuition: the `filter` call *may decrease* the number of data items held

- a filterable container can hold *more or fewer* data items of type  $T$

Examples:

- $\text{Option}[T] \equiv 1 + T$ 
  - ▶ `Some(123).filter(_ > 0)` returns `Some(123)`
  - ▶ `Some(123).filter(_ == 1)` returns `None`
  - ▶ `Some(123).withFilter(_ == 1).map(identity)` returns `None`
- $\text{List}[T] \equiv 1 + T + T \times T + T \times T \times T + \dots$ 
  - ▶ `List(10, 20, 30).filter(_ > 10)` returns `List(20, 30)`
  - ▶ `List(10, 20, 30).filter(_ == 1)` returns `List()`

What we learn from these examples:

- The data type must contain a *disjunction* having different counts of  $T$
- When the predicate `p` returns `false` on some  $T$  values, the remaining data goes to a part of the disjunction that has fewer  $T$  values
- Values `x` are *algebraically* replaced by 1 (a `Unit`) when `p(x) = false`
- The container can become “empty” as a result of filtering

# Examples of filterable functors I

- Consider these business requirements:
  - ▶ An order can be placed on Tuesday and/or on Friday
  - ▶ An order is approved under certain conditions (amount < \$1000, etc.)

```
final case class Orders[A](tue: Option[A], fri: Option[A]) {  
  def withFilter(p: A ⇒ Boolean): Orders[A] =  
    Orders(tue.filter(p), fri.filter(p))  
}  
Orders(Some(500), Some(2000)).withFilter(_ < 1000)  
// returns Orders(Some(500), None) – see example code
```

- This functor type is written as  $F^A = (1 + A) \times (1 + A)$ 
  - ▶ When a value does not pass the filter, the  $A$  is replaced by 1
- Filtering is applied independently to both parts of the product type
- What if additional business requirements were given:
  - ▶ (a) both orders must be approved, or else no orders can be placed  
or
  - ▶ (b) both orders can be placed if at least one of them is approved
- Does this still make sense as “filtering”?
  - ▶ Need mathematical laws to decide this

# Filterable functors: Intuitions II

- Intuition: computations in the functor block should “make sense”
  - ▶ we should be able to reason correctly by looking at the program text
- A schematic example of a functor block program using `map` and `filter`:

```
for { // computations lifted into the List functor
  x ← List(...) // the first line has “←”, other lines do not
  y = f(x) // will become a “map(f)” after compilation
  if p1(y) // will become a “withFilter(p1)”
  if p2(y)
  z = g(x, y)
  if q(x, y, z) // – more conditions, etc.; see example code
} yield // for all x in list, such that conditions hold, compute this:
  k(x, y, z) // all the new values will stay within the container
```

- What we intuitively expect to be true about such programs:
  - ① `y = f(x); if p(y);` is equivalent to `if p(f(x)); y = f(x);`
  - ② `if p1(y); if p2(y);` is equivalent to `if p1(y) && p2(y)`
  - ③ When a filter predicate `p(x)` returns `true` for *all* `x`, we can delete the line “`if p(x)`” from the program with no change to the results
  - ④ When a filter predicate `p(x)` returns `false` for some `x` then *that* `x` will be excluded from computations performed after “`if p(x)`”

## Examples of filterable functors II: Checking the laws

- Properties 1 – 4 are expressed as laws for `filter`  $(p \Rightarrow \text{Boolean}) \Rightarrow F^A \Rightarrow F^A$ :
  - ①  $\text{fmap } f^{A \Rightarrow B} \circ \text{filter } p^{B \Rightarrow \text{Boolean}} = \text{filter } (f \circ p) \circ \text{fmap } f^{A \Rightarrow B}$
  - ②  $\text{filter } p_1^{A \Rightarrow \text{Boolean}} \circ \text{filter } p_2^{A \Rightarrow \text{Boolean}} = \text{filter } (x \Rightarrow p_1(x) \wedge p_2(x))$
  - ③  $\text{filter } (x^A \Rightarrow \text{true}) = \text{id}^{F^A \Rightarrow F^A}$
  - ④  $\text{filter } p \circ \text{fmap } f^{A \Rightarrow B} = \text{filter } p \circ \text{fmap } (f|_p)$  where  $f|_p$  is the *partial function* defined as `{ case x if p(x) => f(x) }` – only works if  $p(x)$  holds
- Can define a type class `Filterable`, method `withFilter`
- Check the laws for the `Orders` functor (see example code)
  - ▶ Laws hold for the `Orders` functor with / without business rule (a)
  - ▶ Another filterable functor:  $F^A \equiv 1 + A \times A$  (“collapsible product”)
- Examples of functors that are *not* filterable:
  - ▶ “Orders” with additional business rule (b) – breaks law 2 for some  $p_{1,2}$
  - ▶  $F^A$  defining `filter` in a special way e.g. for  $A = \text{Int}$  – breaks law 1
  - ▶  $F^A \equiv 1 + A$  defining `filter`  $(p)(x) \equiv 1 + 0$  breaks law 3
  - ▶  $F^A \equiv A$  – must define `filter`  $(p^{A \Rightarrow \text{Boolean}})(x^A) = x$ , breaking law 4
  - ▶  $F^A \equiv A \times (1 + A)$  – unable to remove the first  $A$ , breaking law 4

The equational laws 1–4 specify *rigorously* what it means to “filter data”!

# Worked examples I: Programming with filterables

- 1 John can have up to 3 coupons, and Jill up to 2. *All* of John's coupons must be valid on purchase day, while each of Jill's coupons is checked independently. Implement the filterable functor describing this setup.
- 2 A server received a sequence of requests. Each request must be authenticated. Once a non-authenticated request is found, no further requests are accepted. Is this setup described by a filterable functor?

For each of these functors, determine whether they are filterable, and if so, implement `withFilter` via a type class:

- 3 `final case class P[T](first: Option[T], second: Option[(T, T)])`
- 4  $F^A \equiv \text{Int} + \text{Int} \times A + \text{Int} \times A \times A + \text{Int} \times A \times A \times A$
- 5  $F^A = \text{NonEmptyList}^A$  defined recursively as  $F^A \equiv A + A \times F^A$
- 6  $F^{Z,A} \equiv Z + \text{Int} \times Z \times A \times A$  (with respect to the type parameter  $A$ )
- 7  $F^{Z,A} \equiv 1 + Z + \text{Int} \times A \times \text{List}^A$  (w.r.t. the type parameter  $A$ )
- 8 \* Show that  $C^{Z,A} = A \Rightarrow 1 + Z$  is a filterable *contrafunctor* w.r.t.  $A$  (implement `withFilter` with the same type signature; no law checking)

# Exercises I

- 1 Confucius gave wisdom on each of the 7 days of a week. Sometimes the wise proverbs were hard to remember. If Confucius forgets what he said on a given day, he also forgets what he said on all the previous days of the week. Is this setup described by a filterable functor?
- 2 Define `evenFilter(p)` on an `IndexedSeq[T]` such that a value `x: T` is retained if `p(x)=true` and only if the sequence has an *even* number of elements `y` for which `p(y)=false`. Does this define a filterable functor?

Implement `filter` for these functors if possible (law checking optional):

- 3  $F^A \equiv \text{Int} + \text{String} \times A \times A \times A$
- 4 `final case class Q[A, Z](id: Long, user1: Option[(A, Z)], user2: Option[(A, Z)])` – with respect to the type parameter `A`
- 5  $F^A = \text{MyTree}^A$  defined recursively as  $F^A \equiv 1 + A \times F^A \times F^A$
- 6 `final case class R[A](x: Int, y: Int, z: A, data: List[A])`, where the standard functor `List` already has `withFilter` defined
- 7 \* Show that  $C^A \equiv A + A \times A \Rightarrow 1 + Z$  is a filterable contrafunctor



# Filterable functors: The laws in depth I

Is there a shorter formulation of the laws that is easier to remember?

- Intuition: When  $p(x) = \text{false}$ , replace  $x : A$  by  $1 : \text{Unit}$  in  $F[A]$ 
  - ▶ (1) How to replace  $x$  by  $1$  in  $F[A]$  without breaking the types?
  - ▶ (2) How to transform the resulting type back to  $F[A]$ ?
- We could do (1) if instead of  $F^A$  we had  $F^{1+A}$  i.e.  $F[\text{Option}[A]]$ 
  - ▶ Now use `filter` to replace  $A$  by  $1$  in each item of type  $1 + A$
  - ▶ Get  $F^{1+A}$  from  $F^A$  using `inflate` :  $F^A \Rightarrow F^{1+A} = \text{fmap}(\text{Some}^{A \Rightarrow 1+A})$
  - ▶ Filter  $F^{1+A} \Rightarrow F^{1+A}$  using `fmap` ( $x^{1+A} \Rightarrow \text{filter}_{\text{Opt}}(p^{A \Rightarrow \text{Boolean}})(x)$ )

$$\text{filter } p : F^A \xrightarrow{\text{inflate}} F^{1+A} \xrightarrow{\text{fmap}(\text{filter}_{\text{Opt}} p)} F^{1+A} \xrightarrow{\text{deflate}} F^A$$

- Doing (2) means *defining* a function `deflate`:  $F[\text{Option}[A]] \Rightarrow F[A]$ 
  - ▶ standard library already has `flatten`:  $\text{Seq}[\text{Option}[T]] \Rightarrow \text{Seq}[T]$
- Simplify  $\text{fmap}(\text{Some}^{A \Rightarrow 1+A}) \circ \text{fmap}(\text{filter}_{\text{Opt}} p) = \text{fmap}(\text{bop}(p))$  where we defined `bop` ( $p$ ) :  $(A \Rightarrow 1 + A) \equiv x \Rightarrow \text{Some}(x).\text{filter}(p)$
- In this way, express `filter` through `deflate` (see example code)
  - ▶  $\text{filter } p = \text{fmap}(\text{bop } p) \circ \text{deflate}$ . – Notation:  $\text{bop } p$  is  $\text{bop}(p)$ , like  $\cos x$

$$\text{filter } p : F^A \xrightarrow{\text{fmap}(\text{bop } p)} F^{1+A} \xrightarrow{\text{deflate}} F^A$$

## Filterable functors: Using `deflate`

- So far we have expressed `filter` through `deflate`
- We can also express `deflate` through `filter` (assuming law 4 holds):

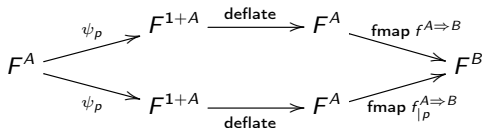
$$\text{deflate} : F^{1+A} \xrightarrow{\text{filter}(\cdot.\text{nonEmpty})} F^{1+A} \xrightarrow{\text{fmap}(\cdot.\text{get})} F^A$$

```
def deflate[F[_],A](foa: F[Option[A]]): F[A] =  
  foa.filter(_._nonEmpty).map(_._get) // _._get is 0 + x^A ⇒ x^A  
  // for F = Seq, this would be foa.collect { case Some(x) ⇒ x }  
  // for arbitrary functor F we need to use the partial function, _._get
```

- This means `deflate` and `filter` are **computationally equivalent**
  - ▶ We could specify filterable functors by implementing `deflate`
    - ★ The implementation of `filter` would then be derived by library
- Use `deflate` to verify that some functors are certainly not filterable:
  - ▶  $F^A = A + A \times A$ . Write  $F^{1+A} = 1 + A + (1 + A) \times (1 + A)$ 
    - ★ cannot map  $F^{1+A} \Rightarrow F^A$  because we do not have  $1 \rightarrow A$
  - ▶  $F^A = \text{Int} \Rightarrow A$ . Write  $F^{1+A} = \text{Int} \Rightarrow 1 + A$ 
    - ★ type signature of `deflate` would be  $(\text{Int} \Rightarrow 1 + A) \Rightarrow \text{Int} \Rightarrow A$
    - ★ cannot map  $F^{1+A} \Rightarrow F^A$  because we do not have  $1 + A \rightarrow A$
- `deflate` is easier to implement and to reason about

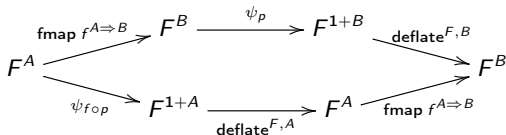
## \* Filterable functors: The laws in depth II

- We were able to define **deflate** only by assuming that law 4 holds
- Now, law 4 is satisfied *automatically* if **filter** is defined via **deflate**!
  - ▶ Denote  $\psi_p^{F^A \Rightarrow F^{1+A}} \equiv \text{fmap}(\text{bop } p)$  for brevity, then  $\text{filter } p = \psi_p \circ \text{deflate}$
  - ▶ Law 4 then becomes:  $\psi_p \circ \text{deflate} \circ \text{fmap } f^{A \Rightarrow B} = \psi_p \circ \text{deflate} \circ \text{fmap } f|_p$



- We would like to interchange **deflate** and **fmap** in both sides
  - ▶ We need a *naturality* law; let's express law 1 through **deflate**:

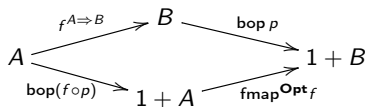
$$\text{fmap } f^{A \Rightarrow B} \circ \psi_p \circ \text{deflate}^{F, B} = \psi_{f \circ p} \circ \text{deflate}^{F, A} \circ \text{fmap } f^{A \Rightarrow B}$$



Can we simplify  $\text{fmap } f \circ \psi_p = \text{fmap } f \circ \text{fmap}(\text{bop } p) = \text{fmap}(f \circ \text{bop } p)$ ?

## \* Filterable functors: The laws in depth III

- Have property:  $f^{A \Rightarrow B} \circ \text{bop} (p^{B \Rightarrow \text{Boolean}}) = \text{bop} (f \circ p) \circ \text{fmap}^{\text{Opt}} f$  (see code)

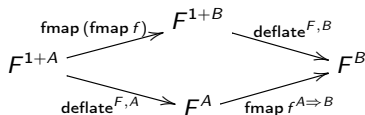


We can now rewrite Law 1 as

$$\text{fmap} (\text{bop} (f \circ p)) \circ \text{fmap} (\text{fmap}^{\text{Opt}} f) \circ \text{deflate} = \text{fmap} (\text{bop} (f \circ p)) \circ \text{deflate} \circ \text{fmap } f$$

Remove common prefix  $\text{fmap} (\text{bop} (f \circ p)) \circ \dots$  from both sides:

$$\text{fmap} (\text{fmap}^{\text{Opt}} f^{A \Rightarrow B}) \circ \text{deflate}^{F, B} = \text{deflate}^{F, A} \circ \text{fmap } f^{A \Rightarrow B} \quad \text{— law 1 for deflate}$$



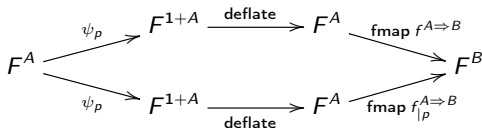
- deflate**:  $F^{1+A} \Rightarrow F^A$  is a **natural transformation** (has naturality law)
  - Example:  $F^A = 1 + A \times A$
  - $F^{1+A} = 1 + (1 + A) \times (1 + A) = 1 + 1 \times 1 + A \times 1 + 1 \times A + A \times A$
- natural transformations map containers  $G^A \Rightarrow H^A$  by rearranging data in them

## \* Filterable functors: The laws in depth IV

- The naturality law for **deflate**:

$$\text{fmap}(\text{fmap}^{\text{Opt}} f^{A \Rightarrow B}) \circ \text{deflate}^{F,B} = \text{deflate}^{F,A} \circ \text{fmap} f^{A \Rightarrow B}$$

Law 4 expressed via **deflate**:



$$\psi_p \circ \text{deflate}^{F,A} \circ \text{fmap} f^{A \Rightarrow B} = \psi_p \circ \text{deflate}^{F,A} \circ \text{fmap} f_{|p}$$

- Use naturality to interchange **deflate** and **fmap** in both sides of law 4:

$$\psi_p \circ \text{fmap}(\text{fmap}^{\text{Opt}} f) \circ \text{deflate}^{F,B} = \psi_p \circ \text{fmap}(\text{fmap}^{\text{Opt}} f_{|p}) \circ \text{deflate}^{F,B}$$

[omit  $\text{deflate}^{F,B}$  from both sides; expand  $\psi_p$ ]

$$\text{bop } p \circ \text{fmap}^{\text{Opt}} f = \text{bop } p \circ \text{fmap}^{\text{Opt}} f_{|p} \quad \text{— check this by hand:}$$

`x  $\Rightarrow$  Some(x).filter(p).map(f)`

`x  $\Rightarrow$  Some(x).filter(p).map { x if p(x)  $\Rightarrow$  f(x) }`

- These functions are equivalent because law 4 holds for **Option**

# Filterable functors: The laws in depth V

Maybe  $\psi_p \circ \text{deflate}$  is easier to handle than **deflate**? Let us define

$$\text{fmapOpt}^{F,A,B}(f^{A \Rightarrow 1+B}) : F^A \Rightarrow F^B = \text{fmap } f \circ \text{deflate}^{F,B}$$

A commutative triangle diagram with  $F^A$  at the bottom-left vertex,  $F^{1+B}$  at the top vertex, and  $F^B$  at the bottom-right vertex. An arrow labeled  $\text{fmap } f^{A \Rightarrow 1+B}$  points from  $F^A$  to  $F^{1+B}$ . An arrow labeled  $\text{deflate}^{F,B}$  points from  $F^{1+B}$  to  $F^B$ . A direct arrow labeled  $\text{fmapOpt } f^{A \Rightarrow 1+B}$  points from  $F^A$  to  $F^B$ .

- **fmapOpt** and **deflate** are *equivalent*:  $\text{deflate}^{F,A} = \text{fmapOpt}^{F,1+A,A}(\text{id}^{1+A \Rightarrow 1+A})$
- Express laws 1 – 3 in terms of **fmapOpt**: do they get simpler?
  - ▶ Express **filter** through **fmapOpt**:  $\text{filter } p = \text{fmapOpt}^{F,A,A}(\text{bop } p)$
  - ▶ Consider the expression needed for law 2:  $x \Rightarrow p_1(x) \wedge p_2(x)$
  - ▶  $\text{bop}(x \Rightarrow p_1(x) \wedge p_2(x)) = x^A \Rightarrow (\text{bop } p_1)(x). \text{flatMap}(\text{bop } p_2)$  – see code
    - ★ Denote this computation by  $\diamond_{\text{Opt}}$  and write

$$q_1^{A \Rightarrow 1+B} \diamond_{\text{Opt}} q_2^{B \Rightarrow 1+C} \equiv x^A \Rightarrow q_1(x). \text{flatMap}(q_2)$$

- ▶ Similar to composition of functions, except the types are  $A \Rightarrow 1 + B$ 
  - ★ This is a particular case of **Kleisli composition**; the general case:  
 $\diamond_M : (A \Rightarrow M^B) \Rightarrow (B \Rightarrow M^C) \Rightarrow (A \Rightarrow M^C)$ ; we set  $M^A \equiv 1 + A$
  - ★ The **Kleisli identity** function:  $\text{id}_{\diamond_{\text{Opt}}}^{A \Rightarrow 1+A} \equiv x^A \Rightarrow \text{Some}(x)$
  - ★ Kleisli composition  $\diamond_{\text{Opt}}$  is associative and respects the Kleisli identity!
  - ★ **fmapOpt** lifts a Kleisli<sub>Opt</sub> function  $f^{A \Rightarrow 1+B}$  into the functor  $F$

# Filterable functors: The laws in depth VI

Simplifying down to two laws

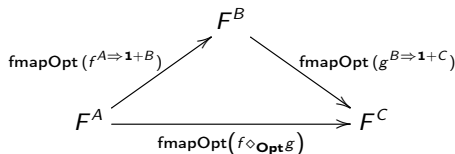
- Only *two* laws are necessary for `fmapOpt`!

- Identity law** (covers old law 3):

$$\text{fmapOpt}(\text{id}^{A \Rightarrow 1+A}_{\diamond_{\text{Opt}}}) = \text{id}^{F^A \Rightarrow F^A}$$

- Composition law** (covers old laws 1 and 2):

$$\text{fmapOpt}(f^{A \Rightarrow 1+B}) \circ \text{fmapOpt}(g^{B \Rightarrow 1+C}) = \text{fmapOpt}(f \diamond_{\text{Opt}} g)$$



- The two laws for `fmapOpt` are very similar to the two functor laws
  - Both of them use more complicated types than the old laws
  - Conceptually, the new laws are simpler (lift  $f^{A \Rightarrow 1+B}$  into  $F^A \Rightarrow F^B$ )

## \* Filterable functors: The laws in depth VII

Showing that old laws 1 – 3 follow from the identity and composition laws for `fmapOpt`

- Old law 3 is *equivalent* to the identity law for `fmapOpt`:

$$\text{filter}(x^A \Rightarrow \text{true}) = \text{fmap}(x^A \Rightarrow 0 + x) \circ \text{deflate} = \text{fmapOpt}(\text{id}_{\diamond_{\text{Opt}}}) = \text{id}^{F^A \Rightarrow F^A}$$

- Derive old law 2: need to work with  $q_{1,2} \equiv \text{bop}(p_{1,2}) : A \Rightarrow 1 + A$

- ▶ The Boolean conjunction  $x \Rightarrow p_1(x) \wedge p_2(x)$  corresponds to  $q_1 \diamond_{\text{Opt}} q_2$
- ▶ Apply the composition law to Kleisli functions of types  $A \Rightarrow 1 + A$ :

$$\begin{aligned}\text{filter } p_1 \circ \text{filter } p_2 &= \text{fmapOpt } q_1 \circ \text{fmapOpt } q_2 \\ &= \text{fmapOpt}(q_1 \diamond_{\text{Opt}} q_2) = \text{fmapOpt}(\text{bop}(x \Rightarrow p_1(x) \wedge p_2(x)))\end{aligned}$$

- Derive old law 1:

- ▶ express `filter` through `fmapOpt`; old law 1 becomes

$$\text{fmap } f \circ \text{fmapOpt}(\text{bop } p) = \text{fmapOpt}(\text{bop}(f \circ p)) \circ \text{fmap } f \quad - \text{eq. } (*)$$

- ▶ lift  $f^{A \Rightarrow B}$  to  $\text{Kleisli}_{\text{Opt}}$  by defining  $k_f^{A \Rightarrow 1+B} = f \circ \text{id}_{\diamond_{\text{Opt}}}$ ; then we have  $\text{fmapOpt}(k_f) = \text{fmap } k_f \circ \text{deflate} = \text{fmap } f \circ \text{fmap } \text{id}_{\diamond_{\text{Opt}}} \circ \text{deflate} = \text{fmap } f$
- ▶ rewrite eq. (\*) as  $\text{fmapOpt}(k_f \diamond_{\text{Opt}} \text{bop } p) = \text{fmapOpt}(\text{bop}(f \circ p) \diamond_{\text{Opt}} k_f)$
- ▶ it remains to show that  $k_f \diamond_{\text{Opt}} \text{bop } p = \text{bop}(f \circ p) \diamond_{\text{Opt}} k_f$
- ▶ use the properties  $k_f \diamond_{\text{Opt}} q = f \circ q$  and  $q \diamond_{\text{Opt}} k_f = q \circ \text{fmap}^{\text{Opt}} f$ , and  $f \circ \text{bop } p = \text{bop}(f \circ p) \circ \text{fmap}^{\text{Opt}} f$  (property from slide 11)



# Summary: The methods and the laws

Filterable functors can be defined via `filter`, `deflate`, or `fmapOpt`

- All three methods are *equivalent* but have different roles:
  - ▶ The easiest to use in program code is `filter` / `withFilter`
  - ▶ The easiest type signature to implement and reason about is `deflate`
  - ▶ Conceptually, the laws are easiest to remember with `fmapOpt`
- \* The 2 laws for `fmapOpt` are the 2 functor laws with a Kleisli “twist”
- \* Category theory accommodates this via a generalized definition of functors as liftings between “twisted” types. Compare:
  - ▶  $\text{fmap} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$  – ordinary container (“endofunctor”)
  - ▶  $\text{contrafmap} : (B \Rightarrow A) \Rightarrow F^A \Rightarrow F^B$  – lifting from reversed functions
  - ▶  $\text{fmapOpt} : (A \Rightarrow 1 + B) \Rightarrow F^A \Rightarrow F^B$  – lifting from  $\text{Kleisli}_{\text{Opt}}$ -functions
- CT gives us some *intuitions* about how to derive better laws:
  - ▶ look for type signatures that resemble a generalized sort of “lifting”
  - ▶ look for natural transformations and use the naturality law
- However, CT does not directly provide any derivations for the laws
  - ▶ you will not find the laws for `filter` or `deflate` in any CT book
  - ▶ CT is abstract, only gives hints about possible further directions
    - ★ investigate functors having “liftings” with different type signatures
    - ★ replace `Option` in the  $\text{Kleisli}_{\text{Opt}}$  construction by another functor

# Structure of filterable functors

How to recognize a filterable functor by its type?

Intuition from `deflate`: reshuffle data in  $F^A$  after replacing some  $A$ 's by 1

- “reshuffling” usually means reusing different parts of a disjunction

Some constructions of exponential-polynomial filterable functors

- 1  $F^A = Z$  (constant functor) for a fixed type  $Z$  (define `fmapOpt f = id`)
  - ▶ Note:  $F^A = A$  (identity functor) is *not* filterable
- 2  $F^A \equiv G^A \times H^A$  for any filterable functors  $G^A$  and  $H^A$
- 3  $F^A \equiv G^A + H^A$  for any filterable functors  $G^A$  and  $H^A$
- 4  $F^A \equiv G^{H^A}$  for *any* functor  $G^A$  and filterable functor  $H^A$
- 5  $F^A \equiv 1 + A \times G^A$  for a filterable functor  $G^A$ 
  - ▶ Note: *pointed* types  $P$  are isomorphic to  $1 + Z$  for some type  $Z$ 
    - ★ Example of non-trivial pointed type:  $A \Rightarrow A$
    - ★ Example of non-pointed type:  $A \Rightarrow B$  when  $A$  is different from  $B$
  - ▶ So  $F^A \equiv P + A \times G^A$  where  $P$  is a pointed type and  $G^A$  is filterable
  - ▶ Also have  $F^A \equiv P + A \times A \times \dots \times A \times G^A$  similarly
- 6  $F^A \equiv G^A + A \times F^A$  (recursive) for a filterable functor  $G^A$
- 7  $F^A \equiv G^A \Rightarrow H^A$  if contrafunctor  $G^A$  and functor  $H^A$  *both filterable*
  - ▶ Note: the functor  $F^A \equiv G^A \Rightarrow A$  is not filterable

## \* Worked examples II: Constructions of filterable functors I

(2) The `fmapOpt` laws hold for  $F^A \times G^A$  if they hold for  $F^A$  and  $G^A$

- For  $f^{A \Rightarrow 1+B}$ , get  $\text{fmapOpt}_F(f) : F^A \Rightarrow F^B$  and  $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define  $\text{fmapOpt}_{F \times G} f \equiv p^{F^A} \times q^{G^A} \Rightarrow \text{fmapOpt}_F(f)(p) \times \text{fmapOpt}_G(f)(q)$
- Identity law:  $f = \text{id}_{\diamond_{\text{Opt}}}$ , so  $\text{fmapOpt}_F f = \text{id}$  and  $\text{fmapOpt}_G f = \text{id}$ 
  - ▶ Hence we get  $\text{fmapOpt}_{F+G}(f)(p \times q) = \text{id}(p) \times \text{id}(q) = p \times q$
- Composition law:

$$\begin{aligned} & (\text{fmapOpt}_{F \times G} f_1 \circ \text{fmapOpt}_{F+G} f_2)(p \times q) \\ &= \text{fmapOpt}_{F \times G}(f_2) (\text{fmapOpt}_F(f_1)(p) \times \text{fmapOpt}_G(f_1)(q)) \\ &= (\text{fmapOpt}_F f_1 \circ \text{fmapOpt}_F f_2)(p) \times (\text{fmapOpt}_G f_1 \circ \text{fmapOpt}_G f_2)(q) \\ &= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(p) \times \text{fmapOpt}_G(f_1 \diamond f_2)(q) \\ &= \text{fmapOpt}_{F \times G}(f_1 \diamond_{\text{Opt}} f_2)(p \times q) \end{aligned}$$

- Exactly the same proof as that for functor property for  $F^A \times G^A$ 
  - ▶ this is because `fmapOpt` corresponds to a generalized functor
- New proofs are necessary only when using non-filterable functors
  - ▶ these are used in constructions 4 – 6

## \* Worked examples II: Constructions of filterable functors II

(5) The `fmapOpt` laws hold for  $F^A \equiv 1 + A \times G^A$  if they hold for  $G^A$

- For  $f^{A \Rightarrow 1+B}$ , get  $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define  $\text{fmapOpt}_F(f)(1 + a^A \times q^{G^A})$  by returning  $0 + b \times \text{fmapOpt}_G(f)(q)$  if the argument is  $0 + a \times q$  and  $f(a) = 0 + b$ , and returning  $1 + 0$  otherwise
- Identity law:  $f = \text{id}_{\text{Opt}}$ , so  $f(a) = 0 + a$  and  $\text{fmapOpt}_G f = \text{id}$ 
  - ▶ Hence we get  $\text{fmapOpt}_F(\text{id}_{\text{Opt}})(1 + a \times q) = 1 + a \times q$
- Composition law: need only to check for arguments  $0 + a \times q$ , and only when  $f_1(a) = 0 + b$  and  $f_2(b) = 0 + c$ , in which case  $(f_1 \diamond_{\text{Opt}} f_2)(a) = 0 + c$ ; then

$$\begin{aligned} & (\text{fmapOpt}_F f_1 \circ \text{fmapOpt}_F f_2)(0 + a \times q) \\ &= \text{fmapOpt}_F(f_2) (\text{fmapOpt}_F(f_1)(0 + a \times q)) \\ &= \text{fmapOpt}_F(f_2) (0 + b \times \text{fmapOpt}_G(f_1)(q)) \\ &= 0 + c \times (\text{fmapOpt}_G f_1 \circ \text{fmapOpt}_G f_2)(q) \\ &= 0 + c \times \text{fmapOpt}_G(f_1 \diamond_{\text{Opt}} f_2)(q) \\ &= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(0 + a \times q) \end{aligned}$$

This is a “greedy filter”: if  $f(a)$  is empty, will delete all data in  $G^A$

## \* Worked examples II: Constructions of filterable functors III

(6) The `fmapOpt` laws hold for  $F^A \equiv G^A + A \times F^A$  if they hold for  $G^A$

- For  $f^{A \Rightarrow 1+B}$ , we have  $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$  and  $\text{fmapOpt}'_F(f) : F^A \Rightarrow F^B$  (for use in recursive arguments as the inductive assumption)
- Define  $\text{fmapOpt}_F(f)(q^{G^A} + a^A \times p^{F^A})$  by returning  $0 + \text{fmapOpt}'_F(f)(p)$  if  $f(a) = 1 + 0$ , and  $\text{fmapOpt}_G(f)(q) + b \times \text{fmapOpt}'_F(f)(p)$  otherwise
- Identity law:  $\text{id}_{\diamond_{\text{Opt}}}(x) \neq 1 + 0$ , so  $\text{fmapOpt}_F(\text{id}_{\diamond_{\text{Opt}}})(q + a \times p) = q + a \times p$
- Composition law:  
 $(\text{fmapOpt}_F(f_1) \circ \text{fmapOpt}_F(f_2))(q + a \times p) = \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(q + a \times p)$
- For arguments  $q + 0$ , the laws for  $G^A$  hold; so assume arguments  $0 + a \times p$ . When  $f_1(a) = 0 + b$  and  $f_2(b) = 0 + c$ , the proof of the previous example will go through. So we need to consider the two cases  $f_1(a) = 1 + 0$  and  $f_1(a) = 0 + b$ ,  $f_2(b) = 1 + 0$
- If  $f_1(a) = 1 + 0$  then  $(f_1 \diamond_{\text{Opt}} f_2)(a) = 1 + 0$ ; to show  $\text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\text{Opt}} f_2)(p)$ , use the inductive assumption about  $\text{fmapOpt}'_F$  on  $p$
- If  $f_1(a) = 0 + b$  and  $f_2(b) = 1 + 0$  then  $(f_1 \diamond_{\text{Opt}} f_2)(a) = 1 + 0$ ; to show  $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\text{Opt}} f_2)(p)$ , rewrite  $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p))$  and again use the inductive assumption about  $\text{fmapOpt}'_F$  on  $p$

This is a “list-like filter”: if  $f(a)$  is empty, will recurse into nested  $F^A$  data

## Worked examples II: Constructions of filterable functors IV

Use known filterable constructions to show that

$F^A \equiv (\text{Int} \times \text{String}) \Rightarrow (1 + \text{Int} \times A + A \times (1 + A) + (\text{Int} \Rightarrow 1 + A + A \times A \times \text{String}))$   
is a filterable functor

- Instead of implementing `Filterable` and verifying laws by hand, we analyze the structure of this data type and use known constructions
- Define some auxiliary functors that are parts of the structure of  $F^A$ ,
  - ▶  $R_1^A = (\text{Int} \times \text{String}) \Rightarrow A$  and  $R_2^A = \text{Int} \Rightarrow A$
  - ▶  $G^A = 1 + \text{Int} \times A + A \times (1 + A)$  and  $H^A = 1 + A + A \times A \times \text{String}$
- Now we can rewrite  $F^A = R_1 [G^A + R_2 [H^A]]$ 
  - ▶  $G^A$  is filterable by construction 5 because it is of the form  $G^A = 1 + A \times K^A$  with filterable functor  $K^A = 1 + \text{Int} + A$
  - ▶  $K^A$  is of the form  $1 + A + X$  with constant type  $X$ , so it is filterable by constructions 1 and 3 with the `Option` functor  $1 + A$
  - ▶  $H^A$  is filterable by construction 5 with  $H^A = 1 + A \times (1 + A \times \text{String})$ , while  $1 + A \times \text{String}$  is filterable by constructions 5 and 1
- Constructions 3 and 4 show that  $R_1 [G^A + R_2 [H^A]]$  is filterable

Note that there are more than one way of implementing `Filterable` here

## \* Exercises II

- 1 Implement a `Filterable` instance for `type F[T] = G[H[T]]` assuming that the functor `H[T]` already has a `Filterable` instance (construction 4). Verify the laws rigorously (i.e. by calculations, not tests).
- 2 For `type F[T] = Option[Int  $\Rightarrow$  Option[(T, T)]]`, implement a `Filterable` instance. Show that the filterable laws hold by using known filterable constructions (avoiding explicit proofs or tests).
- 3 Implement a `Filterable` instance for  $F^A \equiv G^A + \text{Int} \times A \times A \times F^A$  (recursive) for a filterable functor  $G^A$ . Verify the laws rigorously.
- 4 Show that  $F^A = 1 + A \times G^A$  is in general *not* filterable if  $G^A$  is an arbitrary (non-filterable) functor; it is enough to give an example.
- 5 Show that  $F^A = 1 + G^A + H^A$  is filterable if  $1 + G^A$  and  $1 + H^A$  are filterable (even when  $G^A$  and  $H^A$  are by themselves not filterable).
- 6 Show that the functor  $F^A = A + (\text{Int} \Rightarrow A)$  is not filterable.
- 7 Show that one can define `deflate`:  $C^{1+A} \Rightarrow C^A$  for any contrafunctor  $C^A$  (not necessarily filterable), similarly to how one can define `inflate`:  $F^A \Rightarrow F^{1+A}$  for any functor  $F^A$  (not necessarily filterable).

## \* Bonus slide I: Definition of filterable contrafunctors

When is a contrafunctor filterable?

When a contrafunctor  $C^A$  with  $\text{contrafmap} : (B \Rightarrow A) \Rightarrow C^A \Rightarrow C^B$  has also

- **filter/withFilter**:  $(A \Rightarrow \text{Boolean}) \Rightarrow C^A \Rightarrow C^A$  – same as for functors
- **inflate**:  $C^A \Rightarrow C^{1+A}$  and **contrafmapOpt**:  $(B \Rightarrow 1 + A) \Rightarrow C^A \Rightarrow C^B$
- All three functions are computationally equivalent...
  - ▶  $\text{filter}(p^{A \Rightarrow \text{Boolean}}) = \text{inflate}^{C^A \Rightarrow C^{1+A}} \circ \text{contrafmap}(\text{bop } p)$
  - ▶  $\text{inflate}^{C^A \Rightarrow C^{1+A}} = \text{contrafmap}(0 + x^A \Rightarrow x) \circ \text{filter}(\_ \Rightarrow \text{true})$
  - ▶  $\text{contrafmapOpt } f^{B \Rightarrow 1+A} = \text{inflate} \circ \text{contrafmap } f$
  - ▶  $\text{inflate} = \text{contrafmapOpt}(\text{id}^{1+A \Rightarrow 1+A})$
- but have different laws
  - ▶ 4 laws (naturality, conjunction, identity, partial function) for **filter**
  - ▶ 3 laws (naturality, conjunction, identity) for **inflate**
  - ▶ 2 laws (identity, contracomposition) for **contrafmapOpt**
    - ★ as before, **contrafmapOpt** is a “twisted” version of **contrafmap**
- Examples of filterable contrafunctors
  - ▶  $C^A \equiv A \Rightarrow 1 + Z$  where  $Z$  is a fixed type
  - ▶  $C^A \equiv 1 + A \Rightarrow Z$
- Examples of non-filterable contrafunctors
  - ▶  $C^A \equiv A \times F^A \Rightarrow Z$  – cannot implement **inflate**



## \* Bonus slide II: Structure of filterable contrafunctors

How to build up a filterable contrafunctor from parts?

- Filterable contrafunctors “can consume fewer data items”
- The easiest function to consider first is `inflate`

Some constructions of filterable contrafunctors:

- 1  $C^A = Z$  (constant contrafunctor)

Functor constructions (no need to check laws for these):

- 2  $F^A \equiv G^A \times H^A$  for any filterable contrafunctor  $G^A$  and  $H^A$
- 3  $F^A \equiv G^A + H^A$  for any filterable contrafunctor  $G^A$  and  $H^A$
- 4  $F^A \equiv G^{H^A}$  for  $H^A$  a filterable (contra)functor and  $G^A$  any (contra)functor – various combinations possible here
- 5  $F^A \equiv G^A \Rightarrow H^A$  if functor  $G^A$  and contrafunctor  $H^A$  *both filterable*

Special constructions:

- 6  $F^A \equiv 1 + A \times G^A \Rightarrow H^A$  where  $G^A$  and  $H^A$  are filterable
- 7  $F^A \equiv A \times G^A \Rightarrow 1 + H^A$  if  $G^A$  and  $H^A$  are filterable