Chapter 8: Applicative and traversable functors

Part 1: Practical examples

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Motivation for applicative functors

Monads are inconvenient for expressing *independent* effects

- Monads perform effects *sequentially* even if effects are independent:

```
x ← Future { c1 }
y ← Future { c2 }
z ← Future { c3 }
```

```
Future { c1 }.flatMap { x ⇒
  Future { c2 }.flatMap { y ⇒
    Future { c3 }.map { z ⇒ ...}
} }
```

- We would like to parallelize independent computations
- We would like to accumulate *all* errors, rather than stop at the first one
- Changing the order of effects will (generally) change the result:

```
for {
  x ← List(1, 2)
  y ← List(10, 20)
} yield f(x, y)
// f(1, 10), f(1, 20), f(2, 10), f(2, 20)
```

```
for {
  y ← List(10, 20)
  x ← List(1, 2)
} yield f(x, y)
// f(1, 10), f(2, 10), f(1, 20), f(2, 20)
```

- We would like to express a computation where effects are unordered
  - This can be expressed with a method `map2`, *not* defined via `flatMap`: the desired type signature is map2 : $F^A \times F^B \Rightarrow (A \times B \Rightarrow C) \Rightarrow F^C$
  - So we need a functor that has `map2` but is not necessarily a monad

Defining `map2`, `map3`, etc.

Consider 1, 2, 3, ... commutative and independent "effects"

`for { x1 ← c1 } yield f(x1)`	`c1.map(f)`
`for { x1 ← c1 x2 ← c2 } yield f(x1, x2)`	`(c1, c2).map2(f)`
`for { x ← c1 x2 ← c2 x3 ← c3 } yield f(x1, x2, x3)`	`(c1, c2, c3).map3(f)`

- Generalize to `mapN` from

$$\mathrm{map}_1 : F^A \Rightarrow (A \Rightarrow Z) \Rightarrow F^Z$$

$$\mathrm{map}_2 : F^A \times F^B \Rightarrow (A \times B \Rightarrow Z) \Rightarrow F^Z$$

$$\mathrm{map}_3 : F^A \times F^B \times F^C \Rightarrow (A \times B \times C \Rightarrow Z) \Rightarrow F^Z$$

- Can we avoid having to define map_n separately for each n?

Intuition: the `zip` operation on lists

- Simplify $\text{fmap2} : (A \times B \Rightarrow C) \Rightarrow F^A \times F^B \Rightarrow F^C$ by substituting $f = \text{id}^{A \times B \Rightarrow A \times B}$, expecting to obtain a simpler natural transformation:

$$\text{zip} : F^A \times F^B \Rightarrow F^{A \times B}$$

- This is quite similar to `zip` for lists:

```
List(1, 2).zip(List(10, 20)) = List((1, 10), (2, 20))
```

- The functions `zip` and `fmap2` are computationally equivalent:

$$\text{zip} = \text{fmap2}\,(\text{id})$$

$$\text{fmap2}\,(f^{A \times B \Rightarrow C}) = \text{zip} \circ \text{fmap}\, f$$

$$\begin{tikzcd}
 & F^{A \times B} \arrow[dr, "\text{fmap}\, f^{A \times B \Rightarrow C}"] & \\
F^A \times F^B \arrow[ur, "\text{zip}"] \arrow[rr, "\text{fmap2}\,(f^{A \times B \Rightarrow C})"'] & & F^C
\end{tikzcd}$$

- The functor F is "zippable" if such a `zip` exists

Deriving the `ap` operation

- Set $A \equiv B \Rightarrow C$, get $\text{zip}^{[B\Rightarrow C,B]} : F^{B\Rightarrow C} \times F^{B} \Rightarrow F^{(B\Rightarrow C)\times B}$
- Use `eval` : $(B \Rightarrow C) \times B \Rightarrow C$ and $\text{fmap}\,(\text{eval}) : F^{(B\Rightarrow C)\times B} \Rightarrow F^{C}$
- Define $\text{app}^{[B,C]} : F^{B\Rightarrow C} \times F^{B} \Rightarrow F^{C} \equiv \text{zip} \circ \text{fmap}\,(\text{eval})$
- The functions `zip` and `app` are computationally equivalent:
  - use $\text{pair} : (A \Rightarrow B \Rightarrow A \times B) = a^{A} \Rightarrow b^{B} \Rightarrow a \times b$
  - use $\text{fmap}\,(\text{pair}) \equiv \text{pair}^{\uparrow}$ on an $fa^{F^{A}}$, get $(\text{pair}^{\uparrow} fa) : F^{B\Rightarrow A\times B}$; then

$$\text{zip}\,(fa \times fb) = \text{app}\left((\text{pair}^{\uparrow} fa) \times fb\right)$$

$$\text{app}^{[B\Rightarrow C,B]} = \text{zip}^{[B\Rightarrow C,B]} \circ \text{fmap}\,(\text{eval})$$

$$\begin{tikzcd}
 & F^{(B\Rightarrow C)\times B} \arrow[dr, "\text{fmap(eval)}"] & \\
F^{B\Rightarrow C} \times F^{B} \arrow[ur, "\text{zip}"] \arrow[rr, "\text{app}^{[B\Rightarrow C,B]}"'] & & F^{C}
\end{tikzcd}$$

- Rewrite this using curried arguments: $\text{fzip}^{[A,B]} : F^{A} \Rightarrow F^{B} \Rightarrow F^{A\times B}$; $\text{ap}^{[B,C]} : F^{B\Rightarrow C} \Rightarrow F^{B} \Rightarrow F^{C}$; then $\text{ap}\, f = \text{fzip}\, f \circ \text{fmap}\,(\text{eval})$.
- Now $\text{fzip}\, p^{F^{A}} q^{F^{B}} = \text{ap}\,(\text{pair}^{\uparrow} p)\, q$, hence we can write as point-free: $\text{fzip} = \text{pair}^{\uparrow} \circ \text{ap}$. With explicit types: $\text{fzip}^{[A,B]} = \text{pair}^{\uparrow} \circ \text{ap}^{[B,A\Rightarrow B]}$.