

Chapter 9: Traversable functors and contrafunctors

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Motivation for the `traverse` operation

- Consider data of type List^A and processing $f : A \Rightarrow \text{Future}^B$
- Typically, we want to wait until the entire data set is processed
- What we need is $\text{List}^A \Rightarrow (A \Rightarrow \text{Future}^B) \Rightarrow \text{Future}^{\text{List}^B}$
- Generalize: $L^A \Rightarrow (A \Rightarrow F^B) \Rightarrow F^{L^B}$ for some type constructors F, L
- This operation is called `traverse`
 - ▶ How to implement it: for example, a 3-element list is $A \times A \times A$
 - ▶ Consider $L^A \equiv A \times A \times A$, apply map f and get $F^B \times F^B \times F^B$
 - ▶ We will get $F^{L^B} \equiv F^{B \times B \times B}$ if we can apply `zip` as $F^B \times F^B \Rightarrow F^{B \times B}$
- So we need to assume that F is applicative
- In Scala, we have `Future.traverse()` that assumes L to be a sequence
 - ▶ This is the iconic example that fixes the requirements
- Questions:
 - ▶ Which functors L can have this operation?
 - ▶ Can we express `traverse` through a simpler operation?
 - ▶ What are the required laws for `traverse`?
 - ▶ What about contrafunctors or profunctors?

Deriving the `sequence` operation

- The type signature of `traverse` is a complicated “lifting”
 - ▶ A “lifting” is always equivalent to a simpler natural transformation
- To derive it, ask: what is missing from `fmap` to do the job of `traverse`?

$$\text{fmap} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow L^{F^B}$$

- We need F^{L^B} , but the `traverse` operation gives us L^{F^B} instead
 - ▶ What’s missing is a natural transformation `sequence` : $L^{F^B} \Rightarrow F^{L^B}$
- The functions `traverse` and `sequence` are computationally equivalent:

$$\text{trav } f \xrightarrow{A \Rightarrow F^B} = \text{fmap } f \circ \text{seq}$$

$$\begin{array}{ccc} & L^{F^B} & \\ \text{fmap } f \xrightarrow{A \Rightarrow F^B} \nearrow & & \searrow \text{seq} \\ L^A & \xrightarrow{\text{trav } f \xrightarrow{A \Rightarrow F^B}} & F^{L^B} \end{array}$$

Here F is an *arbitrary* applicative functor

- ▶ Keep in mind the example `Future.sequence` : $\text{List}^{\text{Future}^X} \Rightarrow \text{Future}^{\text{List}^X}$
- ▶ Examples: `List`;
- ▶ Non-traversable: $L^A \equiv R \Rightarrow A$; lazy lists (“infinite streams”)
 - ★ Note: We *cannot* have the opposite transformation $F^{L^B} \Rightarrow L^{F^B}$

Polynomial functors are traversable

- Generalize from the example $L^A \equiv A \times A \times A$ to other polynomials
- Polynomial functors have the form

$$L^A \equiv Z \times A \times \dots \times A + Y \times A \times \dots \times A + \dots + Q \times A + P$$

- To implement $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$, consider monomial $L^A \equiv Z \times A \times \dots \times A$
- We have $L^{F^B} = Z \times F^B \times \dots \times F^B$; apply **zip** and get $Z \times F^{B \times \dots \times B}$
- Lift Z into the functor F using $Z \Rightarrow F^A \Rightarrow F^{Z \times A}$ (or with $F.\text{pure}$)
- The result is $F^{Z \times B \times \dots \times B} \equiv F^{L^B}$
 - ▶ For a polynomial L^A , apply this procedure to each monomial
 - ▶ Note that we could apply **zip** in various different orders
- The traversal order is arbitrary, may be application-specific
- Non-polynomial functors are not traversable (see [Bird et al., 2013](#))
 - ▶ Example: $L^A \equiv E \Rightarrow A$, $F^A \equiv 1 + A$; can't have $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$
- All polynomial functors are traversable, and usually in several ways
 - ▶ Therefore it is useful to have a type class for traversable functors

Motivation for the laws of the `traverse` operation

- The “**law of traversals**” paper (2012) argues that `traverse` should “visit each element” of the container L^A exactly once, and evaluate each corresponding “effect” F^B exactly once; then they formulate the laws
- To derive the laws, use the “lifting” intuition for `traverse`,

$$\text{trav} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow F^{L^B}$$

Look for “identity” and “composition” laws:

- ① “Identity” as `pure` : $A \Rightarrow F^A$ must be lifted to `pure` : $L^A \Rightarrow F^{L^A}$
- ② “Identity” as $\text{id}^{A \Rightarrow A}$ with $F^A \equiv A$ (identity functor) lifted to $\text{id}^{L^A \Rightarrow L^A}$
- ③ “Compose” $f : A \Rightarrow F^B$ and $g : B \Rightarrow G^C$ to get $h : A \Rightarrow F^{G^C}$, where F, G are applicative; a traversal with h maps L^A to $F^{G^{L^C}}$ and must be equal to the composition of traversals with f and then with $g^{F\uparrow}$

Questions:

- Are the laws for the `sequence` operation simpler?
- Are all these laws independent?
- What functors L satisfy these laws *for all* applicative functors F ?

Formulation of the laws for `traverse`

- Identity law: for any applicative functor F ,

$$\text{trav}(\text{pure}) = \text{pure}$$

$$L^A \xrightarrow[\text{trav}(\text{pure}^{A \Rightarrow F^A})]{\text{pure}^{L^A \Rightarrow FL^A}} FL^A$$

- ▶ Second identity law: $\text{trav}^{\text{Id}}(\text{id}^A) = \text{id}^{L^A}$ is a consequence with $F = \text{Id}$

- Composition law: for any $f^{A \Rightarrow FB}$ and $g^{B \Rightarrow GC}$,

$$\text{trav } f \circ \text{trav} (g^\uparrow) = \text{trav} (f \circ g^\uparrow)$$

$$\begin{array}{ccc} & L^{FB} & \\ \text{trav}_F f^{A \Rightarrow FB} \nearrow & & \searrow \text{trav}_{FG} (\text{fmap}_{FG})^{FB \Rightarrow FG^C} \\ L^A & \xrightarrow{\text{trav}_{FG} h^{A \Rightarrow FG^C}} & F^{GL^C} \end{array}$$

$$\text{where } h^{A \Rightarrow FG^C} \equiv f \circ g^{F\uparrow}$$

Derivation of the laws for `sequence`

The

Constructions of traversable functors

The

Traversable profunctors

The

Traversability with respect to profunctors

The

“Folding” a traversable functor into a monoid

The

- ① Show that any traversable functor L admits a method

$$\text{consume} : (L^A \Rightarrow B) \Rightarrow L^{F^A} \Rightarrow F^B$$

for any applicative functor F . Show that `traverse` and `consume` are equivalent.