Chapter 6: Computations lifted to a functor context I
Filterable functors, their laws and structure

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Computations within a functor context

- Example:

$$\sum_{x \in \mathbb{Z};\ 0 \le x \le 100;\ \cos x > 0} \sqrt{\cos x} \approx 38.71$$

  Scala code:
  ```
  (0 to 100).map(math.cos(_)).filter(_ > 0).map(math.sqrt).sum
  ```
- Using Scala's `for`/`yield` syntax ("functor block", "`for` comprehension")

  ```
  (for { x ← 0 to 100              (0 to 100).map { x ⇒
      y = math.cos(x)                math.cos(x) }.filter { y ⇒
      if y > 0                       y > 0 }.map { y ⇒
  } yield { math.sqrt(y) }             math.sqrt(y)
  ).sum                            }.sum
  ```

  - "Functor block" is a syntax for manipulating data within a container
    - ⋆ Container must be a functor (has `map` such that the laws hold)
    - ⋆ Data changes but remains within the same container
- A **filterable functor** is a functor that has a `withFilter` method
- Can use "`if`" when `withFilter(p: A⇒Boolean): F[A] ⇒ F[A]` is defined
  - What are the required laws for `withFilter`?
  - What data types are filterable functors?

Filterable functors: Intuitions I

Intuition: the `filter` call *may decrease* the number of data items held

- a filterable container can hold *more or fewer* data items of type T

Examples:

- `Option[T]` $\equiv 1 + T$
  - ▸ `Some(123).filter(_ > 0)` returns `Some(123)`
  - ▸ `Some(123).filter(_ == 1)` returns `None`
  - ▸ `Some(123).withFilter(_ == 1).map(identity)` returns `None`
- `List[T]` $\equiv 1 + T + T \times T + T \times T \times T + ...$
  - ▸ `List(10, 20, 30).filter(_ > 10)` returns `List(20, 30)`
  - ▸ `List(10, 20, 30).filter(_ == 1)` returns `List()`

What we learn from these examples:

- The data type must contain a *disjunction* having different counts of T
- When the predicate `p` returns `false` on some T values, the remaining data goes to a part of the disjunction that has fewer T values
- Values `x` are *algebraically* replaced by 1 (a `Unit`) when `p(x) = false`
- The container can become "empty" as a result of filtering

Examples of filterable functors I

- Consider these business requirements:
  - An order can be placed on Tuesday and/or on Friday
  - An order is approved if requested amount is less than $1,000, etc.

```scala
final case class Orders[A](tue: Option[A], fri: Option[A]) {
  def withFilter(p: A ⇒ Boolean): Orders[A] =
    Orders(tue.filter(p), fri.filter(p))
}
Orders(Some(500), Some(2000)).withFilter(_ < 1000)
// returns Orders(Some(500), None)
```

- The functor type is $F^A = (1 + A) \times (1 + A)$
  - When a value does not pass the filter, the A is replaced by 1
- Filtering is applied to both parts of the product type independently
- What if additional business requirements were given:
  - (a) both orders must be approved, or else no orders can be placed or
  - (b) both orders can be placed if at least one of them is approved
- Does this still make sense as "filtering"?
  - Need mathematical laws to decide this

Filterable functors: Intuitions II

- Intuition: computations in the functor block should "make sense"
  - we should be able to reason correctly by looking at the program text
- A schematic example of a functor block program using `map` and `filter`:

```
for { // computations lifted to the List functor
  x ← List(...) // the first line has "←", other lines do not
  y = f(x) // will become a "map(f)" after compilation
  if p1(y) // will become a "withFilter(p1)"
  if p2(y)
  z = g(x, y)
  if q(x, y, z)
} yield // for all x in list, such that conditions hold, compute this:
  k(x, y, z) // the new values stay within the container
```

- What we intuitively expect to be true about such programs:
  1. `y = f(x); if p(y);` is equivalent to `if p(f(x)); y = f(x);`
  2. `if p1(y); if p2(y);` is equivalent to `if p1(y) && p2(y)`
  3. When a filter predicate `p(x)` returns `true` for *all* `x`, we can delete the line "`if p(x)`" from the program with no change to the results
  4. When a filter predicate `p(x)` returns `false` for some `x` then *that* `x` will be excluded from computations performed after "`if p(x)`"

Examples of filterable functors I: Checking the laws

- Properties 1 – 4 are expressed as laws for `filter`$^{(p \Rightarrow \text{Boolean}) \Rightarrow F^A \Rightarrow F^A}$:
  1. $\text{fmap } f^{A \Rightarrow B} \circ \text{filter } p^{B \Rightarrow \text{Boolean}} = \text{filter } (f \circ p) \circ \text{fmap } f^{A \Rightarrow B}$
  2. $\text{filter } p_1^{A \Rightarrow \text{Boolean}} \circ \text{filter } p_2^{A \Rightarrow \text{Boolean}} = \text{filter } (x \Rightarrow p_1(x) \wedge p_2(x))$
  3. $\text{filter } (x^A \Rightarrow \text{true}) = \text{id}^{F^A \Rightarrow F^A}$
  4. $\text{filter } p \circ \text{fmap } f^{A \Rightarrow B} = \text{filter } p \circ \text{fmap } (f_{|p})$ where $f_{|p}$ is the *partial function* defined as `x => if (p(x)) f(x) else ???`

- Can define a type class `Filterable`, method `withFilter`
- Check the laws for Example I (see code)
  - "Orders" example with / without business rule (a) – laws hold
  - $F^A \equiv 1 + A \times A$ ("collapsible product")
- Examples of functors that are *not* filterable:
  - "Orders" with additional business rule (b) – breaks law 2 for some $p_{1,2}$
  - F^A defining `filter` in a special way for $A = \text{Int}$ (breaks law 1)
  - $F^A \equiv 1 + A$ defining $\text{filter } (p)(x) \equiv 1 + 0$ breaks law 3
  - $F^A \equiv A$ – must define $\text{filter } (p^{A \Rightarrow \text{Boolean}})(x^A) = x$, breaking law 4
  - $F^A \equiv A \times (1 + A)$ – unable to remove the first A, breaking law 4

The equational laws 1–4 specify *rigorously* what it means to "filter data"!

Worked examples I: Programming with filterables

1. John can have up to 3 coupons, and Jill up to 2. *All* of John's coupons must be valid on purchase day, while each of Jill's coupons is checked independently. Implement the filterable functor describing this setup.

2. A server received a sequence of requests. Each request must be authenticated. Once a non-authenticated request is found, no further requests are accepted. Is this setup described by a filterable functor?

   For each of these functors, determine whether they are filterable, and if so, implement `withFilter` via a type class:

3. `final case class P[T](first: Option[T], second: Option[(T, T)])`

4. $F^A \equiv \text{Int} + \text{Int} \times A + \text{Int} \times A \times A + \text{Int} \times A \times A \times A$

5. $F^A = \text{NonEmptyList}^A$ defined recursively as $F^A \equiv A + A \times F^A$

6. $F^{Z,A} \equiv Z + \text{Int} \times Z \times A \times A$ (with respect to the type parameter A)

7. $F^{Z,A} \equiv 1 + Z + \text{Int} \times A \times \text{List}^A$ (w.r.t. the type parameter A)

8. * Show that $C^{Z,A} = A \Rightarrow 1 + Z$ is a filterable *contrafunctor* w.r.t. A (implement `withFilter` with the same type signature; no law checking)

Exercises I

1. Confucius gave wisdom on each of the 7 days of a week. Sometimes the wise proverbs were hard to remember. If Confucius forgets what he said on a given day, he also forgets what he said on all the previous days of the week. Is this setup described by a filterable functor?

2. Define `evenFilter(p)` on an `IndexedSeq[T]` such that a value `x: T` is retained if `p(x)=true` *and* only if the sequence has an *even* number of elements `y` for which `p(y)=false`. Does this define a filterable functor?

   Implement `filter` for these functors if possible (law checking optional):

3. $F^A \equiv \text{Int} + \text{String} \times A \times A \times A$

4. `final case class Q[A, Z](id: Long, user1: Option[(A, Z)], user2: Option[(A, Z)])` – with respect to the type parameter A

5. $F^A = \text{MyTree}^A$ defined recursively as $F^A \equiv 1 + A \times F^A \times F^A$

6. `final case class R[A](x: Int, y:  Int, z: A, data: List[A])`, where the standard functor List already has `withFilter` defined

7. * Show that $C^A \equiv A + A \times A \Rightarrow 1 + Z$ is a filterable contrafunctor

Filterable functors: The laws in depth I

Is there a more elegant formulation of the laws, easier to understand?

- Main intuition: When `p(x) = false`, replace `x: A` by `1: Unit` in `F[A]`
  - (1) How to replace `x` by `1` in `F[A]` without breaking the types?
  - (2) How to transform the resulting type back to `F[A]`?
  - We could do (1) if instead of the type `F[A]` we had `F[Option[A]]`
    - ⋆ Map F^A to F^{1+A} using fmap $(\text{Some}^{A \Rightarrow 1+A}) : F^A \Rightarrow F^{1+A}$
    - ⋆ Now we can replace A by 1 in each item of type $1 + A$
  - Doing (2) means *defining* a function `flatten: F[Option[A]]` $\Rightarrow$ `F[A]`
    - ⋆ standard library has `flatten[T]: Seq[Option[T]]` $\Rightarrow$ `Seq[T]`
  - Express `filter` through `flatten` (see example code):
    - ⋆ filter $(p) = $ fmap $(\text{optB}\,(p)) \circ $ flatten, where we defined `optB` as
    ```
    def optB[T](p: T ⇒ Boolean): T ⇒ Option[T] = ...
    ```
  - Note: the Boolean type is isomorphic to $1 + 1$ or `Option[Unit]`
- Express `flatten` through `filter` (using law 4):
  ```
  def flatten[F[_],T](c: F[Option[T]]): F[T] =
    c.filter(_.nonEmpty).map(_.get) // _.get is 0 + x^A ⇒ x^A
    // for F = Seq, this would be c.collect { case Some(x) ⇒ x }
  ```
- Law 4 is satisfied *automatically* if `filter` is defined via `flatten`!

* Filterable functors: The laws in depth II

Showing that law 4 is satisfied automatically if `filter` is defined via `flatten`

- Denote $\psi^{A \Rightarrow 1+A} \equiv \mathsf{optB}\,(p^{A \Rightarrow 1+1}) = x^A \Rightarrow \mathsf{fmap^{Opt}}\,(\_ \Rightarrow x)\,(p(x))$
  - Have property: $f^{T \Rightarrow A} \circ \mathsf{optB}\,(p^{A \Rightarrow 1+1}) = \mathsf{optB}\,(f \circ p) \circ \mathsf{fmap^{Opt}}\,f$
- Law 4: $\mathsf{fmap}\,\psi \circ \mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f^{T \Rightarrow A} = \mathsf{fmap}\,\psi \circ \mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f_{|p}$
  - We would like to interchange `flatten` and `fmap` here. Use Law 1?
- Reformulate Law 1 in terms of `flatten`:

$$\mathsf{fmap}\,f^{T \Rightarrow A} \circ \mathsf{fmap}\,\psi \circ \mathsf{flatten}^{F,A} = \mathsf{filter}\,(f \circ p) \circ \mathsf{fmap}\,f$$

$$\mathsf{fmap}\,(f^{T \Rightarrow A} \circ \mathsf{optB}(p^{A \Rightarrow 1+A})) \circ \mathsf{flatten}^{F,A} = \mathsf{fmap}\,(\mathsf{optB}\,(f \circ p)) \circ \mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f$$

$$\mathsf{fmap}^F(\mathsf{optB}\,(f \circ p)) \circ \mathsf{fmap}^F(\mathsf{fmap^{Opt}}\,f) = \mathsf{fmap}^F(\mathsf{optB}\,(f \circ p) \circ \mathsf{fmap^{Opt}}\,f)$$

$$[\text{remove common prefix } \mathsf{fmap}\,(\mathsf{optB}\,(f \circ p)) \circ ... \text{ from both sides}]$$

$$\mathsf{fmap}\,(\mathsf{fmap^{Opt}}\,f^{T \Rightarrow A}) \circ \mathsf{flatten}^{F,A} = \mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f \quad - \textbf{ law 1 for flatten}$$

- We can now interchange `flatten` and `fmap` in $\mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f_{|p}^{T \Rightarrow A}$:

$$\mathsf{fmap}\,\psi \circ \mathsf{flatten}^{F,T} \circ \mathsf{fmap}\,f_{|p} = \mathsf{fmap}\,\psi \circ \mathsf{fmap}\,(\mathsf{fmap^{Opt}}\,f_{|p}) \circ \mathsf{flatten}^{F,A}$$

$$= \mathsf{fmap}\,(\psi \circ \mathsf{fmap^{Opt}}\,f) \circ \mathsf{flatten}^{F,A} = \mathsf{fmap}\,(\psi \circ \mathsf{fmap^{Opt}}\,f_{|p}) \circ \mathsf{flatten}^{F,A}$$

$$\psi \circ \mathsf{fmap^{Opt}}\,f = \psi \circ \mathsf{fmap^{Opt}}\,f_{|p} \quad - \text{ check this by hand}$$

Filterable functors: The laws in depth III

Maybe fmap ∘ flatten is easier to handle than `flatten`? Let us define
$$\mathsf{fmapOpt}^{F,A,B}(f^{A \Rightarrow 1+B}) : (A \Rightarrow 1 + B) \Rightarrow F^A \Rightarrow F^B = \mathsf{fmap}\, f \circ \mathsf{flatten}^{F,B}$$

- `fmapOpt` and `flatten` are *equivalent*: $\mathsf{flatten}^{F,A} = \mathsf{fmapOpt}^{F,1+A,A}(\mathsf{id}^{1+A \Rightarrow 1+A})$
- Express laws $1 - 3$ in terms of `fmapOpt` and $\psi^{A \Rightarrow 1+A} \equiv \mathsf{optB}\,(p)$
  - Express `filter` through `fmapOpt`: $\mathsf{filter}\,(p) = \mathsf{fmapOpt}^{F,A,A}\,(\psi)$
  - Consider the expression needed for law 2: $x \Rightarrow p_1(x)$ and $p_2(x)$
    - ★ Written in terms of ψ_1 and ψ_2, this is $x^A \Rightarrow \psi_1(x).\mathsf{flatMap}\,(\psi_2)$
  - Similar to composition of functions, except the types are $A \Rightarrow 1 + B$
    - ★ This is a particular case of **Kleisli composition**; the general case:
      $\diamond_M : (A \Rightarrow M^B) \Rightarrow (B \Rightarrow M^C) \Rightarrow (A \Rightarrow M^C)$; we set $M^A \equiv 1 + A$
    - ★ The **Kleisli identity** function: $\mathsf{id}_{\diamond_{\mathbf{Opt}}}^{A \Rightarrow 1+A} \equiv x^A \Rightarrow \mathsf{Some}\,(x)$
    - ★ Kleisli composition $\diamond_{\mathbf{Opt}}$ is associative and respects the Kleisli identity!
- `fmapOpt` lifts a Kleisli$_{\mathbf{Opt}}$ function $f^{A \Rightarrow 1+B}$ into the functor F
- Only *two* laws are necessary for `fmapOpt`!
  1. **Identity law** (covers old law 3): $\mathsf{fmapOpt}\,(\mathsf{id}_{\diamond_{\mathbf{Opt}}}^{A \Rightarrow 1+A}) = \mathsf{id}^{F^A \Rightarrow F^A}$
  2. **Composition law** (covers old laws 1 and 2):
     $\mathsf{fmapOpt}\,(f^{A \Rightarrow 1+B}) \circ \mathsf{fmapOpt}\,(g^{B \Rightarrow 1+C}) = \mathsf{fmapOpt}\,(f \diamond_{\mathbf{Opt}} g)$
  - The two laws for `fmapOpt` are very similar to the two functor laws

* Filterable functors: The laws in depth IV
Showing that old laws 1 − 3 follow from the identity and composition laws for `fmapOpt`

- Old law 3 is *equivalent* to the identity law for `fmapOpt`:

  $$\text{filter}\,(x^A \Rightarrow 0 + 1) = \text{fmap}\,(x^A \Rightarrow 0 + x) \circ \text{flatten} = \text{fmapOpt}\,(\text{id}_{\diamond_{\textbf{Opt}}}) = \text{id}^{F^A \Rightarrow F^A}$$

- Derive old law 2: need to work with $\psi \equiv \text{optB}\,(p) : A \Rightarrow 1 + A$
  - ▶ The Boolean conjunction $x \Rightarrow p_1(x) \wedge p_2(x)$ corresponds to $\psi_1 \diamond_{\textbf{Opt}} \psi_2$
  - ▶ Apply the composition law to Kleisli functions of types $A \Rightarrow 1 + A$:

  $$\text{filter}\,(p_1) \circ \text{filter}\,(p_2) = \text{fmapOpt}\,(\psi_1) \circ \text{fmapOpt}\,(\psi_2)$$
  $$= \text{fmapOpt}\,(\psi_1 \diamond_{\textbf{Opt}} \psi_2) = \text{fmapOpt}\,(\text{optB}\,(x \Rightarrow p_1(x) \wedge p_2(x)))$$

- Derive old law 1: express `filter` through `fmapOpt`, so law 1 becomes
  - ▶ $\text{fmap}\,f \circ \text{fmapOpt}\,(\text{optB}\,(p)) = \text{fmapOpt}\,(\text{optB}\,(f \circ p)) \circ \text{fmap}\,f$ − eq. (*)
  - ▶ denote $k_f^{A \Rightarrow 1 + A} = x^A \Rightarrow 0 + f(x)$; that is, $k_f = f \circ \text{id}_{\diamond_{\textbf{Opt}}}$; then we have
    $\text{fmapOpt}\,(k_f) = \text{fmap}\,k_f \circ \text{flatten} = \text{fmap}\,f \circ \text{fmap}\,\text{id}_{\diamond_{\textbf{Opt}}} \circ \text{flatten} = \text{fmap}\,f$
  - ▶ rewrite (*) as $\text{fmapOpt}\,(k_f \diamond_{\textbf{Opt}} \text{optB}\,(p)) = \text{fmapOpt}\,(\text{optB}\,(f \circ p) \diamond_{\textbf{Opt}} k_f)$
  - ▶ it remains to show that $k_f \diamond_{\textbf{Opt}} \text{optB}\,(p) = \text{optB}\,(f \circ p) \diamond_{\textbf{Opt}} k_f$
  - ▶ use the properties $k_f \diamond_{\textbf{Opt}} \psi = f \circ \psi$ and $\psi \diamond_{\textbf{Opt}} k_f = \psi \circ \text{fmap}^{\textbf{Opt}} f$, and
    $f \circ \text{optB}\,(p) = \text{optB}\,(f \circ p) \circ \text{fmap}^{\textbf{Opt}} f$ (property from slide 8)

Summary so far

Filterable functors can be defined via `filter`, `flatten`, or `fmapOpt`

- All three are computationally *equivalent* but have different roles:
    - ▶ The easiest to use in program code is `filter` / `withFilter`
    - ▶ The easiest type signature to implement is `flatten`
    - ▶ The easiest to use for checking laws is `fmapOpt`
- The easiest way to derive the laws is to *begin* with simpler laws
- * The 2 laws for `fmapOpt` are functor laws with a Kleisli "twist"
- Category theory accommodates this via a generalized definition of functors as liftings between "twisted" function types. Compare:
    - ▶ fmap : $(A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$ – ordinary container ("endofunctor")
    - ▶ contrafmap : $(B \Rightarrow A) \Rightarrow F^A \Rightarrow F^B$ – lifting from reversed functions
    - ▶ fmapOpt : $(A \Rightarrow 1 + B) \Rightarrow F^A \Rightarrow F^B$ – lifting from Kleisli$_{\mathsf{Opt}}$-functions
- CT gives us an intuition: prefer type signatures that resemble "lifting"
    - ▶ but CT is abstract, does not directly deliver a good formulation of laws
    - ▶ CT gives us no help with derivations when we struggle with the laws

Structure of filterable functors

Intuition from `flatten`: reshuffle data in F^A after replacing some A's by 1

- "reshuffling" means reusing different parts of a disjunction

Construction of exponential-polynomial filterable functors

1. $F^A = Z$ (constant functor) for a fixed type Z (define fmapOpt $f =$ id)
   - Note: $F^A = A$ (identity functor) is *not* filterable
2. $F^A \equiv G^A \times H^A$ for any filterable functors G^A and H^A
3. $F^A \equiv G^A + H^A$ for any filterable functors G^A and H^A
4. $F^A \equiv G^{H^A}$ for *any* functor G^A and filterable functor H^A
5. $F^A \equiv 1 + A \times G^A$ for a filterable functor G^A
   - Note: *pointed* types P are isomorphic to $1 + Z$ for some type Z
     - Example of non-trivial pointed type: $A \Rightarrow A$
     - Example of non-pointed type: $A \Rightarrow B$ when A is different from B
   - So $F^A \equiv P + A \times G^A$ where P is a pointed type and G^A is filterable
   - Also have $F^A \equiv P + A \times A \times ... \times A \times G^A$ similarly
6. $F^A \equiv G^A + A \times F^A$ (recursive) for a filterable functor G^A
7. $F^A \equiv G^A \Rightarrow H^A$ if contrafunctor G^A and functor H^A *both filterable*
   - Note: the functor $F^A \equiv G^A \Rightarrow A$ is not filterable

* Worked examples II: Constructions of filterable functors I

(2) The `fmapOpt` laws hold for $F^A \times G^A$ if they hold for F^A and G^A

- For $f^{A \Rightarrow 1+B}$, get $\mathrm{fmapOpt}_F(f) : F^A \Rightarrow F^B$ and $\mathrm{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\mathrm{fmapOpt}_{F \times G} f \equiv p^{F^A} \times q^{G^A} \Rightarrow \mathrm{fmapOpt}_F(f)(p) \times \mathrm{fmapOpt}_G(f)(q)$
- Identity law: $f = \mathrm{id}_\diamond$, so $\mathrm{fmapOpt}_F f = \mathrm{id}$ and $\mathrm{fmapOpt}_G f = \mathrm{id}$
  - Hence we get $\mathrm{fmapOpt}_{F+G}(f)(p \times q) = \mathrm{id}(p) \times \mathrm{id}(q) = p \times q$
- Composition law:

$$
\begin{aligned}
&(\mathrm{fmapOpt}_{F \times G} f_1 \circ \mathrm{fmapOpt}_{F+G} f_2)(p \times q) \\
&= \mathrm{fmapOpt}_{F \times G}(f_2)\left(\mathrm{fmapOpt}_F(f_1)(p) \times \mathrm{fmapOpt}_G(f_1)(q)\right) \\
&= (\mathrm{fmapOpt}_F f_1 \circ \mathrm{fmapOpt}_F f_2)(p) \times (\mathrm{fmapOpt}_G f_1 \circ \mathrm{fmapOpt}_G f_2)(q) \\
&= \mathrm{fmapOpt}_F(f_1 \diamond f_2)(p) \times \mathrm{fmapOpt}_G(f_1 \diamond f_2)(q) \\
&= \mathrm{fmapOpt}_{F \times G}(f_1 \diamond f_2)(p \times q)
\end{aligned}
$$

- Exactly the same proof as that for functor property for $F^A \times G^A$
  - this is because `fmapOpt` corresponds to a generalized functor
- New proofs are necessary only when using non-filterable functors
  - these are used in constructions $4 - 6$

* Worked examples II: Constructions of filterable functors II

(5) The `fmapOpt` laws hold for $F^A \equiv 1 + A \times G^A$ if they hold for G^A

- For $f^{A \Rightarrow 1+B}$, get $\mathsf{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\mathsf{fmapOpt}_F(f)(1 + a^A \times q^{G^A})$ by returning $0 + b \times \mathsf{fmapOpt}_G(f)(q)$ if the argument is $0 + a \times q$ and $f(a) = 0 + b$, and returning $1 + 0$ otherwise
- Identity law: $f = \mathsf{id}_\diamond$, so $f(a) = 0 + a$ and $\mathsf{fmapOpt}_G f = \mathsf{id}$
  - Hence we get $\mathsf{fmapOpt}_F(\mathsf{id}_\diamond)(1 + a \times q) = 1 + a \times q$
- Composition law: need only to check for arguments $0 + a \times q$, and only when $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, in which case $(f_1 \diamond f_2)(a) = 0 + c$; then

$$(\mathsf{fmapOpt}_F\, f_1 \circ \mathsf{fmapOpt}_F\, f_2)(0 + a \times q)$$
$$= \mathsf{fmapOpt}_F(f_2)\,(\mathsf{fmapOpt}_F(f_1)(0 + a \times q))$$
$$= \mathsf{fmapOpt}_F(f_2)\,(0 + b \times \mathsf{fmapOpt}_G(f_1)(q))$$
$$= 0 + c \times (\mathsf{fmapOpt}_G\, f_1 \circ \mathsf{fmapOpt}_G\, f_2)(q)$$
$$= 0 + c \times \mathsf{fmapOpt}_G(f_1 \diamond f_2)(q)$$
$$= \mathsf{fmapOpt}_F(f_1 \diamond f_2)(0 + a \times q)$$

This is a "greedy filter": if $f(a)$ is empty, will delete all data in G^A

* Worked examples II: Constructions of filterable functors III

(6) The `fmapOpt` laws hold for $F^A \equiv G^A + A \times F^A$ if they hold for G^A

- For $f^{A \Rightarrow 1+B}$, get $\mathrm{fmapOpt}_G(f) : G^A \Rightarrow G^B$ and $\mathrm{fmapOpt}'_F(f) : F^A \Rightarrow F^B$ (for use in recursive arguments as the inductive assumption)

- Define $\mathrm{fmapOpt}_F(f)(q^{G^A} + a^A \times p^{F^A})$ by returning $0 + \mathrm{fmapOpt}'_F(f)(p)$ if $f(a) = 1 + 0$, and $\mathrm{fmapOpt}_G(f)(q) + b \times \mathrm{fmapOpt}'_F(f)(p)$ otherwise

- Identity law: $f(a) = \mathrm{id}_\diamond(a) \neq 1 + 0$, so $\mathrm{fmapOpt}_F(\mathrm{id}_\diamond)(q + a \times p) = q + a \times p$

- Composition law:
  $(\mathrm{fmapOpt}_F(f_1) \circ \mathrm{fmapOpt}_F(f_2))(q + a \times p) = \mathrm{fmapOpt}_F(f_1 \diamond f_2)(q + a \times p)$

- For arguments $q + 0$, the laws for G^A hold; so assume arguments $0 + a \times p$. When $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, the proof of the previous example will go through. So we need to consider the two cases $f_1(a) = 1 + 0$ and $f_1(a) = 0 + b$, $f_2(b) = 1 + 0$

- If $f_1(a) = 1 + 0$ then $(f_1 \diamond f_2)(a) = 1 + 0$; to show $\mathrm{fmapOpt}'_F(f_2)(\mathrm{fmapOpt}'_F(f_1)(p))$ $= \mathrm{fmapOpt}'_F(f_1 \diamond f_2)(p)$, use the inductive assumption about $\mathrm{fmapOpt}'_F$ on p

- If $f_1(a) = 0 + b$ and $f_2(b) = 1 + 0$ then $(f_1 \diamond f_2)(a) = 1 + 0$; to show $\mathrm{fmapOpt}_F(f_2)(0 + b \times \mathrm{fmapOpt}'_F(f_1)(p)) = \mathrm{fmapOpt}'_F(f_1 \diamond f_2)(p)$, rewrite $\mathrm{fmapOpt}_F(f_2)(0 + b \times \mathrm{fmapOpt}'_F(f_1)(p)) = \mathrm{fmapOpt}'_F(f_2)(\mathrm{fmapOpt}'_F(f_1)(p))$ and use the inductive assumption about $\mathrm{fmapOpt}'_F$ on p

This is a "list-like filter": if $f(a)$ is empty, will recurse into nested F^A data

Worked examples II: Constructions of filterable functors IV

Use known filterable constructions to show that
$$F^A \equiv (\text{Int} \times \text{String}) \Rightarrow (1 + \text{Int} \times A + A \times (1 + A) + (\text{Int} \Rightarrow 1 + A + A \times A \times \text{String}))$$
is a filterable functor

- Instead of implementing `Filterable` and verifying laws by hand, we analyze the structure of this data type and use known constructions
- Define some auxiliary functors that are parts of the structure of F^A,
  - $R_1^A = (\text{Int} \times \text{String}) \Rightarrow A$ and $R_2^A = \text{Int} \Rightarrow A$
  - $G^A = 1 + \text{Int} \times A + A \times (1 + A)$ and $H^A = 1 + A + A \times A \times \text{String}$
- Now we can rewrite $F^A = R_1 \left[G^A + R_2 \left[H^A \right] \right]$
  - G^A is filterable by construction 5 because it is of the form
    $G^A = 1 + A \times K^A$ with filterable functor $K^A = 1 + \text{Int} + A$
  - K^A is of the form $1 + A + X$ with constant type X, so it is filterable by constructions 1 and 3 with the `Option` functor $1 + A$
  - H^A is filterable by construction 5 with $H^A = 1 + A \times (1 + A \times \text{String})$, while $1 + A \times \text{String}$ is filterable by constructions 5 and 1
- Constructions 3 and 4 show that $R_1 \left[G^A + R_2 \left[H^A \right] \right]$ is filterable

Note that there are more than one way of implementing `Filterable` here

* Exercises II

1. Implement a `Filterable` instance for `type F[T] = G[H[T]]` assuming that the functor `H[T]` already has a `Filterable` instance (construction 4). Verify the laws rigorously (i.e. by calculations, not tests).

2. For `type F[T] = Option[Int ⇒ Option[(T, T)]]`, implement a `Filterable` instance. Show that the filterable laws hold by using known filterable constructions (avoiding explicit proofs or tests).

3. Implement a `Filterable` instance for $F^A \equiv G^A + \text{Int} \times A \times A \times F^A$ (recursive) for a filterable functor G^A. Verify the laws rigorously.

4. Show that $F^A = 1 + A \times G^A$ is in general *not* filterable if G^A is an arbitrary (non-filterable) functor; it is enough to give an example.

5. Show that $F^A = 1 + G^A + H^A$ is filterable if $1 + G^A$ and $1 + H^A$ are filterable (even when G^A and H^A are by themselves not filterable).

Filterable contrafunctors I: The definition and the laws
When is a contrafunctor filterable?

When a contrafunctor C^A with $\texttt{contrafmap} : (B \Rightarrow A) \Rightarrow C^A \Rightarrow C^B$ has also

- $\texttt{filter}/\texttt{withFilter} : (A \Rightarrow \text{Boolean}) \Rightarrow C^A \Rightarrow C^A$ just like for functors
- $\texttt{inflate} : C^A \Rightarrow C^{1+A}$ and $\texttt{contrafmapOpt} : (B \Rightarrow 1 + A) \Rightarrow C^A \Rightarrow C^B$
- All three functions are computationally equivalent...
  - $\text{filter}(p^{A \Rightarrow \text{Boolean}}) = \text{inflate}^{C^A \Rightarrow C^{1+A}} \circ \text{contrafmap}(\text{optB}\,(p))$
  - $\text{inflate}^{C^A \Rightarrow C^{1+A}} = \text{contrafmap}\,(0 + x^A \Rightarrow x) \circ \text{filter}\,(\_ \Rightarrow \text{true})$
  - $\text{contrafmapOpt}\,f^{B \Rightarrow 1+A} = \text{inflate} \circ \text{contrafmap}\,f$
  - $\text{inflate} = \text{contrafmapOpt}\,(\text{id}^{1+A \Rightarrow 1+A})$
- but have different laws
  - 4 laws (naturality, conjunction, identity, partial function) for $\texttt{filter}$
  - 3 laws (naturality, conjunction, identity) for $\texttt{inflate}$
  - 2 laws (identity, contracomposition) for $\texttt{contrafmapOpt}$
    - ⋆ as before, $\texttt{contrafmapOpt}$ is a "twisted" version of $\texttt{contrafmap}$
- Examples of filterable contrafunctors
  - $C^A \equiv A \Rightarrow 1 + Z$ where Z is a fixed type
  - $C^A \equiv 1 + A \Rightarrow Z$
- Examples of non-filterable contrafunctors
  - $C^A \equiv A \times F^A \Rightarrow Z$ – cannot implement $\texttt{inflate}$

Filterable contrafunctors II: Their structure
How to build up a filterable contrafunctor from parts?

- The easiest function to consider first is `inflate`

Constructions of filterable contrafunctors:

1. $C^A = Z$ (constant contrafunctor)
   Functor constructions (no need to check laws for these):
2. $F^A \equiv G^A \times H^A$ for any filterable contrafunctor G^A and H^A
3. $F^A \equiv G^A + H^A$ for any filterable contrafunctor G^A and H^A
4. $F^A \equiv G^{H^A}$ for H^A a filterable (contra)functor and G^A any (contra)functor – various combinations possible here
5. $F^A \equiv G^A \Rightarrow H^A$ if functor G^A and contrafunctor H^A *both filterable*
   Special constructions:
6. $F^A \equiv 1 + A \times G^A \Rightarrow H^A$ where G^A and H^A are filterable
7. $F^A \equiv A \times G^A \Rightarrow 1 + H^A$ if G^A and H^A are filterable