Generating code with the Curry-Howard correspondence

Type inhabitation at compile time

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The code `val x: T = ...` shows that *we can compute a value* of type `T` as part of our program expression

- Let's denote this *proposition* by $\mathcal{CH}(T)$ – "$\mathcal{C}$ode $\mathcal{H}$as a value of type `T`"
- Correspondence between types and propositions, for a given program:

Type	Proposition	Short notation
`T`	$\mathcal{CH}(T)$	T
`(A, B)`	$\mathcal{CH}(A)$ *and* $\mathcal{CH}(B)$	$A \times B$, $A \wedge B$
`Either[A, B]`	$\mathcal{CH}(A)$ *or* $\mathcal{CH}(B)$	$A + B$, $A \vee B$
`A ⇒ B`	$\mathcal{CH}(A)$ *implies* $\mathcal{CH}(B)$	$A \Rightarrow B$
`Unit`	*True*	1
`Nothing`	*False*	0

- Type parameter `[T]` in a function type means $\forall T$
- Example: `def dupl[A]: A ⇒ (A, A)`. The type of this function corresponds to the (valid) theorem $\forall A : A \Rightarrow A \times A$

The CH correspondence: proposition→type / proof→code

- Any valid theorem can be implemented in code

Proposition	Code
$\forall A : A \Rightarrow A$	`def identity[A](x:A):A = x`
$\forall A : A \Rightarrow 1$	`def toUnit[A](x:A): Unit = ()`
$\forall A \forall B : A \Rightarrow A + B$	`def inLeft[A,B](x:A): Either[A,B] = Left(x)`
$\forall A \forall B : A \times B \Rightarrow A$	`def first[A,B](p:(A,B)):A = p._1`
$\forall A \forall B : A \Rightarrow (B \Rightarrow A)$	`def const[A,B](x:A):B⇒A = (y:B)⇒x`

- Non-theorems *cannot be implemented* in code
  - Examples of non-theorems:
    $\forall A : 1 \Rightarrow A$; $\forall A \forall B : A + B \Rightarrow A$;
    $\forall A \forall B : A \Rightarrow A \times B$; $\forall A \forall B : (A \Rightarrow B) \Rightarrow A$
- Given a type's formula, can we implement it in code?
  - Example: $\forall A \forall B : ((((A \Rightarrow B) \Rightarrow A) \Rightarrow A) \Rightarrow B) \Rightarrow B$
- Constructive (intuitionistic) propositional logic has a decision algorithm
- The curryhoward library implements the IPL prover in a Scala macro

Worked examples I

1. Implement `map` for the Reader monad,

$$\text{map} : (E \Rightarrow A) \Rightarrow (A \Rightarrow B) \Rightarrow (E \Rightarrow B)$$

2. Show that one cannot implement $(E \Rightarrow A) \Rightarrow (E \Rightarrow F) \Rightarrow (F \Rightarrow A)$
3. Implement `map[A,B]: Option[A] ⇒ (A ⇒ B) ⇒ Option[B]`

Often, there is only one useful implementation

The `curryhoward` library tries to generate that implementation automatically

Using the `curryhoward` library

Two main use cases:

1. Define a method and provide an automatic implementation

```
def map[E, A, B](readerA: E ⇒ A, f: A ⇒ B): E ⇒ B = implement
```

2. Automatically build an expression from previously computed values

```
val f: String ⇒ Boolean ⇒ Int = {...}
case class Result(x: Int, name: String)
val result = ofType[Result]("abc", f, true)
```

Features:

- Compile-time code generation via Scala macros
- Supports functions, tuples, sealed trait / case classes / case objects
- Constant types (`Int`, `String`, etc.) are treated as type parameters
- If several implementations are available, chooses “intelligently”

Worked examples II

Demo time

1. Implement `map: Option[A] ⇒ (A ⇒ B) ⇒ Option[B]` that satisfies the identity law: `map(opt)(x ⇒ x) = opt`
2. Show that one cannot implement $(E \Rightarrow A) \Rightarrow (E \Rightarrow F) \Rightarrow (F \Rightarrow A)$
3. Implement the distributive law

$$(A + B) \times C \Leftrightarrow A \times C + B \times C$$

   In Scala: `(Either[A, B], C) ⇔ Either[(A, C), (B, C)]`

4. Implement `point`, `map` and `flatMap` for the Reader and State monads

See test code

Proof search I: Gentzen's calculus LJ (1935)

- A "complete and sound calculus" is a set of axioms and derivation rules that will yield all (and only!) valid theorems of the logic

$$(X \text{ is atomic}) \frac{}{\Gamma, X \vdash X}\, Id \qquad \frac{}{\Gamma \vdash \top}\, \top$$

$$\frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\, L\Rightarrow \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\, R\Rightarrow$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\, L+ \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\, R+_i$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\, L\times_i \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\, R\times$$

- Sequents are nodes in the proof search tree
- Use these rules "bottom-up" to perform a proof search
- Example: $\emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q$

Proof search example I

Root sequent $S_0 : \emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q$

- S_0 with rule $R \Rightarrow$ yields $S_1 : (R \Rightarrow R) \Rightarrow Q \vdash Q$
- S_1 with rule $L \Rightarrow$ yields $S_2 : (R \Rightarrow R) \Rightarrow Q \vdash R \Rightarrow R$ and $S_3 : Q \vdash Q$
- Sequent S_3 follows from the *Id* axiom; it remains to prove S_2
- S_2 with rule $L \Rightarrow$ yields $S_4 : (R \Rightarrow R) \Rightarrow Q \vdash R \Rightarrow R$ and $S_5 : Q \vdash R \Rightarrow R$
  - We are stuck here because $S_4 = S_2$ (we are in a loop)
  - We can prove S_5, but that will not help
  - So we backtrack (erase S_4, S_5) and apply another rule to S_2
- S_2 with rule $R \Rightarrow$ yields $S_6 : (R \Rightarrow R) \Rightarrow Q; R \vdash R$
- Sequent S_6 follows from the *Id* axiom

Therefore we have proved S_0.
$Q.E.D.$

Proof search II: From deduction rules to code

- Proofs are the λ-calculus terms arising from deduction rules
- Proof of a sequent $A, B, C \vdash G \Leftrightarrow$ code/expression $g(a, b, c) : G$
- Each rule has a *proof transformer* function: $\mathrm{PT}_{R\Rightarrow}$, PT_{L+} , etc.
- Example: to prove S_0, start from S_6 backwards:

$$
\begin{array}{rcl}
S_6 : (R \Rightarrow R) \Rightarrow Q; R \vdash R & (\text{axiom } Id) & t_6(rrq, r) : R = r \\
S_2 : (R \Rightarrow R) \Rightarrow Q \vdash (R \Rightarrow R) & \mathrm{PT}_{R\Rightarrow}(t_6) & t_2(rrq) : (R \Rightarrow R) = (r \Rightarrow t_6(rrq, r)) \\
S_3 : Q \vdash Q & (\text{axiom } Id) & t_3(q) : Q = q \\
S_1 : (R \Rightarrow R) \Rightarrow Q \vdash Q & \mathrm{PT}_{L\Rightarrow}(t_2, t_3) & t_1(rrq) : Q = t_3(rrq(t_2(rrq))) \\
S_0 : \emptyset \vdash ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q & \mathrm{PT}_{R\Rightarrow}(t_1) & t_0 = (rrq \Rightarrow t_1(rrq))
\end{array}
$$

- Simplified final code (proof term) having the required type:

$$
t_0 : ((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q = (rrq \Rightarrow rrq\,(r \Rightarrow r))
$$

Proof search III: The calculus LJT

Vorobieff-Hudelmaier-Dyckhoff, 1950-1990

- The Gentzen calculus generates a loop if rule $L \Rightarrow$ is applied ≥ 2 times
- The calculus LJT keeps all rules of LJ except rule $L \Rightarrow$
- Replace rule $L \Rightarrow$ by pattern-matching on A in the premise $A \Rightarrow B$:

$$(X \text{ is atomic}) \; \frac{\Gamma, X, B \vdash D}{\Gamma, X, X \Rightarrow B \vdash D} \; L \Rightarrow_1$$

$$\frac{\Gamma, A \Rightarrow B \Rightarrow C \vdash D}{\Gamma, (A \wedge B) \Rightarrow C \vdash D} \; L \Rightarrow_2$$

$$\frac{\Gamma, A \Rightarrow C, B \Rightarrow C \vdash D}{\Gamma, (A \vee B) \Rightarrow C \vdash D} \; L \Rightarrow_3$$

$$\frac{\Gamma, B \Rightarrow C \vdash A \Rightarrow B \qquad \Gamma, C \vdash D}{\Gamma, (A \Rightarrow B) \Rightarrow C \vdash D} \; L \Rightarrow_4$$

- Rule $L \Rightarrow$ is based on the key theorem:

$$((A \Rightarrow B) \Rightarrow C) \Rightarrow (A \Rightarrow B) \Longleftrightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow B)$$

Proof search IV: The calculus LJT

"*It is obvious that it is obvious*" – a mathematician after thinking for a half-hour

- The key theorem for rule $L \Rightarrow$ is attributed to Vorobieff (1958):

be extracted from Lemma 7 in [22]. One could also go further and make subproofs sensible.

LEMMA 2. $\vdash_{\mathrm{LJ}} \Gamma, (C \supset D) \supset B \Rightarrow C \supset D$ *iff* $\vdash_{\mathrm{LJ}} \Gamma, D \supset B \Rightarrow C \supset D$.
PROOF. Trivial [34].

THEOREM 1. *The systems* LJ *and* LJT *are equivalent.*
PROOF. As noted earlier, it is routine to show that any sequent provable

[R. Dyckhoff, *Contraction-Free Sequent Calculi for Intuitionistic Logic*, 1992]

- A stepping stone to this theorem:

$$((A \Rightarrow B) \Rightarrow C) \Rightarrow B \Rightarrow C$$

Proof (*obviously* trivial): $f \Rightarrow b \Rightarrow f\ (\_ \Rightarrow b)$

Making practical use of the CH correspondence

Implications for actually writing code

What can we do now?

- Given a fully parametric type, decide whether it can be implemented in code ("type is inhabited"); if so, *generate* the code
- Let `curryhoward` fill in the code when it is trivial to do so

What problems cannot be solved with these tools?

- Automatically generate code satisfying properties (e.g. isomorphism)
  - The heuristics will help in some cases
- Express complicated conditions via types (e.g. "array is sorted")
  - Need dependent types for that (Coq, Agda, Idris, ...)

Title, Abstract, Bibliography

Generating code with the Curry-Howard correspondence: Type inhabitation at compile time

I implemented a library for compile-time code generation from Scala type signatures. The library uses (compile-time) reflection, the Curry-Howard correspondence, and a theorem prover for the constructive propositional logic. Using this library, I illustrate how the Curry-Howard correspondence maps types into propositions and proofs into code. I will also explain some details of the algorithm I used for automatic code generation from type signatures. As an illustration of using this library for automatic code generation, I demonstrate working examples such as implementing `map` and `flatMap` for the Reader and State monads.

- D. Galmiche, D. Larchey-Wendling – *Formulae-as-Resources Management for an Intuitionistic Theorem Prover* (1998). In 5th Workshop on Logic, Language, Information and Computation, WoLLIC'98, Sao Paulo.
- R. Dyckhoff – *Contraction-free sequent calculi for intuitionistic logic* (1992), The Journal of Symbolic Logic, Vol. 57, No. 3, (Sep., 1992), pp. 795-807.