

Chapter 7: Computations lifted to a functor context II. Monads

Part 2: Laws and structure of monads and semimonads

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Semimonad laws I: The intuitions

What properties of functor block programs do we expect to have?

- In $x \leftarrow c$, the value of x will *go over items* held in container c
- Manipulating items in container is followed by a generator:

$x \leftarrow \text{cont1}$	$y \leftarrow \text{cont1}$
$y = f(x)$	$\text{.map}(x \Rightarrow f(x))$
$z \leftarrow \text{cont2}(y)$	$z \leftarrow \text{cont2}(y)$

$\text{cont1.flatMap}(x \Rightarrow \text{cont2}(f(x))) = \text{cont1.map}(f).\text{flatMap}(y \Rightarrow \text{cont2}(y))$

- Manipulating items in container is preceded by a generator:

$x \leftarrow \text{cont1}$	$x \leftarrow \text{cont1}$
$y \leftarrow \text{cont2}(x)$	$z \leftarrow \text{cont2}(x)$
$z = f(y)$	$\text{.map}(f)$

$\text{cont1.flatMap}(\text{cont2}).\text{map}(f) = \text{cont1.flatMap}(x \Rightarrow \text{cont2}(x).\text{map}(f))$

- Within a generator, `for {...} yield` can be inlined:

$x \leftarrow \text{cont}$	$yy \leftarrow \text{for } \{ x \leftarrow \text{cont}$
$y \leftarrow p(x)$	$y \leftarrow p(x) \} \text{ yield } y$
$z \leftarrow \text{cont2}(y)$	$z \leftarrow \text{cont2}(yy)$

$\text{cont.flatMap}(x \Rightarrow p(x).\text{flatMap}(\text{cont2})) = \text{cont.flatMap}(p).\text{flatMap}(\text{cont2})$

Semimonad laws II: The laws for `flatMap`

For brevity, write `flm` instead of `flatMap`

A **semimonad** S^A has $\text{flm}^{[A,B]} : (A \Rightarrow S^B) \Rightarrow S^A \Rightarrow S^B$ with 3 laws:

- ❶ $\text{flm} (f^{A \Rightarrow B} \circ g^{B \Rightarrow S^C}) = \text{fmap } f \circ \text{flm } g$ (naturality in A)

$$\begin{array}{ccc} & S^B & \\ \text{fmap } f^{A \Rightarrow B} \nearrow & & \searrow \text{flm } g^{B \Rightarrow S^C} \\ S^A & \xRightarrow{\text{flm } (f^{A \Rightarrow B} \circ g^{B \Rightarrow S^C})} & S^C \end{array}$$

- ❷ $\text{flm} (f^{A \Rightarrow S^B} \circ \text{fmap } g^{B \Rightarrow C}) = \text{flm } f \circ \text{fmap } g$ (naturality in B)

$$\begin{array}{ccc} & S^B & \\ \text{flm } f^{A \Rightarrow S^B} \nearrow & & \searrow \text{fmap } g^{B \Rightarrow C} \\ S^A & \xRightarrow{\text{flm } (f^{A \Rightarrow S^B} \circ \text{fmap } g^{B \Rightarrow C})} & S^C \end{array}$$

- ❸ $\text{flm} (f^{A \Rightarrow S^B} \circ \text{flm } g^{B \Rightarrow S^C}) = \text{flm } f \circ \text{flm } g$ (associativity)

$$\begin{array}{ccc} & S^B & \\ \text{flm } f^{A \Rightarrow S^B} \nearrow & & \searrow \text{flm } g^{B \Rightarrow S^C} \\ S^A & \xRightarrow{\text{flm } (f^{A \Rightarrow S^B} \circ \text{flm } g^{B \Rightarrow S^C})} & S^C \end{array}$$

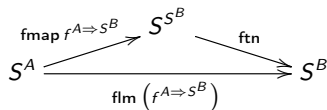
Is there a shorter and clearer formulation of these laws?

Semimonad laws III: The laws for `flatten`

The methods `flatten` (denoted by `ftn`) and `flatMap` are equivalent:

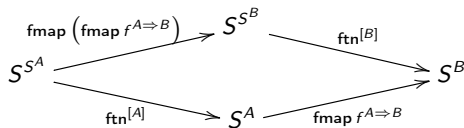
$$\text{ftn}^{[A]} : S^{S^A} \Rightarrow S^A \equiv \text{flm}^{[S^A, A]}(m^{S^A} \Rightarrow m)$$

$$\text{flm}(f^{A \Rightarrow S^B}) \equiv \text{fmap } f \circ \text{ftn}$$

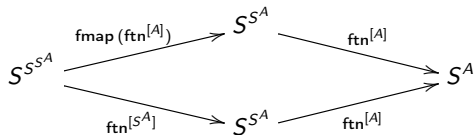


It turns out that `flatten` has only 2 laws:

- ❶ $\text{fmap}(\text{fmap } f^{A \Rightarrow B}) \circ \text{ftn}^{[B]} = \text{ftn}^{[A]} \circ \text{fmap } f$ (naturality)



- ❷ $\text{fmap}(\text{ftn}^{[A]}) \circ \text{ftn}^{[A]} = \text{ftn}^{[S^A]} \circ \text{ftn}^{[A]}$ (associativity)



Equivalence of a natural transformation and a “lifting”

- Equivalence of flm and ftn : $\text{ftn} = \text{flm}(\text{id})$; $\text{flm } f = \text{fmap } f \circ \text{ftn}$
- We saw this before: $\text{deflate} = \text{fmapOpt}(\text{id})$; $\text{fmapOpt } f = \text{fmap } f \circ \text{deflate}$
 - ▶ Is there a general pattern where two such functions are equivalent?
- Let $\text{tr} : F^{G^A} \Rightarrow F^A$ be a natural transformation (F and G are functors)
- Define $\text{ftr} : (A \Rightarrow G^B) \Rightarrow F^A \Rightarrow F^B$ by $\text{ftr } f = \text{fmap } f \circ \text{tr}$
- It follows that $\text{tr} = \text{ftr}(\text{id})$, and we have equivalence between tr and ftr :

$$\text{tr} : F^{G^A} \Rightarrow F^A = \text{ftr}(m^{G^A} \Rightarrow m)$$

$$\text{ftr}(f^{A \Rightarrow G^B}) = \text{fmap } f \circ \text{tr}$$

A commutative triangle diagram illustrating the relationship between the functors F^A , F^{G^B} , and F^B . The diagram consists of three nodes: F^A at the bottom-left, F^{G^B} at the top, and F^B at the bottom-right. An arrow labeled $\text{fmap } f^{A \Rightarrow G^B}$ points from F^A to F^{G^B} . An arrow labeled tr points from F^{G^B} to F^B . A long arrow labeled $\text{ftr}(f^{A \Rightarrow G^B})$ points directly from F^A to F^B , representing the composition of the other two arrows.

- An automatic law for ftr (“naturality in A ”) follows from the definition:
 $\text{fmap } g \circ \text{ftr } f = \text{fmap } g \circ \text{fmap } f \circ \text{tr} = \text{fmap}(g \circ f) \circ \text{tr} = \text{ftr}(g \circ f)$
 - ▶ This is why tr always has *one law fewer* than ftr
- To demonstrate equivalence in the direction $\text{ftr} \rightarrow \text{tr}$: Start with an arbitrary ftr satisfying “naturality in A ”, then obtain $\text{tr} = \text{ftr}(\text{id})$ from it, then verify $\text{ftr } f = \text{fmap } f \circ \text{tr}$ with that tr ; $\text{fmap } f \circ \text{ftr}(\text{id}) = \text{ftr}(f \circ \text{id}) = \text{ftr } f$

Semimonad laws IV: Deriving the laws for `flatten`

Denote for brevity $q^\uparrow \equiv \text{fmap } q$ for any function q (“lifting” $q^{A \Rightarrow B}$ to S)
Express $\text{flm } f = f^\uparrow \circ \text{ftn}$ and substitute that into flm ’s 3 laws:

- ❶ $\text{flm } (f \circ g) = f^\uparrow \circ \text{flm } g$ gives $(f \circ g)^\uparrow \circ \text{ftn} = f^\uparrow \circ g^\uparrow \circ \text{ftn}$
– this law holds automatically due to functor composition law
 - ❷ $\text{flm } (f \circ g^\uparrow) = \text{flm } f \circ g^\uparrow$ gives $(f \circ g^\uparrow)^\uparrow \circ \text{ftn} = f^\uparrow \circ \text{ftn} \circ g^\uparrow$;
using the functor composition law, we reduce this to
 $g^{\uparrow\uparrow} \circ \text{ftn} = \text{ftn} \circ g^\uparrow$ – this is the naturality law
 - ❸ $\text{flm } (f \circ \text{flm } g) = \text{flm } f \circ \text{flm } g$ with functor composition law gives
 $f^\uparrow \circ g^{\uparrow\uparrow} \circ \text{ftn}^\uparrow \circ \text{ftn} = f^\uparrow \circ \text{ftn} \circ g^\uparrow \circ \text{ftn}$; using ftn ’s naturality and omitting
the common factor $f^\uparrow \circ g^{\uparrow\uparrow}$, we get $\text{ftn}^\uparrow \circ \text{ftn} = \text{ftn} \circ \text{ftn}$ – associativity law
- `flatten` has the simplest type signature *and* the fewest laws
 - It is usually easy to check naturality!
 - ▶ **Parametricity theorem:** Any *pure, fully parametric* code for a function of type $F^A \Rightarrow G^A$ will implement a natural transformation
 - Checking `flatten`’s associativity needs *a lot* more work!

The `cats` library has a `FlatMap` type class, defining `flatten` via `flatMap`

Checking the associativity law for standard monads

- Implement `flatten` for these functors and check the laws (see code):
 - ▶ `Option` monad: $F^A \equiv 1 + A$; $\text{ftn} : 1 + (1 + A) \Rightarrow 1 + A$
 - ▶ `Either` monad: $F^A \equiv Z + A$; $\text{ftn} : Z + (Z + A) \Rightarrow Z + A$
 - ▶ `List` monad: $F^A \equiv \text{List}^A$; $\text{ftn} : \text{List}^{\text{List}^A} \Rightarrow \text{List}^A$
 - ▶ `Writer` monad: $F^A \equiv A \times W$; $\text{ftn} : (A \times W) \times W \Rightarrow A \times W$
 - ▶ `Reader` monad: $F^A \equiv R \Rightarrow A$; $\text{ftn} : (R \Rightarrow (R \Rightarrow A)) \Rightarrow R \Rightarrow A$
 - ▶ `State`: $F^A \equiv S \Rightarrow A \times S$; $\text{ftn} : (S \Rightarrow (S \Rightarrow A \times S)) \times S \Rightarrow S \Rightarrow A \times S$
 - ▶ `Continuation` monad: $F^A \equiv (A \Rightarrow R) \Rightarrow R$;
 $\text{ftn} : (((A \Rightarrow R) \Rightarrow R) \Rightarrow R) \Rightarrow (A \Rightarrow R) \Rightarrow R$
- Code implementing these `flatten` functions is *fully parametric* in A
 - ▶ Naturality of these functions follows from parametricity theorem
 - ▶ Associativity needs to be checked for each monad!
- Example of a useful semimonad that is *not* a full monad:
 - ▶ $F^A \equiv A \times V \times W$; $\text{ftn}((a \times v_1 \times w_1) \times v_2 \times w_2) = a \times v_1 \times w_2$
- Examples of *non-associative* (i.e. wrong) implementations of `flatten`:
 - ▶ $F^A \equiv A \times W \times W$; $\text{ftn}((a \times v_1 \times v_2) \times w_1 \times w_2) = a \times w_2 \times w_1$
 - ▶ $F^A \equiv \text{List}^A$, but `flatten` concatenates the nested lists in reverse order

Motivation for monads

- Monads represent values with a “special computational context”
- Specific monads will have methods to create various contexts
- Monadic composition will “combine” the contexts associatively
- It is generally useful to have an “empty context” available:

$$\text{pure} : A \Rightarrow M^A$$

Adding the empty context to another context should be a no-op

- Empty context is followed by a generator:

`y ← pure(x)`

`y = x`

`z ← cont(y)`

`z ← cont(y)`

`pure(x).flatMap(y ⇒ cont(y)) = cont(x)`

`pure ∘ flm f = f` – left identity

- Empty context is preceded by a generator:

`x ← cont`

`x ← cont`

`y ← pure(x)`

`y = x`

`cont.flatMap(x ⇒ pure(x)) = cont`

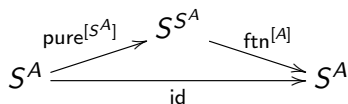
`flm (pure) = id` – right identity

The monad laws formulated in terms of `pure` and `flatten`

- Naturality law for `pure`: $f \circ \text{pure} = \text{pure} \circ \text{fmap } f$



- Left identity: $\text{pure} \circ \text{flm } f = \text{pure} \circ \text{fmap } f \circ \text{ftn} = f \circ \text{pure} \circ \text{ftn} = f$
requires that $\text{pure} \circ \text{ftn} = \text{id}$ (both sides applied to S^A)



- Right identity: $\text{flm}(\text{pure}) = \text{fmap}(\text{pure}) \circ \text{ftn} = \text{id}^{S^A \Rightarrow S^A}$



Formulating laws via Kleisli functions

- Recall: we formulated the laws of filterables via `fmapOpt`
 - type signature of `fmapOpt` : $(A \Rightarrow 1 + B) \Rightarrow S^A \Rightarrow S^B$
 - and then we had to compose functions of types $A \Rightarrow 1 + B$ via \diamond_{Opt}
- Here we have `flm` : $(A \Rightarrow S^B) \Rightarrow S^A \Rightarrow S^B$ instead of `fmapOpt`
- Can we compose **Kleisli functions** with “twisted” types $A \Rightarrow S^B$?
- Use `flm` to define **Kleisli composition**: $f^{A \Rightarrow S^B} \diamond g^{B \Rightarrow S^C} \equiv f \circ \text{flm } g$
- Define **Kleisli identity** id_\diamond of type $A \Rightarrow S^A$ as $\text{id}_\diamond \equiv \text{pure}$
- Composition law: $\text{flm } (f \diamond g) = \text{flm } f \circ \text{flm } g$ (same as for `fmapOpt`)
 - Shows that `flatMap` is a “lifting” of $A \Rightarrow S^B$ to $S^A \Rightarrow S^B$
- These laws are similar to functor “lifting” laws...
 - except that \diamond is used for composing Kleisli functions
- What are the properties of \diamond ?
 - Exactly similar to the properties of function composition $f \circ g$

Reformulate `flm`’s laws in terms of the \diamond operation:

- `flm`’s left and right identity laws: $\text{pure} \diamond f = f$ and $f \diamond \text{pure} = f$
- Associativity law: $(f \diamond g) \diamond h = f \diamond (g \diamond h)$
 - Follows from the `flm` law: $f \circ \text{flm } (g \circ \text{flm } h) = f \circ \text{flm } g \circ \text{flm } h$

* Motivation for categories and functors

Compare different “liftings” seen so far, and generalize

Category	Type $A \rightsquigarrow B$	Identity	Composition
plain functions	$A \Rightarrow B$	$\text{id} : A \Rightarrow A$	$f^{A \Rightarrow B} \circ g^{B \Rightarrow C}$
lifted to F	$F^A \Rightarrow F^B$	$\text{id} : F^A \Rightarrow F^A$	$f^{F^A \Rightarrow F^B} \circ g^{F^B \Rightarrow F^C}$
Kleisli over F	$A \Rightarrow F^B$	$\text{pure} : A \Rightarrow F^A$	$f^{A \Rightarrow F^B} \diamond g^{B \Rightarrow F^C}$

Category theory generalizes this situation

Category: a certain class of “twisted functions” $A \rightsquigarrow B$ called **morphisms**

- For any two morphisms $f^{A \rightsquigarrow B}$ and $g^{B \rightsquigarrow C}$ the **composition** morphism $f \diamond g$ of type $A \rightsquigarrow C$ must exist
- For each type A , the **identity** morphism id_\diamond of type $A \rightsquigarrow A$ must exist
- Composition respects identity: $\text{id}_\diamond \diamond f = f$ and $f \diamond \text{id}_\diamond = f$
- Composition is associative: $(f \diamond g) \diamond h = f \diamond (g \diamond h)$

General **functor**: a map from one category to another

- A functor must **fmap** each morphism from one category to the other
- Functor laws: **fmap** must preserve identity and composition
 - ▶ What we call “functor” is called **endofunctor** in category theory
 - ▶ An endofunctor’s **fmap** goes from plain functions to F -lifted functions

* From Kleisli back to `flatMap`

The Kleisli functions, $A \rightsquigarrow B \equiv A \Rightarrow S^B$, form a category iff S is a monad

- `fmap` and `flatMap` are computationally equivalent to Kleisli composition:
 - ▶ Define `flatMap` through Kleisli: $\text{flm } f^{A \Rightarrow S^B} \equiv \text{id}^{S^A \Rightarrow S^A} \diamond f$
 - ▶ Require two additional laws that connect \diamond , `fmap`, and \circ :
 - ★ Left naturality: $f^{A \Rightarrow B} \circ g^{B \Rightarrow S^C} = (f \circ \text{pure}) \diamond g$
 - ★ Right naturality: $f^{A \Rightarrow S^B} \circ \text{fmap } g^{B \Rightarrow C} = f \diamond (g \circ \text{pure})$
 - ▶ So, can define `fmap` through Kleisli: $\text{fmap } g^{A \Rightarrow B} \equiv \text{id}^{S^A \Rightarrow S^A} \diamond (g \circ \text{pure})$

The laws for `pure` and `flatMap` then follow from category axioms for Kleisli:

- Left and right identity laws follow from $\text{id} \diamond \text{pure} = \text{id}$ and $\text{pure} \diamond f = f$
- Associativity for `flatMap` follows from $(\text{id} \diamond f) \diamond g = \text{id} \diamond (f \diamond g)$
- Use “left naturality”, get: $(f \circ g) \diamond h = (f \circ \text{pure}) \diamond g \diamond h = f \circ (g \diamond h)$
- Naturality for `pure`: $\text{pure} \circ \text{fmap } f = \text{pure} \diamond (f \circ \text{pure}) = f \circ \text{pure}$
- Define `flatten`: $\text{ftn} = \text{id}^{S^{S^A} \Rightarrow S^{S^A}} \diamond \text{id}^{S^A \Rightarrow S^A}$
- Naturality for `flatten`: $\text{ftn} \circ \text{fmap } f = \text{id} \diamond \text{id} \diamond (f \circ \text{pure}) = \text{id} \diamond \text{fmap } f$
and $\text{fmap}(\text{fmap } f) \circ \text{ftn} = \text{id} \diamond ((\text{fmap } f) \circ \text{pure}) \circ \text{id} \diamond \text{id} = \text{id} \diamond \text{fmap } f$

Structure of semigroups and monoids

- Semimonad contexts are combined associatively, as in a semigroup
 - ▶ A full monad includes an “empty” context, i.e. the identity element
 - ▶ Semigroup with an identity element is a monoid

Some constructions of semigroups and monoids (see code):

- 1 Any type Z is a semigroup with operation $z_1 \circledast z_2 = z_1$ (or z_2)
 - 2 $1 + S$ is a monoid if S is (at least) a semigroup (or $S \equiv 0$)
 - 3 List^A is a monoid (for any type A), also Seq^A etc.
 - 4 The function type $A \Rightarrow A$ is a monoid (for any type A)
 - ▶ The operation $f \circledast g$ can be either $f \circ g$ or $g \circ f$
 - 5 Any totally ordered type is a monoid, with \circledast defined as \max or \min
 - 6 $S_1 \times S_2$ is a semigroup (monoid) if S_1, S_2 are semigroups (monoids)
 - 7 $S_1 + S_2$ is a semigroup (monoid) if S_1, S_2 are semigroups (monoids)
 - 8 $\mathbf{M}[S]$ is a monoid if $\mathbf{M}[_]$ is a monad and S is a monoid.
 - 9 $S \times P$ is a semigroup if S is a semigroup that has an **action on** P .
 - ▶ The “action” is $\alpha : S \Rightarrow P \Rightarrow P$ such that $\alpha(s_2) \circ \alpha(s_1) = \alpha(s_1 \circledast s_2)$.
 - ▶ $S \times P$ is a “twisted product.” Examples: $(A \Rightarrow A) \times A$; $\text{Bool} \times (1 + A)$.
- Other examples of monoids: Int (many), String , Set^A , Akka's [Route](#)

Structure of (semi)monads

How to recognize a (semi)monad by its type? Open question!

Intuition from `flatten`: reshuffle data in F^{F^A} to fit into F^A

Some constructions of exponential-polynomial (semi)monads:

- ① $F^A \equiv Z$ (constant functor) for a fixed type Z
 - ▶ For a full monad, need to choose $Z = 1$
- ② $F^A \equiv A \times G^A$ for any functor G^A (a full monad only if $G^A \equiv 1$)
- ③ $F^A \equiv G^A \times H^A$ for any (semi)monads G^A and H^A
 - ▶ but $G^A + H^A$ is generally *not* a semimonad
- ④ $F^A \equiv R \Rightarrow G^A$ is a (semi)monad for any (semi)monad G^A
- ⑤ $F^A \equiv A + G^A$ is a monad for a monad G^A (**free pointed** over G)
- ⑥ $F^A \equiv G^{Z+A \times W}$ is a monad if G is a monad and W a monoid
- ⑦ $F^A \equiv A + G^{F^A}$ (recursive) for any functor G^A (**free monad** over G)

Semimonad-only constructions:

- ⑧ $F^A \equiv G^A + G^{F^A}$ (recursive) for any functor G^A
- ⑨ $F^A \equiv H^A \Rightarrow A \times G^A$ for any contrafunctor H^A and functor G^A
 - ▶ Obtain a full monad only when $G^A \equiv 1$, i.e. $F^A \equiv H^A \Rightarrow A$

Exercises II

- 1 Show that $M[S]$ is a monoid if $M[_]$ is a monad and S is a monoid.
- 2 A framework implements a “route” type R as $R \equiv Q \Rightarrow (E + S)$, where Q is a query, E is an error response, and S is a success response. A server is defined as a “sum” of several routes. For a given query Q , the response is the first route (if it exists) that yields a success. Implement the route “summation” operation and show that it makes R into a semigroup. What would be necessary to make R into a monoid?
- 3 Verify the associativity law for the semimonad $F^A \equiv Z + \text{Bool} \times A$.
- 4 Show that the functor $F^A \equiv \text{Boolean} \times M^A$ (where M^A is an arbitrary monad) can be made into a semimonad but not into a monad.
- 5 If W and R are arbitrary fixed types, which of the functors can be made into a semimonad: $F^A \equiv W \times (R \Rightarrow A)$, $G^A \equiv R \Rightarrow (W \times A)$?
- 6 Show that $F^A \equiv (P \Rightarrow A) + (Q \Rightarrow A)$ is not a semimonad (cannot define `flatMap`) when P and Q are arbitrary, different types.
- 7 Implement the `flatten` and `pure` methods for $D^A \equiv 1 + A \times A$ (`type D[A] = Option[(A, A)]`) in at least two significantly different ways, and show that the monad laws always fail to hold. (D^A is not a monad!)

Exercises II (continued)

- 8 A programmer implemented the `fmap` method for $F^A \equiv A \times (A \Rightarrow Z)$ as
- ```
def fmap[A,B](f: A⇒B): ((A, A⇒Z)) ⇒ (B, B⇒Z) =
 { case (a, az) ⇒ (f(a), (λ_: B) ⇒ az(a)) }
```

Show that this implementation fails to satisfy the functor laws.

- 9 Show that  $P^A \equiv Z + W \times A$  is a (full) monad if  $W$  is a monoid.
- 10 Verify that the full monad laws hold for construction 4.
- 11 Implement `flatten` and `pure` for  $F^A \equiv A + (R \Rightarrow A)$ , where  $R$  is a fixed type, and show that all the monad laws hold.
- 12 For construction 5, show that an identity law would fail if `pure` were defined as `a ⇒ Right(Monad[G].pure(a))` instead of as `Left(a)`.
- 13 Implement the monad methods for  $F^A \equiv (Z \Rightarrow 1 + A) \times \text{List}^A$  using the known monad constructions (no need to check the laws).
- 14 Implement the semimonad construction 2 by discarding the first effect (not the second), and show that the associativity law is still satisfied.
- 15 For semimonad construction 8, show that the associativity law holds.
- 16 Verify the identity laws for the State and Continuation monads.



## Addendum: Miscellaneous remarks

- A non-empty list  $F^A \equiv A \times \text{List}^A$  is a semigroup but not a monoid.
- Contrafunctors cannot be monads or semimonads because if  $H^A$  is a contrafunctor then  $H^{H^A}$  is a *functor*, so a natural transformation between  $H^{H^A}$  and  $H^A$  (in either direction) is impossible.
- Any exponential-polynomial contrafunctor  $H^A$  is equivalent to  $H^A \equiv F^A \Rightarrow Z$  for some exp-poly functor  $F^A$  and some fixed type  $Z$ .