Chapter 7: Computations lifted to a functor context II. Monads

Part 2: Laws and structure of semimonads

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Semimonad laws I: The intuitions

What properties of functor block programs do we expect to have?

- In `x ← c`, the value of `x` will *go over items* held in container `c`
- Manipulating items in container is followed by a generator:

```
x ← cont1                    y ← cont1
y = f(x)                          .map(x ⇒ f(x))
z ← cont2(y)                 z ← cont2(y)
```

```
cont1.flatMap(x ⇒ cont2(f(x))) = cont1.map(f).flatMap(y ⇒ cont2(y))
```

- Manipulating items in container is preceded by a generator:

```
x ← cont1                    x ← cont1
y ← cont2(x)                 z ← cont2(x)
z = f(y)                          .map(f)
```

```
cont1.flatMap(cont2).map(f) = cont1.flatMap(x ⇒ cont2(x).map(f))
```

- Within a generator, `for {...} yield` can be inlined:

```
x ← cont                     yy ← for { x ← cont
y ← p(x)                              y ← p(x) } yield y
z ← cont2(y)                 z ← cont2(yy)
```

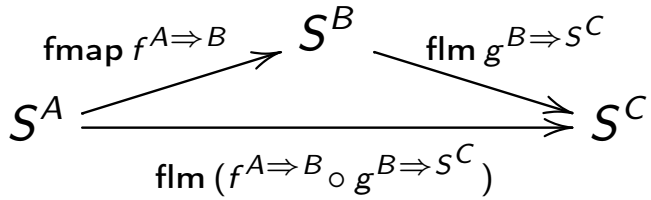
```
cont.flatMap(x ⇒ p(x).flatMap(cont2)) = cont.flatMap(p).flatMap(cont2)
```

Semimonad laws II: The laws for `flatMap`

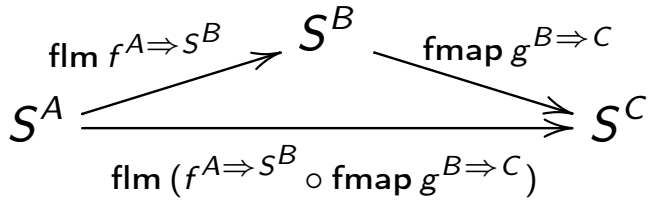
For brevity, write flm instead of `flatMap`

A **semimonad** S^A has $\text{flm}^{[A,B]} : (A \Rightarrow S^B) \Rightarrow S^A \Rightarrow S^B$ with 3 laws:

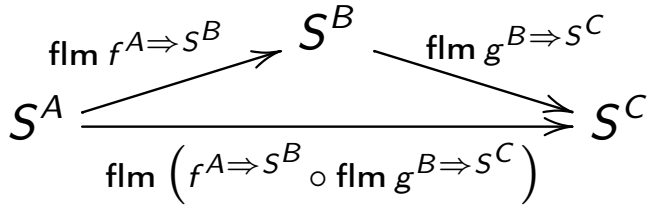
1. $\text{flm}\,(f^{A\Rightarrow B} \circ g^{B\Rightarrow S^C}) = \text{fmap}\, f \circ \text{flm}\, g$ (naturality in A)



2. $\text{flm}\,(f^{A\Rightarrow S^B} \circ \text{fmap}\, g^{B\Rightarrow C}) = \text{flm}\, f \circ \text{fmap}\, g$ (naturality in B)



3. $\text{flm}\,(f^{A\Rightarrow S^B} \circ \text{flm}\, g^{B\Rightarrow S^C}) = \text{flm}\, f \circ \text{flm}\, g$ (associativity)



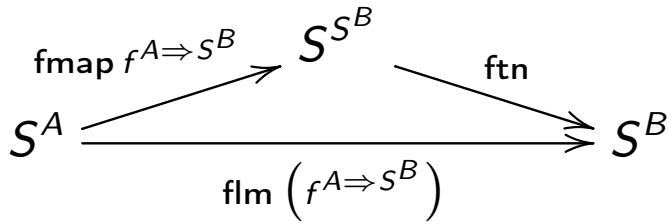
Is there a shorter and clearer formulation of these laws?

Semimonad laws III: The laws for `flatten`

The methods `flatten` (denoted by ftn) and `flatMap` are equivalent:

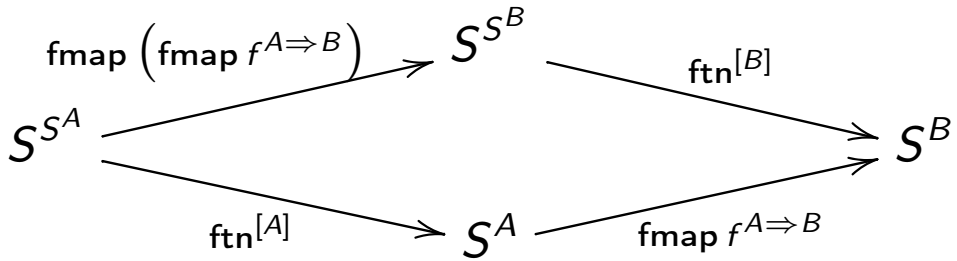
$$\text{ftn}^{[A]} : S^{S^A} \Rightarrow S^A \equiv \text{flm}^{[S^A, A]}(m^{S^A} \Rightarrow m)$$

$$\text{flm}\left(f^{A \Rightarrow S^B}\right) \equiv \text{fmap}\, f \circ \text{ftn}$$

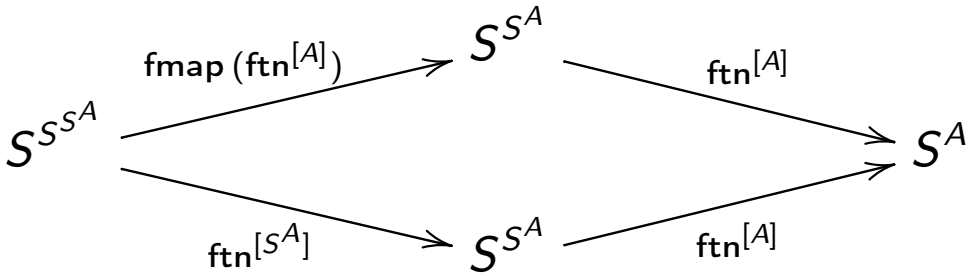


It turns out that `flatten` has only 2 laws:

1. $\text{fmap}\left(\text{fmap}\, f^{A \Rightarrow B}\right) \circ \text{ftn}^{[B]} = \text{ftn}^{[A]} \circ \text{fmap}\, f$ (naturality)



2. $\text{fmap}\left(\text{ftn}^{[A]}\right) \circ \text{ftn}^{[A]} = \text{ftn}^{[S^A]} \circ \text{ftn}^{[A]}$ (associativity)

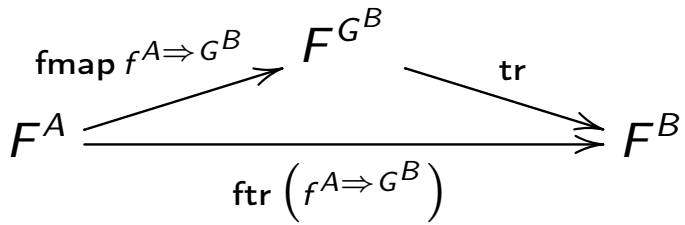


Equivalence of a natural transformation and a "lifting"

- Equivalence of flm and ftn: $\text{ftn} = \text{flm}\,(\text{id})$; $\text{flm}\,f = \text{fmap}\,f \circ \text{ftn}$
- We saw this before: $\text{deflate} = \text{fmapOpt}\,(\text{id})$; $\text{fmapOpt}\,f = \text{fmap}\,f \circ \text{deflate}$
  - Is there a general pattern where two such functions are equivalent?
- Let $\text{tr} : F^{G^A} \Rightarrow F^A$ be a natural transformation (F and G are functors)
- Define $\text{ftr} : (A \Rightarrow G^B) \Rightarrow F^A \Rightarrow F^B$ by $\text{ftr}\,f = \text{fmap}\,f \circ \text{tr}$
- It follows that $\text{tr} = \text{ftr}\,(\text{id})$, and we have equivalence between tr and ftr:

$$\text{tr} : F^{G^A} \Rightarrow F^A = \text{ftr}(m^{G^A} \Rightarrow m)$$

$$\text{ftr}\left(f^{A \Rightarrow G^B}\right) = \text{fmap}\,f \circ \text{tr}$$



- An automatic law for ftr ("naturality in A") follows from the definition:
  $\text{fmap}\,g \circ \text{ftr}\,f = \text{fmap}\,g \circ \text{fmap}\,f \circ \text{tr} = \text{fmap}\,(g \circ f) \circ \text{tr} = \text{ftr}\,(g \circ f)$
  - This is why tr always has *one law fewer* than ftr
- To demonstrate equivalence in the direction ftr → tr: Start with an arbitrary ftr satisfying "naturality in A", then obtain $\text{tr} = \text{ftr}\,(\text{id})$ from it, then verify $\text{ftr}\,f = \text{fmap}\,f \circ \text{tr}$ with that tr; $\text{fmap}\,f \circ \text{ftr}\,(\text{id}) = \text{ftr}\,(f \circ \mathit{id}) = \text{ftr}\,f$

Semimonad laws IV: Deriving the laws for `flatten`

Denote for brevity $q^{\uparrow} \equiv \text{fmap}\, q$ for any function q ("lifting" $q^{A \Rightarrow B}$ to S)
Express $\text{flm}\, f = f^{\uparrow} \circ \text{ftn}$ and substitute that into flm's 3 laws:

1. $\text{flm}\,(f \circ g) = f^{\uparrow} \circ \text{flm}\, g$ gives $(f \circ g)^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ g^{\uparrow} \circ \text{ftn}$
   – this law holds automatically due to functor composition law
2. $\text{flm}\left(f \circ g^{\uparrow}\right) = \text{flm}\, f \circ g^{\uparrow}$ gives $(f \circ h)^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ \text{ftn} \circ h$;
   using the functor composition law, we reduce this to
   $h^{\uparrow} \circ \text{ftn} = \text{ftn} \circ h$ – this is the naturality law
3. $\text{flm}\,(f \circ \text{flm}\, g) = \text{flm}\, f \circ \text{flm}\, g$ with functor composition law gives
   $f^{\uparrow} \circ g^{\uparrow\uparrow} \circ \text{ftn}^{\uparrow} \circ \text{ftn} = f^{\uparrow} \circ \text{ftn} \circ g^{\uparrow} \circ \text{ftn}$; using ftn's naturality and omitting the common factor $f^{\uparrow} \circ g^{\uparrow\uparrow}$, we get $\text{ftn}^{\uparrow} \circ \text{ftn} = \text{ftn} \circ \text{ftn}$ – associativity law

- `flatten` has the simplest type signature *and* the fewest laws
- It is usually easy to check naturality!
  - **Parametricity theorem**: Any *pure, fully parametric* code for a function of type $F^A \Rightarrow G^A$ will implement a natural transformation
- Checking `flatten`'s associativity needs *a lot* more work!

The `cats` library has a `FlatMap` type class, defining `flatten` via `flatMap`

Checking the associativity law for standard monads

- Implement `flatten` for these functors and check the laws (see code):
  - `Option` monad: $F^A \equiv 1 + A$; ftn : $1 + (1 + A) \Rightarrow 1 + A$
  - `Either` monad: $F^A \equiv Z + A$; ftn : $Z + (Z + A) \Rightarrow Z + A$
  - `List` monad: $F^A \equiv \text{List}^A$; ftn : $\text{List}^{\text{List}^A} \Rightarrow \text{List}^A$
  - Writer monad: $F^A \equiv A \times W$; ftn : $(A \times W) \times W \Rightarrow A \times W$
  - Reader monad: $F^A \equiv R \Rightarrow A$; ftn : $(R \Rightarrow (R \Rightarrow A)) \Rightarrow R \Rightarrow A$
  - State: $F^A \equiv S \Rightarrow A \times S$; ftn : $(S \Rightarrow (S \Rightarrow A \times S) \times S) \Rightarrow S \Rightarrow A \times S$
  - Continuation monad: $F^A \equiv (A \Rightarrow R) \Rightarrow R$;
    ftn : $((((A \Rightarrow R) \Rightarrow R) \Rightarrow R) \Rightarrow R) \Rightarrow (A \Rightarrow R) \Rightarrow R$
- Code implementing these `flatten` functions is *fully parametric* in A
  - Naturality of these functions follows from parametricity theorem
  - Associativity needs to be checked for each monad!
- Example of a useful semimonad that is *not* a full monad:
  - $F^A \equiv A \times V \times W$; $\text{ftn}\left((a \times v_1 \times w_1) \times v_2 \times w_2\right) = a \times v_1 \times w_2$
- Examples of *non-associative* (i.e. wrong) implementations of `flatten`:
  - $F^A \equiv A \times W \times W$; $\text{ftn}\left((a \times v_1 \times v_2) \times w_1 \times w_2\right) = a \times w_2 \times w_1$
  - $F^A \equiv \text{List}^A$, but `flatten` concatenates the nested lists in reverse order

Motivation for monads

- Monads represent values with a "special computational context"
- Specific monads will have methods to create various contexts
- Monadic composition will "combine" the contexts associatively
- It is generally useful to have an "empty context" available:

$$\text{pure} : A \Rightarrow M^A$$

Adding the empty context to another context should be a no-op

- Empty context is followed by a generator:

```
y ← pure(x)
z ← cont(y)
```

```
y = x
z ← cont(y)
```

```
pure(x).flatMap(y ⇒ cont(y)) = cont(x)
```

$\text{pure} \circ \text{flm}\, f = f$ – left identity

- Empty context is preceded by a generator:

```
x ← cont
y ← pure(x)
```

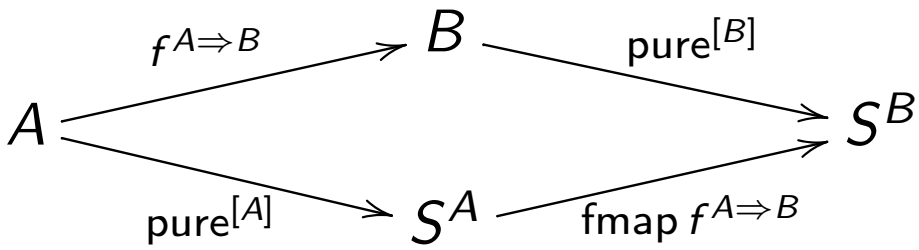
```
x ← cont
y = x
```

```
cont.flatMap(x ⇒ pure(x)) = cont
```

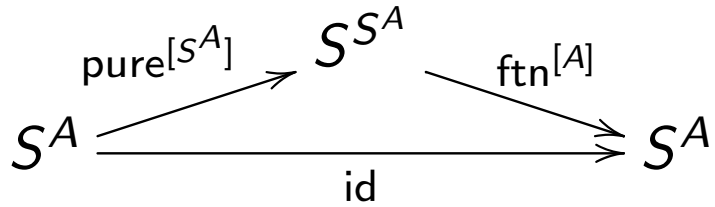
$\text{flm}\,(\text{pure}) = \text{id}$ – right identity

The monad laws formulated in terms of `pure` and `flatten`

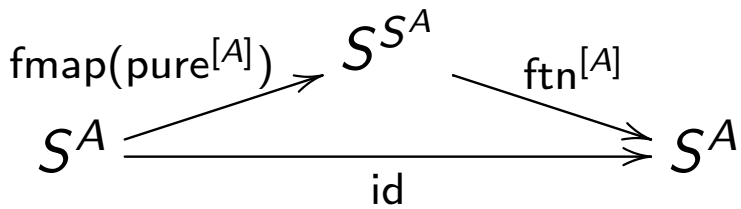
- Naturality law for `pure`: $f \circ \text{pure} = \text{pure} \circ \text{fmap}\, f$



- Left identity: $\text{pure} \circ \text{flm}\, f = \text{pure} \circ \text{fmap}\, f \circ \text{ftn} = f \circ \text{pure} \circ \text{ftn} = f$ requires that $\text{pure} \circ \text{ftn} = \text{id}$ (both sides applied to S^A)



- Right identity: $\text{flm}\,(\text{pure}) = \text{fmap}\,(\text{pure}) \circ \text{ftn} = \text{id}^{S^A \Rightarrow S^A}$



Formulating laws via Kleisli functions

- Recall: we formulated the laws of filterables via `fmapOpt`
  - type signature of fmapOpt : $(A \Rightarrow 1 + B) \Rightarrow S^A \Rightarrow S^B$
  - and then we had to compose functions of types $A \Rightarrow 1 + B$ via $\diamond_{\text{Opt}}$
- Here we have flm : $\left(A \Rightarrow S^B\right) \Rightarrow S^A \Rightarrow S^B$ instead of `fmapOpt`
- Can we compose **Kleisli functions** with "twisted" types $A \Rightarrow S^B$?
- Use flm to define **Kleisli composition**: $f^{A \Rightarrow S^B} \diamond g^{B \Rightarrow S^C} \equiv f \circ \text{flm}\, g$
- Define **Kleisli identity** $\text{id}_\diamond$ of type $A \Rightarrow S^A$ as $\text{id}_\diamond \equiv \text{pure}$
- Composition law: $\text{flm}\,(f \diamond g) = \text{flm}\, f \circ \text{flm}\, g$ (same as for `fmapOpt`)
  - Shows that `flatMap` is a "lifting" of $A \Rightarrow S^B$ to $S^A \Rightarrow S^B$
- These laws are similar to functor "lifting" laws...
  - except that $\diamond$ is used for composing Kleisli functions
- What are the properties of $\diamond$?
  - Exactly similar to the properties of function composition $f \circ g$

Reformulate flm's laws in terms of the $\diamond$ operation:

- flm's left and right identity laws: $\text{pure} \diamond f = f$ and $f \diamond \text{pure} = f$
- Associativity law: $(f \diamond g) \diamond h = f \diamond (g \diamond h)$
  - Follows from the flm law: $f \circ \text{flm}\,(g \circ \text{flm} h) = f \circ \text{flm}\, g \circ \text{flm}\, h$

* Motivation for categories and functors

Compare different "liftings" seen so far, and generalize

Category	Type $A \rightsquigarrow B$	Identity	Composition
plain functions	$A \Rightarrow B$	$\mathrm{id} : A \Rightarrow A$	$f^{A \Rightarrow B} \circ g^{B \Rightarrow C}$
lifted to F	$F^A \Rightarrow F^B$	$\mathrm{id} : F^A \Rightarrow F^A$	$f^{F^A \Rightarrow F^B} \circ g^{F^B \Rightarrow F^C}$
Kleisli over F	$A \Rightarrow F^B$	$\mathrm{pure} : A \Rightarrow F^A$	$f^{A \Rightarrow F^B} \diamond g^{B \Rightarrow F^C}$

Category theory generalizes this situation

Category: a certain class of "twisted functions" $A \rightsquigarrow B$ called **morphisms**

- For any two morphisms $f^{A \rightsquigarrow B}$ and $g^{B \rightsquigarrow C}$ the **composition** morphism $f \diamond g$ of type $A \rightsquigarrow C$ must exist
- For each type A, the **identity** morphism $\mathrm{id}_\diamond$ of type $A \rightsquigarrow A$ must exist
- Composition respects identity: $\mathrm{id}_\diamond \diamond f = f$ and $f \diamond \mathrm{id}_\diamond = f$
- Composition is associative: $(f \diamond g) \diamond h = f \diamond (g \diamond h)$

General **functor**: a map from one category to another

- A functor must `fmap` each morphism from one category to the other
- Functor laws: `fmap` must preserve identity and composition
  - What we call "functor" is called **endofunctor** in category theory
  - An endofunctor's `fmap` goes from plain functions to F-lifted functions

* From Kleisli back to `flatMap`

The Kleisli functions, $A \rightsquigarrow B \equiv A \Rightarrow S^B$, form a category iff S is a monad

- `map` and `flatMap` are computationally equivalent to Kleisli composition:
  - Define `flatMap` through Kleisli: $\text{flm}\, f^{A \Rightarrow S^B} \equiv \text{id}^{S^A \Rightarrow S^A} \diamond f$
  - Require two additional laws that connect $\diamond$, fmap, and $\circ$:
    - ★ Left naturality: $f^{A \Rightarrow B} \circ g^{B \Rightarrow S^C} = (f \circ \text{pure}) \diamond g$
    - ★ Right naturality: $f^{A \Rightarrow S^B} \circ \text{fmap}\, g^{B \Rightarrow C} = f \diamond (g \circ \text{pure})$
  - So, can define `fmap` through Kleisli: $\text{fmap}\, g^{A \Rightarrow B} \equiv \text{id}^{S^A \Rightarrow S^A} \diamond (g \circ \text{pure})$

The laws for `pure` and `flatMap` then follow from category axioms for Kleisli:

- Left and right identity laws follow from $\text{id} \diamond \text{pure} = \text{id}$ and $\text{pure} \diamond f = f$
- Associativity for `flatMap` follows from $(\text{id} \diamond f) \diamond g = \text{id} \diamond (f \diamond g)$
- Use "left naturality", get: $(f \circ g) \diamond h = (f \circ \mathit{pure}) \diamond g \diamond h = f \circ (g \diamond h)$
- Naturality for `pure`: $\text{pure} \circ \text{fmap}\, f = \text{pure} \diamond (f \circ \text{pure}) = f \circ \text{pure}$
- Define `flatten`: $\text{ftn} = \text{id}^{S^{S^A} \Rightarrow S^{S^A}} \diamond \text{id}^{S^A \Rightarrow S^A}$
- Naturality for `flatten`: $\text{ftn} \circ \text{fmap}\, f = \text{id} \diamond \text{id} \diamond (f \circ \text{pure}) = \text{id} \diamond \text{fmap}\, f$ and $\text{fmap}\, (\text{fmap}\, f) \circ \text{ftn} = \text{id} \diamond ((\text{fmap}\, f) \circ \text{pure}) \circ \text{id} \diamond \text{id} = \text{id} \diamond \text{fmap}\, f$

Structure of semigroups and monoids

- Semimonad contexts are combined associatively, as in a semigroup
  - A full monad includes an "empty" context, i.e. the identity element
  - Semigroup with an identity element is a monoid

Some constructions of semigroups and monoids (see code):

1. Any type Z is a semigroup with operation $z_1 \circledast z_2 = z_1$ (or z_2)
2. $1 + S$ is a monoid if S is (at least) a semigroup (or $S \equiv 0$)
3. List^A is a monoid (for any type A), also Seq^A etc.
4. The function type $A \Rightarrow A$ is a monoid (for any type A)
   - The operation $f \circledast g$ can be either $f \circ g$ or $g \circ f$
5. Any totally ordered type is a monoid, with $\circledast$ defined as max or min
6. $S_1 \times S_2$ is a semigroup (monoid) if S_1, S_2 are semigroups (monoids)
7. $S_1 + S_2$ is a semigroup (monoid) if S_1, S_2 are semigroups (monoids)
8. `M[S]` is a monoid if `M[_]` is a monad and `S` is a monoid.
9. $S \times P$ is a semigroup if S is a semigroup that has an **action on** P.
   - The "action" is $\alpha : S \Rightarrow P \Rightarrow P$ such that $\alpha(s_2) \circ \alpha(s_1) = \alpha(s_1 \circledast s_2)$.
   - $S \times P$ is a "twisted product." Examples: $(A \Rightarrow A) \times A$; $\text{Bool} \times (1 + A)$.

- Other examples of monoids: Int (many), String, Set^A, Akka's `Route`

Structure of (semi)monads

How to recognize a (semi)monad by its type? Open question!

Intuition from `flatten`: reshuffle data in F^{F^A} to fit into F^A

Some constructions of exponential-polynomial (semi)monads:

1. $F^A \equiv Z$ (constant functor) for a fixed type Z
   - For a full monad, need to choose $Z = 1$
2. $F^A \equiv A \times G^A$ for any functor G^A (a full monad only if $G^A \equiv 1$)
3. $F^A \equiv G^A \times H^A$ for any (semi)monads G^A and H^A
   - but $G^A + H^A$ is generally *not* a semimonad
4. $F^A \equiv R \Rightarrow G^A$ is a (semi)monad for any (semi)monad G^A
5. $F^A \equiv A + G^A$ is a monad for a semimonad G^A (**free pointed** over G)
6. $F^A \equiv G^{Z + A \times W}$ is a (semi)monad if G is a (semi)monad
   - For a full monad, need W to be a monoid
7. $F^A \equiv A + G^{F^A}$ (recursive) for any functor G^A (**free monad** over G)
8. $F^A \equiv G^A + G^{F^A}$ (recursive) for any functor G^A (semimonad only!)
9. $F^A \equiv H^A \Rightarrow A \times G^A$ for any contrafunctor H^A and functor G^A
   - For a full monad, need to set $G^A \equiv 1$, i.e. $F^A \equiv H^A \Rightarrow A$

Exercises II

1. Show that `M[S]` is a monoid if `M[_]` is a monad and `S` is a monoid.
2. A framework implements a "route" type R as $R \equiv Q \Rightarrow (E + S)$, where Q is a query, E is an error response, and S is a success response. A server is defined as a "sum" of several routes. For a given query Q, the response is the first route (if it exists) that yields a success. Implement the route "summation" operation and show that it makes R into a semigroup. What would be necessary to make R into a monoid?
3. Verify the associativity law for the semimonad $F^A \equiv Z + \text{Bool} \times A$.
4. Show that the functor $F^A \equiv \text{Boolean} \times M^A$ (where M^A is an arbitrary monad) can be made into a semimonad but not into a monad.
5. If W and R are arbitrary fixed types, which of the functors can be made into a semimonad: $F^A \equiv W \times (R \Rightarrow A)$, $G^A = R \Rightarrow (W \times A)$?
6. Show that $F^A = (P \Rightarrow A) + (Q \Rightarrow A)$ is not a semimonad (cannot define `flatMap`) when P and Q are arbitrary, different types.
7. Implement the `flatten` and `pure` methods for $D^A \equiv 1 + A \times A$ (`type D[A] = Option[(A, A)]`) in at least two significantly different ways, and show that the monad laws always fail to hold. (D^A is not a monad!)

Exercises II (continued)

8. Implement `flatten` and `pure` for $F^A \equiv A + (R \Rightarrow A)$, where R is a fixed type, and show that all the monad laws hold.
9. Check the identity laws for monad construction 5, $F^A \equiv A + G^A$, when pure_F is defined as $\text{id}^{A \Rightarrow A} + 0$ (given that G^A is a monad). Show that the identity laws fail if pure_F were defined as $0 + \text{pure}_G$.
10. Implement the monad methods for $F^A \equiv (Z \Rightarrow 1 + A) \times \text{List}^A$ using the known monad constructions (no need to check the laws).
11. Implement the semimonad construction 2 by discarding the first effect (not the second), and show that the associativity law is still satisfied.
12. A programmer implemented the `fmap` method for $F^A \equiv A \times (A \Rightarrow Z)$ as

```
def fmap[A,B](f: A⇒B): ((A, A⇒Z)) ⇒ (B, B⇒Z) =
  { case (a, az) ⇒ (f(a), (_: B) ⇒ az(a)) }
```

Show that this implementation fails to satisfy the functor laws.