

Chapter 1: Values, types, expressions, functions

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What is “Functional Programming”?

Functional programming...

- treats programs as mathematical expressions
- uses age-old mathematical intuition to design software
- is natural and effective in OCaml, Haskell, F#, Scala, Swift, etc.
- ... but not in C, C++, JavaScript, Java (before version 8), or Python!

We will be using Scala for all examples...

- ...but the same material looks very similar in the other languages

Examples of functional programs

Compute the factorial of a natural number:

$$n! = \prod_{k=1}^n k$$

Check whether a natural number is a prime:

$$\text{prime}(n) = \forall i \text{ such that } 2 \leq i < n : n \not\equiv 0 \pmod i$$

Count how many even numbers there are in a given set S of integers:

$$\text{count_even} = \sum_{k \in S} \text{is_even}(k)$$

$$\text{where we defined } \text{is_even}(k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

- Scala programs implementing this are similar to the math
 - ▶ Programs in Java or Python are *not* similar to the math!

What exactly is “math-like” about that code?

- The code represents a *mathematical expression* that we want to compute
- Each value is immutable and has a fixed *type* (integer, set, function, etc.)
- The code can define new names or new functions *within an expression*
- There are no loops, no “goto” or “repeat”
 - ▶ Have you ever seen a math book that says things such as,
“*now change k to $k - 1$ and repeat Equation 123 until $k = 0$* ” ?
or
“*at this point, if $x > 0$ then go back to page 208, else assign $x = 0$* ”?

Adapting math to programming I

Values, expressions, and types

In math, there are two kinds of “variables”:

- named constant values,

```
val a = 123
```

- function arguments,

```
def f(x: Int, y: Int) = x + y - 1
```

Math texts never try to “modify” a value, so...

- “`val`”s and function arguments are immutable

Each value has a fixed type (`Int`, `Boolean`, `Set[Int]`, etc.)

- Type represents the set of possible values of the function argument
- Type is automatically assigned to named constants

Adapting math to programming II

Anonymous functions vs. named functions

There are two ways of defining a function in Scala:

- **named** function – using `def` with a name

```
def merge(x: Int, y: Int): Int = { x + y - 1 }
```

- **anonymous** function – in math notation, $x \mapsto f(x)$:

```
(x: Int, y: Int) => x + y - 1
```

Anonymous functions are *values*:

- they are immutable and have a fixed type, e.g. `(Int, Int) => Int`
- they can be assigned a name and used later in an expression:

```
val double: (Int => Int) = { x => x * 2 }; double(y)
```

- or they can be used directly as arguments of other functions:

```
(1 to 100).map(x => x * x)
```

Some collections in Scala

What is `(1 to 100)`? What type does it have?

```
scala> (1 to 100)
res0: scala.collection.immutable.Range.Inclusive = Range 1 to 100
```

- Sequence, `Seq` and its subtypes: `List`, `IndexedSeq`, etc.

```
val a: Seq[Int] = Seq(2, 4, 6, 8)
val b = a(0) // now b: Int == 2
```

- Set: `Set`

```
val b: Set[String] = Set("x", "y", "z")
```

- Dictionary: `Map`

```
val c: Map[String, Int] = Map("x" -> 5, "y" -> 3, "z" -> 1)
```

Note the **parameterized** types `Seq[Int]` and `Map[String, Int]`

Collections can have any element type: `String`, `Boolean`, `Set[Seq[Int]]`, ...

Adapting math to programming III

Encoding sums, products, quantifiers using anonymous functions

The methods `map`, `filter`, `forall`, `exists` are defined on all collections

The methods `sum`, `product` are defined on collections of *numbers*

<i>Mathematical notation</i>	<i>Scala code</i>
$x \mapsto \sqrt{x^2 + 1}$	<code>x => math.sqrt(x * x + 1)</code>
sequence $[1, 2, \dots, n]$	<code>(1 to n)</code>
sequence $[f(1), \dots, f(n)]$	<code>(1 to n).map(k => f(k))</code>
$\sum_{k=1}^n k^2$	<code>(1 to n).map(k => k*k).sum</code>
$\prod_{k=1}^n f(k)$	<code>(1 to n).map(f).product</code>
$\forall k$ such that $1 \leq k \leq n : p(k)$ holds	<code>(1 to n).forall(k => p(k))</code>
$\exists k, 1 \leq k \leq n$ such that $p(k)$ holds	<code>(1 to n).exists(k => p(k))</code>
$\sum_{k \in S: p(k) \text{ holds}} f(k)$	<code>s.filter(p).map(f).sum</code>

Adapting math to programming IV

Higher-order functions

Derivatives and integrals could be implemented as functions:

```
def deriv(f: Double => Double): (Double => Double) = { ??? }  
def integ(f: Double => Double, range: (Double, Double)): Double = ???
```

Higher-order functions take function arguments and/or return functions

- Many computations with sequences, sets, dictionaries can be done using higher-order functions, *without loops*, concisely and error-free
- The Scala standard library has many more higher-order methods for collections
 - ▶ `size`, `reduce`, `zip`, `zipWithIndex`, `flatten`, `flatMap`, `foldLeft`, `foldRight`, `scanLeft`, `collect`, `distinct`, `groupBy`, ...
- Write code by formulating the problem as a mathematical expression

Examples

- Using both `def` and `val`, define a function that...
 - adds 20 to its integer argument
 - takes an integer `x`, and returns a *function* that adds `x` to *its* argument
 - takes an integer `x` and returns `true` iff `x + 1` is prime
- What are the types of the functions in Examples 1 - 3?
- Compute the average of all numbers in a sequence of type `Seq[Double]`. Use `sum` and `size` but no loops.
- Given n , compute the Wallis product truncated up to $\frac{2n}{2n+1}$:

$$\frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \cdots \frac{2n}{2n+1}.$$

Use a sequence of `Int` or `Double` numbers, `map`, and `product`.

- Given `s: Seq[Set[Int]]`, compute the sequence containing the sets of size at least 3. Use `map`, `filter`, `size`. The result must be again of type `Seq[Set[Int]]`.

Summary

- What problems can we solve now?
 - ▶ Compute mathematical expressions involving sums, products, and quantifiers, based on integer ranges (such as $\sum_{k=1}^n f(k)$ etc.)
 - ▶ Implement functions that take or return other functions
 - ▶ Work on collections using `map` and other library methods
- What kinds of problems are not solved with these tools?
 - ▶ Compute the smallest n such that $f(f(f(\dots f(1)\dots)) > 1000$, where the function $f()$ is applied n times.
 - ▶ Find the k -th largest element in an (unsorted) array of integers.
 - ▶ Find the median element in a sorted array of integers.
- Why can't we solve such problems yet?
 - ▶ Because we can't yet put *mathematical induction* into code

Exercises

- 1 Define a function of type `Seq[Double] => Seq[Double]` that “normalizes” the sequence: it finds the element having the max. absolute value and, if that value is nonzero, divides all elements by that factor.
- 2 Define a function of type `Seq[Seq[Int]] => Seq[Seq[Int]]` that adds 20 to every element of every inner sequence.
- 3 An integer n is called “two-factor” if it is divisible by only two different integers j such that $2 \leq j < n$. Compute the set of all “two-factor” integers n among $n \in [1, \dots, 100]$.
- 4 Given a function f of type `Int => Boolean`, an integer n is called “two- f ” if there are only two different integers $j \in [1, \dots, n]$ such that $f(j)$ returns `true`. Define a function that takes f as an argument and returns a sequence of all “two- f ” integers among $n \in [1, \dots, 100]$. What is the type of that function? Rewrite Exercise 3 using that function.
- 5 Define a function that takes two functions $f: \text{Int} \Rightarrow \text{Double}$ and $g: \text{Double} \Rightarrow \text{String}$ as arguments, and returns a new function that computes the functional composition of f and g .