Chapter 7: Computations lifted to a functor context II

Semimonads and monads

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Example:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(i,j,k)$$

Using Scala's `for`/`yield` syntax ("functor block")

```
(for { i ← 1 to n
    j ← 1 to n
    k ← 1 to n
  } yield f(i, j, k)
).sum
```

```
(1 to m).flatMap { i ⇒
  (1 to n).flatMap { j ⇒
    (1 to n).map { k ⇒
      f(i, j, k)
}}}.sum
```

- `map` replaces the last left arrow, `flatMap` replaces other left arrows
- Container must have both `map` (i.e. be a functor) and `flatMap`
  - When the functor is *also* filterable, we can use "`if`" as well
- Standard library defines `flatMap()` as equivalent of `map()` ∘ `flatten`
  - `(1 to n).flatMap(j ⇒ ...)` is `(1 to n).map(j ⇒ ...).flatten`
- `flatten: F[F[A]]⇒F[A]` can be expressed through `flatMap` as well:
  - `(xss: Seq[Seq[A]]).flatten = xss.flatMap { (xs: Seq[A]) ⇒ xs }`
- Functors having `flatMap`/`flatten` are "squashable" or **semimonads**

What is `flatMap` doing with the data in a collection?

Consider this schematic code using `Seq` as the container:

```
val result = for {
  i ← 1 to m
  j ← 1 to n
  x = f(i, j)
  k ← 1 to p
  y = g(i,j,k)
  ...
} yield h(x,y)
```

Computations are repeated for all i, for all j, etc.

- The computation processes all elements from each collection
  - The number of resulting data items is $m * n * p$
    - All the resulting data items must fit within *the same* container type!
    - The set of *container capacity counts* is closed under multiplication
- What container types have this property?
  - `Seq`, `NonEmptyList` – can hold *any* number of elements $\geq$ min. count
  - `Option`, `Either`, `Try`, `Future` – can hold 0 or 1 elements
  - "Tree-like" containers, e.g. binary tree (holds 1, 2, 4, 8, 16, ... elements)
  - "Non-standard" containers: $F^A \equiv \text{String} \Rightarrow A$; $F^A \equiv (A \Rightarrow \text{Int}) \Rightarrow \text{Int}$

Typical tasks solved with "list-like" semimonads:

- Create a list of all combinations or all permutations of a sequence
- Traverse a "solution tree" with DFS and filter out incorrect solutions
  - Can use eager (`Seq`) or lazy (`Iterator`, `Stream`) evaluation strategies

Examples: see code

1. All permutations of `Seq("a", "b", "c")`
2. All subsets of `Set("a", "b", "c")`
3. All subsequences of length 3 out of the sequence `(1 to m)`
4. All solutions of the "8 queens" problem
5. Generalize examples 1-3 to support arbitrary length n instead of 3
6. Generalize example 4 to solve n-queens problem
7. Transform Boolean formulas between CNF and DNF

Intuitions for "single-value" semimonads

`Option`, `Either`, `Try`, `Future`

- The container can hold $n = 1$ or $n = 0$ values of type A
- Computations may yield a result ($n = 1$), or may fail ($n = 0$)
- The functor block chains several such computations
- Once any computation fails, the entire functor block fails ($0 * n = 0$)
- Only if *all* computations succeed, the functor block returns one value
- Filtering can also make the entire expression fail

A schematic example of a functor block program using the `Try` functor:

```
val result: Try[A] = for { // computations in the Try functor
  x ← Try(...) // first computation; may fail
  y = f(x) // no possibility of failure in this line
  if p(y) // the entire expression will fail if this is false
  z ← Try(g(x, y)) // may fail here
  r ← Try(...) // may fail here as well
} yield r //  r is of type A, so result is of type Try[A]
```

- This "flat" code replaces a long chain of nested `if/else` or `match/case`
  - Computations are sequential (also when using the `Future` functor!)

Examples of filterable functors II: Checking the laws

- Properties 1 – 4 are expressed as laws for `filter`$^{(p\Rightarrow \text{Boolean})\Rightarrow F^A \Rightarrow F^A}$:
  1. $\text{fmap}\, f^{A\Rightarrow B} \circ \text{filter}\, p^{B\Rightarrow \text{Boolean}} = \text{filter}\,(f \circ p) \circ \text{fmap}\, f^{A\Rightarrow B}$
  2. $\text{filter}\, p_1^{A\Rightarrow \text{Boolean}} \circ \text{filter}\, p_2^{A\Rightarrow \text{Boolean}} = \text{filter}\,(x \Rightarrow p_1(x) \wedge p_2(x))$
  3. $\text{filter}\,\left(x^A \Rightarrow \text{true}\right) = \text{id}^{F^A \Rightarrow F^A}$
  4. $\text{filter}\, p \circ \text{fmap}\, f^{A\Rightarrow B} = \text{filter}\, p \circ \text{fmap}\,\left(f_{|p}\right)$ where $f_{|p}$ is the *partial function* defined as `{ case x if p(x) ⇒ f(x) }` – only works if $p(x)$ holds
- Can define a type class `Filterable`, method `withFilter`
- Check the laws for the `Orders` functor (see example code)
  - Laws hold for the `Orders` functor with / without business rule (a)
  - Another filterable functor: $F^A \equiv 1 + A \times A$ ("collapsible product")
- Examples of functors that are *not* filterable:
  - "Orders" with additional business rule (b) – breaks law 2 for some $p_{1,2}$
  - F^A defining `filter` in a special way e.g. for $A = \text{Int}$ – breaks law 1
  - $F^A \equiv 1 + A$ defining $\text{filter}\,(p)\,(x) \equiv 1 + 0$ breaks law 3
  - $F^A \equiv A$ – must define $\text{filter}\left(p^{A\Rightarrow \text{Boolean}}\right)\left(x^A\right) = x$, breaking law 4
  - $F^A \equiv A \times (1 + A)$ – unable to remove the first A, breaking law 4

The equational laws 1–4 specify *rigorously* what it means to "filter data"!

Worked examples I: Programming with filterables

1. John can have up to 3 coupons, and Jill up to 2. *All* of John's coupons must be valid on purchase day, while each of Jill's coupons is checked independently. Implement the filterable functor describing this setup.
2. A server received a sequence of requests. Each request must be authenticated. Once a non-authenticated request is found, no further requests are accepted. Is this setup described by a filterable functor?

   For each of these functors, determine whether they are filterable, and if so, implement `withFilter` via a type class:
3. `final case class P[T](first: Option[T], second: Option[(T, T)])`
4. $F^A \equiv \text{Int} + \text{Int} \times A + \text{Int} \times A \times A + \text{Int} \times A \times A \times A$
5. $F^A = \text{NonEmptyList}^A$ defined recursively as $F^A \equiv A + A \times F^A$
6. $F^{Z,A} \equiv Z + \text{Int} \times Z \times A \times A$ (with respect to the type parameter A)
7. $F^{Z,A} \equiv 1 + Z + \text{Int} \times A \times \text{List}^A$ (w.r.t. the type parameter A)
8. * Show that $C^{Z,A} = A \Rightarrow 1 + Z$ is a filterable *contrafunctor* w.r.t. A (implement `withFilter` with the same type signature; no law checking)

Exercises I

1. Confucius gave wisdom on each of the 7 days of a week. Sometimes the wise proverbs were hard to remember. If Confucius forgets what he said on a given day, he also forgets what he said on all the previous days of the week. Is this setup described by a filterable functor?
2. Define `evenFilter(p)` on an `IndexedSeq[T]` such that a value `x: T` is retained if `p(x)=true` *and* only if the sequence has an *even* number of elements `y` for which `p(y)=false`. Does this define a filterable functor?

   Implement `filter` for these functors if possible (law checking optional):
3. $F^A \equiv \text{Int} + \text{String} \times A \times A \times A$
4. `final case class Q[A, Z](id: Long, user1: Option[(A, Z)], user2: Option[(A, Z)])` – with respect to the type parameter A
5. $F^A = \text{MyTree}^A$ defined recursively as $F^A \equiv 1 + A \times F^A \times F^A$
6. `final case class R[A](x: Int, y: Int, z: A, data: List[A])`, where the standard functor List already has `withFilter` defined
7. * Show that $C^A \equiv A + A \times A \Rightarrow 1 + Z$ is a filterable contrafunctor

Filterable functors: The laws in depth I

Is there a shorter formulation of the laws that is easier to remember?

- Intuition: When `p(x) = false`, replace `x: A` by `1: Unit` in `F[A]`
  - (1) How to replace `x` by `1` in `F[A]` without breaking the types?
  - (2) How to transform the resulting type back to `F[A]`?
- We could do (1) if instead of F^A we had F^{1+A} i.e. `F[Option[A]]`
  - Now use `filter` to replace A by 1 in each item of type $1+A$
  - Get F^{1+A} from F^A using $\text{inflate} : F^A \Rightarrow F^{1+A} = \text{fmap}\left(\text{Some}^{A\Rightarrow 1+A}\right)$
  - Filter $F^{1+A} \Rightarrow F^{1+A}$ using $\text{fmap}\left(x^{1+A} \Rightarrow \text{filter}_{\text{Opt}}(p^{A\Rightarrow\text{Boolean}})(x)\right)$

$$\text{filter}\,p: \; F^A \xrightarrow[\text{inflate}]{} F^{1+A} \xrightarrow[\text{fmap}\left(\text{filter}_{\text{Opt}}\,p\right)]{} F^{1+A} \xrightarrow[\text{deflate}]{} F^A$$

- Doing (2) means *defining* a function `deflate: F[Option[A]] ⇒ F[A]`
  - standard library already has `flatten[T]: Seq[Option[T]] ⇒ Seq[T]`
- Simplify $\text{fmap}(\text{Some}^{A\Rightarrow 1+A}) \circ \text{fmap}\left(\text{filter}_{\text{Opt}}\,p\right) = \text{fmap}\left(\text{bop}\left(p\right)\right)$ where we defined `bop` $(p) : (A \Rightarrow 1 + A) \equiv$ `x ⇒ Some(x).filter(p)`
- In this way, express `filter` through `deflate` (see example code)
  - $\text{filter}\,p = \text{fmap}\left(\text{bop}\,p\right) \circ \text{deflate}$. – Notation: $\text{bop}\,p$ is $\text{bop}\left(p\right)$, like $\cos x$

$$\text{filter}\,p: \; F^A \xrightarrow[\text{fmap}(\text{bop}\,p)]{} F^{1+A} \xrightarrow[\text{deflate}]{} F^A$$

Filterable functors: Using `deflate`

- So far we have expressed `filter` through `deflate`
- We can also express `deflate` through `filter` (assuming law 4 holds):

$$\text{deflate}: \; F^{1+A} \xrightarrow[\text{filter(.nonEmpty)}]{} F^{1+A} \xrightarrow[\text{fmap(.get)}]{} F^{A}$$

```
def deflate[F[_],A](foa: F[Option[A]]): F[A] =
  foa.filter(_.nonEmpty).map(_.get) // _.get is 0 + x^A ⇒ x^A
// for F = Seq, this would be foa.collect { case Some(x) ⇒ x }
// for arbitrary functor F we need to use the partial function, _.get
```

- This means `deflate` and `filter` are **computationally equivalent**
  - We could specify filterable functors by implementing `deflate`
    - ★ The implementation of `filter` would then be derived by library
- Use `deflate` to verify that some functors are certainly not filterable:
  - $F^A = A + A \times A$. Write $F^{1+A} = 1 + A + (1 + A) \times (1 + A)$
    - ★ cannot map $F^{1+A} \Rightarrow F^A$ because we do not have $1 \rightarrow A$
  - $F^A = \text{Int} \Rightarrow A$. Write $F^{1+A} = \text{Int} \Rightarrow 1 + A$
    - ★ type signature of `deflate` would be $(\text{Int} \Rightarrow 1 + A) \Rightarrow \text{Int} \Rightarrow A$
    - ★ cannot map $F^{1+A} \Rightarrow F^A$ because we do not have $1 + A \rightarrow A$
- `deflate` is easier to implement and to reason about

* Filterable functors: The laws in depth II

- We were able to define `deflate` only by assuming that law 4 holds
- Now, law 4 is satisfied *automatically* if `filter` is defined via `deflate`!
  - Denote $\psi_p^{F^A \Rightarrow F^{\mathbf{1}+A}} \equiv \text{fmap}\,(\text{bop}\ p)$ for brevity, then $\text{filter}\ p = \psi_p \circ \text{deflate}$
  - Law 4 then becomes: $\psi_p \circ \text{deflate} \circ \text{fmap}\ f^{A \Rightarrow B} = \psi_p \circ \text{deflate} \circ \text{fmap}\ f_{|p}$

$$
\begin{tikzcd}
 & F^{\mathbf{1}+A} \arrow[r, "\text{deflate}"] & F^A \arrow[dr, "\text{fmap}\ f^{A\Rightarrow B}"] & \\
F^A \arrow[ur, "\psi_p"] \arrow[dr, "\psi_p"'] & & & F^B \\
 & F^{\mathbf{1}+A} \arrow[r, "\text{deflate}"'] & F^A \arrow[ur, "\text{fmap}\ f_{|p}^{A\Rightarrow B}"'] &
\end{tikzcd}
$$

- We would like to interchange `deflate` and `fmap` in both sides
  - We need a *naturality* law; let's express law 1 through `deflate`:

$$\text{fmap}\ f^{A \Rightarrow B} \circ \psi_p \circ \text{deflate}^{F,B} = \psi_{f \circ p} \circ \text{deflate}^{F,A} \circ \text{fmap}\ f^{A \Rightarrow B}$$

$$
\begin{tikzcd}
 & F^B \arrow[r, "\psi_p"] & F^{\mathbf{1}+B} \arrow[dr, "\text{deflate}^{F,B}"] & \\
F^A \arrow[ur, "\text{fmap}\ f^{A\Rightarrow B}"] \arrow[dr, "\psi_{f\circ p}"'] & & & F^B \\
 & F^{\mathbf{1}+A} \arrow[r, "\text{deflate}^{F,A}"'] & F^A \arrow[ur, "\text{fmap}\ f^{A\Rightarrow B}"'] &
\end{tikzcd}
$$

Can we simplify $\text{fmap}\ f \circ \psi_p = \text{fmap}\ f \circ \text{fmap}\,(\text{bop}\ p) = \text{fmap}\,(f \circ \text{bop}\ p)$?

* Filterable functors: The laws in depth III

- Have property: $f^{A\Rightarrow B} \circ \text{bop}\left(p^{B\Rightarrow \text{Boolean}}\right) = \text{bop}\,(f \circ p) \circ \text{fmap}^{\text{Opt}}\, f$ (see code)

$$\begin{tikzcd}
& B \arrow[dr, "\text{bop}\,p"] & \\
A \arrow[ur, "f^{A\Rightarrow B}"] \arrow[dr, "\text{bop}(f\circ p)"'] & & 1+B \\
& 1+A \arrow[ur, "\text{fmap}^{\text{Opt}} f"'] &
\end{tikzcd}$$

We can now rewrite Law 1 as

$\text{fmap}\,(\text{bop}\,(f \circ p)) \circ \text{fmap}\,(\text{fmap}^{\text{Opt}} f) \circ \text{deflate} = \text{fmap}\,(\text{bop}\,(f \circ p)) \circ \text{deflate} \circ \text{fmap}\, f$

Remove common prefix $\text{fmap}\,(\text{bop}\,(f \circ p)) \circ ...$ from both sides:

$\text{fmap}\,(\text{fmap}^{\text{Opt}} f^{A\Rightarrow B}) \circ \text{deflate}^{F,B} = \text{deflate}^{F,A} \circ \text{fmap}\, f^{A\Rightarrow B}$ – **law 1 for deflate**

$$\begin{tikzcd}
& F^{1+B} \arrow[dr, "\text{deflate}^{F,B}"] & \\
F^{1+A} \arrow[ur, "\text{fmap}\,(\text{fmap}\,f)"] \arrow[dr, "\text{deflate}^{F,A}"'] & & F^{B} \\
& F^{A} \arrow[ur, "\text{fmap}\,f^{A\Rightarrow B}"'] &
\end{tikzcd}$$

- `deflate`: $F^{1+A} \Rightarrow F^A$ is a **natural transformation** (has naturality law)
  - Example: $F^A = 1 + A \times A$
  - $F^{1+A} = 1 + (1+A) \times (1+A) = 1 + 1 \times 1 + A \times 1 + 1 \times A + A \times A$
- natural transformations map containers $G^A \Rightarrow H^A$ by rearranging data in them

* Filterable functors: The laws in depth IV

- The naturality law for `deflate`:

$$\text{fmap}\,(\text{fmap}^{\text{Opt}} f^{A\Rightarrow B}) \circ \text{deflate}^{F,B} = \text{deflate}^{F,A} \circ \text{fmap}\, f^{A\Rightarrow B}$$

Law 4 expressed via `deflate`:

$$\begin{tikzcd}
 & F^{1+A} \arrow[r, "\text{deflate}"] & F^A \arrow[dr, "\text{fmap}\ f^{A\Rightarrow B}"] & \\
F^A \arrow[ur, "\psi_p"] \arrow[dr, "\psi_p"'] & & & F^B \\
 & F^{1+A} \arrow[r, "\text{deflate}"'] & F^A \arrow[ur, "\text{fmap}\ f_{|p}^{A\Rightarrow B}"'] &
\end{tikzcd}$$

$$\psi_p \circ \text{deflate}^{F,A} \circ \text{fmap}\, f^{A\Rightarrow B} = \psi_p \circ \text{deflate}^{F,A} \circ \text{fmap}\, f_{|p}$$

- Use naturality to interchange `deflate` and `fmap` in both sides of law 4:

$$\psi_p \circ \text{fmap}\,(\text{fmap}^{\text{Opt}} f) \circ \text{deflate}^{F,B} = \psi_p \circ \text{fmap}\,(\text{fmap}^{\text{Opt}} f_{|p}) \circ \text{deflate}^{F,B}$$

$$[\text{omit deflate}^{F,B} \text{ from both sides; expand } \psi_p]$$

$$\text{bop}\, p \circ \text{fmap}^{\text{Opt}} f = \text{bop}\, p \circ \text{fmap}^{\text{Opt}} f_{|p} \quad - \quad \text{check this by hand:}$$

```
x ⇒ Some(x).filter(p).map(f)
x ⇒ Some(x).filter(p).map { x if p(x) ⇒ f(x) }
```

- These functions are equivalent because law 4 holds for `Option`

Filterable functors: The laws in depth V

Maybe $\psi_p \circ$ deflate is easier to handle than `deflate`? Let us define

$$\text{fmapOpt}^{F,A,B}(f^{A\Rightarrow\mathbf{1}+B}) : F^A \Rightarrow F^B = \text{fmap}\, f \circ \text{deflate}^{F,B}$$

$$\begin{tikzcd}
 & F^{\mathbf{1}+B} \arrow[dr, "\text{deflate}^{F,B}"] & \\
F^A \arrow[ur, "\text{fmap}\, f^{A\Rightarrow\mathbf{1}+B}"] \arrow[rr, "\text{fmapOpt}\, f^{A\Rightarrow\mathbf{1}+B}"'] & & F^B
\end{tikzcd}$$

- `fmapOpt` and `deflate` are *equivalent*: $\text{deflate}^{F,A} = \text{fmapOpt}^{F,\mathbf{1}+A,A}(\text{id}^{\mathbf{1}+A\Rightarrow\mathbf{1}+A})$
- Express laws $1-3$ in terms of `fmapOpt`: do they get simpler?
    - Express `filter` through `fmapOpt`: $\text{filter}\, p = \text{fmapOpt}^{F,A,A}\,(\text{bop}\, p)$
    - Consider the expression needed for law 2: $x \Rightarrow p_1(x) \wedge p_2(x)$
    - $\text{bop}\,(x \Rightarrow p_1(x) \wedge p_2(x)) = x^A \Rightarrow (\text{bop}\, p_1)\,(x).\text{flatMap}\,(\text{bop}\, p_2)$ – see code
        - ★ Denote this computation by $\diamond_{\text{Opt}}$ and write

$$q_1^{A\Rightarrow\mathbf{1}+B} \diamond_{\text{Opt}} q_2^{B\Rightarrow\mathbf{1}+C} \equiv x^A \Rightarrow q_1(x).\text{flatMap}\,(q_2)$$

    - Similar to composition of functions, except the types are $A \Rightarrow 1 + B$
        - ★ This is a particular case of **Kleisli composition**; the general case: $\diamond_M : (A \Rightarrow M^B) \Rightarrow (B \Rightarrow M^C) \Rightarrow (A \Rightarrow M^C)$; we set $M^A \equiv 1 + A$
        - ★ The **Kleisli identity** function: $\text{id}_{\diamond_{\text{Opt}}}^{A\Rightarrow\mathbf{1}+A} \equiv x^A \Rightarrow \text{Some}\,(x)$
        - ★ Kleisli composition $\diamond_{\text{Opt}}$ is associative and respects the Kleisli identity!
        - ★ `fmapOpt` lifts a $\text{Kleisli}_{\text{Opt}}$ function $f^{A\Rightarrow\mathbf{1}+B}$ into the functor F

Filterable functors: The laws in depth VI

Simplifying down to two laws

- Only *two* laws are necessary for `fmapOpt`!

1. **Identity law** (covers old law 3):

$$\text{fmapOpt}\,(\text{id}^{A\Rightarrow\mathbf{1}+A}_{\diamond_{\mathbf{Opt}}}) = \text{id}^{F^A\Rightarrow F^A}$$

2. **Composition law** (covers old laws 1 and 2):

$$\text{fmapOpt}\,(f^{A\Rightarrow\mathbf{1}+B}) \circ \text{fmapOpt}\,(g^{B\Rightarrow\mathbf{1}+C}) = \text{fmapOpt}\,(f \diamond_{\mathbf{Opt}} g)$$

$$\begin{tikzcd}
& F^B \arrow[dr, "\text{fmapOpt}\,(g^{B\Rightarrow\mathbf{1}+C})"] & \\
F^A \arrow[ur, "\text{fmapOpt}\,(f^{A\Rightarrow\mathbf{1}+B})"] \arrow[rr, "\text{fmapOpt}\left(f\diamond_{\mathbf{Opt}}g\right)"'] & & F^C
\end{tikzcd}$$

- The two laws for `fmapOpt` are very similar to the two functor laws
  - Both of them use more complicated types than the old laws
  - Conceptually, the new laws are simpler (lift $f^{A\Rightarrow 1+B}$ into $F^A \Rightarrow F^B$)

* Filterable functors: The laws in depth VII

Showing that old laws 1 – 3 follow from the identity and composition laws for `fmapOpt`

- Old law 3 is *equivalent* to the identity law for `fmapOpt`:

$$\text{filter}\,(x^A \Rightarrow \text{true}) = \text{fmap}\,(x^A \Rightarrow 0 + x) \circ \text{deflate} = \text{fmapOpt}\,(\text{id}_{\diamond_{\textbf{Opt}}}) = \text{id}^{F^A \Rightarrow F^A}$$

- Derive old law 2: need to work with $q_{1,2} \equiv \text{bop}\,(p_{1,2}) : A \Rightarrow 1 + A$
  - The Boolean conjunction $x \Rightarrow p_1(x) \wedge p_2(x)$ corresponds to $q_1 \diamond_{\textbf{Opt}} q_2$
  - Apply the composition law to Kleisli functions of types $A \Rightarrow 1 + A$:

$$\text{filter}\,p_1 \circ \text{filter}\,p_2 = \text{fmapOpt}\,q_1 \circ \text{fmapOpt}\,q_2$$
$$= \text{fmapOpt}\,(q_1 \diamond_{\textbf{Opt}} q_2) = \text{fmapOpt}\,(\text{bop}\,(x \Rightarrow p_1(x) \wedge p_2(x)))$$

- Derive old law 1:
  - express `filter` through `fmapOpt`; old law 1 becomes

$$\text{fmap}\,f \circ \text{fmapOpt}\,(\text{bop}\,p) = \text{fmapOpt}\,(\text{bop}\,(f \circ p)) \circ \text{fmap}\,f \quad - \text{eq. } (*)$$

  - lift $f^{A \Rightarrow B}$ to $\text{Kleisli}_{\text{Opt}}$ by defining $k_f^{A \Rightarrow 1+B} = f \circ \text{id}_{\diamond_{\textbf{Opt}}}$; then we have $\text{fmapOpt}\,(k_f) = \text{fmap}\,k_f \circ \text{deflate} = \text{fmap}\,f \circ \text{fmap}\,\text{id}_{\diamond_{\textbf{Opt}}} \circ \text{deflate} = \text{fmap}\,f$
  - rewrite eq. (*) as $\text{fmapOpt}\,(k_f \diamond_{\textbf{Opt}} \text{bop}\,p) = \text{fmapOpt}\,(\text{bop}\,(f \circ p) \diamond_{\textbf{Opt}} k_f)$
  - it remains to show that $k_f \diamond_{\textbf{Opt}} \text{bop}\,p = \text{bop}\,(f \circ p) \diamond_{\textbf{Opt}} k_f$
  - use the properties $k_f \diamond_{\textbf{Opt}} q = f \circ q$ and $q \diamond_{\textbf{Opt}} k_f = q \circ \text{fmap}^{\text{Opt}} f$, and $f \circ \text{bop}\,p = \text{bop}\,(f \circ p) \circ \text{fmap}^{\text{Opt}}\,f$ (property from slide 11)

Summary: The methods and the laws

Filterable functors can be defined via `filter`, `deflate`, or `fmapOpt`

- All three methods are *equivalent* but have different roles:
  - The easiest to use in program code is `filter` / `withFilter`
  - The easiest type signature to implement and reason about is `deflate`
  - Conceptually, the laws are easiest to remember with `fmapOpt`
- * The 2 laws for `fmapOpt` are the 2 functor laws with a Kleisli "twist"
- * Category theory accommodates this via a generalized definition of functors as liftings between "twisted" types. Compare:
  - $\text{fmap} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$ – ordinary container ("endofunctor")
  - $\text{contrafmap} : (B \Rightarrow A) \Rightarrow F^A \Rightarrow F^B$ – lifting from reversed functions
  - $\text{fmapOpt} : (A \Rightarrow 1 + B) \Rightarrow F^A \Rightarrow F^B$ – lifting from $\text{Kleisli}_{\text{Opt}}$-functions
- CT gives us some *intuitions* about how to derive better laws:
  - look for type signatures that resemble a generalized sort of "lifting"
  - look for natural transformations and use the naturality law
- However, CT does not directly provide any derivations for the laws
  - you will not find the laws for `filter` or `deflate` in any CT book
  - CT is abstract, only gives hints about possible further directions
    - ★ investigate functors having "liftings" with different type signatures
    - ★ replace `Option` in the $\text{Kleisli}_{\text{Opt}}$ construction by another functor

Structure of filterable functors

How to recognize a filterable functor by its type?

Intuition from `deflate`: reshuffle data in F^A after replacing some A's by 1

- "reshuffling" usually means reusing different parts of a disjunction

Some constructions of exponential-polynomial filterable functors

1. $F^A = Z$ (constant functor) for a fixed type Z (define $\text{fmapOpt}\, f = \text{id}$)
   - Note: $F^A = A$ (identity functor) is *not* filterable
2. $F^A \equiv G^A \times H^A$ for any filterable functors G^A and H^A
3. $F^A \equiv G^A + H^A$ for any filterable functors G^A and H^A
4. $F^A \equiv G^{H^A}$ for *any* functor G^A and filterable functor H^A
5. $F^A \equiv 1 + A \times G^A$ for a filterable functor G^A
   - Note: *pointed* types P are isomorphic to $1 + Z$ for some type Z
     - ⋆ Example of non-trivial pointed type: $A \Rightarrow A$
     - ⋆ Example of non-pointed type: $A \Rightarrow B$ when A is different from B
   - So $F^A \equiv P + A \times G^A$ where P is a pointed type and G^A is filterable
   - Also have $F^A \equiv P + A \times A \times ... \times A \times G^A$ similarly
6. $F^A \equiv G^A + A \times F^A$ (recursive) for a filterable functor G^A
7. $F^A \equiv G^A \Rightarrow H^A$ if contrafunctor G^A and functor H^A *both filterable*
   - Note: the functor $F^A \equiv G^A \Rightarrow A$ is not filterable

* Worked examples II: Constructions of filterable functors I

(2) The `fmapOpt` laws hold for $F^A \times G^A$ if they hold for F^A and G^A

- For $f^{A \Rightarrow 1+B}$, get $\text{fmapOpt}_F(f) : F^A \Rightarrow F^B$ and $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\text{fmapOpt}_{F\times G} f \equiv p^{F^A} \times q^{G^A} \Rightarrow \text{fmapOpt}_F(f)(p) \times \text{fmapOpt}_G(f)(q)$
- Identity law: $f = \text{id}_{\diamond_{\text{Opt}}}$, so $\text{fmapOpt}_F f = \text{id}$ and $\text{fmapOpt}_G f = \text{id}$
  - Hence we get $\text{fmapOpt}_{F+G}(f)(p \times q) = \text{id}(p) \times \text{id}(q) = p \times q$
- Composition law:

$$
\begin{aligned}
& (\text{fmapOpt}_{F\times G}\, f_1 \circ \text{fmapOpt}_{F+G}\, f_2)(p \times q) \\
&= \text{fmapOpt}_{F\times G}(f_2)\left(\text{fmapOpt}_F(f_1)(p) \times \text{fmapOpt}_G(f_1)(q)\right) \\
&= (\text{fmapOpt}_F\, f_1 \circ \text{fmapOpt}_F\, f_2)(p) \times (\text{fmapOpt}_G\, f_1 \circ \text{fmapOpt}_G\, f_2)(q) \\
&= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(p) \times \text{fmapOpt}_G(f_1 \diamond f_2)(q) \\
&= \text{fmapOpt}_{F\times G}(f_1 \diamond_{\text{Opt}} f_2)(p \times q)
\end{aligned}
$$

- Exactly the same proof as that for functor property for $F^A \times G^A$
  - this is because `fmapOpt` corresponds to a generalized functor
- New proofs are necessary only when using non-filterable functors
  - these are used in constructions $4 - 6$

* Worked examples II: Constructions of filterable functors II

(5) The `fmapOpt` laws hold for $F^A \equiv 1 + A \times G^A$ if they hold for G^A

- For $f^{A \Rightarrow 1+B}$, get $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$
- Define $\text{fmapOpt}_F(f)(1 + a^A \times q^{G^A})$ by returning $0 + b \times \text{fmapOpt}_G(f)(q)$ if the argument is $0 + a \times q$ and $f(a) = 0 + b$, and returning $1 + 0$ otherwise
- Identity law: $f = \text{id}_{\diamond_{\textbf{Opt}}}$, so $f(a) = 0 + a$ and $\text{fmapOpt}_G f = \text{id}$
  - Hence we get $\text{fmapOpt}_F(\text{id}_{\diamond_{\textbf{Opt}}})(1 + a \times q) = 1 + a \times q$
- Composition law: need only to check for arguments $0 + a \times q$, and only when $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, in which case $(f_1 \diamond_{\text{Opt}} f_2)(a) = 0 + c$; then

$$
\begin{aligned}
&(\text{fmapOpt}_F\, f_1 \circ \text{fmapOpt}_F\, f_2)(0 + a \times q) \\
&= \text{fmapOpt}_F(f_2)\,(\text{fmapOpt}_F(f_1)(0 + a \times q)) \\
&= \text{fmapOpt}_F(f_2)\,(0 + b \times \text{fmapOpt}_G(f_1)(q)) \\
&= 0 + c \times (\text{fmapOpt}_G\, f_1 \circ \text{fmapOpt}_G\, f_2)(q) \\
&= 0 + c \times \text{fmapOpt}_G(f_1 \diamond_{\text{Opt}} f_2)(q) \\
&= \text{fmapOpt}_F(f_1 \diamond_{\text{Opt}} f_2)(0 + a \times q)
\end{aligned}
$$

This is a "greedy filter": if $f(a)$ is empty, will delete all data in G^A

* Worked examples II: Constructions of filterable functors III

(6) The `fmapOpt` laws hold for $F^A \equiv G^A + A \times F^A$ if they hold for G^A

- For $f^{A\Rightarrow 1+B}$, we have $\text{fmapOpt}_G(f) : G^A \Rightarrow G^B$ and $\text{fmapOpt}'_F(f) : F^A \Rightarrow F^B$ (for use in recursive arguments as the inductive assumption)
- Define $\text{fmapOpt}_F(f)(q^{G^A} + a^A \times p^{F^A})$ by returning $0 + \text{fmapOpt}'_F(f)(p)$ if $f(a) = 1 + 0$, and $\text{fmapOpt}_G(f)(q) + b \times \text{fmapOpt}'_F(f)(p)$ otherwise
- Identity law: $\text{id}_{\diamond_{\textbf{Opt}}}(x) \neq 1 + 0$, so $\text{fmapOpt}_F(\text{id}_{\diamond_{\textbf{Opt}}})(q + a \times p) = q + a \times p$
- Composition law:
  $(\text{fmapOpt}_F(f_1) \circ \text{fmapOpt}_F(f_2))(q + a \times p) = \text{fmapOpt}_F(f_1 \diamond_{\textbf{Opt}} f_2)(q + a \times p)$
- For arguments $q + 0$, the laws for G^A hold; so assume arguments $0 + a \times p$. When $f_1(a) = 0 + b$ and $f_2(b) = 0 + c$, the proof of the previous example will go through. So we need to consider the two cases $f_1(a) = 1 + 0$ and $f_1(a) = 0 + b$, $f_2(b) = 1 + 0$
- If $f_1(a) = 1 + 0$ then $(f_1 \diamond_{\textbf{Opt}} f_2)(a) = 1 + 0$; to show $\text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\textbf{Opt}} f_2)(p)$, use the inductive assumption about $\text{fmapOpt}'_F$ on p
- If $f_1(a) = 0 + b$ and $f_2(b) = 1 + 0$ then $(f_1 \diamond_{\textbf{Opt}} f_2)(a) = 1 + 0$; to show $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_1 \diamond_{\textbf{Opt}} f_2)(p)$, rewrite $\text{fmapOpt}_F(f_2)(0 + b \times \text{fmapOpt}'_F(f_1)(p)) = \text{fmapOpt}'_F(f_2)(\text{fmapOpt}'_F(f_1)(p))$ and again use the inductive assumption about $\text{fmapOpt}'_F$ on p

This is a "list-like filter": if $f(a)$ is empty, will recurse into nested F^A data

Worked examples II: Constructions of filterable functors IV

Use known filterable constructions to show that

$F^A \equiv (\text{Int} \times \text{String}) \Rightarrow (1 + \text{Int} \times A + A \times (1 + A) + (\text{Int} \Rightarrow 1 + A + A \times A \times \text{String}))$

is a filterable functor

- Instead of implementing `Filterable` and verifying laws by hand, we analyze the structure of this data type and use known constructions
- Define some auxiliary functors that are parts of the structure of F^A,
  - $R_1^A = (\text{Int} \times \text{String}) \Rightarrow A$ and $R_2^A = \text{Int} \Rightarrow A$
  - $G^A = 1 + \text{Int} \times A + A \times (1 + A)$ and $H^A = 1 + A + A \times A \times \text{String}$
- Now we can rewrite $F^A = R_1\left[G^A + R_2\left[H^A\right]\right]$
  - G^A is filterable by construction 5 because it is of the form $G^A = 1 + A \times K^A$ with filterable functor $K^A = 1 + \text{Int} + A$
  - K^A is of the form $1 + A + X$ with constant type X, so it is filterable by constructions 1 and 3 with the `Option` functor $1 + A$
  - H^A is filterable by construction 5 with $H^A = 1 + A \times (1 + A \times \text{String})$, while $1 + A \times \text{String}$ is filterable by constructions 5 and 1
- Constructions 3 and 4 show that $R_1\left[G^A + R_2\left[H^A\right]\right]$ is filterable

Note that there are more than one way of implementing `Filterable` here

* Exercises II

1. Implement a `Filterable` instance for `type F[T] = G[H[T]]` assuming that the functor `H[T]` already has a `Filterable` instance (construction 4). Verify the laws rigorously (i.e. by calculations, not tests).
2. For `type F[T] = Option[Int ⇒ Option[(T, T)]]`, implement a `Filterable` instance. Show that the filterable laws hold by using known filterable constructions (avoiding explicit proofs or tests).
3. Implement a `Filterable` instance for $F^A \equiv G^A + \text{Int} \times A \times A \times F^A$ (recursive) for a filterable functor G^A. Verify the laws rigorously.
4. Show that $F^A = 1 + A \times G^A$ is in general *not* filterable if G^A is an arbitrary (non-filterable) functor; it is enough to give an example.
5. Show that $F^A = 1 + G^A + H^A$ is filterable if $1 + G^A$ and $1 + H^A$ are filterable (even when G^A and H^A are by themselves not filterable).
6. Show that the functor $F^A = A + (\text{Int} \Rightarrow A)$ is not filterable.
7. Show that one can define `deflate`: $C^{1+A} \Rightarrow C^A$ for any contrafunctor C^A (not necessarily filterable), similarly to how one can define `inflate`: $F^A \Rightarrow F^{1+A}$ for any functor F^A (not necessarily filterable).

* Bonus slide I: Definition of filterable contrafunctors

When is a contrafunctor filterable?

When a contrafunctor C^A with $\text{contrafmap} : (B \Rightarrow A) \Rightarrow C^A \Rightarrow C^B$ has also

- `filter`/`withFilter`: $(A \Rightarrow \text{Boolean}) \Rightarrow C^A \Rightarrow C^A$ – same as for functors
- `inflate`: $C^A \Rightarrow C^{\mathbb{1}+A}$ and `contrafmapOpt`: $(B \Rightarrow \mathbb{1} + A) \Rightarrow C^A \Rightarrow C^B$
- All three functions are computationally equivalent...
  - $\text{filter}(p^{A\Rightarrow\text{Boolean}}) = \text{inflate}^{C^A\Rightarrow C^{\mathbb{1}+A}} \circ \text{contrafmap}(\text{bop}\, p)$
  - $\text{inflate}^{C^A\Rightarrow C^{\mathbb{1}+A}} = \text{contrafmap}\left(0 + x^A \Rightarrow x\right) \circ \text{filter}\left(\_ \Rightarrow \text{true}\right)$
  - $\text{contrafmapOpt}\, f^{B\Rightarrow\mathbb{1}+A} = \text{inflate} \circ \text{contrafmap}\, f$
  - $\text{inflate} = \text{contrafmapOpt}\left(\text{id}^{\mathbb{1}+A\Rightarrow\mathbb{1}+A}\right)$
- but have different laws
  - 4 laws (naturality, conjunction, identity, partial function) for `filter`
  - 3 laws (naturality, conjunction, identity) for `inflate`
  - 2 laws (identity, contracomposition) for `contrafmapOpt`
    - as before, `contrafmapOpt` is a "twisted" version of `contrafmap`
- Examples of filterable contrafunctors
  - $C^A \equiv A \Rightarrow 1 + Z$ where Z is a fixed type
  - $C^A \equiv 1 + A \Rightarrow Z$
- Examples of non-filterable contrafunctors
  - $C^A \equiv A \times F^A \Rightarrow Z$ – cannot implement `inflate`

* Bonus slide II: Structure of filterable contrafunctors

How to build up a filterable contrafunctor from parts?

- Filterable contrafunctors "can consume fewer data items"
- The easiest function to consider first is `inflate`

Some constructions of filterable contrafunctors:

1. $C^A = Z$ (constant contrafunctor)
   Functor constructions (no need to check laws for these):
2. $F^A \equiv G^A \times H^A$ for any filterable contrafunctor G^A and H^A
3. $F^A \equiv G^A + H^A$ for any filterable contrafunctor G^A and H^A
4. $F^A \equiv G^{H^A}$ for H^A a filterable (contra)functor and G^A any (contra)functor – various combinations possible here
5. $F^A \equiv G^A \Rightarrow H^A$ if functor G^A and contrafunctor H^A *both filterable*
   Special constructions:
6. $F^A \equiv 1 + A \times G^A \Rightarrow H^A$ where G^A and H^A are filterable
7. $F^A \equiv A \times G^A \Rightarrow 1 + H^A$ if G^A and H^A are filterable