Chapter 9: Traversable functors and contrafunctors

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Motivation for the `traverse` operation

- Consider data of type List^A and processing $f : A \Rightarrow \text{Future}^B$
- Typically, we want to wait until the entire data set is processed
- What we need is $\text{List}^A \Rightarrow \left(A \Rightarrow \text{Future}^B\right) \Rightarrow \text{Future}^{\text{List}^B}$
- Generalize: $L^A \Rightarrow \left(A \Rightarrow F^B\right) \Rightarrow F^{L^B}$ for some type constructors F, L
- This operation is called `traverse`
  - How to implement it: for example, a 3-element list is $A \times A \times A$
  - Consider $L^A \equiv A \times A \times A$, apply map f and get $F^B \times F^B \times F^B$
  - We will get $F^{L^B} \equiv F^{B \times B \times B}$ if we can apply `zip` as $F^B \times F^B \Rightarrow F^{B \times B}$
- So we need to assume that F is applicative
- In Scala, we have `Future.traverse()` that assumes L to be a sequence
  - This is the iconic example that fixes the requirements
- Questions:
  - Which functors L can have this operation?
  - Can we express `traverse` through a simpler operation?
  - What are the required laws for `traverse`?
  - What about contrafunctors or profunctors?

Deriving the `sequence` operation

- The type signature of `traverse` is a complicated "lifting"
  - A "lifting" is always equivalent to a simpler natural transformation
- To derive it, ask: what is missing from `fmap` to do the job of `traverse`?

$$\text{fmap} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow L^{F^B}$$

- We need F^{L^B}, but the `traverse` operation gives us L^{F^B} instead
  - What's missing is a natural transformation `sequence` : $L^{F^B} \Rightarrow F^{L^B}$
- The functions `traverse` and `sequence` are computationally equivalent:

$$\text{trav}\ f^{\underline{A \Rightarrow F^B}} = \text{fmap}\ f \circ \text{seq}$$

$$\begin{tikzcd} & L^{F^B} \arrow[dr, "\text{seq}"] & \\ L^A \arrow[ur, "\text{fmap}\ f^{\underline{A\Rightarrow F^B}}"] \arrow[rr, "\text{trav}\ f^{\underline{A\Rightarrow F^B}}"'] & & F^{L^B} \end{tikzcd}$$

Here F is an *arbitrary* applicative functor

- Keep in mind the example `Future.sequence` : $\text{List}^{\text{Future}^X} \Rightarrow \text{Future}^{\text{List}^X}$
- Examples: List;
- Non-traversable: $L^A \equiv R \Rightarrow A$; lazy lists ("infinite streams")
  - ★ Note: We *cannot have* the opposite transformation $F^{L^B} \Rightarrow L^{F^B}$

Polynomial functors are traversable

- Generalize from the example $L^A \equiv A \times A \times A$ to other polynomials
- Polynomial functors have the form

$$L^A \equiv Z \times A \times ... \times A + Y \times A \times ... \times A + ... + Q \times A + P$$

- To implement $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$, consider monomial $L^A \equiv Z \times A \times ... \times A$
- We have $L^{F^B} = Z \times F^B \times ... \times F^B$; apply `zip` and get $Z \times F^{B \times ... \times B}$
- Lift Z into the functor F using $Z \Rightarrow F^A \Rightarrow F^{Z \times A}$ (or with F.pure)
- The result is $F^{Z \times B \times ... \times B} \equiv F^{L^B}$
  - For a polynomial L^A, apply this procedure to each monomial
  - Note that we could apply `zip` in various different orders
- The traversal order is arbitrary, may be application-specific
- Non-polynomial functors are not traversable (see Bird et al., 2013)
  - Example: $L^A \equiv E \Rightarrow A$, $F^A \equiv 1 + A$; can't have $\text{seq} : L^{F^B} \Rightarrow F^{L^B}$
- All polynomial functors are traversable, and usually in several ways
  - Therefore it is useful to have a type class for traversable functors

Motivation for the laws of the `traverse` operation

- The "law of traversals" paper (2012) argues that `traverse` should "visit each element" of the container L^A exactly once, and evaluate each corresponding "effect" F^B exactly once; then they formulate the laws
- To derive the laws, use the "lifting" intuition for `traverse`,

$$\text{trav} : (A \Rightarrow F^B) \Rightarrow L^A \Rightarrow F^{L^B}$$

Look for "identity" and "composition" laws:

1. "Identity" as `pure` $: A \Rightarrow F^A$ must be lifted to `pure` $: L^A \Rightarrow F^{L^A}$
2. "Identity" as $\text{id}^{\underline{A \Rightarrow A}}$ with $F^A \equiv A$ (identity functor) lifted to $\text{id}^{\underline{L^A \Rightarrow L^A}}$
3. "Compose" $f : A \Rightarrow F^B$ and $g : B \Rightarrow G^C$ to get $h : A \Rightarrow F^{G^C}$, where F, G are applicative; a traversal with h maps L^A to $F^{G^{L^C}}$ and must be equal to the composition of traversals with f and then with $g^{F\uparrow}$

Questions:

- Are the laws for the `sequence` operation simpler?
- Are all these laws independent?
- What functors L satisfy these laws *for all* applicative functors F?

Formulation of the laws for `traverse`

- Identity law: For any applicative functor F,

$$\text{trav}\,(\text{pure}) = \text{pure}$$

$$\begin{tikzcd} L^A \arrow[rrr, "\text{pure}^{:L^A\Rightarrow F^{L^A}}", "\text{trav}\,(\text{pure}^{:A\Rightarrow F^A})"'] & & & F^{L^A} \end{tikzcd}$$

  - Second identity law: $\text{trav}^{\text{Id}}(\text{id}^A) = \text{id}^{L^A}$ is a consequence with $F = \text{Id}$
    - ★ So, we need only one identity law

- Composition law: For any $f^{:A\Rightarrow F^B}$ and $g^{:B\Rightarrow G^C}$, & applicative F and G,

$$\text{trav}\,f \circ (\text{trav}\,g)^{\uparrow} = \text{trav}\left(f \circ g^{\uparrow}\right)$$

$$\begin{tikzcd} & F^{L^B} \arrow[dr, "\text{fmap}_F\left(\text{trav}_G\ g\right)^{:L^B\Rightarrow G^{L^C}}"] & \\ L^A \arrow[ur, "\text{trav}_F\ f^{:A\Rightarrow F^B}"] \arrow[rr, "\text{trav}_{FG}\ h^{:A\Rightarrow F^{G^C}}"'] & & F^{G^{L^C}} \end{tikzcd}$$

where $h^{:A\Rightarrow F^{G^C}} \equiv f \circ g^{F\uparrow}$

Derivation of the laws for `sequence`

The

Constructions of traversable functors

The

Traversable profunctors

The

Traversability with respect to profunctors

The

“Folding” a traversable functor into a monoid

The

Exercises

1. Show that any traversable functor L admits a method

$$\text{consume} : (L^A \Rightarrow B) \Rightarrow L^{F^A} \Rightarrow F^B$$

for any applicative functor F. Show that `traverse` and `consume` are equivalent.