Example 1: Fitting the parameters to the model using simulated data

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Background

A generalized SEIR model is used to simulate an epidemic breakout. Seven different states are considered in the following in a similar fashion as by [1], which is derived from SEIQR models [2] and has several similarities with mathematical models for SARS transmission, e.g. [3] :

1. Susceptibles cases S(t)
2. Insusceptible cases P(t)
3. Exposed cases E(t)
4. Infectious cases I(t)
5. Quarantined cases Q(t)
6. Recovered cases R(t)
7. Dead cases D(t)

The parameters are as follows:

- alpha: protection rate
- beta: infection rate
- gamma: Inverse of the average latent time
- delta: rate at which infectious people enter in quarantine
- lambda: time-dependant recovery rate
- kappa: time-dependant mortality rate

The population is assumed constant, i.e. the births and natural death are not modelled. The cure rate and mortality rate are here time-dependent but they need some empirical coefficients to tune the time-dependency of these parameters.

The generalized SEIR model [1] is:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\alpha S(t) - \beta \frac{S(t)I(t)}{N_{pop}}$$
$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\gamma E(t) + \beta \frac{S(t)I(t)}{N_{pop}}$$
$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \gamma E(t) - \delta I(t)$$
$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \delta I(t) - \lambda(t)Q(t) - \kappa(t)Q(t)$$
$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \lambda(t)Q(t)$$
$$\frac{\mathrm{d}D(t)}{\mathrm{d}t} = \kappa(t)Q(t)$$
$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \alpha S(t)$$

Recovery and mortality rates

The content presented hereafter is **not** proposed by [1]. Therefore, it can significantly differ from their work. In the following, the mortality rate $\kappa(t)$ is modelled as

$$\kappa(t) = \frac{\kappa_0}{\exp(\kappa_1(t-\tau_\kappa)) + \exp(-\kappa_1(t-\tau_\kappa))}$$

or as

$$\kappa(t) = \kappa_0 \exp\left(-[\kappa_1(t-\tau_\kappa)]^2\right)$$

or as

$$\kappa(t) = \kappa_0 + \exp(-\kappa_1(t+\tau_\kappa))$$

where κ_0, κ_1 and τ_κ are parameters to be empricially determined. The parameters κ_0 and κ_1 have the dimension of the inverse of a time and τ_κ has the dimension of a time.

The recovery rate $\lambda(t)$ is either modelled as

$$\lambda(t) = \frac{\lambda_0}{1 + \exp(-\lambda_1(t-\tau_\lambda))}$$

or as

$$\lambda(t) = \lambda_0 + \exp(-\lambda_1(t+\tau_\lambda))$$

where λ_0, λ_1 and τ_λ are parameters to be empricially determined. The parameters λ_0 and λ_1 have the dimension of the inverse of a time and τ_λ has the dimension of a time.

The choice of best approximation for $\lambda(t)$ and $\kappa(t)$ is done automatically inside the function fit_SEIRQDP based on a preliminary assessment of the recovery rate and the mortality rate. The idea behind these functions is that the mortality rate should become close to zero (or a constant value κ_0) at time increases while the recovery rate converges toward a constant value λ_0.

Numerical solutions

I also decided to re-write the system of ODEs in a matrix form for the sake of clarity:

$$\frac{\mathrm{dY}}{\mathrm{dt}} = A * Y + F$$

where

$$Y = [S, E, I, Q, R, D, P]^\top$$

$$A = \begin{bmatrix} -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & -\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & -\kappa(t) - \lambda(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa(t) & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = S(t) \cdot \ I(t) \cdot \begin{bmatrix} -\frac{\beta}{N_{\mathrm{pop}}} \\ \frac{\beta}{N_{\mathrm{pop}}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The equation $\frac{\mathrm{dY}}{\mathrm{dt}} = A * Y + F$ is then solved using the classic 4th order Runge-Kutta method.

Important note

The initial number of exposed E_0 and infectious I_0 is not known. In the present version, these values are not estimated by least-square fit altough the choice of E_0 and I_0 could affect the godness of the fit and the estimated parameters.

References

[1] Peng, L., Yang, W., Zhang, D., Zhuge, C., & Hong, L. (2020). Epidemic analysis of COVID-19 in China by dynamical modeling. *arXiv preprint arXiv:2002.06563*.

[2] Feng, Z., & Thieme, H. R. (1995). Recurrent outbreaks of childhood diseases revisited: the impact of isolation. *Mathematical Biosciences*, *128*(1-2), 93-130.

[3] Lipsitch, M., Cohen, T., Cooper, B., Robins, J. M., Ma, S., James, L., ... & Fisman, D. (2003). Transmission dynamics and control of severe acute respiratory syndrome. *Science*, *300*(5627), 1966-1970.

Example

Initialisation

Case of an imaginary epidemy outbreak that took place on 2010-01-01. The simulation time is set to 6 months.

```
clearvars;close all;clc;

% Time definition
dt = 0.1; % time step
time1 = datetime(2010,01,01,0,0,0):dt:datetime(2010,06,01,0,0,0);
N = numel(time1);
t = [0:N-1].*dt;
```

Generate the data

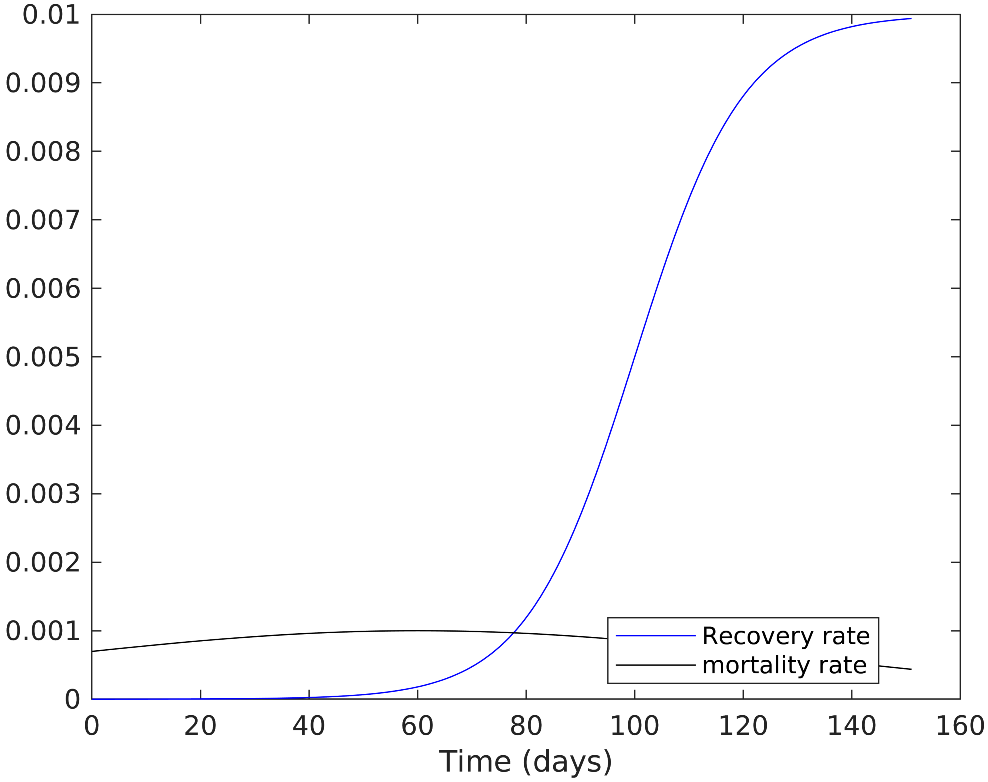
```
Npop= 60e6; % population (60 millions)
Q0 = 200; % Initial number of infectious that have bee quanrantined
I0 = Q0; % Initial number of infectious cases non-quarantined
E0 = 0; % Initial number of exposed
R0 = 10; % Initial number of recovereds
D0 = 10; % Initial number of deads
alpha = 0.08; %  protection rate
beta = 0.9; %  infection rate
gamma= 0.2; % inverse of average latent time
delta= 0.5; % rate at which infectious people enter in quarantine
Lambda = [0.01 0.1 100]; % cure rate (time dependant)
Kappa =  [0.001 0.01,60]; % mortality rate (time dependant)

% Choice of a particular form for lambda(t)
lambdaFun0 = @(a,t) a(1)./(1+exp(-a(2)*(t-a(3))));
kappaFun0 = @(a,t) a(1).*exp(-(a(2)*(t-a(3))).^2);

[S,E,I,Q,R,D,P] = SEIQRDP(alpha,beta,gamma,delta,Lambda,Kappa,Npop,E0,I0,Q0,R0,D0,t,lam

figure
```

```
plot(t,lambdaFun0(Lambda,t),'b',t,kappaFun0(Kappa,t),'k')
legend('Recovery rate','mortality rate','location','best');
xlabel('Time (days)')
```




```
set(gcf,'color','w')
```

Fit the data

The fitting is done using the time histories of the number of quarantined $Q(t)$, recovered $R(t)$ and deads $D(t)$ only. The number of exposed, susceptible, insusceptible and infectious is computed in the model but not used as target.

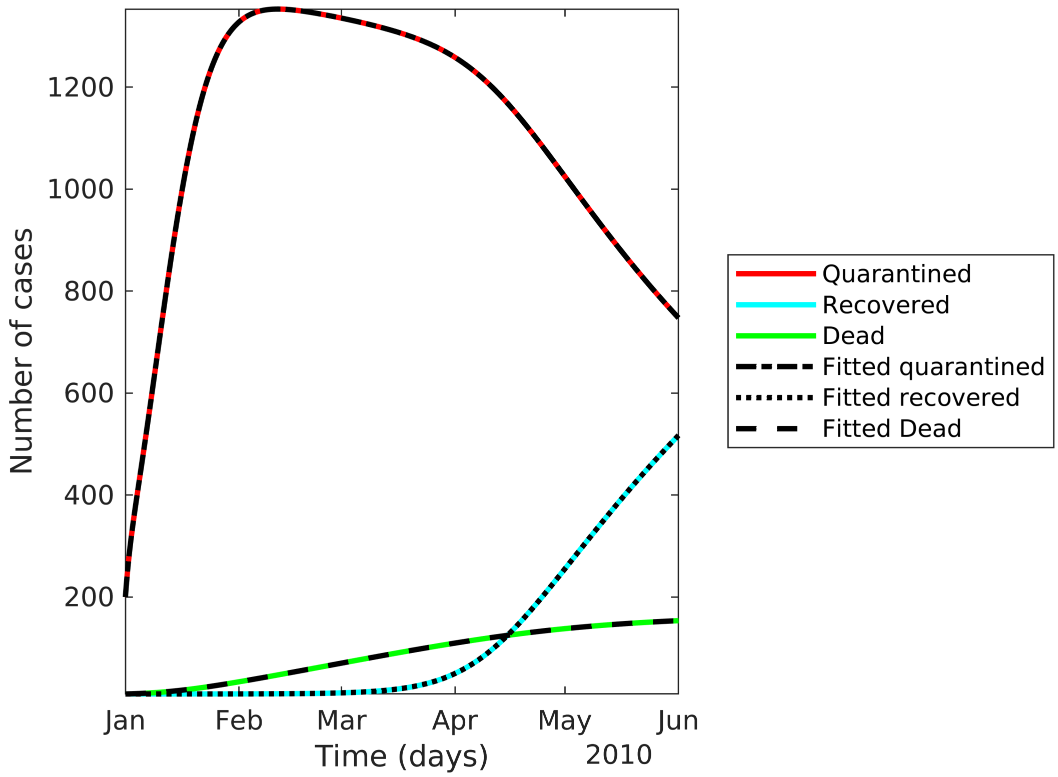
```
lambdaGuess = [0.01 0.5 0.1];
kappaGuess = [0.01 0.1,10];
alphaGuess = 0.05;
betaGuess = 0.7;
deltaGuess = 0.2;
gammaGuess = 0.3;
guess = [alphaGuess,betaGuess,deltaGuess,gammaGuess,lambdaGuess,kappaGuess]; % initial

[alpha1,beta1,gamma1,delta1,Lambda1,Kappa1,lambdaFun,kappaFun0] = fit_SEIQRDP(Q,R,D,Npo
[S1,E1,I1,Q1,R1,D1,P1] =...
    SEIQRDP(alpha1,beta1,gamma1,delta1,Lambda1,Kappa1,Npop,E0,I0,Q0,R0,D0,t,lambdaFun,k
```

Comparison beween fitted and generated time histories

```
figure
```

```
clf;close all;
plot(time1,Q,'r',time1,R,'c',time1,D,'g','linewidth',2);
hold on
plot(time1,Q1,'k-.',time1,R1,'k:',time1,D1,'k--','linewidth',2);
% ylim([0,1.1*Npop])
ylabel('Number of cases')
xlabel('Time (days)')
leg = {'Quarantined','Recovered','Dead','Fitted quarantined','Fitted recovered','Fitted
legend(leg{:},'location','eastoutside')
set(gcf,'color','w')
axis tight
```



Case where the recovered (R) and quarantined (Q) data are not available separately

The number of qarantined Q(t) and recovered cases R(t) is unknwon, However, Q(t) + R(t) is known.

```
guess = [alphaGuess,betaGuess,deltaGuess,gammaGuess,lambdaGuess,kappaGuess]; % initial

[alpha1,beta1,gamma1,delta1,Lambda1,Kappa1,lambdaFun,kappaFun] = ...
    fit_SEIQRDP(Q+R,[],D,Npop,E0,I0,time1,guess,'Display','off');
```

```
Warning:  No data available for "Recovered"
```

```
[S1,E1,I1,Q1,R1,D1,P1] =...
    SEIQRDP(alpha1,beta1,gamma1,delta1,Lambda1,Kappa1,...
    Npop,E0,I0,Q0,R0,D0,t,lambdaFun,kappaFun);
```

```
figure
clf;close all;
plot(time1,Q+R,'r',time1,D,'g','linewidth',2);
hold on
plot(time1,Q1+R1,'k-.',time1,D1,'k--','linewidth',2);
% ylim([0,1.1*Npop])
ylabel('Number of cases')
xlabel('Time (days)')
leg = {'Tested positive minus the deceased cases','Deceased cases','Fitted Tested posit
legend(leg{:},'location','southoutside')
set(gcf,'color','w')
axis tight
```