The Solutions

B—bookwork, A—application, E—new example, T—new theory

1. Random variables.

a)

i) Y=1 and Y=0 are all the possible values and their corresponding probabilities are both equal to ½. So, $E(Y) = 1 * P(Y = 1) + 0 * P(Y = 0) = 1 * \frac{1}{2} + 0 * \frac{1}{2} = 1/2$. [3E]

Y is called a Bernoulli random variable. [3E]

ii) The probability of having j successes knowing that you have flipped the coin for n times is the probability of j successes in any n independent experiments. If we denote the X as the number of successes, then random variable X obeys a binomial distribution: [3E]

$$P(X = j) = \binom{n}{j} p^{j}(1-p)^{(n-j)}$$

$\binom{n}{j}$ (read as "n choose j") is the binomial coefficient, the number of j-combinations from a given set of n elements is also denoted by a variation such as C_n^j. [3E]

iii) If random variable X represents the number of flips, the x=k mean that the "success" comes at the k-th flips and all the k-1 flips before are "failed", so the probability of P(x=k) is [3E]

$$p(1-p)^{k-1}, k = 1,2, \ldots.,$$ [3E]

and it is a Geometric distribution.

b) Proof:

$$\Pr(X \leq x) = \Pr(F^{-1}(U) \leq x) = \Pr(U \leq F(x)) = F(x)$$ [4T]

where the first equality is by definition, second due to monotonicity of F(x), and third because U is uniform. [3T]

2. Estimation and sequences of random variables.

a)

In this problem, n = 3. $\bar{x} = \frac{x_1+\cdots+x_n}{n} = 4$ [1E]

$$f(\boldsymbol{x}, c) = c^{4n}(x_1 \dots x_n)^3 e^{-c(x_1+\cdots+x_n)} = c^{4n}(x_1 \dots x_n)^3 e^{-cn\bar{x}} \quad [3E]$$

$$\frac{\partial f(\boldsymbol{x}, c)}{\partial c} = (\frac{4n}{c} - n\bar{x}) f(\boldsymbol{x}, c) = 0 \quad [3E]$$

$$\hat{c} = \frac{4}{\bar{x}} = 1 \quad [3E]$$

b)

i) Markov

$$P(|X| > a) \leq \frac{E(|X|)}{a} = \frac{\sqrt{\frac{2}{\pi}}\sigma}{a} = \frac{\sqrt{\frac{2}{\pi}}\sigma}{3\sigma} = 0.266 \quad [5A]$$

ii) Chebyshev

$$P(|X| > a) \leq \frac{\sigma^2}{a^2} = \frac{1}{9} = 0.111 \quad [5A]$$

iii) Chernoff

$$P(|X| > a) = 2P(X > a) \leq 2e^{-\frac{a^2}{2\sigma^2}} = 2e^{-4.5} = 0.0222 \quad [5A]$$

3. Random processes.

a)

i) The probability

$$P = P[N(4) - N(0) \le 1, N(8) - N(4) \ge 2, N(12) - N(8) \le 1\ ]$$

is required. Recall the number of arrivals $N(t_1, t_2)$ in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt.

Due to the properties of Poisson processes, [2E]

$$P = P[N(4) - N(0) \le 1]P[N(8) - N(4) \ge 2]P[N(12) - N(8) \le 1\ ]$$

$$= P[N(4) - N(0) \le 1]P[N(8) - N(4) \ge 2]P[N(12) - N(8) \le 1\ ]$$

$$= P[N(4) \le 1]P[N(4) \ge 2]P[N(4) \le 1\ ]$$

Since

$$P[N(4) \le 1] = P[N(4) = 0] + P[N(4) = 1]$$

and [2E]

$$P[N(4) = 0] = e^{-0.25\times 4} = e^{-1} = 0.368$$

$$P[N(4) = 1] = 0.25 \times 4 \times e^{-0.25\times 4} = e^{-1}$$

we get [2E]

$$P[N(4) \le 1] = 2 \times 0.368 = 0.736.$$

Of course [2E]

$$P[N(4) \ge 2] = 1 - P[N(4) \le 1] = 0.264$$

Finally, the required probability [2E]

$$P = 0.736 \times 0.264 \times 0.736 = 0.143$$

ii) This means that there are at most two failures in 4 hours. Since the number of failures in 4 hours is a Poisson random variable with parameter 1, [2E]

$$P[t_3 \ge 4] = P[N(4) \le 2] = P[N(4) = 0] + P[N(4) = 1] + P[N(4) = 2]$$
$$= e^{-1} + e^{-1} + \frac{1}{2}e^{-1} = 2.5 \times 0.368 = 0.92$$

[3E]

b)

i) matched filter

$$h(t) = s(t_0 - t) = \begin{cases} \sin(2\pi(1-t)) = -\sin(2\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

[5E]

ii) maximum SNR

$$(SNR)_{\max} = \frac{E_s}{N_0} = \frac{\int_0^{+\infty} s(t)^2 dt}{N_0}$$

$$= \int_0^1 [\sin(2\pi t)]^2 dt$$

$$= \int_0^1 \frac{1-\cos(4\pi t)}{2} dt$$

$$= 1/2$$

[5E]

4. Markov chains and martingales.

a) Firstly, notice that N = 50. After the first 10 flips, the gambler's capital is i = 12. [3A]

i) For a fair coin, his ruin probability is

$$P = \frac{N-i}{N} = \frac{38}{50} = \frac{19}{25}$$

[3A]

ii) For p=1/3,

$$P = \frac{1-\left(\frac{p}{q}\right)^{N-i}}{1-\left(\frac{p}{q}\right)^{N}} = \frac{1-(1/2)^{N-i}}{1-(1/2)^{N}} = \frac{1-(1/2)^{38}}{1-(1/2)^{50}} \approx 1$$

[4A]

b)

i) Just use definition

$E[Y_n|Y_{n-1}, Y_{n-2}, \ldots, Y_1] = Y_{n-1} + E[X_n\lambda^n] = Y_{n-1}$ [3T]

ii) Firstly, each term has characteristic function

$$E\left[e^{j\omega X_i\lambda^i}\right] = \frac{1}{2}[e^{j\omega\lambda^i} + e^{-j\omega\lambda^i}] = \cos(\omega\lambda^i)$$

Since X_i's are i.i.d., the c.f. of Y_n is simply the product

$$\prod_{i=1}^{n}\cos(\omega\lambda^i)$$ [4T]

iii) Note that

$Y_{n+1} = X_0 + \sum_{i=1}^{n+1} X_i\lambda^i = X_0 + \lambda\sum_{i=1}^{n+1} X_i\lambda^{i-1}.$

Let us denote $Y_n{}' = \sum_{i=1}^{n+1} X_i\lambda^{i-1}$, which has exactly the same distribution as $Y_n = \sum_{i=0}^{n} X_i\lambda^i$, again because X_i's are i.i.d.

Now we can write

$$Y_{n+1} = X_0 + \lambda Y_n' = T(Y_n') = \begin{cases} 1 + \lambda Y_n' = T_1(Y_n') & if\ X_0 = 1 \\ -1 + \lambda Y_n' = T_2(Y_n') & otherwise \end{cases}$$

Obviously,

$$\begin{aligned} P(Y_{n+1} \in E) &= P(T(Y_n') \in E) \\ &= \frac{1}{2}P(T(Y_n') \in E|X_0 = 1) + \frac{1}{2}P(T(Y_n') \in E|X_0 = -1) \\ &= \frac{1}{2}P(T_1(Y_n') \in E) + \frac{1}{2}P(T_2(Y_n') \in E) \\ &= \frac{1}{2}P\left(Y_n' \in T_1^{-1}(E)\right) + \frac{1}{2}P\left(Y_n' \in T_2^{-1}(E)\right) \\ &= \frac{1}{2}P\left(Y_n \in T_1^{-1}(E)\right) + \frac{1}{2}P\left(Y_n \in T_2^{-1}(E)\right) \end{aligned}$$

[4T]

iv) The limiting distribution must satisfy

$$P(Y \in E) = \frac{1}{2} P(Y \in T_1^{-1}(E)) + \frac{1}{2} P(Y \in T_2^{-1}(E))$$ [4T]

By inspection (it's best to draw a figure of function T defined above), we find it is a uniform distribution on [-2, 2].

NB: $\lambda = ½$ is a lucky case. The general problem for $\lambda \in (0, 1)$ is difficult and remains a major open question in number theory, except some special cases. This problem is called Bernoulli convolution; interested students may visit webpage

https://www.dpmms.cam.ac.uk/~pv270/bernoulli.html