DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

********* Solutions *********

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate, f, of a signal in Hz may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output	IIR	Infinite Impulse Response
CTFT	Continuous-Time Fourier Transform	LTI	Linear Time-Invariant
DCT	Discrete Cosine Transform	MDCT	Modified Discrete Cosine Transform
DFT	Discrete Fourier Transform	PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform	SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response		

A datasheet is included at the end of the examination paper.

Key: B=bookwork, U=unseen example, T=Novel theory

********* Questions and Solutions *********

1. (a) (i) The geometric way of seeing this is to derive the convolution $v[n]$ by flipping $y[n]$ and then sliding it along $x[n]$. We can divide the convolution into three portions:

I. part of $y[n]$ extends before $x[0]$,

II. $y[n]$ fits entirely within $x[n]$,

III. part of $y[n]$, extends past $x[N-1]$.

Only when $y[n]$ fits entirely within $x[n]$ will $v[n] = w[n]$. This corresponds to $M - 1 \leq n \leq N - 1$.

To derive this algebraically, we note that for $v[n] = w[n]$, we need $y[n-r] = y[(n-r)_{\text{mod } N}]$ for the summing range $r = 0, \ldots, N-1$. We know that $y[i] = y[i_{\text{mod } N}]$ for $0 \leq i \leq N-1$ since the "mod N " then has no effect. Less obviously, it will also be true when $M - N \leq i \leq -1$ since for these values $y[i] = y[i+N] = 0$. So overall $y[i] = y[i_{\text{mod } N}]$ whenever $M - N \leq i \leq N - 1$.

So, when $r = 0$, $y[n-r] = y[(n-r)_{\text{mod } N}]$ requires $n - r = n \leq N - 1$. When $r = N - 1$, $y[n-r] = y[(n-r)_{\text{mod } N}]$ requires $n - r = n - N + 1 \geq M - N \Rightarrow n \geq M - 1$. Combining these gives $M - 1 \leq n \leq N - 1$. [3]

(ii) $v[1] = 1 \times 2 + 2 \times 5 = 12$ and $w[1] = 1 \times 2 + 2 \times 5 + 3 \times 7 = 33$.
For completeness, $v[n] = [5,12,22,16,24,26,21]$ and $w[n] = [31,33,22,16,24]$. [2]

(b) (i) An LTI system is BIBO stable if a bounded input sequence, $x[n]$ always gives a bounded output sequence $y[n]$. That is, $|x[n]| < B \; \forall n \Rightarrow |y[n]| < f(B) \; \forall n$. [2]

(ii) Define $x[n] = +1$ if $h[-n] \geq 0$ and $x[n] = -1$ otherwise. Then $y[0] = \sum_{r=-\infty}^{\infty} h[r]x[0-r] = \sum_{r=-\infty}^{\infty} |h[n]|$ which is finite from the BIBO assumption since $x[n] \leq 1$ and thus is bounded. [3]

(c) (i) $W = 0$ when the numerator is zero but the denominator is non-zero which is when $0.5(M+1)\omega = k\pi$ for $k \neq 0$. This happens first when $k = 1$ giving $\omega = \frac{2\pi}{M+1}$. [1]

(ii) Using the small angle formula $\sin\theta = \theta$ gives $W = \frac{0.5(M+1)\omega}{(M+1)\times 0.5\omega}$ dB for small ω (including ω=0). Equivalently, you can use L'Hôpital's rnule. [2]

(iii) At $\omega = \frac{3\pi}{M+1}$, $|W| = \frac{1}{(M+1)\sin\frac{1.5\pi}{M+1}} \approx \frac{1}{1.5\pi} = 0.212 = -13.5\text{dB}$. [2]

(d) (i) $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=\frac{M+1}{2}}^{M} h[n]e^{-j\omega n}$.

Substituting $r = M - n$ in the second summation and reversing the order of summation gives:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{r=0}^{\frac{M-1}{2}} h[M-r]e^{j\omega(M-r)}$$

Since $h[M-r] = h[r]$ we can combine the sums to give

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n](e^{-j\omega n} + e^{j\omega(M-r)}) = e^{-\frac{j\omega M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega\left(n-\frac{M}{2}\right)} + e^{j\omega\left(n-\frac{M}{2}\right)}\right)$$

$$H(e^{j\omega}) = 2e^{-\frac{j\omega M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h[n]\cos\left(n - \frac{M}{2}\right)\omega$$

Hence, $A = 2$ and $\theta(\omega) = -\frac{M}{2}\omega$. [4]

(ii) $\angle H(e^{j\omega}) = \theta(\omega) = -\frac{M}{2}\omega$ so $\tau_H(e^{j\omega}) = \frac{M}{2}$ [1]

(e) From the block diagram we can write $Y = (z^{-1} + a)V$ and $V = X - az^{-1}V$. From the second of these, we get $X = (z^{-1} + a)V$ and combining the two equations (to eliminate V) then gives $\frac{Y}{X} = \frac{z^{-1}+a}{1+az^{-1}}$ which is an allpass filter. [5]

(f) (i) For the causal signal we have:

$$X[z] = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \; |x| < 1$$

Therefore, $X[z] = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \left|\frac{a}{z}\right| < 1, |z| > |a|$

For the anticausal signal $-\gamma^n u[-n-1]$, we have

$$X[z] = \sum_{n=-\infty}^{\infty} -\gamma^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n$$

$$= -\frac{z}{\gamma}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right)\left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right]$$

Therefore, $X[z] = -\left(\frac{z}{\gamma}\right)\frac{1}{1-\frac{z}{\gamma}} = \frac{z}{z-\gamma}, \left|\frac{z}{\gamma}\right| < 1, |z| < |\gamma|$ [2]

(ii) We wish to write $H[z] = \frac{b}{2-z^{-1}} + \frac{c}{1+4z^{-1}} = \frac{(b+2c)+(4b-c)z^{-1}}{(2-z^{-1})(1+4z^{-1})}$.

By matching coefficients, we obtain $b + 2c$=2 and $4b - c = 17$ from which $b = 4$ and $c = -1$.

$$H[z] = \frac{2}{2 - z^{-1}} - \frac{1}{1 + 4z^{-1}}$$

The corresponding poles are $z = 0.5$ and $z = -4$ so the sequence we need is $2 \times 0.5^n u[n] - (-(-4)^n u[-n-1]) = 2^{1-n} u[n] + (-4)^n u[-n-1]$. [3]

(g) (i) Taking z-transforms: $Y(z) = az^{-1}Y(z) + (1-a)X(z)$ from which $H[z] = \frac{Y[z]}{X[z]} = \frac{1-a}{1-az^{-1}}$.

$h[-1] = 0, h[0] = 1 - a, h[1] = (1-a)a, h[2] = (1-a)a^2$. [2]

(ii) The system function is $H[z] = \frac{1-a}{1-az^{-1}} = \frac{(1-a)z}{z-a}$. This has a pole at $z = a$ and a zero at $z = 0$. [2]

(iii) The system function is $|H(e^{j\omega})|^2 = H(e^{j\omega}) \times H^*(e^{j\omega}) = 0.5$. Thus

$$0.5 = \frac{1-a}{1-ae^{-j\omega}} \times \frac{1-a}{1-ae^{-j\omega}} = \frac{(1-a)^2}{1-2a\cos\omega + a^2}$$

From this, $\cos\omega = \frac{1+a^2-2(1-a)^2}{2a} = \frac{4a-1-a^2}{2a} = 1 - \frac{(1-a)^2}{2a}$ and so

$\omega_{3\text{dB}} = \cos^{-1}\left(\frac{4a-1-a^2}{2a}\right) = \cos^{-1}\left(1 - \frac{(1-a)^2}{2a}\right)$ [3]

(iii) For $n \geq 0$, we have from part (i) that $(1-a)a^n$. This can also be proved by induction from the given recurrence relation assuming that $x[n]$ is an impulse at $n = 0$. If we now substitute $t = \frac{n}{f_s}$ (or, equivalently $t = nT$ where the sample period is $T = \frac{1}{f_s}$) into $g(t) = Ae^{-\frac{t}{\tau}}$ we obtain $h[n] = g\left(\frac{n}{f_s}\right) = Ae^{-\frac{n}{\tau f_s}}$ from which $A = 1-a$ and $a = e^{-\frac{1}{\tau f_s}}$. Rearranging the later equation gives $\tau = \frac{-1}{f_s \ln a}$. [3]

1. (a) (i) We can write

$$1+\frac{(|\hat{z}|^2-1)(1-\lambda^2)}{|1-\lambda\hat{z}|^2}=\frac{(1-\lambda(\hat{z}+\hat{z}^*)+\lambda^2|\hat{z}|^2)+|\hat{z}|^2-\lambda^2|\hat{z}|^2-1+\lambda^2}{|1-\lambda\hat{z}|^2}$$

$$=\frac{|\hat{z}|^2-\lambda(\hat{z}+\hat{z}^*)+\lambda^2}{|1-\lambda\hat{z}|^2}=\frac{|\hat{z}|^2-\lambda(\hat{z}+\hat{z}^*)+\lambda^2}{|1-\lambda\hat{z}|^2}=\frac{(\hat{z}-\lambda)(\hat{z}^*-\lambda)}{|1-\lambda\hat{z}|^2}$$

$$=\frac{|\hat{z}-\lambda|^2}{|1-\lambda\hat{z}|^2}=|z|^2$$

Since $|\lambda|<1$, the numerator term $(1-\lambda^2)$ must be strictly positive. In addition, the denominator term satisfies $|1-\lambda\hat{z}|^2\geq 0$. Hence, assuming for the moment that $1-\lambda\hat{z}\neq 0$, the sign of the fraction is equal to the sign of $(|\hat{z}|^2-1)$ and is positive or negative according to whether $|\hat{z}|>1$ or $|\hat{z}|<1$. Clearly $|\hat{z}|=1$ makes the fraction zero and hence $|z|=1$. Putting all this together, we have shown that $|\hat{z}|<1\Rightarrow|z|<1$ and $|\hat{z}|\geq 1\Rightarrow|z|\geq 1$ which is equivalent to $|z|<1\Rightarrow|\hat{z}|<1$. The special case, $1-\lambda\hat{z}=0$, arises when $\hat{z}=\lambda^{-1}>1$. In this case, the numerator of the fraction is strictly positive and $|z|=+\infty\nless 1$ so the proposition is satisfied. [4]

(ii) The property implies that the unit circle maps into itself; this means that if the bilinear transformation is used to transform a filter, the frequency response of a transformed filter is the same as that of the original filter but with a distorted frequency axis. A filter is stable iff all its poles lie strictly inside the unit circle. If this transformation is applied to a stable filter, the property proved in part i) ensures that the transformed filter is also stable. In the same way, it also ensures that a minimum phase filter will transform into another minimum phase filter and that a causal filter will transform into a causal filter. [2]

(b) (i) For $\omega=0$, the filter gain is $G(e^{j\omega})=1+(e^{j\omega})^{-1}=2$. At $\omega_G=\frac{\pi}{2}$, the filter gain is $G(e^{j\omega_G})=1+(e^{j\omega_G})^{-1}=1+(j)^{-1}=1-j$. Hence $|G(e^{j\omega_G})|=\sqrt{2}=\frac{G(1)}{\sqrt{2}}$. [2]

(ii) For $z=e^{j\omega}$, we can write $z^{1/2}G(z)=z^{1/2}+z^{-1/2}$ and $e^{j\omega/2}G(e^{j\omega})=e^{j\omega/2}+e^{-j\omega/2}=2\cos(\omega/2)$.
Taking the magnitude of each side gives $|G(e^{j\omega})|=2\cos(\omega/2)$ for $|\omega|\leq\pi$.
The dashed line shows $|G(e^{j\omega_G})|=\sqrt{2}$. [4]

(iii) We want a lowpass-to-lowpass transformation with $\omega_0=\frac{\pi}{2}$ and $\widehat{\omega}_1=0.2$. So

$$\lambda=\frac{\sin\left(\frac{\omega_0-\widehat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\widehat{\omega}_1}{2}\right)}=\frac{\sin 0.6854}{\sin 0.8854}=\frac{0.633}{0.774}=0.8176$$

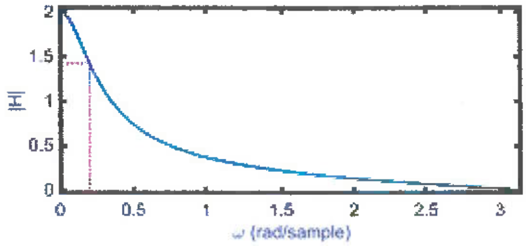
Substituting for

$$H(\hat{z})=1+\frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}=\frac{1-\lambda\hat{z}^{-1}+\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}=(1-\lambda)\frac{1+\hat{z}^{-1}}{1-\lambda\hat{z}^{-1}}$$

$$=0.1824\frac{1+\hat{z}^{-1}}{1-0.8176\hat{z}^{-1}}$$

[5]

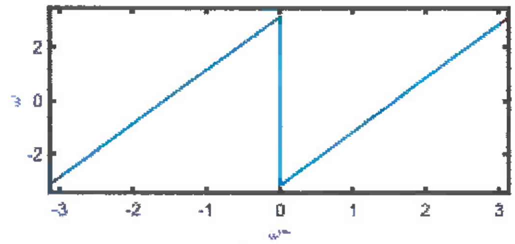
(iv) The frequency axis has been distorted non-linearly but the gains at $\omega = \{0, \pi\}$ are preserved.



[2]

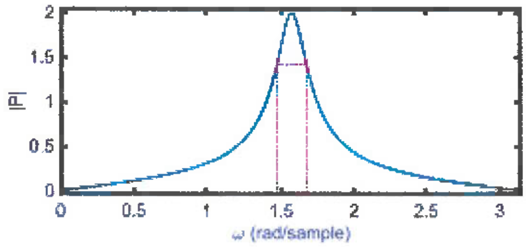
(c) (i) We see that $|\tilde{z}| < 1 \Leftrightarrow |\tilde{z}|^2 < 1 \Leftrightarrow |\tilde{z}^2| < 1 \Leftrightarrow |-\tilde{z}^2| < 1 \Leftrightarrow |z| < 1$. Hence the transformation preserves stability. [2]

(ii) If $z = -\tilde{z}^2$ then $e^{j\omega} = -e^{j2\tilde{\omega}} = e^{j\pi} \times e^{j2\tilde{\omega}} = e^{j(2\tilde{\omega}+\pi)}$ from which $\omega = (2\tilde{\omega} + \pi)_{\text{mod}2\pi}$ (or equivalently $\omega = (2\tilde{\omega} - \pi)_{\text{mod}2\pi}$). [2]



(iii) We have $H(z) = (1-\lambda)\frac{1+z^{-1}}{1-\lambda z^{-1}}$. Making the substitution $z^{-1} = -\tilde{z}^{-2}$ gives $P(\tilde{z}) = (1-\lambda)\frac{1-\tilde{z}^{-2}}{1+\lambda\tilde{z}^{-2}} = 0.1824\frac{1-\tilde{z}^{-2}}{1+0.817\tilde{z}^{-2}}$. [3]

(iv)



$|H(e^{j\omega})| = \sqrt{2}$ for $\omega = \pm 0.2$.Hence, $|P(e^{j\tilde{\omega}})| = \sqrt{2}$ when $(2\tilde{\omega} + \pi)_{\text{mod}2\pi} = \omega = \pm 0.2$. Solving this equation gives

$$2\tilde{\omega} + \pi = \pm 0.2 + 2n\pi$$

$$\tilde{\omega} = \pm 0.1 + \left(n - \frac{1}{2}\right)\pi = \cdots, -\frac{3\pi}{2} \pm 0.1, -\frac{\pi}{2} \pm 0.1, \frac{\pi}{2} \pm 0.1, \frac{3\pi}{2} \pm 0.1, \ldots$$

The two values of ω in the range $0 \le \omega \le \pi$ for which $|P(e^{j\tilde{\omega}})| = \sqrt{2}$ are therefore, $\omega = \frac{\pi}{2} \pm 0.1 = \{1.4708, 1.6708\}$. This is illustrated by the dotted line on the graph. The bandwidth of the filter is 0.2 and is equal to ω_H, the cutoff frequency of $H(z)$. [4]

3. a) Consider the multirate signal processing system shown in the block diagram of Fig. 3.1. This system multiplies the input sample rate by $\frac{P}{Q}$. In the following, let P and Q be coprime with $P < Q$.

i) Explain why the cutoff frequency of the lowpass filter $H(z)$ should be placed at the Nyquist rate of the output signal, $y[m]$ and give the normalized cutoff frequency, ω_0, in rad/sample in terms of P and/or Q.

Determine the required filter order M in terms of P and/or Q if the stop-band attenuation in dB is $a = 60$ and the normalized transition bandwidth is $\Delta\omega = 0.1\omega_0$. You may use the approximation formula $M \approx \frac{a}{3.5\Delta\omega}$, in which a and $\Delta\omega$ denote the stopband attenuation and the normalized transition bandwidth respectively.

[4]

The lowpass filter must eliminate the images introduced by the upsampler and the alias components introduced by the down sampler and must therefore eliminate all frequencies above the lower of the input and output Nyquist frequencies. Since $Q > P$, the output Nyquist frequency is $\frac{\pi}{Q}$. The normalized cutoff frequency is therefore $\omega_0 = \frac{\pi}{\max(P,Q)} = \frac{\pi}{Q}$.

We have $M = \frac{60}{3.5 \times 0.1\omega_0} = \frac{60Q}{3.5 \times 0.1\pi} = 54.6Q$.

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ii) Estimate the average number of multiplications per input sample, $x[n]$, needed to implement the system in the form of Fig. 3.1 using the value of M from part a) i). [2]

The filter requires $M+1$ multiplications per filter output sample, $v[r]$, which equals $(M+1)P$ per input sample, $x[n]$ (since there are P times as many samples of $v[r]$ as there are of $x[n]$). Substituting $M = 54.6Q$ gives $(54.6Q+1)P \approx 54.6PQ$ multiplications per input sample.

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iii) The filter $H(z)$ has a symmetrical impulse response

$$h[r] = g[r]w[r]$$

for $0 \le r \le M$ where $g[r]$ is the impulse response of an ideal lowpass filter with cutoff frequency ω_0 and $w[r]$ is a symmetrical window function.

Derive an expression for the ideal response, $g[r]$, in terms of ω_0, M and r.

[4]

The ideal response (centered on $r = 0$) is $G(e^{j\omega}) = 1$ for $|\omega| < \omega_0$ and zero otherwise. To this ideal response, we need to add a delay of $\frac{M}{2}$ samples which corresponds to a phase shift of $e^{-j0.5M\omega}$ (or, equivalently, we can design a centred filter and then delay the coefficients by $0.5M$ samples to *make it causal). Using the inverse DTFT (available in the formula sheet)*

$$\begin{aligned} g[r] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{-j0.5M\omega}e^{j\omega r}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(r-0.5M)}d\omega = \frac{1}{j2(r-0.5M)\pi}\left[e^{j\omega(r-0.5M)}\right]_{-\omega_0}^{\omega_0} \\ &= \frac{1}{j2(r-0.5M)\pi}\times 2j\sin((r-0.5M)\omega_0) = \frac{\sin((r-0.5M)\omega_0)}{\pi(r-0.5M)} \end{aligned}$$

We can also write this as $g[r] = \frac{\omega_0}{\pi}\operatorname{sinc}((r-0.5M)\omega_0)$ *where* $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$.

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b) It is intended to implement rhe filter $H(z)$ as a polyphase filter with commutated coefficients using the structure shown in Fig. 3.2.

i) Determine, in samples, the length of the filter impulse response $h_0[n]$ in terms of M, P and/or Q. Give an expression for the coefficients $h_0[n]$ in terms of the coefficients $h[r]$. [2]

The length of the filter $h_0[n]$ *is* $\frac{M+1}{P}$ *rounded up to the nearest integer (the order of* $h_0[n]$ *is one less than this). The coefficients are given by* $h_0[n] = h[nP]$.

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ii) If $x[n] = 0$ for $n < 0$, give expressions for $v[0]$, $v[1]$, $v[2P+1]$ in terms of the input $x[n]$ and the coefficients $h_p[n]$. [2]

Since $x[n]$ *is causal,* $v[0] = h_0[0]x[0]$, $v[1] = h_1[0]x[0]$ *and* $v[2P+1] = h_1[0]x[2] + h_1[1]x[1] + h_1[2]x[0]$.

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iii) Consider a particular example with $P = 5$ and $Q = 7$. Describe how the output decimator can be eliminated by changing both the sequence and rate at which the coefficient sets, $h_p[n]$ are accessed, and determine the new coefficient set order for this case. [3]

The output decimator selects every Q^{th} *sample of* $v[r]$ *and discards the others. Therefore if we access the coefficient sets in the order* $p = (mQ) \mod P$ *for* $m = 0, 1, \cdots$ *and reduce the rate by a factor of Q will will generate only the wanted output samples.*

For the specific values $P = 5$ *and* $Q = 7$, *so the coefficient set sequence becomes*

$h_0[n], h_{7 \bmod 5}[n], h_{14 \bmod 5}[n], h_{21 \bmod 5}[n], h_{28 \bmod 5}[n]$

which equals $h_0[n], h_2[n], h_4[n], h_1[n], h_3[n]$.

Since $Q \mod P = 2$ *the value of p increments each time either by 2 or by* $2 - P = -3$

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iv) Determine the number of multiplications per input sample for the system of part b) iii). You may assume that $M+1$ is a multiple of P. [2]

Because of symmetry, the number of distinct coefficients is only $\frac{M+1}{2}$ although it is not necessarily easy to take advantage of this symmetry to reduce the storage requirements.

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c) Consider an input signal, $x[n]$, with sampling rate 22 kHz. Consider also that the system is implemented as in part b) iii) with the values of a and $\Delta\omega$ as given in part a) i).

Determine the values of P, Q and M when the sample rate of the output, $y[m]$, is: (i) 10 kHz and (ii) 10.1 kHz.

For each of these cases estimate the number of multiplications per input sample and the number of distinct coefficients that must be stored. [5]

(i) For an output sample rate of $10\,\text{kHz} = \frac{5}{11} \times 22\,\text{kHz}$, $P = 5$ *and* $Q = 11$. $M = 54.6Q = 601$. *The number of multiplications per input sample is therefore* $\frac{M+1}{Q} = 54.7$. *The total number of distinct coefficients is* $\frac{M+1}{2} = 301$.

(ii) For an output sample rate of $10.1\,\text{kHz} = \frac{101}{220} \times 22\,\text{kHz}$, $P = 101$ *and* $Q = 220$. $M = 54.6Q = 12012$. *The number of multiplications per input sample is therefore* $\frac{M+1}{Q} = 54.6$ *(virtually unchanged). The total number of distinct coefficients is* $\frac{M+1}{2} = 6007$ *(much increased).*

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d) Consider a Farrow filter employing a low-order polynomial $f_n(t)$ where

$$t = \frac{p}{P} \text{ for } 0 \leq p \leq P-1.$$

i) Using any relevant diagrams, give a brief explanation of the operation of the Farrow filter and explain clearly how the coefficients, $h_p[n]$, are obtained. [2]

ii) Give an expression for the target value of $f_0(t)$ in terms of t, M, P and Q for the case that a rectangular window, $w[r] \equiv 1$, is used in the design of $H(z)$ and that $\omega_0 = \frac{\pi}{P}$. [2]

From the answer to part a)iii) $h[r] = g[r] = \frac{\sin(r-0.5M)\omega_0}{\pi(r-0.5M)}$ *so* $h_p[0] = g[p] = \frac{\sin((p-0.5M)\omega_0)}{\pi(p-0.5M)} = f_0\left(\frac{p}{P}\right)$*. We now substitute* $p \to Pt$ *and* $\omega_0 = \frac{\pi}{Q}$ *to get* $f_0(t) = g[Pt] = \frac{\sin\left((Pt-0.5M)\frac{\pi}{Q}\right)}{\pi(Pt-0.5M)}$ *for* $0 \leq t \leq 1$.

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iii) If the polynomials, $f_n(v)$, are of order $K = 5$, determine the number of coefficients that must be stored for each of the cases defined in part c). [2]

The $\frac{M+1}{P}$ *polynomials* $f_n(t)$ *each require* $K = 6$ *coefficients, so we require a total of* $\frac{6(M+1)}{P}$ *coefficients. For the two cases, this gives (i)* $\frac{6\times 492}{5} = 590$ *(somewhat larger than before) and (ii)* $\frac{6\times 9823}{101} = 584$ *(much less than before).*

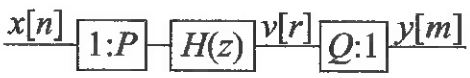


Figure 3.1

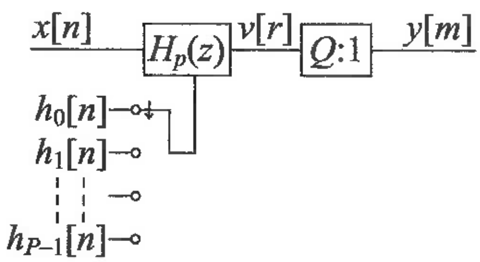


Figure 3.2

4. a) Let

$$x(t) = a(t)e^{j\phi(t)}$$

denote a complex-valued frequency-modulated signal with carrier frequency of 0 Hz and for which the peak frequency deviation is $d = 75\,\text{kHz}$. The amplitude of $a(t)$ is approximately constant with $a(t) \approx 1$. The phase of $a(t)$ is given by

$$\phi(t) = k\int_0^t m(\tau)d\tau$$

where k is a constant and $m(t)$ is a baseband audio signal with bandwidth $b = 15\,\text{kHz}$.

Let $x[n]$ denote the discrete-time signal formed from $x(t)$ sampled with a sampling frequency of 400 kHz.

i) The bandwidth of a double-sideband FM signal B is given by Carson's rule as

$$B = 2(d+b).$$

Starting from Carson's rule, determine the single-sided bandwidth, ω_0, of the discrete-time signal $x[n]$ in radians/sample.

[2]

From Carson's rule, $B = 180\,\text{kHz}$. This bandwidth includes both sidebands, so $\omega_0 = 2\pi \times \frac{90\,\text{kHz}}{400\,\text{kHz}} = 0.45\pi = 1.41\,\text{rad/samp}$.

---

ii) Show that $m(t) = k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right)$ where $\Im\,()$ denotes the imaginary part.

[4]

From the definition of $\phi(t)$, $m(t) = k^{-1}\frac{d\phi}{dt}$. Differentiating $x(t)$ gives $\frac{dx}{dt} = \frac{da}{dt}e^{j\phi(t)} + ja(t)\frac{d\phi}{dt}e^{j\phi(t)}$ from which (since $a(t)$ is real-valued), $x^*(t)\frac{dx(t)}{dt} = a(t)\frac{da}{dt} + ja^2(t)\frac{d\phi}{dt}$ and hence $\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = a^2(t)\frac{d\phi}{dt}$. It follows that $k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = m(t)$ as required.

Note that since $x(t) = a(t)e^{j\phi(t)}$, we could alternatively write $x^*(t) = a(t)e^{-j\phi(t)} = a^2(t)x^{-1}(t)$ which results in $m(t) = k^{-1}\Im\left(x^{-1}(t)\frac{dx(t)}{dt}\right)$. However this expression is harder to implement because it involves taking the reciprocal, $x^{-1}(t)$, of a rapidly varying complex number instead of the reciprocal, $a^{-2}(t)$, of a slowly varying real number whose value is always close to 1.

---

iii) The discrete-time system shown in the block diagram of Fig. 4.2 is designed to implement the equation of part ii). Complex-valued signals are shown as bold connections. The block labelled "Conj" outputs the complex conjugate of its input. The block labelled $D(z)$ performs differentiation and is designed as an FIR filter using the window method with a target response

$$\bar{D}(e^{j\omega}) = \begin{cases} jc\omega & \text{for } |\omega| \le \omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a scaling constant.

Starting with the inverse DTFT, determine a simplified expression for the impulse response $\bar{d}[n]$ of $\overline{D}(z)$. [4]

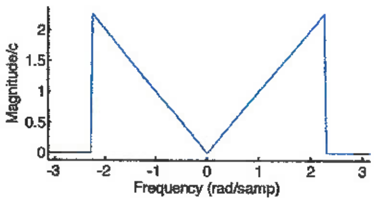
From the inverse DTFT formula (included in the formula sheet) we use integration by parts to obtain

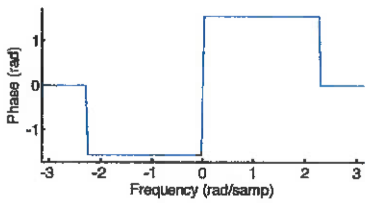
$$\begin{aligned}
\bar{d}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{D}\left(e^{j\omega}\right)e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\int_{-\omega_1}^{\omega_1}\omega e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\left[\frac{\omega}{jn}e^{j\omega n}-\frac{1}{(jn)^2}e^{j\omega n}\right]_{-\omega_1}^{\omega_1} \\
&= \frac{jc}{2\pi}\left(\frac{\omega_1}{jn}2\cos n\omega_1-\frac{1}{(jn)^2}2j\sin n\omega_1\right) \\
&= \frac{c}{\pi n^2}\left(n\omega_1\cos n\omega_1-\sin n\omega_1\right).
\end{aligned}$$

For interest only, the above formula gives $\bar{d}[-5:5]=[-0.07, 0.18, -0.19, -0.02, 0.71, 0, -0.7$
This is fairly similar to the simplest zero-phase differentiator which would be $\bar{d}[-1:1]=[0.5, 0, -0.5]$.

Assuming that $\omega_1=\frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\overline{D}(e^{j\omega})$ over the range $-\pi\le\omega\le\pi$. [4]

Since $\omega_0=0.45\pi=1.41$, $\omega_1=\frac{\omega_0+\pi}{2}=2.28$. *For* $\omega>\omega_1$, $\overline{D}(e^{j\omega})=0$ *and so the phase is indeterminate (shown here as zero). For* $|\omega|\le\omega_1$, *we can write* $\left|\overline{D}(e^{j\omega})\right|=|jc\omega|=c\times|\omega|$ *and* $\angle\overline{D}(e^{j\omega})=\angle jc\omega=\frac{\pi}{2}[-\pi\ if\ \omega<0]=\frac{\pi}{2}\operatorname{sgn}(\omega)$ *assuming* $c>0$.





The transition in the response of $D(e^{j\omega})$ *near the discontinuity in* $\overline{D}(e^{j\omega})$ *near* $\omega=\omega_1=2.28$ *will extend for* $\frac{18}{M+1}$ *either side of the discontinuity for a total width of* $\frac{36}{M+1}$. *So we need* $\frac{36}{M+1}\le\pi-\omega_0=0.55\pi$ *from which* $M\ge\frac{36}{\pi-\omega_0}-1=\frac{36}{0.55\pi}-1=\frac{36}{1.728}-1=20.83-1=19.83\approx 20$.

Let $s[n]$ denote the output of the differentiation block, $D(z)$, in Fig. ??. Consider the case of a practical implementation in which it is necessary to satisfy the constraint that $|s[n]| \leq \frac{1}{\sqrt{2}}$. What is the maximum value of c that will ensure this constraint is satisfied. Clearly state all assumptions made.

[4]

Use the assumptions that $a(t) \equiv 1$ and that $D(e^{j\omega}) = \overline{D}(e^{j\omega})$. Then, at the maximum frequency deviation of 75 kHz, $x[n] = e^{j\omega_f n}$ where $\omega_f = 2\pi \times \frac{75}{400} = 1.18$. To ensure $|s[n]| \leq \frac{1}{\sqrt{2}}$, we require $\left|D(e^{j\omega_f})\right| = c\omega_f \leq \frac{1}{\sqrt{2}}$. Hence $c\sqrt{2} \leq \frac{1}{\omega_f} = \frac{400}{2\pi\times 75} = 0.849$ so that $c \leq 0.5975$.

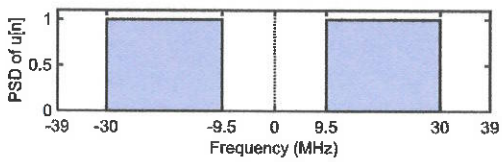
---

b) The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies ± 100 kHz around a centre frequency of $c \times 100$ kHz where the channel index, c, is an integer in the range $876 \leq c \leq 1079$. Figure 4.2 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

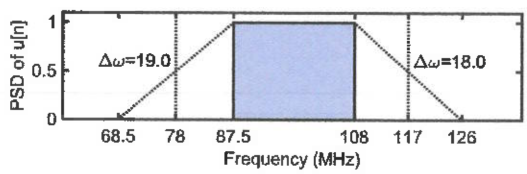
i) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of $u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing.

[3]

The FM band of 87.5 to 108 Mhz will be aliased down by the sample frequency to an image covering $87.5 - 78 = 9.5$ to $108 - 78 = 30$ Mhz.



Frequencies of $78 - 9.5 = 68.5$ Mhz and $2 \times 78 - 30 = 126$ Mhz will be aliased onto the edges of this image and so the widest possible transition bands for the bandpass filter (BPF) are $68.5 - 87.5 = 19$ Mhz and $108 - 129 = 18$ Mhz. These transition widths can also be deduced from the spectrum plot above as $2 \times (9.5 - 0) = 19$ Mhz and $2 \times (39 - 30) = 18$ Mhz since 0 Mhz and 39 Mhz are aliased down from 78 Mhz and 119 Mhz respectively. Although not asked by the question, these correspond to $\Delta\omega = \{1.53, 1.45\}$.



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ii) In Figure 4.2, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

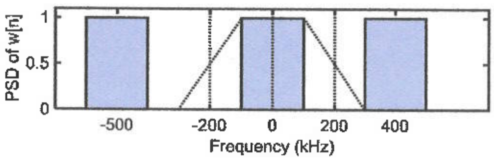
Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [3]

The original unnormalized centre frequency is $\Omega_c = 2\pi c \times 10^5$ (in the range 87.5 to 107.9 MHz) but the aliasing has reduced this by 78 MHz to $\Omega'_c = 2\pi(c-780)\times 10^5$ (in the range 19.5 to 29.9 MHz) so the normalized centre frequency is $\omega_c = \frac{\Omega'_c}{f_s} = \frac{2\pi(c-780)\times 10^5}{78\times 10^6} = \frac{2\pi}{780}c - 2\pi$ meaning that $v[n] = e^{-j\omega_c n} = e^{-j2\pi\frac{(c-780)n}{780}} = e^{-j2\pi\frac{cn}{780}}$. Note that the equivalence $e^{-j2\pi\frac{(c-780)n}{780}} = e^{-j2\pi\frac{cn}{780}}$ means that we can ignore the frequency offset of 2π due to aliasing.

Multiplying a real number, $u[n]$, by a complex number, $v[n]$, requires two multiplications and so the multiplication rate is $2f_s = 156\times 10^6 = 1.56\times 10^8$.

Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range −700 to +700 kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

When $c = 1000$, the spectrum of $u[n]$ is shifted down by 100 MHz to become that of $w[n]$. The shifted centre frequencies of the active FM channels are −0.5, 0 and +0.4 MHz. Also marked on the sketch below, but not requested in the question, is the gain of $H(z)$ and ± the Nyquist frequency, 200 kHz, of the sample rate at $y[n]$.



Explain the purpose of the lowpass FIR filter, $H(z)$ in Figure 4.2. [3]

The lowpass filter must remove frequencies outside the range ±200 kHz (which contain unwanted FM channels) in order to prevent aliasing by the downsampler.

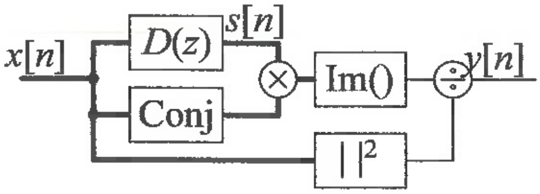


Figure 4.1

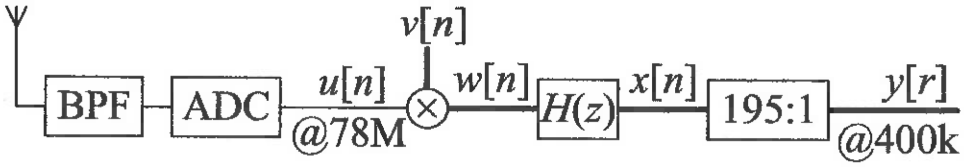


Figure 4.2

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N}X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

$$-\boxed{H(z)}-\boxed{1{:}Q}- \;=\; -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample $v[n]$ by Q: $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if} Q \mid r \\ 0 & \text{if} Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators