8: IIR Filter Transformations

Continuous Time Filters

Classical continuous-time filters optimize tradeoff:
passband ripple v stopband ripple v transition width

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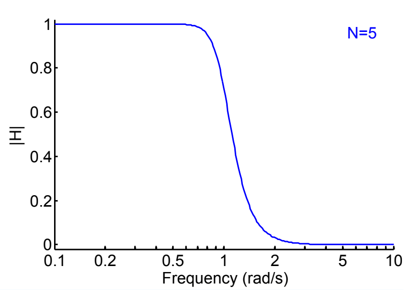

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IIR Filter Design: Butterworth Filter

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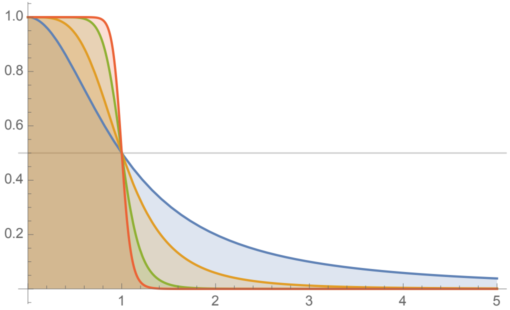

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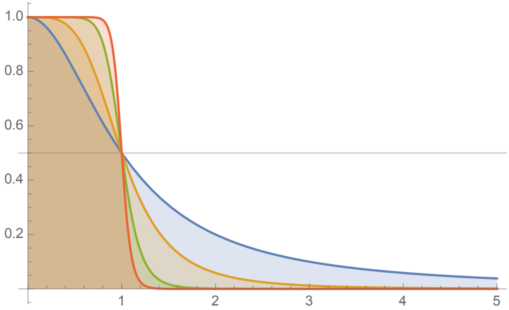

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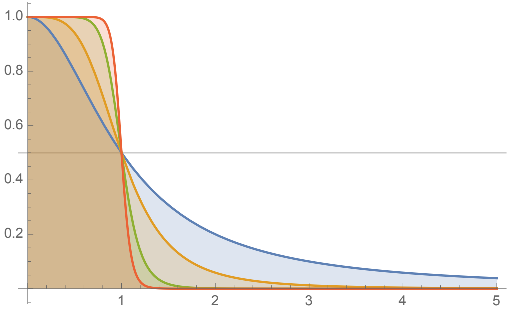

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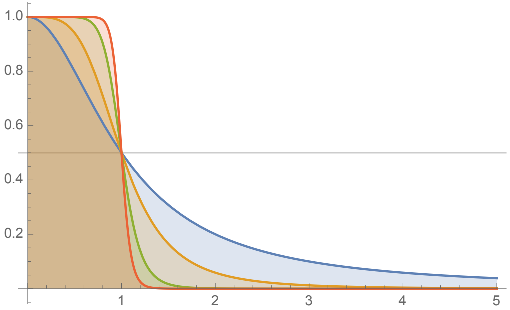
4. It is an all-pole filter.



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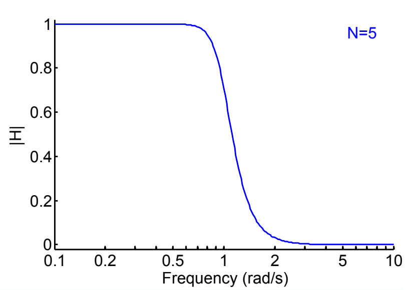
Classical continuous-time filters optimize tradeoff: passband ripple v stopband ripple v transition width



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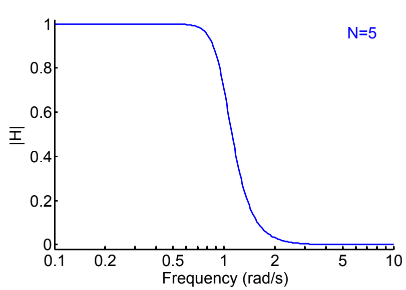

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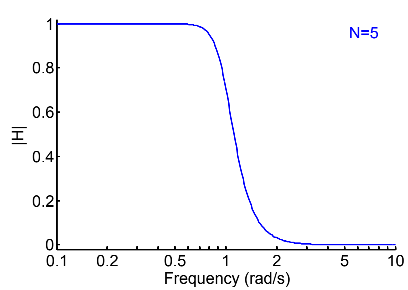

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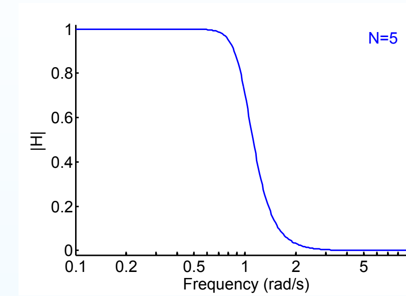

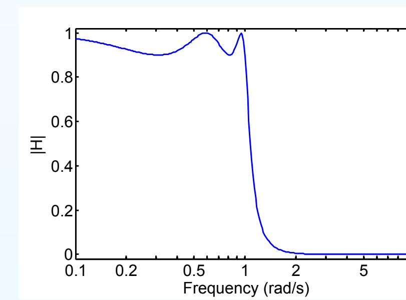

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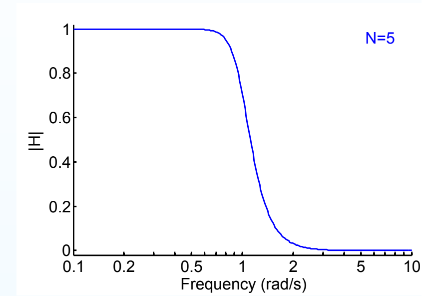

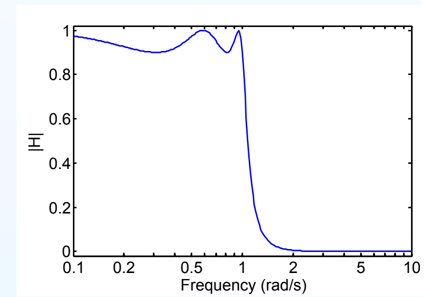

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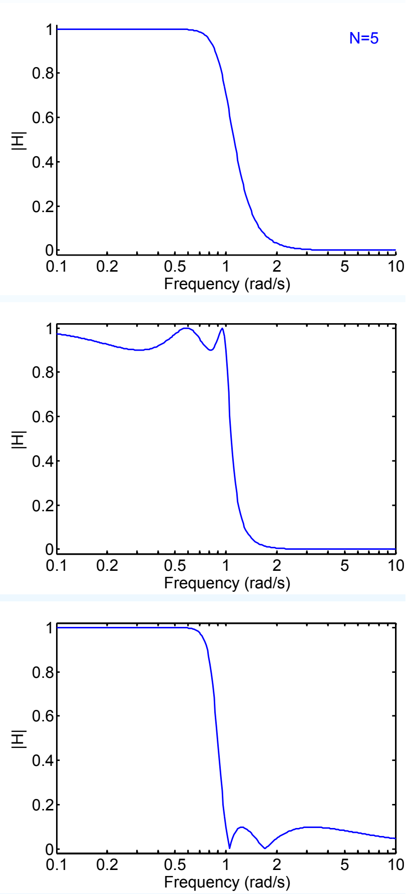

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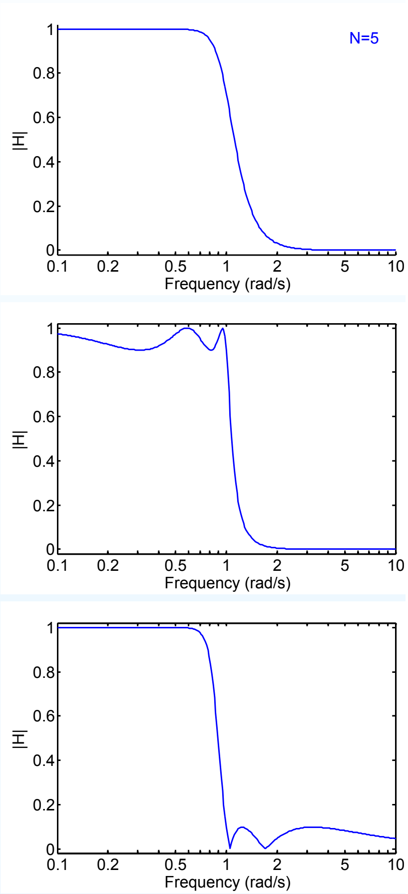

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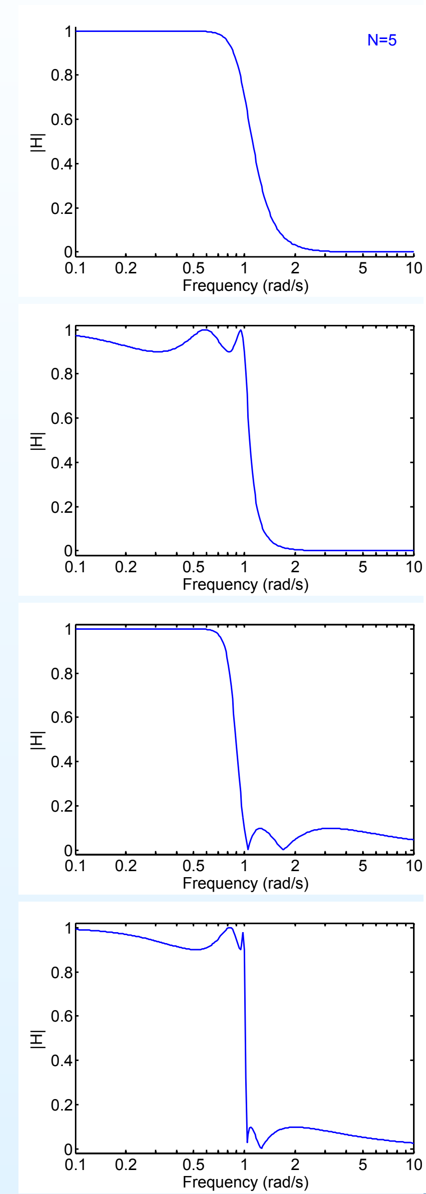

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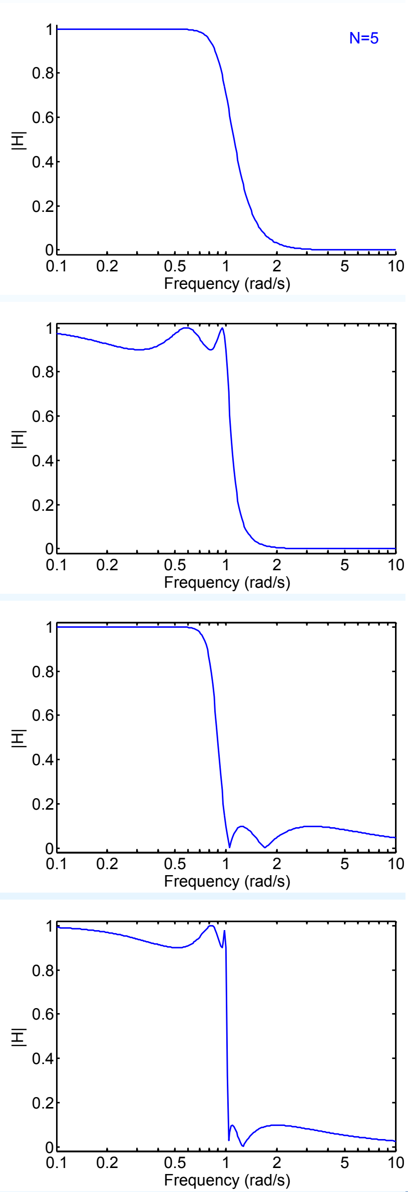

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There are explicit formulae for pole/zero positions.

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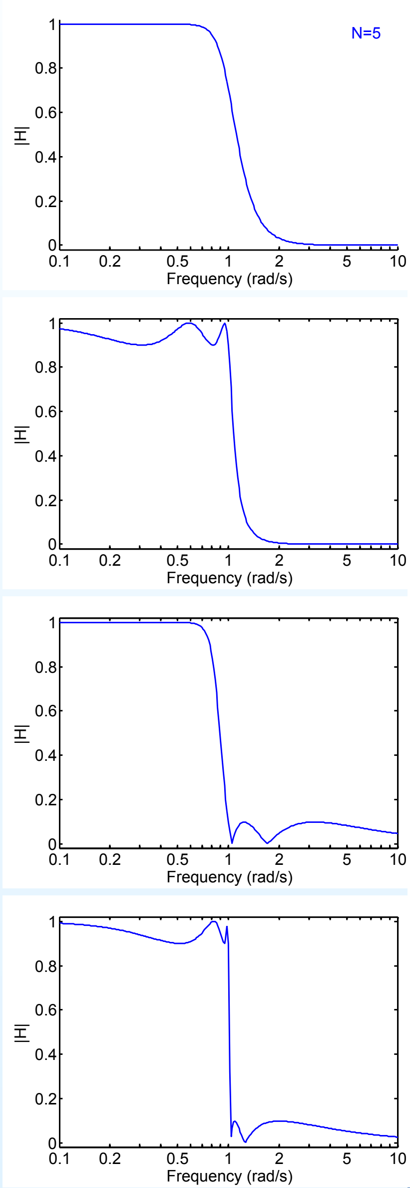
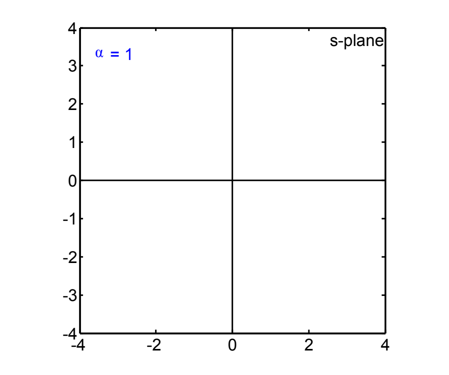
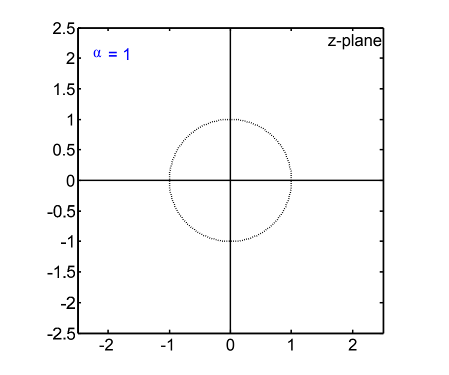


Bilinear Mapping

Change variable: $z = \frac{\alpha+s}{\alpha-s}$





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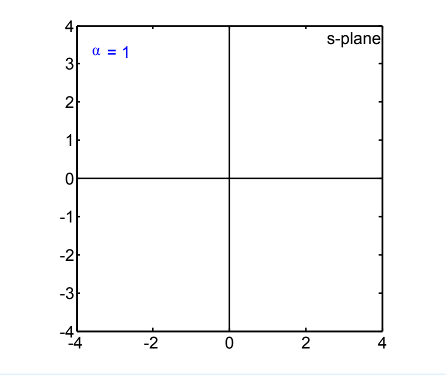

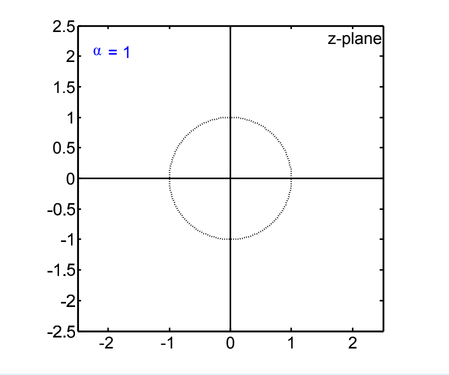

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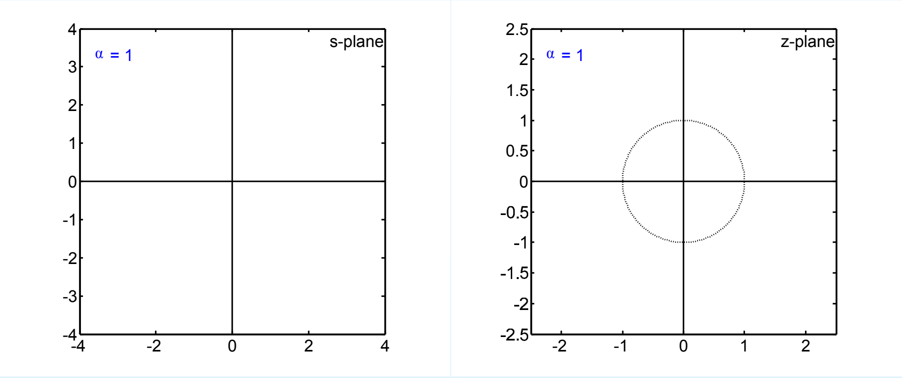

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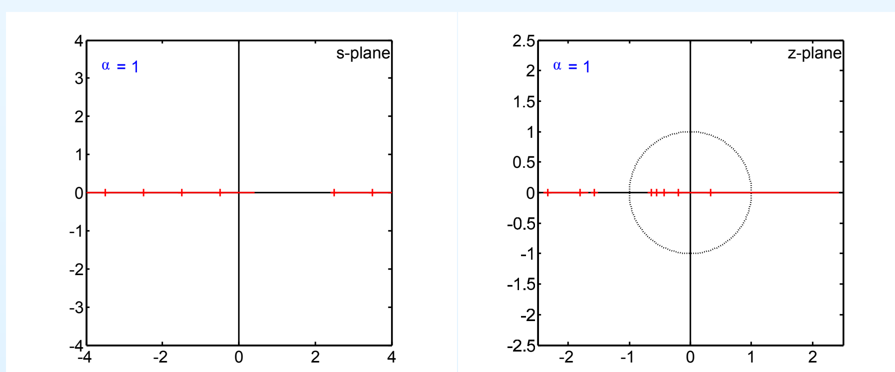

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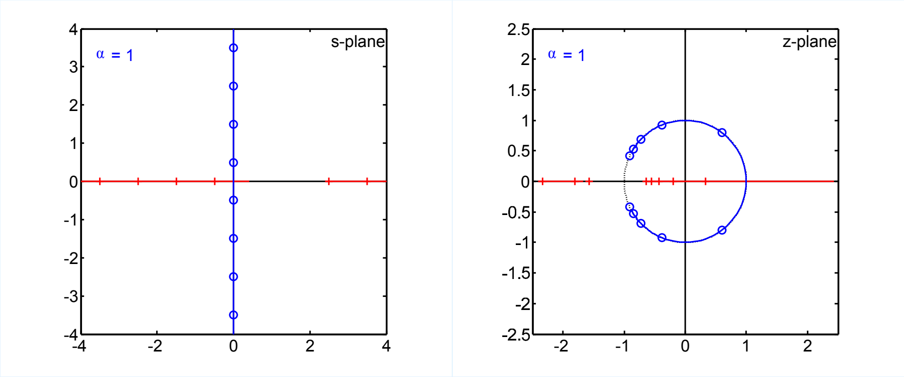

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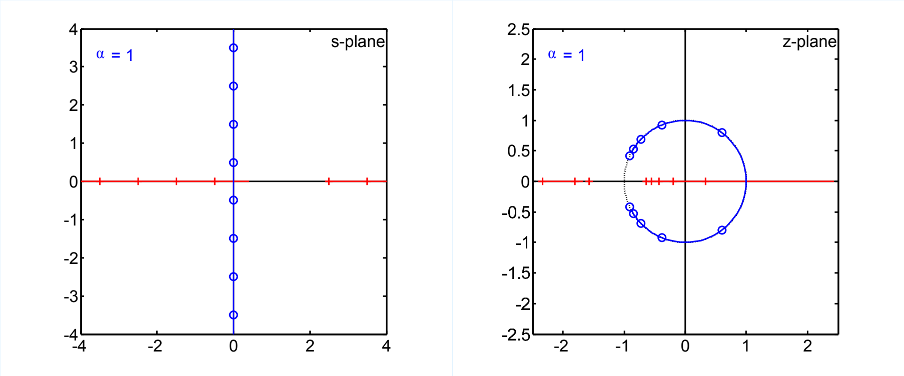

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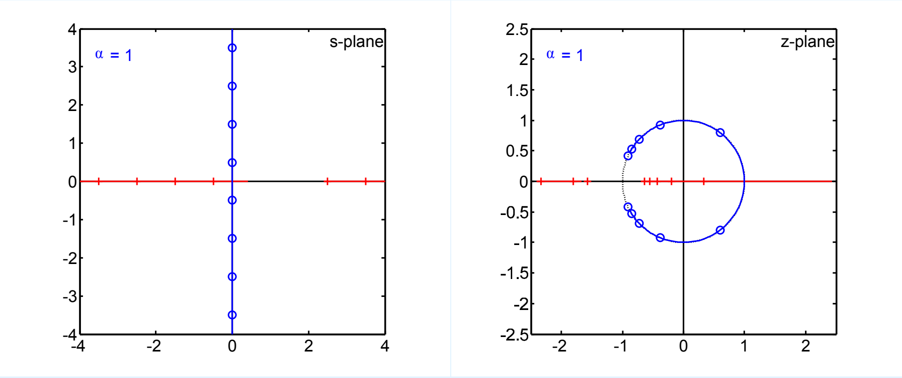

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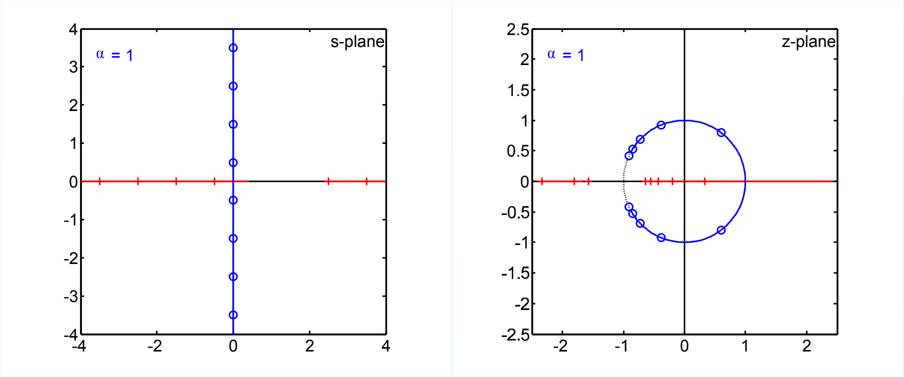

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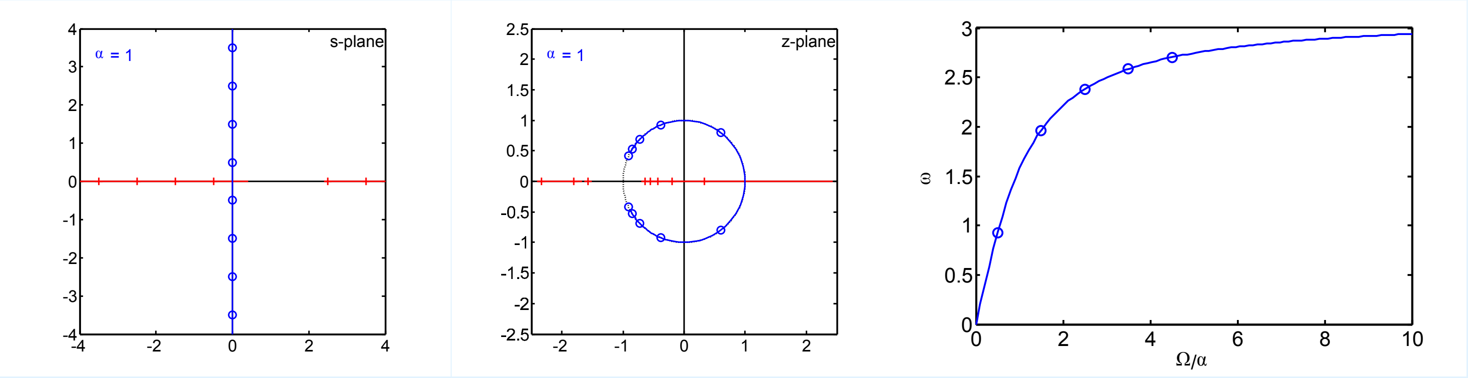

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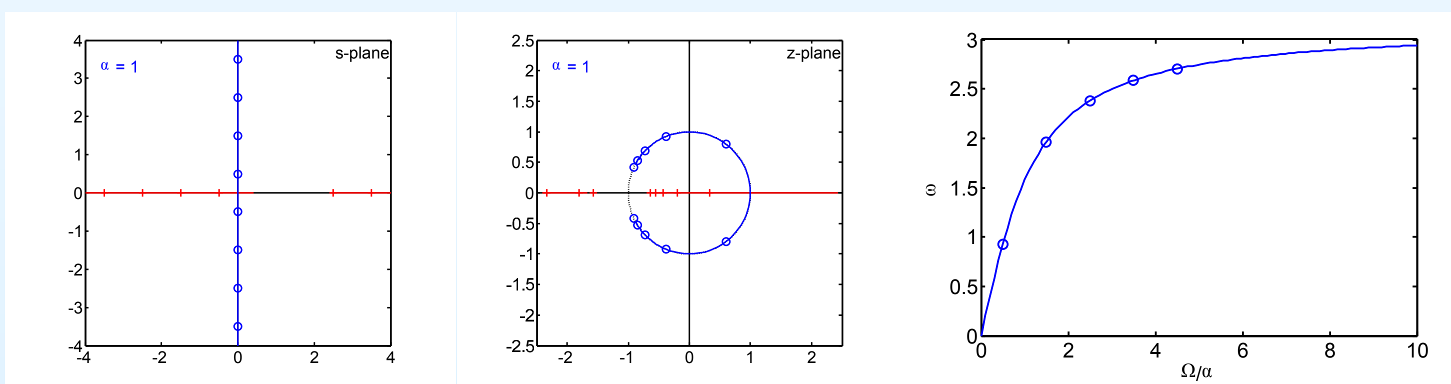

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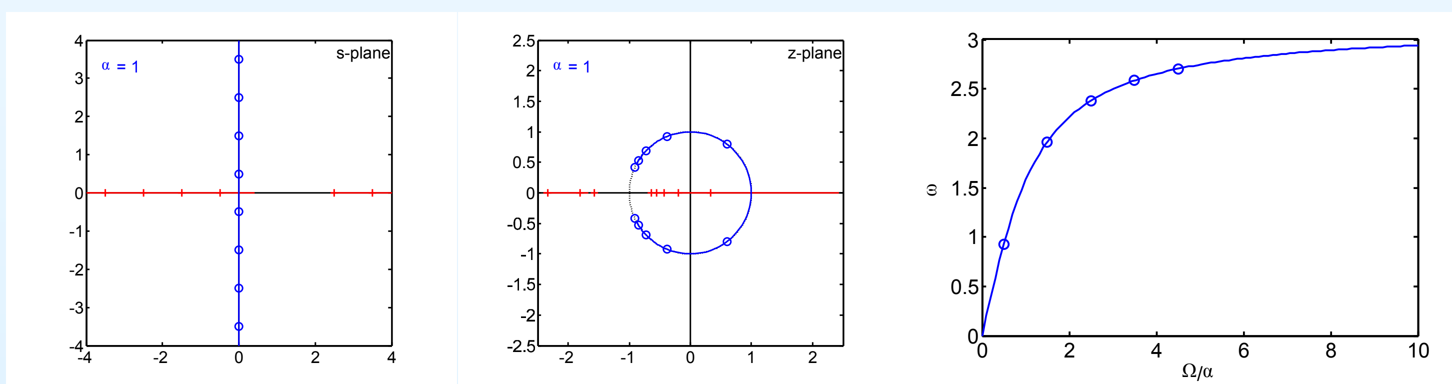

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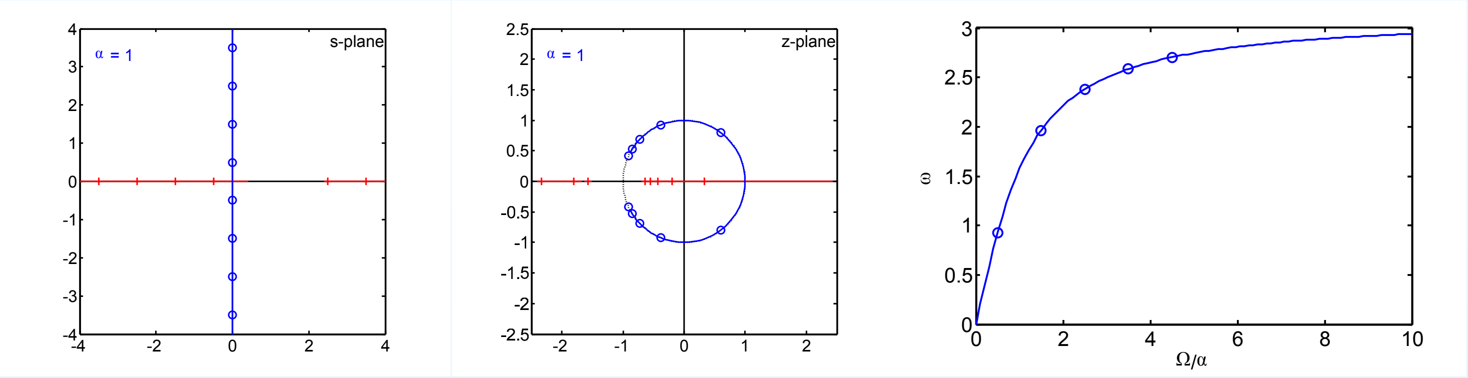

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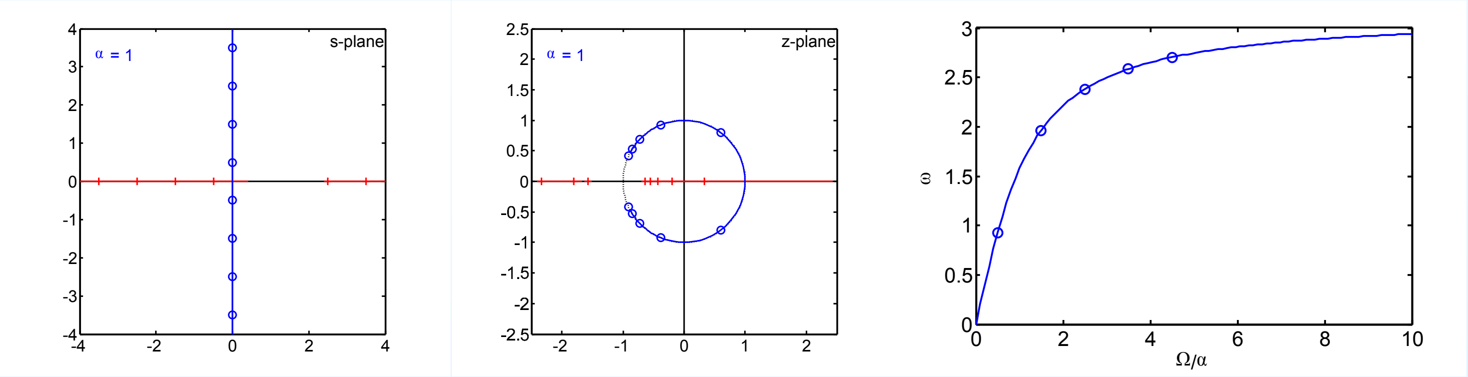

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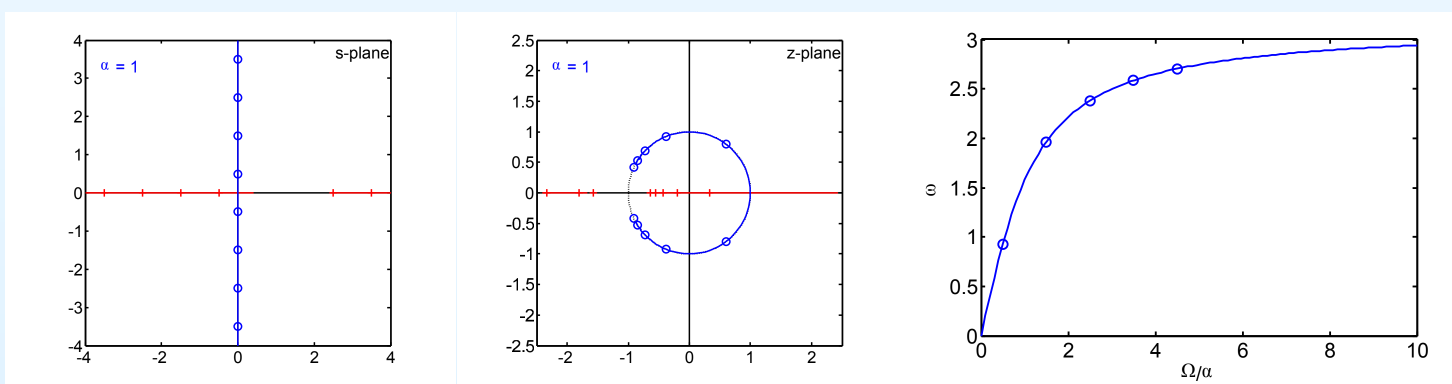
  $$= \frac{\alpha^2+2\alpha x+x^2+y^2}{\alpha^2-2\alpha x+x^2+y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2+y^2}$$

  $$x < 0 \Leftrightarrow |z| < 1$$



Bilinear Mapping

Change variable: $z = \frac{\alpha+s}{\alpha-s} \Leftrightarrow s = \alpha\frac{z-1}{z+1}$: a one-to-one invertible mapping

- $\Re$ axis $(s) \leftrightarrow \Re$ axis (z)
- $\Im$ axis $(s) \leftrightarrow$ Unit circle (z)

  Proof: $z = e^{j\omega} \Leftrightarrow s = \alpha\frac{e^{j\omega}-1}{e^{j\omega}+1} = \alpha\frac{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}} = j\alpha\tan\frac{\omega}{2} = j\Omega$
- Left half plane$(s) \leftrightarrow$inside of unit circle (z)

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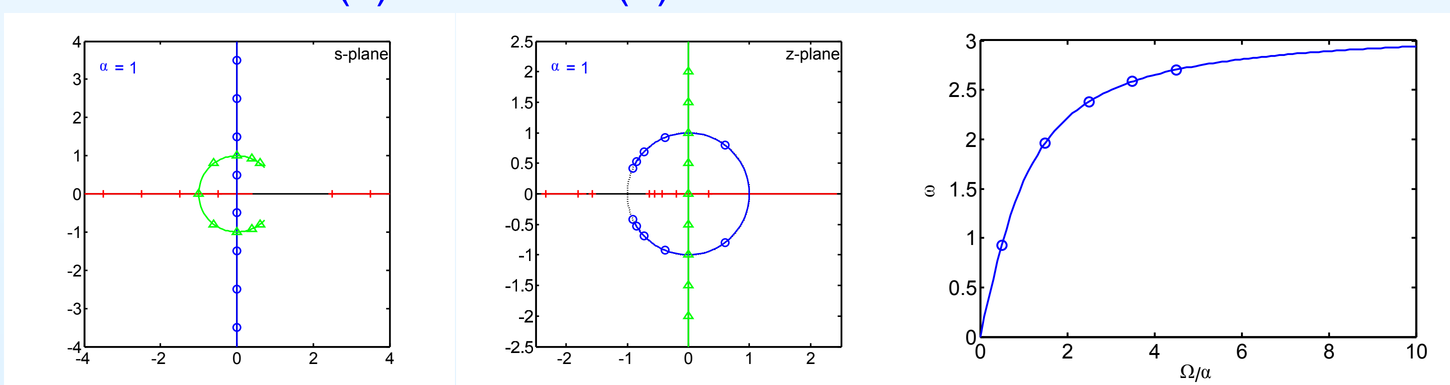
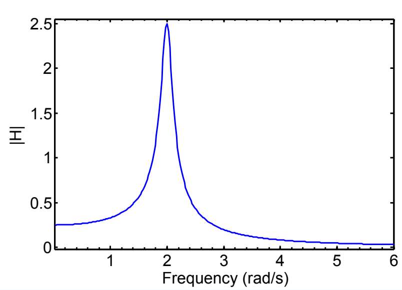
  $$= \frac{\alpha^2+2\alpha x+x^2+y^2}{\alpha^2-2\alpha x+x^2+y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2+y^2}$$
  $$x < 0 \Leftrightarrow |z| < 1$$
- Unit circle $(s) \leftrightarrow \Im$ axis (z)



Continuous Time Filters

Take $\widetilde{H}(s) = \frac{1}{s^2+0.2s+4}$ and choose $\alpha = 1$



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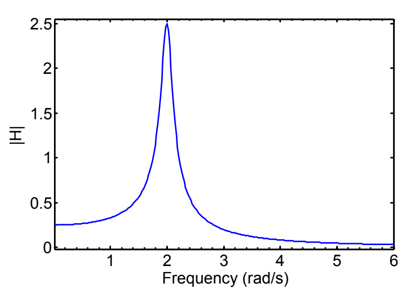

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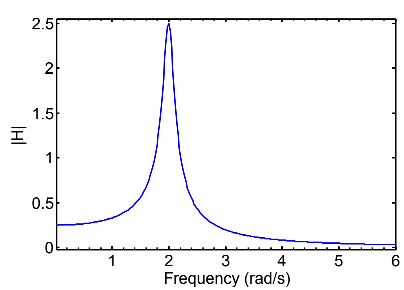

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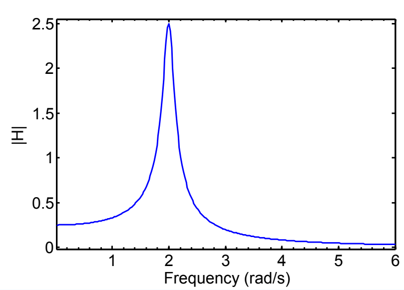

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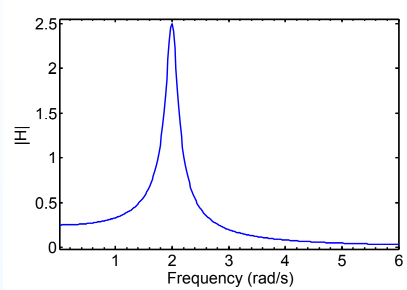

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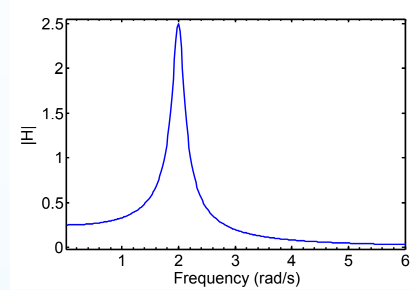

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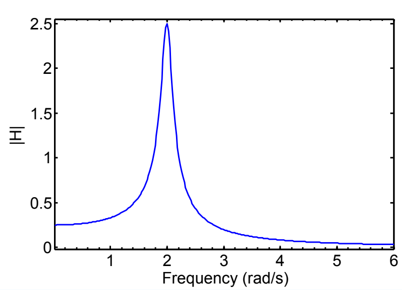

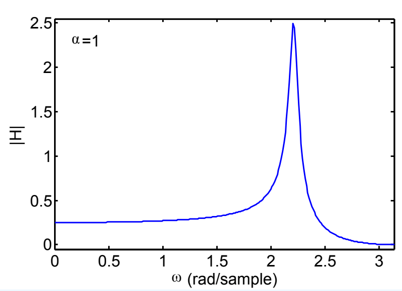

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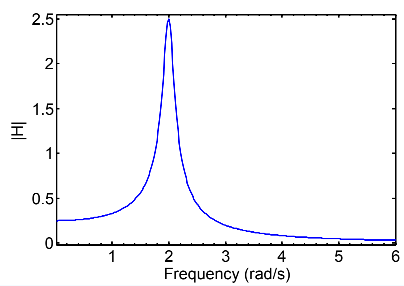

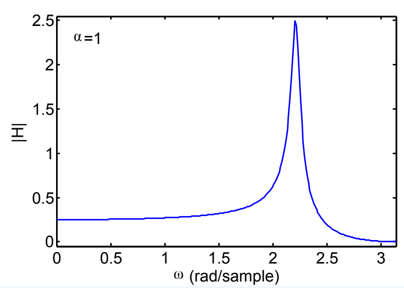

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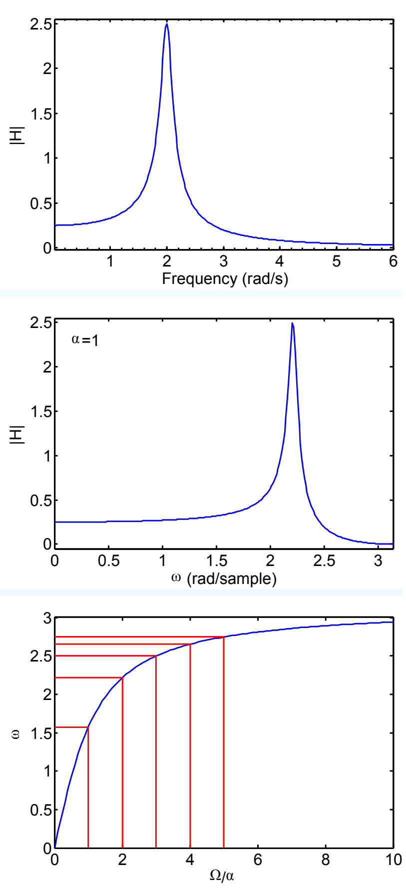

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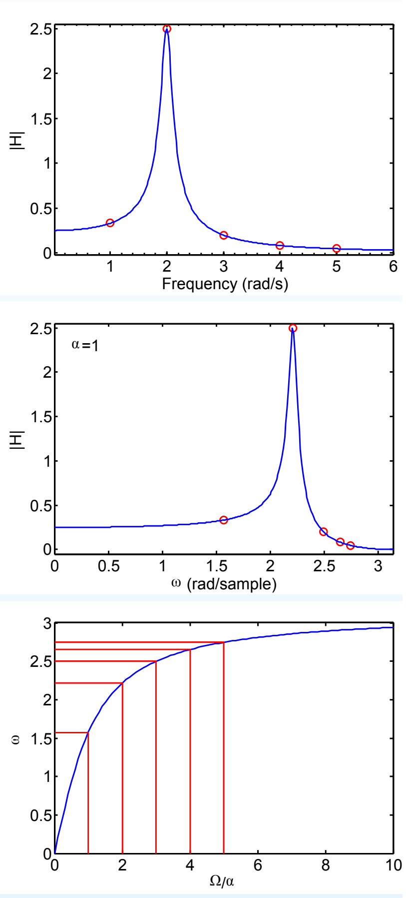

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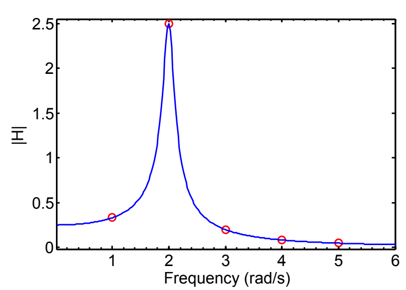

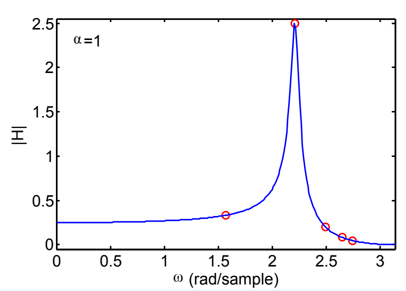

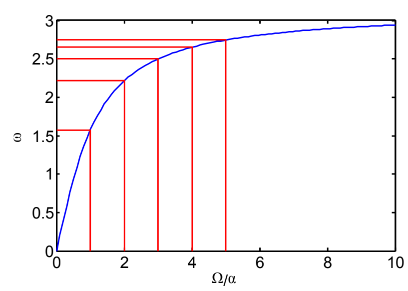

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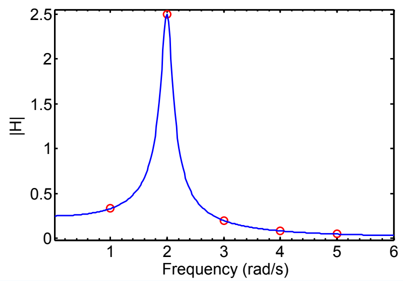
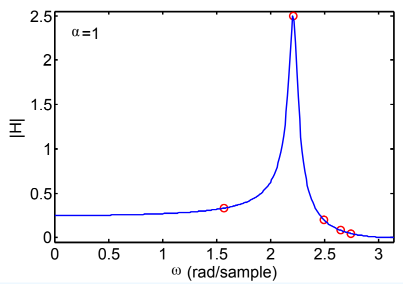
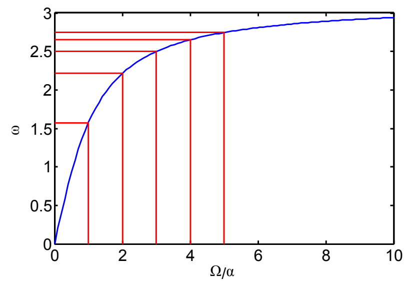
Set $\alpha = 2f_s = \frac{2}{T}$ to map low frequencies to themselves







Mapping Poles and Zeros

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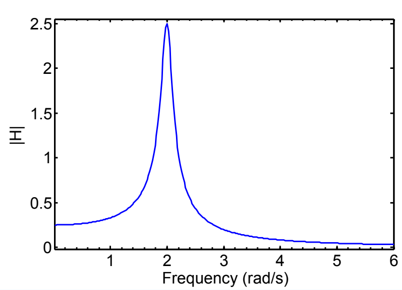

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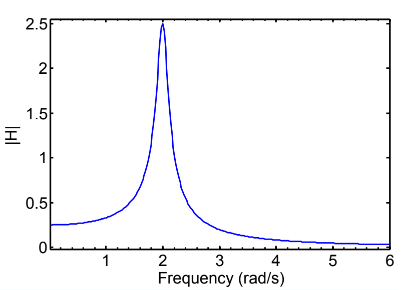

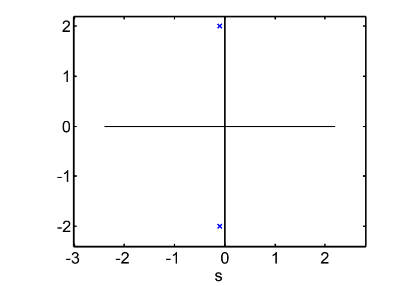

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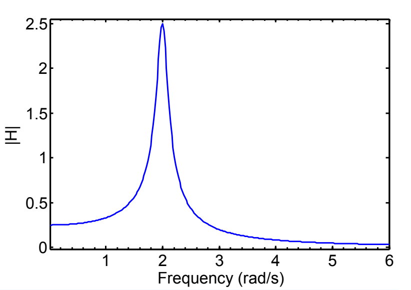

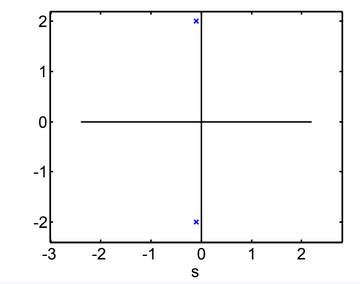

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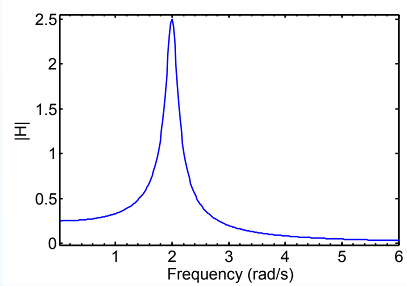

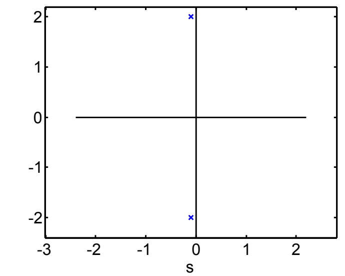

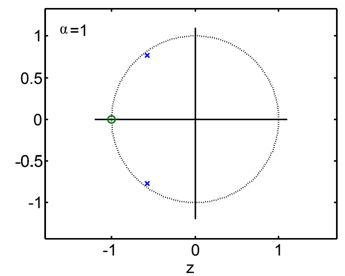

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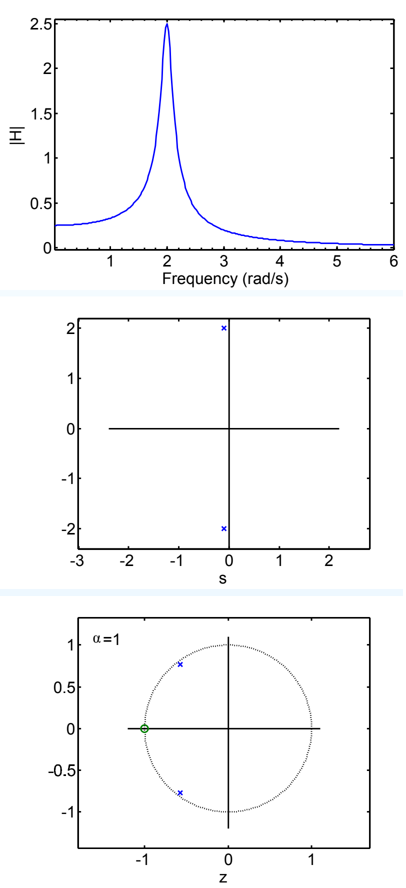

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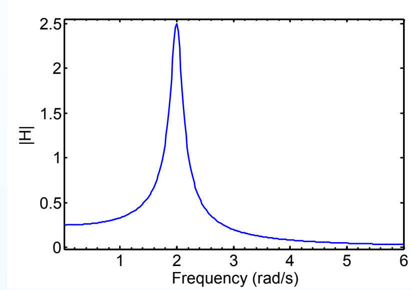

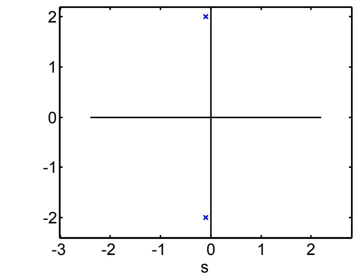

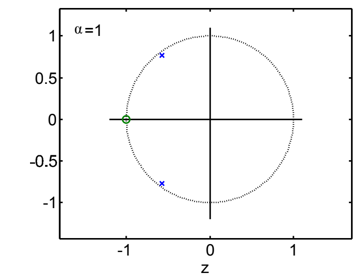

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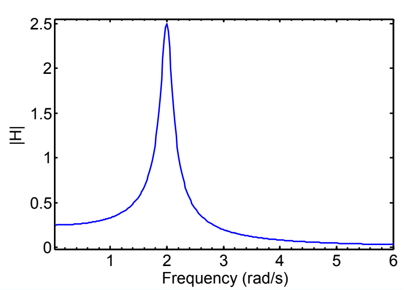

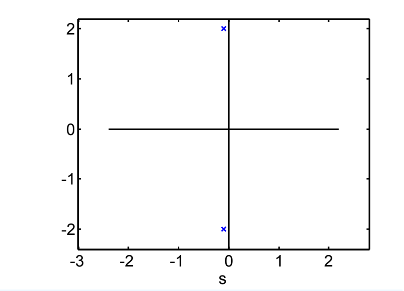

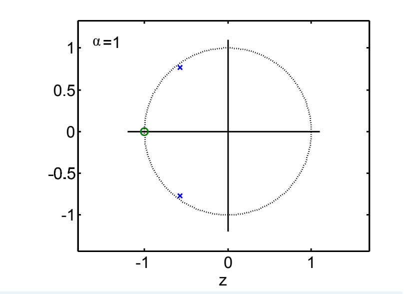

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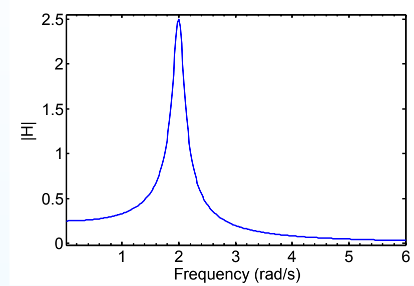

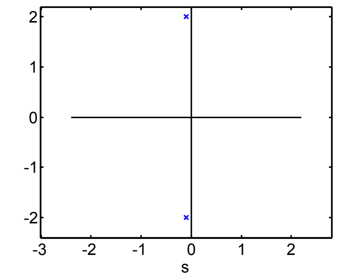

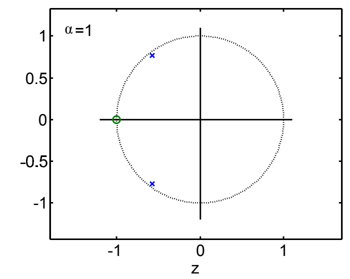

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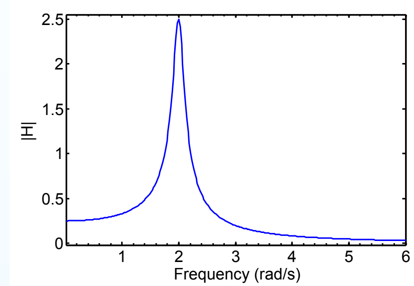

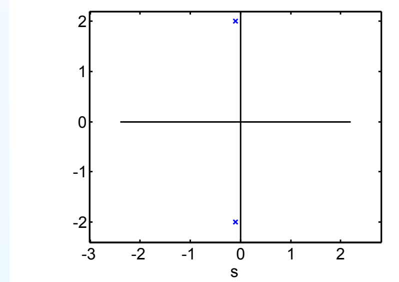

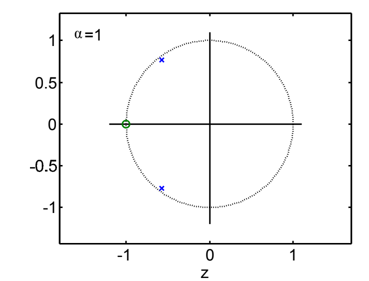

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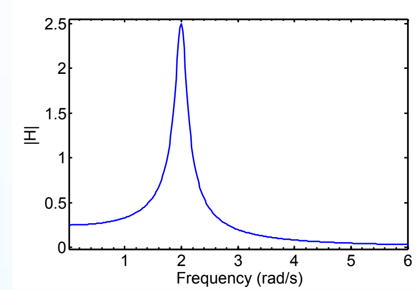

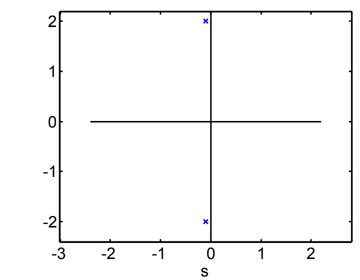

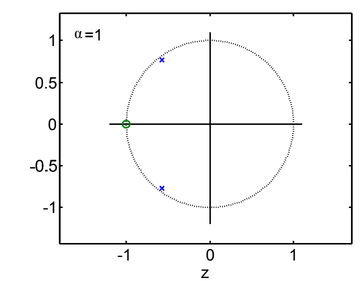

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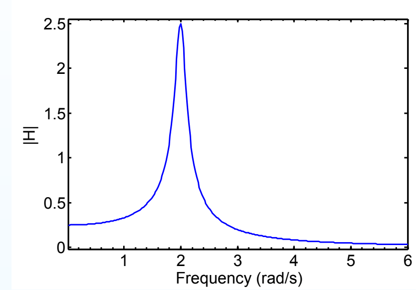

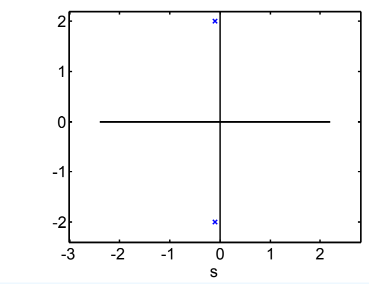

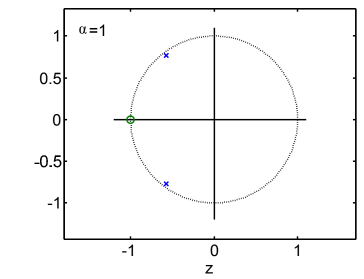

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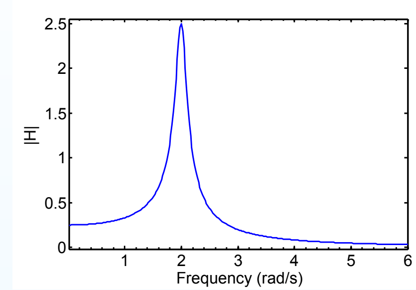

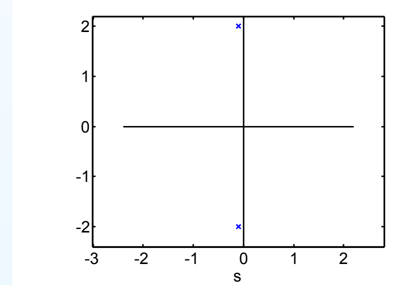

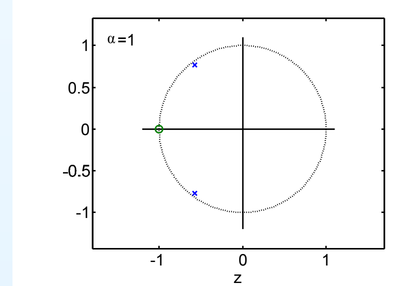

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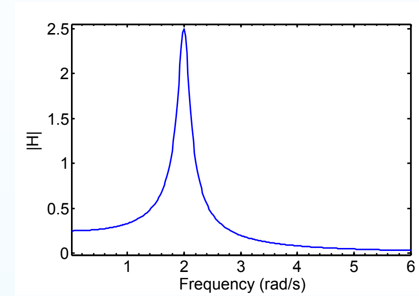

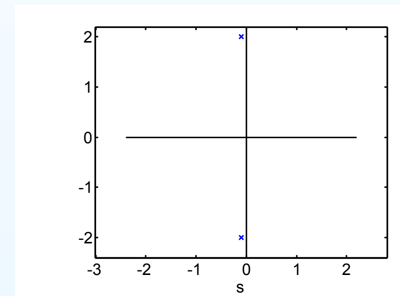

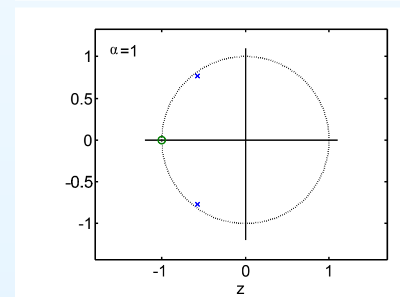

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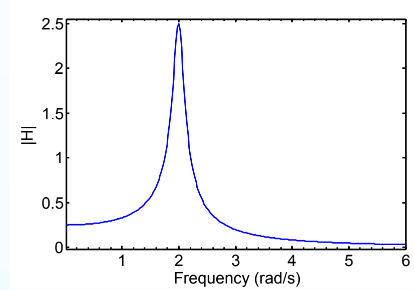

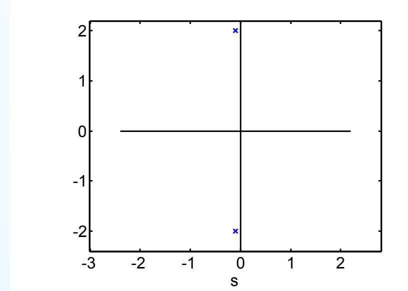

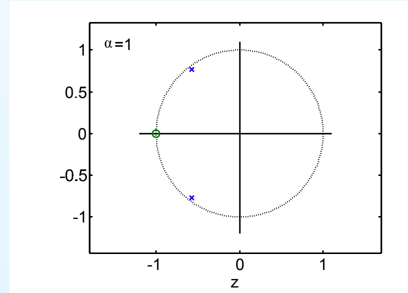

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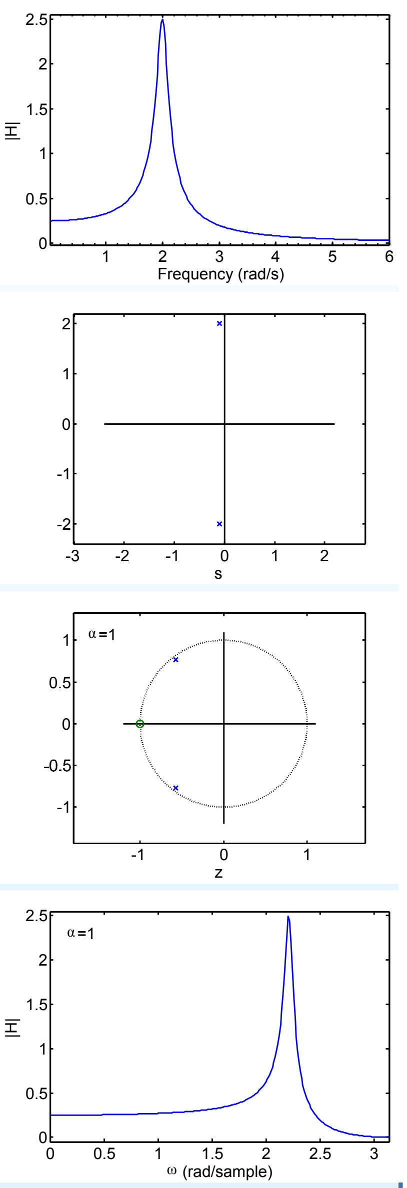

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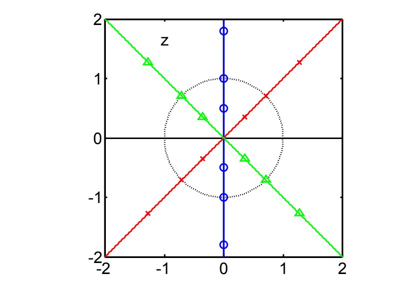

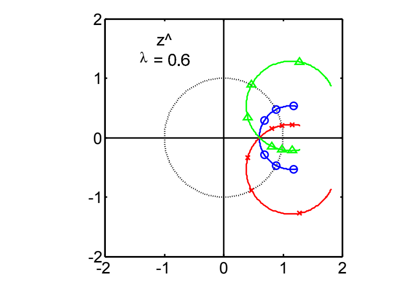

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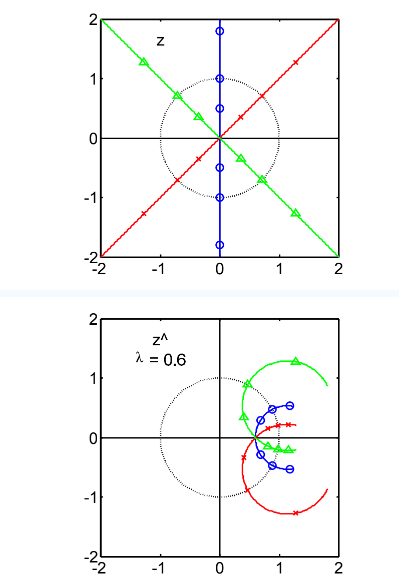

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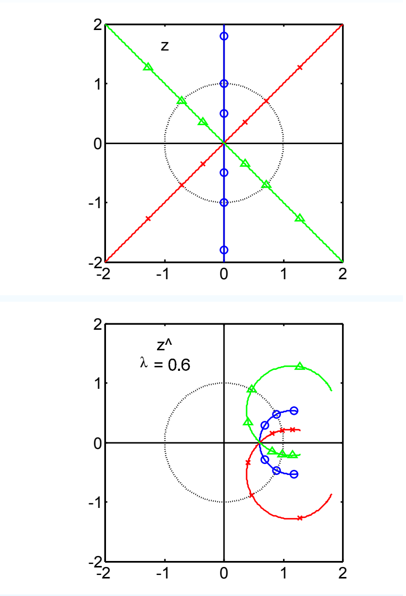

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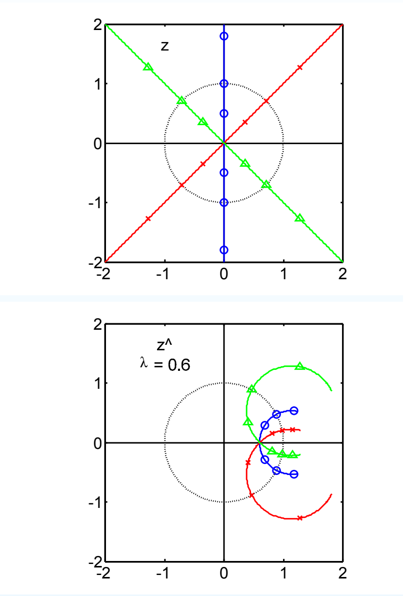

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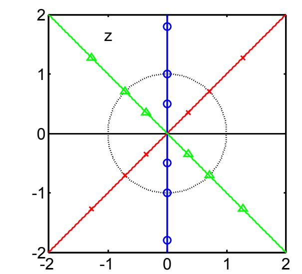

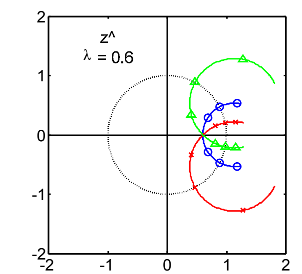

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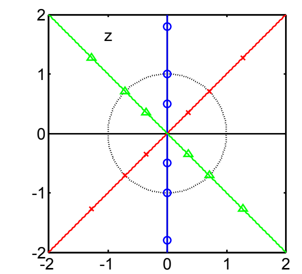
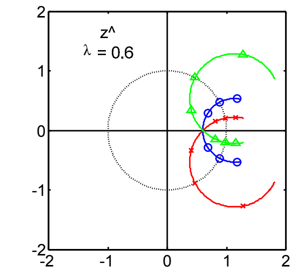
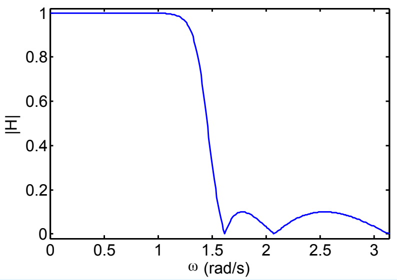
$$\omega_0 = \frac{\pi}{2} = 1.57$$







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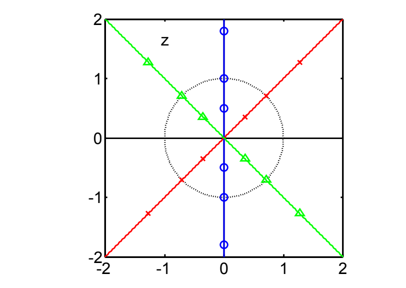

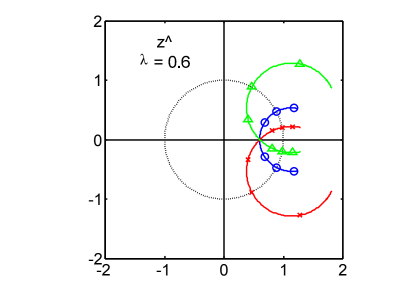

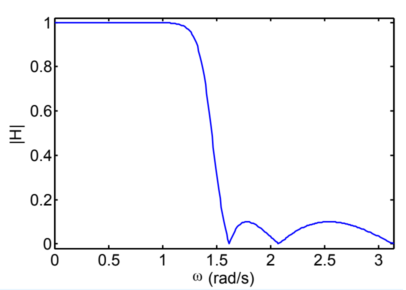

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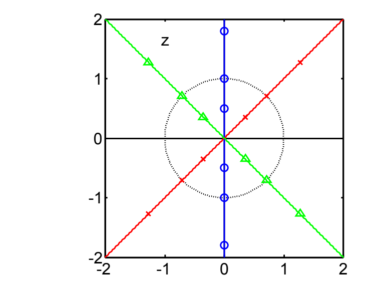

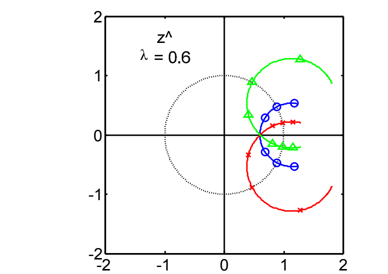

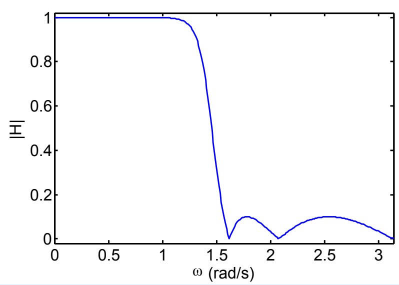

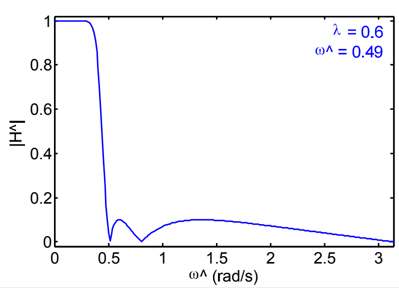

Explicit Computation

$\lambda = 4/10$ and $\omega_0 = \pi/2$.

$$\tan\left(\frac{\omega}{2}\right) = \frac{1+\lambda}{1-\lambda}\tan\left(\frac{\widehat{\omega}}{2}\right)$$

$$\rightarrow \underbrace{\tan\left(\frac{\pi}{4}\right)}_{=1} = \underbrace{\frac{1+0.6}{0.4}}_{=4}\tan\left(\frac{\widehat{\omega}}{2}\right)$$

$$\Rightarrow \widehat{\omega} = 2\tan^{-1}\left(\frac{1}{4}\right) \approx \frac{49}{100}$$

Constantinides Transformations

Transform any lowpass filter with cutoff frequency ω_0 to:

Target	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Constantinides Transformations

Transform any lowpass filter with cutoff frequency ω_0 to:

Target	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
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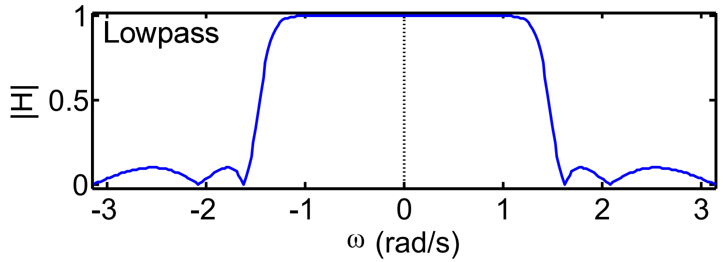
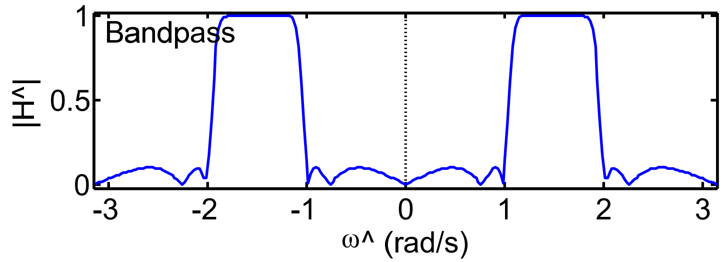
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Bandpass and bandstop transformations are quadratic and so will double the order:





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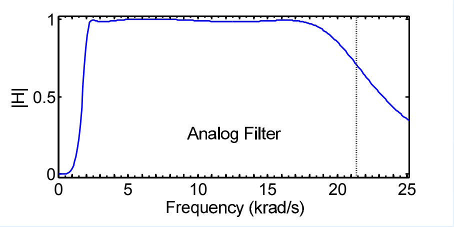

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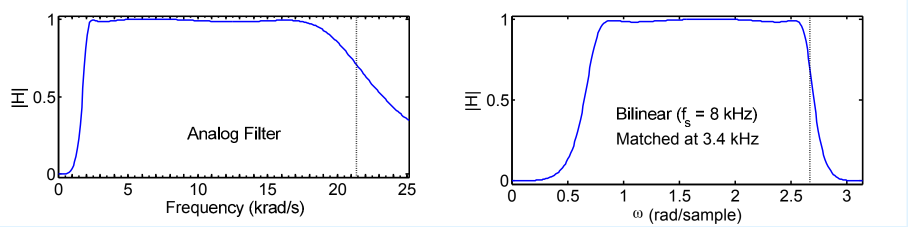

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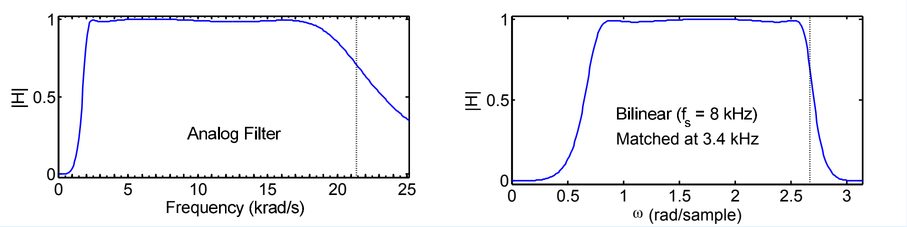

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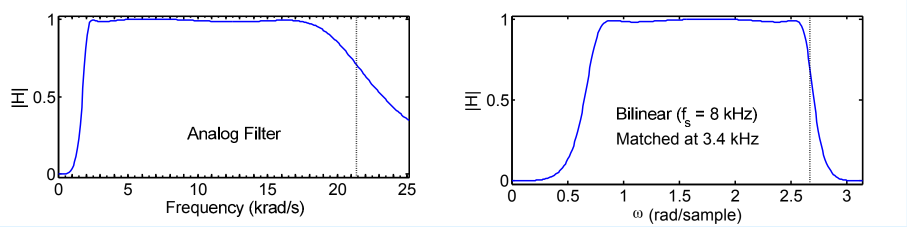

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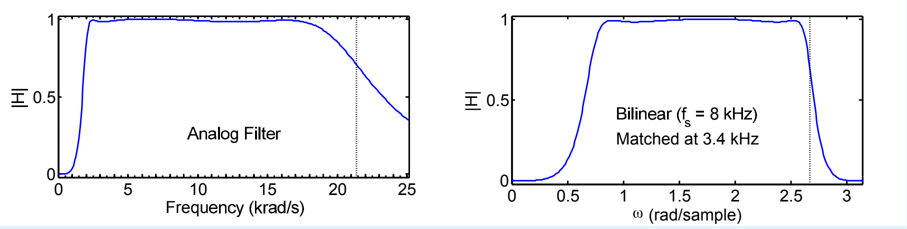

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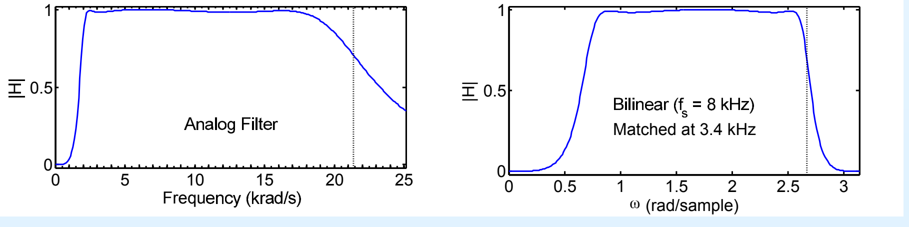

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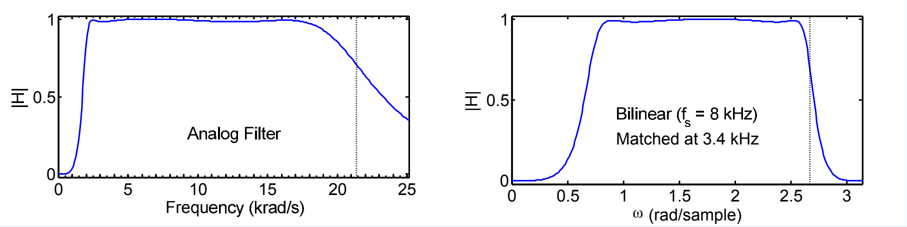

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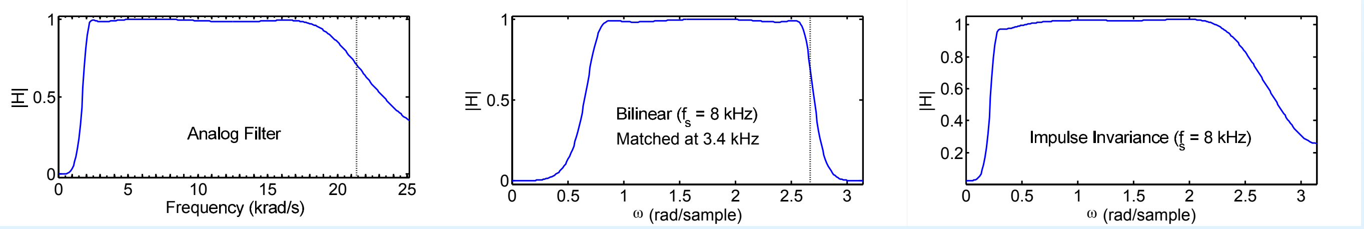

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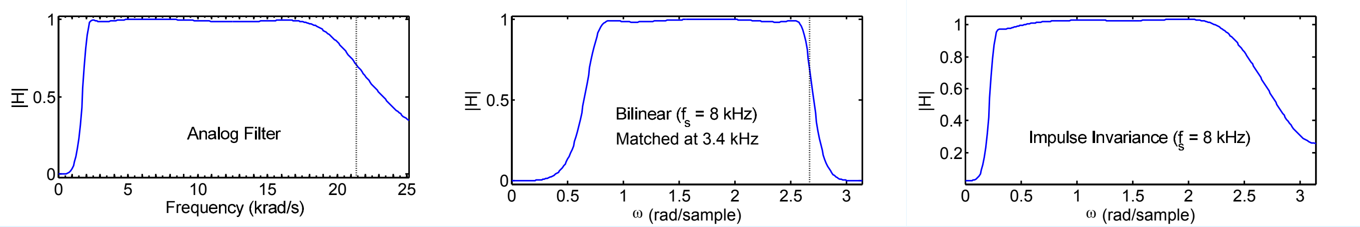

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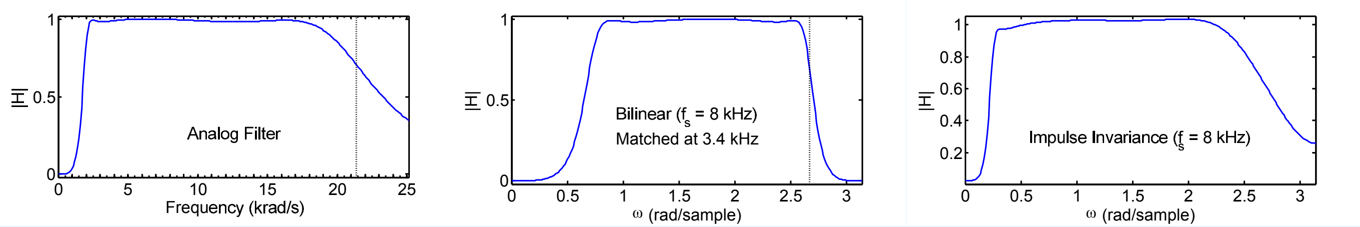

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Properties:

☺ Impulse response correct. ☺ No distortion of frequency axis.

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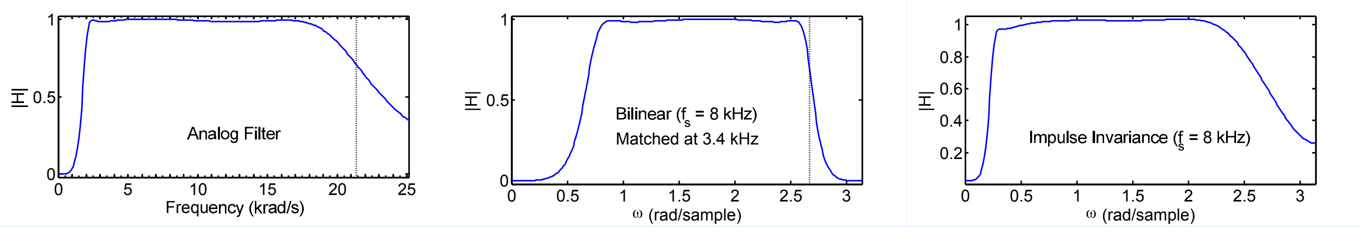

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Properties:

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☹ Frequency response is aliased.

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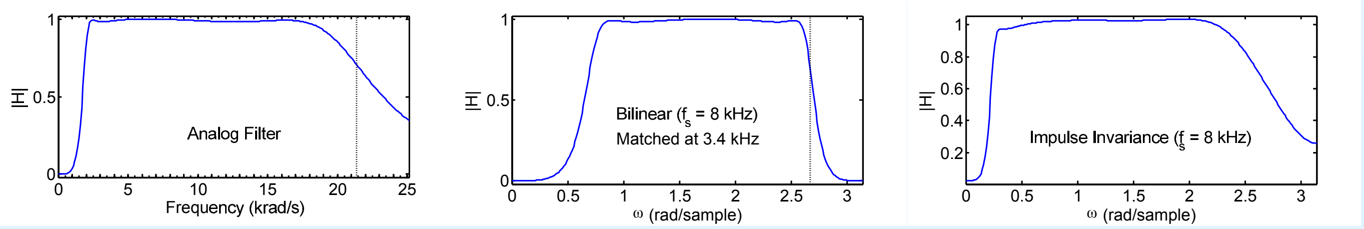

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For further details see Mitra: 9.

MATLAB routines

bilinear	Bilinear mapping
impinvar	Impulse invariance
butter butterord	Analog or digital Butterworth filter
cheby1 cheby1ord	Analog or digital Chebyshev filter
cheby2 cheby2ord	Analog or digital Inverse Chebyshev filter
ellip ellipord	Analog or digital Elliptic filter