1. Random Variables

a) $X \sim \mathcal{N}(5, 4)$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi \times 4}} \cdot e^{-\frac{(x-5)^2}{2\times 4}}$$

$$= \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{(x-5)^2}{8}}$$

$Y = 2X + 4$

As X is Gaussian, Y is also Gaussian

$$\mathbb{E}[Y] = \mathbb{E}[2X+4]$$
$$= 2\mathbb{E}[X] + 4$$
$$= 14$$

$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$
$$= \mathbb{E}[(2X+4)^2] - 14^2$$
$$= \mathbb{E}[4X^2 + 16X + 16] - 196$$
$$= 4\mathbb{E}[X^2] + 16\mathbb{E}[X] + 16 - 196$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$\Rightarrow \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2$$
$$= 4 + 25$$
$$= 29$$

$$Var(Y) = 4\mathbb{E}[X^2] + 16\mathbb{E}[X] - 196 + 16$$
$$= 4 \times 29 + 16 \times 5 + 16 - 196$$
$$= 16$$

As Y is Gaussian

$$f_Y(y) = \frac{1}{\sqrt{2\pi \times 16}} \cdot e^{-\frac{(y-14)}{2\times 16}}$$
$$= \frac{1}{4\sqrt{2\pi}} \cdot e^{-\frac{(y-14)}{32}}$$

b)

A_1: the check is in 1st one

A_2: the check is in 2nd one

A_3: the check is in 3rd one

B_1: bet 1st one, open 2nd one and it's empty

B_2: bet 1st one, open 3rd one and it's empty

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$$

$$P(B_1|A_1) = \frac{1}{2}, \quad P(B_1|A_2) = 0$$
$$P(B_1|A_3) = 1$$

$$P(B_1) = P(B_1|A_1)\cdot P(A_1) + P(B_1|A_2)\cdot P(A_2) + P(B_1|A_3)\cdot P(A_3)$$
$$= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3}$$
$$= \frac{1}{2}$$

$$P(A_1|B_1) = \frac{P(B_1|A_1)\cdot P(A_1)}{P(B)}$$
$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(A_3|B_1) = \frac{P(B_1|A_3)\cdot P(A_3)}{P(B)}$$
$$= \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

As $P(A_1|B_1) < P(A_3|B_1)$,

I will switch to the third one

c)

$f_X(x) = e^{-x}, \quad x>0$

$f_Y(y) = e^{-y}, \quad y>0$

i) $Z = X + Y$

$F_Z(z) = P\{X+Y \le z\}$

$= \int_0^{+\infty}\int_0^{z-y} f_{XY}(x,y)\,dx\,dy$

As X and Y are independent,

$f_{XY}(x,y) = f_X(x)\cdot f_Y(y) = e^{-(x+y)}, \quad x>0, y>0$

$F_Z(z) = \int_0^{+\infty}\int_0^{z-y} e^{-(x+y)}\,dx\,dy$

$= \int_0^{+\infty}(-e^{-z} + e^{-y})\,dy$

$= 1 - e^{-z}, \quad z>0$

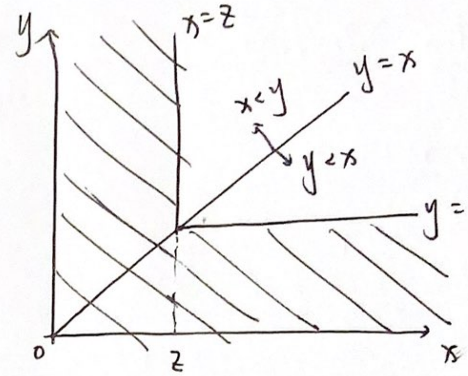
$f_Z(z) = \frac{dF_Z(z)}{dz} = e^{-z}, \quad z>0$

ii) $Z = \min(X, Y)$

$F_Z(z) = P\{\min(X,Y) \le z\}$

$\min(X,Y) = \begin{cases} X, & 0 \le X \le Y \\ Y, & 0 \le Y \le X \end{cases}$

Therefore the regim indicated by the inequality $\min(X,Y) \le z$ is



$\Rightarrow$

$F_Z(z) =$

$\int_0^{+\infty}\int_0^{z} f_{XY}(x,y)\,dx\,dy + \int_0^{z}\int_z^{+\infty} f_{XY}(x,y)\,dx\,dy$

$= \int_0^{+\infty}\int_0^{z} e^{-(x+y)}\,dx\,dy + \int_0^{z}\int_z^{+\infty} e^{-(x+y)}\,dx\,dy$

$= \int_0^{+\infty} -e^{-(z+y)} + e^{-y}\,dy + \int_0^{z} e^{-(z+y)}\,dy$

$\# = 1 - e^{-z} + (-e^{-2z} + e^{-z})$

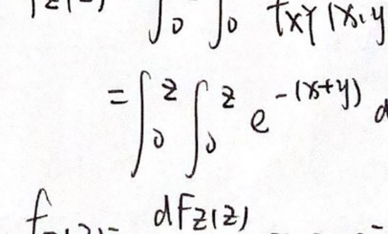
$= 1 - e^{-2z}, \quad z>0$

$f_Z(z) = \frac{dF_Z(z)}{dz} = 2\cdot e^{-2z}, \quad z>0$

iii) $Z = \max(X, Y)$

$F_Z(z) = P\{\max(X,Y) \le z\}$

Similiarly, the region is



$F_Z(z) = \int_0^{z}\int_0^{z} f_{XY}(x,y)\,dx\,dy$

$= \int_0^{z}\int_0^{z} e^{-(x+y)}\,dx\,dy = 1 - 2e^{-z} + e^{-2z}$

$f_Z(z) = \frac{dF_Z(z)}{dz} = 2\cdot e^{-z} - 2\cdot e^{-2z}$

$= 2\cdot(e^{-z} - e^{-2z})$

2. Estimation and Sequence of random variables

a) $f(x) = c^4 \cdot x^3 \cdot e^{-cx}$

The likelihood function is

$f_X(x_1, x_2, x_3; c)$

As x_1, x_2 and x_3 are i.i.d samples

$f_X(x_1, x_2, x_3; c) = f_X(x_1; c) \cdot f_X(x_2; c) \cdot f_X(x_3; c)$

$= c^4 x_1^3 e^{-cx_1} \cdot c^4 \cdot x_2^3 e^{-cx_2} \cdot c^4 x_3^3 \cdot e^{-cx_3}$

$= c^{12} \cdot (x_1 x_2 x_3)^3 \cdot e^{-c(x_1+x_2+x_3)}$

$\ln(f_X(x_1, x_2, x_3; c))$

$= 12 \cdot \ln c + 3 \cdot \ln(x_1 x_2 x_3) - c \cdot (x_1 + x_2 + x_3)$

$$\frac{\partial \ln(f_X(x_1, x_2, x_3; c))}{\partial c} = 0$$

$$\Rightarrow \frac{12}{c} - (x_1 + x_2 + x_3) = 0$$

$$c = \frac{12}{x_1 + x_2 + x_3}$$

$$= \frac{12}{3.1 + 3.4 + 3.3}$$

$$= \frac{60}{49}$$

b) $X_i \sim U(0, 1)$

$\Rightarrow F_{X_i}(x_i) = x_i. \quad 0 \le x_i \le 1$

$Y = \max X_i$

$F(y) = P\{Y \le y\}$

$= P\{\max X_i \le y\}$

(continue)

$= P\{X_1 \le y, X_2 \le y, X_3 \le y, \cdots, X_n \le y\}$

$= P\{X_1 \le y\} \cdot P\{X_2 \le y\} \cdots P\{X_n \le y\}$

$= F_{X_1}(y) \cdot F_{X_2}(y) \cdot \cdots F_{X_n}(y)$

$= \underbrace{y \cdot y \cdot \cdots \cdot y}_{n}$

$= y^n \quad (0 \le y \le 1)$

c)

As the linear MMSE estimate of $S(t - \frac{T}{2})$ is $a \cdot S(t) + b \cdot S(t-T)$, we have the error

$$\varepsilon = S(t - \tfrac{T}{2}) - [aS(t) + bS(t-T)]$$

And there is orthogonal principle

$E[\varepsilon \cdot S^*(t)] = 0$ ①, $E[\varepsilon \cdot S^*(t-T)] = 0$ ②

$E[\varepsilon \cdot S^*(t)]$

$= E[S(t-\frac{T}{2}) \cdot S^*(t) - a \cdot S(t) \cdot S^*(t) - b \cdot S(t-T) \cdot S^*(t)]$

$= E[S(t-\frac{T}{2}) \cdot S^*(t)] - a \cdot E[S(t) \cdot S^*(t)] - b \cdot E[S(t-T) \cdot S^*(t)]$

$= R_S(\frac{T}{2}) - a \cdot R_S(0) - b \cdot R_S(T)$

$= I \cdot (e^{-\frac{1}{2}} - a - b \cdot e^{-1})$

According to ①, we have

$e^{-\frac{1}{2}} - a - b \cdot e^{-1} = 0$ ③

(continue)

Similarly, based on equation ②,

we have

$$R_S(-\tfrac{T}{2}) - a\cdot R_S(-T) - b\cdot R_S(0) = 0$$

$$e^{-\frac{1}{2}} - a\cdot e^{-1} - b = 0 \quad ④$$

$$\begin{cases} e^{-\frac{1}{2}} - a - b\cdot e^{-1} = 0 & ③ \\ e^{-\frac{1}{2}} - a\cdot e^{-1} - b = 0 & ④ \end{cases}$$

$$\Rightarrow \begin{cases} a = \dfrac{\sqrt{e^3 - e}}{e^2-1} = \sqrt{\dfrac{e}{e^2-1}} \\ b = \dfrac{\sqrt{e^3 - e}}{e^2-1} = \sqrt{\dfrac{e}{e^2-1}} \end{cases}$$

The mean square error is

$$E[|\varepsilon|^2] = E[\varepsilon\cdot\varepsilon^*]$$

$$= E\left[\left(S(t-\tfrac{T}{2}) - aS(t) - b\cdot S(t-T)\right)\cdot\left(S(t-\tfrac{T}{2}) - aS(t) - bS(t-T)\right)^*\right]$$

$$= R_S(0) - a\cdot R_S(\tfrac{T}{2}) - b\cdot R_S(-\tfrac{T}{2}) - a\cdot R_S(-\tfrac{T}{2}) + a^2\cdot R_S(0) + ab\cdot R_S(-T) - b\cdot R_S(\tfrac{T}{2})$$
$$+ ab\cdot R_S(T) + b^2\cdot R_S(0)$$

$$= 1 - \frac{4}{\sqrt{e^2-1}} + \frac{2}{e-1}$$

3. Random Processes

a)

i)

In fair-coin experiment.

$P\{\text{head shows}\} = \frac{1}{2}$

$P\{\text{tail shows}\} = \frac{1}{2}$

$E[X(t)]$

$= P\{\text{head shows}\} \times \sin \pi t$

$\quad + P\{\text{tail shows}\} \times 2t$

$= \frac{\sin \pi t + 2t}{2}$

ii)

$F(x,t) = P\{X(t) \le x\}$

$= P\{X(t) \le x \mid \text{head shows}\}$

$\quad + P\{X(t) \le x \mid \text{tail shows}\}$

$= P\{\sin \pi t \le x\} \times \frac{1}{2} + P\{2t \le x\} \times \frac{1}{2}$

1° For $t = 0.25$

$F(x, 0.25) = P\{\frac{\sqrt{2}}{2} \le x\} \times \frac{1}{2} + P\{0.5 \le x\} \times \frac{1}{2}$

$\Rightarrow F(x, 0.25) = \begin{cases} 0 & x < 0.5 \\ \frac{1}{2} & 0.5 \le x < \frac{\sqrt{2}}{2} \\ 1 & x \ge \frac{\sqrt{2}}{2} \end{cases}$

(continue)

2° For $t = 0.5$

$F(x, 0.5) = P\{1 \le x\} \times \frac{1}{2} + P\{1 \le x\} \times \frac{1}{2}$

$\Rightarrow F(x, 0.5) = \begin{cases} 0 & x < 1 \\ 1 & x \ge 1 \end{cases}$

3° For $t = 1$

$F(x, 1) = P\{0 \le x\} \times \frac{1}{2} + P\{2 \le x\} \times \frac{1}{2}$

$\Rightarrow F(x, 1) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$

b)

i) According to Generalized Markov Inequality,

we have $P(x \ge a) = P(g(x) \ge g(a)) \le \frac{E[g(x)]}{g(a)}$

Let $g(x) = x^2$

$\Rightarrow P\{|X(t+2) - X(t)| \ge a\}$

$= P\{(X(t+2) - X(t))^2 \ge a^2\} \le \frac{E[(X(t+2) - X(t))^2]}{a^2}$

$E[(X(t+2) - X(t))^2]$

$= E[X^2(t+2) - 2X(t) \cdot X(t+2) + X^2(t)]$

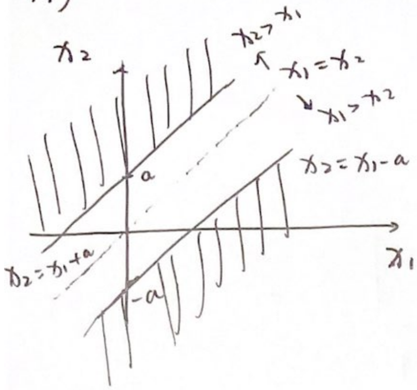
$= E[X^2(t+2)] - 2E[X(t) \cdot X(t+2)] + E[X^2(t)]$

$\xlongequal{X(t)\text{ is real}} R(0) - 2R(2) + R(0)$

$= 2[R(0) - R(2)]$

$\Rightarrow P\{|X(t+2) - X(t)| \ge a\} \le \frac{2[R(0) - R(2)]}{a^2}$

ii)



$P\{|X(t+\tau) - X(t)| \geq a\}$

$= P\{|X_2 - X_1| \geq a\}$

The region reflected by this expression is shown above. Therefore, we have

$P\{|X(t+\tau) - X(t)| \geq a\}$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{x_2 - a} f(x_1, x_2; \tau)\, dx_1 dx_2$$

$$+ \int_{-\infty}^{+\infty} \int_{x_2 + a}^{+\infty} f(x_1, x_2; \tau)\, dx_1 dx_2$$

c)

$$Y'(t) + 2Y(t) = X(t)$$

According to the properties of Fourier Transform, we have.

$$jw \cdot \hat{Y}(w) + 2\hat{Y}(w) = \hat{X}(w)$$

As this is a linear system, the transfer function

$$\Rightarrow H(w) = \frac{\hat{Y}(w)}{\hat{X}(w)} = \frac{1}{2 + jw}$$

~~⇒ The tran~~

(continue)

$$\Rightarrow h(t) = u(t) \cdot e^{-2t}$$

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot X(t-\tau)\, d\tau$$

$$= \int_{0}^{+\infty} e^{-2\tau} \cdot X(t-\tau)\, d\tau$$

$E[Y^2(t)]$

$$= E\left[\int_0^{+\infty} \int_0^{+\infty} e^{-2\tau} \cdot X(t-\tau) \cdot e^{-2k} \cdot X(t-k)\, d\tau dk\right]$$

$$= E\left[\int_0^{+\infty} \int_0^{+\infty} e^{-2\tau - 2k} \cdot X(t-\tau) \cdot X(t-k)\, d\tau dk\right]$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-2\tau - 2k} \cdot E[X(t-\tau) \cdot X(t-k)]\, d\tau dk$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-2\tau - 2k} \cdot 5 \cdot \delta(\tau - k)\, d\tau dk$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-2k - 2k} \cdot 5\delta(\tau - k)\, d\tau dk$$

$$= 5 \cdot \int_0^{+\infty} e^{-4k} \cdot dk$$

$$= \frac{5}{4}$$

$$\cancel{R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) * h^*(t_2)}$$

$$\cancel{= 5 \cdot \delta(t_2 - t_1) * h^*(t_2)}$$

$$\cancel{= \int_{-\infty}^{+\infty} h^*(\tau) \cdot 5 \cdot \delta(t_2 - \tau - t_1)\, d\tau}$$

$$Y(t) = \int_0^{+\infty} e^{-2\tau} \cdot X(t-\tau)\, d\tau$$

$$Y(t - t_0) = \int_0^{+\infty} e^{-2\tau} \cdot X(t - t_0 - \tau)\, d\tau$$

Therefore, this system is ~~time~~ Time-Invariant System.

(continue)

For an LTI system,

if $X(t)$ is w.s.s, $Y(t)$ is w.s.s

Assume $\tau = t_1 - t_2$, we have

$$R_{XY}(t_1,t_2) = R_{XY}(\tau)$$

$$= R_{XX}(\tau) * h^*(-\tau)$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau-k)\cdot h^*(-k)\cdot dk$$

$$= 5\cdot\int_{-\infty}^{+\infty} \delta(\tau-k)\cdot h^*(-k)\, dk$$

$$= 5\cdot\int_{-\infty}^{+\infty} \delta(\tau-k)\cdot h^*(-\tau)\, dk$$

$$= 5\cdot h^*(-\tau)$$

$$= 5\cdot u(\tau)\cdot e^{-2\tau}$$

$$R_{YY}(t_1,t_2) = \cancel{R_{XY}}\ R_{YY}(\tau)$$

$$= R_{XY}(\tau) * h(\tau)$$

$$= \int_{-\infty}^{+\infty} R_{XY}(\tau-k)\cdot h(k)\cdot dk$$

$$= 5\cdot\int_{-\infty}^{+\infty} u(\tau-k)\cdot e^{-2(\tau-k)}\cdot u(k)\cdot e^{-2k}\, dk$$

$$= 5\cdot\int_{-\infty}^{+\infty} u(\tau-k)\cdot u(k)\cdot e^{-2\tau}\, dk$$

$$= 5\cdot e^{-2\tau}\cdot\int_{-\infty}^{+\infty} u(\tau-k)\cdot u(k)\, dk$$

$$= 5\cdot u(\tau)\cdot \tau\cdot e^{-2\tau}$$

d)

The average power of $Y(t)$

$$E[Y^2(t)] = R_{YY}(0) = \cancel{\oint}\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_Y(w)\, dw$$

$$S_Y(w) = S_X(w)\cdot|\hat{H}(w)|^2$$

$$\Rightarrow E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(w)\cdot|\hat{H}(w)|^2\, dw$$

If $\int_{-\infty}^{+\infty} S_X(w)\cdot|\hat{H}(w)|^2\, dw$ is maximized, the value of $E[Y^2(t)]$ is maximized.

As $S_X(w) \geq 0$, $|\hat{H}(w)|^2 \geq 0$, the we have

$$\int_{-\infty}^{+\infty} S_X(w)\,|\hat{H}(w)|^2\, dw \geq 0$$

Therefore, we can maximize $\left|\int_{-\infty}^{+\infty} S_X(w)|\hat{H}(w)|^2\, dw\right|^2$

According to Cauchy-Schwarz inequality,

$$\left|\int_{-\infty}^{+\infty} S_X(w)|\hat{H}(w)|^2 dw\right|^2 \leq \int_{-\infty}^{+\infty} S_X^2(w)\, dw\cdot\int_{-\infty}^{+\infty}|\hat{H}(w)|^4 dw$$

And the equality is obtained when

$$S_X(w) = \lambda\cdot|\hat{H}(w)|^2 \quad ①$$

Another constraint of $S_X(w)$ is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(w)\, dw = R_{XX}(0) = E[X^2(t)] = 10 \quad ②$$

$$\cancel{\Rightarrow \lambda\int_{-\infty}^{+\infty}|\hat{H}(w)|^2 dw = 20\pi}$$

$$\cancel{\hat{H}(w) = H(s)\Big|_{s=jw} = \frac{1}{(jw)^2+2jw+5}}$$

$$\cancel{= \frac{1}{(-w^2+5)+2jw}}$$

(continue)

According to equation ① and ②, we have

$$\frac{\lambda}{2\pi}\int_{-\infty}^{+\infty} \left|\hat{H}(w)\right|^2 dw = 10 \quad ③$$

Based on Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\hat{H}(w)\right|^2 dw = \int_{-\infty}^{+\infty} h^2(t)\, dt$$

Therefore, we have

$$\lambda \cdot \int_{-\infty}^{+\infty} h^2(t)\, dt = 10 \quad ④$$

The transfer function is

$$H(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$$

$$\Rightarrow h(t) = \cancel{e^{-2t}}\ \frac{1}{2}\cdot(e^{-t}\sin 2t)\,u(t)$$

$$\Rightarrow \frac{\lambda}{4}\int_{-\infty}^{+\infty} e^{-2t}\sin^2 2t \cdot u^2(t)\, dt = 10$$

$$\frac{\lambda}{4}\int_{0}^{+\infty} e^{-2t}\sin^2 2t\, dt = 10$$

$$\frac{\lambda}{4}\cdot\int_{0}^{+\infty} e^{-2t}\cdot\frac{1}{2}(1-\cos 4t)\, dt = 10$$

$$\frac{\lambda}{8}\cdot\left(\int_{0}^{+\infty} e^{-2t} dt - \int_{0}^{+\infty} e^{-2t}\cos 4t\, dt\right) = 10$$

$$\frac{\lambda}{8}\left(-\frac{1}{2}e^{-2t} + \frac{1}{10}\cos 4t\, e^{-2t} - \frac{1}{5}\sin 4t\, e^{-2t}\right)\Bigg|_{t=0}^{+\infty} = 10$$

$$\frac{\lambda}{20} = 10$$

$$\lambda = 200$$

Therefore,

$$S_X(w) = \lambda\cdot\left|\hat{H}(w)\right|^2 = 200\left|\hat{H}(w)\right|^2$$

$$\hat{H}(w) = H(s)\Big|_{s=jw} = \frac{1}{(-w^2+5) + j\cdot 2w}$$

$$\left|\hat{H}(w)\right|^2 = \frac{1}{(w^2-3)^2+16}$$

$$S_X(w) = \frac{200}{(w^2-3)^2+16}$$

4. Random Processes and Markov Chains ~~iii)~~

a)

i) $Y_n = \sum_{i=0}^{n} X_i \lambda^i$

we have

$$Y_{n+1} = \sum_{i=0}^{n+1} X_i \lambda^i$$

$$= \sum_{i=0}^{n} X_i \lambda^i + X_{n+1} \cdot \lambda^{n+1} = Y_n + X_{n+1} \cdot \lambda^{n+1}$$

$$\mathbb{E}\{Y_{n+1} \mid Y_n, Y_{n-1}, \cdots Y_1, Y_0\}$$

$$= \mathbb{E}\{Y_n + X_{n+1} \cdot \lambda^{n+1} \mid Y_n, \cdots, Y_0\}$$

$$= \mathbb{E}\{Y_n \mid Y_n, \cdots, Y_0\} + \mathbb{E}\{X_{n+1} \cdot \lambda^{n+1} \mid Y_n, \cdots, Y_0\}$$

$$= Y_n + \lambda^{n+1} \cdot \mathbb{E}\{X_{n+1}\}$$

$$= Y_n + \lambda^{n+1} \times \left(\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times (-1)\right)$$

$$= Y_n$$

Therefore, $\{Y_n\}$ is martingale

ii) The characteristic function of Y_n is

$$\Phi_{Y_n}(w) = \mathbb{E}\left[e^{jY_n w}\right]$$

$$= \mathbb{E}\left[e^{j\sum_{i=0}^{n} X_i \lambda^i w}\right]$$

$$= \mathbb{E}\left[\prod_{i=0}^{n} e^{jX_i \lambda^i w}\right]$$

As $X_0, X_1, X_2 \cdots$ are independent

$$\Phi_{Y_n}(w) = \prod_{i=0}^{n} \mathbb{E}\left[e^{jX_i \lambda^i w}\right]$$

$$= \prod_{i=0}^{n} \left(P(X_i = 1) \cdot e^{j\lambda^i w} + P(X_i = -1) e^{-j\lambda^i w} \right)$$

$$= \prod_{i=0}^{n} \tfrac{1}{2}\left(e^{j\lambda^i w} + e^{-j\lambda^i w}\right)$$

$$= \prod_{i=0}^{n} \cos \lambda^i w$$

iii) $P(Y_{n+1} \in E)$

$$= P\left(\sum_{i=0}^{n+1} X_i \lambda^i \in E\right)$$

$$= P\left(\sum_{i=1}^{n+1} X_i \lambda^i + X_0 \in E\right)$$

~~$=P($~~

$$= P\left(\sum_{i=1}^{n+1} X_i \lambda^i + X_0 \in E \mid X_0 = 1\right) \cdot P(X_0 = 1)$$

$$+ P\left(\sum_{i=1}^{n+1} X_i \lambda^i + X_0 \in E \mid X_0 = -1\right) \cdot P(X_0 = -1)$$

$$= \tfrac{1}{2} P\left(\sum_{i=1}^{n+1} X_i \lambda^i + 1 \in E\right) + \tfrac{1}{2} P\left(\sum_{i=1}^{n+1} X_i \lambda^i - 1 \in E\right)$$

As $X_0, X_1, X_2 \cdots$ have the same distribution,

$$P\left(\sum_{i=1}^{n+1} X_i \lambda^i + 1 \in E\right)$$

$$= P\left(\sum_{i=1}^{n+1} X_{i-1} \cdot \lambda^i + 1 \in E\right)$$

$$= P\left(\lambda \cdot \sum_{i=1}^{n+1} X_{i-1} \cdot \lambda^{i-1} + 1 \in E\right)$$

$$= P\left(\lambda \cdot \sum_{i=0}^{n} X_i \lambda^i + 1 \in E\right)$$

$$= P(\lambda Y_n + 1 \in E)$$

Similiarly,

~~$P(\sum_{i=1}^{n+1} X_i \lambda^i +1 \in$~~

$$P\left(\sum_{i=1}^{n+1} X_i \lambda^i - 1 \in E\right) = P(\lambda Y_n - 1 \in E)$$

Therefore,

$$P(Y_{n+1} \in E)$$

$$= \tfrac{1}{2} P(\lambda Y_n + 1 \in E) + \tfrac{1}{2} P(\lambda Y_n - 1 \in E)$$

$$= \tfrac{1}{2} P(Y_n \in T_1^{-1}(E)) + \tfrac{1}{2} P(Y_n \in T_2^{-1}(E))$$

where $T_1(x) = \lambda x + 1$, $T_2(x) = \lambda x - 1$

iv) $\Phi_{Y_n}(w) = \prod_{i=0}^{n} \cos \lambda^i w$

As $\lambda = \frac{1}{2}$,

$\Phi_{Y_n}(w) = \prod_{i=0}^{n} \cos \frac{w}{2^i}$

$\sin w = 2 \cdot \cos\frac{w}{2} \cdot \sin\frac{w}{2}$

$= 2^2 \cdot \cos\frac{w}{2} \cdot \cos\frac{w}{2^2} \cdot \sin\frac{w}{2^2}$

$\vdots$

$= 2^n \cdot \left(\prod_{i=1}^{n} \cos\frac{w}{2^i}\right) \cdot \sin\frac{w}{2^n}$

$\prod_{i=1}^{n} \cos\frac{w}{2^i} = \frac{\sin w}{2^n \cdot \sin\frac{w}{2^n}}$

Therefore,

$\Phi_{Y_n}(w) = \cos w \cdot \prod_{i=1}^{n} \cos\frac{w}{2^i}$

$= \frac{\cos w \cdot \sin w}{2^n \cdot \sin\frac{w}{2^n}}$

$\lim_{n\to\infty} \Phi_{Y_n}(w) = \lim_{n\to\infty} \frac{\cos w \cdot \sin w}{2^n \cdot \sin\frac{w}{2^n}}$

$= \cos w \cdot \sin w \cdot \lim_{n\to\infty} \frac{1}{2^n \cdot \sin\frac{w}{2^n}}$

$= \cos w \cdot \sin w \cdot \lim_{n\to\infty} \frac{2^{-n}}{\sin(w \cdot 2^{-n})}$

$\overset{t=2^{-n}}{=\!=} \cos w \cdot \sin w \cdot \lim_{t\to 0} \frac{t}{\sin wt}$

$\overset{k=wt}{=\!=} \cos w \cdot \sin w \cdot \lim_{k\to 0} \frac{1}{w} \cdot \frac{k}{\sin k}$

$= \frac{1}{w} \cos w \cdot \sin w \cdot \lim_{k\to 0} \frac{k}{\sin k}$

$= \frac{1}{w} \cdot \cos w \cdot \sin w$

$= \frac{1}{2w} \sin 2w$

$= \frac{1}{4wj} \left(e^{j\cdot 2w} - e^{-j\cdot 2w}\right)$

$\lim_{n\to\infty} \Phi_{Y_n}(w) = \frac{1}{4wj} \cdot \left(e^{j\cdot 2w} - e^{-j\cdot 2w}\right)$

~~This~~

This is the characteristic function of a uniform distribution in the interval $[-2, 2]$

Therefore, the limitation distribution of Y_n is $U(-2, 2)$

b)

i) $p=\frac{1}{3}$. $q=\frac{2}{3}$

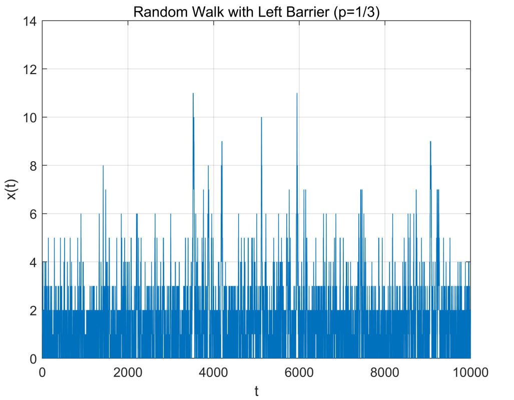


Figure 1

ii) $p=\frac{1}{2}$, $q=\frac{1}{2}$

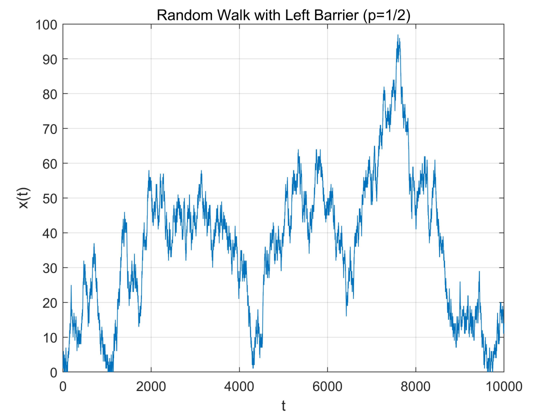


Figure 2

iii) $p=\frac{2}{3}$, $q=\frac{1}{3}$

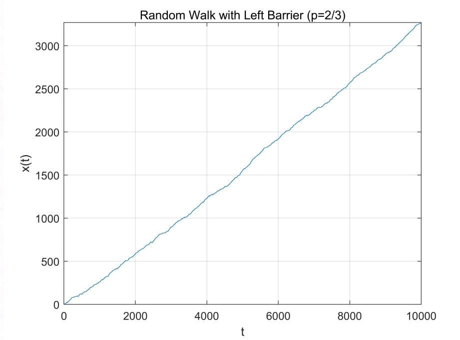


Figure 3

iv) Discussion:

1° For the case where $p=\frac{1}{3}$, we have $p<q$. Therefore, the states are more likely to go back. All the states in this case are recurrent positive. As shown is Figure 1, the states ~~[illegible]~~ merely oscillate within a small scope with low value.

2° For the case where $p=\frac{1}{2}$, we have $p=q$. In this case, all the states are recurrent null, which means that the states must go back with in a sufficient long time but the expection of the time is ∞. The Figure 2 shows this trend, where the curve has rich ups and downs.

3° For the case where $p=\frac{2}{3}$, we have $p>q$. In this case, all the states are transient, which means the states may not go back anymore. In Figure 3, we can observe that the curve seems like a straight and ~~[illegible]~~ has a clear growth trend.