2: Three Different Fourier Transforms

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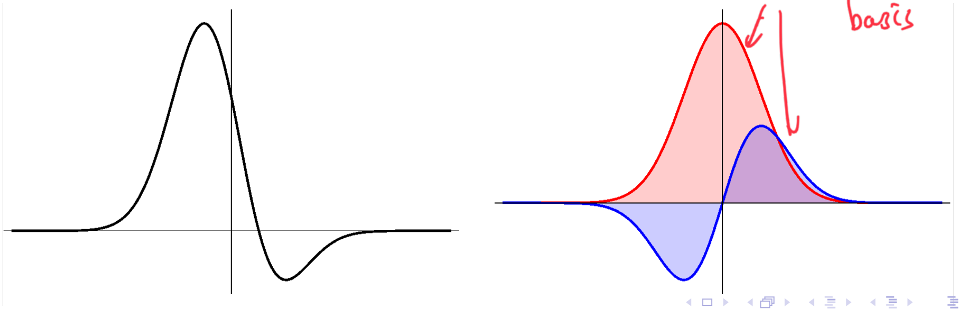
combination of two basis



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$$f(x) = c_0 \operatorname{sinc}(x) + c_{\pm 1} \operatorname{sinc}(x \pm 1) + c_{\pm 2} \operatorname{sinc}(x \pm 2) + \cdots, \qquad c_k = f(k), k \in \mathbb{Z}$$

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$$f(x) = c_0 + c_{\pm 1} \exp(\pm \jmath \omega_0 t) + c_{\pm 2} \exp(\pm \jmath 2 \omega_0 t) + \cdots, \qquad c_k = \frac{1}{T} \int_0^T f(x) e^{-\jmath \frac{2\pi}{T} kt} dt, k \in \mathbb{Z}$$

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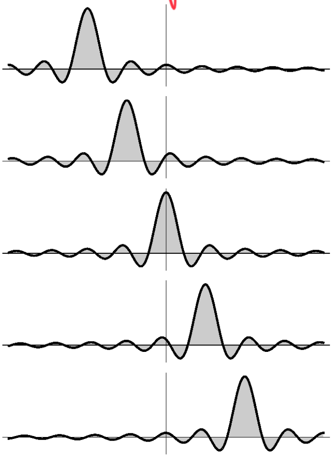
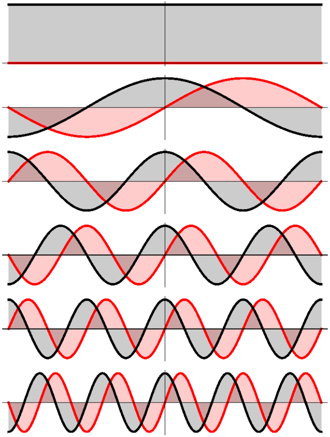




Signal Representation in Orthogonal Basis.

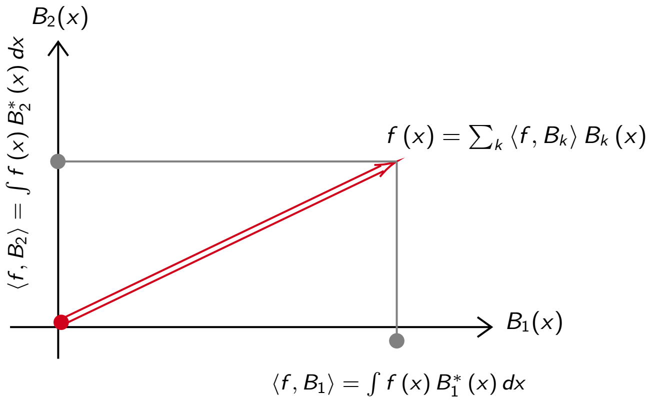
When the basis functions are orthogonal, that is,

$$\langle B_m, B_n \rangle = \int B_m(x)\, B_n^*(x)\, dx = \boxed{\delta_{m-n}}$$

then the function can be represented by the expansion,

$$f(x) = \sum_k \langle f, B_k \rangle B_k(x).$$

This simple yet powerful method is at the heart of the Shannon's Sampling Series as well as the Fourier Series.



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$$\langle B_m, B_n \rangle = \int B_m(x)\, B_n^*(x)\, dx = \frac{1}{T} \int_0^T e^{\jmath \frac{2\pi}{T} mx} e^{-\jmath \frac{2\pi}{T} nx} dx = \frac{1}{T} \int_0^T e^{\jmath \frac{2\pi}{T} (m-n)x} dx = \delta_{m-n}.$$

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3. Because of this, we can write,

$$f(x) = \sum_k \langle f, B_k \rangle B_k(x).$$

where, the Fourier Series coefficients are given by,

$$\langle f, B_k \rangle = \int f(x) B_k^*(x)\, dx = \int f(x)\, e^{-\jmath \frac{2\pi}{T} kx} dx = \widehat{f}\left(\frac{2\pi}{T} k \right).$$

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We use Ω for "real" and $\omega = \Omega T$ for "normalized" angular frequency.
Nyquist frequency is at $\Omega_{\text{Nyq}} = 2\pi\frac{f_s}{2} = \frac{\pi}{T}$ and $\omega_{\text{Nyq}} = \pi$.

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For "power signals" (energy $\propto$ duration), CTFT & DTFT are unbounded.
Fix this by normalizing:

$$X(j\Omega) = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} x(t)e^{-j\Omega t}dt$$

$$X(e^{j\omega}) = \lim_{A\to\infty} \frac{1}{2A+1}\sum_{-A}^{A} x[n]e^{-j\omega n}$$

Convergence of DTFT

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ does not converge for all $x[n]$.

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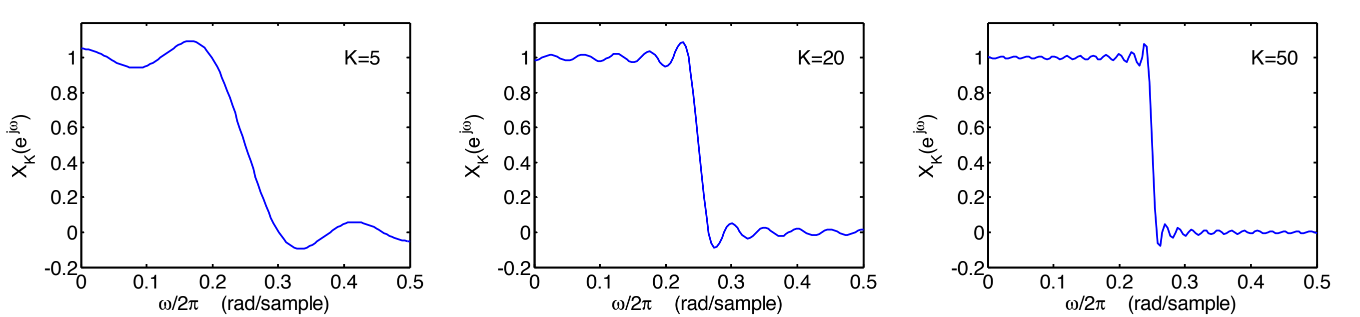

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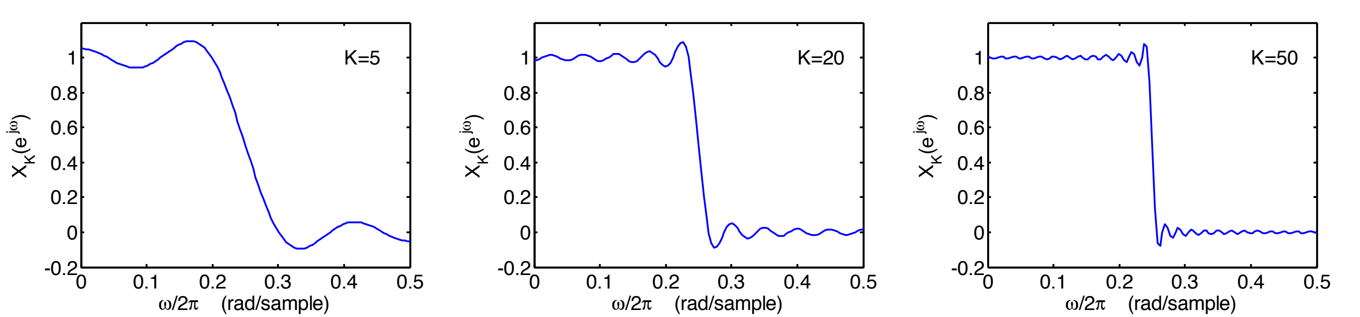

Gibbs phenomenon:

Converges at each ω as $K \to \infty$ but peak error does not get smaller.

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$$\stackrel{\text{(i)}}{=} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\omega\frac{t}{T}}dt$$

(i) OK if $\sum_{-\infty}^{\infty} |x[n]| < \infty$.

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DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

- DTFT is periodic in ω: $X(e^{j(\omega+2m\pi)}) = X(e^{j\omega})$ for integer m.
- DTFT is the z-Transform evaluated at the point $e^{j\omega}$:
  $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
  DTFT converges iff the ROC includes $|z| = 1$.
- DTFT is the same as the CTFT of a signal comprising impulses at the sample times (Dirac δ functions) of appropriate heights:
  $x_\delta(t) = \sum x[n]\delta(t-nT) = x(t) \times \sum_{-\infty}^{\infty} \delta(t-nT)$
  Equivalent to multiplying a continuous $x(t)$ by an impulse train.

Proof: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT)e^{-j\omega\frac{t}{T}}dt$$

$$\overset{\text{(i)}}{=} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\omega\frac{t}{T}}dt$$

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If $x[n]$ has a special property then $X(e^{j\omega})$and $X[k]$ will have corresponding properties as shown in the table (and vice versa):

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The inverse DFT matrix is $\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^H$
equivalently, $\mathbf{G} = \frac{1}{\sqrt{N}}\mathbf{F}$ is a unitary matrix with $\mathbf{G}^H\mathbf{G} = \mathbf{I}$.

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DTFT: Convolution $\rightarrow$ Product

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$$x[n] = g[n] * h[n] = \underbrace{\sum_{k=-\infty}^{\infty} g[k]h[n-k]}_{\text{weighted sum of shifted sequence}}$$

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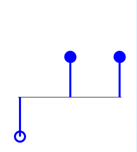
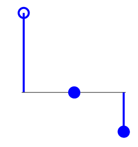
$g[n]$:  $h[n]$: 

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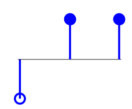
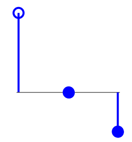
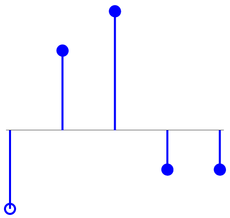
$g[n]:$  $h[n]:$  $g[n] * h[n]:$ 

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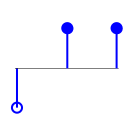
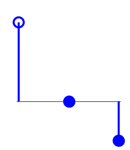
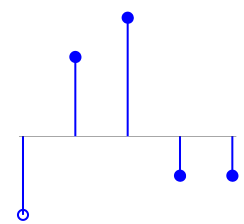
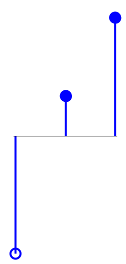
$g[n]:$  $h[n]:$  $g[n] * h[n]:$  $g[n] \circledast_3 h[n]$ 

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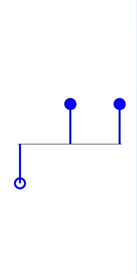
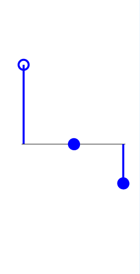
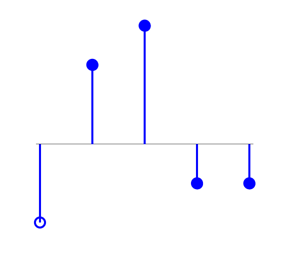
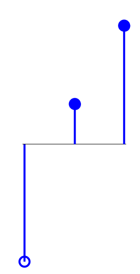
$g[n]$:  $h[n]$:  $g[n] * h[n]$:  $g[n] \circledast_3 h[n]$ 

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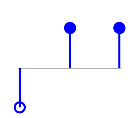
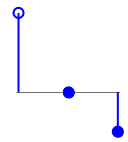
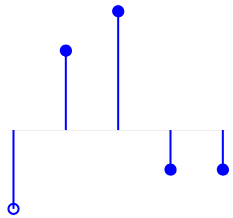
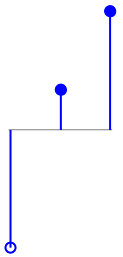
$g[n]:$  $h[n]:$  $g[n] * h[n]:$  $g[n] \circledast_3 h[n]$ 

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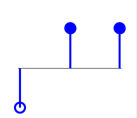
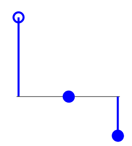
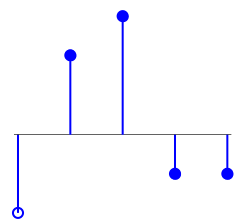
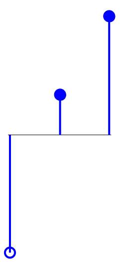
$g[n]$:  $h[n]$:  $g[n] * h[n]$:  $g[n] \circledast_3 h[n]$ 

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$$\Rightarrow X(e^{j\omega}) = G(e^{j\omega})H(e^{j\omega})$$

DFT: Circular convolution$\rightarrow$ Product

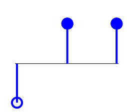
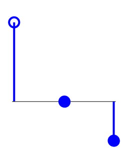
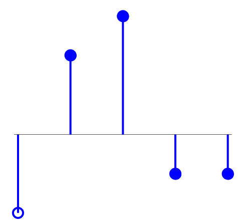
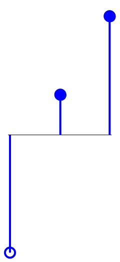
$$x[n] = g[n] \circledast_N h[n] = \sum_{k=0}^{N-1} g[k]h[(n-k)_{\text{mod}N}]$$
$$\Rightarrow X[k] = G[k]H[k]$$

DTFT: Product$\rightarrow$ Circular Convolution $\div 2\pi$

$$y[n] = g[n]h[n]$$
$$\Rightarrow Y(e^{j\omega}) = \frac{1}{2\pi}G(e^{j\omega}) \circledast_\pi H(e^{j\omega}) =$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

DFT: Product$\rightarrow$ Circular Convolution $\div N$

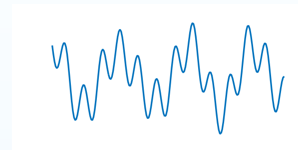
$$y[n] = g[n]h[n]$$
$$\Rightarrow Y[k] = \frac{1}{N}G[k] \circledast_N H[k]$$

$g[n]:$  $h[n]:$  $g[n] * h[n]:$  $g[n] \circledast_3 h[n]$ 

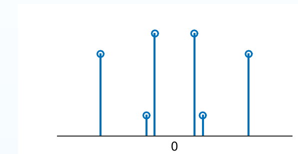
Sampling Process

Time | Time | Frequency

Analog



CTFT $\longrightarrow$



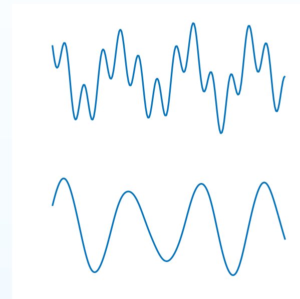
Sampling Process

Time

Time

Frequency

Analog



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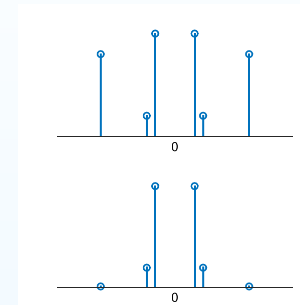
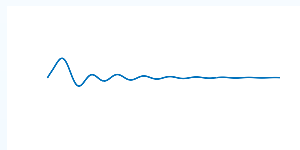


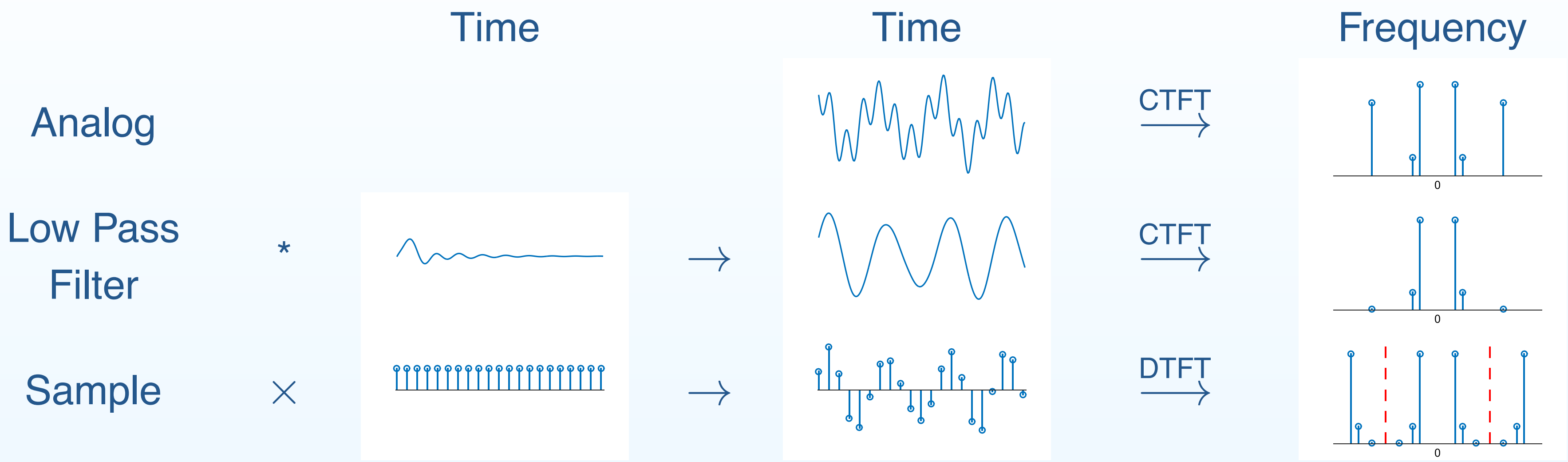
Low Pass Filter $*$



$\longrightarrow$

CTFT $\longrightarrow$

Sampling Process



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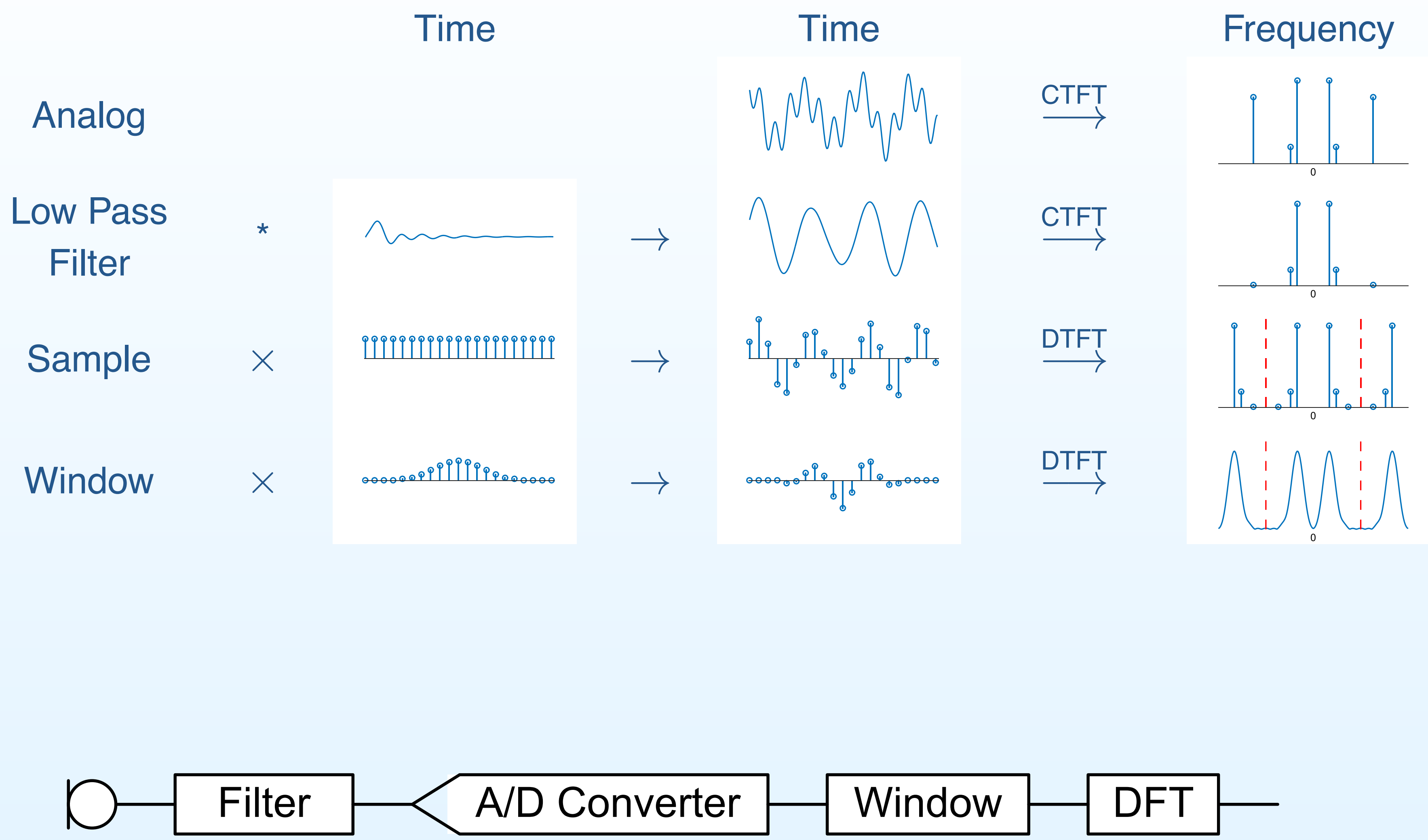

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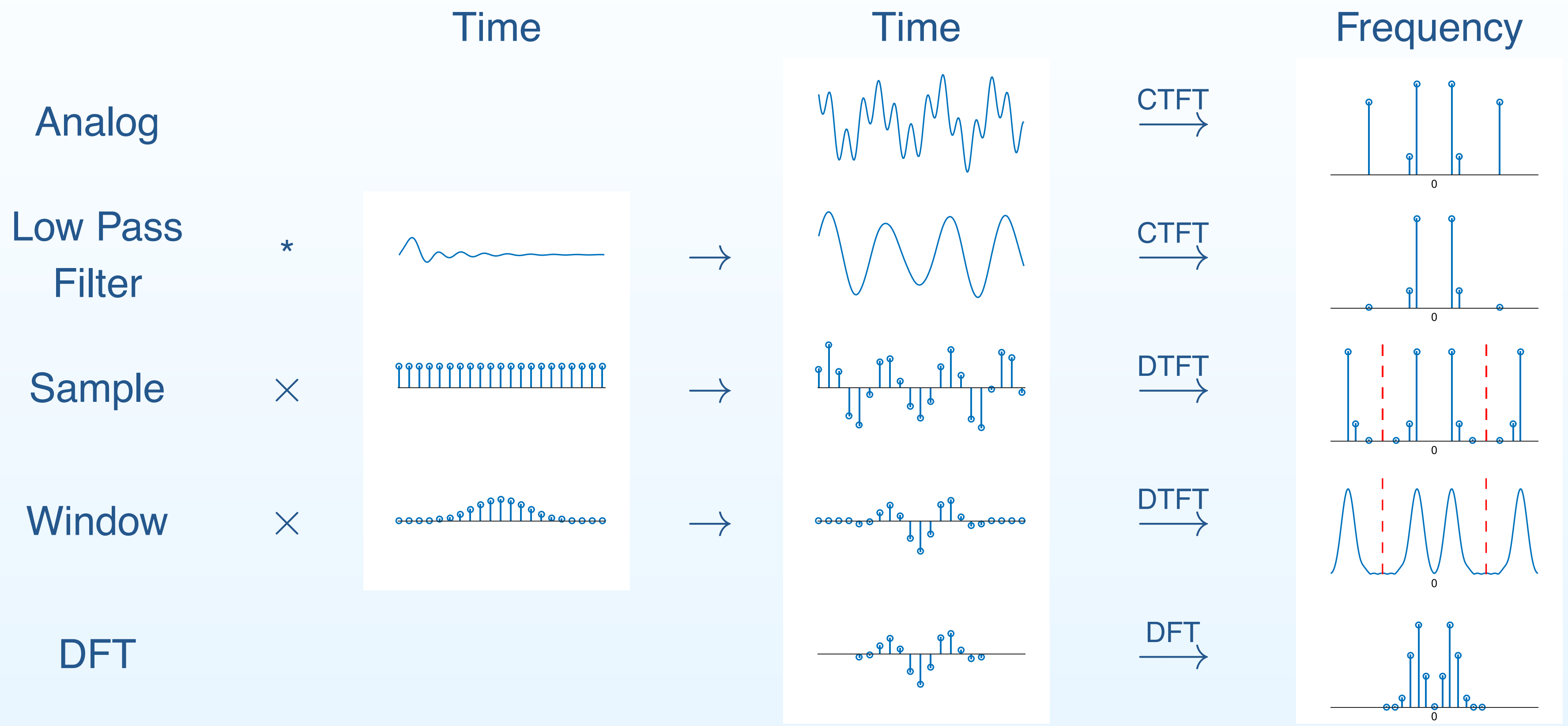
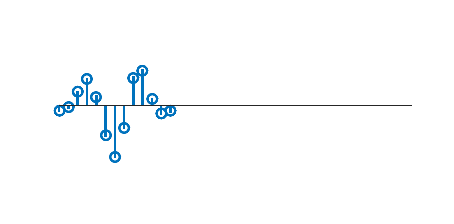


Zero-Padding

Zero padding means added extra zeros onto the end of $x[n]$ before performing the DFT.

Time $x[n]$

Windowed Signal



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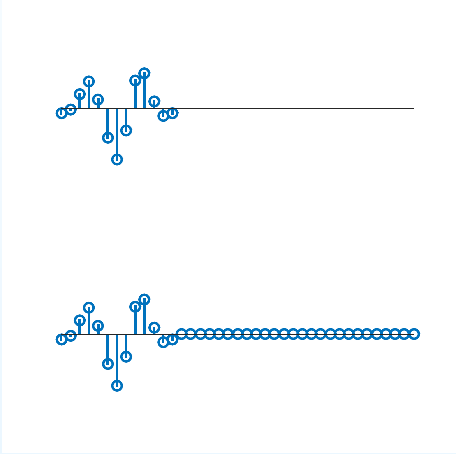
Windowed Signal

With zero-padding



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Frequency $|X[k]|$

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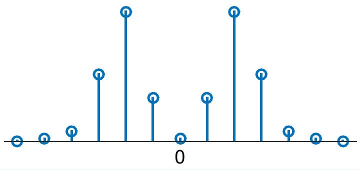
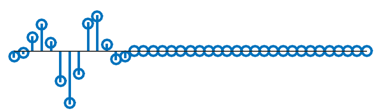


With zero-padding



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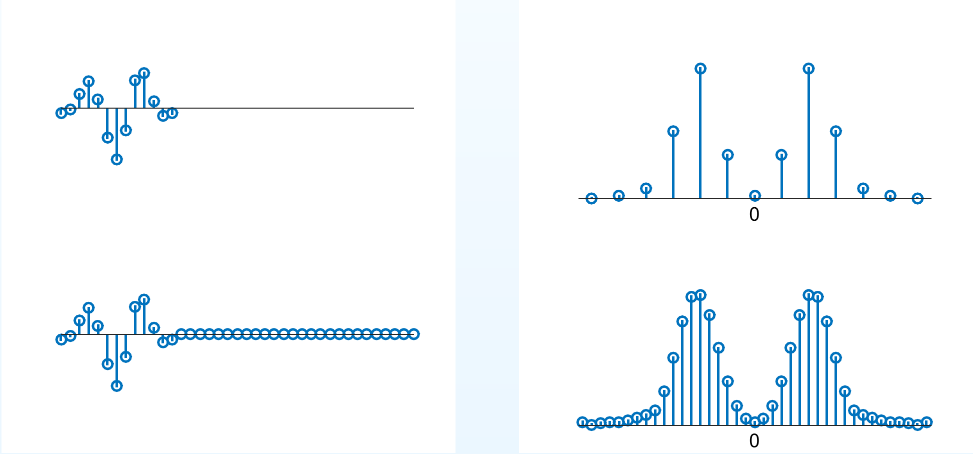

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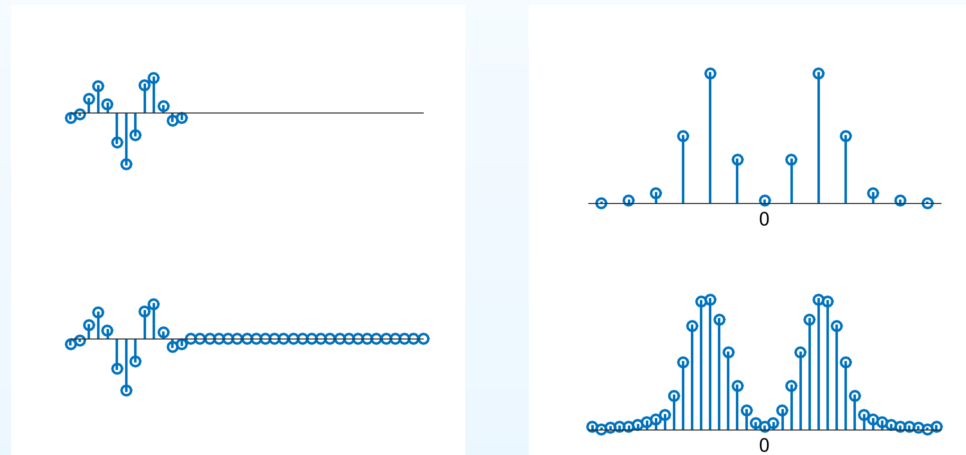

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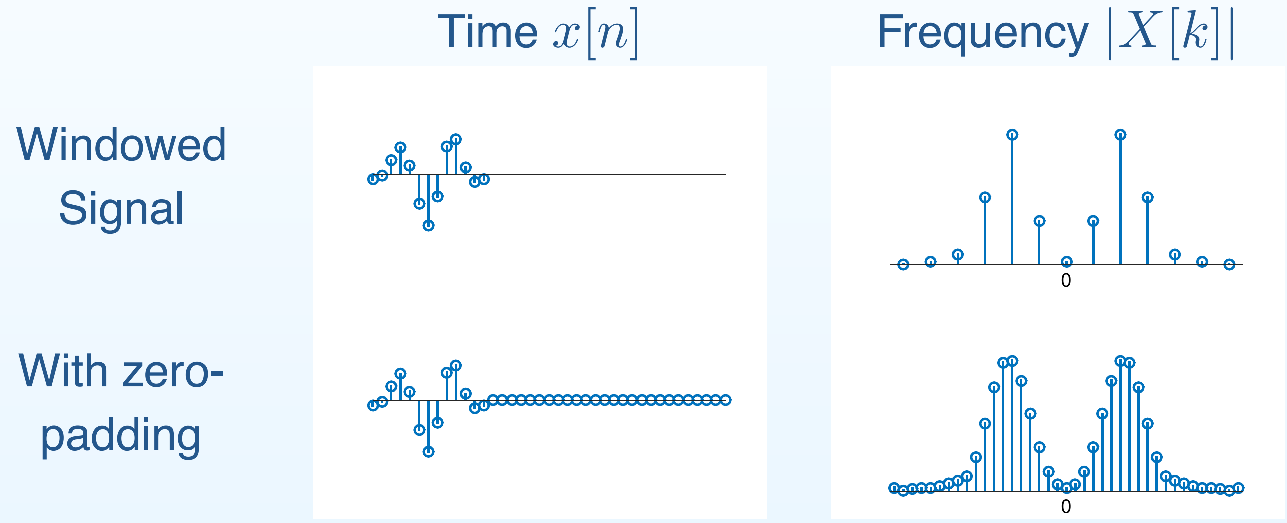

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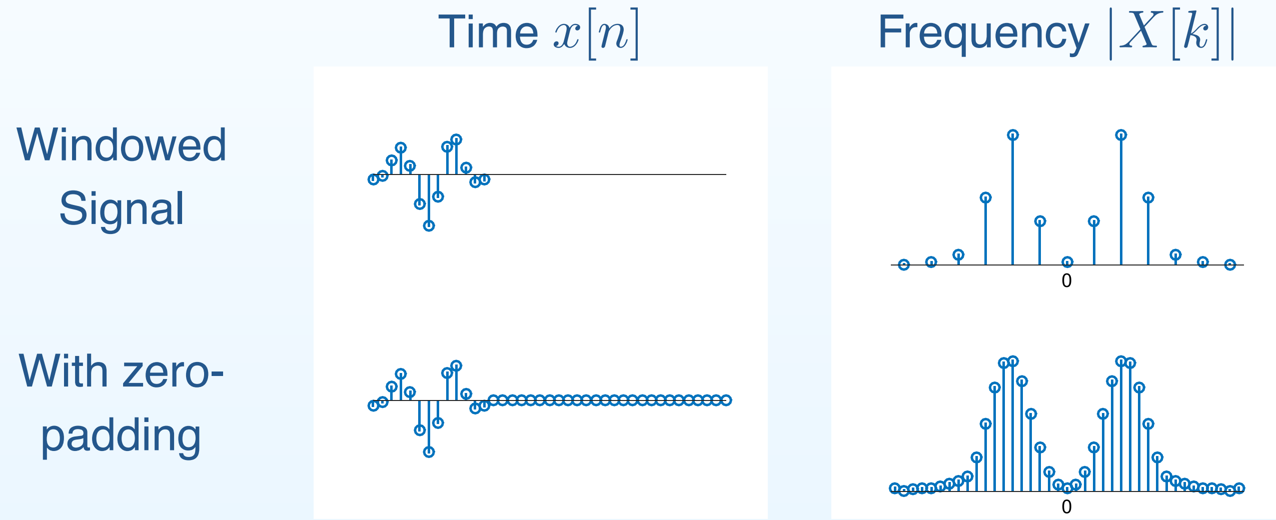
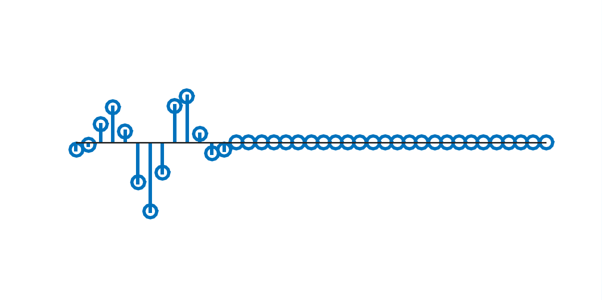


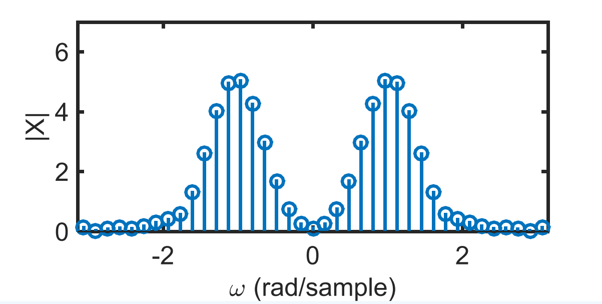
- Zero-padding causes the DFT to evaluate the DTFT at more values of ω_k. Denser frequency samples.
- Width of the peaks remains constant: determined by the length and shape of the window.
- Smoother graph but increased frequency resolution is an illusion.

Phase Unwrapping

Phase of a DTFT is only defined to within an integer multiple of 2π.



$x[n]$



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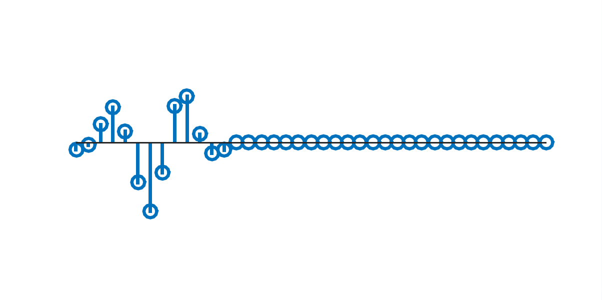

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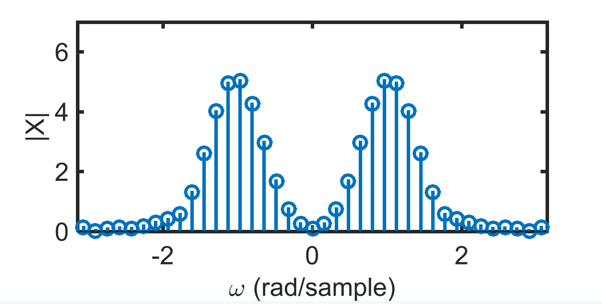

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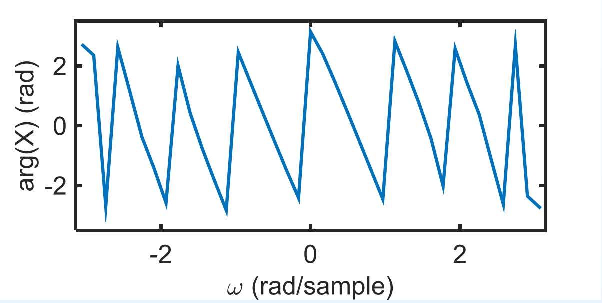

$\angle X[k]$

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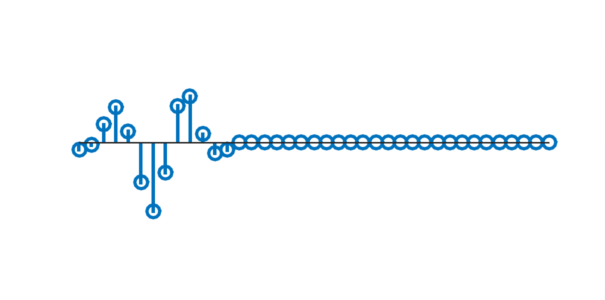

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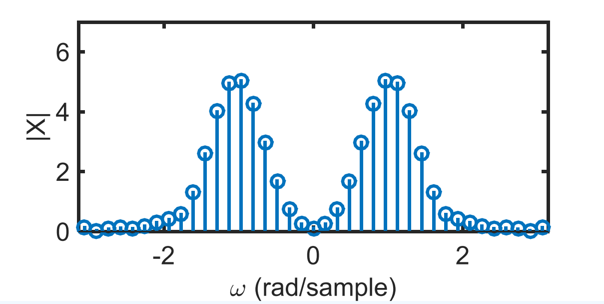
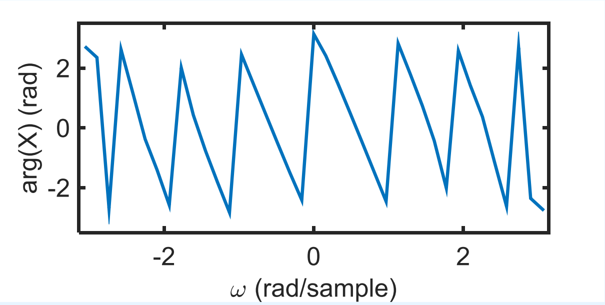


$|X[k]|$



$\angle X[k]$

Phase unwrapping adds multiples of 2π onto each $\angle X[k]$ to make the phase as continuous as possible.

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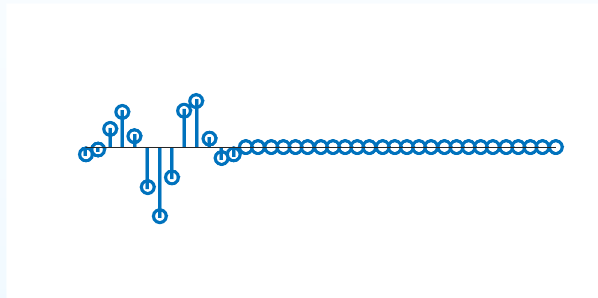

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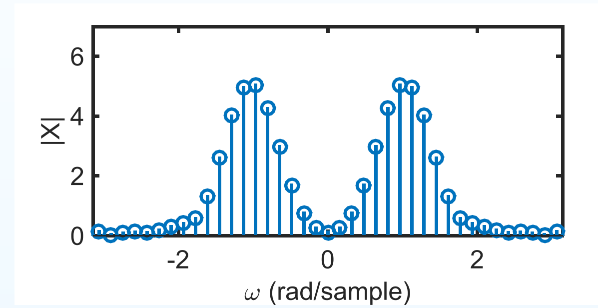
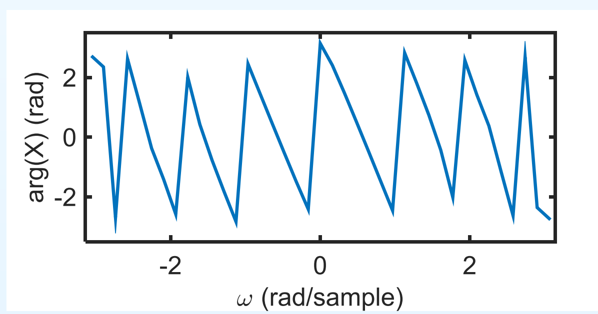


$|X[k]|$



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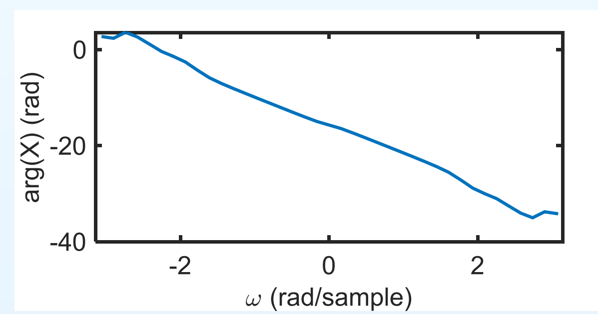

$\angle X[k]$ unwrapped

Phase unwrapping adds multiples of 2π onto each $\angle X[k]$ to make the phase as continuous as possible.

Uncertainty principle

CTFT uncertainty principle: $\left(\frac{\int t^2|x(t)|^2 dt}{\int |x(t)|^2 dt}\right)^{\frac{1}{2}} \left(\frac{\int \omega^2|X(j\omega)|^2 d\omega}{\int |X(j\omega)|^2 d\omega}\right)^{\frac{1}{2}} \geq \frac{1}{2}$

Fourier Uncertainty Relation

A non-zero signal can not be simultaneously localized in time and frequency domains.

[1] "Uncertainty measures and uncertainty relations for angle observables," *Foundations of Physics*, vol. 15, pp. 353–364, 1983.

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For an intuitive understanding, consider the case of a Gaussian function. This is an example of a self-Fourier transform. That is,

$$g_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \xrightarrow{\text{Fourier}} \widehat{g}_\sigma(\omega) = e^{-\frac{\sigma^2\omega^2}{2}}.$$

And hence,

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Shrinking σ in one domain leads to fattening in another domain.

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Shrinking σ in one domain leads to fattening in another domain.

- Interesting topic with historical roots and interpretations in Quantum mechanics.
- There is a variant for discrete sequences that can measure "localization" and goes back to the work of Breitenberger [1].

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The "localization" or spread is measured by,

$$\text{spread}^2 = \int_{\mathbb{R}} (t - \text{center}) f(t) dt \qquad \text{where} \qquad \text{center} = \int_{\mathbb{R}} t f(t) dt.$$

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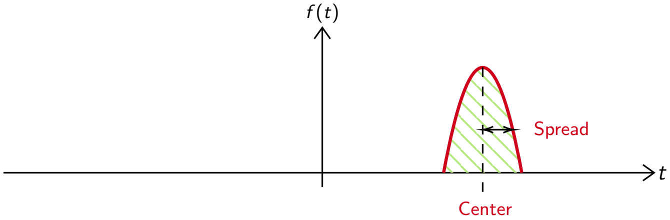

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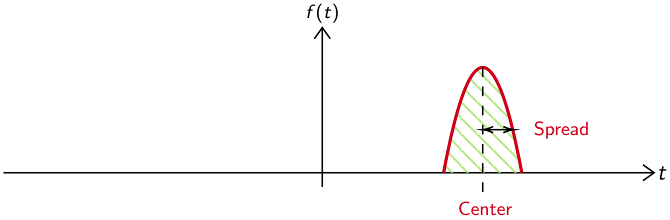

Not surprisingly, for $f(t) = e^{-\frac{(t-\mu)^2}{2\sigma^2}}$ (a Gaussian function), its center and spread are its mean and variance, respectively,

$$\mu \triangleq \int_{\mathbb{R}} t f(t)\, dt \qquad \text{and} \qquad \sigma^2 \triangleq \int_{\mathbb{R}} (t-\mu)^2 f(t)\, dt.$$

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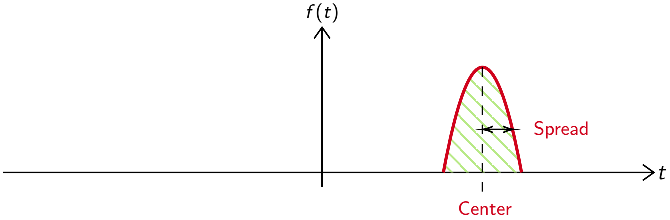

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When dealing with Fourier Uncertainty Principles, we work with squared functions, that is, $|f(t)|^2$.

Uncertainty principle

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The first term measures the "width" of $x(t)$ around $t = 0$.

It is like σ if $|x(t)|^2$ was a zero-mean probability distribution.

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A signal cannot be concentrated in both time and frequency.

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$= \left(\int t^2x^2dt\right)\left(\int |v(t)|^2\, dt\right)$

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The second term is similarly the "width" of $X(j\omega)$ in frequency.

A signal cannot be concentrated in both time and frequency.

Proof Outline:

Assume $\int |x(t)|^2 dt = 1 \Rightarrow \int |X(j\omega)|^2 d\omega = 2\pi$ [Parseval]

Set $v(t) = \frac{dx}{dt} \Rightarrow V(j\omega) = j\omega X(j\omega)$ [by parts]

Now $\int tx\frac{dx}{dt}dt = \frac{1}{2}tx^2(t)\Big|_{t=-\infty}^{\infty} - \int \frac{1}{2}x^2 dt = 0 - \frac{1}{2}$ [by parts]

So $\frac{1}{4} = \left|\int tx\frac{dx}{dt}dt\right|^2 \leq \left(\int t^2x^2dt\right)\left(\int \left|\frac{dx}{dt}\right|^2 dt\right)$ [Schwartz]

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Uncertainty principle

CTFT uncertainty principle: $\left(\frac{\int t^2 |x(t)|^2 dt}{\int |x(t)|^2 dt}\right)^{\frac{1}{2}} \left(\frac{\int \omega^2 |X(j\omega)|^2 d\omega}{\int |X(j\omega)|^2 d\omega}\right)^{\frac{1}{2}} \ge \frac{1}{2}$

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No exact equivalent for DTFT/DFT but a similar effect is true

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For further details see Mitra: 3 & 5.

MATLAB routines

fft, ifft	DFT with optional zero-padding
fftshift	swap the two halves of a vector
conv	convolution or polynomial multiplication (not circular)
$x[n] \circledast y[n]$	real(ifft(fft(x).*fft(y)))
unwrap	remove 2π jumps from phase spectrum