EE4-10
EE9-CS5-1
EE9-SC3
EE9-FPN2-02

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2019

MSc and EEE PART IV: MEng and ACGI

**Corrected copy**

**PROBABILITY AND STOCHASTIC PROCESSES**

Friday, 3 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.** **All questions carry equal marks.**

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : C. Ling
Second Marker(s) : D. Angeli

**Information for students**

*Each of the four questions has 25 marks.*

## The Questions

1. Random variables.

a) Let $X$ and $Y$ be independent random variables, uniform in the interval $(0, 1)$. Find the probability density function of

i) $XY$; [6]

ii) $X / Y$; [6]

iii) $X / (X+Y)$; [6]

b) Let $X$ be a non-negative random variable. Show that

$$E[X] = \int_0^\infty P(X > x)\, dx.$$

[7]

2. Estimation and sequences of random variables.

a) Alice and Bob agree to meet in h-bar after their Friday lectures. They arrive at times that are independent and uniformly distributed between 5:00pm and 5:30pm. Each is prepared to wait $s$ minutes before leaving. Find a minimal $s$ such that the probability that they meet is at least $3/4$.

[13]

b) The random variable $X$ has the Gamma distribution

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0, \alpha = 4, \beta > 0.$$

We observe the i.i.d. samples $x_i$ = 5.6, 6.4, 6.3, 5.7. Find the maximum-likelihood estimate of parameter $\beta$. [12]

3. Random processes.

a) In a fair-coin experiment, we define the random process $X(t)$ as follows:

$X(t) = \sin \pi t$ if head shows;

$X(t) = 2t$ if tail shows.

i) Find the mean $E[X(t)]$. [4]

ii) Find the autocorrelation function of $X(t)$. [4]

iii) Is this a stationary process? [2]

b) Consider a Poisson process in which students arrive at an office at a rate of 1 student/5 minutes.

i) The service time is exactly 10 minutes and each student is served immediately upon arrival. Find the probability mass function of the number of students in service in the office. [5]

ii) Find the probability mass function if the service time is either 10 minutes or 20 minutes, with equal probability. [10]

4. Markov chains.

a) Consider a Markov chain with states $\{0, 1, 2, 3, 4\}$. Suppose that $P_{0,4} = 1$, and that when the chain is in state $i$, $i > 0$, the next state is equally likely to be any state in $\{0, 1, \ldots, i - 1\}$. Find the limiting distribution.

[15]

b) Consider a symmetric random walk on the integers $\{0, \pm 1, \pm 2, \ldots\}$. This Markov chain is a sequence $\{X_n\}$ where $\{X_{n+1} = X_n + 1\} = P\{X_{n+1} = X_n - 1\} = \frac{1}{2}$. Suppose the chain starts at the origin $X_0 = 0$.

i) Derive the probability $P\{X_{2n} = 0\}$. [4]

ii) Using the Stirling formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, determine whether the origin is a recurrent or transient state. [6]

Hint: Being recurrent requires $\sum_n P\{X_{2n} = 0\} = \infty$, while being transient requires $\sum_n P\{X_{2n} = 0\} < \infty$.