9: Optimal IIR Design

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However if $W_E(\omega) = \frac{W_S(\omega)}{|A(e^{j\omega})|}$, then $|E_E(\omega)| = |E_S(\omega)|$

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This is a <u>very</u> widely used technique.

Least Squares Minimization

Given a system of linear equations, in vector-matrix form, say,

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,M} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,M} \end{bmatrix}}_{\mathbf{A}\in\mathbb{C}^{N\times M}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix}}_{\mathbf{x}\in\mathbb{C}^{M}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}}_{\mathbf{b}\in\mathbb{C}^{N}} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

in many applications, it may be of interest to find the "unknown" vector $\mathbf{x}$.

- This may happen when, for example, there is noise in the measurements or even when there are more equations than unknowns.
- In filter design problems, typically, we are given a case when there are more equations than unknowns. This calls for finding the "best" (in least-squares sense) solution $\mathbf{x}$ which minimizes the error,

$$\mathbf{e} = \mathbf{Ax} - \mathbf{b}.$$

In the slides to follow,

- We will re-visit the vector-matrix notation for setting up a linear system of equations.
- Look at a simple case of minimization when vector $\mathbf{x}$ contains 2 unknowns.
- After developing an insight from the simple case, we will solve for the more general case of M unknowns.

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4. Note that,

$$\begin{aligned} \mathbf{e}^\top \mathbf{e} = (\mathbf{A}\mathbf{x} - \mathbf{b})^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) &= \left((\mathbf{A}\mathbf{x})^\top - \mathbf{b}^\top\right)(\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= \left(\mathbf{x}^\top \mathbf{A}^\top - \mathbf{b}^\top\right)(\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} \underbrace{- \mathbf{x}^\top \mathbf{A}^\top \mathbf{b} - \mathbf{b}^\top \mathbf{A}\mathbf{x}}_{-2\mathbf{b}^\top \mathbf{A}^\top \mathbf{x}} - \underbrace{\mathbf{b}^\top \mathbf{b}}_{\|\mathbf{b}\|^2} = \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} - \|\mathbf{b}\|^2 \end{aligned}$$

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Least Squares

1. We have $\|\mathbf{e}\|^2 = \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{b}^\top \mathbf{A}^\top \mathbf{x} - \|\mathbf{b}\|^2$.

   We want to find $\mathbf{x}$ that minimizes the error $\mathbf{e}$.

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4. To find the $\mathbf{x} = (x_1, x_2)$ that minimizes the above, we differentiate with respect to x_1 and x_2.

$$\frac{d}{dx_1}\mathbf{e}^\top\mathbf{e} = 2\alpha x_1 + \gamma x_2 - 2p = 0 \qquad \frac{d}{dx_2}\mathbf{e}^\top\mathbf{e} = 2\beta x_2 + \gamma x_1 - 2q = 0$$

Least Squares (General Case)

1. For the general case, we should expect,

$$\frac{d}{d\mathbf{x}}\|\mathbf{e}\|^2 = \begin{bmatrix} \frac{d}{dx_1}\|\mathbf{e}\|^2 \\ \frac{d}{dx_2}\|\mathbf{e}\|^2 \\ \vdots \\ \frac{d}{dx_M}\|\mathbf{e}\|^2 \end{bmatrix} \xrightarrow{\mathbf{x}\in\mathbb{R}^2} \begin{bmatrix} \frac{d}{dx_1}\mathbf{e}^\top\mathbf{e} = 2\alpha x_1 + \gamma x_2 - 2p \\ \frac{d}{dx_2}\mathbf{e}^\top\mathbf{e} = 2\beta x_2 + \gamma x_1 - 2q \end{bmatrix}.$$

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$$\frac{d}{d\mathbf{x}}\|\mathbf{e}\|^2 = \mathbf{0} \Rightarrow 2\mathbf{A}^\top\mathbf{A}\mathbf{x} - 2\mathbf{A}^\top\mathbf{b} = \mathbf{0}, \quad \text{or,} \quad \mathbf{A}^\top\mathbf{A}\mathbf{x} = \mathbf{A}^\top\mathbf{b} \Rightarrow \boxed{\mathbf{x} = \left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\mathbf{b}}$$

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4. $\mathbf{A}^\dagger = \left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top$

   This is a generalization of the "inverse of a matrix" also known as Moore–Penrose inverse.

   Roger Penrose received the 2020 Nobel Prize in Physics.

Frequency Sampling

For every ω we want: $0 = W(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$

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$$= W(\omega)\left(\sum_{m=0}^{M} b[m]e^{-jm\omega} - D(\omega)\left(1 + \sum_{n=1}^{N} a[n]e^{-jn\omega}\right)\right)$$

IIR Optimization

Writing error $E_E(\omega) = W_E(\omega)\left(B\left(e^{\jmath\omega}\right) - D\left(e^{\jmath\omega}\right)A\left(e^{\jmath\omega}\right)\right)$ in vector-matrix form.

1. Let $u_\omega^z = e^{-\jmath z\omega}$, then, we can write,

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$$D\left(e^{\jmath\omega}\right)A\left(e^{\jmath\omega}\right) \Leftrightarrow \underbrace{\begin{bmatrix} D\left(e^{\jmath\omega_1}\right) & & & \\ & D\left(e^{\jmath\omega_2}\right) & & \\ & & \ddots & \\ & & & D\left(e^{\jmath\omega_K}\right) \end{bmatrix}}_{\text{Diagonal Matrix}} \mathbf{A} = \mathbf{D}\underbrace{\mathbf{U}_{K,N+1}\mathbf{a}}_{\mathbf{A}}.$$

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4. Due to linearity of the matrix, we can also write,

$$\mathbf{U}_{K,M+1}\mathbf{b} - \mathbf{DU}_{K,N+1}\mathbf{a} \Leftrightarrow \left[\begin{array}{c|c} \mathbf{U}_{K,M+1} & -\mathbf{DU}_{K,N+1} \end{array}\right] \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}.$$

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2. Again writing W_E as a diagonal matrix, we have,

$$W_E(\omega)\left(B\left(e^{\jmath\omega}\right) - D\left(e^{\jmath\omega}\right)A\left(e^{\jmath\omega}\right)\right) = 0 \Leftrightarrow \left[\begin{array}{c|c} \mathbf{W}_E\mathbf{U}_{K,M+1} & -\mathbf{W}_E\mathbf{D}\mathbf{U}_{K,N+1} \end{array}\right]\left[\begin{array}{c} \mathbf{b} \\ \mathbf{a} \end{array}\right] = \mathbf{0}.$$

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$$\operatorname{vec}\left(\mathbf{W}_E\mathbf{D}\right) = \left[\begin{array}{cccc} W_E\left(e^{\jmath\omega_1}\right)D\left(e^{\jmath\omega_1}\right) & W_E\left(e^{\jmath\omega_2}\right)D\left(e^{\jmath\omega_2}\right) & \cdots & W_E\left(e^{\jmath\omega_K}\right)D\left(e^{\jmath\omega_K}\right) \end{array}\right]^\top$$

IIR Optimization

Writing error $E_E(\omega) = W_E(\omega)\left(B\left(e^{\jmath\omega}\right) - D\left(e^{\jmath\omega}\right)A\left(e^{\jmath\omega}\right)\right)$ in vector-matrix form.

1. Imposing $a[0] = 1$ (which we did not do before), we get,

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2. In order to force real valued coefficients,

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Frequency Sampling

For every ω we want: $0 = W(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$

$$= W(\omega)\left(\sum_{m=0}^{M} b[m]e^{-jm\omega} - D(\omega)\left(1 + \sum_{n=1}^{N} a[n]e^{-jn\omega}\right)\right)$$

$$\Rightarrow \begin{pmatrix} \mathbf{u}(\omega)^T & \mathbf{v}(\omega)^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = W(\omega)D(\omega)$$

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[K equations, $M + N + 1$ unkowns]

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Replace pole p_i by $(p_i^*)^{-1}$ whenever $|p_i| \geq 1$

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But for faster convergence use Newton-Raphson . . .

Newton-Raphson

Newton: To solve $f(x) = 0$ given an initial guess x_0, we write

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

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So for each ω_k, we can write (omitting the ω and $e^{j\omega}$ arguments)

$$E_S \approx W_S \left(\frac{B_0}{A_0} - D \right) + \frac{W_S}{A_0}(B - B_0) - \frac{W_S B_0}{A_0^2}(A - A_0)$$

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From which we get a linear equation for each ω_k :

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where $W = \frac{W_S}{A_0}$

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Newton: To solve $f(x) = 0$ given an initial guess x_0, we write

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

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Then use linear least squares to minimize the linearized E_S using the above equation replicated for each of the ω_k.

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This result is a consequence of the Hilbert Relations.

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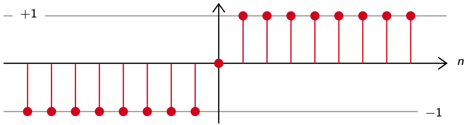

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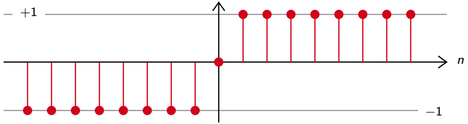

4. Based, on this, we have,

$$h_O[n] = h_E[n]\, t[n] \leftrightarrow H_O\left(e^{\jmath\omega}\right) = \underbrace{\mathrm{Re}\left(H\left(e^{\jmath\omega}\right)\right) \odot -\jmath \cot\left(\frac{\omega}{2}\right) = -\jmath\, \mathrm{Im}\left(H\left(e^{\jmath\omega}\right)\right)}_{\text{Relation between Real and Imaginary Parts of Spectrum}}.$$

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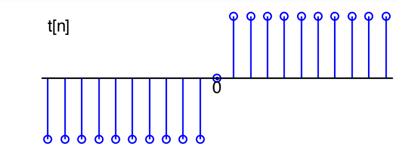

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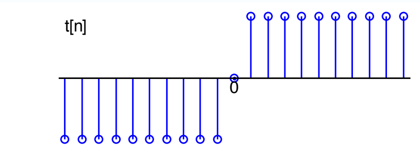

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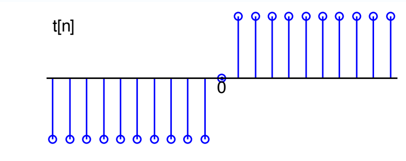

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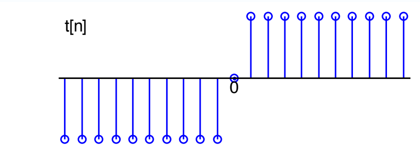

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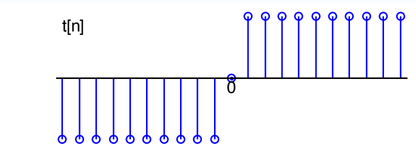

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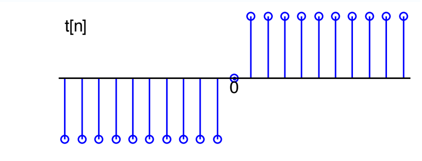

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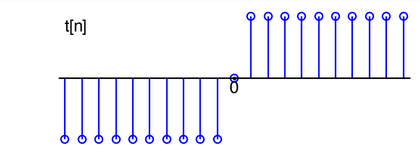

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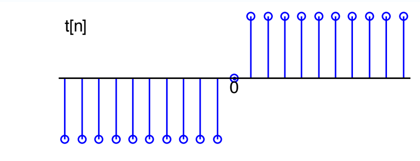

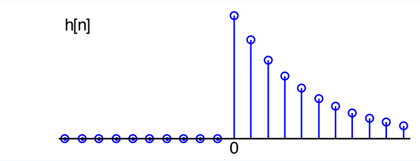

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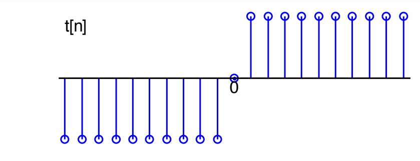

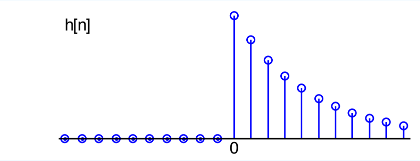

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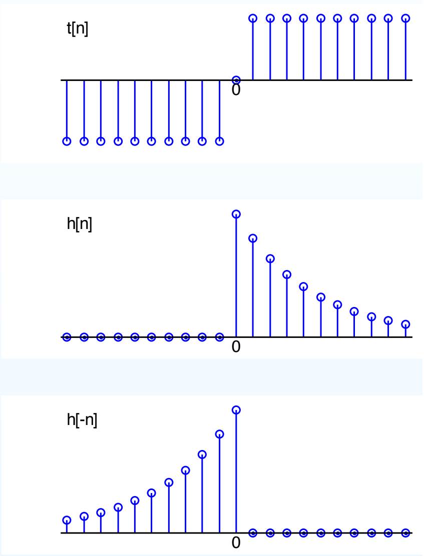

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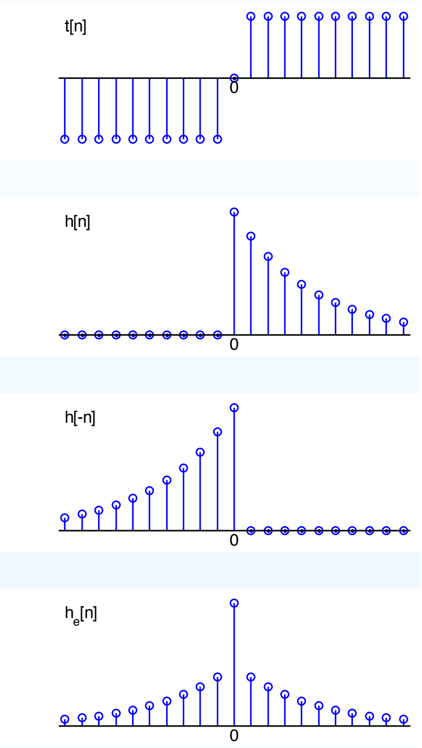

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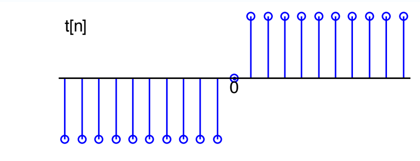

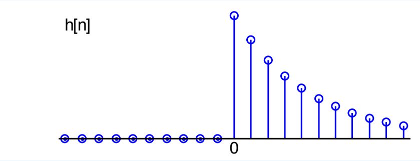

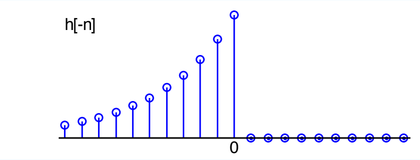

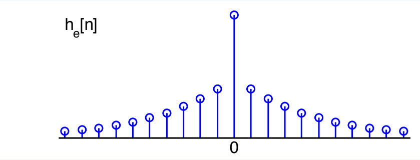

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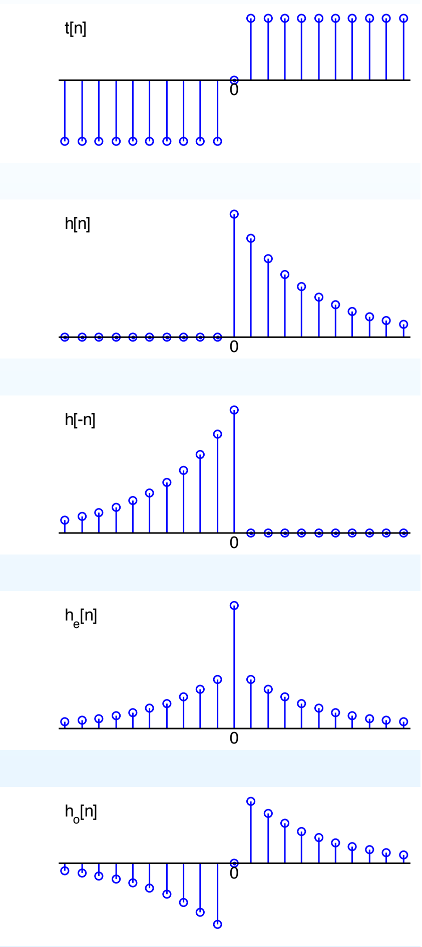

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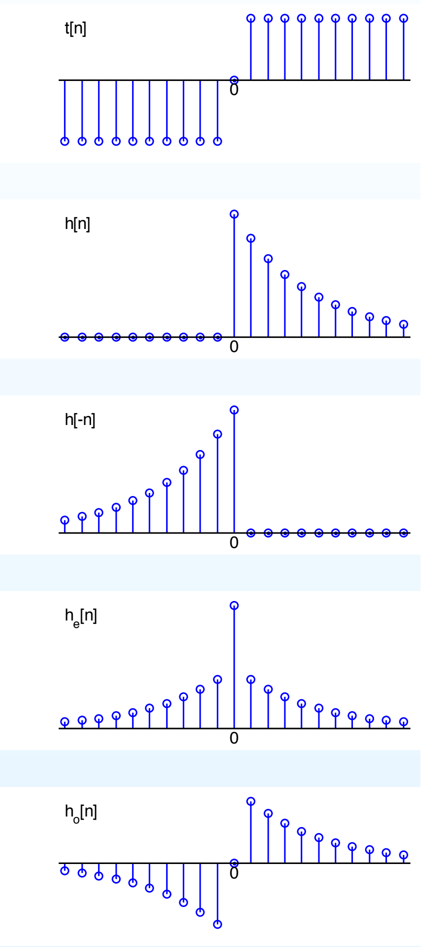

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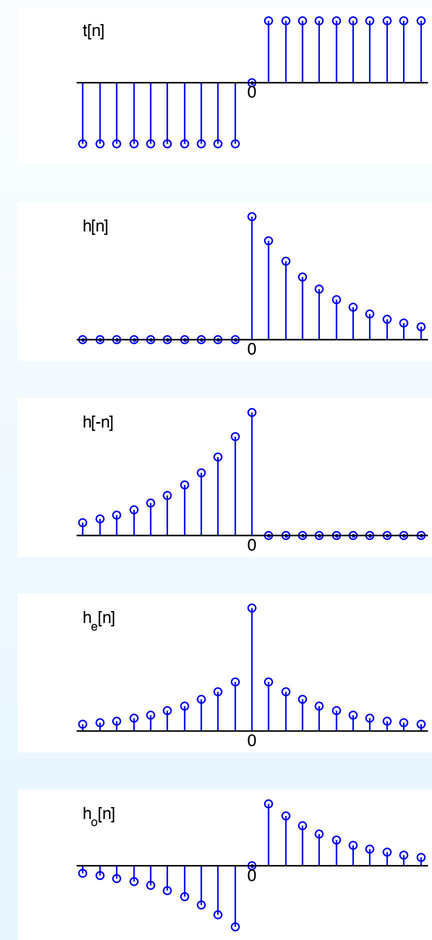

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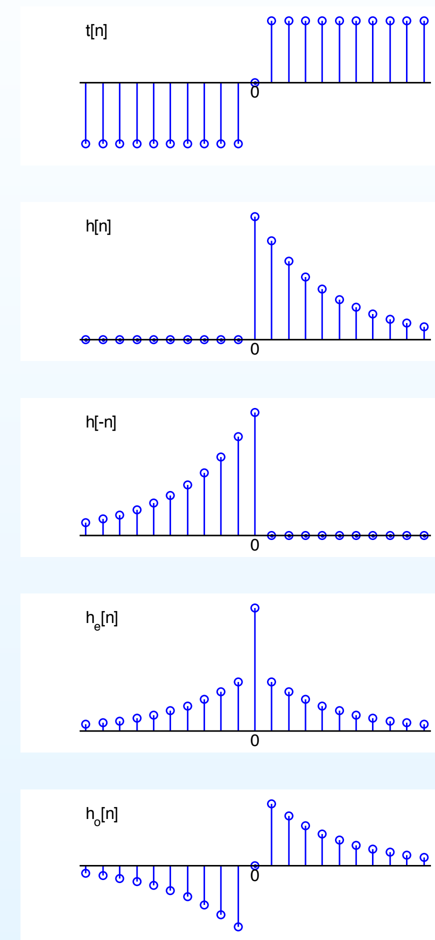

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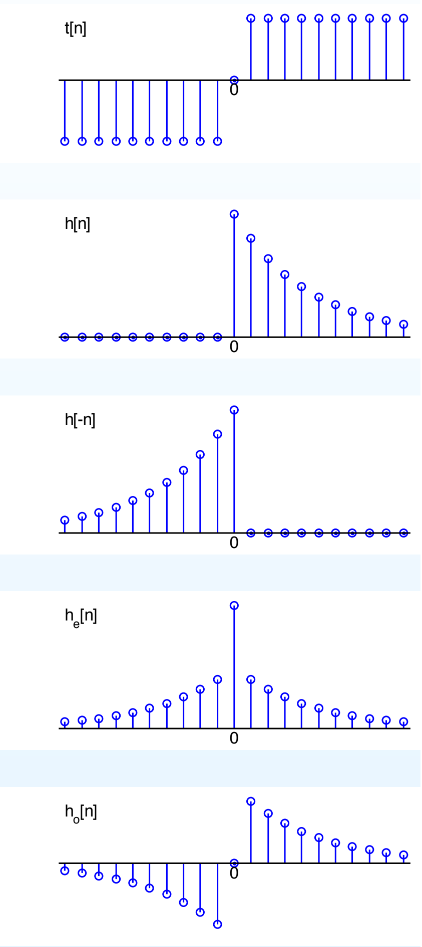

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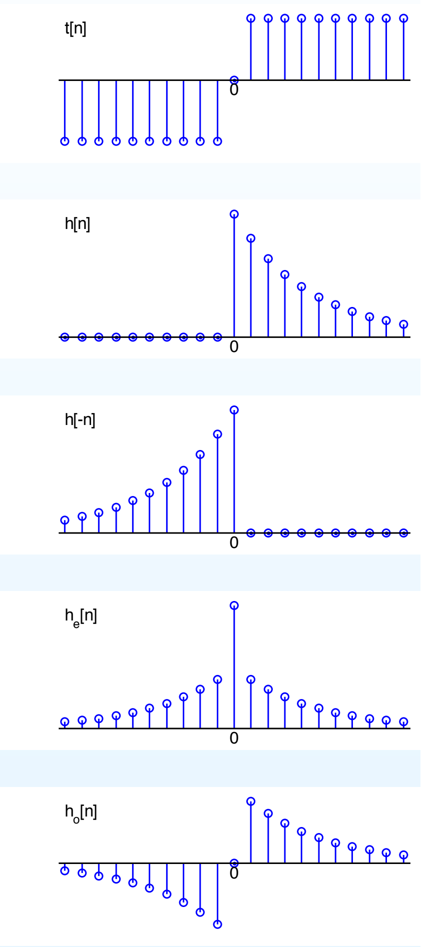

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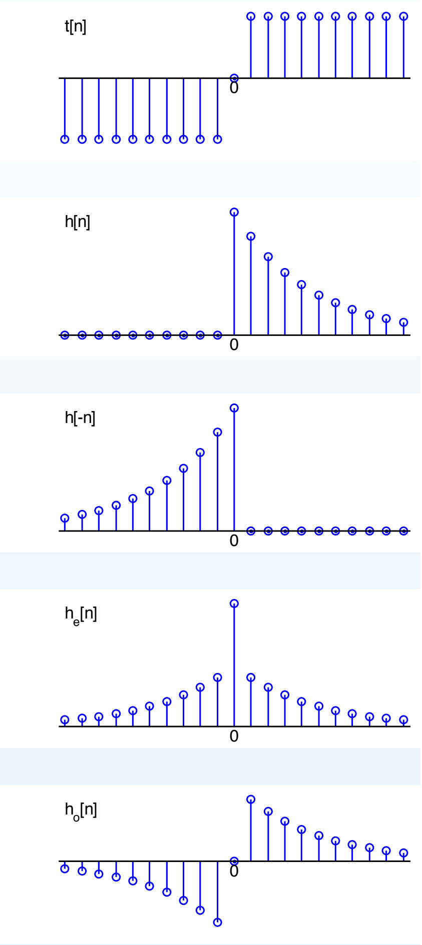

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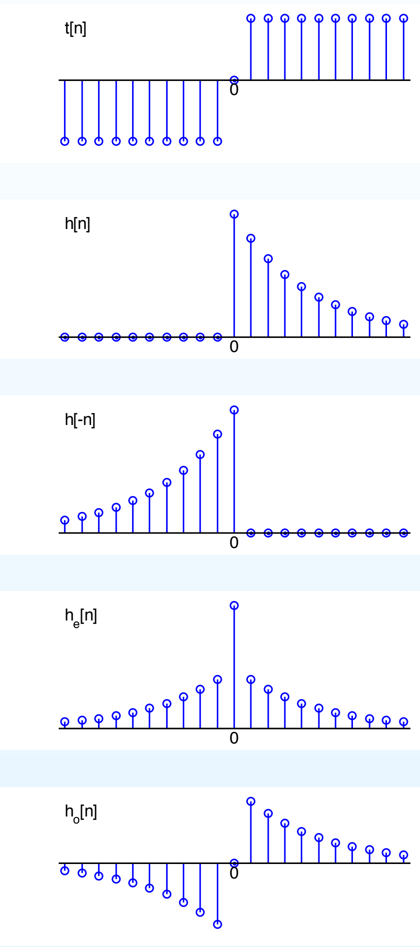

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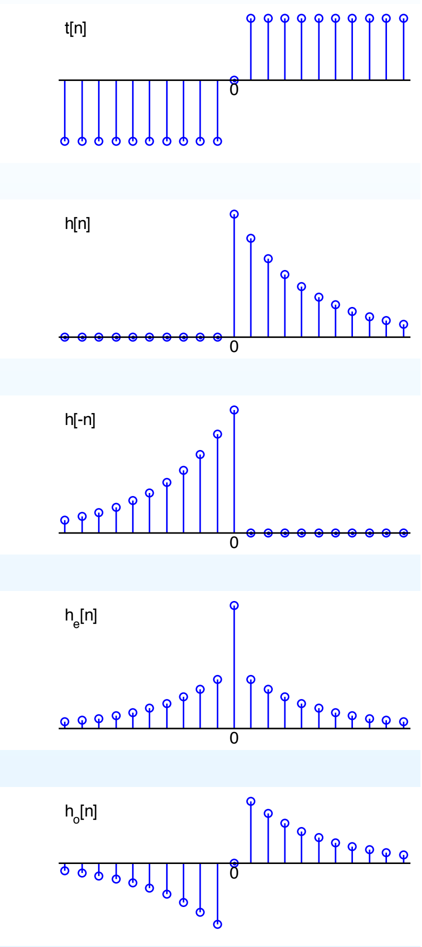

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Fourier Transform of Causal Sequences

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$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X\left(e^{\jmath\omega}\right) \qquad \widehat{x}[n] \overset{\text{DTFT}}{\longleftrightarrow} \widehat{X}\left(e^{\jmath\omega}\right)$$

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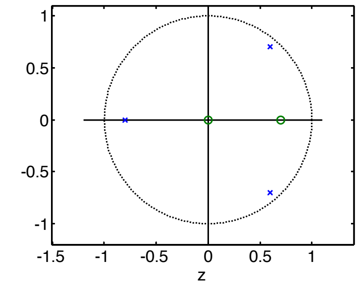

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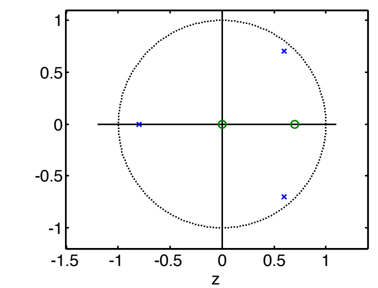

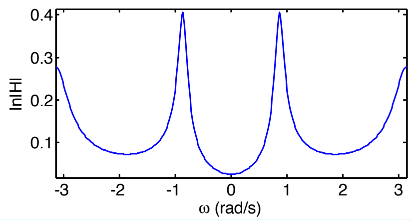

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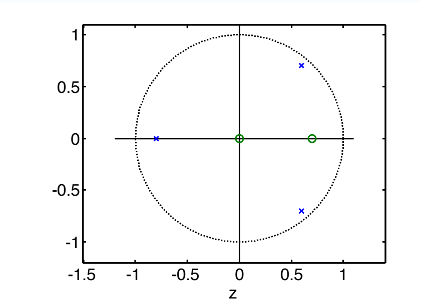

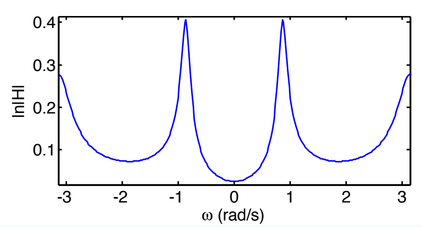

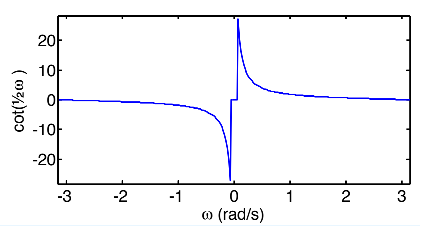

Magnitude $\leftrightarrow$ Phase Relation

Given $H(z) = g\frac{\prod(1-q_m z^{-1})}{\prod(1-p_n z^{-1})}$

$$\begin{aligned}\ln H(z) &= \ln(g) + \sum \ln\left(1 - q_m z^{-1}\right) \\ &\quad - \sum \ln\left(1 - p_n z^{-1}\right) \\ &= \ln\left|H(z)\right| + j\angle H(z)\end{aligned}$$

Taylor Series:

$$\ln\left(1 - az^{-1}\right) = -az^{-1} - \frac{a^2}{2}z^{-2} - \frac{a^3}{3}z^{-3} - \ldots$$

causal and stable provided $|a| < 1$

So, if $H(z)$ is minimum phase (all p_n and q_m inside unit circle) then $\ln H(z)$ is the z-transform of a stable causal sequence and:

$$\angle H(e^{j\omega}) = -\ln\left|H(e^{j\omega})\right| \circledast \cot\frac{\omega}{2}$$
$$\ln\left|H(e^{j\omega})\right| = \ln|g| + \angle H(e^{j\omega}) \circledast \cot\frac{\omega}{2}$$

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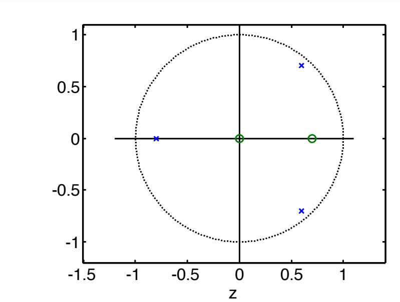

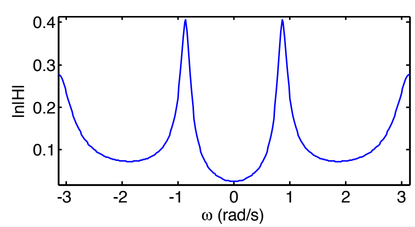

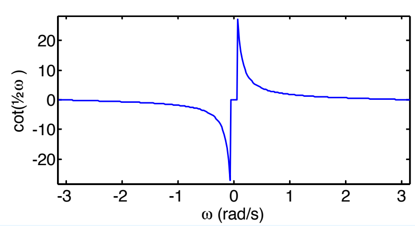

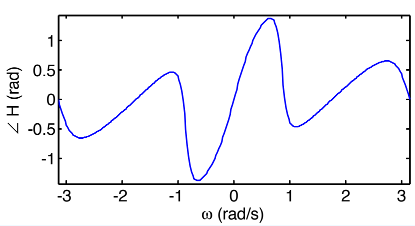

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  - $E_S(\omega) = W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right)$

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For further details see Mitra: 9.

MATLAB routines

invfreqz	IIR design for complex response