Digital Signal Processing and Digital Filters

Imperial College London

Problem Set for Course Revision

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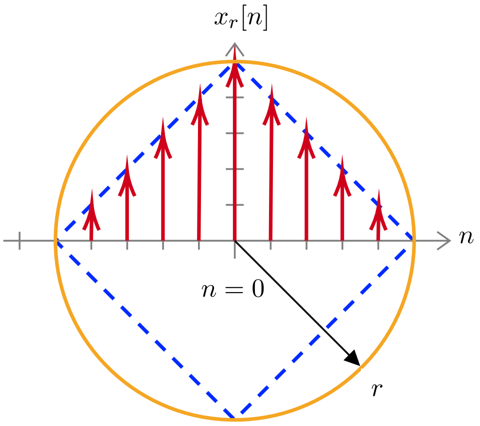
The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Consider the discrete-time sequence $x_r[n]$ given in the figure:



(a) Plot the modulated sequence $y_r[n] = \cos(\pi n) x_r[n]$.

(b) For an arbitrary integer $r \geq 1$ evaluate the energy of $x_r[n]$ (in terms of r). If needed, use the formula for simplification,

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}. \tag{1}$$

(c) How is the energy of $x_r[n]$ related to the energy of $y_r[n]$?

2) As we have seen in the course, many linear-time-invariant systems are described by difference or differential equations. For instance, consider the filter described by differential equation,

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y_c(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x_c(t) \tag{2}$$

where $x_c(t)$ is the continuous-time input and $y_c(t)$ is the corresponding output. In order to work with discrete-time systems, a standard approach is to replace the derivative operator by finite differences. To this end, let us denote the central difference by, $\Delta^n x[k]$ where n is the order of difference and the difference operation is defined by the recursive operations,

$$\begin{cases} \Delta^0 x[k] = x[k] \\ \Delta^1 x[k] = \frac{1}{2}\left(x[k+1] - x[k-1]\right) \\ \Delta^n x[k] = \Delta^1\left(\Delta^{(n-1)} x[k]\right) \end{cases}.$$

Hence the continuous-time filter can be written as a discrete-time filter,

$$\sum_{n=0}^{N} a_n \left(\Delta^n y[k]\right) = \sum_{m=0}^{M} b_m \left(\Delta^m x[k]\right). \tag{3}$$

i) Let H_c and H_d be the transfer-functions of the continuous and discrete-time filters, respectively. How are the two transfer functions related?

ii) What is the relation between continuous-filter frequency and digital-filter frequency?

3) **Analysis of the Kolmogorov–Zurbenko filter.**

A. N. Kolmogorov was a celebrated Russian mathematician who has made several contributions to the field of mathematics. One such contribution is related to the study of turbulence. Let $x[n], n \in \mathbb{Z}$ be a discrete-time sequence. The Kolmogorov–Zurbenko filter output is given by,

$$y[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} c_{M,K}[l]\, x[n+l] \tag{4}$$

where M (odd integer) and K are filter parameters and the filter coefficient $c_{M,K}[l]$ is given by,

$$c_{M,K}[l] = M^{-K} p_{M,K}[l] \tag{5}$$

where $p_{M,K}[l]$ is the solution to the equation,

$$\sum_{k=0}^{K(M-1)} z^k p_{M,K}\left[k - K\frac{M-1}{2}\right] = \left(\sum_{m=0}^{M-1} z^m\right)^K. \tag{6}$$

(a) What is the time-domain impulse response of the Kolmogorov–Zurbenko filter?

(b) What is $c_{M,1}[l]$?
Plot the filter corresponding to $c_{M,1}[l]$.

(c) When $K = 1$, what is the order of the Kolmogorov–Zurbenko filter?

(d) What is the magnitude frequency response of the Kolmogorov–Zurbenko filter? What is the effect on the output of increasing K?

4) **Review of Sampling Theory and Fourier Series**

Suppose $f(t)$ is a function with maximum frequency Ω_0. Shannon's sampling formula allows us to write,

$$f(t) = \sum_{n=-\infty}^{n=\infty} f(nT)\,\mathrm{sinc}\left(\frac{\pi t}{T} - n\pi\right), \qquad T = \frac{\pi}{\Omega_0} \tag{7}$$

where $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$ and T is the sampling rate. Suppose that the same function is sampled with sampling rate $T_0 \leqslant T$. We can interpret samples $f(nT_0)$ as Fourier Series coefficients, $c_n = f(nT_0)$ of some periodic function. Indeed for all n, c_n represent the coefficients of the periodic function,

$$\widehat{C}(\omega) = \frac{1}{T_0} \sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath \omega n T_0}$$

and "windowing" this with a rectangular window leads to the digital-to-analog reconstruction in (7).

(a) Show that $\widehat{C}(\omega)$ is a periodic function. Identify its time-period.

(b) Show the link between $\widehat{C}(\omega)$ and the Fourier transform of $f(t)$.

(c) Show that "windowing" $\widehat{C}(\omega)$ results in (7) in the time domain.

(d) Explain how the choice of rectangular window, in the above, leads to practical problems with implementation of (7), digital-to-analog reconstruction.

(e) To be able to use more general window functions than the rectangular window, what conditions should be imposed on T_0?

(f) Sketch an example of a windowing function that is suitable for reconstruction of $f(t)$ from its samples. Explain the reasons behind your choice.

(g) What modifications should be made to formula (7) so that the new window function you have designed above can recover $f(t)$ from samples $f(nT_0)$?

5) **Multirate System**

Consider the following system with input $x[n]$ and output $y[n]$.

$$x[n] \rightarrow \boxed{1:3} \rightarrow \boxed{2:1} \rightarrow \boxed{\frac{z^{-6}}{\alpha - z^{-6} + \beta z^{-12}}} \rightarrow \boxed{2:1} \rightarrow \boxed{1:3} \rightarrow y[n]$$

Suppose that an equivalent representation of the above system is given by,

$$x[n] \rightarrow \boxed{4:1} \rightarrow \boxed{\frac{B(z)}{A(z)}} \rightarrow \boxed{1:9} \rightarrow y[n]\,.$$

Identify α and β if $A(z)$ has roots at $1/4$ and $1/3$.

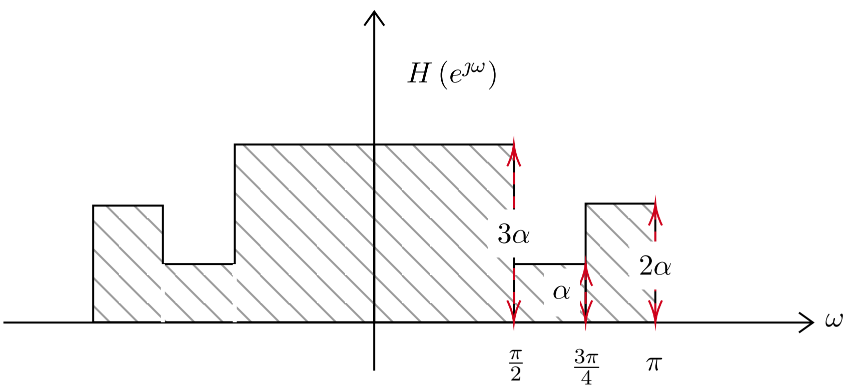
6) Let $|\lambda| < 1$. Consider the LSI system given by,

$$h[-n] \Longrightarrow \boxed{h[n] = \lambda^n u[n]} \Longrightarrow g[n]$$

(a) What is the frequency response of the filter defined by $g[n]$? For $\lambda > 0$, show that the frequency response is maximized at $\omega = 0$.

(b) What is the value of λ for which the group delay of a filter $h[n]$ is equal to zero at $\omega = \pi/3$?

7) Let $H\left(e^{j\omega}\right)$ be a symmetric frequency response filter whose absolute value is depicted in the plot below.



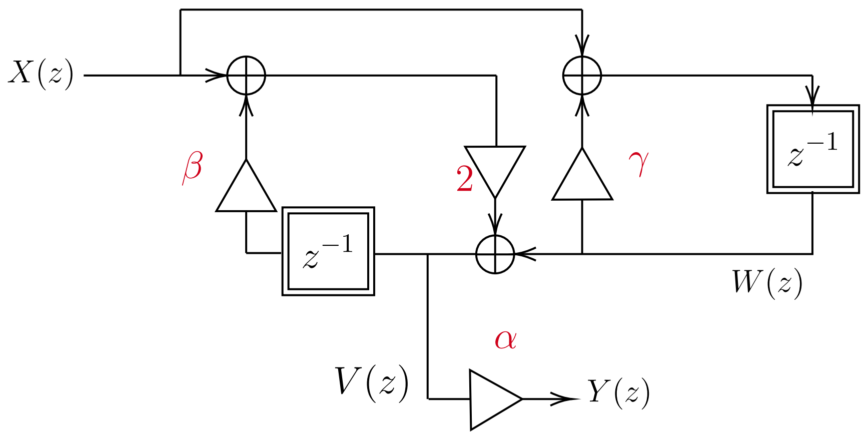
(a) Give the analytical expression of the impulse response $h[n]$.

(b) Find the expression of $\sum_{n\in\mathbb{Z}} |h[n]|^2$.

(c) Compute α such that $\sum_{n\in\mathbb{Z}} |h[n]|^2 = 1$.

(d) Compute $\sum_{n\in\mathbb{Z}} h[n] \operatorname{sinc}\left(\frac{2\pi}{3}n\right)$.

8) Let $H(z) = \frac{Y(z)}{X(z)}$ be a system represented by the diagram below



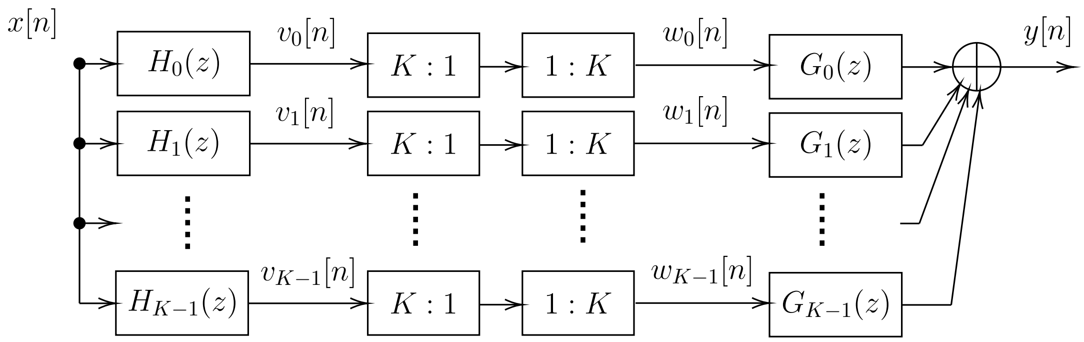
(a) Assume $\alpha = 5$, $\beta = 1/2$, $\gamma = 4/10$. Derive the expression of $W(z)$.

(b) Derive the expression of $H(z)$ via $W(z)$ and $V(z)$.

(c) Assume $\alpha = 5$, $\beta = \frac{1}{\sqrt{2}}$, $\gamma = 4/10$. Let $\widetilde{H}(z)$ denote the system transfer function when β increases by 10%. Compute the frequency response change $\frac{|\widetilde{H}(e^{j\omega})|}{|H(e^{j\omega})|}$ in dB at $\omega = \frac{\pi}{4}$.

(d) For $\alpha = 5$, $\beta = \frac{1}{2\sqrt{2}}$, $\gamma = 4/10$, how should α be changed such that $\frac{|\widetilde{H}(e^{j\omega})|}{|H(e^{j\omega})|} = 1$ at $\omega = \frac{\pi}{2}$?

9) Consider the K-band filter bank below.



(a) Show that

$$w_k[n] = \frac{1}{K}\sum_{m=0}^{K-1} e^{\frac{2\pi jmn}{K}} v_k[n], k = 0, \ldots, K-1. \tag{8}$$

(b) Compute $W_k(z)$ as a function of $V_k(z)$.

(c) Compute $Y(z)$ as a function of $X(z)$.

(d) What are the conditions that would ensure perfect reconstruction?

10) Multiply a Hamming window with an ideal low-pass filter to obtain a FIR causal bandpass filter of even order M and pass band $\omega \in [2,3]$ rad/s. A Hamming window for even M satisfies

$$w[n] = \frac{54}{100} + \frac{46}{100}\cos\left(\frac{2\pi n}{M+1}\right), \quad n = -\frac{M}{2}, \ldots, \frac{M}{2}. \tag{9}$$

11) Assume a FIR filter is designed by windowing an ideal low-pass filter $g[n]$ with window $w[n]$. We know that $G\left(e^{j\omega}\right) = 1_{[-1,1]}(\omega)$, where $G\left(e^{j\omega}\right)$ is the frequency response of $g[n]$. Moreover, we know that $\int_0^\pi W\left(e^{j\omega}\right)d\omega = \frac{\pi}{2}$ and that $W\left(e^{j\omega_n}\right) = 0$, for $\omega_n = \frac{n}{10}, n \in \mathbb{Z}, n \geq 1$. Let $h[n]$ denote the filter impulse response.

(a) Compute $H\left(e^{j\omega}\right)$, the frequency response of $h[n]$.

(b) Using that $W\left(e^{j\omega}\right) \approx 0$ for $\omega > 2$, show that, in the vicinity of $\omega = 1$, $H\left(e^{j\omega}\right)$ satisfies

$$H\left(e^{j\omega}\right) \approx \frac{1}{4} - \frac{1}{\pi}\int_0^{\omega-1} W\left(e^{j\zeta}\right)d\zeta. \tag{10}$$

(c) By finding the local maximum and minimum of $H\left(e^{j\omega}\right)$ around $\omega = 1$, compute the transition width $\Delta\omega$ of filter $H\left(e^{j\omega}\right)$. For guidance, the frequency responses of $h[n]$ and $w[n]$ are depicted below.

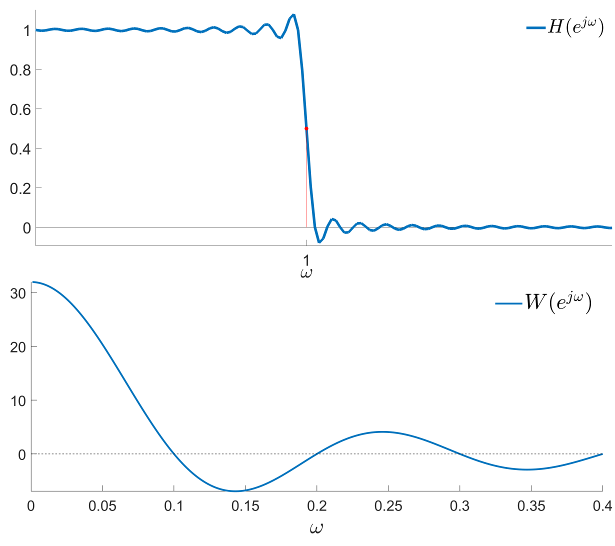


Fig. 1: The frequency responses of the FIR filter and window.