Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 2

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The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Evaluating DTFT.

(a) Consider a sequence $h[n]$ such that $h[n] = 0, n > 0$ and $h[n] = \alpha^n, n \leq 0$.

- What is the DTFT of $h[n]$?

$$H\left(e^{\jmath\omega}\right) = \sum_{n\leqslant 0} \alpha^n e^{-\jmath\omega n} = \sum_{n\geqslant 0} \alpha^{-n} e^{\jmath\omega n} = \frac{1}{1-\frac{e^{\jmath\omega}}{\alpha}}.$$

- Explain what happens when $|\alpha| \leq 1$.
- Given that $\left|H\left(e^{\jmath\omega}\right)\right|_{\omega=0} = 4$, what is α?

Since,

$$\left|H\left(e^{\jmath\omega}\right)\right|_{\omega=0} = \frac{1}{\sqrt{\left(1-\frac{1}{\alpha}\right)^2}}$$

we get,

$$\alpha = \left\{\frac{4}{5}, \frac{4}{3}\right\}.$$

(b) A system is described by the difference equation,

$$y[n] - \frac{1}{3} y[n-1] = x[n] - x[n-2].$$

Find $y[n]$ when $x[n] = \delta[n]$.

Using DTFT, we have,

$$Y\left(e^{\jmath\omega}\right) - \frac{e^{-\jmath\omega}}{3} Y\left(e^{\jmath\omega}\right) = X\left(e^{\jmath\omega}\right) - e^{-2\jmath\omega} X\left(e^{\jmath\omega}\right).$$

We are given $X = 1$ and hence,

$$Y\left(e^{\jmath\omega}\right) = \frac{1}{1-\frac{e^{-\jmath\omega}}{3}} - \frac{e^{-2\jmath\omega}}{1-\frac{e^{-\jmath\omega}}{3}}.$$

Now since,

$$Y\left(e^{\jmath\omega}\right) = \frac{1}{1-\frac{e^{-\jmath\omega}}{3}} \to \left(\frac{1}{3}\right)^n u[n],$$

we obtain,

$$y[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^{n-2} u[n-2].$$

(c) In the classes to come, we learn about the "moving average filter" defined by,

$$h[n] = \sum_{l=0}^{L-1} \delta[n-l].$$

Identify its frequency response or the DTFT.

Simple application of the geometric series.

$$\exp\left(-\jmath \frac{\omega}{2} L\right) \frac{\sin\left(\frac{\omega}{2}(L+1)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

2) Recursive Convolutions.

Let us define a set $\mathcal{S} = [0,1]$ and the sequence,

$$s_0[k] = \begin{cases} 1 & n \in \mathcal{S} \\ 0 & n \notin \mathcal{S}. \end{cases}$$

Furthermore, let us define,

$$\begin{array}{l} s_2\,[k] = (s_0 * s_0)\,[k] \\ s_3\,[k] = (s_0 * s_0 * s_0)\,[k] \\ s_4\,[k] = (s_0 * s_0 * s_0 * s_0)\,[k] \\ \vdots \\ s_N\,[k] = \underbrace{(s_0 * \cdots * s_0)}_{N \text{ convolutions}}\,[k] \end{array}.$$

- Show that,

$$s_N\,[k] = \frac{N!}{k!\,(N-k)!}.$$

- Find the expressions for the magnitude spectrum $|S_N\,(e^{\jmath\omega})|$?

3) Using tools from Fourier Analysis, show that,

$$\sum_{n=-\infty}^{n=+\infty} \mathrm{sinc}\,(x-n) = 1.$$

4) Deriving Shannon's Sampling Formula.

When the basis functions $\{B_k\}_k, k = 0, \pm 1, \pm 2, \cdots$ are orthogonal, that is,

$$\langle B_m, B_n \rangle = \int B_m\,(x)\,B_n^*\,(x)\,dx = \delta_{m-n}$$

then the function in the span of the basis functions, can be represented by the expansion,

$$f\,(x) = \sum_{k\in\mathbb{Z}} \langle f, B_k \rangle\, B_k\,(x).$$

- What is the Fourier Transform of,

$$B_k\,(t) = \mathrm{sinc}\left(\frac{t}{T} - k\right)?$$

- Show that the basis functions are orthogonal.
- Show that when f is a bandlimted function, the coefficients $\langle f, B_k \rangle$ are equivalent to the samples of $f(t)$.

5) Fourier Uncertainty.

Let $f\,(t) = \left(2\pi\sigma^2\right)^{-\frac{1}{4}} e^{-\frac{(t-\mu)^2}{4\sigma^2}}$. We are given that $\int_{\mathbb{R}} |f(t)|^2 dt = 1$.

- Verify that,

$$\mu = \int\limits_{\mathbb{R}} t|f(t)|^2 dt.$$

Let $c_0 \triangleq \frac{1}{\sqrt{2\pi\sigma^2}}$. We then have,

$$\int_{\mathbb{R}} t|f(t)|^2 dt = c_0 \int_{\mathbb{R}} te^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \xrightarrow{x=t-\mu} c_0 \int_{\mathbb{R}} (x+\mu)\, e^{-\frac{x^2}{2\sigma^2}} dx = c_0 I_1 + \mu I_2$$

where $I_1 = \int_{\mathbb{R}} xe^{-\frac{x^2}{2\sigma^2}} dx$ and $I_2 = \int_{\mathbb{R}} |f(x)|^2 dx$. Since $I_2 = 1$ (given) it remains to show that $I_1 = 0$. To show that this is indeed true, let us start by noting that $\frac{d}{dx} e^{-\frac{x^2}{2\sigma^2}} = -\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \Rightarrow xe^{-\frac{x^2}{2\sigma^2}} = -\sigma^2 \frac{d}{dx} e^{-\frac{x^2}{2\sigma^2}}$. This leads to,

$$I_1 = \int_{\mathbb{R}} \frac{d}{dx} e^{-\frac{x^2}{2\sigma^2}} dx = \left(-\sigma^2\right) e^{-\frac{x^2}{2\sigma^2}} \Big|_{-\infty}^{+\infty} = 0.$$

This verifies the result.

- Verify that,

$$\sigma^2 = \int\limits_{\mathbb{R}} (t-\mu)^2 |f(t)|^2 dt.$$

Similar approach as above and uses integration by parts. Start with $t - \mu \to x$ and then $\int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \int_{\mathbb{R}} \underbrace{x}_{u} \underbrace{xe^{-\frac{x^2}{2\sigma^2}}}_{dv} dx$.