Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 3

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The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Which of the following are true?

(a) Antiperiodic signals are not periodic

(b) Antiperiodic signals are periodic

(c) The DCT does not preserve input energy but the orthogonal DCT does

(d) The orthogonal DCT does not preserve input energy but the DCT does

$X[k]$ is antiperiodic if $X[k+2N] = -X[K]$. It follows that $X[k+4N] = -X[K+2N] = X[k]$, and thus $X[k]$ is periodic with period $4N$. The orthogonal DCT includes a normalization step that ensures the input energy is preserved. Therefore (b) and (c) are correct.

2) The DCT is computed by applying the DFT to a modified version of the original signal; the modified signal is generated by repeating the original signal in reverse order, and inserting a zero in between every pair of samples. Based on this knowledge, why is the DFT always real in this case?

The modified signal is symmetric and real, which implies that its DFT, which is the DCT of the original signal, is always real.

3) Let $x[n] = [1, 1, 1, 1]$.

(a) Compute the DCT $X[k]$.

(b) What is the minimal number of samples required to encode the DCT accurately?

(c) Compute the energy of $x[n]$ and $X[k]$. By what amount is the energy of the DCT increased/decreased?

(d) Calculate the orthogonal DCT, which we denote $X_\perp[k]$, and show that it preserves the energy of the original singal.

$$(a) \quad X[k] = \sum_{n=0}^{3} \cos \frac{2\pi(2n+1)k}{16} = \sum_{n=0}^{3} \text{Real}\left\{ e^{j\frac{2\pi(2n+1)k}{16}} \right\}. \tag{1}$$

(b) We have that $X[0] = 4$ and, for $k = 1, 2, 3$, we can see that the terms above can always be grouped in two pairs, such that in every pair the complex numbers are symmetric around the imaginary axis, as shown in Figure 1. Then each pair cancels out, and we are left with $X[k] = 4\delta_{k0}$. Therefore the DCT can be encoded by $X[0]$, and the correct answer is one sample.

(c) The energy of the DCT is thus 4 times higher.

(d) $X_\perp[k] = 2\delta_{k0}, \sum_{k=0}^{3} |X_\perp[k]|^2 = 4$

4) Let $x[n] = [x_1, x_2, x_3, x_4]$. Find all possible values for $x_i, i = 1, \ldots, 4$, that guarantee that $X[k] = 0, k > 0$.

The problem to solve is the under-determined linear system

$$\boldsymbol{Mx} = [0, 0, 0]^T, \tag{2}$$

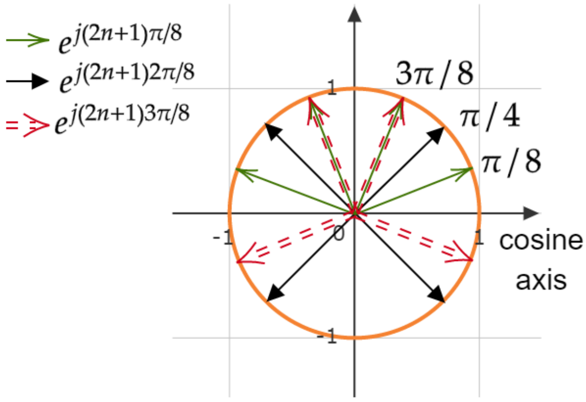

Fig. 1: Terms of $X[k]$ canceling out on the real axis of the complex plane for $k \geq 1$.

where $\cdot^T$ denotes the matrix transposition and

$$\boldsymbol{M} = \begin{bmatrix} \cos\frac{\pi}{8}, & \cos 3\frac{\pi}{8} & -\cos 3\frac{\pi}{8} & -\cos\frac{\pi}{8} \\ \cos\frac{\pi}{4}, & -\cos\frac{\pi}{4} & -\cos\frac{\pi}{4} & \cos\frac{\pi}{4} \\ \cos 3\frac{\pi}{8}, & -\cos\frac{\pi}{8} & \cos\frac{\pi}{8} & -\cos 3\frac{\pi}{8}. \end{bmatrix} \tag{3}$$

Given that $\mathrm{Rank}(\boldsymbol{M}) = 3$, it follows that the solution is of the form $\boldsymbol{x} = \alpha \boldsymbol{x}_0, \alpha \in \mathbb{R}$. We know that the system admits the solution $\boldsymbol{x}_0 = [1, 1, 1, 1]^T$, and thus the only solutions are $\boldsymbol{x} = \alpha [1, 1, 1, 1]^T, \alpha \in \mathbb{R}$.

Answers.

1) Correct options: $(b), (c)$.

2) Because symmetry and real values in the time domain guarantee that the DFT is also real-valued.

3) (a) $X[k] = 4 \cdot \delta_{k0}$.

(b) One sample.

(c) $\sum_{n=0}^{N-1} (x[n])^2 = 4$, $\sum_{k=0}^{N-1} (X[k])^2 = 16$. The energy is increased by 4.

(d) $X_{\perp}[k] = 2 \cdot \delta_{k0}$, $\sum_{k=0}^{N-1} (X_{\perp}[k])^2 = 4$.

4) $x[n] = \alpha [1, 1, 1, 1], \forall \alpha \in \mathbb{R}$.