7: Optimal FIR filters

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   Why? From the basic theory of Fourier series, we have seen that the rectangular window gives the best least-squares approximation to a desired frequency response for a given value of M, because,

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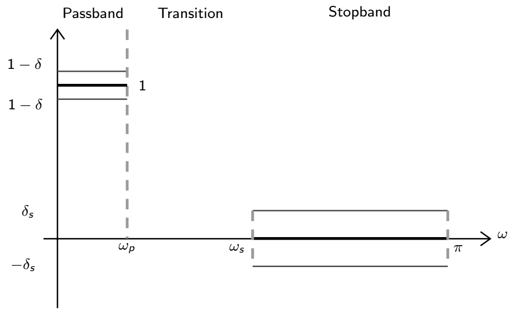
5. Our goal here is to study algorithmic procedures that overcome the above disadvantages.

1. In our design, we will considering even-symmetric, causal filters.
   These boil down to polynomials of the form,

$$H\left(e^{\jmath\omega}\right) = h\left[0\right] + 2\sum_{m=1}^{M/2} h\left[m\right]\cos\left(m\omega\right).$$

   To make it causal, multiply it with $\exp(-\jmath\omega M/2)$. shift by M/2
2. The goal is to satisfy the design constraints based on $M, \delta, \delta_s, \omega_p, \omega_s$.



3. Several algorithms have been proposed since the 70s. Namely, Herrmann (1970) and Hofstetter, Oppenheim and Siegel (1971) (M, δ, δ_s fixed but ω_p, ω_s are variable).
4. Parks and McClellan (1972 onward) — Minimax Criterion.
   Developed the most flexible design: $M, \delta, \delta_s\omega_p, \omega_s$ are all variable.

Overall idea!

1. Write higher order cosine frequencies in terms of "Chebyshev polynomial" or $\cos(m\omega) = T_m(\cos\omega)$.
   Example: $\cos(2\omega) = 2\cos^2(\omega) - 1 \Leftrightarrow T_2(x) = 2x^2 - 1$.
   Can be obtained recursively.

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2. This transformation allows us to write:

$$H(e^{\jmath\omega}) = h[0] + 2\sum_{m=1}^{M/2} h[m]\cos(m\omega) \leftrightarrow \sum_{m=0}^{M/2} h_m(\cos(\omega))^m \leftrightarrow \underbrace{\sum_{m=0}^{M/2} h_m x^m}_{\text{Polynomial}}$$

$h[m] \neq h_m$

or, in explicit polynomial form,

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$$E(\omega) = \arg\min_{h[n]} W(\omega)\left(H_d\left(e^{\jmath\omega}\right) - H\left(e^{\jmath\omega}\right)\right) \equiv \min_{h[m],0\leqslant m\leqslant M/2}\left(\max_{\omega}|E(\omega)|\right).$$

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4. Turns out that we do not have to know the relationship between polynomial coefficients h_m and the filter impulse response $h[m]$.

Basic statement of the alteration theorem.

- Suppose that X_p is a closed subset consisting of the disjoint union of closed subsets of the real axis x.
- Let $P(x)$ be a polynomial of degree r, namely,

$$P(x) = \sum_{m=0}^{r} p_m x^m.$$

- Let us define the weighted error on each interval/subset X_p as,

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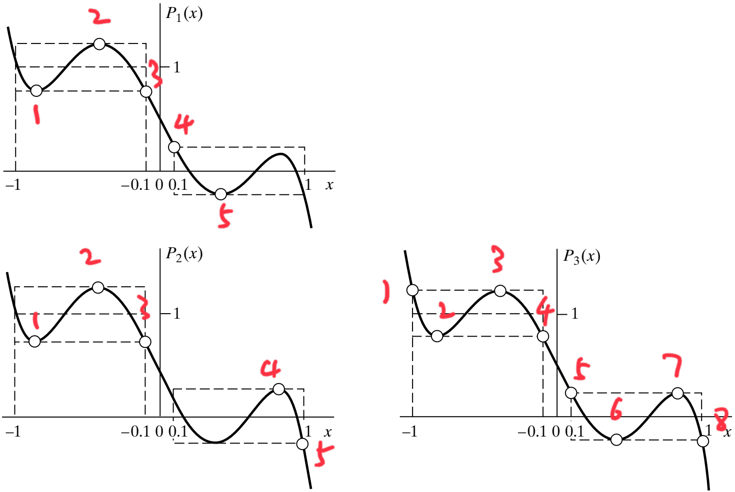
The sufficient and necessary condition that $P(x)$ is a unique minimizer of error $\|E\| = \max\limits_{x \in X_p} |E(x)|$ is that there are at least $r+2$ sign alterations or,

$$x_k \in X_p, \qquad x_1 < x_2 < \cdots < x_{r+2} \text{ such that } E_p(x_k) = -E_p(x_{k+1}) = \pm\|E\| = \max_{x \in X_p} |E(x)|$$

Example. Consider $r = 5$. $X_1 = [-1, -0.1]$ and $X_2 = [0.1, 1]$. Let D_p be defined by,

$$D_p(x) \begin{cases} 1 & x \in X_1 \\ 0 & x \in X_2. \end{cases}$$

From "Alteration Theorem" *at least* $r + 2 = 7$ sign changes! $P_1(x)$ and $P_2(x)$ do not satisfy this condition. $P_3(x)$ is the correct 5th order polynomial.



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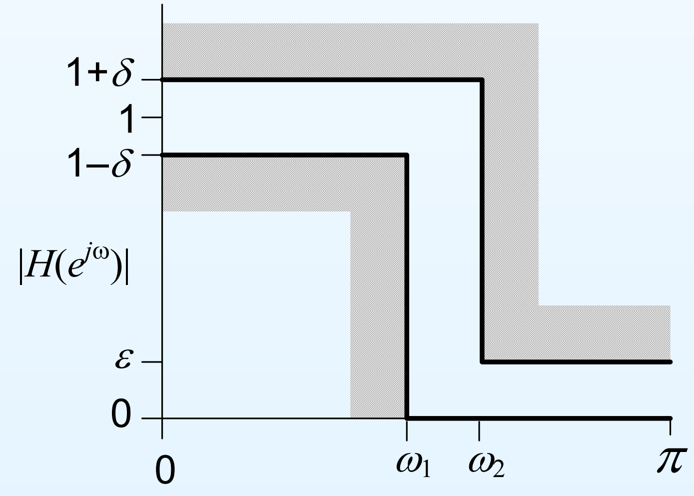

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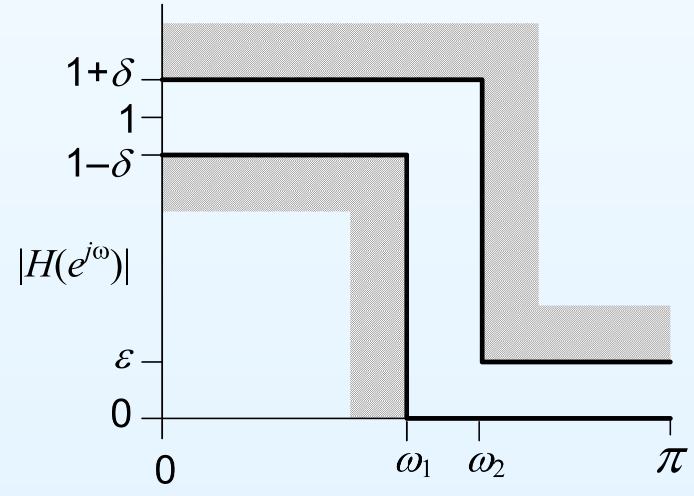

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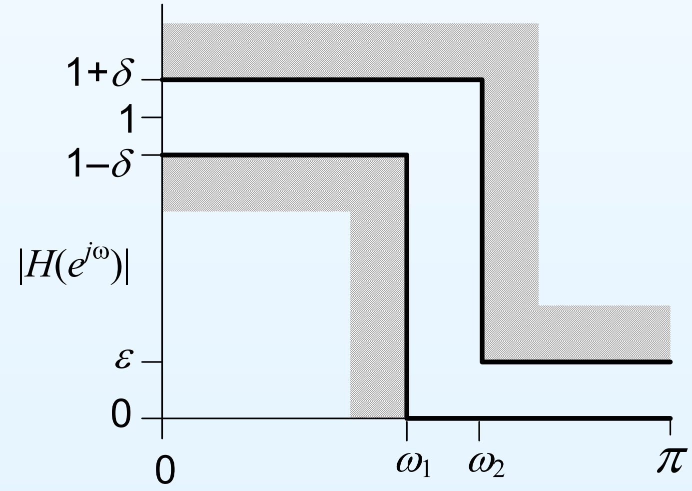

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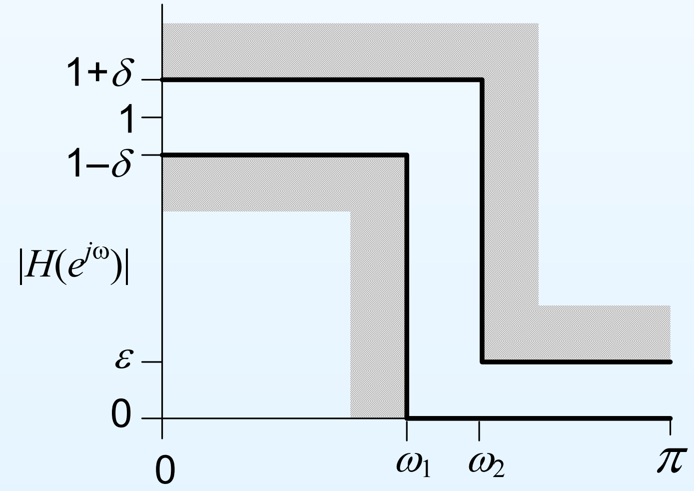

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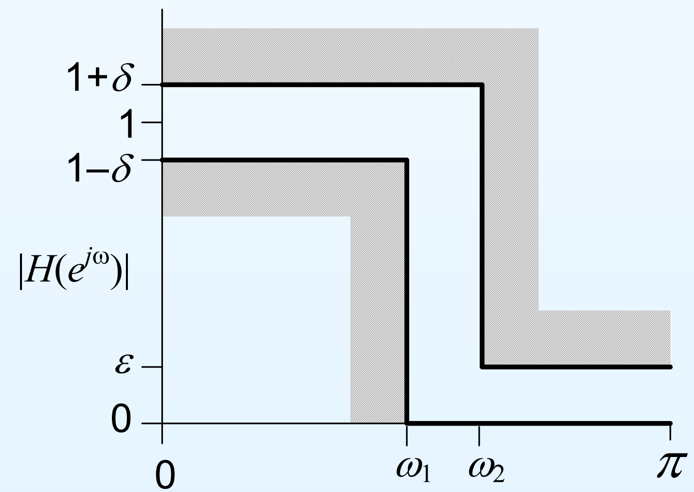

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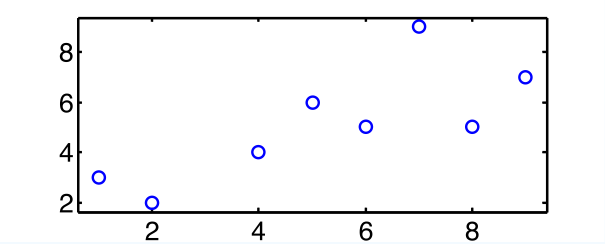
Minimax criterion: $h[n] = \arg\min_{h[n]} \max_{\omega} |e(\omega)|$: minimize max error

Alternation Theorem

Want to find the best fit line: with the smallest maximal error.



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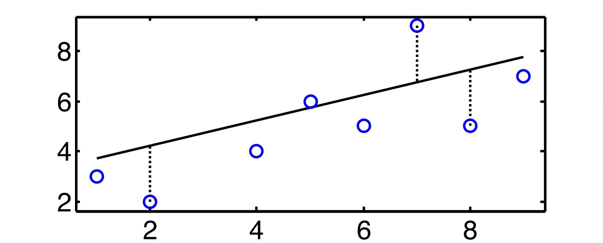

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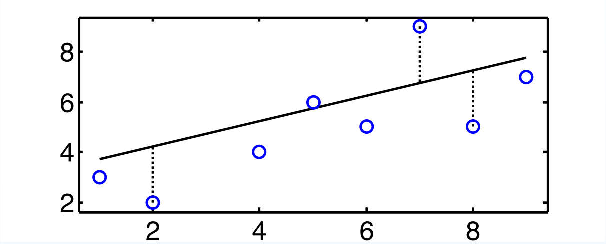

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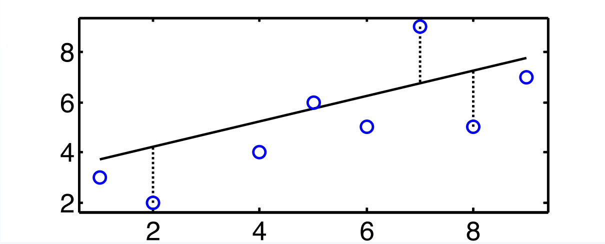

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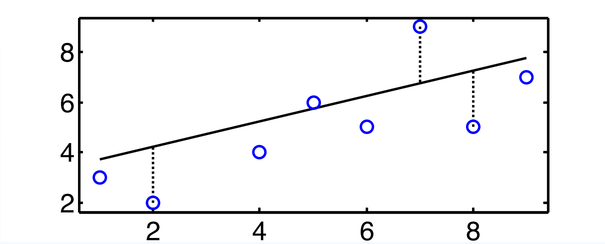

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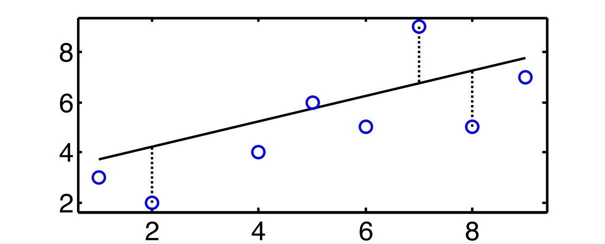

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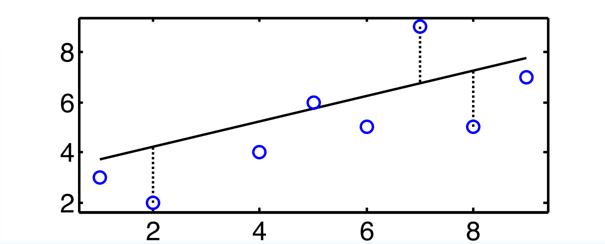

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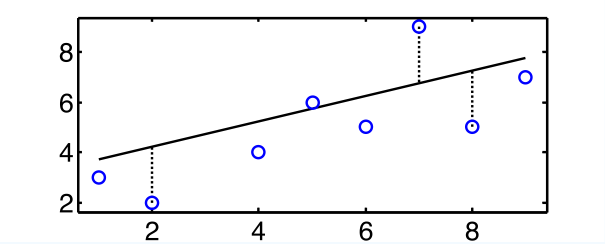

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Fitting to a continuous function is the same as to an infinite number of points.

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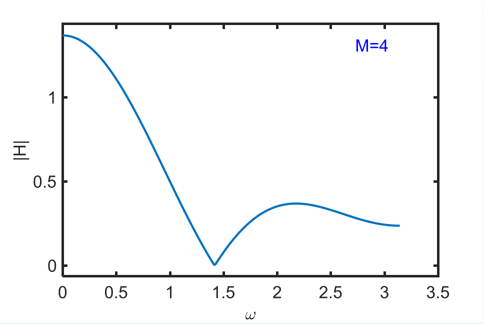

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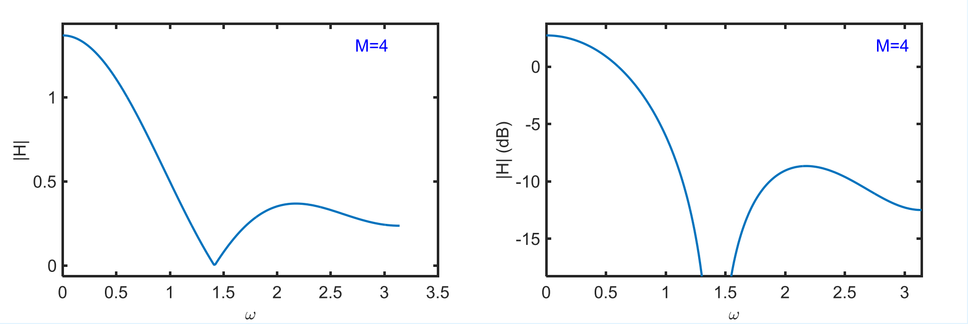

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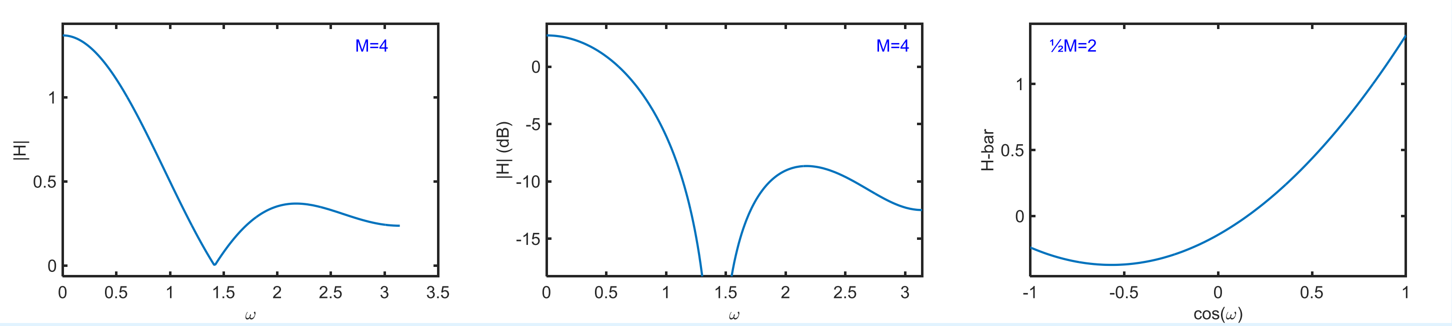

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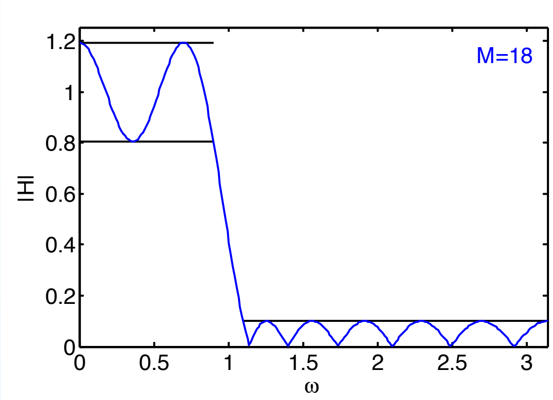

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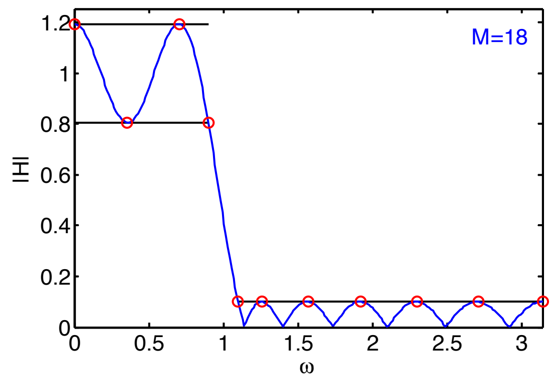

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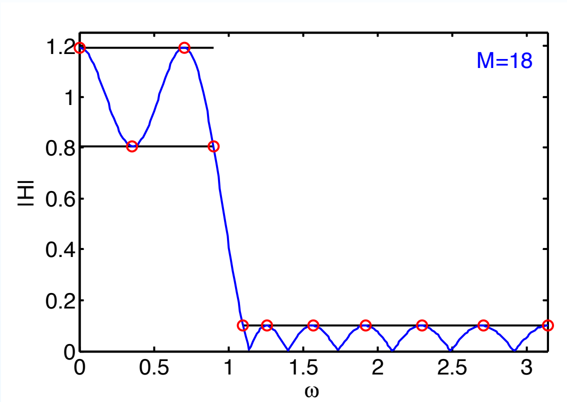

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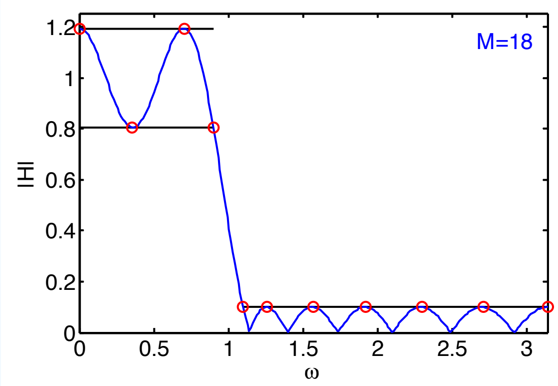

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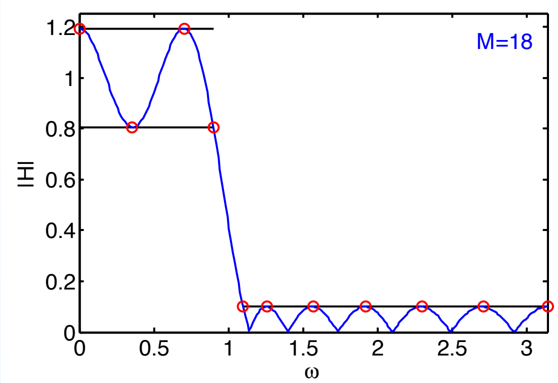

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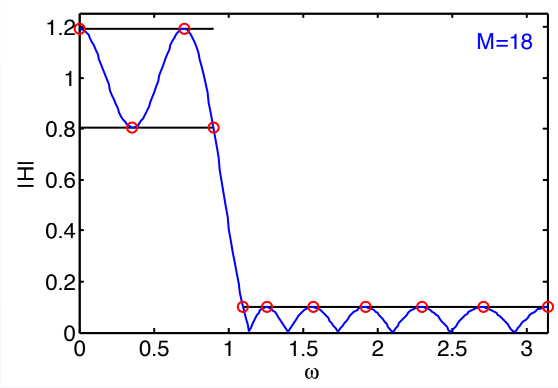

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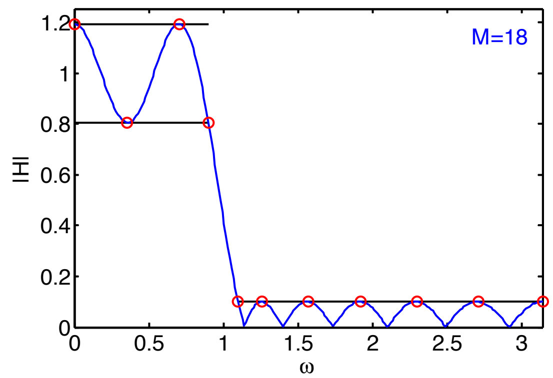

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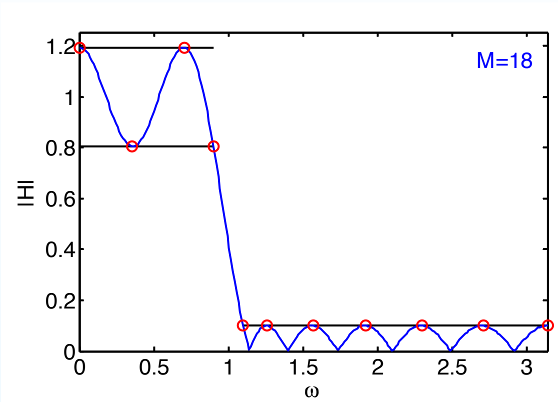

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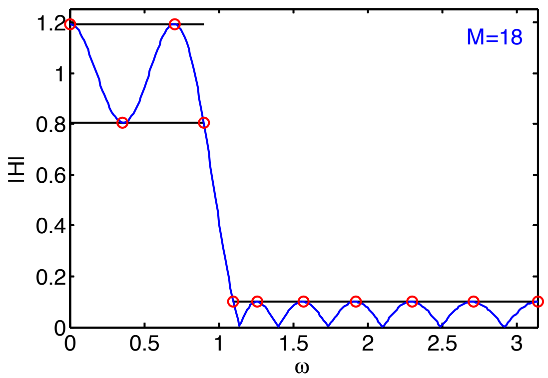

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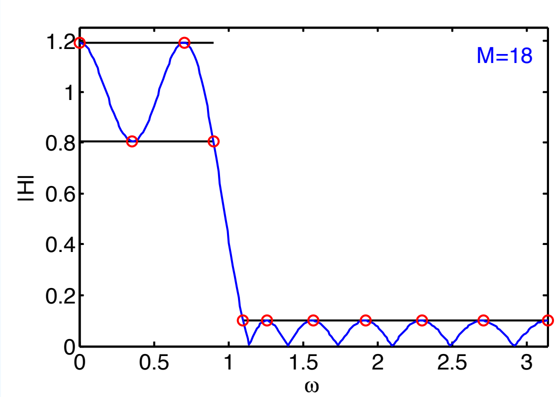

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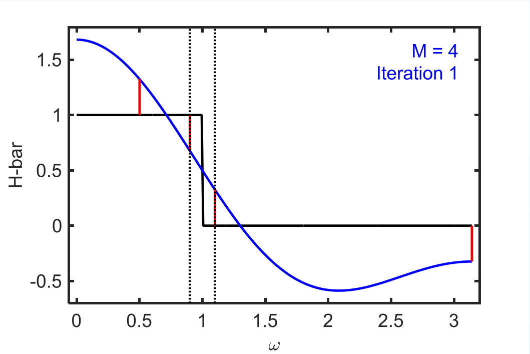

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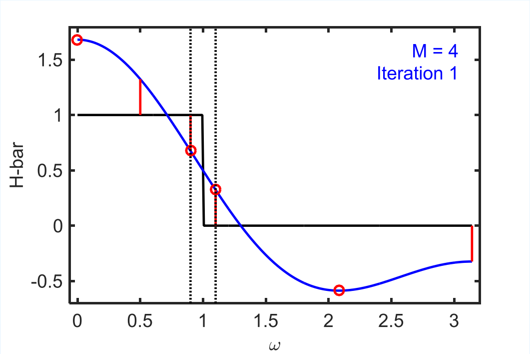

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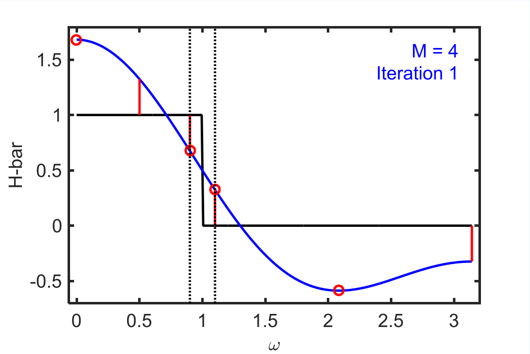

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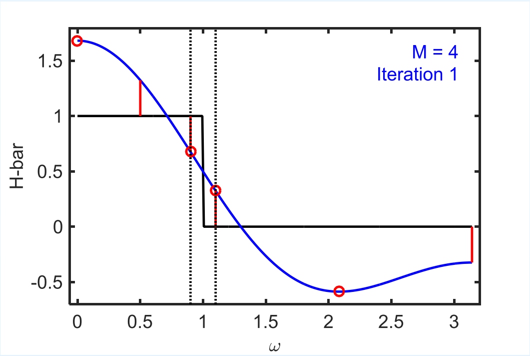

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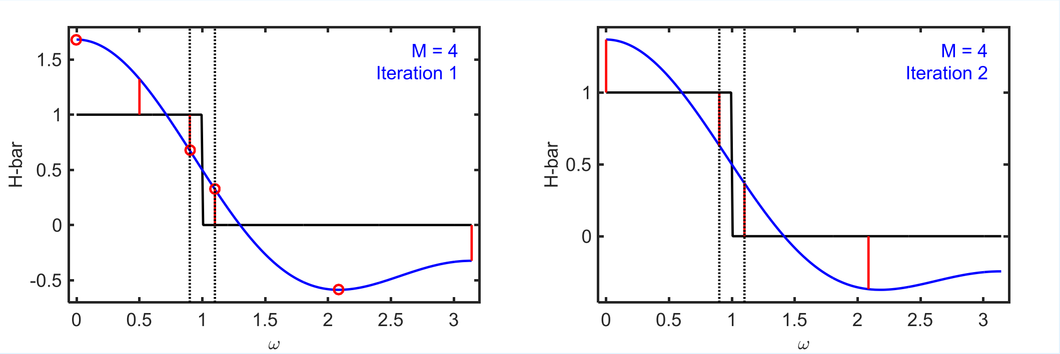

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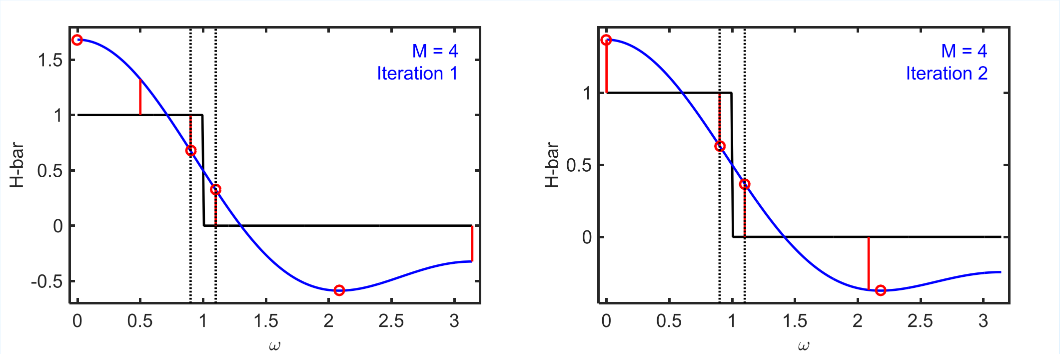

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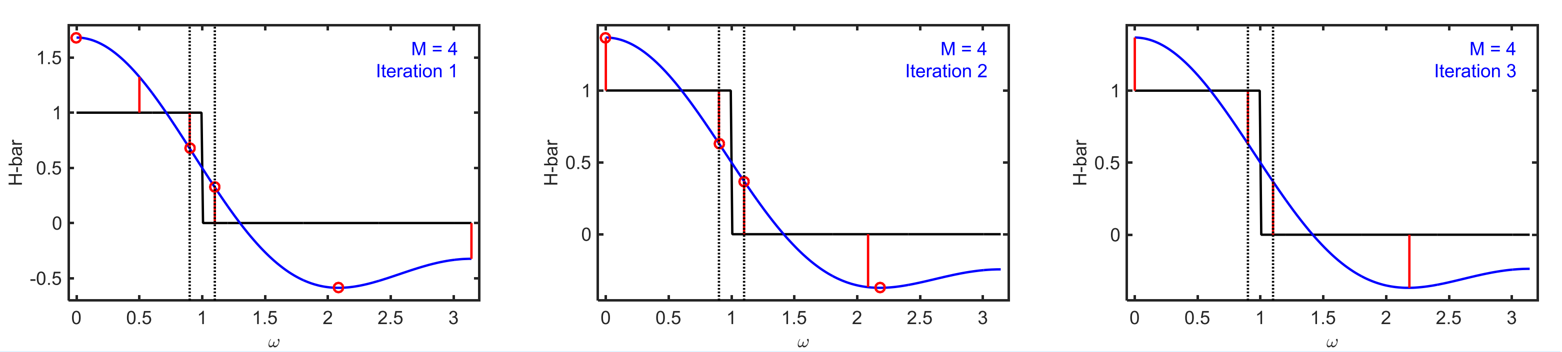

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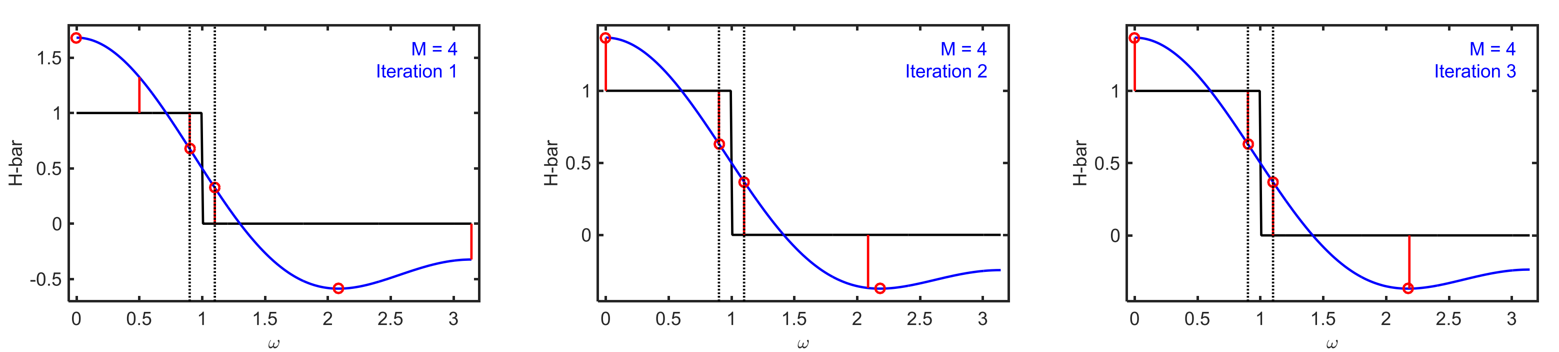

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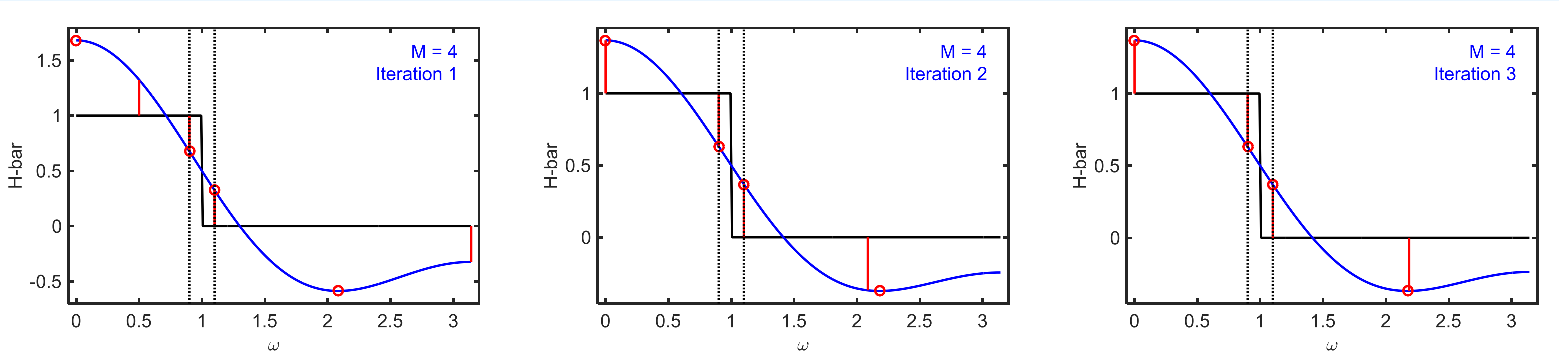

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Lagrange Interpolation.

$$\underbrace{\left.\begin{bmatrix} x_0 & y_0 \\ \vdots & \vdots \\ x_K & y_K \end{bmatrix}\right\}}_{\text{Set of Points in } \mathbb{R}^2} \xrightarrow{\text{Lagrange Interpolation}} L_K(x) = \sum_{k=0}^{K} y_k \ell_k(x)$$

where,

$$\ell_k(x) = \prod_{m \in [0,K], m \neq n} \frac{x - x_m}{x_n - x_m}.$$

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Hence, we can write,

$$\begin{bmatrix} \omega_0 & \overline{H}(e^{\jmath\omega_0}) \\ \vdots & \vdots \\ \omega_K & \overline{H}(e^{\jmath\omega_K}) \end{bmatrix} \xrightarrow{\text{Lagrange Interpolation}} \begin{array}{c} L_K(\omega) = \sum\limits_{k=0}^{K} \overline{H}(e^{\jmath\omega_k}) \, \ell_k(\omega) \\ \downarrow \\ \ell_k(\omega) = \prod\limits_{m \in [0,K], m \neq n} \frac{\cos(\omega) - \cos(\omega_m)}{\cos(\omega_n) - \cos(\omega_m)} \end{array}$$

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Example Design

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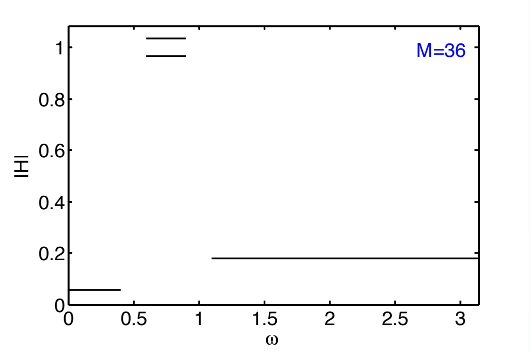

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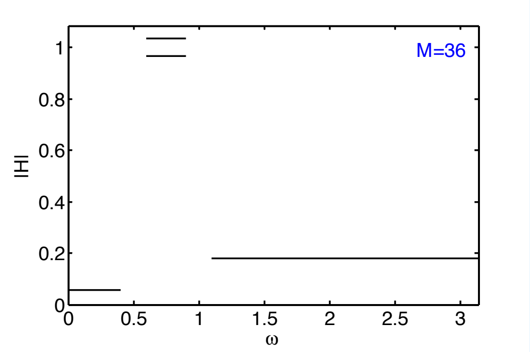

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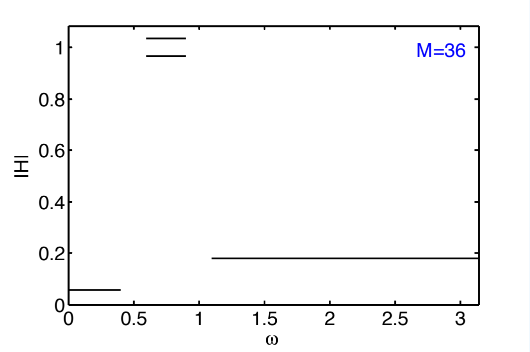

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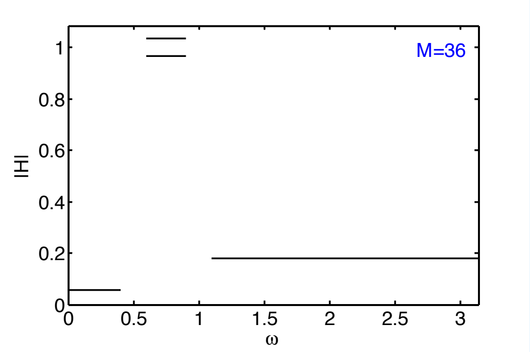

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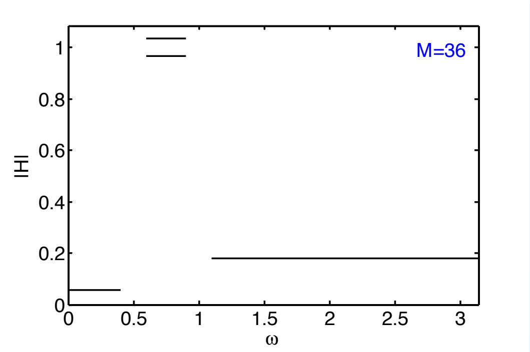

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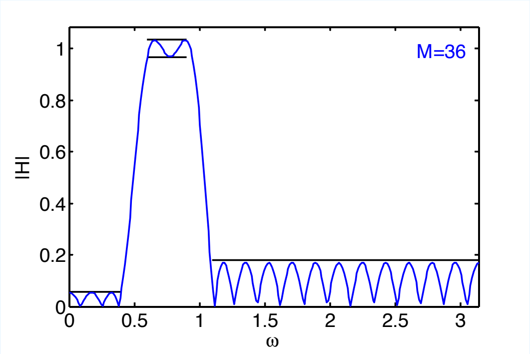

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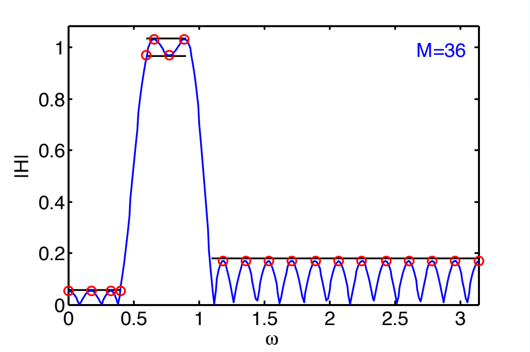

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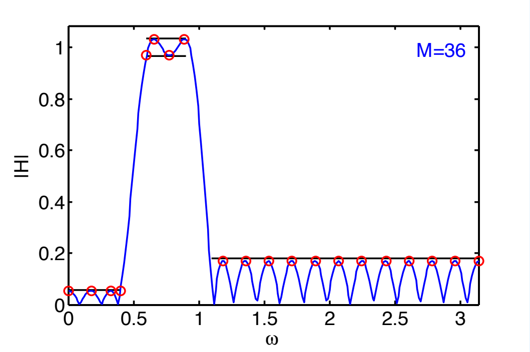

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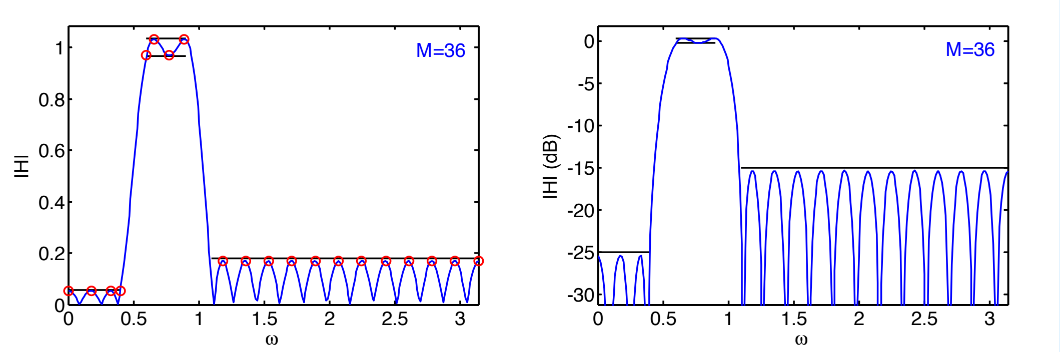

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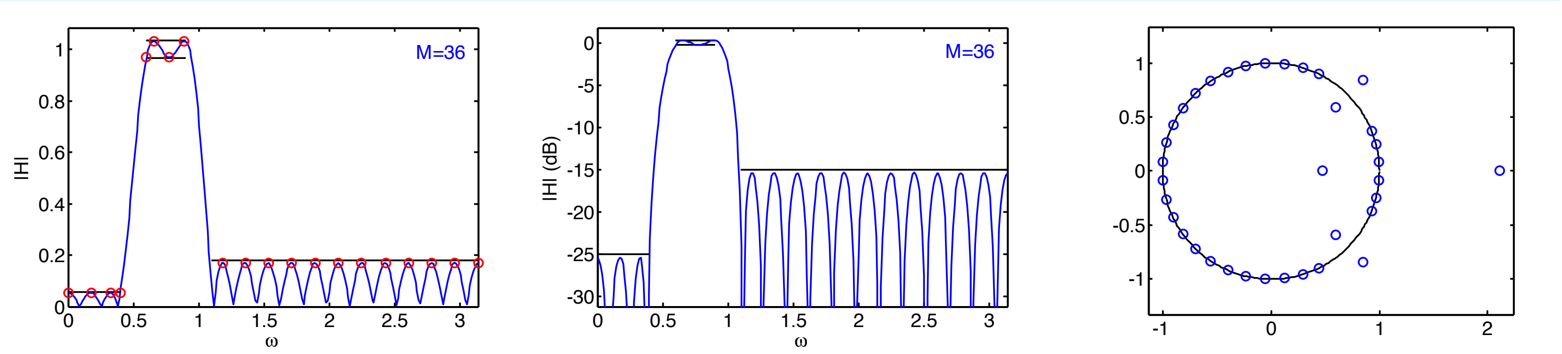

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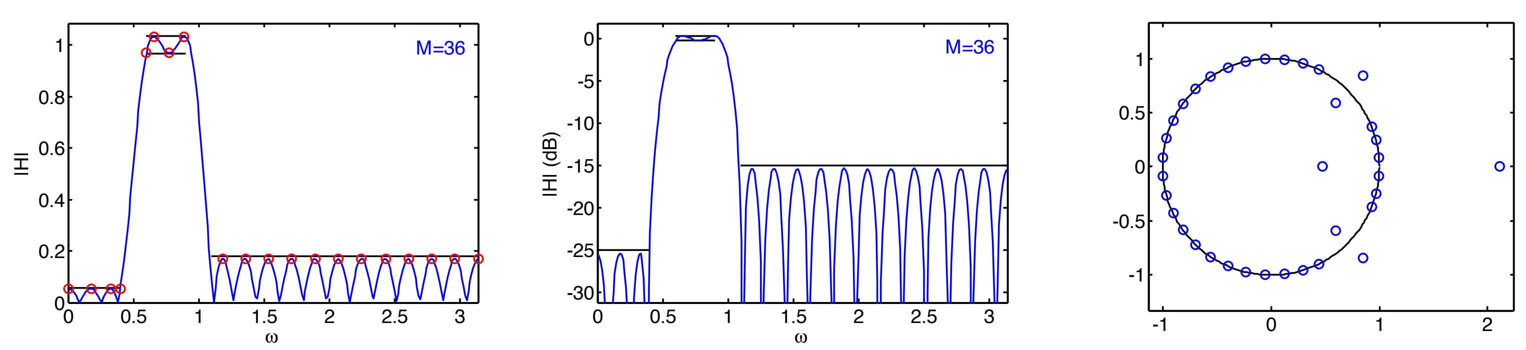

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  - no envelope distortion, all frequencies have the same delay ☺
  - symmetric or antisymmetric: $h[n] = h[-n] \forall n$ or $-h[-n] \forall n$
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Remez Exchange Algorithm (also known as Parks-McLellan Algorithm)

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Optimal Filters: minimax error criterion

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For further details see Mitra: 10.

MATLAB routines

firpm	optimal FIR filter design
firpmord	estimate require order for firpm
cfirpm	arbitrary-response filter design
remez	[obsolete] optimal FIR filter design