EE401: Advanced Communication Theory

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Multi-Antenna Wireless Communications

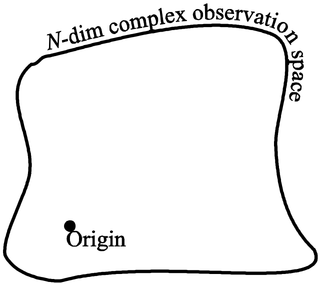
SIMO, MISO and MIMO Antenna Array Comms

Table of Contents

Notation

$\underline{a}, \underline{A}$	denotes a column vector
$\mathbb{A}$ (or $\underline{\underline{A}}$)	denotes a matrix
$\mathbb{I}_N$	$N \times N$ identity matrix
$\underline{1}_N$	vector of N ones
$\underline{0}_N$	vector of N zeros
$\mathbb{O}_{N,M}$	$N \times M$ matrix of zeros
$(.)^T$	transpose
$(.)^H$	Hermitian transpose
$\mathbb{A}^{\#}$	pseudo-inverse of $\mathbb{A}$
$\odot, \oslash$	Hadamard product, Hadamard division
$\otimes$	Kronecker product
$\boxtimes$	Khatri-Rao product (column by column Kronecker product)
$\exp(\underline{a}), \exp(\mathbb{A})$	element by element exponential
$\mathcal{L}[\mathbb{A}]$	linear space/subspace spanned by the columns of $\mathbb{A}$
$\mathcal{L}[\mathbb{A}]^{\perp}$	complement subspace to $\mathcal{L}[\mathbb{A}]$
$\mathcal{P}[\mathbb{A}]$ (or $\mathbb{P}_{\mathbb{A}}$)	projection operator on to $\mathcal{L}[\mathbb{A}]$
$\mathcal{P}[\mathbb{A}]^{\perp}$ (or $\mathbb{P}_{\mathbb{A}}^{\perp}$)	projection operator on to $\mathcal{L}[\mathbb{A}]^{\perp}$

- The expression "N-**dimensional complex (or real) observation space** " is denoted by the symbol $\mathcal{H}$ and pictorially represented as follows

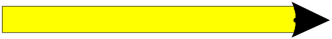


Note that any vector in this space has N elements.

- The expression "one-dimensional subspace spanned by the $(N \times 1)$ vector $\underline{a}$" is mathematically denoted by $\mathcal{L}[\underline{a}]$ and pictorially represented by

$\mathcal{L}[\underline{a}]$

- The expression "M-dimensional subspace (with $M \geq 2$) spanned by the columns of the $(N \times M)$ matrix $\mathbb{A}$" is mathematically denoted by $\mathcal{L}[\mathbb{A}]$ and pictorially represented by



$\mathcal{L}[\mathbb{A}]$

- Note that any vector $\underline{x} \in \mathcal{L}[\mathbb{A}]$ can be written as a linear combination of the columns of the matrix $\mathbb{A}$
  i.e.

$$\underline{x} = \mathbb{A}.\underline{a} \tag{2}$$

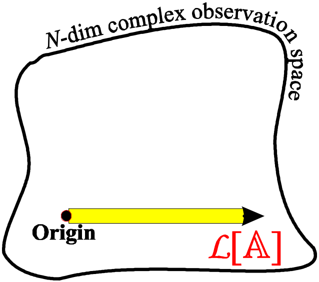
  where $\underline{a}$ is an $(M \times 1)$ vector with elements that are the coefficients of this linear combination.

Common Symbols

N	number of Rx array elements
ϕ	elevation angle (Direction-of-Arrival)
θ	azimuth angle (Direction-of-Arrival)
$\underline{u}$	$[\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi]^T$ (3×1) real unit-vector pointing towards the direction (θ, ϕ) $\underline{u}^T\underline{u} = 1$
c	velocity of light
F_c	carrier frequency
λ	wavelength
k	wavenumber
$\underline{k}$	wavevector

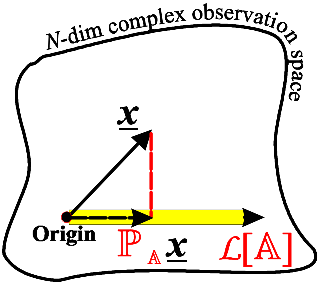
The Concept of the Projection Operator

- Consider an $(N \times M)$ matrix $\mathbb{A}$ with $M \leq N$ (i.e. the matrix has M columns)
- Let the columns of $\mathbb{A}$ be linearly independent (i.e. a column of $\mathbb{A}$ cannot be written as a linear combination of the remaining $M-1$ columns)
- Then the columns of $\mathbb{A}$ span a subspace $\mathcal{L}[\mathbb{A}]$ of dimensionality M (i.e. $\dim\{\mathcal{L}[\mathbb{A}]\}=M$) lying in an N-dimensional space $\mathcal{H}$ (observation space), and this is shown below:



- Any vector $\underline{x} \in \mathcal{H}$ can be projected on to $\mathcal{L}[\mathbb{A}]$ by using the concept of the projection operator $\mathcal{P}[\mathbb{A}]$ (or $\mathbb{P}_{\mathbb{A}}$). That is:

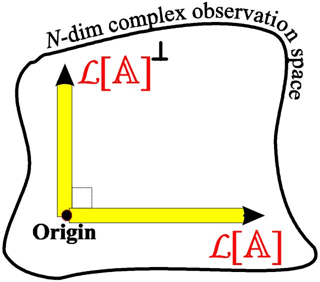
$$\begin{aligned} \mathcal{P}[\mathbb{A}] &= \mathbb{P}_{\mathbb{A}} = \text{projection operator on to } \mathcal{L}[\mathbb{A}] && (3) \\ &= \mathbb{A}(\mathbb{A}^H\mathbb{A})^{-1}\mathbb{A}^H && (4) \end{aligned}$$



- Properties of Projection Operator

$$\mathbb{P}_{\mathbb{A}} : \begin{cases} (N \times N) \text{ matrix} \\ \mathbb{P}_{\mathbb{A}}\mathbb{P}_{\mathbb{A}} = \mathbb{P}_{\mathbb{A}} \\ \mathbb{P}_{\mathbb{A}} = \mathbb{P}_{\mathbb{A}}^H \end{cases} \qquad (5)$$

- $\mathcal{L}[\mathbb{A}]^{\perp}$ denotes the complement subspace to $\mathcal{L}[\mathbb{A}]$

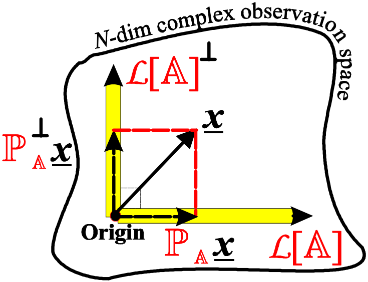


$$\dim\left(\mathcal{L}[\mathbb{A}]\right) = M \tag{6}$$
$$\dim\left(\mathcal{L}[\mathbb{A}]^{\perp}\right) = N - M \tag{7}$$

- $\mathbb{P}_{\mathbb{A}}^{\perp}$ represents the projection operator of $\mathcal{L}[\mathbb{A}]^{\perp}$ and is defined as

$$\mathbb{P}_{\mathbb{A}}^{\perp} = \mathbb{I}_N - \mathbb{P}_{\mathbb{A}} \tag{8}$$

- Any vector $\underline{x} \in \mathcal{H}$ can be projected on to $\mathcal{L}[\mathbb{A}]^{\perp}$ and $\mathcal{L}[\mathbb{A}]$ as follows



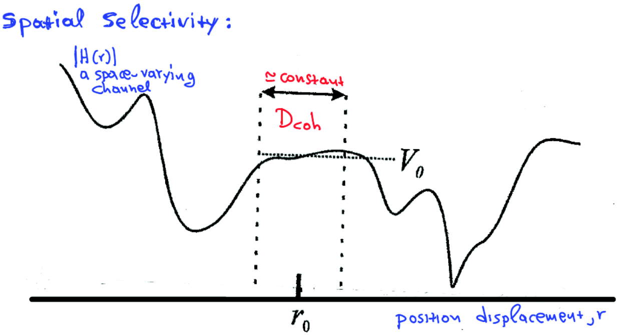
- Comment:

$$\text{if} \quad \underline{x} \in \mathcal{L}[\mathbb{A}] \quad \text{then} \quad \begin{cases} \mathbb{P}_{\mathbb{A}}\underline{x} = \underline{x} \\ \mathbb{P}_{\mathbb{A}}^{\perp}\underline{x} = \underline{0}_{N-M} \end{cases} \tag{9}$$

Space-Selective Fading

- A wireless receiver is located (and moves) in our 3D real space.
- In addition to delay-spread (causing frequency-selective fading) and Doppler-spread (causing time-selective fading) there is also angle-spread
- Angle Spread causes Space-Selective fading
- Note that a channel has
  - space-selectivity if it **varies** (i.e. its transfer function varies) as a function of space and
  - spatial-coherence if its transfer function **does not vary** as a function of space over a specified distance (D_{coh}) of interest
  where

$$D_{\text{coh}} = \text{is known as coherence distance.} \tag{10}$$

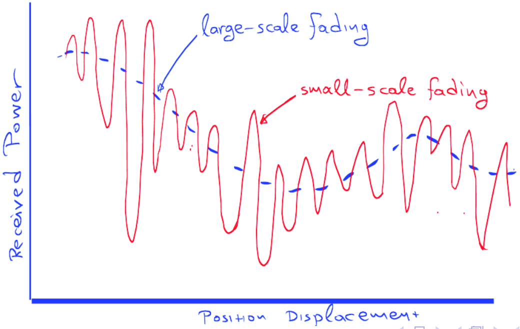


$$|H(r)| \approx V_0 \text{ for } |r - r_0| \leq \frac{D_{\mathrm{coh}}}{2} \tag{11}$$

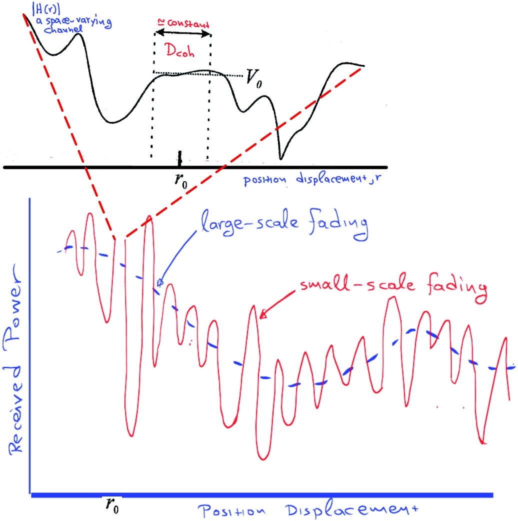
- In other words a wireless channel has spatial coherence if the envelope of the carrier remains constant over a spatial displacement of the receiver.
- D_{coh} represents the largest distance that a wireless receiver can move with the channel appearing to be static/constant.

Small-Scale and Large-Scale fading

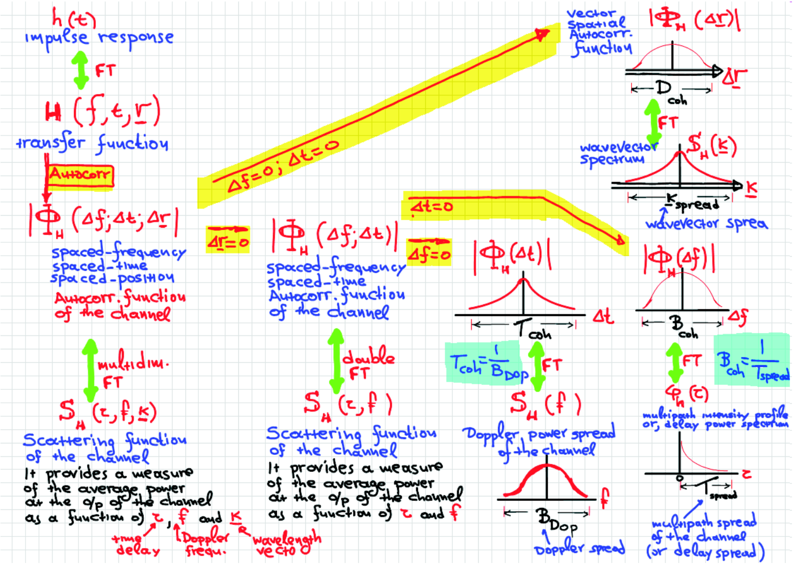
- If the displacement of the receiver is greater than D_{coh} then the channel experiences **small-scale fading** (this is due to multipaths added constructively or destructively).
- If this displacement is very large (i.e. corresponds to a very large number of wavelengths) then the channel experiences **large-scale fading** (this is due to path loss over large distances and shadowing by large objects - it is typically independent of frequency)



Spatial Selectivity:



Scattering Function - Wireless Channel Analysis



Wavenumber Spectrum

- If the transfer function of the channel includes "space" then we have:

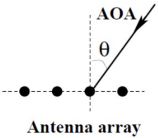
$$\begin{array}{ccccc}
\underbrace{H(f,t,\underline{r})}_{Transfer-function} & \overset{corr}{\Longrightarrow} & \underbrace{\Phi_H(\Delta f,\Delta t,\Delta \underline{r})}_{\substack{Autocorrelation-function \\ (frequency,time,space)}} & \overset{FT}{\Longrightarrow} & \underbrace{S_H(\tau,\mathfrak{f},\underline{k})}_{Scattering-function} \\
 & & \downarrow & \downarrow & \downarrow \\
 & & \downarrow & \downarrow & \downarrow \\
 & & \downarrow & \downarrow & \downarrow \\
 & & \underbrace{\Phi_H(\Delta \underline{r})}_{\substack{Autocorrelation-function \\ (space)}} & \overset{FT}{\Longrightarrow} & \underbrace{S_H(\underline{k})}_{\substack{wavenumber \\ spectrum}} \\
 & & \boxed{\text{(known as spatial Autocor function)}} & &
\end{array} \tag{12}$$

and the scattering function $S_H(\underline{k})$ is known as the wavenumber spectrum

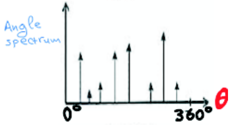
$$\text{where} \left\{ \begin{array}{lllcll}
f & frequency & & ; & \tau & delay \\
t & time & & ; & \mathfrak{f} & Doppler\ frequency \\
\underline{r} & location & (\text{x,y,z}) & ; & \underline{k} & wavenumber\ vector = \|\underline{k}\| . \underline{u}(\theta,\phi)
\end{array} \right\} \tag{13}$$

Angle Spectrum

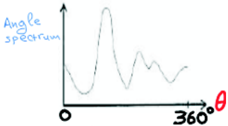
- Angle Spectrum is the average power as a function of direction-of-arrival (θ, ϕ) where θ represents the azimuth angle and ϕ is the elevation angle (in the case of Tx-array this is a function of the direction-of-departure).
- Examples of Angle Spectrum for $\phi = 0^\circ$:



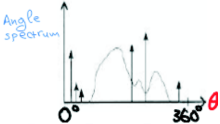
1. Angle Spectrum (**Specular**):



2. Angle Spectrum (**Diffused**):



3. Angle Spectrum (**Combination**) :



- For $\phi = 0$, the mean angle of azimuth and rms angle spread σ_θ are functions of angle spectrum $p(\theta)$ as follows:

$$\text{mean} \quad : \quad \theta_{mean} \triangleq \begin{cases} \triangleq \dfrac{\int_0^{\theta_{\max}} \theta . p(\theta) d\theta}{\int_0^{\theta_{\max}} p(\theta) d\theta} & = \text{cont. case} \\ \\ \triangleq \dfrac{\sum_{i=1}^{L} \theta_i . p(\theta_i)}{\sum_{i=1}^{L} p(\theta_i)} & = \text{discrete case} \end{cases} \tag{14}$$

$$\text{rms} \quad : \quad \sigma_\theta \triangleq \begin{cases} \triangleq \sqrt{\dfrac{\int_0^{\theta_{\max}} (\theta - \theta_{mean})^2 . p(\theta) d\theta}{\int_0^{\theta_{\max}} p(\theta) d\theta}} & = \text{cont. case} \\ \\ \triangleq \sqrt{\dfrac{\sum_{i=1}^{L} (\theta - \theta_{mean})^2 p(\theta)}{\sum_{i=1}^{L} p(\theta)}} & = \text{discrete case} \end{cases} \tag{15}$$

- NB:
  - **angle spectrum** causes **space selective fading** $\Longrightarrow$ signal amplitude depends on spatial location of antennas/signals
  - **Coherence distance** D_{coh} is the spatial separation for which the autocorr. coef. of spatial fading drops to 0.7
  - **Coherence distance** D_{coh} is inversely proportional to σ_θ, i.e.

$$D_{coh} \propto \frac{1}{\sigma_\theta} \tag{16}$$

- The angle spectrum, denoted by $p(\theta, \phi)$, is related to the wavenumber spectrum $S_H(\underline{k})$ via the "ring mapping":

$$\underbrace{H(f, t, \underline{r})}_{Transfer-function} \overset{corr}{\Longrightarrow} \underbrace{\Phi_H(\Delta f, \Delta t, \Delta \underline{r})}_{\substack{Autocorrelation-function \\ (frequency, time, space)}} \overset{FT}{\Longrightarrow} \underbrace{S_H(\tau, \mathfrak{f}, \underline{k})}_{Scattering-function}$$

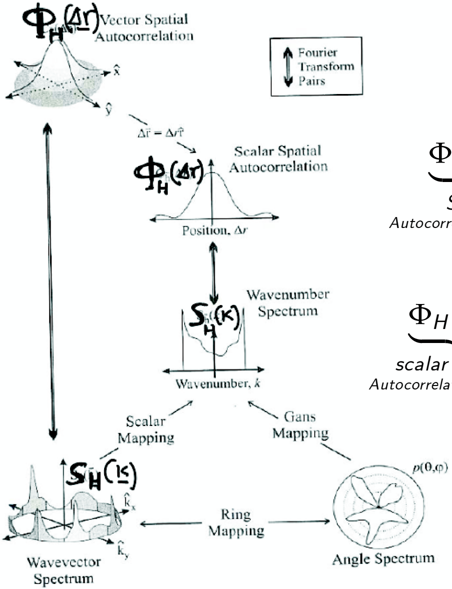
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\underbrace{\Phi_H(\Delta \underline{r})}_{\substack{Spatial \\ Autocorrelation-function}} \overset{FT}{\Longrightarrow} \boxed{\underbrace{S_H(\underline{k})}_{\substack{wavenumber \\ spectrum}}} \underset{ring\ mapping}{\longleftrightarrow} \boxed{\underbrace{p(\theta, \phi)}_{\substack{angle \\ spectrum}}} \tag{17}$$

That is,

$$\boxed{S_H(\underline{k}) = \frac{(2\pi)^3 \delta(\|\underline{k}\| - \frac{2\pi}{\lambda})}{(\frac{2\pi}{\lambda})^2} \times p(\theta, \phi)} \tag{18}$$

- Remember that if $\Delta f = 0$ and $\Delta t = 0$ then



$$\underbrace{\Phi_H(\Delta\underline{r})}_{\substack{Spatial\\ Autocorrelation-function}} \overset{FT}{\Longrightarrow} \underbrace{S_H(\underline{k})}_{\substack{Scattering\ \ function\\ (wavenumber\ \ spectrum)}} \tag{19}$$

$$\underbrace{\Phi_H(\Delta r)}_{\substack{scalar\ \ spatial\\ Autocorrelation-function}} \overset{FT}{\Longrightarrow} \underbrace{S_H(k)}_{\substack{Scattering\ \ function\\ (scalar\ \ wavenumber\ spectrum)}} \tag{20}$$

Correspondence between Frequ, Time & Space Parameters

- Consider a wireless channel transfer function $H(f, t, \underline{r})$

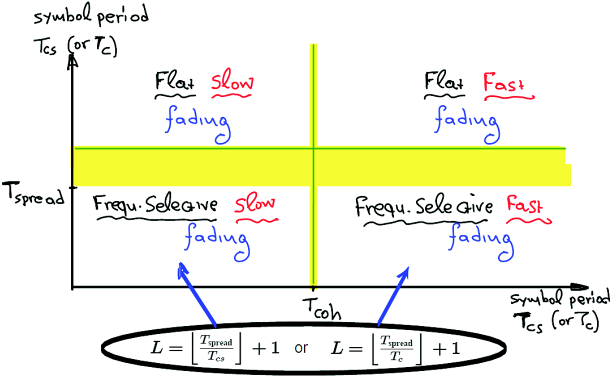
	Frequency	Time	Space
Dependency	f	t	$\underline{r}$
Coherence	B_{coh} coherence bandwidth	T_{coh} coherence time	D_{coh} coherence distance
Spectral domain	τ delay	$\mathfrak{f}$ Doppler freq	$\underline{k}$ wavevector
Spectral width	T_{spread} delay spread	B_{Dop} Doppler spread	$\underline{k}_{spread}$ wavevector spread

N.B.: Remember the various types of coherence.

$$\begin{cases} \textit{temporal} \text{ coherence} & \text{-coherence time } T_{\text{coh}} \\ \textit{frequency} \text{ coherence} & \text{-coherence bandwidth } B_{\text{coh}} \\ \textit{spatial} \text{ coherence} & \text{-coherence distance } D_{\text{coh}} \end{cases}$$

The Relationship between Coherence-Time and Bandwidth

- We have seen that one of the wireless channels classification is "slow" and "fast fading - as this is shown below:



- A trade-off between "slow" and "fast" fading is when $T_{\text{coh}} \simeq T_{cs}$ (or, for CDMA, $T_{\text{coh}} \simeq T_c$). This implies

$$\boxed{T_{\text{coh}} \times B \simeq 1 \text{ (or, for CDMA: } T_{\text{coh}} \times B_{ss} \simeq 1)} \quad (21)$$

- An electromagnetic wave of frequency F_c (carrier frequency) travelling for the time T_{coh} with velocity c (speed of light, i.e. $3 \times 10^8 m/s$) will cover a distance d, where

$$\begin{aligned} d &= c.T_{\text{coh}} &(22)\\ &= \underbrace{F_c \lambda_c}_{=c} T_{\text{coh}} \Longrightarrow T_{\text{coh}} = \frac{d}{F_c \lambda_c} &(23) \end{aligned}$$

Using Equations 21 and 23 we have

$$\frac{d}{F_c \lambda_c} B \simeq 1 \Longrightarrow d \simeq \frac{F_c}{B} \lambda_c (= \frac{c}{B}) \tag{24}$$

- However, if an electromagnetic wave is travelling for the time T_{cs} with velocity c will cover a distance d_{cs},where

$$d_{cs} = c.T_{cs} \tag{25}$$

- We have seen that in **slow fading region** we have $T_{\text{coh}} \gtrsim T_{cs}$ (or, for CDMA, $T_{\text{coh}} \gtrsim T_c$). In this case:

$$T_{\text{coh}} \gtrsim T_{cs} \Longrightarrow cT_{\text{coh}} \gtrsim cT_{cs} \Longrightarrow d \gtrsim d_{cs}$$

i.e.

$$\text{slow fading: } d_{cs} \lesssim d \simeq \frac{F_c}{B}\lambda_c (= \frac{c}{B}) \tag{26}$$

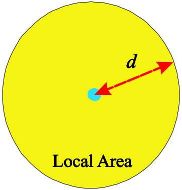
N.B. - In summary (for slow fading):

$$\boxed{T_{\text{coh}} \gtrsim T_{cs} \text{ (i.e. } T_{\text{coh}} \times B \gtrsim 1) \Longrightarrow d_{cs} \lesssim d \simeq \frac{c}{B}} \tag{27}$$

□

The Concept of the "Local Area"

- "Local Area" is the largest volume of free-space (i.e. $\frac{4}{3}\pi d^3$) about a specific point in space $\underline{r}_o = \left[r_{x_0}, r_{y_0}, r_{z_0}\right]^T$ (e.g. the reference point of the Rx-array) in which the wireless channel can be modelled as the summation of homogeneous planewaves, where

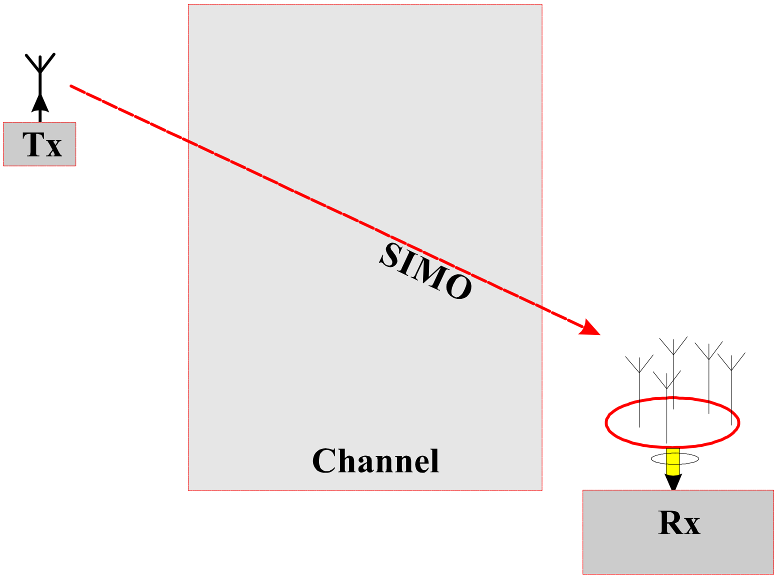


$$\boxed{\text{radius of local area} \lesssim d = \frac{c}{B}} \qquad (28)$$

- Example: The radius of the "local area" for WiFi electromagnetic waves ($F_c = 2.4\text{GHz}$, $B = 5\text{MHz}$) is:

$$\text{radius} \lesssim \frac{c}{B} \Longrightarrow \text{radius} \lesssim \frac{3 \times 10^8}{5 \times 10^6} = 60m$$

Wireless SIMO Channels



Antenna Array

- Consider a single path from Tx single antenna system to an array of N antennas
- An array system is a collection of $N > 1$ sensors (transducing elements, receivers, antennas, etc) distributed in the 3-dimensional Cartesian space, with a common reference point.
- Consider an antenna-array Rx with locations given by the matrix

$$\left[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\right] = \left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right]^T \quad (3 \times N) \tag{29}$$

  where $\underline{r}_k$ is a 3×1 real vector denoting the location of the k^{th} sensor $\forall k = 1, 2, \ldots, N$
  and $\underline{r}_x$,$\underline{r}_y$ and $\underline{r}_z$ are $N \times 1$ vectors with elements the x, y and z coordinates of the N antennas
- The region over which the sensors are distributed is called the aperture of the array. In particular the array aperture is defined as follows

$$\text{array aperture} \doteq \max_{ij} \left\| \underline{r}_i - \underline{r}_j \right\| \tag{30}$$

Array Manifold Vector

- It is also know as Array Response Vector.
- Modelling of Array Manifold Vectors (see Chapter-1 of my book):

$$\begin{aligned} \underline{S}(\theta,\phi) &= \exp(-j[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]^T \underline{k}(\theta,\phi)) & (31)\\ &\stackrel{or}{=} \exp(-j[\underline{r}_x, \underline{r}_y, \underline{r}_z]\underline{k}(\theta,\phi)) & (32)\\ &= (N \times 1) \text{ complex vector} \end{aligned}$$

where

$$\begin{aligned} \underline{k}(\theta,\phi) &= \begin{cases} \frac{2\pi F_c}{c}.\underline{u}(\theta,\phi) = \frac{2\pi}{\lambda_c}.\underline{u}(\theta,\phi) & \text{in meters} \\ \pi.\underline{u}(\theta,\phi) & \text{in units of halfwavelength} \end{cases} \\ &= \text{wavenumber vector} \end{aligned}$$

$$\begin{aligned} \underline{u}(\theta,\phi) &= [\cos\theta\cos\phi,\ \ \sin\theta\cos\phi,\ \ \sin\phi]^T & (33)\\ &= (3 \times 1) \text{ real unit-vector pointing towards the direction } (\theta,\phi) \\ \|\underline{u}(\theta,\phi)\| &= 1 & (34) \end{aligned}$$

- In many cases the signals are assumed to be on the (x,y) plane (i.e. $\phi = 0^\circ$). In this case the manifold vector is simplified to

$$\begin{aligned} \underline{S}(\theta) &= \exp(-j[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]^T \underline{k}(\theta, 0^\circ)) \\ &= \exp(-j\pi(\underline{r}_x \cos\theta + \underline{r}_y \sin\theta)) \end{aligned} \tag{35}$$

- A popular class of arrays is that of linear arrays: $\underline{r}_y = \underline{r}_z = \underline{0}_N$ in this case, Equation 31 is simplified to

$$\underline{S}(\theta) = \exp(-j\pi\underline{r}_x \cos\theta) \tag{36}$$

- Summary: An array maps one or more real directional parameters p or (p, q) to an $(N \times 1)$ complex vector $\underline{S}(p)$ or $\underline{S}(p, q)$, known as array manifold vector, or array response vector, or source position vector. That is

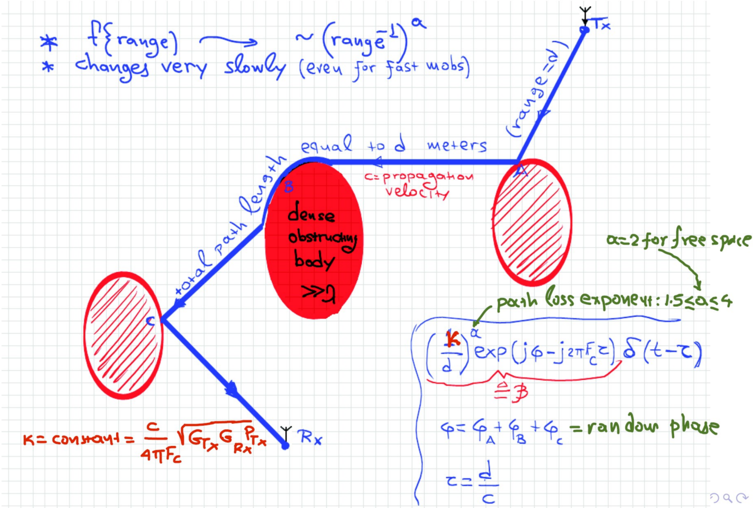
$$p \in \mathcal{R}^1 \stackrel{\underline{r}}{\longmapsto} \underline{S}(p) \in \mathcal{C}^N \tag{37}$$

$$\text{or } (p, q) \in \mathcal{R}^2 \stackrel{\underline{r}}{\longmapsto} \underline{S}(p, q) \in \mathcal{C}^N \tag{38}$$

- Note: Ideally Equations 37 and 38 should be an '*one-to-one*' mapping

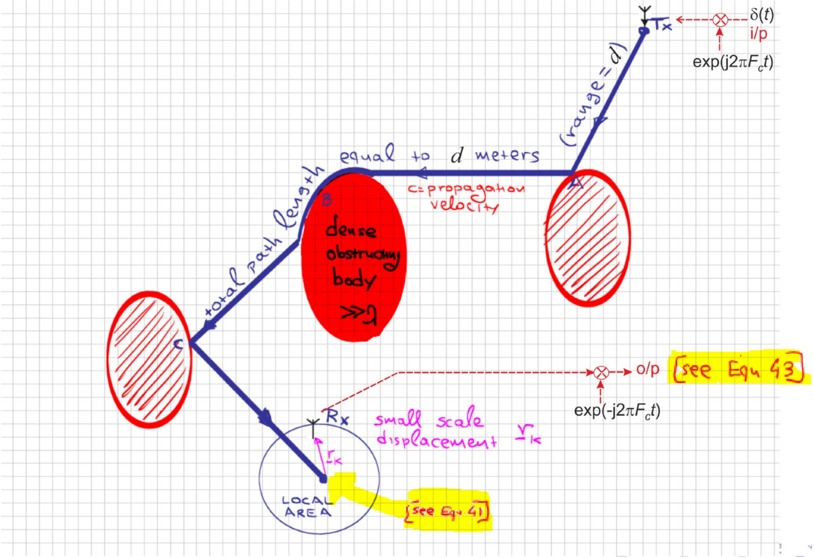
Single-path SIMO Channel Modelling

Single Path/Ray of an Electromagnetic Wave:



Single-path SIMO Channel Modelling

Impulse Response (Including the Carrier):



- With reference to the previous figure, consider at the baseband-input a delta-line $\delta(t)$
- Then the antenna will transmit the signal $\exp(j2\pi F_c t).\delta(t)$, i.e.

$$\text{at the Tx} = \exp\left(j2\pi F_c t\right)\delta(t) \tag{39}$$

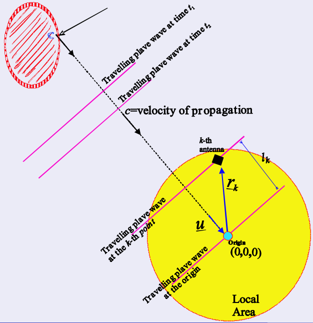
in the form of an electromagnetic wave that travels with the velocity of light c for time τ and covers a distance d arriving at the Rx's reference point (see figure) and modelled as follows:

$$\begin{aligned}
\text{at the Rx's ref. point} &= \left(\frac{K}{d}\right)^{\alpha}\exp\left(j\phi\right)\exp\left(j2\pi F_c(t-\frac{d}{c})\right)\delta(t-\frac{d}{c}) \\
&= \underbrace{\left(\frac{K}{d}\right)^{\alpha}\exp\left(j\phi\right)\exp\left(-j2\pi F_c\frac{d}{c}\right)}_{\triangleq\beta}\exp\left(j2\pi F_c t\right)\delta(t-\frac{d}{c}) \\
&\qquad \left[\text{where}\begin{cases} K: & \text{constant} = \frac{c}{4\pi F_c}g \text{ (known)} \\ & \text{with } g = \text{antenna gain factor (known)} \\ \alpha: & \text{path loss exponent; } 1.5 \leq \alpha \leq 4 \text{ (known)}\end{cases}\right] \quad (40) \\
&= \beta.\exp\left(j2\pi F_c t\right).\delta(t-\frac{d}{c}) \quad (41)
\end{aligned}$$

- If the k-th antenna is not at the reference point of the array but there is a displacement $\underline{r}_k$ (within the Rx "Local Area"), then the electromagnetic wave will travel with the velocity of light c for a time $\Delta\tau_k$ (+ve or -ve) which corresponds to the distance $\underline{r}_k^T \underline{u}(\theta, \phi)$ between the ref. point and the k-th antenna's location. That is,

$$\Delta\tau_k = \frac{\underline{r}_k^T \underline{u}(\theta, \phi)}{c} \tag{42}$$

Proof of 42:



With reference to the LHS figure (and $\underline{u} \triangleq \underline{u}(\theta, \phi)$) we have:

$$\begin{aligned}\Delta\tau_k &= \frac{\sqrt{\underline{r}_k^T \underline{u}(\underline{u}^T \underline{u})^{-1} \underline{u}^T \underline{r}_k}}{c} \\ &= \frac{\sqrt{\underline{r}_k^T \underline{u}\,\underline{u}^T \underline{r}_k}}{c} \\ &= \frac{\sqrt{\left(\underline{r}_k^T \underline{u}\right)^2}}{c} = \frac{\underline{r}_k^T \underline{u}}{c}\end{aligned}$$

- Note: $\underline{r}_k \in R^{3\times 1}$ represents the Cartesian coords of the k-th Rx-antenna (in m)
- Thus, at the o/p (k^{th} Rx-antenna) we have

$$\begin{aligned}
\underset{\text{(baseband)}}{\text{o/p}} &= \begin{bmatrix} \text{Equation-41} \\ \text{with displacement: } \Delta\tau_k \end{bmatrix} \times \exp\left(-j2\pi F_c t\right) \\
&= \beta \exp\left(j2\pi F_c (t-\Delta\tau_k)\right) \exp\left(-j2\pi F_c t\right) .\delta(t-\Delta\tau_k-\frac{d}{c}) \\
&= \beta \exp\left(-j2\pi F_c \Delta\tau_k\right) .\delta(t-\Delta\tau_k-\frac{d}{c}) \\
&= \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\underline{u}(\theta,\phi)\right) .\delta(t-\frac{\overbrace{d+\underline{r}_k^T\underline{u}(\theta,\phi)}^{\simeq d \text{ (as } d>>>\|\underline{r}_k\|)}}{c}) \\
&= \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\underline{u}(\theta,\phi)\right) .\delta(t-\frac{d}{c}) \\
&= \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\underline{u}(\theta,\phi)\right) .\delta(t-\underset{=\frac{d}{c}}{\underset{\uparrow}{\tau}}) \qquad (43)
\end{aligned}$$

- That is, the Tx planewave $\exp(j2\pi F_c t)\,\delta(t)$ arrives at each antenna of the array and produces, at time $t=\tau$, a constant-amplitude voltage-vector as follows:

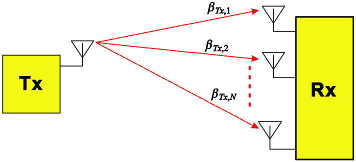
$$\begin{bmatrix} 1^{\text{st}} \text{ ant.} \\ 2^{\text{nd}} \text{ ant.} \\ \vdots \\ N^{\text{th}} \text{ ant.} \end{bmatrix} = \begin{bmatrix} \text{Equ.43, for } k=1 \\ \text{Equ.43, for } k=2 \\ \vdots \\ \text{Equ.43, for } k=N \end{bmatrix}$$

$$= \begin{bmatrix} \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \\ \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \\ \vdots \\ \beta \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\underline{u}(\theta,\phi)\right).\delta(t-\tau) \end{bmatrix}$$

$$= \beta \underbrace{\begin{bmatrix} \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\underline{u}(\theta,\phi)\right) \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\underline{u}(\theta,\phi)\right) \\ \vdots \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\underline{u}(\theta,\phi)\right) \end{bmatrix}}_{\triangleq \textit{array manifold vector } \underline{S}(\theta,\phi)} \delta(t-\tau)$$

$$= \beta.\underline{S}(\theta,\phi).\delta(t-\tau) \tag{44}$$

- i.e.



Non-parametric Model

$$\Leftrightarrow \underbrace{\begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix}}_{\text{Non-parametric}} = \underbrace{\beta . \underline{S}(\theta, \phi)}_{\text{Parametric}} \tag{45}$$

- where

$$\underline{S}(\theta, \phi) = \begin{bmatrix} \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_1^T\,\underline{u}(\theta, \phi)\right) \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_2^T\,\underline{u}(\theta, \phi)\right) \\ \cdots\cdot \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_k^T\,\underline{u}(\theta, \phi)\right) \\ \cdots\cdot \\ \exp\left(-j\frac{2\pi F_c}{c}\underline{r}_N^T\,\underline{u}(\theta, \phi)\right) \end{bmatrix} \tag{46}$$

$$= \exp\left(-j\frac{2\pi F_c}{c}\left[\underline{r}_1, \quad \underline{r}_2, \quad \ldots \quad \underline{r}_N\right]^T \underline{u}(\theta, \phi)\right) \tag{47}$$

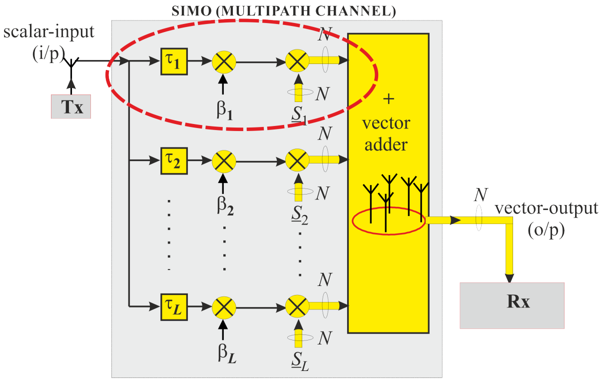
$$= \exp\left(-j\frac{2\pi F_c}{c}\left[\underline{r}_x, \quad \underline{r}_y, \quad \underline{r}_z\right] \underline{u}(\theta, \phi)\right) \tag{48}$$

Multi-path SIMO Channel Modelling

- Let us assume that the transmitted signal arrives at the reference point of an array receiver via L paths (multipaths).
- Consider that the ℓ^{th} path arrives at the array from direction (θ_ℓ, ϕ_ℓ) with channel propagation parameters β_ℓ and τ_ℓ representing the complex path gain and path-delay, respectively.
- Note that θ_ℓ and ϕ_ℓ represent the azimuth and elevation angles respectively associated with ℓ-th path.
- Let us assume that the L paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{49}$$

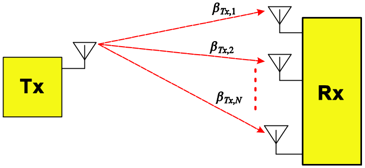
- Furthermore, the path coefficients β_ℓ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter powers.



- The vector $\underline{S}_\ell = \underline{S}(\theta_\ell, \phi_\ell) \in C^N$, is the array manifold vector of the ℓ-th path .
- The impulse response (vector) of the SIMO multipath channel is

$$\boxed{\text{SIMO: } \underline{h}(t) = \sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t - \tau_\ell)} \quad (50)$$

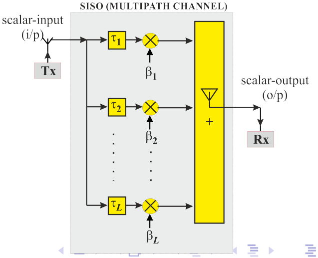
- i.e. multipath SIMO channel



Non-parametric

$$\Leftrightarrow \underbrace{\begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix}}_{\text{Non-parametric}} = \underbrace{\sum_{\ell=1}^{L} \beta_\ell . \underline{S}(\theta_\ell, \phi_\ell)}_{\text{Parametric}} \tag{51}$$

- Remember:
  SISO Multipath Channel Modelling



Modelling of the Received Vector-Signal

- Consider a single Tx transmitting a baseband signal $m(t)$ via an L-path SIVO channel.
- The received $(N \times 1)$ vector-signal $\underline{x}(t)$ can be modelled as follows:

$$\begin{aligned}
\underline{x}(t) &= \underline{h}(t) * m(t) + \underline{\mathrm{n}}(t) \\
&= \left(\sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t-\tau_\ell) \right) * m(t) + \underline{\mathrm{n}}(t) \\
\Rightarrow \underline{x}(t) &= \sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.m(t-\tau_\ell) + \underline{\mathrm{n}}(t) \\
&= \mathbb{S}\underline{m}(t) + \underline{\mathrm{n}}(t) \qquad (52)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{S} &= \left[\underline{S}_1, \quad \underline{S}_2, \quad \cdots, \quad \underline{S}_L \right], \text{ with } \underline{S}_\ell \triangleq \underline{S}(\theta_\ell, \phi_\ell), \ell = 1, .., L \qquad (53) \\
\underline{m}(t) &= \left[\beta_1 m(t-\tau_1), \quad \beta_2 m(t-\tau_2), \quad \cdots, \quad \beta_L m(t-\tau_L) \right]^T \qquad (54) \\
\underline{\mathrm{n}}(t) &= \left[\mathrm{n}_1(t), \quad \mathrm{n}_2(t), \quad \cdots, \quad \mathrm{n}_N(t) \right]^T \qquad (55)
\end{aligned}$$

Multi-user SIMO

- Next consider an Rx array of N antennas operating the the presence of M co-channel transmitters/users.
- In this case we have added the subscript i to refer to the i-th Tx.
- The received $(N \times 1)$ vector-signal $\underline{x}(t)$ from all M transmitters/users (transmitting at the same time on the same frequency band) can be modelled as follows:.

$$\begin{aligned} \underline{x}(t) &= \sum_{i=1}^{M} \sum_{\ell=1}^{L} \underline{S}_{i\ell}.\beta_{i\ell}.m_i(t-\tau_\ell) + \underline{\mathrm{n}}(t) \\ &= \mathbb{S}\underline{m}(t) + \underline{\mathrm{n}}(t) \end{aligned} \quad (56)$$

with

$$\underline{S}_{il} \triangleq \underline{S}(\theta_{il}, \phi_{il})$$

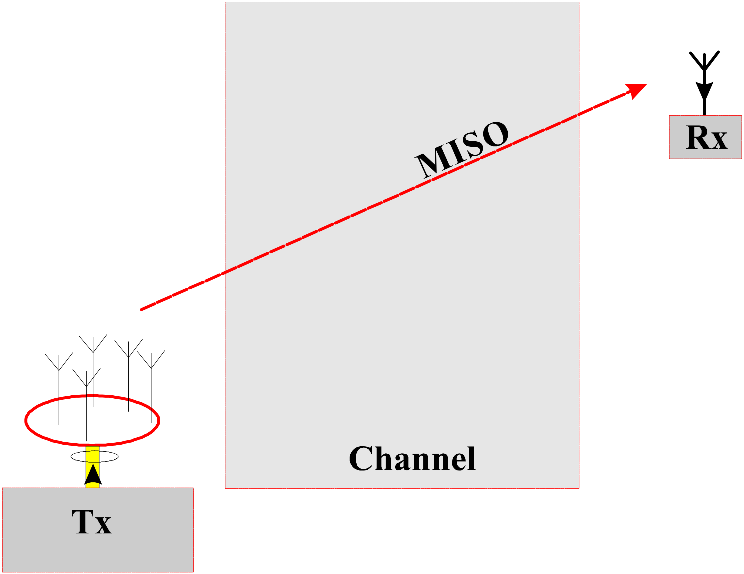
where $\mathbb{S}$, $\underline{m}(t)$ and $\underline{\mathrm{n}}(t)$ defined as follows:↘↘

$$\mathbb{S} \triangleq \left[\overbrace{\underline{S}_{11}, \underline{S}_{12}, ..., \underline{S}_{1L}}^{\triangleq \mathbb{S}_1}, \overbrace{\underline{S}_{21}, \underline{S}_{22}, ..., \underline{S}_{2L}}^{\triangleq \mathbb{S}_2}, ..., \overbrace{\underline{S}_{M1}, \underline{S}_{M2}, ..., \underline{S}_{ML}}^{\triangleq \mathbb{S}_M} \right] \quad (57)$$

$$\underline{m}(t) \triangleq \begin{bmatrix} \left.\begin{matrix} \beta_{11} m_1(t-\tau_{11}) \\ \beta_{12} m_1(t-\tau_{12}) \\ \vdots \\ \beta_{1L} m_1(t-\tau_{1L}) \end{matrix}\right\} \triangleq \underline{m}_1(t) \\ \beta_{21} m_2(t-\tau_{21}) \quad \vdots \\ \vdots \\ \left.\begin{matrix} \beta_{M1} m_M(t-\tau_{M1}) \\ \beta_{M2} m_M(t-\tau_{M2}) \\ \vdots \\ \beta_{ML} m_M(t-\tau_{ML}) \end{matrix}\right\} \triangleq \underline{m}_M(t) \end{bmatrix} \quad (58)$$

$$\underline{\mathrm{n}}(t) \triangleq \left[\mathrm{n}_1(t), \quad \mathrm{n}_2(t), \quad ..., \quad \mathrm{n}_N(t)\right]^T \quad (59)$$

Wireless MISO Channels



Reciprocity Theorem

- Antenna characteristics are independent of the direction of energy flow.
  - The impedance & radiation pattern are the same when the antenna radiates a signal and when it receives it.
- The Tx and Rx array-patterns are the same.
- The Tx-array is an array of $\overline{N}$ elements/sensors/antennas with locations $\underline{\underline{\overline{\mathbf{r}}}}$

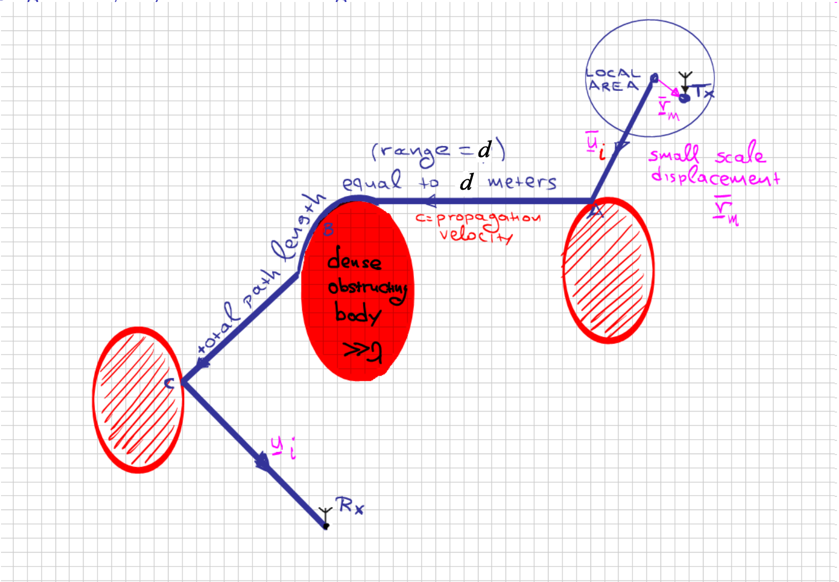
$$\underline{\underline{\overline{\mathbf{r}}}} = [\underline{\overline{r}}_1, \underline{\overline{r}}_2, ..., \underline{\overline{r}}_{\overline{N}}] = [\underline{\overline{r}}_x, \underline{\overline{r}}_y, \underline{\overline{r}}_z]^T \ (3 \times \overline{N})$$

with $\underline{\overline{r}}_m$ denoting the location of the m^{th} Tx-sensor $\forall m = 1, 2, ..., \overline{N}$

- Notation:
  - the bar at the top of a symbol, i.e. $\overline{(.)}$, denotes a Tx-parameter.

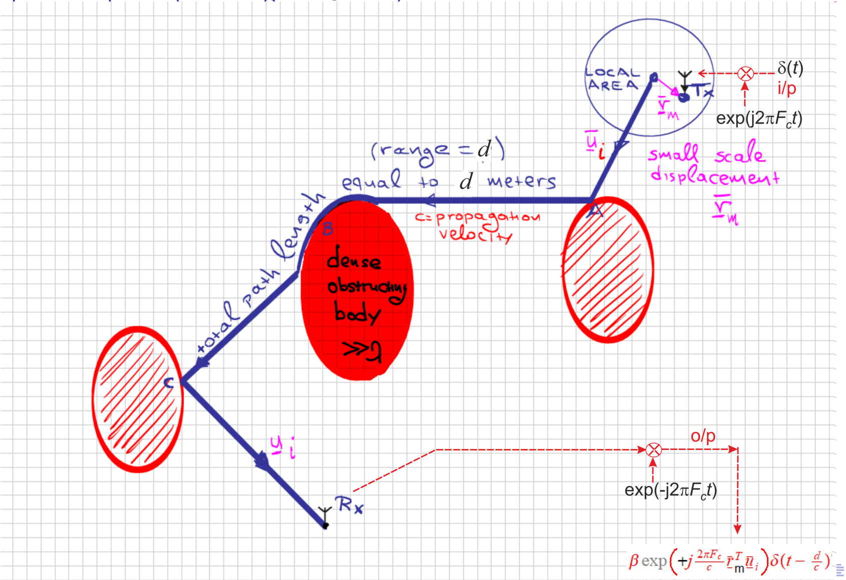
Single-path MISO Channel Modelling

Single Path/Ray of an Electromagnetic Wave:



Single-path MISO Channel Modelling

Impulse Response (Including the Carrier):



- That is, if the Tx is displaced at a specific point $\underline{\overline{r}}_m$ (within its local area $\overline{L}_A$) and the direction of the planewave propagation is described by the vector $\underline{\overline{u}}_i = \underline{u}(\overline{\theta}, \overline{\phi})$ where

$$\underline{u}_i(\overline{\theta},\overline{\phi}) = \left[\cos\overline{\theta}\cos\overline{\phi}, \quad \sin\overline{\theta}\cos\overline{\phi}, \quad \sin\overline{\phi}\right]^T$$

- **If** the Tx employs an array of $\overline{N}$ elements/sensors/antennas with locations $\underline{\underline{\overline{\mathbf{r}}}}$
  then the channel impulse response (single path) is

$$h(t) = \beta \underline{\overline{S}}^H \underline{\delta}(t - \frac{d}{c}) \tag{60}$$

  - where

$$\begin{aligned} \underline{\overline{S}}(\theta,\phi) &= \exp(+j[\underline{\overline{r}}_1, \underline{\overline{r}}_2, \ldots, \underline{\overline{r}}_N]^T \underline{k}(\overline{\theta},\overline{\phi})) && (61) \\ &\overset{or}{=} \exp(+j[\underline{\overline{r}}_x, \underline{\overline{r}}_y, \underline{\overline{r}}_z]\underline{k}(\overline{\theta},\overline{\phi})) && (62) \\ &= (\overline{N} \times 1) \text{ complex vector} \end{aligned}$$

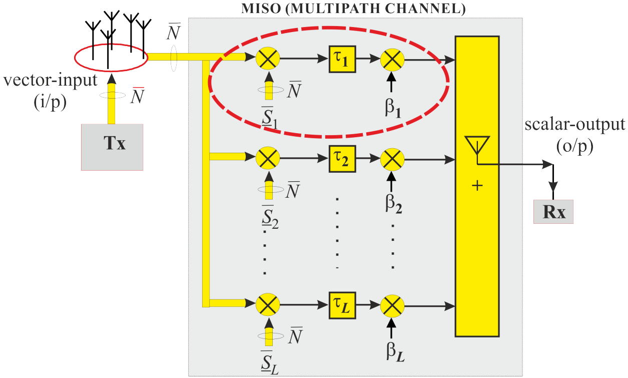
$$\underline{k}(\overline{\theta},\overline{\phi}) = \begin{cases} \frac{2\pi F_c}{c}.\underline{u}(\overline{\theta},\overline{\phi}) = \frac{2\pi}{\lambda_c}.\underline{u}(\overline{\theta},\overline{\phi}) & \text{in meters} \\ \pi.\underline{u}(\overline{\theta},\overline{\phi}) & \text{in units of halfwavelength} \end{cases}$$

Multi-path MISO Channel

- Let us assume that the transmitted signal(s) from the Tx-array arrives at the receiver via L resolvable paths (multipaths).
- Consider that the ℓ^{th} path's direction-of-departure is $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and propagates to the Rx with channel propagation parameters β_ℓ and τ_ℓ representing the complex path gain and path-delay, respectively.
- Let us assume that the L paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{63}$$

- remember that the path coefficients β_ℓ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter power



- The vector $\underline{S}_\ell = \underline{S}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in C^{\overline{N}}$ is the array manifold vector of the ℓ-th path .
- The impulse response of the MISO channel is

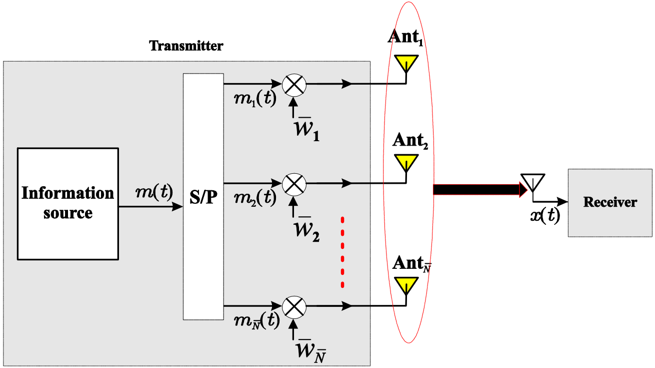
$$\boxed{\text{MISO: } h(t) = \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell^H \underline{\delta}(t - \tau_\ell)} \tag{64}$$

Modelling of the Rx Scalar-Signal x(t)

- Consider a Tx-array of $\overline{N}$ antennas transmitting a baseband signal $m(t)$ via MISO multipath channel of L resolvable paths (frequency selective MISO). We will consider the following two cases:
  - Case-1: The $m(t)$ is demultiplexed to $\overline{N}$ different signals (one signal per antenna element) forming the vector $\underline{m}(t)$.
  - Case-2: All Tx-array elements transmit the same signal $m(t)$.
- In both cases the transmitted signals may, or may not, be weighted.

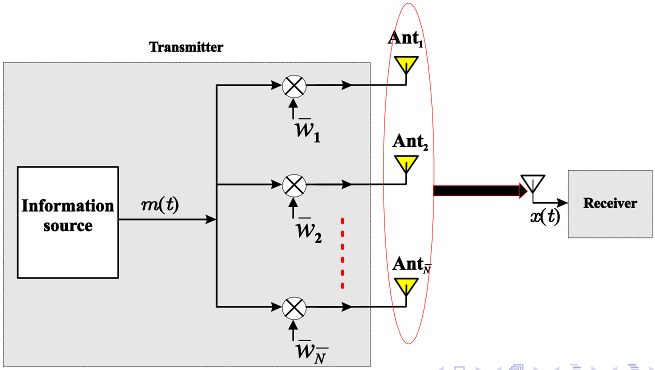
- Case-1: The received scalar-signal $x(t)$ can be modelled as follows:

$$\begin{aligned} x(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}^H(\overline{\theta}_\ell, \overline{\phi}_\ell).\left(\underline{\delta}(t-\tau_\ell) \circledast (\underline{w} \odot \underline{m}(t))\right) + \mathrm{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell^H.\left(\underline{w} \odot \underline{m}(t-\tau_\ell)\right) + \mathrm{n}(t) \end{aligned} \tag{65}$$



- Case-2: The signal is "copied" to each antenna and the received scalar-signal $x(t)$ can be modelled as follows:

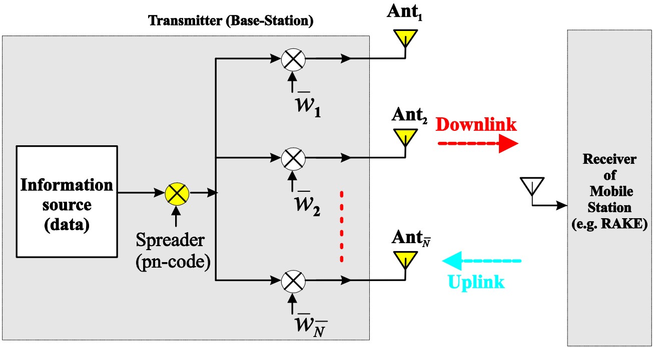
$$\begin{aligned} x(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{\overline{S}}^H(\overline{\theta}_\ell, \overline{\phi}_\ell).\left(\underline{\delta}(t-\tau_\ell) \circledast (\underline{\overline{w}} m(t))\right) + \mathrm{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underline{\overline{S}}_\ell^H \underline{\overline{w}}.m(t-\tau_\ell) + \mathrm{n}(t) \end{aligned} \tag{66}$$



Transmit Diversity

- Provides diversity benefits to a mobile using base station antenna array for frequency division duplexing (FDD) schemes. Cost is shared among different users.
- Order of diversity can be increased when used with other conventional forms of diversity (e.g. multipath diversity).
- Two main types of diversity combining techniques in 3G:
  - Transmit diversity with feedback from receiver (close loop)
  - Transmit diversity without feedback from receiver (open loop)

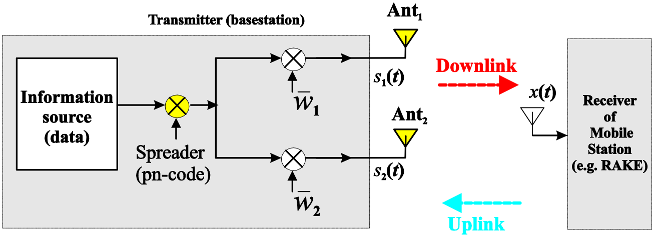
Transmit Diversity: "Close Loop"



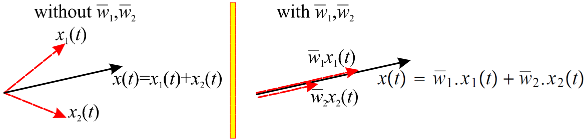
- The transmitter transmits some pilot signals
- The mobile (based on this pilot signals) estimates the Channel State Information (CSI), i.e. channel parameters.
- The mobile transmits the CSI to the BS (uplink)
- The base station generates the weights and transmits data to the mobile.

UMTS 3GPP Standard (Close Loop)

- $\overline{N} = 2$ **Tx-antennas**



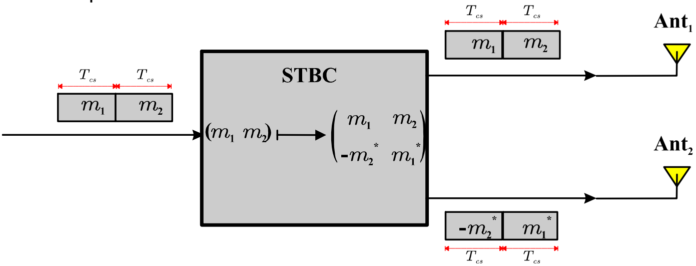
- where
  - $\overline{w}_1, \overline{w}_2$ are adjusted such as $|x(t)|^2$ is maximised, e.g.



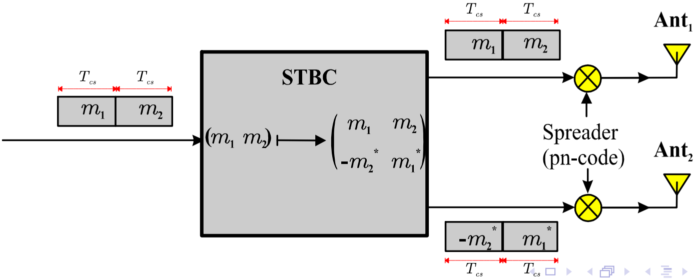
  - $\overline{w}_1, \overline{w}_2$ are adjusted based on the feedback information from the receiver

Transmit Diversity: "Open Loop"

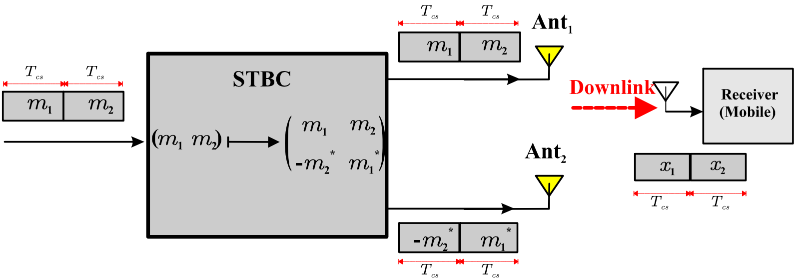
- without spreader:



- with spreader:



- Example of STBC Downlink Equations (without spreader):
  - STBC = Space-Time Block Code
  - No geometric/space information is used.



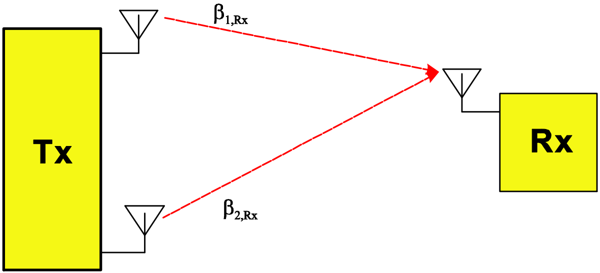
  - Receiver input (L unresolvable multipaths - flat fading):

$$\begin{cases} \text{1st interval: } x_1 = \sum_{j=1}^{L} \left(\beta_{1j,Rx} m_1 - \beta_{2j,Rx} m_2^* \right) + n_1 \\ \text{2nd interval: } x_2 = \sum_{j=1}^{L} \left(\beta_{1j,Rx} m_2 + \beta_{2j,Rx} m_1^* \right) + n_2 \end{cases} \tag{67}$$

$$\Rightarrow \begin{cases} x_1 = \beta_{1,Rx} m_1 - \beta_{2,Rx} m_2^* + \mathrm{n}_1 \\ x_2 = \beta_{1,Rx} m_2 + \beta_{2,Rx} m_1^* + \mathrm{n}_2 \end{cases} \tag{68}$$

where

$$\beta_{1,Rx} \equiv \sum_{j=1}^{L} \beta_{1j,Rx} \quad \text{and} \quad \beta_{2,Rx} \equiv \sum_{j=1}^{L} \beta_{2j,Rx} \tag{69}$$



- Equivalently,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_1, & -m_2^* \\ m_2, & m_1^* \end{bmatrix} \underbrace{\begin{bmatrix} \beta_{1,Rx} \\ \beta_{2,Rx} \end{bmatrix}}_{\triangleq \underline{\beta}_{Rx}} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{70}$$

or, in an alternative format:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_{1,Rx}, & -\beta_{2,Rx} \\ \beta_{2,Rx}^*, & \beta_{1,Rx}^* \end{bmatrix}}_{\triangleq \mathbb{H}} \underbrace{\begin{bmatrix} m_1 \\ m_2^* \end{bmatrix}}_{\triangleq \underline{m}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\triangleq \underline{n}} \tag{71}$$

i.e.

$$\underline{x} = \mathbb{H}\underline{m} + \underline{n} \tag{72}$$

where

$$\mathbb{H}^H\mathbb{H} = \underbrace{\left(\left| \beta_{1,Rx} \right|^2 + \left| \beta_{2,Rx} \right|^2 \right)}_{= \left\| \underline{\beta}_{Rx} \right\|^2} \mathbb{I}_2$$

- Decoder (Rx): This is denoted by the matrix $\mathbb{H}$

$$\text{decoder's o/p}: \underline{G} = \mathbb{H}^H \underline{x} \tag{73}$$

$$= \mathbb{H}^H \mathbb{H} \underline{m} + \underbrace{\mathbb{H}^H \underline{\mathrm{n}}}_{\triangleq \underline{\tilde{\mathrm{n}}}} \tag{74}$$

- That is, the decision variables are

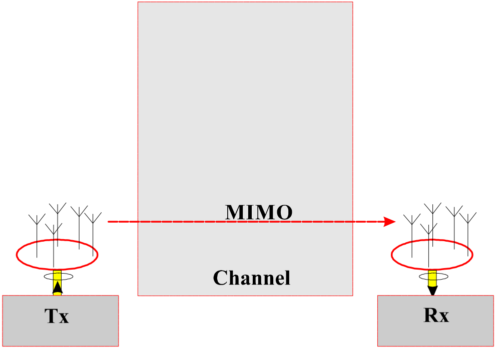
$$\underline{G} = \left\|\underline{\beta}_{Rx}\right\|^2 \underline{m} + \underline{\tilde{\mathrm{n}}} \tag{75}$$

i.e.

$$\left\{ \begin{array}{l} G_1 = \left\|\underline{\beta}_{Rx}\right\|^2 m_1 + \tilde{\mathrm{n}}_1 \\ G_2 = \left\|\underline{\beta}_{Rx}\right\|^2 m_2^* + \tilde{\mathrm{n}}_2 \end{array} \right| \tag{76}$$

- Note:
  1. the receiver needs to know (estimate) the channel weights $\beta_{1,Rx}$ and $\beta_{2,Rx}$ but there is no need to send them back to the transmitter (i.e. open loop)
  2. $\beta_{1,Rx}$ and $\beta_{2,Rx}$ can be estimated by transmitting some pilot symbols as m_1 and m_2 and, then, using Equation 70

Wireless MIMO Channels



- Consider a single path from a Tx-array of $\overline{N}$ antennas to an Rx-array of N antennas with locations given by the matrices

$$\text{Tx-array:} \qquad \underline{\underline{\overline{\mathbf{r}}}} = [\underline{\overline{r}}_1, \underline{\overline{r}}_2, \ldots, \underline{\overline{r}}_{\overline{N}}] = \left[\underline{\overline{r}}_{\mathbf{x}}, \underline{\overline{r}}_{\mathbf{y}}, \underline{\overline{r}}_{\mathbf{z}}\right]^T \quad (3 \times \overline{N})$$

$$\text{Rx-array:} \qquad \underline{\underline{\mathbf{r}}} = [\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N] = \left[\underline{r}_{\mathbf{x}}, \underline{r}_{\mathbf{y}}, \underline{r}_{\mathbf{z}}\right]^T \quad (3 \times N)$$

- If the direction-of-departure of this single path planewave propagation is $(\overline{\theta}, \overline{\phi})$ and the direction-of-arrival is (θ, ϕ) then the impulse response is

$$\underline{h}(t) = \beta \underline{S}(\theta, \phi).\underline{\overline{S}}^H(\theta, \phi).\underline{\delta}(t - \frac{\rho}{c})$$

where

$$\text{Tx:} \qquad \underline{\overline{S}} = \underline{\overline{S}}\left(\overline{\theta}, \overline{\phi}\right) = \exp\left(+j\underline{\underline{\overline{\mathbf{r}}}}^T \underline{k}(\overline{\theta}, \overline{\phi})\right) \tag{77}$$

$$\text{Rx:} \qquad \underline{S} = \underline{S}\left(\theta, \phi\right) = \exp\left(-j\underline{\underline{\mathbf{r}}}^T \underline{k}(\theta, \phi)\right) \tag{78}$$

with $\underline{k}(\overline{\theta}, \overline{\phi})$ and $\underline{k}(\theta, \phi)$ denote the wavenumber vectors of the Tx-array and Rx-array respectively

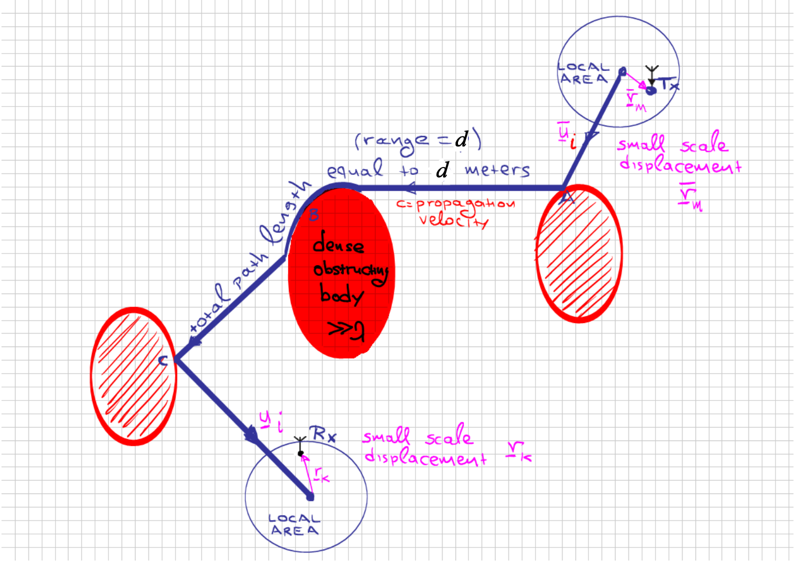
- For instance $\underline{k}(\overline{\theta}, \overline{\phi})$ is defined as

$$\underline{k}(\overline{\theta}, \overline{\phi}) = \left\{ \begin{array}{ll} \frac{2\pi F_c}{c}.\underline{u}(\overline{\theta}, \overline{\phi}) = \frac{2\pi}{\lambda_c}.\underline{u}(\overline{\theta}, \overline{\phi}) & \text{in meters} \\ \pi.\underline{u}(\overline{\theta}, \overline{\phi}) & \text{in units of halfwavelength} \end{array} \right\}$$

$$\underline{u}\left(\overline{\theta}, \overline{\phi}\right) = \left[\cos\overline{\theta}\cos\overline{\phi}, \quad \sin\overline{\theta}\cos\overline{\phi} \quad \sin\overline{\phi}\right]$$

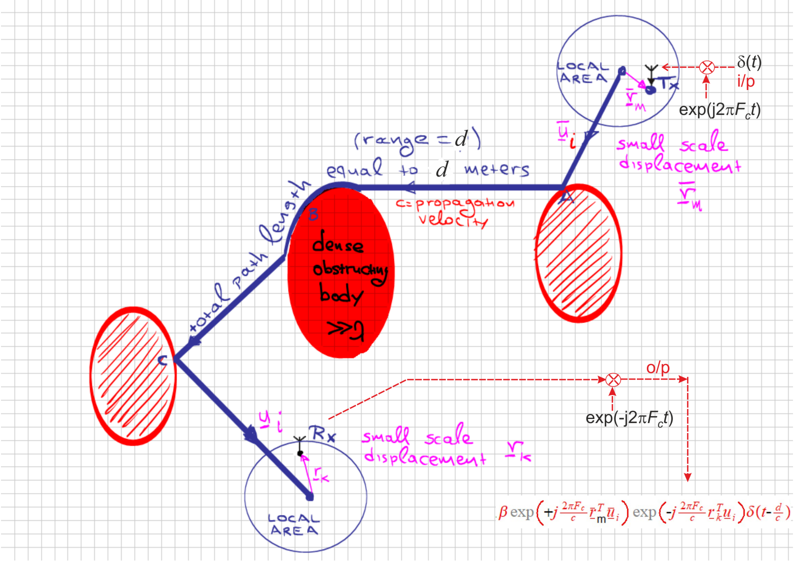
Single-path MIMO Channel Modelling

Single Path/Ray of an Electromagnetic Wave:



Single-path MIMO Channel Modelling

Impulse Response (Including the Carrier):

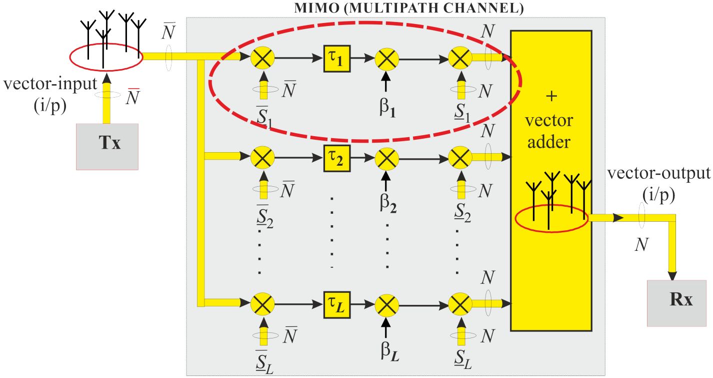


Multipath MIMO Channel

- Let us assume that the transmitted signal(s) from the Tx-array arrives at the Rx-array via L resolvable paths (multipaths).
- Consider that the ℓ^{th} path's direction-of-departure is $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and propagates to the Rx with channel propagation parameters β_ℓ and τ_ℓ representing the complex path gain and path-delay, respectively.
- Let us assume that the L paths are arranged such that

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_\ell \leq \ldots \leq \tau_L \tag{79}$$

- once again, remember that the path coefficients β_ℓ model the effects of path losses and shadowing, in addition to random phase shifts due to reflection; they also encompass the effects of the phase offset between the modulating carrier at the transmitter and the demodulating carrier at the receiver, as well as differences in the transmitter power.
- In the following figure the vectors $\overline{\underline{S}}_\ell = \overline{\underline{S}}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in C^{\overline{N}}$ and $\underline{S}_\ell = \underline{S}(\theta_\ell, \phi_\ell) \in C^N$ are the Tx- and Rx- array manifold vectors of the ℓ-th path.



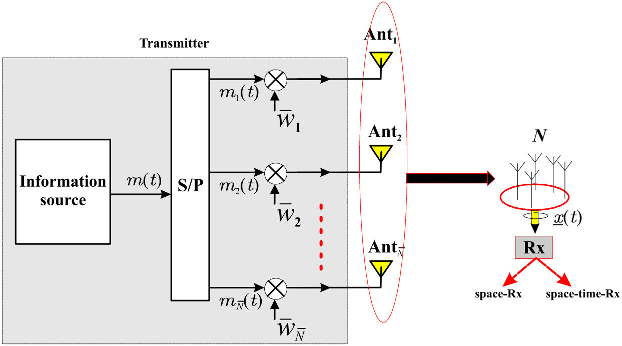
- The impulse response of the MIMO channel is

$$\boxed{\text{MIMO: } \underline{h}(t) = \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \underline{\bar{S}}_\ell^H \delta(t - \tau_\ell)} \tag{80}$$

Modelling of the Rx Vector-Signal x(t)

- Consider a Tx-array of $\overline{N}$ antennas transmitting a baseband signal $m(t)$ via MIMO multipath channel of L resolvable paths (frequency selective MIMO).
- We will consider the following two cases:
  - Case-1: The $m(t)$ is demultiplexed to $\overline{N}$ different signals (one signal per antenna element) forming the vector $\underline{m}(t)$.
  - Case-2: All Tx-array elements transmit the same signal $m(t)$.
- In both cases the transmitted signals may, or may not, be weighted. The Rx is also equipped with an array of N antennas.

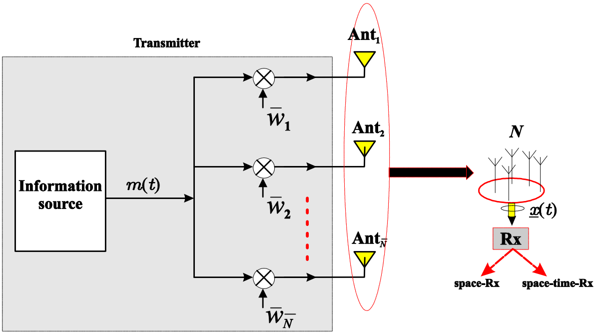
Case-1:



- The received vector-signal $\underline{x}(t)$ can be modelled as follows:

$$\begin{aligned} \underline{x}(t) &= \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \overline{\underline{S}}_\ell^H . \left(\underline{\delta}(t-\tau_\ell) \circledast (\overline{\underline{w}} \odot \underline{m}(t)) \right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell . \underline{S}_\ell . \overline{\underline{S}}_\ell^H . \left(\overline{\underline{w}} \odot \underline{m}(t-\tau_\ell) \right) + \underline{n}(t) \end{aligned} \tag{81}$$

Case-2:



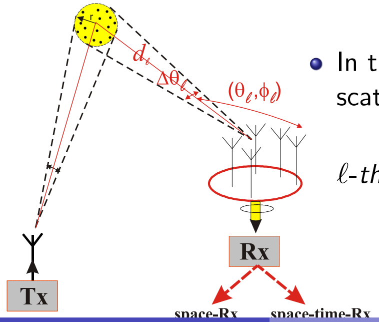
- In this case the signal is "copied" to each antenna and may be weighted by a complex weight.
- The received vector-signal $\underline{x}(t)$ can be modelled as follows:

$$\begin{aligned}\underline{x}(t) &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell \overline{\underline{S}}_\ell^H . \left(\underline{\delta}(t-\tau_\ell) \circledast (\underline{w} m(t))\right) + \underline{n}(t) \\ &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell \overline{\underline{S}}_\ell^H \underline{w}. m(t-\tau_\ell) + \underline{n}(t) \end{aligned} \tag{82}$$

SIMO Channels: Multipath Clusterring

- Consider a scatterer (ℓ-*th* scatterer, say) which can be seen as a large number of paths (L_{scat}, say) around the direction (θ_ℓ, ϕ_ℓ). That is, the direction-of-arrival of the k-th path satisfies the condition

$$(\theta_\ell, \phi_\ell) - \frac{(\Delta\theta_\ell, \Delta\phi_\ell)}{2} \leq (\theta_{\ell k}, \phi_{\ell k}) \leq (\theta_\ell, \phi_\ell) + \frac{(\Delta\theta_\ell, , \Delta\phi_\ell)}{2}, \forall k \quad (83)$$



- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-}th \text{ scatterer} = \sum_{k=1}^{L_{scat}} \underline{S}(\theta_{\ell k}, \phi_{\ell k}).\beta_{\ell k}.\delta(t - \tau_{\ell k}) \quad (84)$$

- If $\boxed{\Delta\theta_\ell = \text{relatively small}}$ then

$$(\theta_{\ell 1}, \phi_{\ell 1}) \simeq (\theta_{\ell 2}, \phi_{\ell 2}) \simeq \ldots \simeq (\theta_{\ell L_{scat}}, \phi_{\ell L_{scat}}) \triangleq (\theta_\ell, \phi_\ell) \quad (85)$$

$$\tau_{\ell 1} \simeq \tau_{\ell k} \simeq \ldots \simeq \tau_{\ell L_{scat}} \triangleq \tau_\ell \quad (86)$$

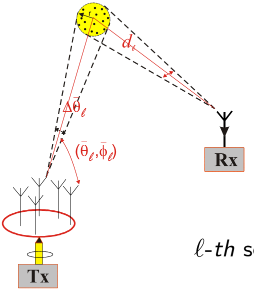
and, thus,

$$\begin{aligned}
\ell\text{-}th \text{ scatterer} &= \sum_{k=1}^{L_{scat}} \underline{S}(\theta_{\ell k}, \phi_{\ell k}).\beta_{\ell k}.\delta(t - \tau_{\ell k}) \\
&= \sum_{k=1}^{L_{scat}} \underline{S}(\theta_\ell, \phi_\ell).\beta_{\ell k}.\delta(t - \tau_\ell) \\
&= \underline{S}(\theta_\ell, \phi_\ell).\delta(t - \tau_\ell) \underbrace{\sum_{k=1}^{L_{scat}} \beta_{\ell k}}_{\triangleq \beta_\ell} \\
&= \underline{S}(\theta_\ell, \phi_\ell).\beta_\ell.\delta(t - \tau_\ell) \\
&= \underline{S}_\ell.\beta_\ell.\delta(t - \tau_\ell) \quad (87)
\end{aligned}$$

MISO Channels: Multipath Clustering

- Consider a scatterer (ℓ-*th* scatterer, say) which can be seen as a large number of paths (L_ℓ, say) around the direction $(\overline{\theta}_\ell, \overline{\phi}_\ell)$. That is, the direction-of-arrival of the k-th path satisfies the condition

$$(\overline{\theta}_\ell, \overline{\phi}_\ell) - \frac{(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)}{2} \leq (\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) \leq (\overline{\theta}_\ell, \overline{\phi}_\ell) + \frac{(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)}{2}, \forall k \quad (88)$$



- L_ℓ= number of paths (ℓ^{th} scatterer)
- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-}th \text{ scatterer} = \sum_{k=1}^{L_\ell} \beta_{\ell k} . \underline{S}^H(\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) . \underline{\delta}(t - \tau_{\ell k}) \quad (89)$$

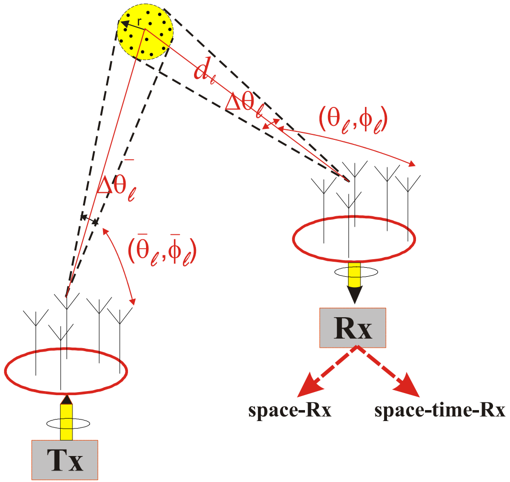
- **If** $\overline{\Delta\theta_\ell}$ = relatively small **then**

$$
\begin{aligned}
(\overline{\theta}_{\ell 1},\overline{\phi}_{\ell 1}) &\simeq (\overline{\theta}_{\ell 2},\overline{\phi}_{\ell 2}) \simeq \ldots \simeq (\overline{\theta}_{\ell L_\ell},\overline{\phi}_{\ell L_\ell}) \triangleq (\overline{\theta}_{\ell},\overline{\phi}_{\ell}) && \text{(90a)}\\
\tau_{\ell 1} &\simeq \tau_{\ell k} \simeq \ldots \simeq \tau_{\ell L_\ell} \triangleq \tau_\ell && \text{(90b)}
\end{aligned}
$$

and, thus,

$$
\begin{aligned}
\ell\text{-}th \text{ scatterer} &= \sum_{k=1}^{L_\ell} \beta_{\ell k} \underline{\overline{S}}^H(\overline{\theta}_{\ell k},\overline{\phi}_{\ell k}).\underline{\delta}(t-\tau_{\ell k})\\
&= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell})\\
&= \underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell}) \underbrace{\sum_{k=1}^{L_\ell} \beta_{\ell k}}_{\triangleq \beta_\ell}\\
&= \underline{\overline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell})\underline{\delta}(t-\tau_{\ell}).\beta_\ell\\
&= \beta_\ell.\underline{\overline{S}}_\ell^H.\underline{\delta}(t-\tau_{\ell}) && \text{(91)}
\end{aligned}
$$

MIMO Channels: Multipath Clustering



- Consider a scatterer (ℓ-*th* scatterer, say) which can be seen as a large number of paths (L_ℓ, say) with directions-of-departure around the direction $(\bar{\theta}_\ell, \bar{\phi}_\ell)$ and directions-of-arrival around the direction (θ_ℓ, ϕ_ℓ).

- That is, the direction-of-departure and direction-of-arrival of the k-th path satisfies the condition

$$(\overline{\theta}_\ell, \overline{\phi}_\ell) - \frac{(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)}{2} \leq (\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) \leq (\overline{\theta}_\ell, \overline{\phi}_\ell) + \frac{(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)}{2}, (\forall k) \tag{92}$$

$$(\theta_\ell, \phi_\ell) - \frac{(\Delta\theta_\ell, \Delta\phi_\ell)}{2} \leq (\theta_{\ell k}, \phi_{\ell k}) \leq (\theta_\ell, \phi_\ell) + \frac{(\Delta\theta_\ell, \Delta\phi_\ell)}{2}, (\forall k) \tag{93}$$

- In this case the impulse response for this scatterer can be written as follows:

$$\ell\text{-th scatterer} = \sum_{k=1}^{L_\ell} \beta_{\ell k} . \underline{S}(\theta_{\ell k}, \phi_{\ell k}) \underline{\overline{S}}^H (\overline{\theta}_{\ell k}, \overline{\phi}_{\ell k}) . \underline{\delta}(t - \tau_{\ell k}) \tag{94}$$

- If $\boxed{\Delta\theta_\ell \text{ and } \overline{\Delta\theta}_\ell \text{ are relatively small}}$ then

$$(\overline{\theta}_{\ell 1}, \overline{\phi}_{\ell 1}) \simeq (\overline{\theta}_{\ell 2}, \overline{\phi}_{\ell 2}) \simeq \ldots \simeq (\overline{\theta}_{\ell L_\ell}, \overline{\phi}_{\ell L_\ell}) \triangleq (\overline{\theta}_\ell, \overline{\phi}_\ell) \tag{95}$$

$$(\theta_{\ell 1}, \phi_{\ell 1}) \simeq (\theta_{\ell 2}, \phi_{\ell 2}) \simeq \ldots \simeq (\theta_{\ell L_\ell}, \phi_{\ell L_\ell}) \triangleq (\theta_\ell, \phi_\ell) \tag{96}$$

$$\tau_{\ell 1} \simeq \tau_{\ell k} \simeq \ldots \simeq \tau_{\ell L_\ell} \triangleq \tau_\ell \tag{97}$$

- Thus, if L_ℓ denotes the No. of paths of the ℓ-*th* scatterer,

$$\begin{aligned}
\ell\text{-}th\text{ scatterer} &= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{S}(\theta_{\ell k},\phi_{\ell k})\overline{\underline{S}}^H(\overline{\theta}_{\ell k},\overline{\phi}_{\ell k}).\underline{\delta}(t-\tau_{\ell k}) \\
&= \sum_{k=1}^{L_\ell} \beta_{\ell k}.\underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell}) \\
&= \underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell})\underbrace{\sum_{k=1}^{L_\ell} \beta_{\ell k}}_{\triangleq\beta_\ell} \qquad (98) \\
&= \beta_\ell.\underline{S}(\theta_{\ell},\phi_{\ell})\overline{\underline{S}}^H(\overline{\theta}_{\ell},\overline{\phi}_{\ell}).\underline{\delta}(t-\tau_{\ell}) \\
&= \beta_\ell.\underline{S}_\ell.\overline{\underline{S}}_\ell^H.\underline{\delta}(t-\tau_{\ell}) \qquad (99)
\end{aligned}$$

Summary

- From Equations 87, 91 and 99 a scatterer can be seen as a single path
  - (for SIMO) with direction of arrival (θ_ℓ, ϕ_ℓ) and fading coefficient the term β_ℓ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e. $\beta_\ell = \sum_{k=1}^{L_{scat}} \beta_{\ell k}$.
  - (for MISO) with direction of departure $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and fading coefficient the term β_ℓ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e. $\beta_\ell = \sum_{k=1}^{L_\ell} \beta_{\ell k}$.
  - (for MIMO) with directions of departure $(\overline{\theta}_\ell, \overline{\phi}_\ell)$ and (θ_ℓ, ϕ_ℓ) as well as with fading coefficient the term β_ℓ that represents the addition/combination of the fading coefficients of all paths of this scatterer i.e. $\beta_\ell = \sum_{k=1}^{L_\ell} \beta_{\ell k}$.

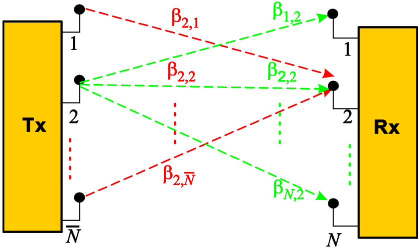
N.B.:

- Another way to represent a scatterer is using Taylor Series Expansion. This will involve:
  - the manifold vector $\underline{S}_\ell$, its first derivative $\dot{\underline{S}}$ and $(\Delta\theta_\ell, \Delta\phi_\ell)$, for SIMO;
  - the manifold vector $\overline{\underline{S}}_\ell$, its first derivative $\dot{\overline{\underline{S}}}$ and $(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)$, for MISO;
  - the two manifold vectors $\underline{S}_\ell$, $\overline{\underline{S}}_\ell$ and their first derivatives $\dot{\underline{S}}$, $\dot{\overline{\underline{S}}}$ as well as $(\Delta\theta_\ell, \Delta\phi_\ell)$ and $(\overline{\Delta\theta}_\ell, \overline{\Delta\phi}_\ell)$ for MIMO.

$\square$

MIMO Systems (without geometric information)

- Let us consider a comm. system with multiple antennas at both the Tx and the Rx.



$$\mathbb{H} = \begin{array}{c} \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,\overline{N}} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,\overline{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N,1} & \beta_{N,2} & \cdots & \beta_{N,\overline{N}} \end{bmatrix} \\ \begin{array}{cccc} \underline{\beta}_{1,\mathrm{Tx}} & \underline{\beta}_{2,\mathrm{Tx}} & & \underline{\beta}_{\overline{N},\mathrm{Tx}} \end{array} \end{array}$$

- $\beta_{i,j}$ = gain from the j^{th} Tx-antenna to the i^{th} Rx-antenna
  $\underline{\beta}_{j,Tx}$ = gain-vector with its i^{th} element the gain β_{ij}
  - (i.e. from the j-th Tx antenna to all the Rx antennas)
- If Rx is synchronised to the Tx then for the n^{th} data symbol interval then we have the following received vector-signal:

$$\underline{x}[n] = \mathbb{H}\underline{m}[n] + \underline{\mathrm{n}}[n] \quad (N \times 1)$$

  where

$$\mathbb{H} = \left[\underline{\beta}_{1,Tx}, \underline{\beta}_{2,Tx}, \ldots, \underline{\beta}_{\overline{N},Tx}\right]$$

- Second order statistics (covariance matrix) of $\underline{x}[n]$ is as follows:

$$\mathbb{R}_{xx} = \mathbb{H}\mathbb{R}_{mm}\mathbb{H}^H + \underbrace{\mathbb{R}_{\mathrm{nn}}}_{\sigma_n^2 \mathbb{I}_N} \quad (N \times N)$$

- In a MIMO system it is assumed that the Matrix $\mathbb{H}$ is known
- Note:
  - The matrix $\mathbb{H}$ and the manifold vectors are related according to the following expression

$$\begin{aligned}\mathbb{H} &= \sum_{\ell=1}^{L} \beta_\ell \underline{S}_\ell \underline{\overline{S}}_\ell^H && (100)\\ &= \mathbb{S}\begin{bmatrix}\beta_1, & 0, & ..., & 0\\ 0, & \beta_2, & ..., & 0\\ ..., & ..., & ..., & ...\\ 0, & 0, & ..., & \beta_L\end{bmatrix}\overline{\mathbb{S}}^H && (101)\end{aligned}$$

where

$$\begin{aligned}\mathbb{S} &= \left[\underline{S}_1, \quad \underline{S}_2, \quad ..., \quad \underline{S}_L\right]\\ \underline{\overline{\mathbb{S}}} &= \left[\underline{\overline{S}}_1, \quad \underline{\overline{S}}_2, \quad ..., \quad \underline{\overline{S}}_L\right]\end{aligned}$$

Capacity - of MIMO Channels (without space info)

- SISO Capacity:

$$C = B\log_2\left(1 + \text{SNIR}_{in}\right) \text{ bits/s} \tag{102}$$

- General MIMO Capacity expression:

$$C = B\log_2\left(\frac{\det\left(\mathbb{R}_{xx}\right)}{\det\left(\mathbb{R}_{\text{nn}}\right)}\right) \text{ bits/s} \tag{103}$$

- Based on Equation 103 it can be easily proven that

$$C = B\log_2\left(\det\left(\mathbb{I}_N + \frac{1}{\sigma_n^2}\mathbb{H}\mathbb{R}_{mm}\mathbb{H}^H\right)\right) \text{ bits/s} \tag{104}$$

- Furthermore, for independent parallel channels (i.e. using a multiplexer at Tx, Case-1) :

$$\mathbb{R}_{mm} = \text{diagonal} = \begin{bmatrix} P_1, & 0, & \ldots, & 0 \\ 0, & P_2, & \ldots, & 0 \\ \ldots, & \ldots, & \ldots, & \ldots \\ 0, & 0, & \ldots, & P_{\overline{N}} \end{bmatrix}$$

  and thus Equation 104 is simplified to

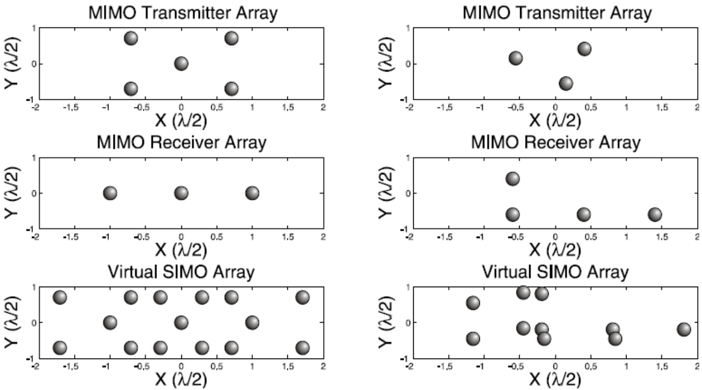
$$C = B\log_2\left(\prod_{j=1}^{\overline{N}}\left(1 + \frac{\left\|\underline{\beta}_{j,Tx}\right\|^2 P_j}{\sigma_n^2}\right)\right) \tag{105}$$

$$= B\sum_{j=1}^{\overline{N}}\log_2\left(1 + \frac{\left\|\underline{\beta}_{j,Tx}\right\|^2 P_j}{\sigma_n^2}\right) \text{ bits/sec} \tag{106}$$

Equivalence between MIMO and SIMO

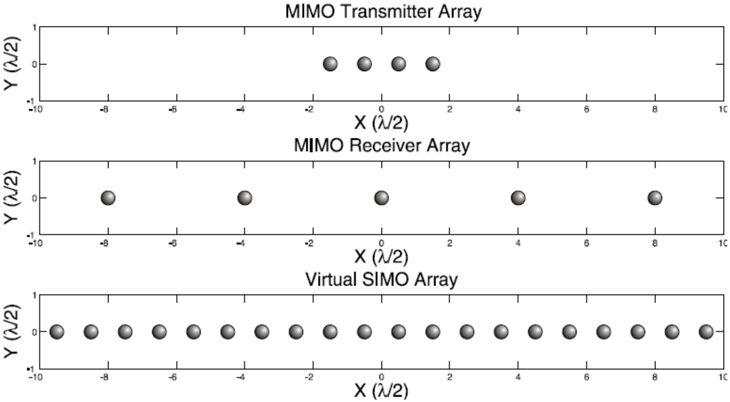
Spatial Convolution and Virtual Antenna Array

MIMO Tx antennas: $\overline{N}$, Rx antennas: N	$\Longleftrightarrow$	virtual-SIMO $\overline{N} \times N$ Rx /Tx antennas



Equivalence between MIMO and SIMO (cont.)

Spatial Convolution and Virtual Antenna Array



Spatial Convolution and Virtual Antenna Array

$$\text{Tx-array:} \quad \underline{\underline{\overline{\mathbf{r}}}} \triangleq [\underline{\overline{r}}_1, \underline{\overline{r}}_2, \ldots, \underline{\overline{r}}_{\overline{N}}] = [\underline{\overline{r}}_{\mathbf{x}}, \underline{\overline{r}}_{\mathbf{y}}, \underline{\overline{r}}_{\mathbf{z}}]^T \quad (3 \times \overline{N})$$

$$\text{Rx-array:} \quad \underline{\underline{\mathbf{r}}} \triangleq [\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N] = [\underline{r}_{\mathbf{x}}, \underline{r}_{\mathbf{y}}, \underline{r}_{\mathbf{z}}]^T \quad (3 \times N)$$

$$\text{virtual-array:} \quad \underline{\underline{\mathbf{r}}}_{\text{virtual}} \triangleq \underline{\underline{\overline{\mathbf{r}}}} \otimes \underline{1}_N^T + \underline{1}_{\overline{N}}^T \otimes \underline{\underline{\mathbf{r}}} \tag{107}$$

$$\Updownarrow$$

$$\underline{S}_{\text{virtual}} \triangleq \underline{\overline{S}}^* \otimes \underline{S} \tag{108}$$

N.B.:

1. Equation 107 is known as "spatial convolution"
2. The concept of the "virtual array", is also applicable to MISO
3. A MIMO is equivalent to a "virtual-MISO" (by virtually transfering the Tx antenna array to the Rx) or to a "virtual-SIMO" (by virtually transfering the Rx antenna array to the Tx).