10: Digital Filter Structures

Direct Forms

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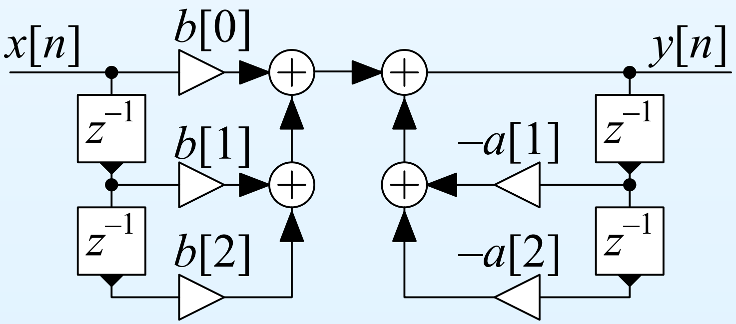

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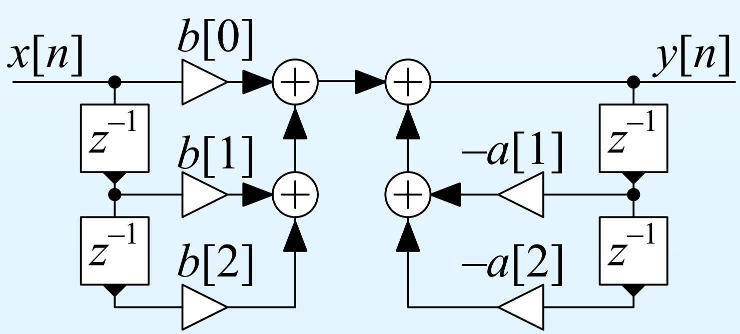

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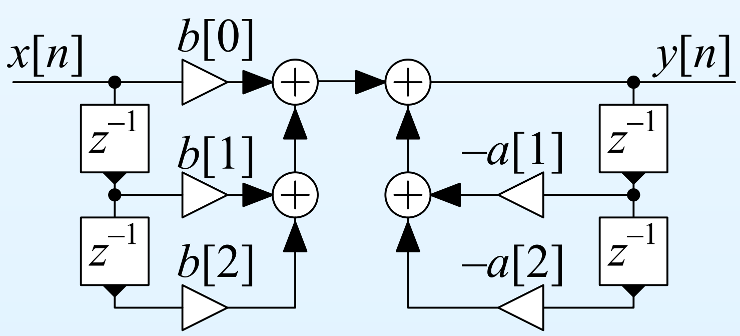

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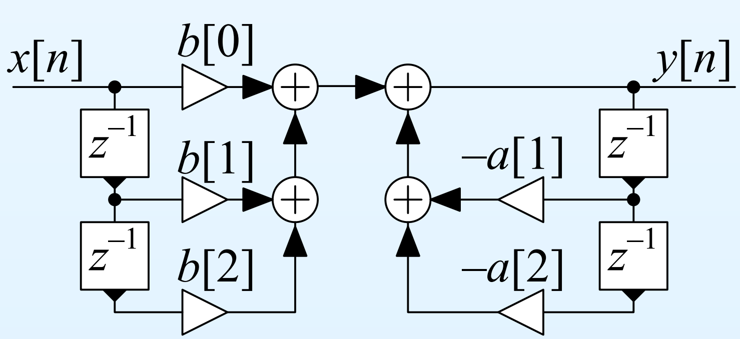

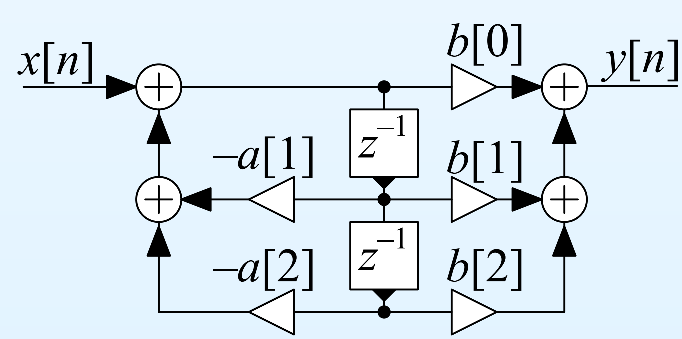

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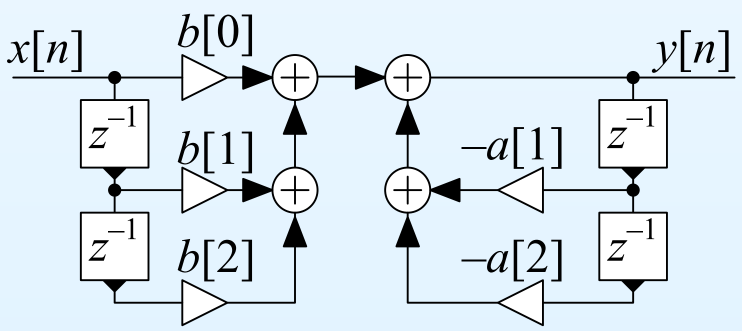

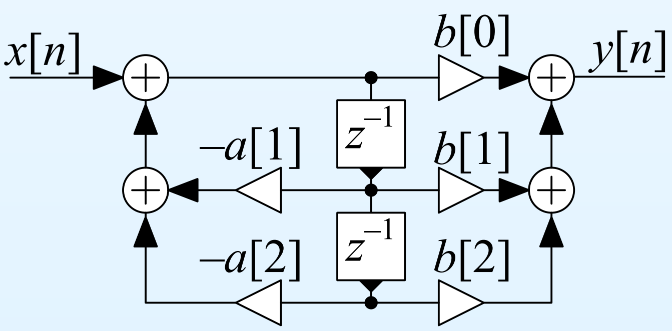

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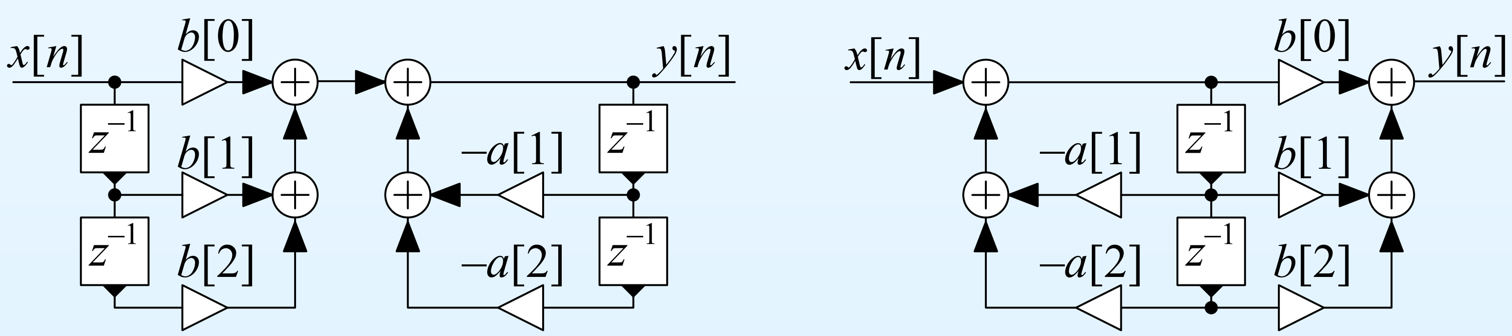

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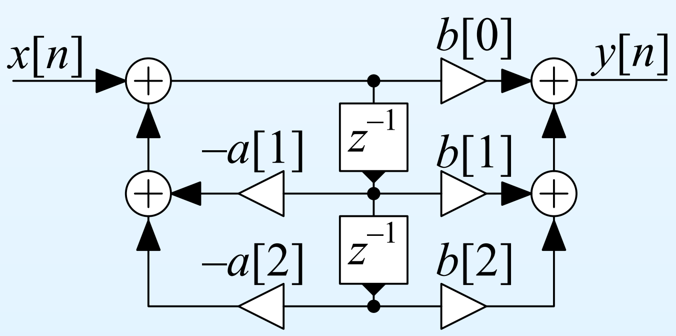

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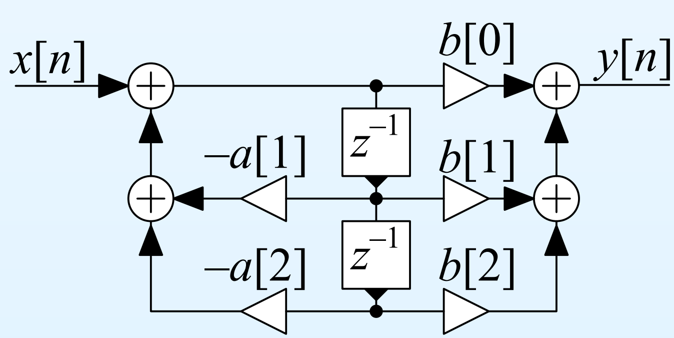

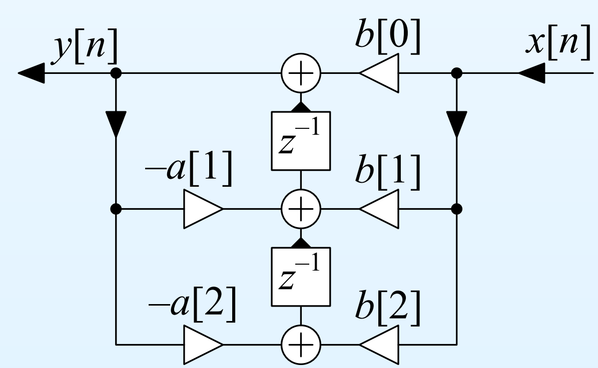

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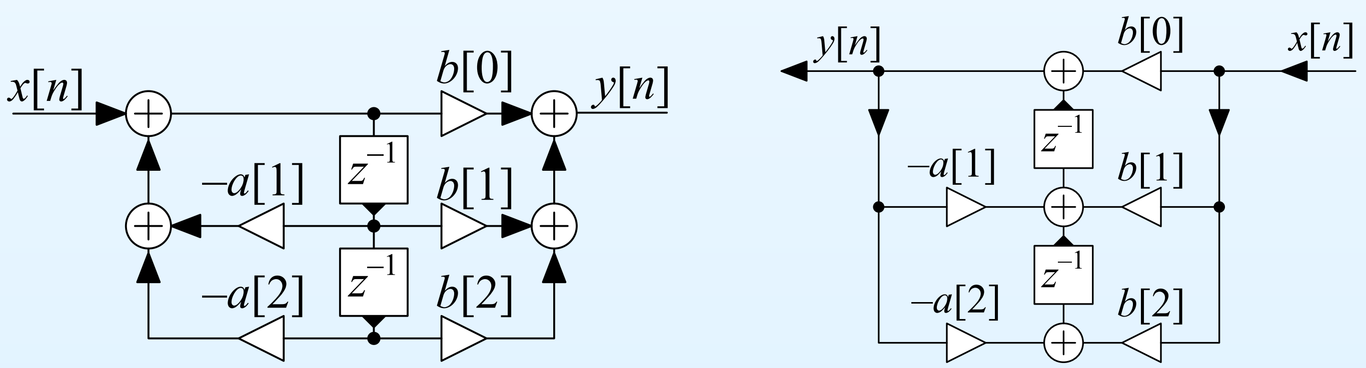

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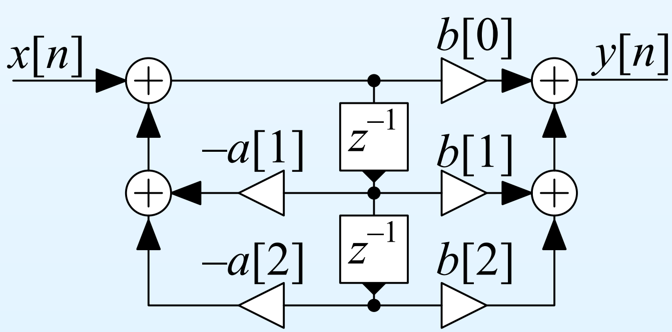
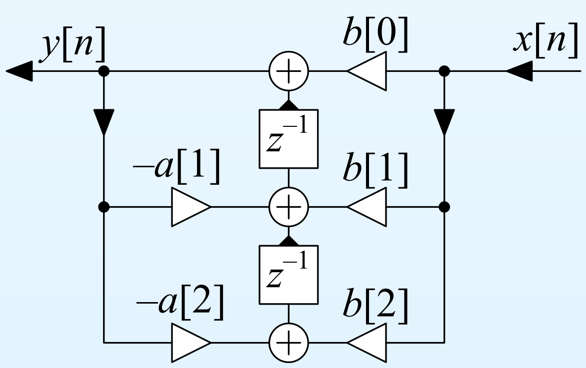
Note: A valid block diagram must never have any feedback loops that don't go through a delay (z^{-1} block).





State Space

$\mathbf{v}[n]$ is a vector of delay element outputs

Can write: $\mathbf{v}[n+1] = \mathbf{P}\mathbf{v}[n] + \mathbf{q}x[n]$

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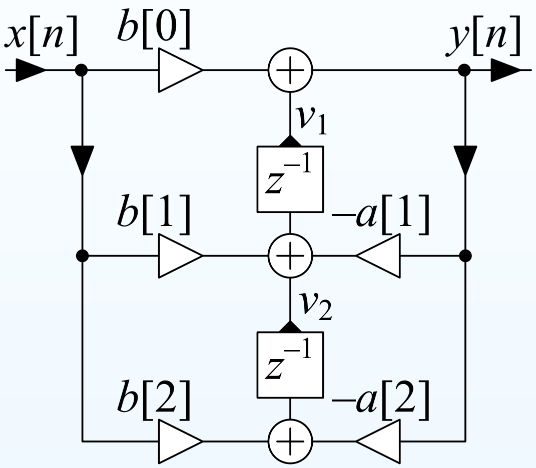

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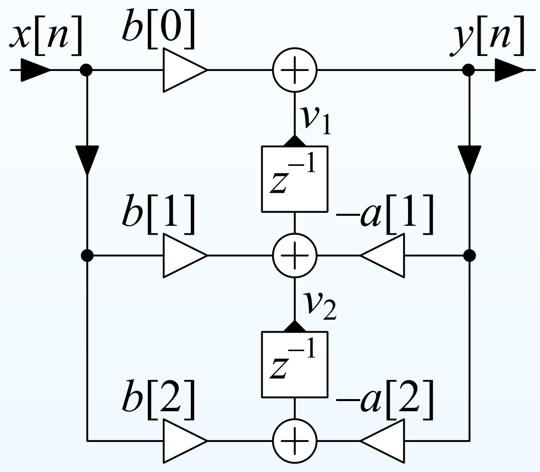

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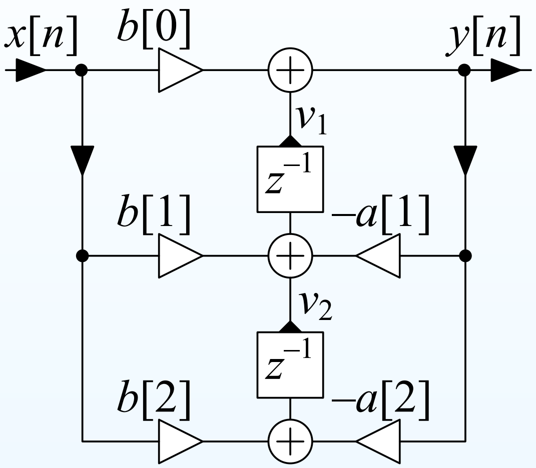

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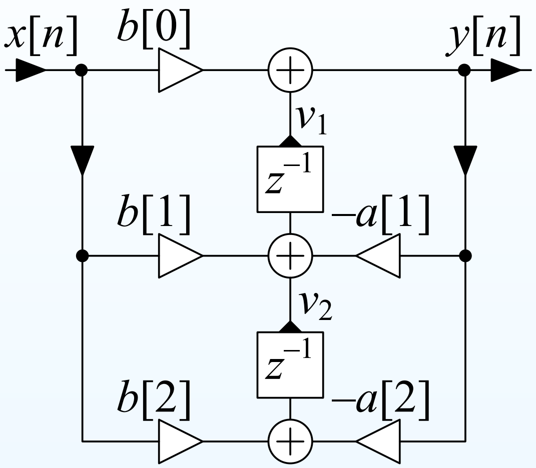

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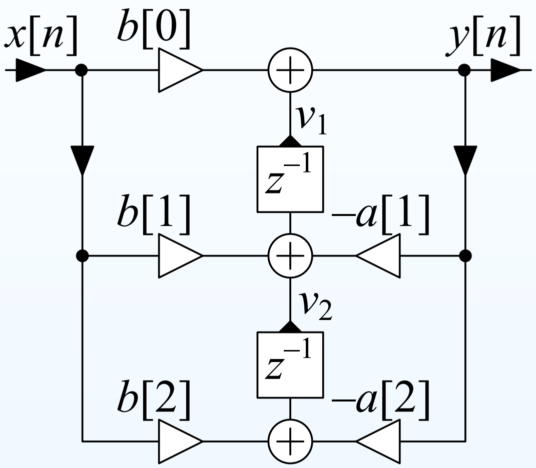

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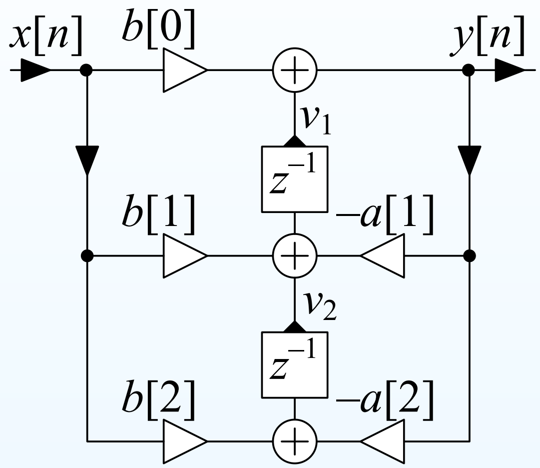

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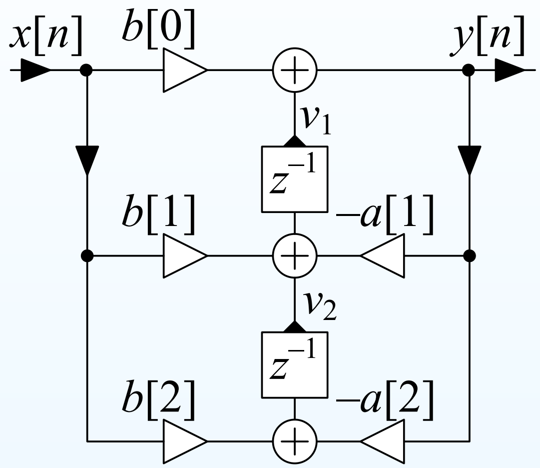

The transfer function is given by:

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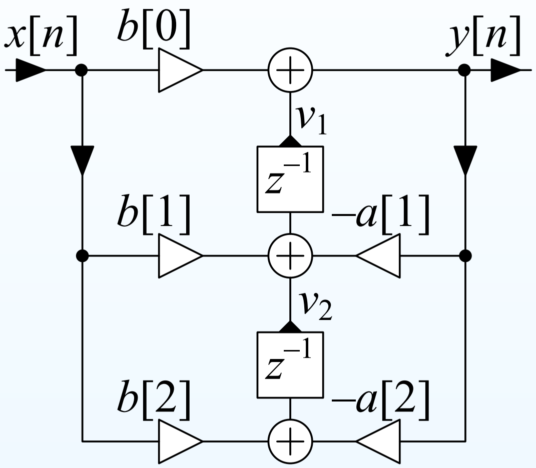

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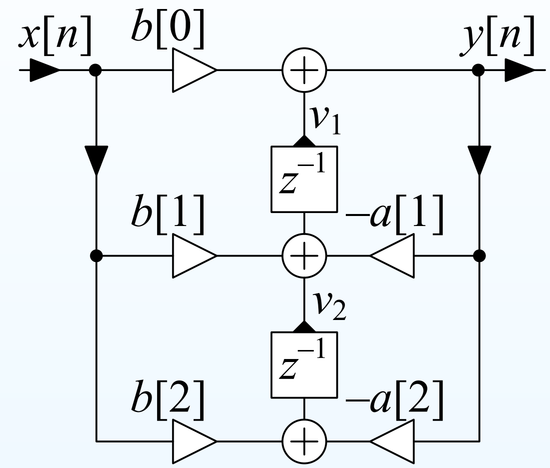

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The transposed form has $\mathbf{P} \to \mathbf{P}^T$ and $\mathbf{q} \leftrightarrow \mathbf{r} \quad \Rightarrow \quad$ same $H(z)$

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  Arithmetic errors introduce noise that is then filtered by the transfer function between the point of noise creation and the output.

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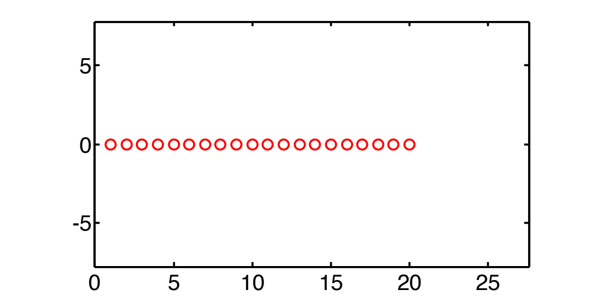

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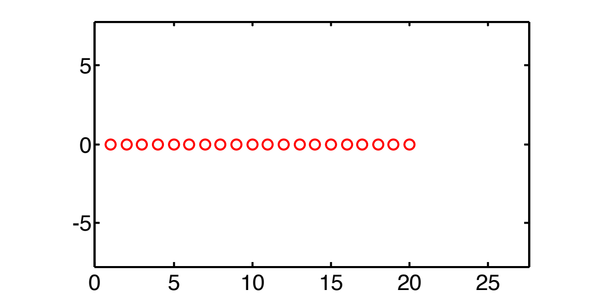

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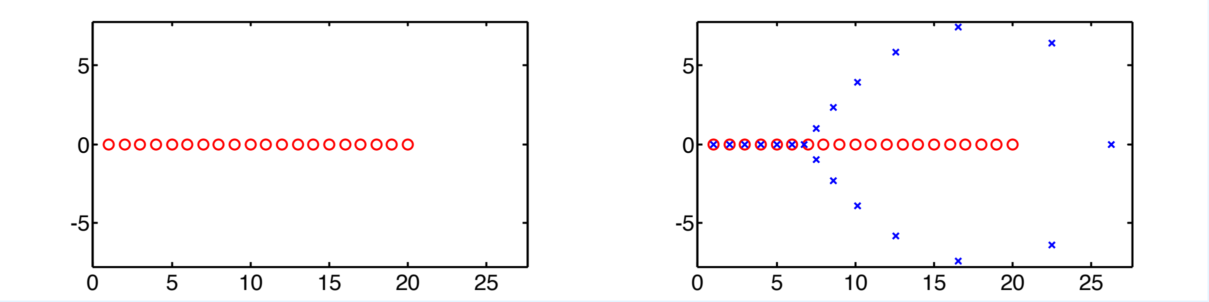

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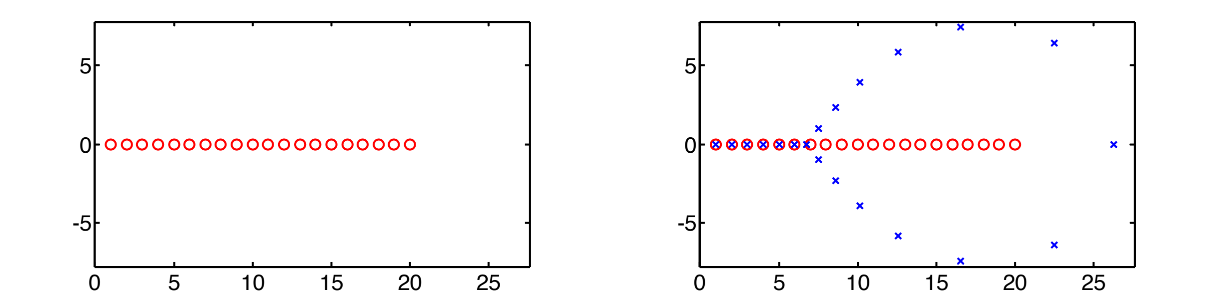

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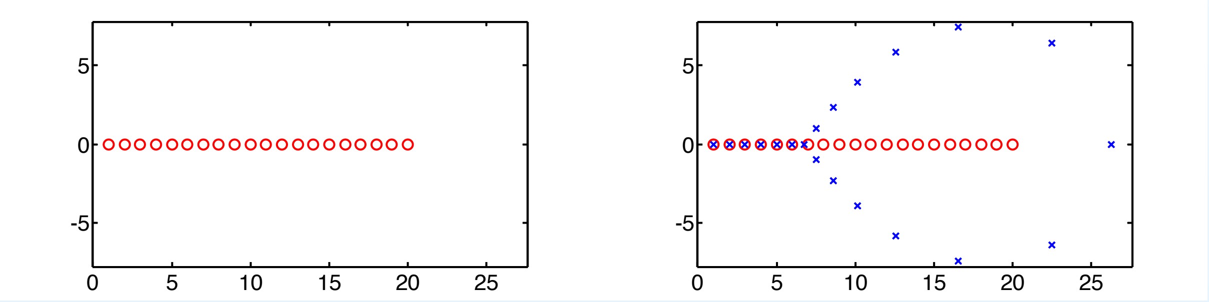

Moral: Avoid using direct form for filters orders over about 10.

Cascaded Biquads

Avoid high order polynomials by factorizing into quadratic terms:

$$\frac{B(z)}{A(z)} = g\frac{\prod\left(1+b_{k,1}z^{-1}+b_{k,2}z^{-2}\right)}{\prod\left(1+a_{k,1}z^{-1}+a_{k,2}z^{-2}\right)}$$

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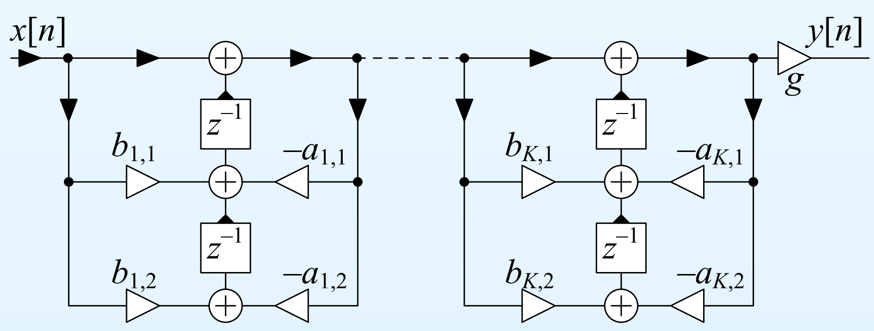

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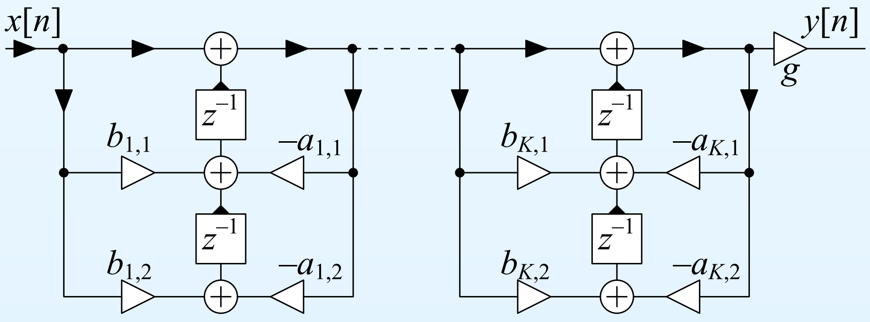

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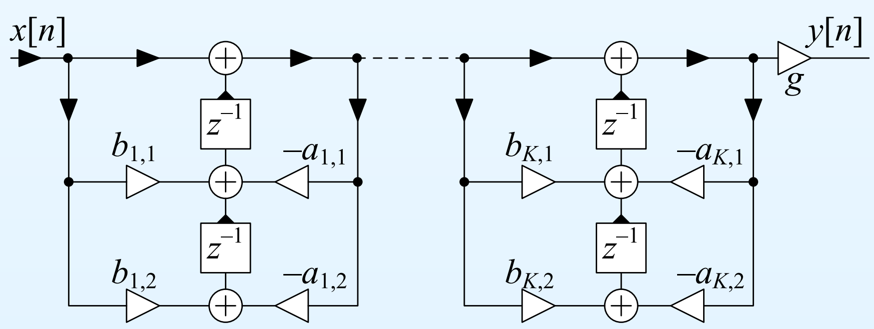
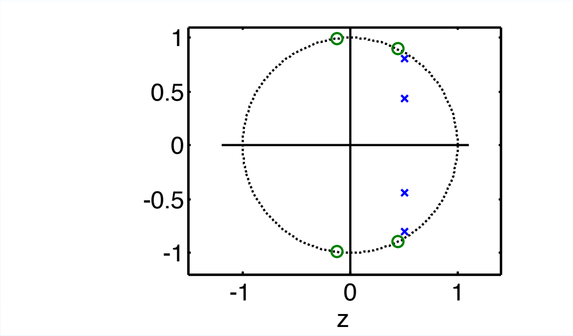
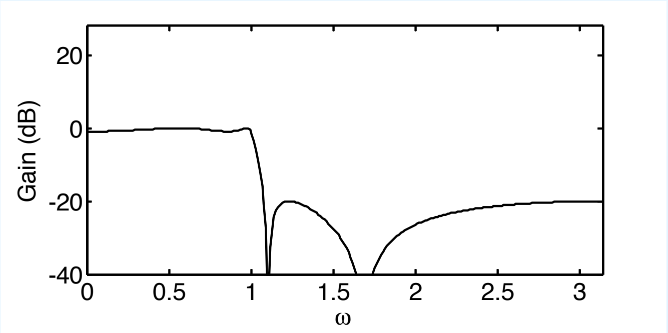


Pole-zero Pairing/Ordering

Example: Elliptic lowpass filter





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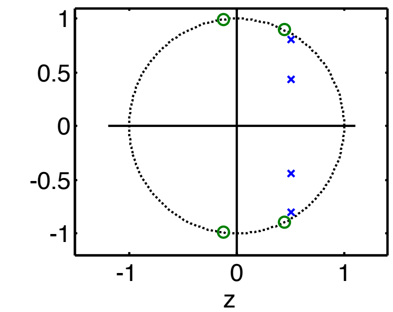
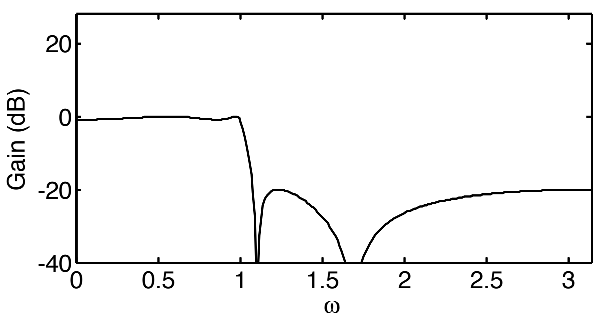
2 pole pairs and 2 zero pairs





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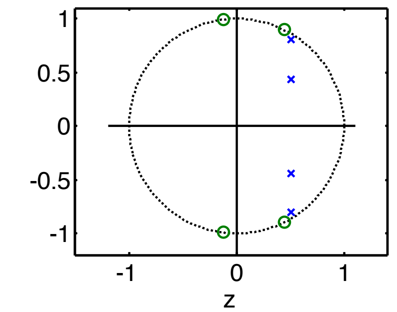

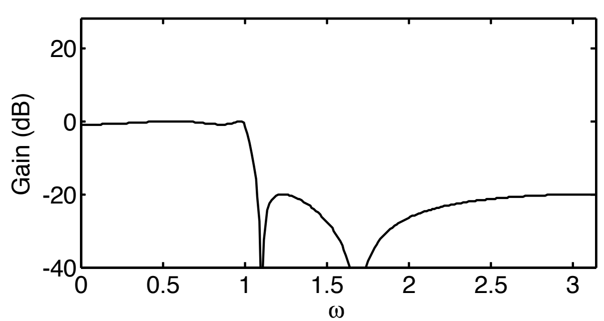

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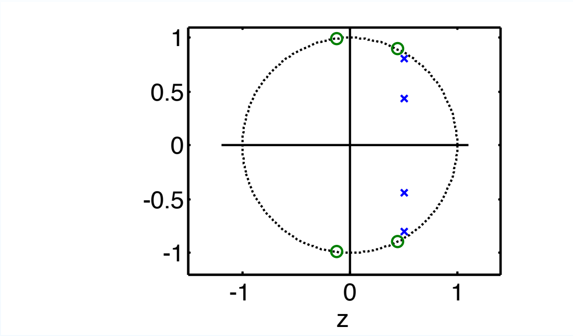

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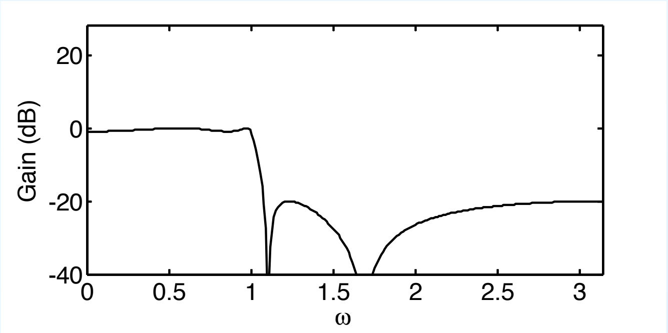

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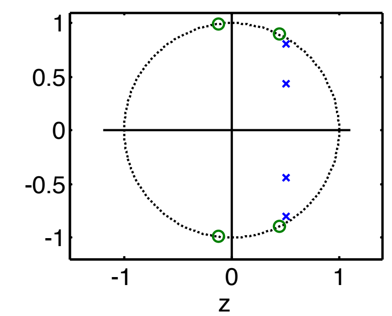

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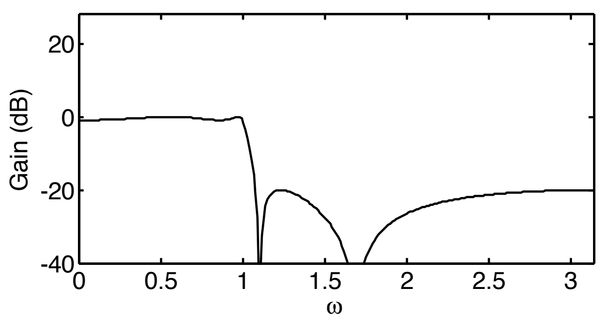

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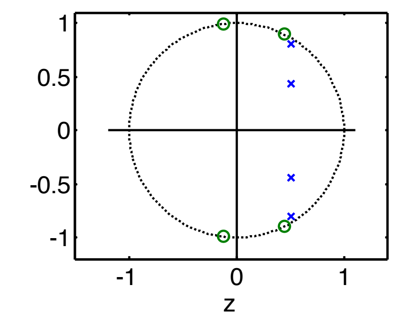

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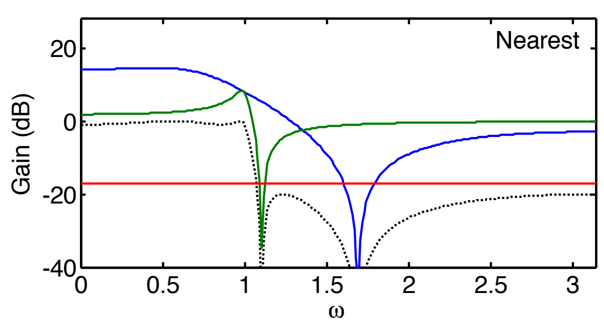

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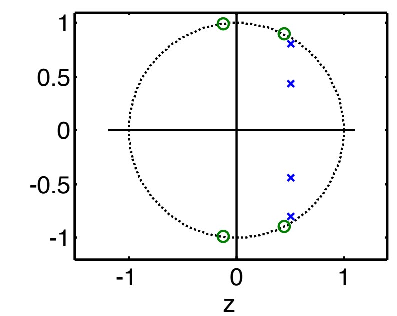

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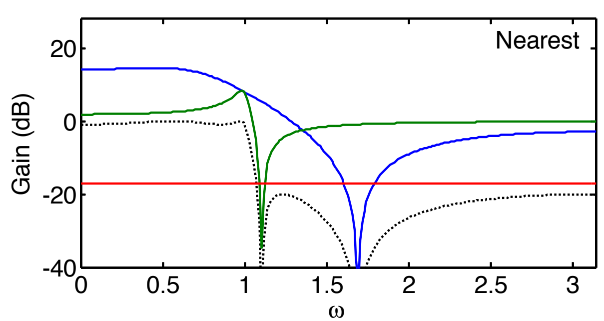

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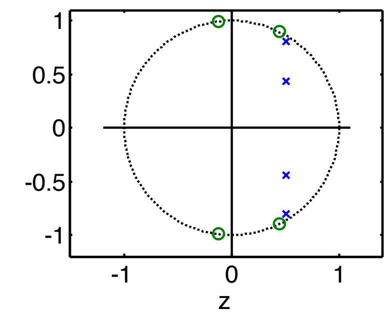

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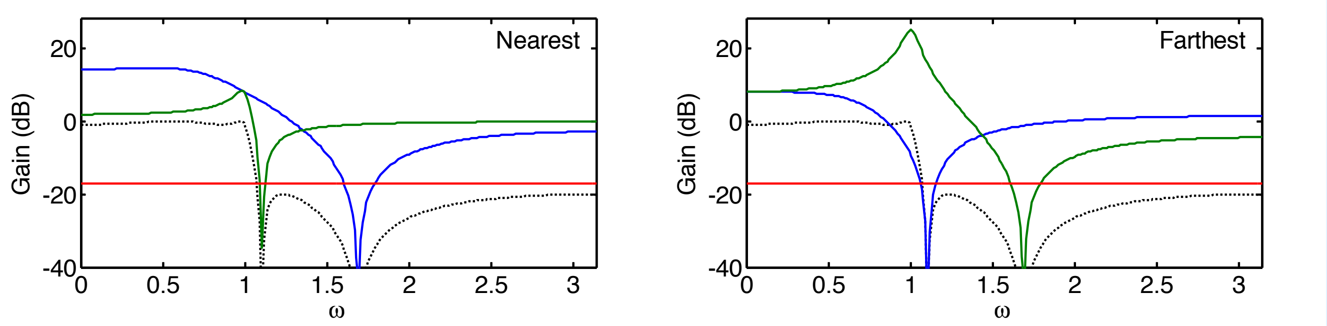

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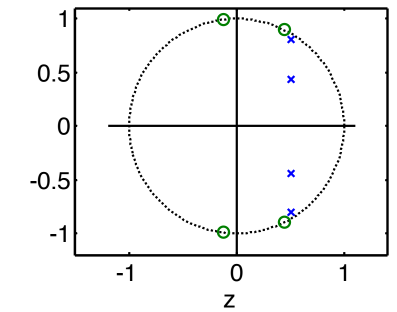

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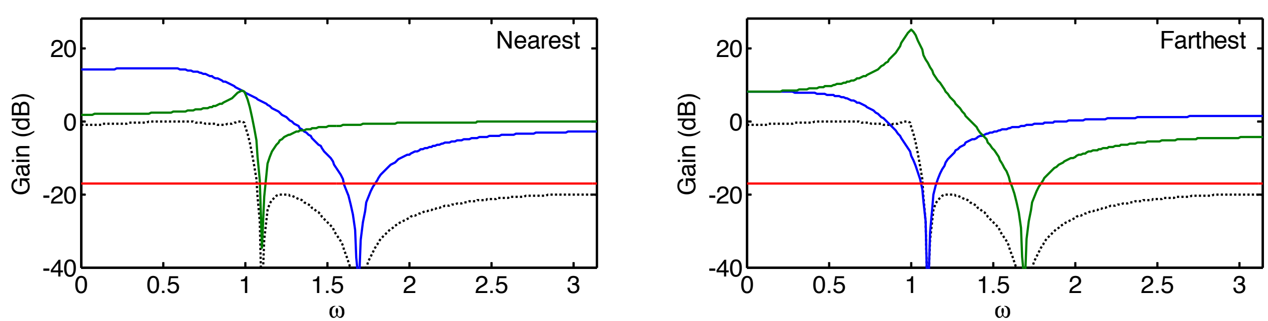

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We still need M additions and M delays.

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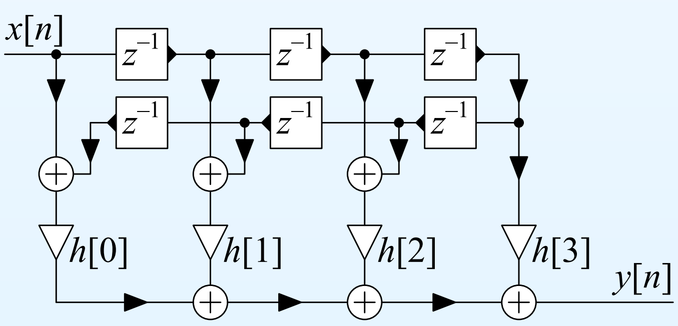

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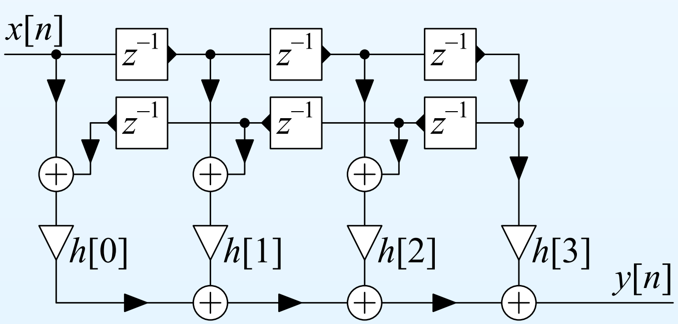

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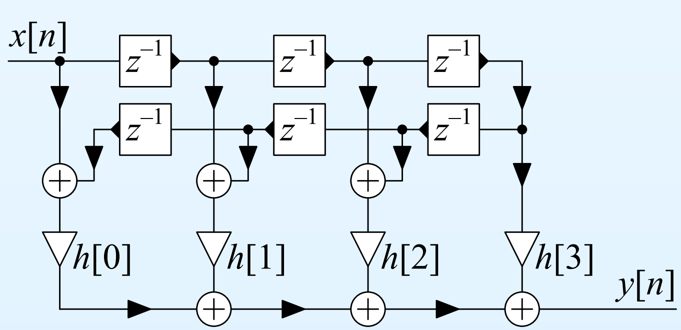
Software Implementation:

All that matters is the total number of multiplies and adds



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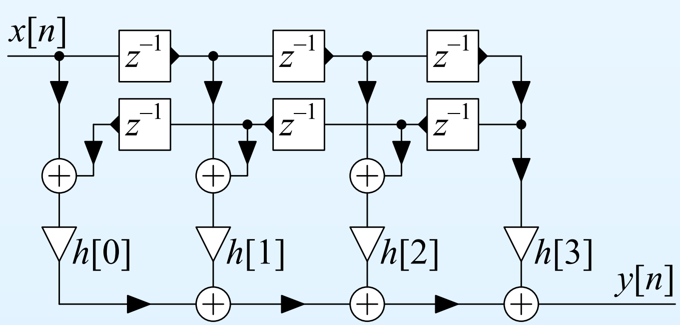

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Direct form: Maximum sequential delay = $4a + m$

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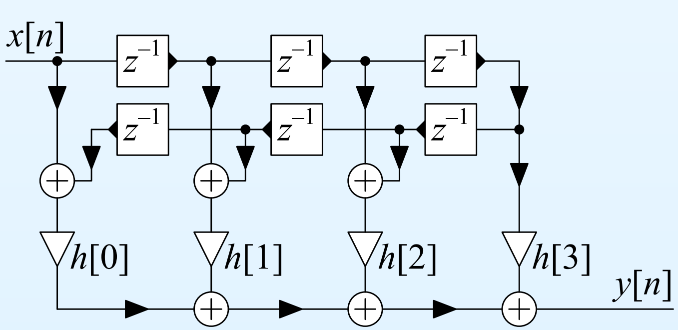

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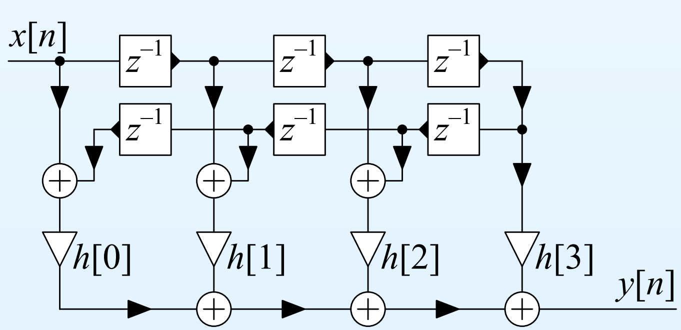

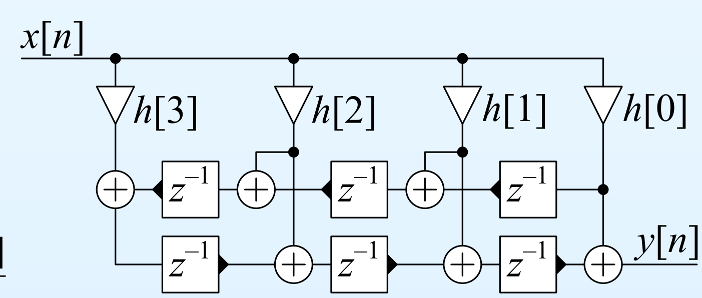

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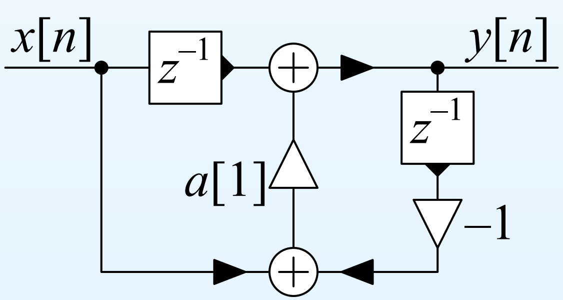

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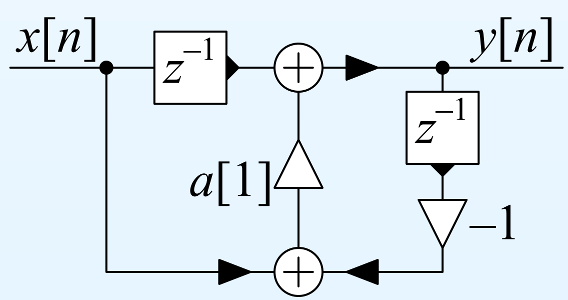

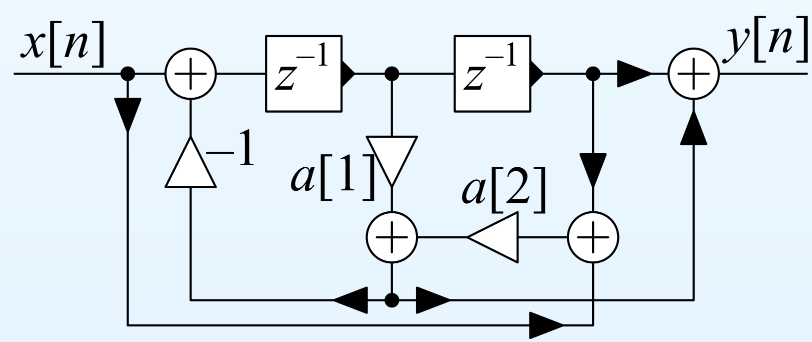

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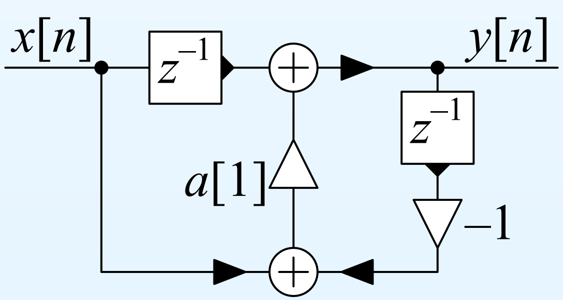

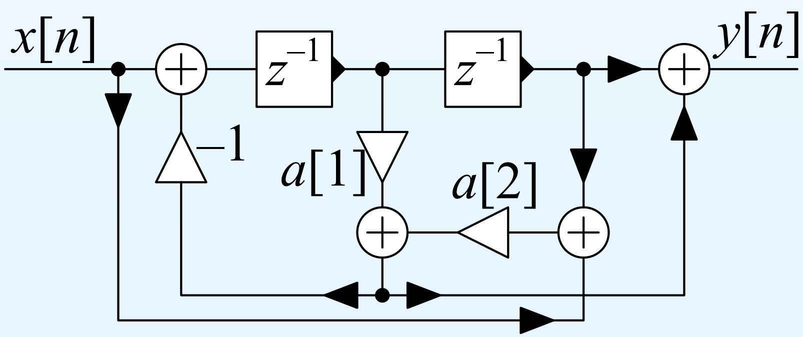

Allpass filters have a gain magnitude of 1 even with coefficient errors.

Lattice Stage

+

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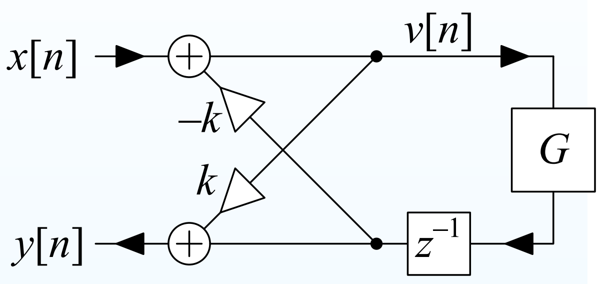

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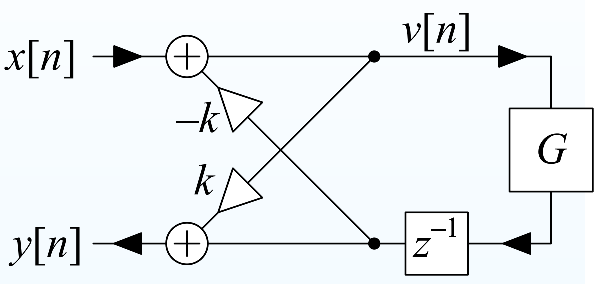

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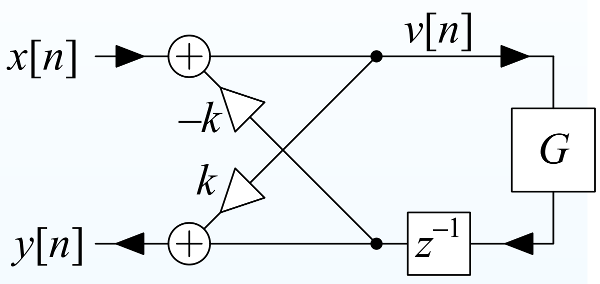

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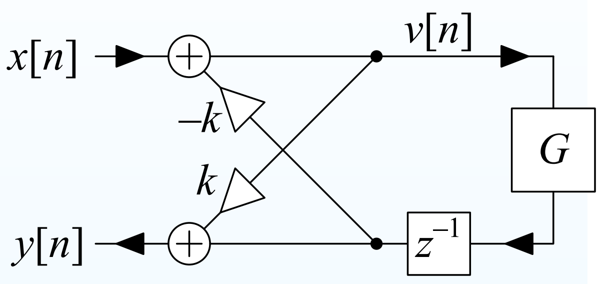

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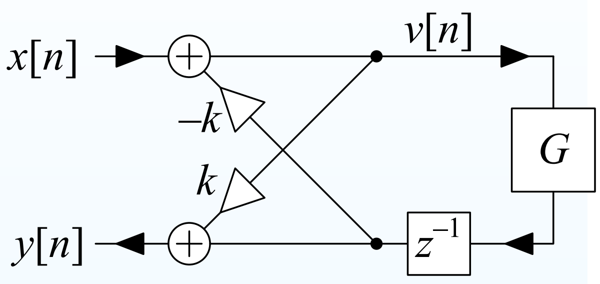

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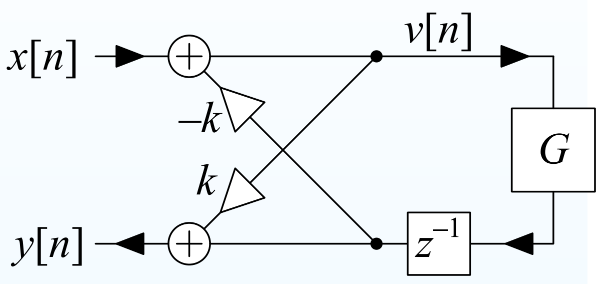
Lattice Stage

Suppose G is allpass: $G(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$



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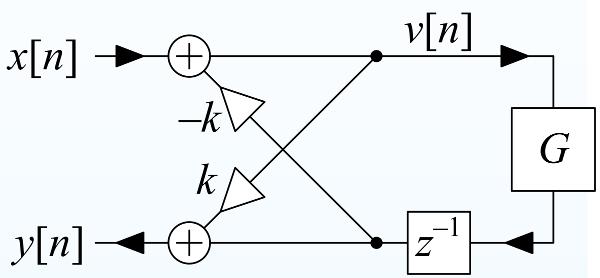

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Lattice Stage

+

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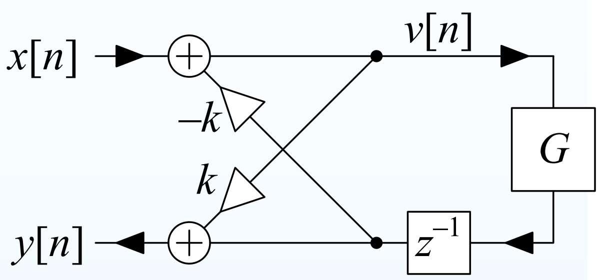

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$$d[n] = \begin{cases} 1 & n = 0 \\ a[n] + ka[N+1-n] & 1 \le n \le N \\ k & n = N+1 \end{cases}$$

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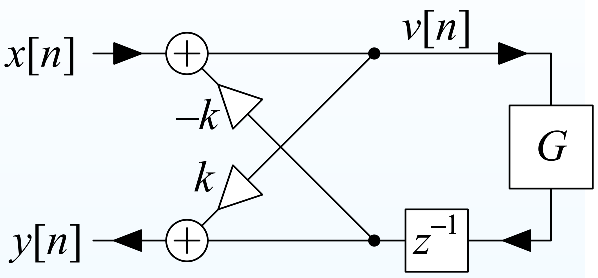

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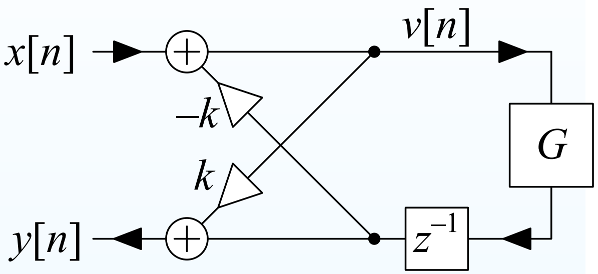

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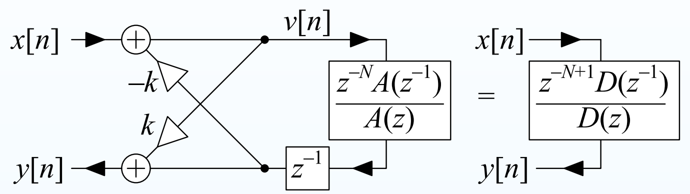
$$k = d[N+1] \qquad a[n] = \frac{d[n]-kd[N+1-n]}{1-k^2}$$

If $G(z)$ is stable then $\frac{Y(z)}{X(z)}$ is stable if and only if $|k| < 1$ (see note)

Suppose $N = 3$, $k = 0.5$ and
$A(z) = 1 + 4z^{-1} - 6z^{-2} + 10z^{-3}$



Example $A(z) \leftrightarrow D(z)$

Suppose $N = 3$, $k = 0.5$ and
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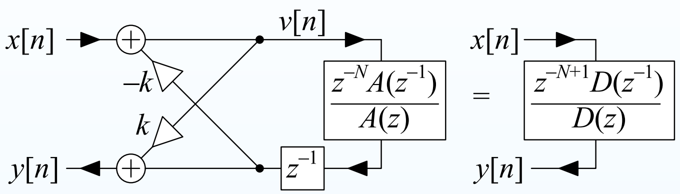

$A(z) \to D(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$A(z)$	1	4	-6	10	
$z^{-4}A(z^{-1})$		10	-6	4	1
$D(z) = A(z) + kz^{-4}A(z^{-1})$	1	9	-9	12	0.5

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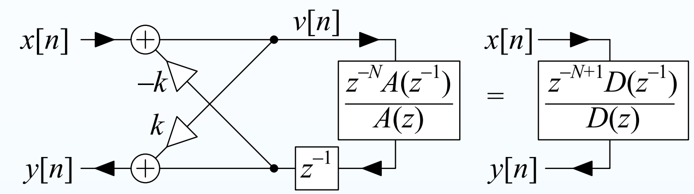

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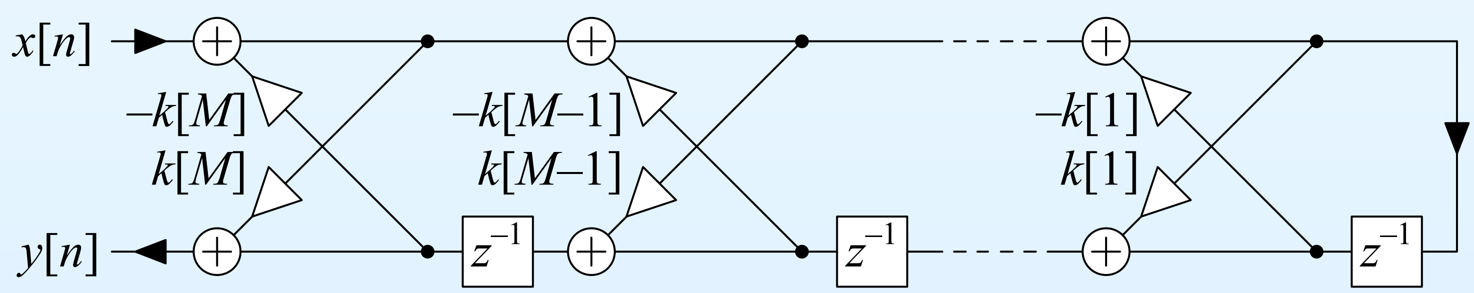
$D(z) \to A(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$D(z)$	1	9	-9	12	0.5
$k = d[N+1]$					0.5
$z^{-4}D(z^{-1})$	0.5	12	-9	9	1
$D(z) - kz^{-4}D(z^{-1})$	0.75	3	-4.5	7.5	0
$A(z) = \frac{D(z) - kz^{-4}D(z^{-1})}{1-k^2}$	1	4	-6	10	0

Allpass Lattice

We can implement any allpass filter $H(z) = \frac{z^{-M}A(z^{-1})}{A(z)}$ as a lattice filter with M stages:



Allpass Lattice

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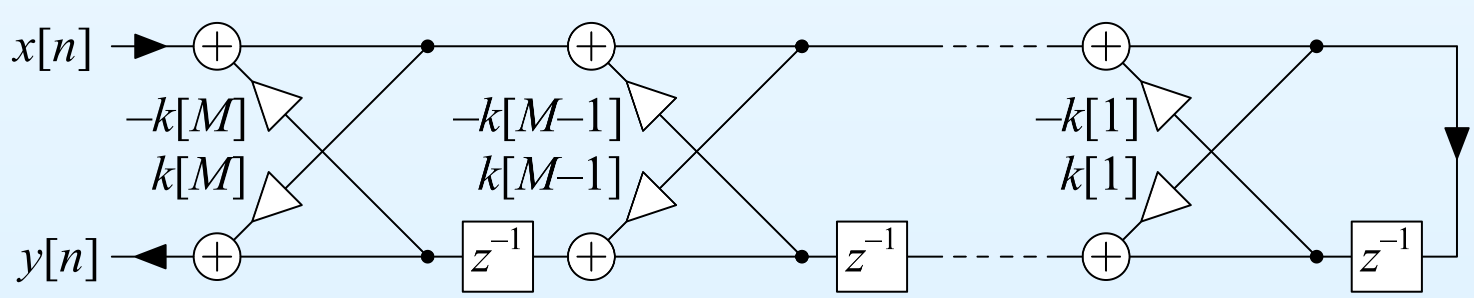

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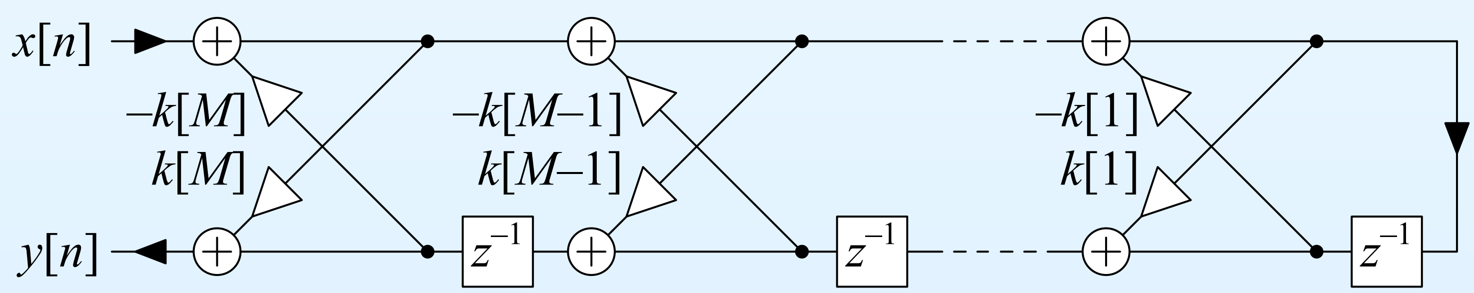

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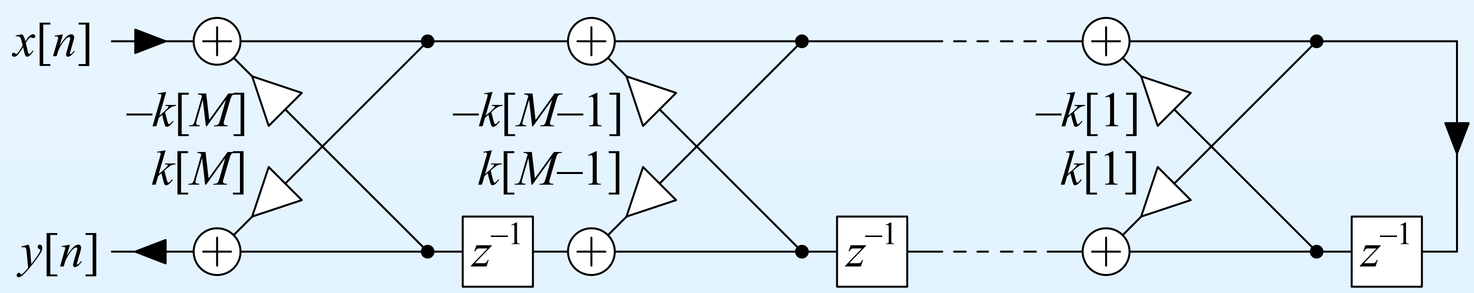

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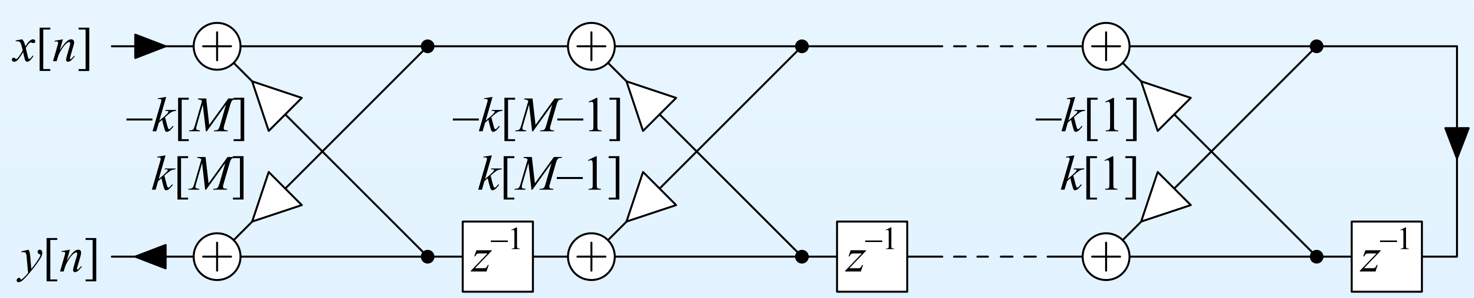

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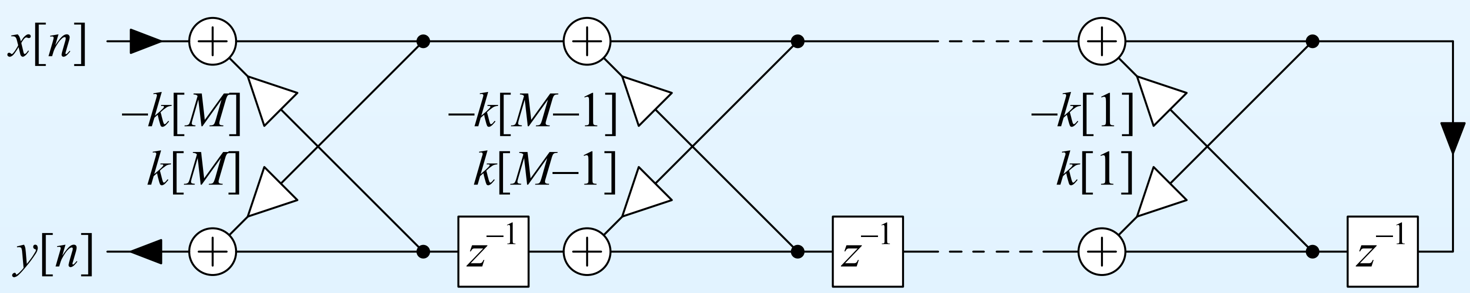

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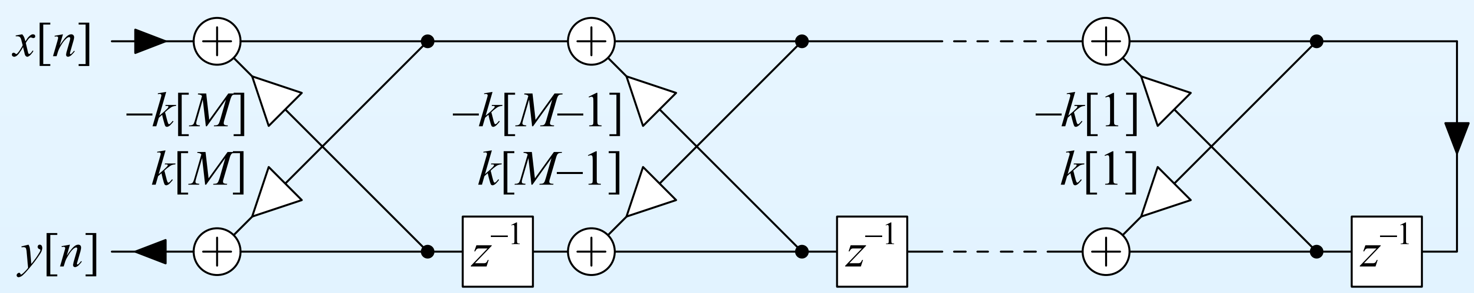
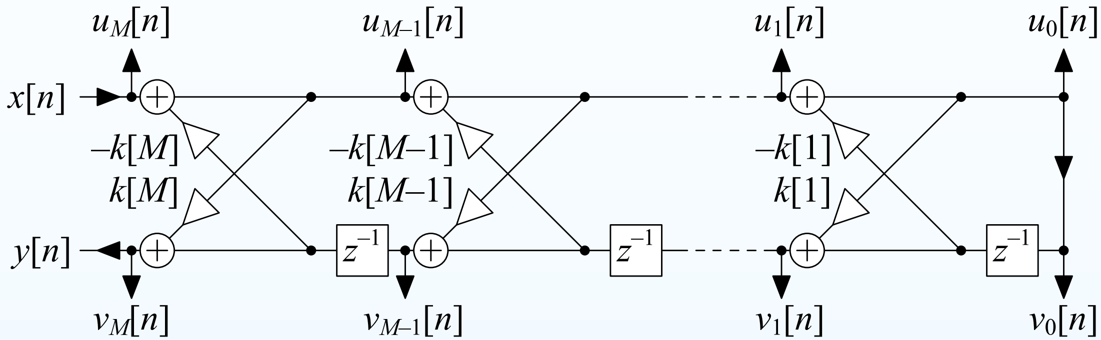
$A(z)$ is stable iff $|k[m]| < 1$ for all m (good stability test)

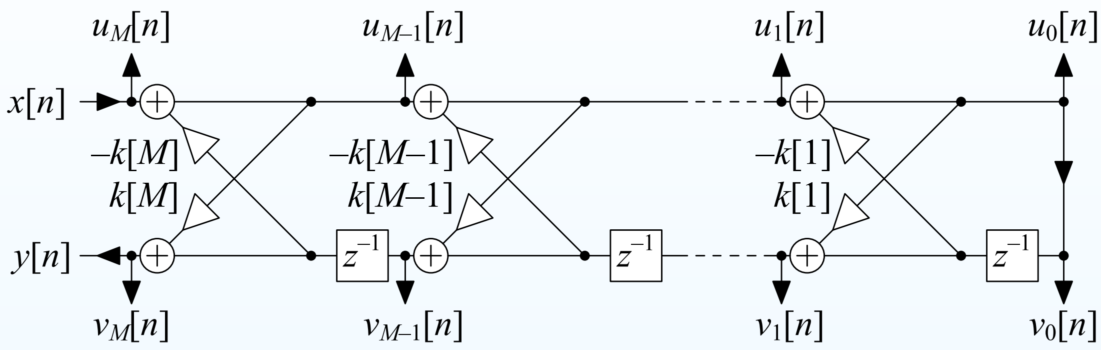


Lattice Filter



Label outputs $u_m[n]$ and $v_m[n]$ and define $H_m(z) = \frac{V_m(z)}{U_m(z)} = \frac{z^{-m}A_m(z^{-1})}{A_m(z)}$

Lattice Filter



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From earlier slide (slide 12):

$$\frac{U_{m-1}(z)}{U_m(z)} = \frac{1}{1+k[m]z^{-1}H_{m-1}(z)}$$

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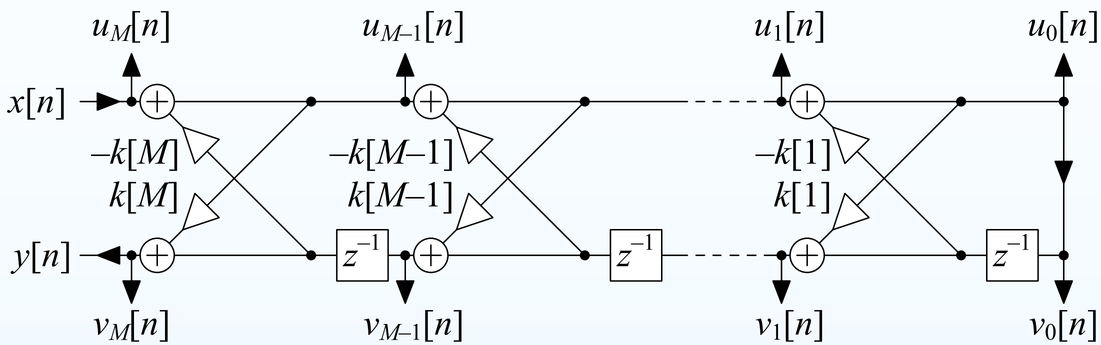

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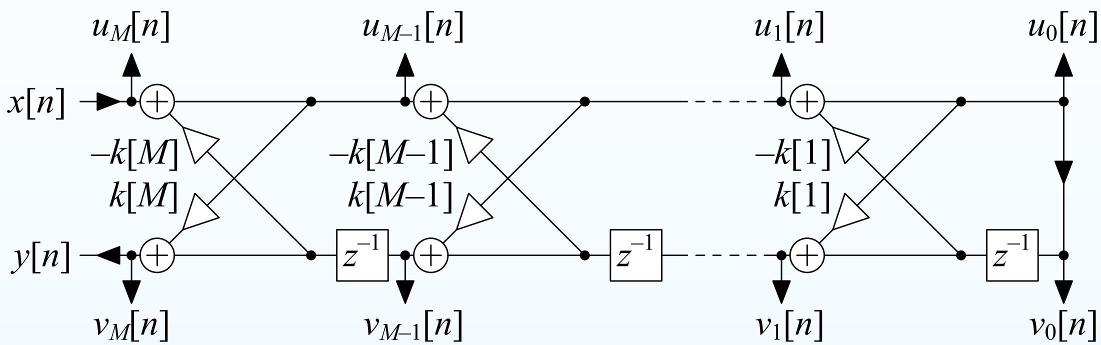

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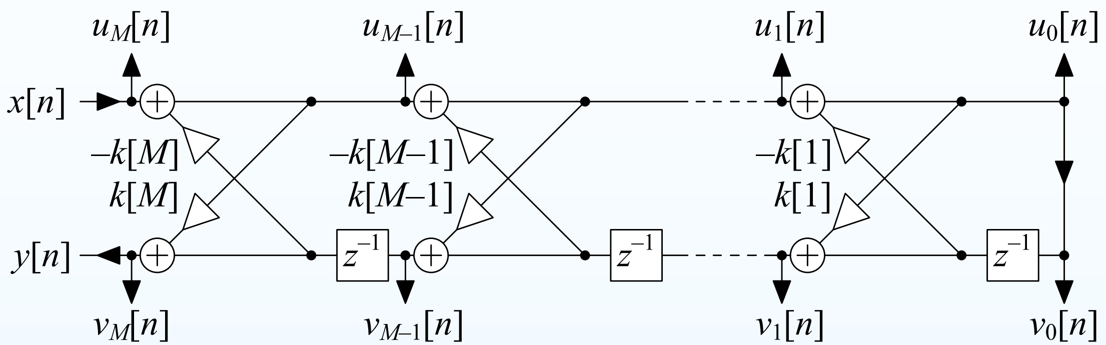

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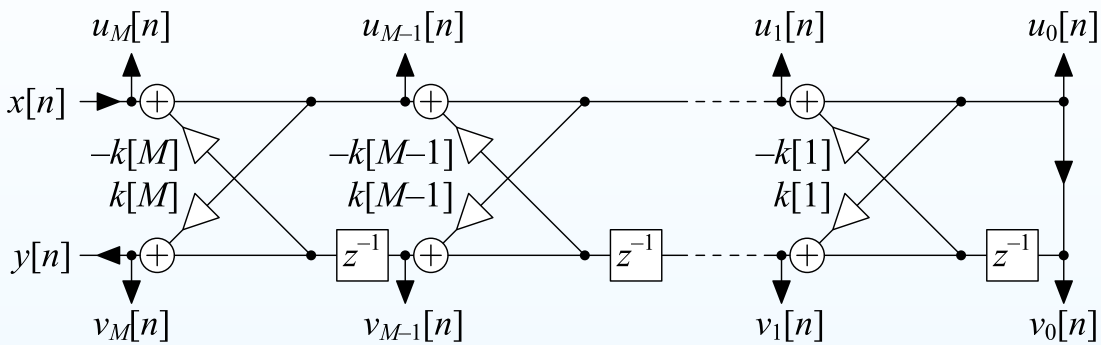
Hence:

$$\frac{U_m(z)}{X(z)} = \frac{A_m(z)}{A(z)}$$

Lattice Filter



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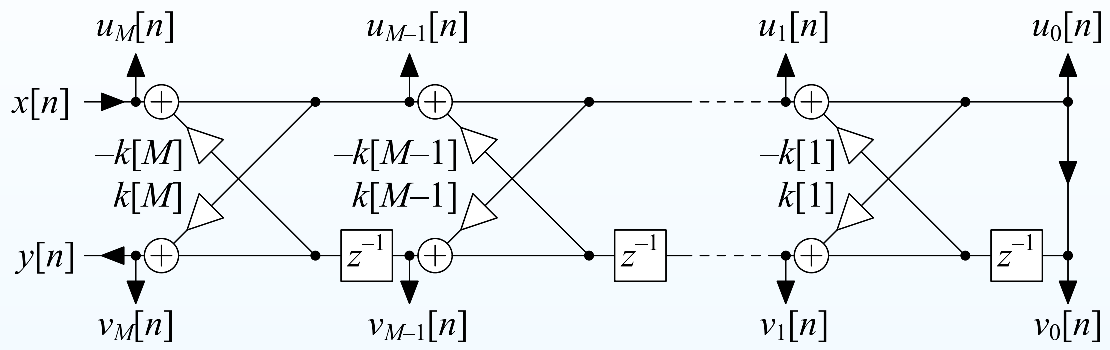
Lattice Filter



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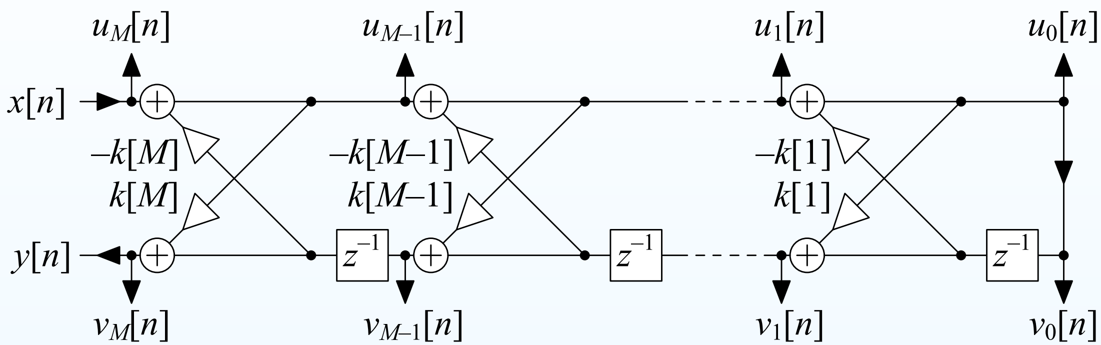

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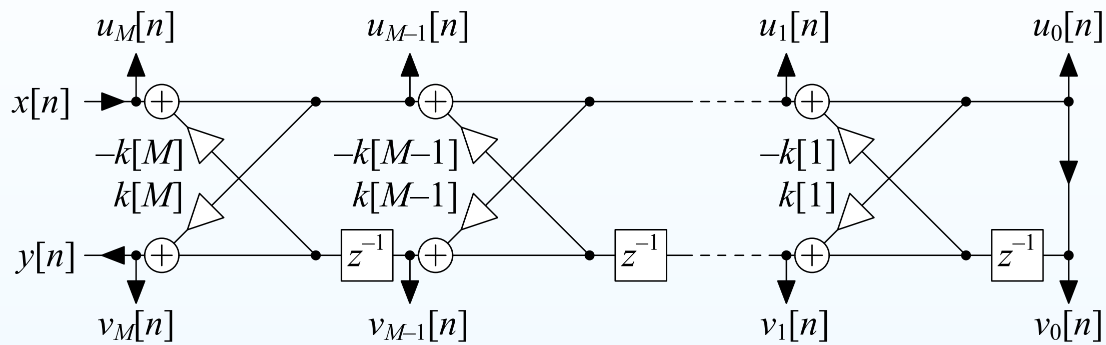
The numerator of $\frac{V_m(z)}{X(z)}$ is of order m so you can create any numerator of order M by summing appropriate multiples of $V_m(z)$:

$$w[n] = \sum_{m=0}^{M} c_m v_m[n]$$

Lattice Filter



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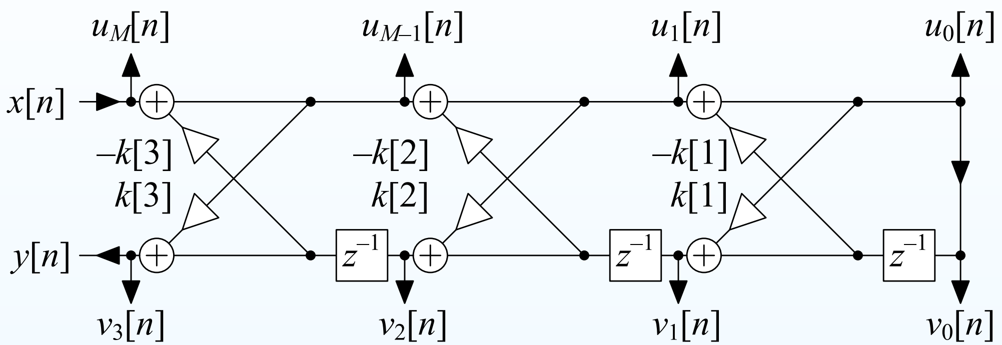
Hence:

$$\frac{U_m(z)}{X(z)} = \frac{A_m(z)}{A(z)} \qquad \text{and} \qquad \frac{V_m(z)}{X(z)} = \frac{U_m(z)}{X(z)} \times \frac{V_m(z)}{U_m(z)} = \frac{z^{-m}A_m(z^{-1})}{A(z)}$$

The numerator of $\frac{V_m(z)}{X(z)}$ is of order m so you can create any numerator of order M by summing appropriate multiples of $V_m(z)$:

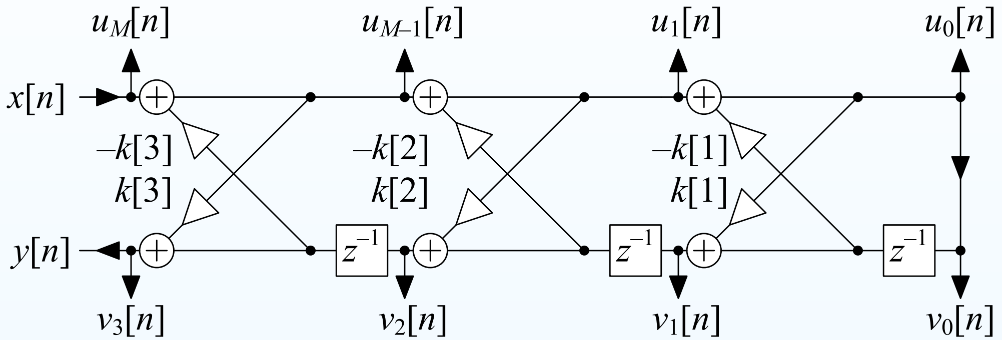
$$w[n] = \sum_{m=0}^{M} c_m v_m[n] \quad \Rightarrow \quad W(z) = \frac{\sum_{m=0}^{M} c_m z^{-m} A_m(z^{-1})}{A(z)}$$

Lattice Example



$$A(z) = A_3(z) = 1 + 0.2z^{-1} - 0.23z^{-2} + 0.2z^{-3}$$

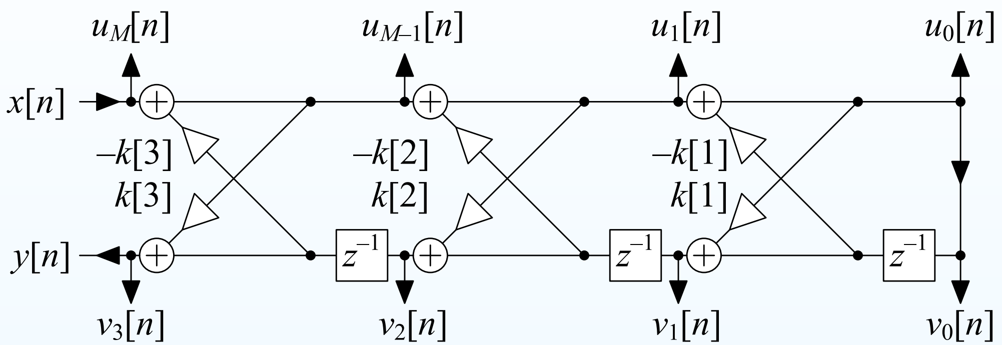
Lattice Example



$$A(z) = A_3(z) = 1 + 0.2z^{-1} - 0.23z^{-2} + 0.2z^{-3}$$

- $k[3] = 0.2 \Rightarrow a_2[\,] = \frac{[1,\,0.2,\,-0.23]-0.2[0.2,\,-0.23,\,0.2]}{1-0.2^2} = [1,\ 0.256,\ -0.281]$

Lattice Example



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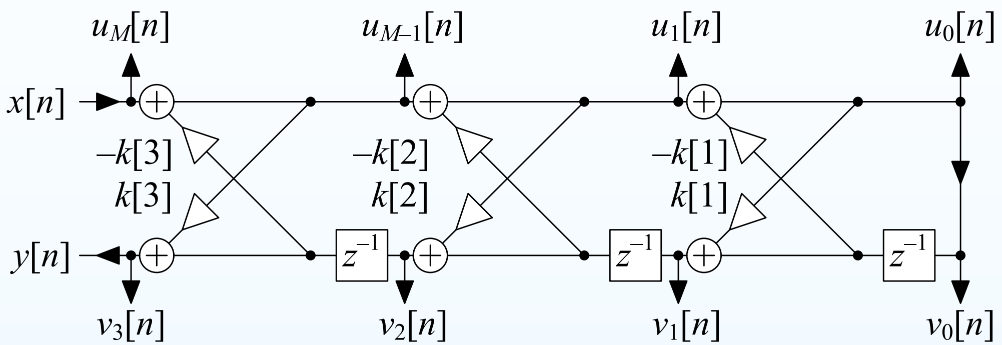
Lattice Example



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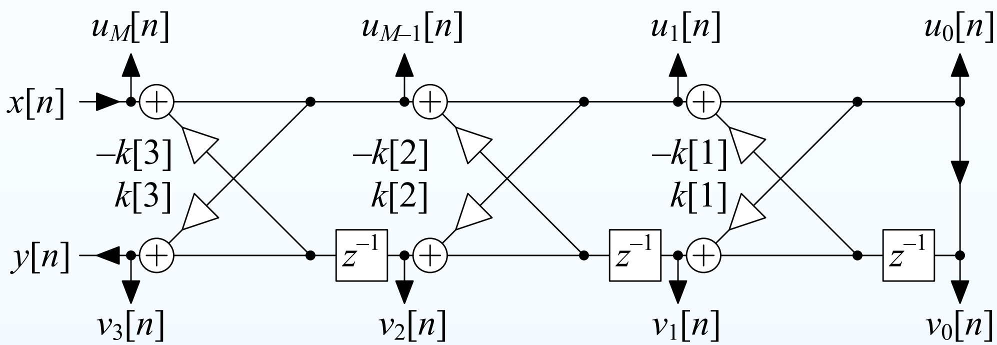

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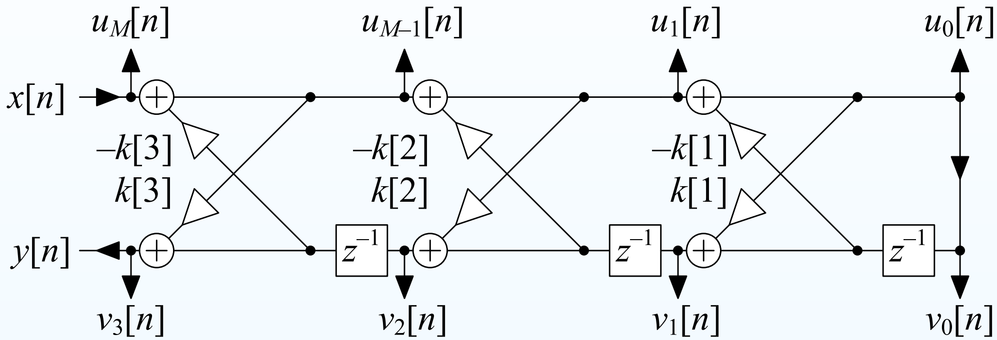

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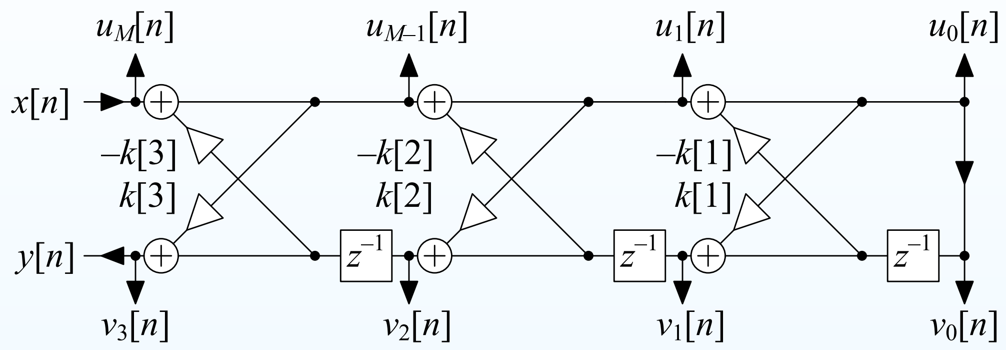

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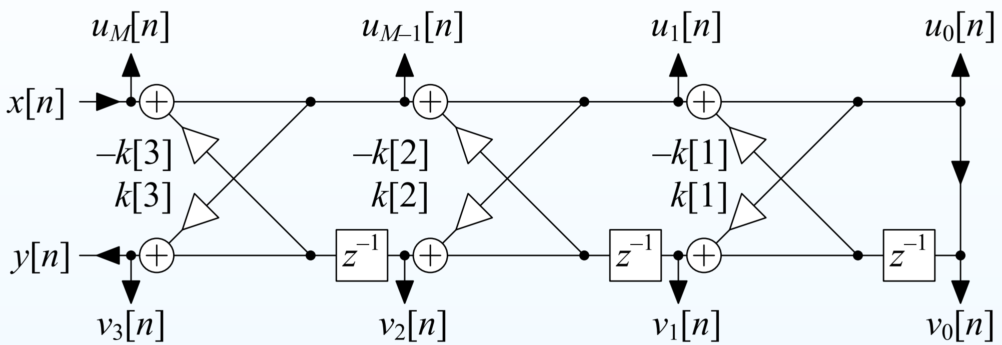

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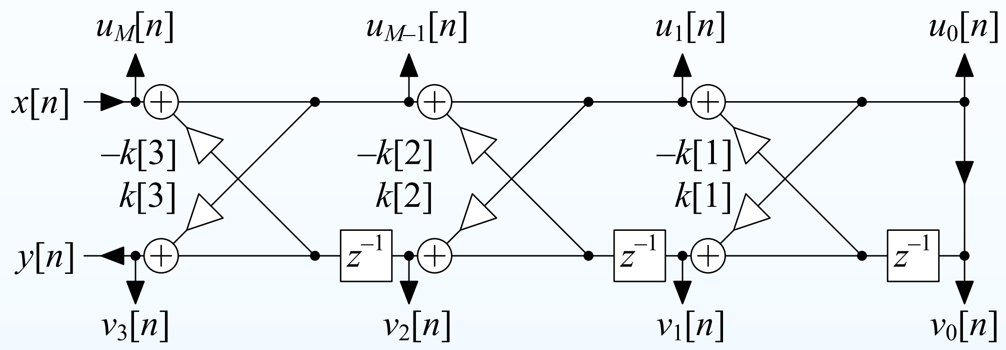

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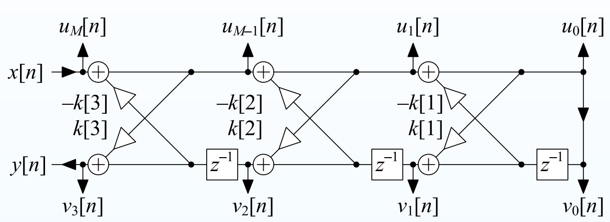
Add together multiples of $\frac{V_m(z)}{X(z)}$ to create an arbitrary $\frac{B(z)}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$

Lattice Example Numerator

Form a new output signal as $w[n] = \sum_{m=0}^{M} c_m v_m[n]$



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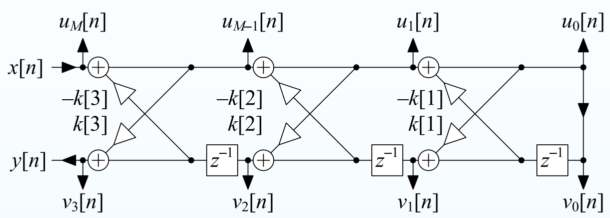

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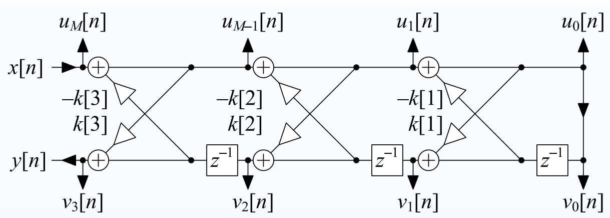

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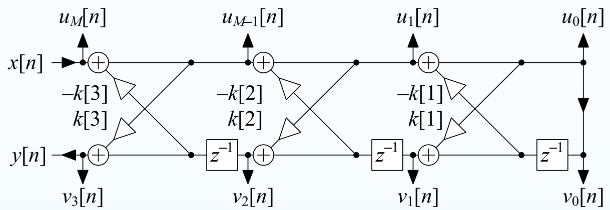

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We have $\begin{pmatrix} b[0] \\ b[1] \\ b[2] \\ b[3] \end{pmatrix} = \begin{pmatrix} 1 & 0.357 & -0.281 & 0.2 \\ 0 & 1 & 0.256 & -0.23 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$

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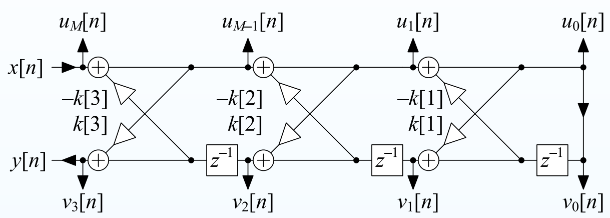

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Hence choose c_m as $\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0.357 & -0.281 & 0.2 \\ 0 & 1 & 0.256 & -0.23 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b[0] \\ b[1] \\ b[2] \\ b[3] \end{pmatrix}$

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For further details see Mitra: 8.

MATLAB routines

residuez	$\frac{b(z^{-1})}{a(z^{-1})} \rightarrow \sum_k \frac{r_k}{1-p_k z^{-1}}$
tf2sos,sos2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2sos,sos2zp	$\{z_m, p_k, g\} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{\in 1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2ss,ss2zp	$\{z_m, p_k, g\} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
tf2ss,ss2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
poly	$\text{poly}(\mathbf{A}) = \det(z\mathbf{I}-\mathbf{A})$