IMPERIAL COLLEGE LONDON

EE4-10
EE9-CS5-1
EE9-SC3
EE9-FPN2-02

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2020

MSc and EEE PART IV: MEng and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Monday, 18 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : C. Ling
Second Marker(s) : D. Angeli

Information for students

Each of the four questions has 25 marks.

The Questions

1. Random variables.

a) Consider an experiment of flipping a fair coin.

i) Let Y be a random variable such that $Y = 1$ if the outcome is "head" and $Y = 0$ otherwise. What is the expected value of Y? [6]

ii) What is the probability of having j heads knowing that you have flipped the coin for n times? [6]

iii) Suppose that we flip a coin until the first heads comes up. What is the probability distribution of the number of flips? [6]

b) Suppose we wish to generate samples of a random variable with cumulative distribution function $F(x)$. This can be done by using an inverse transform: firstly generate random variable U uniform in the interval [0, 1], then apply inverse transform to obtain $X = F^{-1}(U)$ where F^{-1} denotes the inverse of F. Show that X has the correct distribution.

[7]

2. Estimation and sequences of random variables.

a) The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx},\ \ x > 0$. We observe the i.i.d. samples x_i = 4.1, 3.7, 4.2. Find the maximum-likelihood estimate of parameter c.

[10]

b) Let X be a Gaussian random variable with zero mean and variance σ^2. Estimate the tail probability $P(|X| > a)$ where $a = 3\sigma$ using

i) Markov inequality [5]

ii) Chebyshev inequality [5]

iii) Chernoff bound [5]

Hint: $E(|X|^n) = \begin{cases} 1 \cdot 3 \cdots (n-1)\sigma^n, & n \ \text{even}, \\ 2^k k! \sigma^{2k+1}\sqrt{2/\pi}, & n = (2k+1), \ \text{odd}. \end{cases}$

3. Random processes.

a) The number of failures $N(t)$, which occur in a computer network over the time interval $[0, t)$, can be modelled by a Poisson process $\{N(t), t \geqslant 0\}$. On average, there is a failure after every 4 hours, i.e. the intensity of the process is equal to $\lambda = 0.25$.

i) What is the probability of at most 1 failure in [0, 4), while at the same time at least 2 failures occur in [4, 8), and at most 1 failure in [8, 12) ? (time unit: hour) [10]

ii) What is the probability that the third failure occurs after 4 hours? [5]

Formula: Poisson distribution $P(X = k) = e^{-\lambda}\dfrac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots, \infty$

b) Consider the matched filter shown in the following figure:

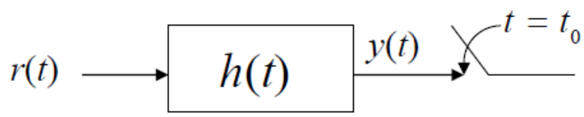

Fig. 3.1. Matched filter.

Let $r(t)$ represent a deterministic signal $s(t)$ corrupted by noise

$$r(t) = s(t) + w(t), \qquad 0 < t < t_0 = 1$$

where $w(t)$ is additive white noise with power spectral density $N_0 = 1$. Let $s(t)$ be a sinusoidal signal

$$s(t) = \begin{cases} \sin(2\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

i) Find the expression of the optimum matched filter $h(t)$. [5]

ii) Find the maximum output signal-to-noise ratio (SNR). [5]

4. Markov chains and martingales.

a) Let p be the probability that the coin turns "head", and let $q = 1 - p$. Gambler's ruin probability is given by

$$P_i = \begin{cases} \dfrac{1-\left(p/q\right)^{N-i}}{1-\left(p/q\right)^{N}}, & \text{if } p \neq \dfrac{1}{2} \\ \dfrac{N-i}{N}, & \text{if } p = \dfrac{1}{2} \end{cases}$$

A coin is flipped by a gambler with initial money £10. He wins £1 if the coin turns "head", and loses £1 if the coin turns "tail". The gambler will stop playing when he either has lost all of his money or he reaches £50. Of the first 10 flips, there are 6 "heads" and 4 "tails".

Calculate the probability that he will lose all of his money, given the results of the first 10 flips,

i) if the coin is fair, i.e., $p = 1/2$; [5]

ii) if the coin is unfair with $p = 1/3$. [5]

b) Fix a parameter $\lambda \in (0, 1)$ and let $X_0, X_1, X_2, \ldots$ be a sequence of independent random variables, whose distribution satisfies $P(X_j = -1) = P(X_j = 1) = 1/2$. Consider the following random sequence

$$Y_n = \sum_{i=0}^{n} X_i \lambda^i, \qquad n = 0,1,2,3, \ldots$$

i) Show $\{Y_n\}$ is a martingale. [3]

ii) Derive the characteristic function of Y_n. [4]

iii) Show that for any set E

$$P(Y_{n+1} \in E) = \frac{1}{2} P(Y_n \in T_1^{-1}(E)) + \frac{1}{2} P(Y_n \in T_2^{-1}(E))$$

where $T_1(x) = \lambda x + 1$ and $T_2(x) = \lambda x - 1$. [4]

iv) What is the limiting distribution of Y_n as $n \to \infty$ if $\lambda = 1/2$? [4]