Imperial College London

Department of Electronic and Electrical Engineering

Examinations 2020

Digital Signal Processing and Digital Filters
MSc and EEE/EIE PART IV

Monday, 11 May, 2020 • Exam Duration: Three Hours

There are FOUR questions on this paper. Answer Question 1 and any TWO other questions.
Question 1 is worth 40% of the marks and other questions are worth 30%.

Markers: Ayush Bhandari and Tania Stathaki.

Question 1 [40 Marks]

a) Which of the following statements is true and why?

i) All discrete-time signals are digital signals.
ii) All digital signals are discrete-time signals.
iii) Some discrete-time signals are digital signals.
iv) Some digital signals are discrete-time signals. **[2 Marks]**

By definition, digital signals are discrete in both amplitude and time. Hence, (ii) and (iii) are correct.

b) What is the value of z_0 below,

$$z_0 = \sum_{n=-\infty}^{n=+\infty} \mathrm{sinc}\,(t-n)$$

where t denotes continuous-time. (Hint: Use Nyquist–Shannon Sampling Theorem). **[2 Marks]**

According to the sampling theorem, we have, $f(t) = \sum_{n\in\mathbb{Z}} f(nT)\,\mathrm{sinc}\,(t/T - n)$. When comparing this with above, we see that $f(nT) = 1$ and $T = 1$. Hence $f(t) = 1$ which is the value of z_0.

c) It is typically assumed that the linear convolution operation is *associative* or,

$$f * (g * h) = f * g * h = (f * g) * h$$

but it was shown in the class that this is only when the sums related to the convolution operations converge. Provide a counter-example when the convolution is <u>not</u> associative. (Hint: you can use one of the sequences as constant and another one as unit-step.) **[2 Marks]**

Consider the following setting,

$$\begin{aligned} f[n] &= u[n] \\ g[n] &= \delta[n+1] - \delta[n]. \\ h[n] &= 1 \end{aligned}$$

Clearly, $f * g = \delta[n]$ while $g * h = 0$. Hence $f * (g * h) = 0$ while $(f * g) * h = 1$ showing that convolution is non-associative.

d) Suppose we are given two sequences of lengths L_1 and L_2. Under what condition are the linear and circular convolutions equivalent? (Just write the condition) **[2 Marks]**

e) Let $h[n] = \alpha^n u[n]$ be a given filter. What is the z–transform of M-times downsampled version of the same filter? **[2 Marks]**

First, we note that,

$$H(z) = \sum_{n\in\mathbb{Z}} h[n]\, z^{-n} = \sum_{n\geqslant 0} \left(\alpha z^{-1}\right)^n = \frac{1}{1-\alpha z^{-1}}.$$

Next, we have the following result,

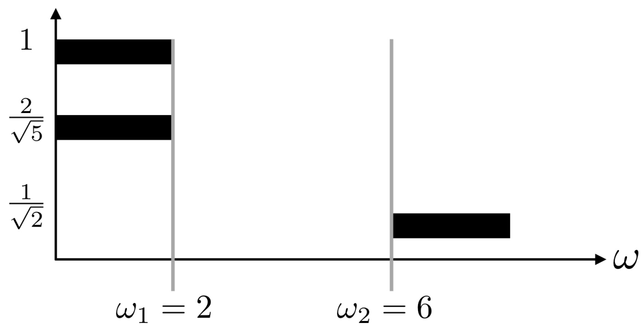
$$H(z) \rightarrow \boxed{\boxed{M:1}} \rightarrow H_M(z) = \frac{1}{M}\sum_{m=0}^{M-1} H\left(e^{-\jmath\frac{2\pi m}{M}} z^{\frac{1}{M}}\right).$$

Combining the two results, we obtain,

$$H_M(z) = \frac{1}{M}\sum_{m=0}^{M-1} \frac{1}{1-\alpha e^{\jmath\frac{2\pi m}{M}} z^{-\frac{1}{M}}}.$$

f) Let $h(t) = \kappa e^{-\lambda t} u(t)$ be a continuous-time impulse response of some filter where κ is the gain, λ is the filter parameter, and $u(t)$ is the usual unit-step function. Find a range of values for κ and λ so that, the frequency-domain gain at zero frequency is unity and the following filter specifications are met.

[4 Marks]



For the continuous-time filter, let us evaluate its Fourier Transform,

$$H(\omega) = \kappa \int_0^\infty e^{-\lambda t} e^{-\jmath\omega t} dt = \kappa \int_0^\infty e^{-(\lambda+\jmath\omega)t} dt = \kappa \int_0^\infty e^{-st} dt.$$

The above can be simplified using the elementary integral,

$$\int e^{-st} dt = -\frac{1}{s} e^{-st} \Rightarrow \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_{t=\infty} + \frac{1}{s} e^{-st}\Big|_{t=0} = \frac{1}{s}$$

giving,

$$H(\omega) = \frac{\kappa}{\lambda + \jmath\omega}.$$

According to the first condition, we obtain, $H(\omega)|_{\omega=0} = 1 \Rightarrow \kappa = \lambda$. According the filter specification in the diagram, we have the following constraints,

$$\frac{2}{\sqrt{5}} \leqslant |H(2)| \leqslant 1 \Rightarrow \frac{2}{\sqrt{5}} \leqslant \frac{\lambda}{\sqrt{\lambda^2+4}} \leqslant 1 \Rightarrow 4\lambda^2 + 16 \leqslant 5\lambda^2$$

$$|H(6)| \leqslant \frac{1}{\sqrt{2}} \Rightarrow \frac{\lambda}{\sqrt{\lambda^2+36}} \leqslant \frac{1}{\sqrt{2}} \Rightarrow 2\lambda^2 \leqslant \lambda^2 + 36.$$

Hence, $4 \leqslant \lambda \leqslant 6$ and $\kappa = \lambda$.

g) Let us define a sequence by,

$$x[n] = (r_1)^n u[n] - (r_2)^n u[-(n+1)] \tag{1}$$

where $r_1 = -1/3$ and $r_2 = 1/2$. On the z-plane, plot the poles and zeros together with the region-of-convergence.

[2 Marks]

In the above sequence in (1), what happens if we exchange r_1 and r_2? **[2 Marks]**

For the first part,

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-(n+1)].$$

Due to linearity of the z-transform, we evaluate the individual terms as follows,

$$\underbrace{\left(-\frac{1}{3}\right)^n u[n]}_{r_1^n u[n]} \xrightarrow{\text{Z-domain}} \frac{1}{1+\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$$

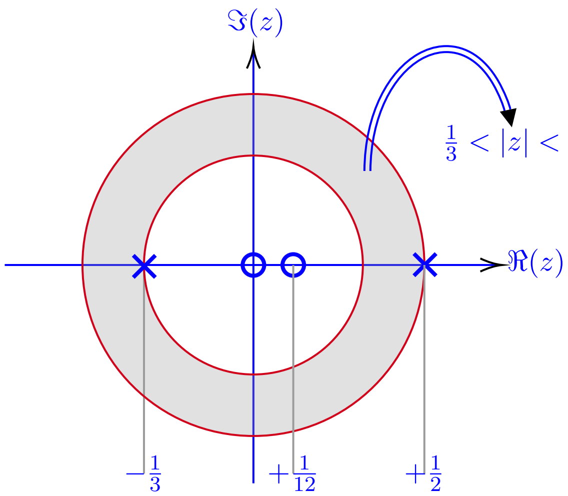
and,

$$\underbrace{-\left(\frac{1}{2}\right)^n u[-(n+1)]}_{-r_2^n u[-(n+1)]} \xrightarrow{\text{Z-domain}} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2}.$$

Hence,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z(z-1/12)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$

We see two zeros at $z = 0$ and $z = 1/12$ and two poles at $z = -1/3$ and $z = 1/2$. The RoC is marked as follows.



Inter-changing the roles of r_1 and r_2 results in,

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-(n+1)].$$

In the z-transform domain, this takes the form of,

$$X(z) = \underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{|z|<\frac{1}{3}} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}.$$

Since $[0, 1/3) \cap (1/2, \infty) = \emptyset$, the z-transform does not exist.

b) As we have seen in the course, many linear-time-invariant systems are described by difference or differential equations. For instance, consider the filter described by differential equation,

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y_c(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x_c(t) \tag{2}$$

where $x_c(t)$ is the continuous-time input and $y_c(t)$ is the corresponding output. In order to work with discrete-time systems, a standard approach is to replace the derivative operator by finite differences. To this end, let us denote the central difference by, $\Delta^n x[k]$ where n is the order of difference and the difference operation is defined by the recursive operations,

$$\begin{cases} \Delta^0 x[k] = x[k] \\ \Delta^1 x[k] = \frac{1}{2}(x[k+1] - x[k-1]) \\ \Delta^n x[k] = \Delta^1 \left(\Delta^{(n-1)} x[k]\right) \end{cases}.$$

Hence the continuous-time filter can be written as a discrete-time filter,

$$\sum_{n=0}^{N} a_n (\Delta^n y[k]) = \sum_{m=0}^{M} b_m (\Delta^n x[k]). \tag{3}$$

i) Let H_c and H_d be the transfer-functions of the continuous and discrete-time filters, respectively. How are the two transfer functions related? [4 **Marks**]

ii) What is the relation between continuous-filter frequency and digital-filter frequency? [2 **Marks**]

Part i). We start with the frequency domain representation of the equation in (2),

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y_c(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x_c(t) \xrightarrow{\text{Fourier Domain}} \widehat{y}_c(\omega) \sum_{n=0}^{N} a_n (\jmath\omega)^n = \widehat{x}_c(\omega) \sum_{m=0}^{M} b_m (\jmath\omega)^m.$$

Based on this, we can write the continuous-time response,

$$H_c(\omega) = \frac{\widehat{y}_c(\omega)}{\widehat{x}_c(\omega)} = \frac{\sum_{m=0}^{M} b_m (\jmath\omega)^m}{\sum_{n=0}^{N} a_n (\jmath\omega)^n}.$$

To relate the same with its discrete-time version in (3), we develop the z-transform,

$$y_1[k] = \left(\Delta^1 x\right)[k] = \frac{1}{2}(x[n+1] - x[n-1]) \xrightarrow{\text{Z-Transform}} Y_1(z) = \left(\frac{z - z^{-1}}{2}\right) X(z).$$

Now since, $y_2[k] = \left(\Delta^2 x\right)[k] = \left(\Delta^1 y_1\right)[k]$, we have,

$$Y_2(z) = \left(\frac{z - z^{-1}}{2}\right) Y_1(z) = \left(\frac{z - z^{-1}}{2}\right)^2 X(z).$$

Repeating the same process for n–times, it is not difficult to see that,

$$Y_n(z) = \left(\frac{z - z^{-1}}{2}\right)^n X(z).$$

We can use this to simplify (3) as follows,

$$\sum_{n=0}^{N} a_n (\Delta^n y[k]) = \sum_{m=0}^{M} b_m (\Delta^n x[k]) \xrightarrow{\text{Z-Transform}}$$
$$Y(z) \sum_{n=0}^{N} a_n \left(\frac{z - z^{-1}}{2}\right)^n = X(z) \sum_{m=0}^{M} b_m \left(\frac{z - z^{-1}}{2}\right)^m.$$

Hence, the discrete-time system's transfer function is,

$$H_d\left(z\right) = \frac{Y\left(z\right)}{X\left(z\right)} = \frac{\sum\limits_{m=0}^{M} b_m \left(\frac{z-z^{-1}}{2}\right)^m}{\sum\limits_{n=0}^{N} a_n \left(\frac{z-z^{-1}}{2}\right)^n}.$$

Comparing this with $H_c\left(\omega\right)$, we conclude that,

$$H_d\left(z\right) = H_c\left(\omega\right)\Big|_{\omega=\jmath^{-1}\left(\frac{z-z^{-1}}{2}\right)}.$$

Part ii). From the previous part, we have, $\jmath\omega = \left(\frac{z-z^{-1}}{2}\right)$. Evaluating the same at $z = e^{\jmath\Omega}$ gives the desired relation, $\omega = \sin\left(\Omega\right)$.

i) For some linear-time-invariant system, the transfer function is given by,

$$H\left(z\right) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}.$$

Suppose $x\left[n\right]$ is the input to the system and $y\left[n\right]$ is the output. Derive the difference equation that is satisfied by $x\left[n\right]$ and $y\left[n\right]$. [4 **Marks**]

The transfer function of a system $H\left(z\right)$ is the ratio of the output to the input. Hence,

$$H\left(z\right) = \frac{Y\left(z\right)}{X\left(z\right)} = \frac{1+\frac{2}{z}+\frac{1}{z^2}}{1+\frac{1}{4z}-\frac{3}{8z^2}} \Rightarrow X\left(z\right)\left(1+\frac{2}{z}+\frac{1}{z^2}\right) = Y\left(z\right)\left(1+\frac{1}{4z}-\frac{3}{8z^2}\right).$$

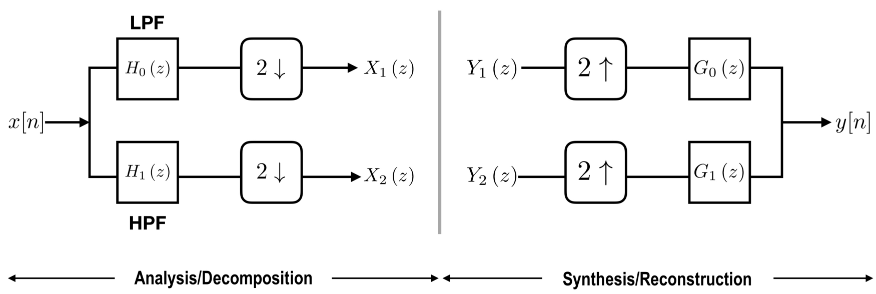
The above can be directly converted to the impulse-response format,

$$x\left[n\right] + 2x\left[n-1\right] + x\left[n-2\right] = y\left[n\right] + \tfrac{1}{4}y\left[n-1\right] - \tfrac{3}{8}y\left[n-2\right]$$
$$\Updownarrow$$
$$\sum_{k=0}^{2} a_k x\left[n-k\right] = \sum_{k=0}^{2} b_k y\left[n-k\right]$$

where,

	$k=0$	$k=1$	$k=2$
a_k	1	2	1
b_k	1	$\frac{1}{4}$	$-\frac{3}{8}$

j) Depicted below is a system that is describing what is known as the two-channel quadrature-mirror filterbank (QMF).



The input signal $x[n]$ undergoes filtering and decimation or downsampling ($\mathbf{2}\downarrow$). This is the signal analysis part. There on, given sequences or transfer functions, the synthesis part reconstructs the original signal by upsampling ($\mathbf{2}\uparrow$) followed by filtering. In the above H_0 and H_1 are low-pass and high-pass filters, respectively.

i) Identify outputs $X_1(z)$ and $X_2(z)$ given $x[n]$, $H_0(z)$ and $H_1(z)$. [2 Marks]

ii) How is $y[n]$ related to inputs transfer functions $Y_1(z)$ and $Y_2(z)$? [2 Marks]

iii) Suppose that one system's output becomes another system's input, that is,

$$Y_1(z) = X_1(z) \quad \text{and} \quad Y_2(z) = X_2(z).$$

What is the frequency response of $y[n]$? [2 Marks]

Under what conditions (in frequency domain), $y[n] = x[n]$? [2 Marks]

Let $h[n]$ be the impulse response of some filter $H(z)$. Furthermore, let $H_0(\omega) = H(\omega)$ and $H_1(\omega) = H(\omega - \pi)$. Identify the impulse response of $h_1[n]$, $g_0[n]$ and $g_1[n]$ in terms of $h[n]$. [2 Marks]

This is a classic textbook example also covered in details the notes and in the lecture, hence, short explanations are provided.

i) Here,

$$X_m(z) = \frac{1}{2}\left(V_m\left(z^{\frac{1}{2}}\right) + V_m\left(-z^{\frac{1}{2}}\right)\right), \qquad k = 0, 1$$

where $V_m(z) = H_m(z)\, X(z)\, , m = 0, 1$.

ii)

$$Y(z) = Y_1\left(z^2\right) G_0(z) + Y_2\left(z^2\right) G_1(z).$$

iii) When $Y_1(z) = X_1(z)$ and $Y_2(z) = X_2(z)$, we have,

$$\begin{aligned} Y &= \frac{1}{2}\left[\begin{array}{cc} X & \overline{X} \end{array}\right]\left[\begin{array}{cc} H_0 & H_1 \\ \overline{H}_0 & \overline{H}_1 \end{array}\right]\left[\begin{array}{c} G_0 \\ G_1 \end{array}\right] \\ &= \frac{1}{2}\left(X\left(H_0 G_0 + H_1 G_1\right) + \overline{X}\left(\overline{H}_0 G_0 + \overline{H}_1 G_1\right)\right) \end{aligned}$$

where $\overline{X}(z) = X(-z)$. Substituting $z = e^{\jmath\omega}$, we obtain the frequency response.

From the above part, we have a direct relation between $Y(z)$ and $X(z)$. For $y[n] = x[n]$ (perfect reconstruction), we can impose the follwing conditions,

$$Y = X\frac{1}{2}\underbrace{\left(H_0 G_0 + H_1 G_1\right)}_{\propto z^{-d}} + \overline{X}\frac{1}{2}\underbrace{\left(\overline{H}_0 G_0 + \overline{H}_1 G_1\right)}_{=0}$$

and hence,

$$\frac{1}{2}\left(H_0(z)\, G_0(z) + H_1(z)\, G_1(z)\right) = z^{-d} \quad \text{and} \quad \frac{1}{2}\left(\overline{H}_0(z)\, G_0(z) + \overline{H}_1(z)\, G_1(z)\right) = 0.$$

Again, replacing z by $e^{\jmath\omega}$ results in the desired conditions in the frequency domain.

In short, the filters are identified as,

$$\left[\begin{array}{cc} h_0[n] & h_1[n] \\ g_0[n] & g_1[n] \end{array}\right] = \left[\begin{array}{cc} h[n] & (-1)^n h[n] \\ 2h[n] & -2(-1)^n h[n] \end{array}\right].$$

Detailed explanation is below.

We are given,

$$\begin{aligned} H_0(\omega) &= H(\omega) \Rightarrow H_0(z) = H(z) \\ H_1(\omega) &= H(\omega - \pi) \Rightarrow H_1(z) = H(-z) \end{aligned}.$$

From this, we obtain, $h_0[n] = h[n]$ and $h_1[n] = (-1)^n h[n]$, which is because,

$$H_1(z) = H(-z) = \sum\nolimits_n h_n(-z)^{-n} = \begin{cases} +h[n]z^{-n} & n \text{ is even} \\ -h[n]z^{-n} & n \text{ is odd} \end{cases}.$$

To compute the coefficients of filters g_0 and g_1, we make use of the equation,

$$Y = X\frac{1}{2}\underbrace{(H_0G_0 + H_1G_1)}_{\propto z^{-d}} + \overline{X}\frac{1}{2}\underbrace{(\overline{H}_0G_0 + \overline{H}_1G_1)}_{=0}$$

which can be re-written as,

$$\frac{1}{2}\begin{bmatrix} H_0 & H_1 \\ \overline{H}_0 & \overline{H}_1 \end{bmatrix}\begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = \begin{bmatrix} z^{-d} \\ 0 \end{bmatrix} \xrightarrow{H_1=\overline{H}_0, H_0=H} \frac{1}{2}\begin{bmatrix} H & \overline{H} \\ \overline{H} & H \end{bmatrix}\begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = \begin{bmatrix} z^{-d} \\ 0 \end{bmatrix}.$$

The above system of equations can be solved as simultaneous equations and this yields,

$$\begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = \frac{2z^{-d}}{HH - \overline{H}\overline{H}}\begin{bmatrix} H & -\overline{H} \\ -\overline{H} & H \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} G_0 \\ G_1 \end{bmatrix} \propto \begin{bmatrix} +2H \\ -2\overline{H} \end{bmatrix}.$$

Hence $g_0[n] = 2h[n]$ and $g_1[n] = -2(-1)^n h[n]$.

k) Identify parameters p and q in the filter, $h[n] = pq^n u[n]$ given that $h[1] = 1$ and the group delay at $\omega = \pi$ is $-1/3$. **[2 Marks]**

We are given, $h[1] = 1$. For this to be true, we must have, $pq = 1$. Further, we are given that group delay at $\omega_0 = \pi$ is $-1/3$. To make use of this, we first note that the z-transform of $pq^n u[n]$ is,

$$H(z) = \frac{p}{1 - qz^{-1}}.$$

According to the definition of the group delay,

$$\tau_H = \Re\left(-\frac{z}{H(z)}\partial_z H(z)\right) \quad \text{at } z = e^{\jmath\omega}$$

and in the above (using derivative chain rule),

$$\partial_z H(z) = -\frac{pq}{(1 - qz^{-1})^2 z^2} \xrightarrow{\text{Simplification}} -\frac{z}{H(z)}\partial_z H(z) = \frac{q}{z - q}.$$

Finally, the group delay simplifies to,

$$\tau_H(\omega) = \Re\left(\frac{q}{e^{\jmath\omega} - q}\right) = \frac{q\cos(\omega) - q^2}{1 + q^2 - 2q\cos(\omega)}.$$

We are given, $\tau_H(\pi) = -\frac{1}{3}$. We use this equation to solve for q,

$$\tau_H(\pi) = -\frac{q}{1 + q} = -\frac{1}{3} \Rightarrow q = \frac{1}{2}.$$

Hence, $p = 2$.

Question 2 [30 Marks]

a) **Analysis of the Kolmogorov–Zurbenko filter.**

A. N. Kolmogorov was a celebrated Russian mathematician who has made several contributions to the field of mathematics. One such contribution is related to the study of turbulence. Let $x[n], n \in \mathbb{Z}$ be a discrete-time sequence. The Kolmogorov–Zurbenko filter output is given by,

$$y[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} c_{M,K}[l]\, x[n+l] \tag{4}$$

where M (odd integer) and K are filter parameters and the filter coefficient $c_{M,K}[l]$ is given by,

$$c_{M,K}[l] = M^{-K} p_{M,K}[l] \tag{5}$$

where $p_{M,K}[l]$ is the solution to the equation,

$$\sum_{k=0}^{K(M-1)} z^k p_{M,K}\left[k - K\frac{M-1}{2}\right] = \left(\sum_{m=0}^{M-1} z^m\right)^K. \tag{6}$$

i) What is the time-domain impulse response of the Kolmogorov–Zurbenko filter? **[4 Marks]**

We are required to find a filter h such that,

$$x[n] \to \boxed{h[n]} \to y[n].$$

By inspection, we note that,

$$y[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} c_{M,K}[l]\, x[n+l] = (\bar{c}_{M,K} * x)[n]$$

where $\bar{c}_{M,K}[n] = c_{M,K}[-n]$. And hence,

$$h[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} \bar{c}_{M,K}[l]\, \delta[n-l].$$

We note that $c_{M,K}$ is a symmetric filter.

ii) What is $c_{M,1}[l]$?

Plot the filter corresponding to $c_{M,1}[l]$. **[2 Marks]**

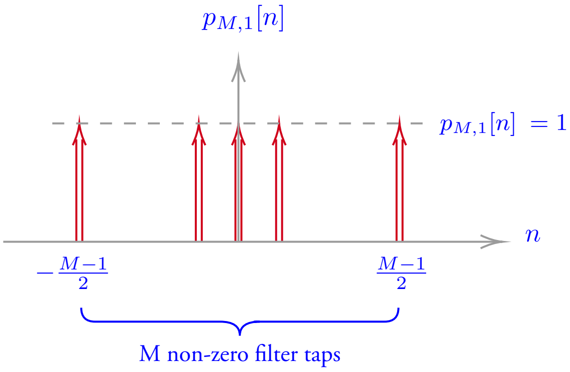
Setting $K = 1$, we observe that, $c_{M,1}[l] = \frac{1}{M} p_{M,1}[l]$. Accordingly, we have have,

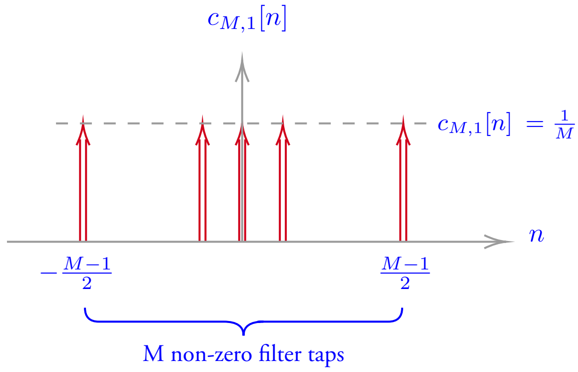
$$\sum_{k=0}^{(M-1)} z^k p_{M,1}\left[k - \frac{M-1}{2}\right] = \left(\sum_{m=0}^{M-1} z^m\right) = \underbrace{1 + z + z^2 + \cdots + z^{M-1}}_{\text{All one sequence of length M}}$$

and hence, all the values taken by the filter p are one as shown by its z-transform. This simplifies to what is known as the moving average filter,

$$p_{M,1}[l] = \sum_{m=0}^{M-1} \delta\left[l - \frac{M-1}{2}\right] \quad \text{and} \quad c_{M,1}[l] = \frac{1}{M}\sum_{m=0}^{M-1} \delta\left[l - \frac{M-1}{2}\right].$$

The filters are plotted below,





iii) When $K = 1$, what is the order of the Kolmogorov–Zurbenko filter? [4 **Marks**]

Since the size of the filter is given by,

$$\sum_{k=0}^{(M-1)} z^k p_{M,1}\left[k - \frac{M-1}{2}\right] = \left(\sum_{m=0}^{M-1} z^m\right) = \underbrace{1 + z + z^2 + \cdots + z^{M-1}}_{\text{All one sequence of length M}}$$

we observe that this filter has M taps.

iv) What is the magnitude frequency response of the Kolmogorov–Zurbenko filter? What is the effect of increasing K on the output? [9+1 **Marks**]

We have already seen that the filter response in the time-domain is given by,

$$h[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} \bar{c}_{M,K}[l]\,\delta[n-l]$$

which comprises of KM non-zero, weighted impulses. For simplicity, let us define the set,

$$\mathcal{L} = \left[-K\left(\tfrac{M-1}{2}\right), \cdots, K\left(\tfrac{M-1}{2}\right)\right].$$

The frequency response simplifies to,

$$\begin{aligned} H(\omega) &= \sum_{n\in\mathbb{Z}} h[n]\, e^{-\jmath\omega n} = \sum_{l\in\mathcal{L}} \bar{c}_{M,K}[l] \underbrace{\sum_{n\in\mathbb{Z}} e^{-\jmath\omega n}\delta[n-l]}_{=0, n\neq l} = \sum_{l\in\mathcal{L}} \bar{c}_{M,K}[l]\, e^{-\jmath\omega l} \\ &= \sum_{l\in\mathcal{L}} c_{M,K}[l]\, e^{\jmath\omega l} = \frac{1}{M^K}\sum_{l\in\mathcal{L}} p_{M,K}[l]\, e^{\jmath\omega l}. \end{aligned}$$

When comparing the above with (6), we observe that, $H(\omega)$ is nothing but (6) evaluated at $z = e^{\jmath\omega}$. Towards this end, we first simplify the right hand side, which yields,

$$\left(\sum_{m=0}^{M-1} z^m\right)^K = \left(\frac{1-z^M}{1-z}\right)^K.$$

and further to this, we have,

$$\left.\frac{1-z^M}{1-z}\right|_{z=e^{\jmath\omega}} = \frac{1-e^{\jmath\omega M}}{1-e^{\jmath\omega}} = \left(\frac{e^{-\jmath\frac{\omega}{2}M}}{e^{-\jmath\frac{\omega}{2}}}\right)\left(\frac{e^{+\jmath\frac{\omega}{2}M} - e^{-\jmath\frac{\omega}{2}M}}{e^{+\jmath\frac{\omega}{2}} - e^{-\jmath\frac{\omega}{2}}}\right) = \left(\frac{e^{-\jmath\frac{\omega}{2}M}}{e^{-\jmath\frac{\omega}{2}}}\right)\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}.$$

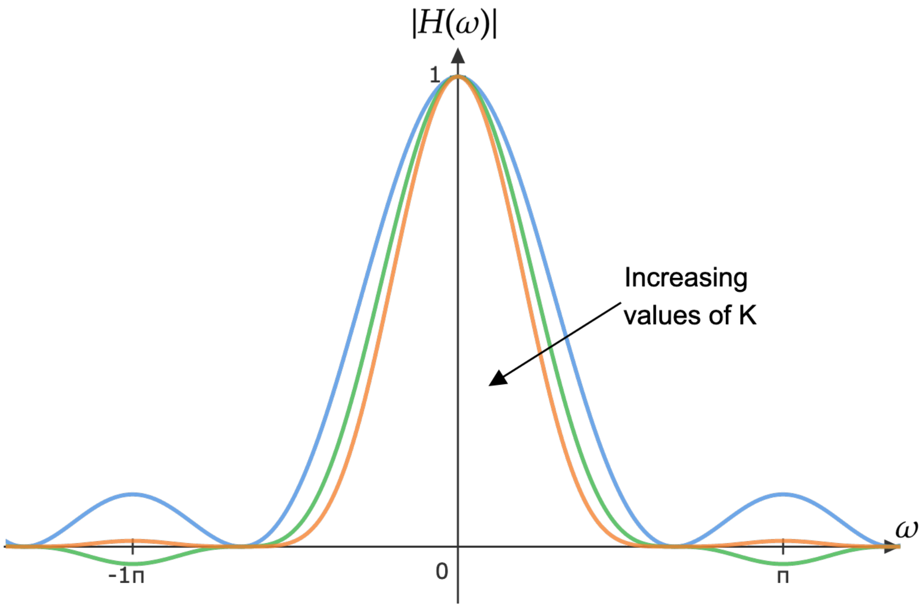
And hence,

$$H(\omega) = \sum_{l \in \mathcal{L}} c_{M,K}[l] e^{\jmath\omega l} = \frac{1}{M^K} e^{\jmath K\left(\frac{M-1}{2}\right)} \sum_{l \in \mathcal{L}} p_{M,K}[l] e^{\jmath\omega l}$$
$$= \left(\frac{1}{M}\sum_{m=0}^{M-1} e^{\jmath\omega m}\right)^K = \frac{1}{M^K}\left(\frac{e^{-\jmath\frac{\omega}{2}M}}{e^{-\jmath\frac{\omega}{2}}}\right)^K \left(\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right)^K.$$

And hence, the magnitude frequency-response is given by,

$$|H(\omega)| = \left|\frac{1}{M}\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|^K.$$

As we can see from the magnitude response, as K increases, the high-pass frequencies are attenuated sharply and this leads to greater smoothing of the input sequence. This is shown in the figure below.



b) **Optimal FIR Filter Design**

Let f be a function with maximum frequency Ω_0 with derivative,

$$g(t) = \frac{d}{dt} f(t).$$

Suppose that we want to approximate g with linear combination of f, that is,

$$g(t) \approx \alpha_1 f(t) + \alpha_2 f(t - T)$$

where T is the sampling rate and $\{\alpha_1, \alpha_2\}$ are the coefficients of a 2-tap filter. As we have seen in one of the earlier questions, continuous derivative operators (e.g. (2)) are often replaced by finite differences (e.g. (3)) when working with FIR filters. In that case, $\{\alpha_1, \alpha_2\}$ are fixed and do not depend on signal's bandwidth. For a given bandwidth Ω_0, $\{\alpha_1, \alpha_2\}$ can be obtained by minimizing the least-squares error between the derivative operator and the 2-tap FIR-filter. Since we are working with least-squares error, the same operations can also be performed in the Fourier domain. Show that if the target filter is $\jmath\omega$ (in the Fourier domain), then, the optimal values of the filter taps are,

$$\alpha_2 = -\Omega_0 \frac{\sin\theta - \theta\cos\theta}{\theta^2 - \sin^2\theta} \quad \text{and} \quad \alpha_1 = -\alpha_2 \frac{\sin\theta}{\theta} \quad \text{where} \quad \theta = \Omega_0 T.$$

[10 Marks]

To ease the computations, we provide the following integrals that will be useful. In each case, $\theta = \Omega_0 T$.

$$\int_{-\Omega_0}^{\Omega_0} \cos(\omega T)\, d\omega = 2\tfrac{\sin\theta}{T} \qquad \int_{-\Omega_0}^{\Omega_0} \cos^2(\omega T)\, d\omega = +\tfrac{\sin(2\theta)}{2T} + \Omega_0$$

$$\int_{-\Omega_0}^{\Omega_0} \omega\sin(\omega T)\, d\omega = 2\tfrac{\sin\theta - \theta\cos\theta}{T^2} \qquad \int_{-\Omega_0}^{\Omega_0} \sin^2(\omega T)\, d\omega = -\tfrac{\sin(2\theta)}{2T} + \Omega_0$$

This is an *optimal filter design* problem where we can use least-squares criterion,

$$\min_{\alpha_k} \int |E(t)|^2 dt$$

where the error term is,

$$E(t) = g(t) - (\alpha_1 f(t) + \alpha_2 f(t-T)).$$

This also allows us to directly manipulate the equations in the Fourier domain because energy in the time domain is the same as the energy in the frequency domain. In the Fourier domain, the error term reads,

$$\widehat{E}(\omega) = \underbrace{\jmath\omega}_{\text{Target Filter}} - \underbrace{\left(\alpha_1 + \alpha_2 e^{\jmath\omega T}\right)}_{\text{FIR Filter}}$$

which translates to,

$$\min_{\alpha_k} \int_{-\Omega_0}^{+\Omega_0} \left|\widehat{E}(\omega)\right|^2 d\omega = \min_{\alpha_k} \int_{-\Omega_0}^{+\Omega_0} \left|\jmath\omega - \alpha_1 - \alpha_2 e^{\jmath\omega T}\right|^2 d\omega.$$

To find the best coefficient α_1, α_2, we can evaluate this integral and minimize the same with respect to unknowns α_1, α_2. To do so, first note that,

$$\left|\jmath\omega - \alpha_1 - \alpha_2 e^{\jmath\omega T}\right|^2 = \underbrace{\omega^2}_{T_1} + \underbrace{\alpha_1^2 + \alpha_2^2}_{T_2} + \underbrace{2\alpha_1\alpha_2\cos(\omega T)}_{T_3} + \underbrace{2\alpha_2\omega\sin(\omega T)}_{T_4}$$

Integrating each term $T_1, \ldots, T_4$ using the table of integral provided, we see that the individual integration components are,

$$I_1 = \int_{-\Omega_0}^{+\Omega_0} \omega^2 d\omega = \tfrac{2}{3}\Omega_0^3$$

$$I_2 = \int_{-\Omega_0}^{+\Omega_0} \left(\alpha_1^2 + \alpha_2^2\right) d\omega = 2\left(\alpha_1^2 + \alpha_2^2\right)\Omega_0$$

$$I_3 = 2\alpha_1\alpha_2 \int_{-\Omega_0}^{+\Omega_0} \cos(\omega T)\, d\omega = 4\alpha_1\alpha_2 \tfrac{\sin(\Omega_0 T)}{T}$$

$$I_4 = 2\alpha_2 \int_{-\Omega_0}^{+\Omega_0} \omega\sin(\omega T)\, d\omega = 4\alpha_2 \left(\tfrac{\sin(\Omega_0 T) - (\Omega_0 T)\cos(\Omega_0 T)}{T^2}\right)$$

and now we have,

$$I = \int_{-\Omega_0}^{+\Omega_0} \left| \widehat{E}(\omega) \right|^2 d\omega = I_1 + I_2 + I_3 + I_4.$$

To find the values of α_k which minimize I, we differentiate I with respect to α_k, yielding,

$$\begin{aligned}
\frac{dI}{d\alpha_1} &= 4\alpha_1\Omega_0 + 4\alpha_2 \frac{\sin(\Omega_0 T)}{T} \\
\frac{dI}{d\alpha_2} &= 4\alpha_2\Omega_0 + 4\alpha_1 \frac{\sin(\Omega_0 T)}{T} + 4\frac{\sin(\Omega_0 T) - (\Omega_0 T)\cos(\Omega_0 T)}{T^2}.
\end{aligned}$$

Setting each derivative to zero (for optimal value of α_1 and α_2), we obtain a system of equations with two unknowns, namely, α_1 and α_2,

$$\frac{dI}{d\alpha_k} = 0 \iff \begin{bmatrix} \Omega_0 & \frac{\sin(\Omega_0 T)}{T} \\ \frac{\sin(\Omega_0 T)}{T} & \Omega_0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sin(\Omega_0 T) - (\Omega_0 T)\cos(\Omega_0 T)}{T^2} \end{bmatrix}$$

From the first equation, gives the result for α_1,

$$\Omega_0 \alpha_1 = -\alpha_2 \frac{\sin(\Omega_0 T)}{T} \Rightarrow \boxed{\boxed{\alpha_1 = -\alpha_2 \frac{\sin(\Omega_0 T)}{\Omega_0 T}}}.$$

In short hand notation, let us write, $S = \sin(\Omega_0 T)$ and $C = \cos(\Omega_0 T)$. Simplification of the second equation directly yields the result,

$$\begin{aligned}
&\alpha_1 \frac{S}{T} + \Omega_0 \alpha_2 = -\frac{S - (\Omega_0 T) C}{T^2} \\
&\Rightarrow -\alpha_2 \frac{S^2}{\Omega_0 T^2} + \Omega_0 \alpha_2 = -\frac{S - (\Omega_0 T) C}{T^2} \\
&\Rightarrow \alpha_2 \frac{\Omega_0^2 T^2 - S^2}{\Omega_0} = -S + (\Omega_0 T) C \\
&\Rightarrow \boxed{\boxed{\alpha_2 = \Omega_0 \frac{(\Omega_0 T)\cos(\Omega_0 T) - \sin(\Omega_0 T)}{\Omega_0^2 T^2 - \sin^2(\Omega_0 T)}}}.
\end{aligned}$$

Question 3 [30 Marks]

Windowing, Low-Pass Filtering and Digital-to-Analog Reconstruction

a) Windowing is an important aspect of filter design. In this context, one typically multiplies a function or a sequence by a window to obtain a truncated function or sequence, respectively. Examples of window functions include, rectangular window, Hanning, Hamming, Blackman–Harris and Kaiser windows.

As we have seen, most of the windows have two common properties, (a) they are positive and (b) there are non-zero on a finite interval. Suppose $w(t) \geqslant 0$ is a window function that is non-zero only on the interval $[-L, L]$. Then prove that its Fourier Transform,

$$\widehat{w}(\omega) = \int w(t) e^{-\jmath\omega t} dt$$

is always maximum at the zero-frequency or mathematically, $|\widehat{w}(\omega)| \leqslant \widehat{w}(0)$. **[4 Marks]**

The solution to this question is based on the following elementary inequality,

$$\left|\int_{x_0}^{x_1} f(x)\,dx\right| \leqslant \int_{x_0}^{x_1} |f(x)|\,dx \quad \text{and hence} \quad \left|\int_{x_0}^{x_1} f(x)\,g(x)\,dx\right| \leqslant \int_{x_0}^{x_1} |f(x)|\,|g(x)|\,dx$$

According to the definition of the Fourier transform,

$$\widehat{w}(0) = \int_{t\in[-L,L]} w(t)\,e^{-\jmath 0 t}dt = \int_{t\in[-L,L]} w(t)\,dt.$$

Hence, starting with,

$$|\widehat{w}(\omega)| = \left|\int_{t\in[-L,L]} w(t)\,e^{-\jmath\omega t}dt\right| \leqslant \int_{t\in[-L,L]} |w(t)|\underbrace{\left|e^{-\jmath\omega t}\right|}_{=1}dt \leqslant \int_{t\in[-L,L]} w(t)\,dt = \widehat{w}(0)\,.$$

b) Suppose $f(t)$ is a function with maximum frequency Ω_0. Shannon's sampling formula allows us to write,

$$f(t) = \sum_{n=-\infty}^{n=\infty} f(nT)\,\text{sinc}\left(\frac{t}{T} - n\right), \qquad T = \frac{\pi}{\Omega_0} \tag{7}$$

where T is the sampling rate. Suppose that the same function is sampled with sampling rate $T_0 \leqslant T$. We can interpret samples $f(nT_0)$ as Fourier Series coefficients, $c_n = f(nT_0)$ of some periodic function. Indeed for all n, c_n represent the coefficients of the periodic function,

$$\widehat{C}(\omega) = \frac{1}{T_0}\sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath\omega n T_0}$$

and "windowing" this with a rectangular window leads to the digital-to-analog reconstruction in (7).

- Show that $\widehat{C}(\omega)$ is a periodic function. Identify its time-period. **[2 Marks]**

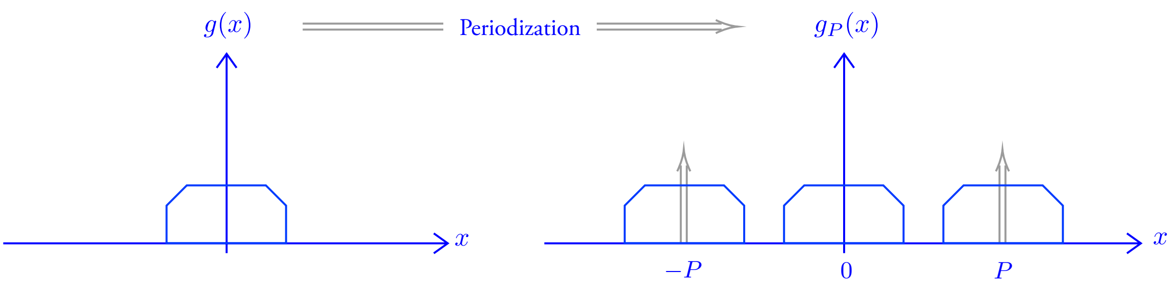
  For any integer k, we have,

$$\widehat{C}\left(\omega + \frac{2\pi k}{T_0}\right) = \frac{1}{T_0}\sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath\left(\omega + \frac{2\pi k}{T_0}\right)nT_0} = \frac{1}{T_0}\sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath\omega n T_0}\underbrace{e^{-\jmath 2\pi k n}}_{=1,k,n\in\mathbb{Z}} = \widehat{C}(\omega)\,.$$

  Hence, $\widehat{C}(\omega)$ is indeed a periodic function with least period $2\pi/T_0$.

- Show the link between $\widehat{C}(\omega)$ and the Fourier transform of $f(t)$. **[6 Marks]**

  Suppose that $g_P(x) = g_P(x+P) = \sum_{m\in\mathbb{Z}} g(x+mP)$ is a P–periodic function. Note that g_P can be obtained by periodizing any function g,



Such a periodic function $g_p(x)$ admits a Fourier series of the form,

$$g_P(x) = g_P(x+P) = \sum_{n\in\mathbb{Z}} G_k e^{\jmath xn\left(\frac{2\pi}{P}\right)}$$

where the Fourier series coefficients are given by,

$$G_n = \frac{1}{P}\int_0^P g_P(x)\, e^{-\jmath xn\left(\frac{2\pi}{P}\right)} dx.$$

Since we are given, $c_n = \int F_P(\omega)\, e^{\jmath\omega nT_0} d\omega = f(nT_0)$ with $P = 2\pi/T_0$ (from the above question), we see that,

$$\widehat{C}(\omega) = \frac{1}{T_0}\sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath\omega nT_0} = F_{\frac{2\pi}{T_0}}(\omega) = \sum_{k\in\mathbb{Z}} \widehat{f}\left(\omega + \frac{2\pi k}{T_0}\right)$$

or, $\widehat{C}(\omega)$ is the periodized version of the Fourier transform of $f(t)$.

- Show that "windowing" $\widehat{C}(\omega)$ results in (7) in the time domain. **[6 Marks]**

  From the above example, we observe that

$$\widehat{C}(\omega) = \sum_{k\in\mathbb{Z}} \widehat{f}\left(\omega + \frac{2\pi k}{T_0}\right).$$

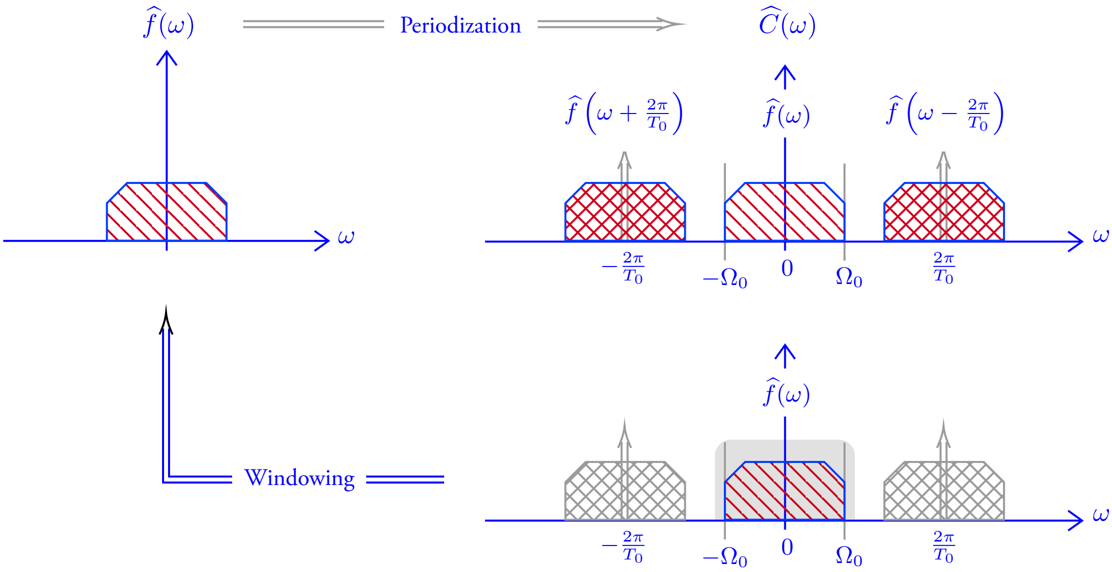
Indeed, if,

$$\left|\frac{2\pi}{T_0} - \Omega_0\right| \geqslant |\Omega_0|$$

the spectral copies of $f(t)$ do not alias and we can window $\widehat{C}(\omega)$ to obtain $\widehat{f}(\omega)$ using $\widehat{f}(\omega) = \widehat{C}(\omega)\,\widehat{h}(\omega)$ where the window is defined by,

$$\widehat{h}(\omega) = \begin{cases} 1 & |\omega| \leq |\Omega_0| \\ \text{arbitrary} & |\omega| \in \left[\Omega_0, \frac{2\pi}{T_0} - \Omega_0\right) \\ 0 & \text{elsewhere} \end{cases}.$$

Now since $\widehat{f}(\omega) = \widehat{C}(\omega)\,\widehat{h}(\omega)$, the inverse Fourier transform of $\widehat{C}(\omega)\,\widehat{h}(\omega)$ reconstructs $f(t)$. The windowing operation is shown below.



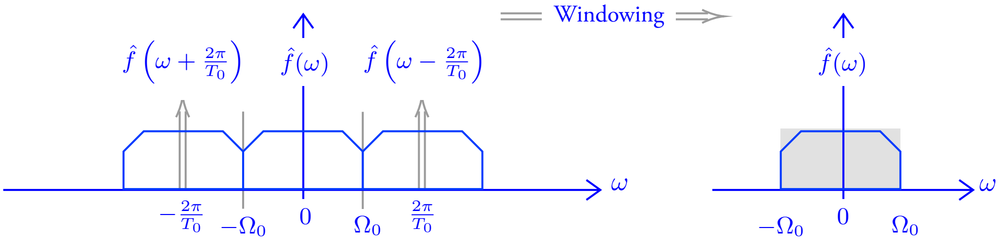
- Explain how the choice of rectangular window, in the above, leads to practical problems with implementation of (7), digital-to-analog reconstruction.

  [2 **Marks**]

  When the sampling rate T_0 is chosen such that,

$$\left|\frac{2\pi}{T_0} - \Omega_0\right| = |\Omega_0|$$

  the spectral copies exactly overlap.



  In this case, the rectangular window is the only choice. This is equivalent to convolution with the sinc filter in the time-domain. Such a filter has a sharp transition bandwidth resulting a filter that would require a high number of filter taps.
  Equivalently, since,

$$\text{sinc}\,(t) = \frac{\sin(\pi t)}{\pi t}$$

  it decays with the envelope of $1/t$ which implies that samples far away from origin affect the reconstruction at origin. Hence local reconstruction is not possible unless very long filters are used, resulting in implementation complexities.

- To be able to use more general window functions than the rectangular window, what conditions should be imposed on T_0? [4 **Marks**]

  By using T_0 such that,

$$\left|\frac{2\pi}{T_0} - \Omega_0\right| > |\Omega_0| \Rightarrow T_0 \leqslant \frac{\pi}{\Omega_0}$$

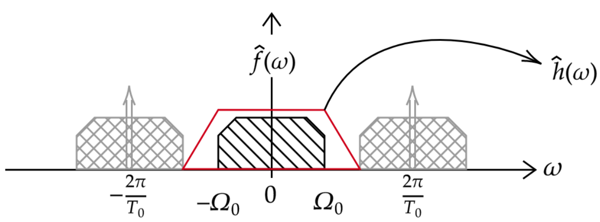
  we see that smaller the value of T_0, the farther away are the spectral copies and the easier it is to design a windowing filter.

- Sketch an example of a windowing function that is suitable for reconstruction of $f(t)$ from its samples. Explain the reasons behind your choice. [4 **Marks**]

  An example of the window that is based on,

$$\widehat{h}(\omega) = \begin{cases} 1 & |\omega| \leq |\Omega_0| \\ \text{arbitrary} & |\omega| \in \left[\Omega_0, \frac{2\pi}{T_0} - \Omega_0\right) \\ 0 & \text{elsewhere} \end{cases}.$$

  is sketched below.



  By relaxing the transition bandwidth of the filter, we can implement it much more efficiently as the filter is localized in the time domain (equivalent to lesser number of filter taps).

- What modifications should be made to formula (7) so that the new window function you have designed above can recover $f(t)$ from samples $f(nT_0)$? **[2 Marks]**

  Since the rectangular window is replaced by a more general window function, in the reconstruction formula (7), we replace sinc function by a more general filter $h(t)$ which is the inverse Fourier transform of $\widehat{h}(\omega)$. Equivalently, we have,

$$f(t) = \sum_{n=-\infty}^{n=\infty} f(nT)\, h\left(\frac{t}{T} - n\right).$$

Question 4 [30 Marks]

Engineering and Implementation Aspects

a) **Linear Phase Filters**

Let $M = 8$ be the filter order. In the context of linear phase filter design,

- Write the equation for transfer function, $H(z)$ corresponding to a linear phase filter. **[2 Marks]**
- Show the diagram for **Direct Form** implementation. **[2 Marks]**
- Show the diagram for **Transpose Form** implementation. **[2 Marks]**
- The maximum sequential delay (MSD) is attributed to the number of additions and multiplications involved in the implementation of the filter. Compare the MSD for **Direct Form** and **Transpose Form** implementations. Argue which implementation is better. **[2 Marks]**

This question is a classic text book example.

- Linear phase filter has symmetric/anti-symmetric coefficients.
  Here, $M = 8$, which is even, and hence,

$$H(z) = \sum_{m\in[0,M]} h[m]\, z^{-m} = h\left[\frac{M}{2}\right] z^{-\frac{M}{2}} + \sum_{m\in\left[0,\frac{M}{2}-1\right]} h[m]\left(z^{-m} + z^{m-M}\right).$$

  This gives,

$$\begin{aligned} H(z) &= h[4]\, z^{-4} + \sum_{m\in[0,3]} h[m]\left(z^{-m} + z^{m-8}\right) \\ &= h[4]\, z^{-4} + h[0]\left(1 + z^{-8}\right) + h[1]\left(z^{-1} + z^{-7}\right) + h[2]\left(z^{-2} + z^{-6}\right) + h[3]\left(z^{-3} + z^{-5}\right). \end{aligned}$$

- The direct and transpose form have been discussed in the course notes and the case of $M = 8$ can be directly adapted from the example of $M = 3$ in the notes and is shown in Fig. 1.
- Direct form: Maximum sequential delay $= 4a + m$ while, Transpose form: Maximum sequential delay $= a + m$, where a and m are the delays of adder and multiplier respectively. Hence, the transpose form implementation is better as it requires lesser resources.

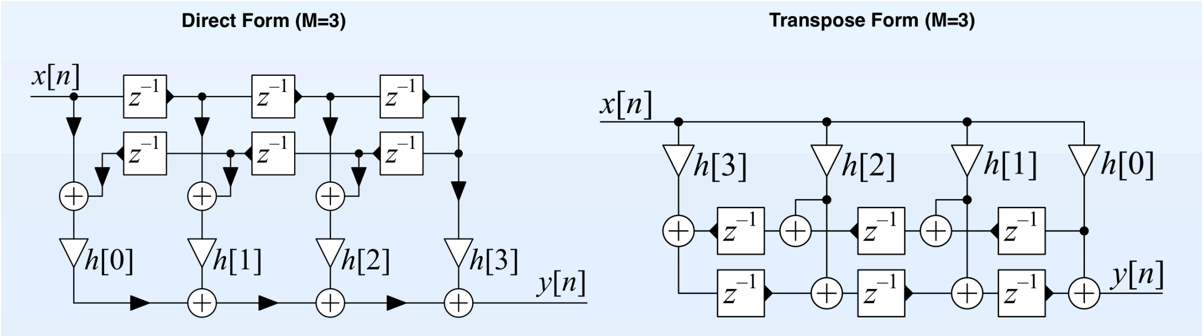


Fig. 1: Direct form and transpose form implementation example.

b) **Inverse Filter Design**

Suppose that a function is given by,

$$\phi(t) = \begin{cases} \frac{2}{3} - |t|^2 + \frac{|t|^3}{2} & 0 \leqslant |t| < 1 \\ \frac{(2-|t|)^3}{6} & 1 \leqslant |t| < 2 \\ 0 & 2 \leqslant |t| \end{cases}.$$

- Is $\phi(t)$ a symmetric function? Argue by plotting this function. [2 Marks]
- Convert $\phi(t)$ into an FIR filter by sampling it at integer points.
  Let $\Phi(z)$ be the z-transform of this FIR filter. Write the explicit form of $\Phi(z)$. [2 Marks]
- Inverse filter design. Suppose that the FIR filter is defined by

  $$p[k] = \phi(t)|_{t=k}, k = \mathbb{Z} \qquad \text{(that is, } k \text{ takes integer values).}$$

  Then, we say that $p_{\mathsf{inv}}[k]$ is an inverse-filter when,

  $$p_{\mathsf{inv}}[k] * p[k] = \delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

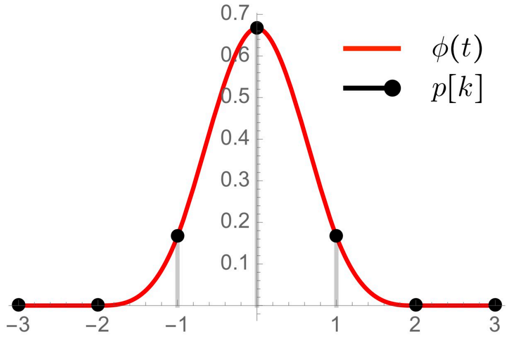
  Identify the transfer function of $p_{\mathsf{inv}}[k]$ in terms of $\Phi(z)$. [2 Marks]
  Write down the impulse-response of $p_{\mathsf{inv}}[k]$ given the definition of $\phi(t)$. [4 Marks]
  Is $p_{\mathsf{inv}}[k]$ an FIR or IIR filter? [1 Marks]
  Plot the impulse response of $p_{\mathsf{inv}}[k]$. [1 Marks]

- This is a symmetric filter because in the definition of $\phi(t)$, the intervals are defined in terms of $|t|$ and hence, $\phi(t) = \phi(-t)$. The function is plotted in Fig. 2.
- By sampling on integer points, we obtain,

  $$\phi(k)|_{k\in\mathbb{Z}} = \frac{1}{6}(\delta[k+1] + 4\delta[k] + \delta[k-1]) = p[k].$$

  Accordingly, its z-transform is given by,

  $$\Phi(z) = \frac{1}{6}\left(z^{-1} + 4 + z\right).$$



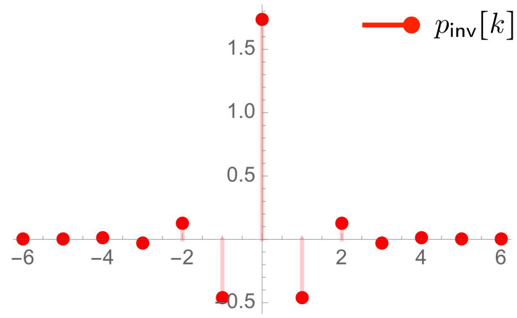


Fig. 2: Filter and Inverse Filter.

- By the definition of the inverse filter, we have,

$$p[k] * p_{\text{inv}}[k] = \delta[k] \xrightarrow{\text{z-domain}} \Phi(z)\,\Phi_{\text{inv}}(z) = 1.$$

And hence, we have,

$$\Phi_{\text{inv}}(z) = \frac{1}{\Phi(z)} = \frac{6}{z^{-1}+4+z}.$$

where Φ_{inv} is the z-transform of p_{inv}.
Let $b_0 = \sqrt{3}-2$ be the smallest root of $z^{-1}+4+z$. Then, we have,

$$\Phi_{\text{inv}}(z) = \frac{-6b_0}{(1-b_0z^{-1})(1-b_0z)} = \frac{-6b_0}{1-b_0^2}\left(\frac{1}{(1-b_0z^{-1})} + \frac{1}{(1-b_0z)} - 1\right).$$

This is a sum of first-order filters and its impulse response is given by,

$$p_{\text{inv}}[k] = \left(\frac{-6b_0}{1-b_0^2}\right) b_0^{|k|}.$$

This is an IIR filter and is plotted in Fig. 2.

c) **Multirate System**

Consider the following system with input $x[n]$ and output $y[n]$.

$$x[n] \to \boxed{1:3} \to \boxed{2:1} \to \boxed{\frac{z^{-6}}{\alpha - z^{-6} + \beta z^{-12}}} \to \boxed{2:1} \to \boxed{1:3} \to y[n]$$

Suppose that an equivalent representation of the above system is given by,

$$x[n] \to \boxed{4:1} \to \boxed{\frac{B(z)}{A(z)}} \to \boxed{1:9} \to y[n].$$

Identify α and β if $A(z)$ has roots at $1/4$ and $1/3$. **[5 Marks]**

The following sequence leads to the solution,

$$x[n] \rightarrow \underbrace{\boxed{1:3} \rightarrow \boxed{2:1}}_{\text{Exchange}} \rightarrow \underbrace{\boxed{H(z)} \rightarrow \boxed{2:1}}_{\text{Use Noble Identity}} \rightarrow \boxed{1:3} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{2:1} \rightarrow \underbrace{\boxed{1:3} \rightarrow \boxed{2:1}}_{\text{Exchange}} \rightarrow \boxed{H_1(z)} \rightarrow \boxed{1:3} \rightarrow y[n]$$

$$x[n] \rightarrow \underbrace{\boxed{2:1} \rightarrow \boxed{2:1}}_{\text{Merge}} \rightarrow \underbrace{\boxed{1:3} \rightarrow \boxed{H_1(z)}}_{\text{Exchange using Noble Identity}} \rightarrow \boxed{1:3} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{4:1} \rightarrow \boxed{H_2(z)} \rightarrow \underbrace{\boxed{1:3} \rightarrow \boxed{1:3}}_{\text{Merge}} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{4:1} \rightarrow \boxed{H_2(z)} \rightarrow \boxed{1:9} \rightarrow y[n]$$

where,

$$H_1(z) = \frac{z^{-3}}{\alpha - z^{-3} + \beta z^{-6}}$$

and,

$$H_2(z) = \frac{z^{-1}}{\alpha - z^{-1} + \beta z^{-2}}$$

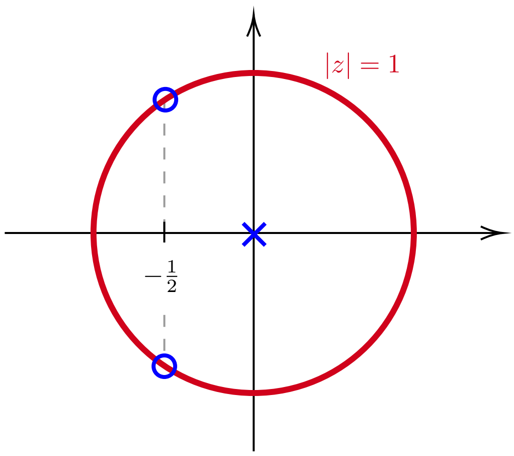
yielding, $B(z) = z^{-1}$ and $A(z) = \alpha - z^{-1} + \beta z^{-2}$. We are given that,

$$A\left(\frac{1}{3}\right) = 0 \Rightarrow \alpha + 9\beta - 3 = 0 \text{ and } A\left(\frac{1}{4}\right) = 0 \Rightarrow \alpha + 16\,\beta - 4 = 0$$

and hence, $\alpha = 12/7$ and $\beta = 1/7$.

d) **Filter Identification**

The pole-zero plot of a discrete filter is given below. When the input $x[n] = 1$ for all n, the output is exactly the same. What is the impulse response of such a filter?



[5 Marks]

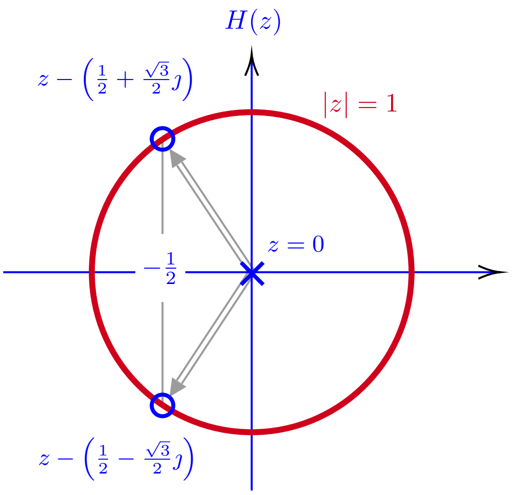
The filter is identified as,

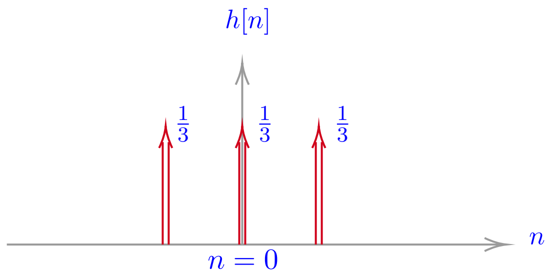
$$h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right).$$

From the provided pole-zero plot, we observes the following poles and zeros,

Pole(s)	Zero(s)
$z_p = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$	$z_0 = 0$

.





Based on this information, we can construct the z-transform as follows,

$$H(z) = \frac{Y(z)}{X(z)} = K \frac{\left(z - \frac{1}{2} + \jmath\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - \jmath\frac{\sqrt{3}}{2}\right)}{z}.$$

Also we are given that $\forall n, x[n] = 1 \Rightarrow y[n] = 1$.

This implies, $X(z)|_{z=1} = Y(z)|_{z=1} \Rightarrow H(1) = 1$.

Since $H(z) = z^{-1}K\left(z + 1 + z^{-1}\right)$, we obtain, $H(1) = 1 = 3K$ or $K = 1/3$ and,

$$H(z) = \frac{1}{3}\frac{z + 1 + z^{-1}}{z} \longleftrightarrow h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right).$$