4: Linear Time Invariant Systems

LTI Systems

Linear Time-invariant (LTI) systems have two properties:

Linear: $\mathscr{H}\left(\alpha u[n]+\beta v[n]\right)=\alpha\mathscr{H}\left(u[n]\right)+\beta\mathscr{H}\left(v[n]\right)$
Time Invariant: $y[n]=\mathscr{H}\left(x[n]\right)\Rightarrow y[n-r]=\mathscr{H}\left(x[n-r]\right)\forall r$

The behaviour of an LTI system is completely defined by its impulse response: $h[n]=\mathscr{H}\left(\delta[n]\right)$

Proof:

We can always write $x[n]=\sum_{r=-\infty}^{\infty}x[r]\delta[n-r]$

Hence

$$\begin{aligned}\mathscr{H}\left(x[n]\right)&=\mathscr{H}\left(\sum_{r=-\infty}^{\infty}x[r]\delta[n-r]\right)\\&=\sum_{r=-\infty}^{\infty}x[r]\mathscr{H}\left(\delta[n-r]\right)\\&=\sum_{r=-\infty}^{\infty}x[r]h[n-r]\\&=x[n]*h[n]\end{aligned}$$

Convolution Properties

Convolution: $x[n] * v[n] = \sum_{r=-\infty}^{\infty} x[r]v[n-r]$

Convolution obeys normal arithmetic rules for multiplication:

Commutative: $x[n] * v[n] = v[n] * x[n]$

Proof: $\sum_r x[r]v[n-r] \overset{\text{(i)}}{=} \sum_p x[n-p]v[p]$
(i) substitute $p = n - r$

Associative: $x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$
$\Rightarrow x[n] * v[n] * w[n]$ is unambiguous

Proof: $\sum_{r,s} x[n-r]v[r-s]w[s] \overset{\text{(i)}}{=} \sum_{p,q} x[p]v[q-p]w[n-q]$
(i) substitute $p = n - r$, $q = n - s$

Distributive over $+$:
$x[n] * (\alpha v[n] + \beta w[n]) = (x[n] * \alpha v[n]) + (x[n] * \beta w[n])$
Proof: $\sum_r x[n-r] (\alpha v[r] + \beta w[r]) =$
$\alpha \sum_r x[n-r]v[r] + \beta \sum_r x[n-r]w[r]$

Identity: $x[n] * \delta[n] = x[n]$

Proof: $\sum_r \delta[r]x[n-r] \overset{\text{(i)}}{=} x[n]$ (i) all terms zero except $r = 0$.

BIBO Stability

BIBO Stability: Bounded Input, $x[n] \Rightarrow$ Bounded Output, $y[n]$

The following are equivalent:
(1) An LTI system is BIBO stable
(2) $h[n]$ is absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
(3) $H(z)$ region of absolute convergence includes $|z| = 1$.

Proof $(1) \Rightarrow (2)$:

Define $x[n] = \begin{cases} 1 & h[-n] \geq 0 \\ -1 & h[-n] < 0 \end{cases}$

then $y[0] = \sum x[0-n]h[n] = \sum |h[n]|$.

But $|x[n]| \leq 1 \forall n$ so BIBO $\Rightarrow y[0] = \sum |h[n]| < \infty$.

Proof $(2) \Rightarrow (1)$:

Suppose $\sum |h[n]| = S < \infty$ and $|x[n]| \leq B$ is bounded.

Then $|y[n]| = \left|\sum_{r=-\infty}^{\infty} x[n-r]h[r]\right|$

$$\leq \sum_{r=-\infty}^{\infty} |x[n-r]|\,|h[r]|$$

$$\leq B \sum_{r=-\infty}^{\infty} |h[r]| \leq BS < \infty$$

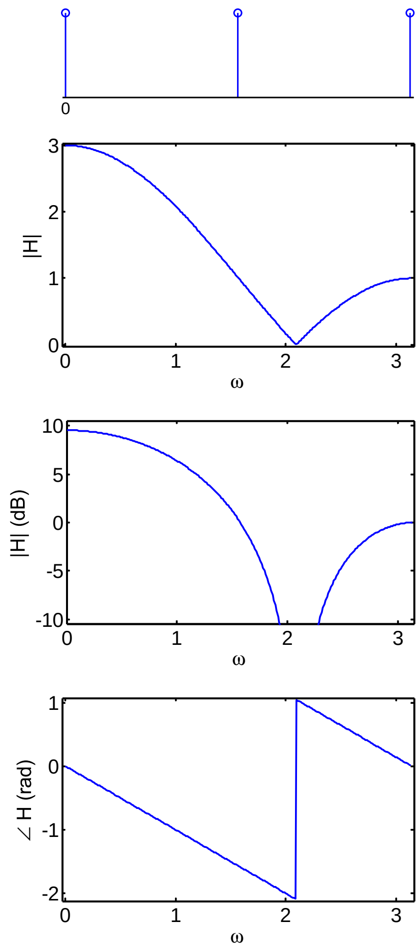
Frequency Response

For a BIBO stable system $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
where $H(e^{j\omega})$is the DTFT of $h[n]$ i.e. $H(z)$ evaluated at $z = e^{j\omega}$.

Example: $h[n] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega}$$
$$= e^{-j\omega}\left(1 + 2\cos\omega\right)$$

$$\left|H(e^{j\omega})\right| = \left|1 + 2\cos\omega\right|$$
$$\angle H(e^{j\omega}) = -\omega + \pi\frac{1-\operatorname{sgn}(1+2\cos\omega)}{2}$$



Sign change in $(1 + 2\cos\omega)$ at $\omega = 2.1$ gives
(a) gradient discontinuity in $|H(e^{j\omega})|$
(b) an abrupt phase change of $\pm\pi$.

Group delay is $-\frac{d}{d\omega}\angle H(e^{j\omega})$: gives delay of the modulation envelope at each ω.
Normally varies with ω but for a symmetric filter it is constant: in this case $+1$ samples.
Discontinuities of $\pm k\pi$ do not affect group delay.

Causal System: cannot see into the future

i.e. output at time n depends only on inputs up to time n.

Formal definition:

If $v[n] = x[n]$ for $n \leq n_0$ then $\mathscr{H}(v[n]) = \mathscr{H}(x[n])$ for $n \leq n_0$.

The following are equivalent:

(1) An LTI system is causal

(2) $h[n]$ is causal $\Leftrightarrow h[n] = 0$ for $n < 0$

(3) $H(z)$ converges for $z = \infty$

Any right-sided sequence can be made causal by adding a delay.

All the systems we will deal with are causal.

Conditions on $h[n]$ and $H(z)$

Summary of conditions on $h[n]$ for LTI systems:

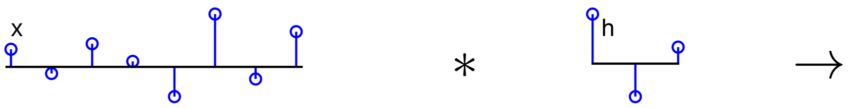
Causal	$\Leftrightarrow$	$h[n] = 0$ for $n < 0$
BIBO Stable	$\Leftrightarrow$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$

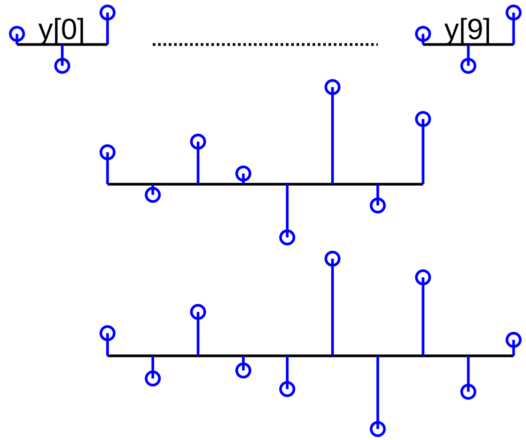
Summary of conditions on $H(z)$ for LTI systems:

Causal	$\Leftrightarrow$	$H(\infty)$ converges
BIBO Stable	$\Leftrightarrow$	$H(z)$ converges for $\lvert z \rvert = 1$
Passive	$\Leftrightarrow$	$\lvert H(z) \rvert \le 1$ for $\lvert z \rvert = 1$
Lossless or Allpass	$\Leftrightarrow$	$\lvert H(z) \rvert = 1$ for $\lvert z \rvert = 1$

Convolution Complexity

$y[n] = x[n] * h[n]$: convolve $x[0:N-1]$ with $h[0:M-1]$





$N = 8,\ M = 3$
$M + N - 1 = 10$

Convolution sum:

$y[n] = \sum_{r=0}^{M-1} h[r]x[n-r]$

$y[n]$ is only non-zero in the range
$$0 \le n \le M + N - 2$$
Thus $y[n]$ has only
$M + N - 1$ non-zero values

Algebraically:

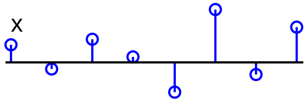
$$x[n-r] \neq 0 \Rightarrow 0 \le n - r \le N - 1$$
$$\Rightarrow n + 1 - N \le r \le n$$

Hence: $y[n] = \sum_{r=\max(0,n+1-N)}^{\min(M-1,n))} h[r]x[n-r]$

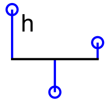
We must multiply each $h[n]$ by each $x[n]$ and add them to a total
$\Rightarrow$ total arithmetic complexity ($\times$ or $+$ operations) $\approx 2MN$

Circular Convolution

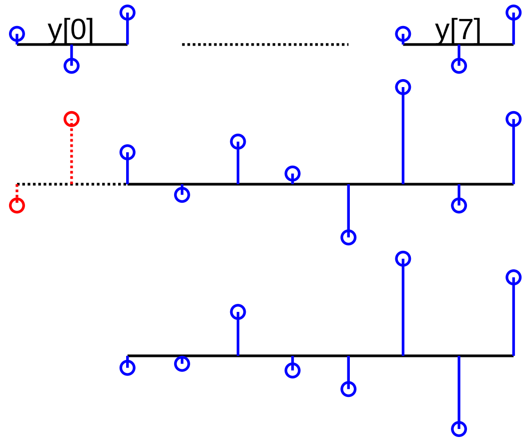
$y_{\circledast}[n] = x[n] \circledast_N h[n]$: circ convolve $x[0:N-1]$ with $h[0:M-1]$



$\circledast_N$



$\rightarrow$



Convolution sum:

$y_{\circledast_N}[n] = \sum_{r=0}^{M-1} h[r]x[(n-r)_{\text{mod } N}]$

$y_{\circledast_N}[n]$ has period N
$\Rightarrow y_{\circledast_N}[n]$ has N distinct values

$N = 8,\ M = 3$

- Only the first $M-1$ values are affected by the circular repetition:
  $y_{\circledast_N}[n] = y[n]$ for $M-1 \leq n \leq N-1$
- If we append $M-1$ zeros (or more) onto $x[n]$, then the circular repetition has no effect at all and:
  $y_{\circledast_{N+M-1}}[n] = y[n]$ for $0 \leq n \leq N+M-2$

Circular convolution is a necessary evil in exchange for using the DFT

Idea: Use DFT to perform circular convolution - less computation

(1) Choose $L \geq M + N - 1$ (normally round up to a power of 2)

(2) Zero pad $x[n]$ and $h[n]$ to give sequences of length L: $\tilde{x}[n]$ and $\tilde{h}[n]$

(3) Use DFT: $\tilde{y}[n] = \mathcal{F}^{-1}(\tilde{X}[k]\tilde{H}[k]) = \tilde{x}[n] \circledast_L \tilde{h}[n]$

(4) $y[n] = \tilde{y}[n]$ for $0 \leq n \leq M + N - 2$.

Arithmetic Complexity:

DFT or IDFT take $4L\log_2 L$ operations if L is a power of 2
(or $16L\log_2 L$ if not).

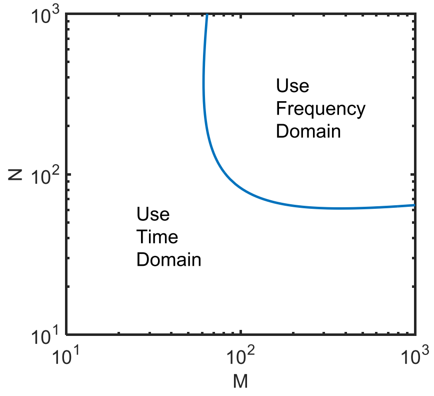
Total operations: $\approx 12L\log_2 L \approx 12\,(M+N)\log_2(M+N)$

Beneficial if both M and N are $>\sim 70$.

Example: $M = 10^3$, $N = 10^4$:

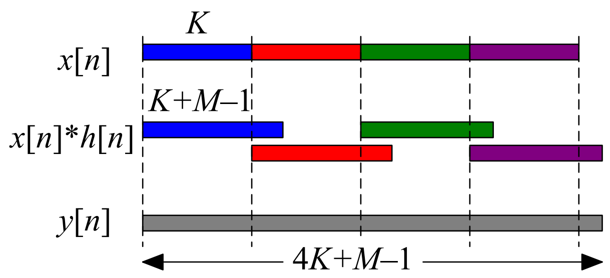
Direct: $2MN = 2 \times 10^7$

with DFT: $= 1.8 \times 10^6$ ☺



But: (a) DFT may be very long if N is large
(b) No outputs until all $x[n]$ has been input.

If N is very large:
(1) chop $x[n]$ into $\frac{N}{K}$ chunks of length K
(2) convolve each chunk with $h[n]$
(3) add up the results



Each output chunk is of length $K+M-1$ and overlaps the next chunk

Operations: $\approx \frac{N}{K} \times 8\left(M+K\right)\log_2\left(M+K\right)$

Computational saving if $\approx 100 < M \ll K \ll N$

Example: $M = 500$, $K = 10^4$, $N = 10^7$

Direct: $2MN = 10^{10}$

single DFT: $12\left(M+N\right)\log_2\left(M+N\right) = 2.8 \times 10^9$

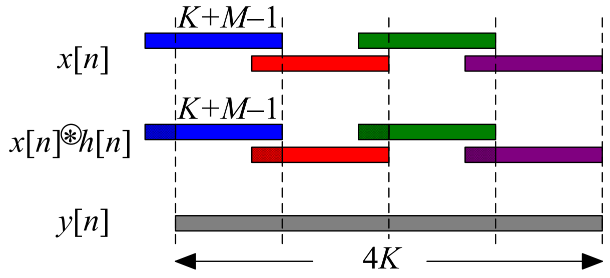
overlap-add: $\frac{N}{K} \times 8\left(M+K\right)\log_2\left(M+K\right) = 1.1 \times 10^9$ ☺

Other advantages:
(a) Shorter DFT
(b) Can cope with $N = \infty$
(c) Can calculate $y[0]$ as soon as $x[K-1]$ has been read:
algorithmic delay $= K-1$ samples

Overlap Save

Alternative method:

(1) chop $x[n]$ into $\frac{N}{K}$ overlapping chunks of length $K+M-1$
(2) $\circledast_{K+M-1}$ each chunk with $h[n]$
(3) discard first $M-1$ from each chunk
(4) concatenate to make $y[n]$



The first $M-1$ points of each output chunk are invalid

Operations: slightly less than overlap-add because no addition needed to create $y[n]$

Advantages: same as overlap add

Strangely, rather less popular than overlap-add

Summary

- LTI systems: impulse response, frequency response, group delay
- BIBO stable, Causal, Passive, Lossless systems
- Convolution and circular convolution properties
- Efficient methods for convolution
  - single DFT
  - overlap-add and overlap-save

For further details see Mitra: 4 & 5.

MATLAB routines

fftfilt	Convolution using overlap add
$x[n] \circledast y[n]$	real(ifft(fft(x).*fft(y)))