Logbook BC

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Ex-1

1° According to the sampling theory,

for a band base signal $x_b(t)$ with band-limit $\frac{W}{2}$

$$x_b(t) = \sum_n x[n]\, \text{sinc}(Wt - n)$$

where $x[n] = x_b(\frac{n}{W})$ (Sampling with $\frac{1}{W}$ rate)

2° Baseband receive signal $y_b(t)$

$$y_b(t) = \sum_i a_i^b(t)\, x_b(t - \tau_i(t)) + w(t)$$

where $a_i^b(t) = a_i(t)\, e^{-j2\pi f_c \tau_i(t)}$

3° Substitution of $x_b(t)$ in $y_b(t)$

$$y_b(t) = \sum_n x[n] \sum_i a_i^b(t)\, \text{sinc}(Wt - W\tau_i(t) - n)$$

4°. Sample $y_b(t)$ with $T_s = \frac{1}{W}$

$$y[m] = y_b(\tfrac{m}{W}) = \sum_n x[n] \sum_i a_i^b(\tfrac{m}{W}) \cdot \text{sinc}(m - n - \tau_i(\tfrac{m}{W}) \cdot W) + w[m]$$

5° Let $l = m - n$

$$y[m] = \sum_l x[m-l] \cdot \sum_i a_i^b(\tfrac{m}{W}) \cdot \text{sinc}(l - \tau_i(\tfrac{m}{W}) \cdot W) + w[m]$$

Let $h_l[m] = \sum_i a_i^b(\frac{m}{W}) \cdot \text{sinc}(l - \tau_i(\frac{m}{W}) \cdot W)$

$$y[m] = \sum_l h_l[m] \cdot x[m-l] + w[m]$$

Frequency selective channel $\longrightarrow$ several subchannel $\longrightarrow$ OFDM

$y[m] = \sum_{l=0}^{L-1} h_l x[m-L] + w[m]$ $\qquad\qquad$ $y_n[i] = h_n[i]\, d_n[i] + w_n[i]$

$$\underbrace{\hspace{8cm}}_{\text{DFT}}$$

Ex-2

The optimization problem

$$\max_{P_0,\dots,P_{Nc-1}} \sum_{n=0}^{Nc-1} \log\left(1+\frac{P_n|h_n|^2}{N_0}\right)$$

$$\text{s.t.} \begin{cases} \sum_{n=0}^{Nc-1} P_n = P \\ P_n \geq 0,\ n=0,\dots,Nc-1 \end{cases}$$

Using Lagrange method to address this optimization

$$L(\lambda, P_0, \dots, P_{Nc-1}) = \sum_{n=0}^{Nc-1} \log\left(1+\frac{P_n|h_n|^2}{N_0}\right) - \lambda\left(\sum_{n=0}^{Nc-1} P_n - P\right)$$

$$\frac{\partial L}{\partial P_n} = \frac{|h_n|^2}{(N_0 + P_n|h_n|^2)\cdot \ln 2}$$

To find the KKT condition of this optimization problem with $P_n \geq 0$,

Let $L_0(u, \lambda, P_0, \dots, P_{Nc-1}) = L(\lambda, P_0, \dots, P_{Nc-1}) - \sum_{n=0}^{Nc-1} u_n(-P_n)$

The KKT conditions are

$$\nabla_{P_n} L_0 = 0,\quad u_n \geq 0,\quad u_n P_n = 0,\quad -P_n \leq 0,\quad \sum_{n=0}^{Nc-1} P_n - P = 0$$

$$\frac{\partial L_0}{\partial P_n} = \frac{\partial L}{\partial P_n} + u_n = 0$$

$$\Rightarrow \frac{\partial L}{\partial P_n} = -u_n$$

($u_n P_n = 0 \Rightarrow u_n = 0$)

1° If $P_n > 0$, $u_n = 0$ satisfies all the KKT conditions

$$\frac{\partial L}{\partial P_n} = -u_n = 0$$

2° If $P_n = 0$, $u_n \geq 0$ satisfies all the KKT conditions

$$\frac{\partial L}{\partial P_n} = -u_n \leq 0$$

($u_n \geq 0$, $P_n = 0$, $u_n P_n = 0 \Rightarrow u_n \geq 0$)

Therefore, for this optimization problems,

the KKT conditions are

$$\begin{cases} \frac{\partial L}{\partial P_n} = 0, & \text{if } P_n > 0 \\ \frac{\partial L}{\partial P_n} \leq 0, & \text{if } P_n = 0 \end{cases}$$

Step 2

Note: if the P_n is negative by calculation, P_n is set to 0

For $P_n > 0$,

$$\frac{\partial L}{\partial P_n} = \frac{|h_n|^2}{(N_0 + P_n|h_n|^2)\ln 2} - \lambda = 0$$

$$\Rightarrow \boxed{P_n = \frac{1}{\lambda \cdot \ln 2} - \frac{N_0}{|h_n|^2}}$$

$$P_n = \frac{1}{\lambda \ln 2} - \frac{N_0}{|h_n|^2}$$

$$P = \sum_{n=0}^{Nc-1} P_n = \frac{Nc}{\lambda \ln 2} - N_0 \cdot \sum_{n=0}^{Nc-1} \frac{1}{|h_n|^2}$$

$$\Rightarrow \lambda = \frac{Nc}{\left(P + N_0 \sum_{n=0}^{Nc-1} \frac{1}{|h_n|^2}\right) \cdot \ln 2}$$

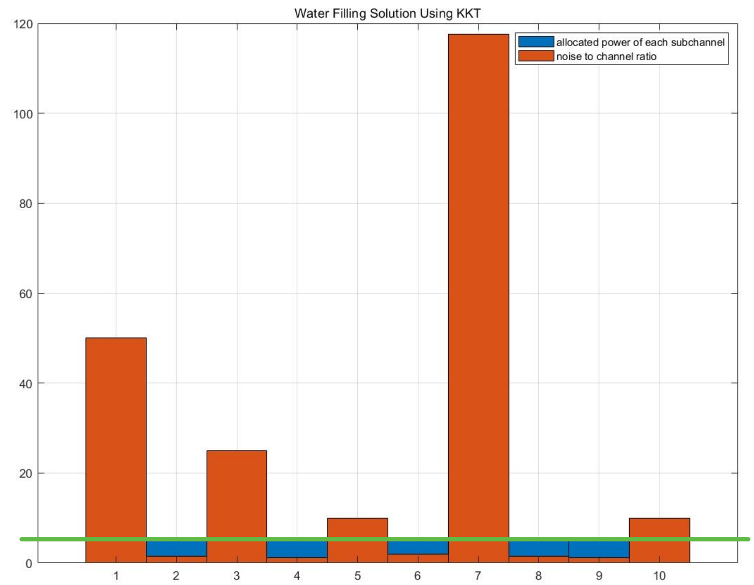
```
%% initialization
Nc = 10;
N = 1;
lambda = 0.28;
h = [.1+.1i, .2+.8i, .01+.2i, .1+.9i, .3+.1i, .1+.7i, .09+.02i, .1+.8i, .4+.8i, .1+.3i];
h_2 = abs(h).^2;

%% calaulate the power allocation
P = 1/(lambda * log(2)) - N./h_2;
P(P<=0) = 0;
% check validation of the result by assuming Pn is zero
validation = h_2./(N * log(2)) - lambda;

%% plotting
carrierNoise = N ./ h_2;
bar(P+carrierNoise, 1);
hold on;
bar(carrierNoise, 1);
grid on;
title('Water Filling Solution Using KKT')
legend('allocated power of each subchannel', 'noise to channel ratio');
```

Step 3

$\lambda = 0.128$



The result figure is shown on the left.

The orange part shows the noise to channel ratio, which is calculated by $\frac{N_0}{|h_n|^2}$.

The noise to channel ratio indicates the badness of the subchannel. The higer noise to channel ratio is, the worse the channel state is.

The blue parts are the power allocated to each subchannel.

It is noticable that the subchannels allocated with powe have the bars of the same height, which is shown by the green line in the figure.

It is just like fill the water to a "bowel" consisting of the orange bars.

The parts with lower bars (the better channel state) will be filled by water firstly.

In this case, we always try to exploit the subchannel with better state to transmit more information, so that the maximum capacity can be obtained.

EX 3

the optimization problem

$$\max_{P_0,\cdots,P_{N_C-1}} \sum_{n=0}^{N_C-1} \log\left(1+\frac{P_n|h_n|^2}{N_0}\right)$$

$$\text{s.t.} \begin{cases} \sum_{n=0}^{N_C-1} P_n = P \\ P_n \geq 0 \\ \sum_{n=0}^{N_C-1} \mathbb{E}[|y_n|^2] = P_d \end{cases}$$

$$\sum_{n=0}^{N_C-1} \mathbb{E}[|y_n|^2] = \sum_{n=0}^{N_C-1} |h_n|^2 P_n + N_0 \geq P_d$$

Step 1

$$\boxed{L(\lambda, u, P_0, \cdots, P_{N_C-1}) = \sum_{n=0}^{N_C-1} \log\left(1+\frac{P_n|h_n|^2}{N_0}\right) - \lambda\left(\sum_{n=0}^{N_C-1} P_n - P\right) - u\left(-\sum_{n=0}^{N_C-1} |h_n|^2 P_n - N_0 + P_d\right)}$$

$$L_0(\lambda, u, k, P_0, \cdots, P_{N_C-1}) = L(\lambda, u, P_0, \cdots, P_{N_C-1}) - \sum_{n=0}^{N_C-1} k_n \cdot (-P_n)$$

The KKT conditions are

$\nabla_{P_n} L_0 = 0, \quad \sum_{n=0}^{N_c-1} P_n - P = 0 \quad -\sum_{n=0}^{N_c-1} |h_n|^2 P_n - N_0 + P_d = 0, \quad \boxed{k_n \geq 0, \quad k_n \cdot P_n = 0, \quad -P_n \leq 0}$

$$\frac{\partial L_0}{\partial P_n} = \frac{|h_n|^2}{(N_0 + P_n|h_n|^2)\ln 2} - \lambda + u|h_n|^2 + k_n = 0$$

$$\Rightarrow \frac{\partial L}{\partial P_n} + k_n = 0$$

$$\frac{\partial L}{\partial P_n} = -k_n$$

the same as Ex 2

we can get

$$\boxed{\begin{array}{l} \text{If } P_n = 0, \ k_n \geq 0, \Rightarrow \frac{\partial L}{\partial P_n} \leq 0 \\ \text{If } P_n > 0, \ k_n = 0, \Rightarrow \frac{\partial L}{\partial P_n} = 0 \end{array}}$$

For $P_n > 0$, $\frac{\partial L}{\partial P_n} = 0$

$$\Rightarrow \frac{|h_n|^2}{(N_0 + P_n|h_n|^2)\ln 2} - \lambda + u|h_n|^2 = 0$$

$$\frac{|h_n|^2}{N_0 + P_n|h_n|^2} = (\lambda - u|h_n|^2) \cdot \ln 2$$

$$\boxed{P_n = \frac{1}{(\lambda - u|h_n|^2)\ln 2} - \frac{N_0}{|h_n|^2}}$$

Step 2)

optimal power allocation

two unknowns are needed to be solved.

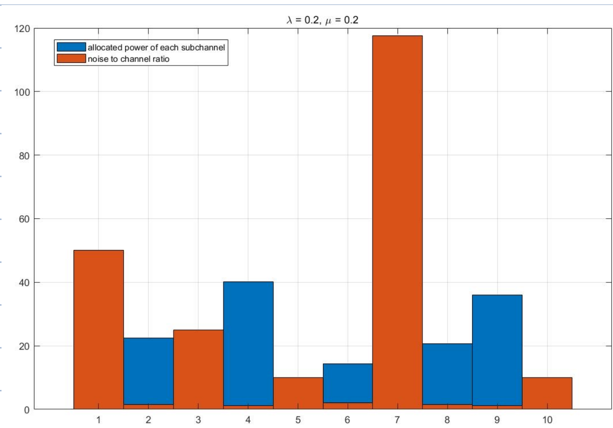
$$\sum_{n=0}^{N_c-1} P_n = \sum_{n=0}^{N_c-1} \left[\frac{1}{(\lambda - u|h_n|^2)\ln 2} - \frac{N_0}{|h_n|^2}\right] = P$$

$$\mathbb{E}[|y_n|^2] = \sum_{n=0}^{N_c-1} |h_n|^2 P_n + N_0 = P_d$$

by using the constraints at transmitter and receiver, the λ and u can be determined.

Step 3)

For $\lambda = 0.2$, $u = 0.2$



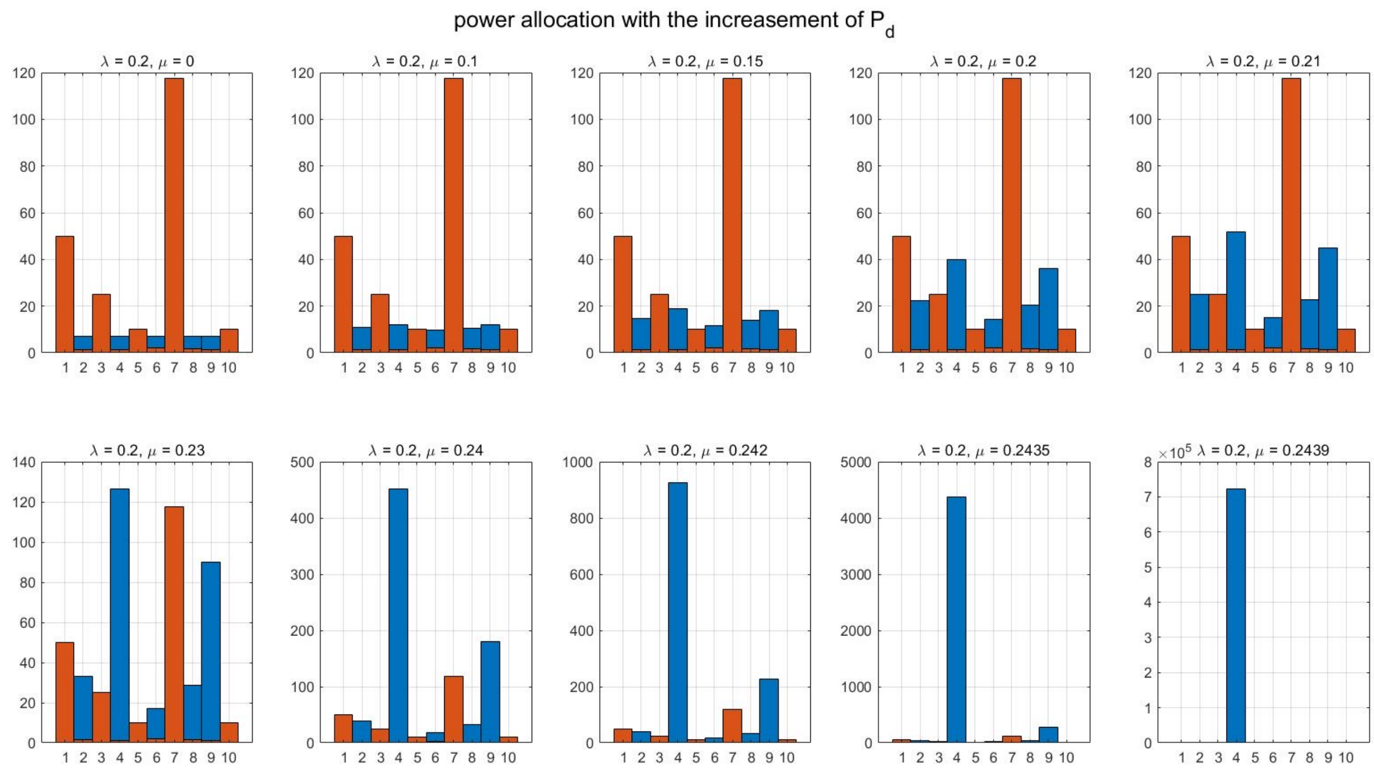
The allocation result is shown on the left.

We can notice that the sub-channels with better conditions (lower noise to channel ratio) are allocated with more power.

By observing the two constraints, we notice that: Step 4)

For the fixed λ, if P_d increases, the value of u also increase, but the $\lambda - u|h_n|^2 > 0$ should be guaranteed.

Therefore, by increasing u, we can analysis the impact of increasement of P_d on the allocation



We can notice that with the increasement of P_d, the subchannel with better condition will be allocated with more percentage of the whole transmit power P.

For $P_d = 0$ ($u=0$), there is no constraint on the receive power, so the power is spread among the subchannels as water filling.

In contrast, in the final figure (the largest P_d in the figures), almost all the power is allocated to the sub-channel with the best condition (the 4th subchannel)

In my view, I suppose we can understand the power allocation in this way:

At the extreme condition where the P_d is maximized, we need to transmit all the power in the subchannel with the best channel conditions, because it has the higher channel gain. This can also be proved in mathmatic

If $|h_k|^2$ is the highest channel gain

$$P_d = \sum_{i=1}^{n} |h_i|^2 P_i + N_0 \le \sum_{i=1}^{n} |h_k|^2 P_i + N_0 = \boxed{|h_k|^2 \sum_{i=1}^{n} P_i} + N_0 = |h_k|^2 P + N_0$$

all the power is allocated to the best channel