3: Discrete Cosine Transform

DFT Problems

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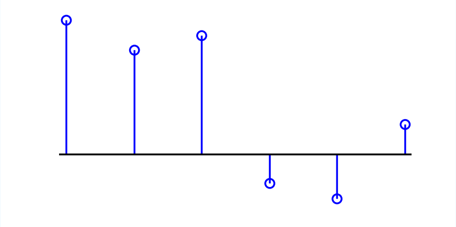
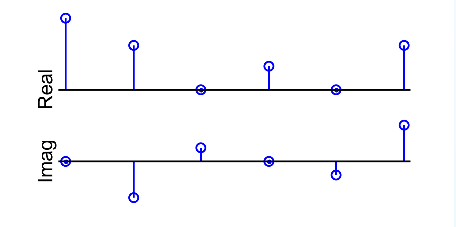
 $\rightarrow$ 

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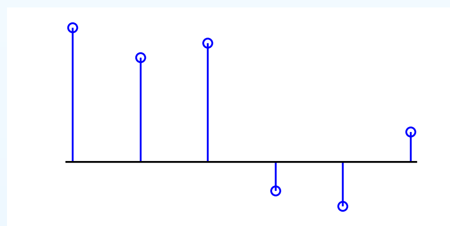

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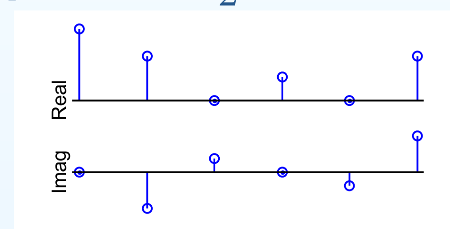

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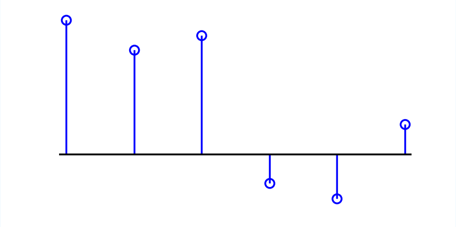

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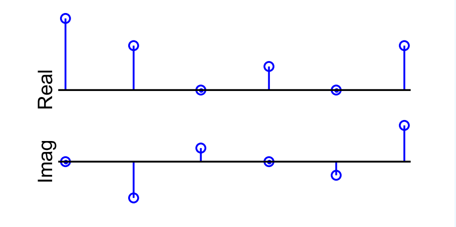

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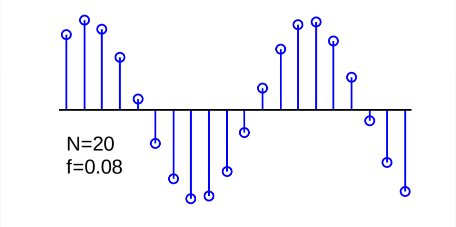

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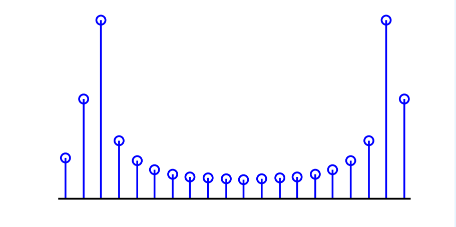

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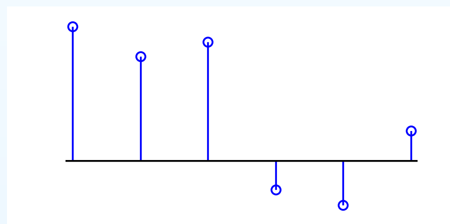

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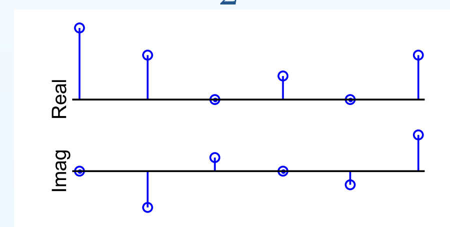

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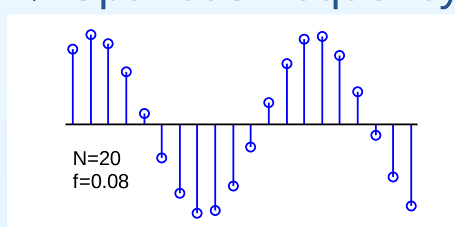

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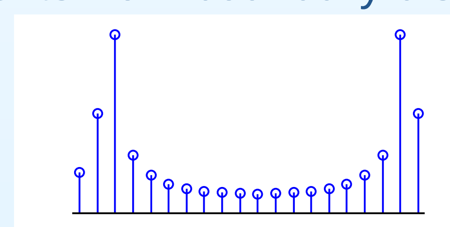
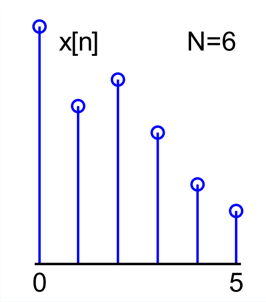


The Discrete Cosine Transform (DCT) overcomes these problems.

DCT

To form the Discrete Cosine Transform (DCT), replicate $x[0 : N-1]$ but in reverse order and insert a zero between each pair of samples:



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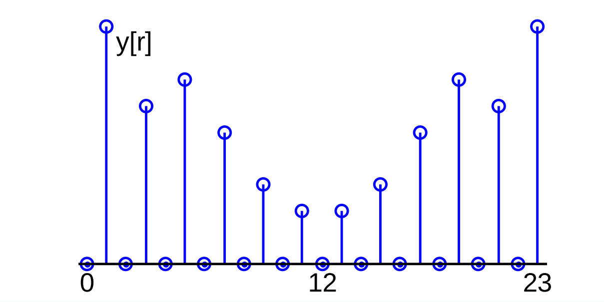

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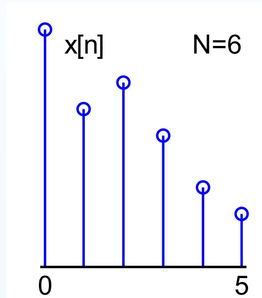

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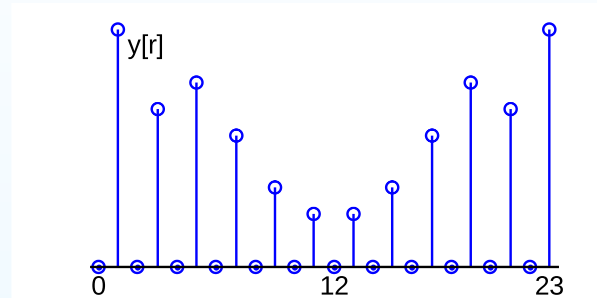
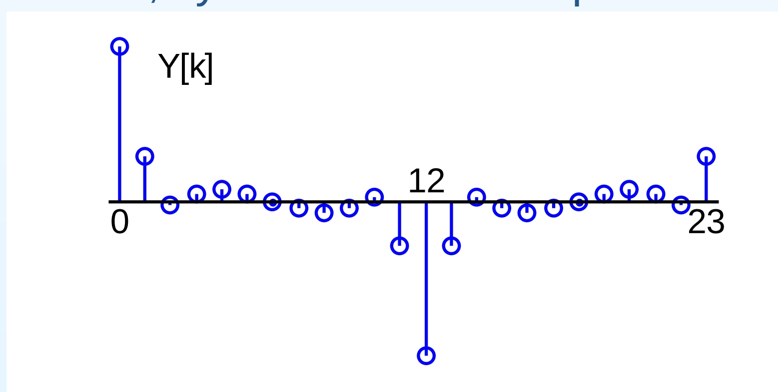


Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.

Result is real, symmetric and anti-periodic:



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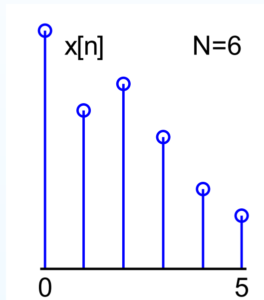

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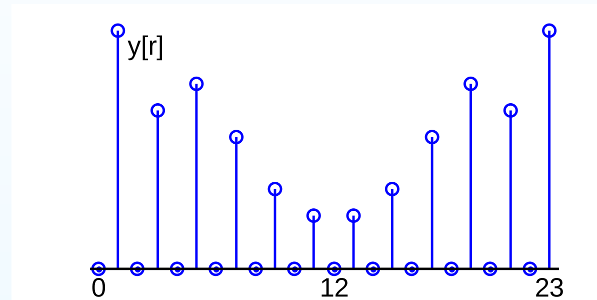

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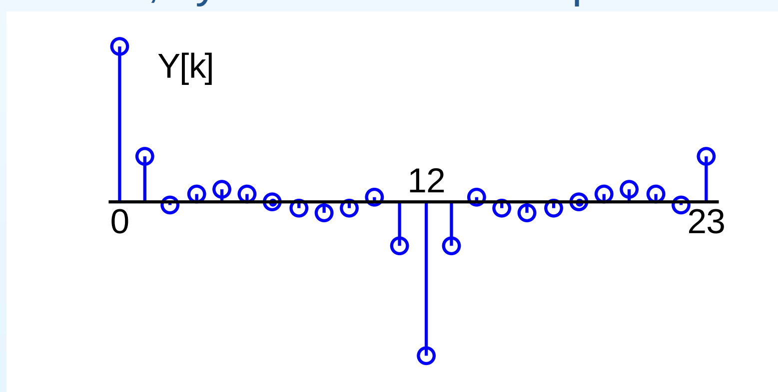
Result is real, symmetric and anti-periodic: only need first N values



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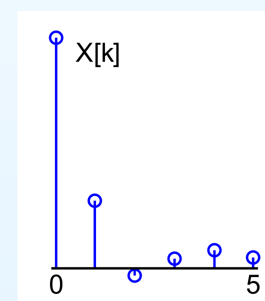

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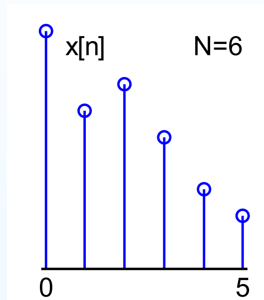

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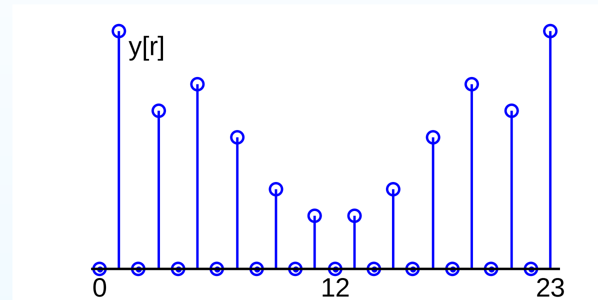

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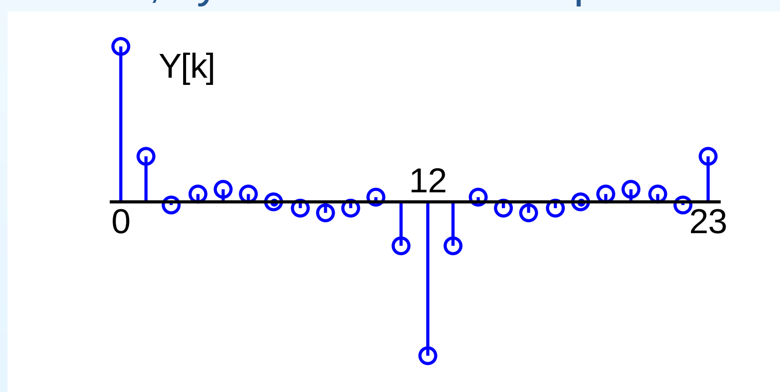

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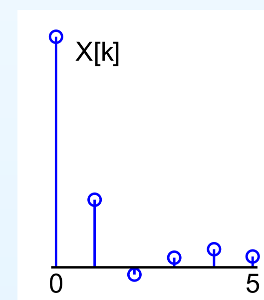

Forward DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ for $k = 0 : N-1$

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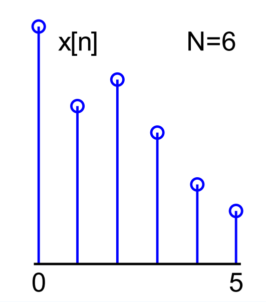

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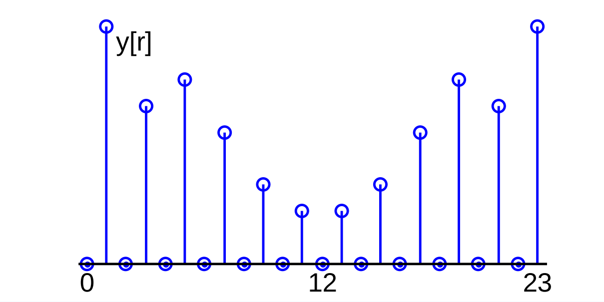

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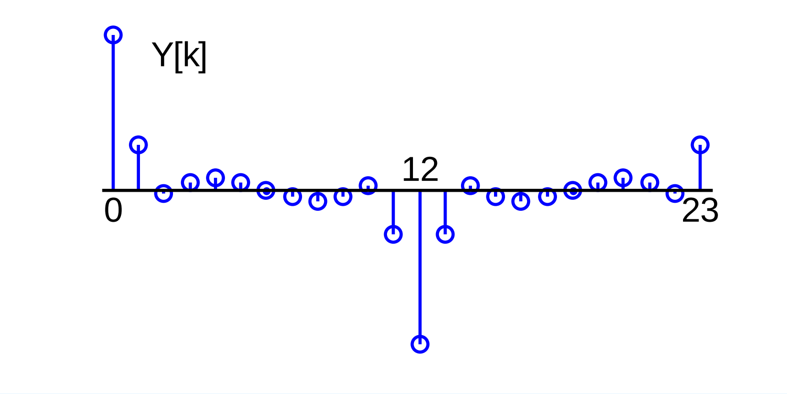

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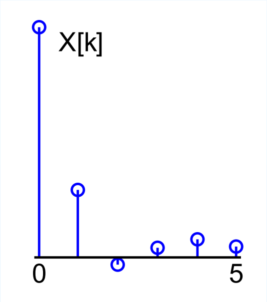

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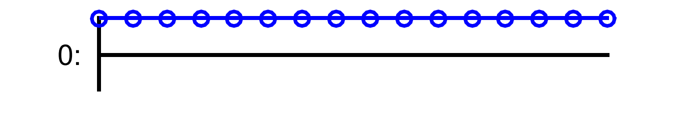

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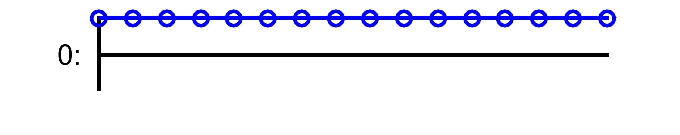

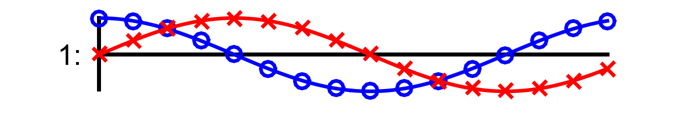

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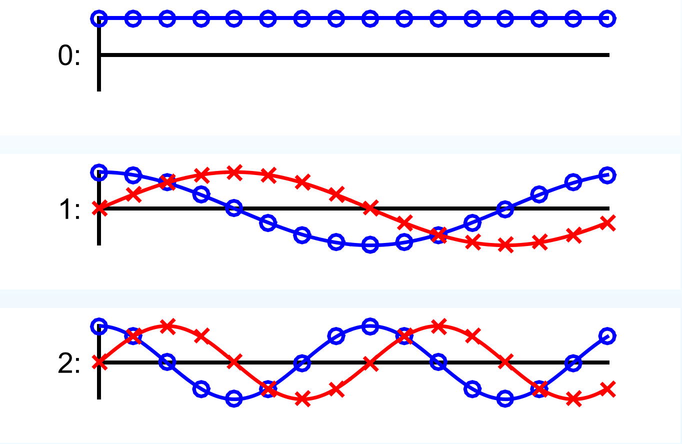

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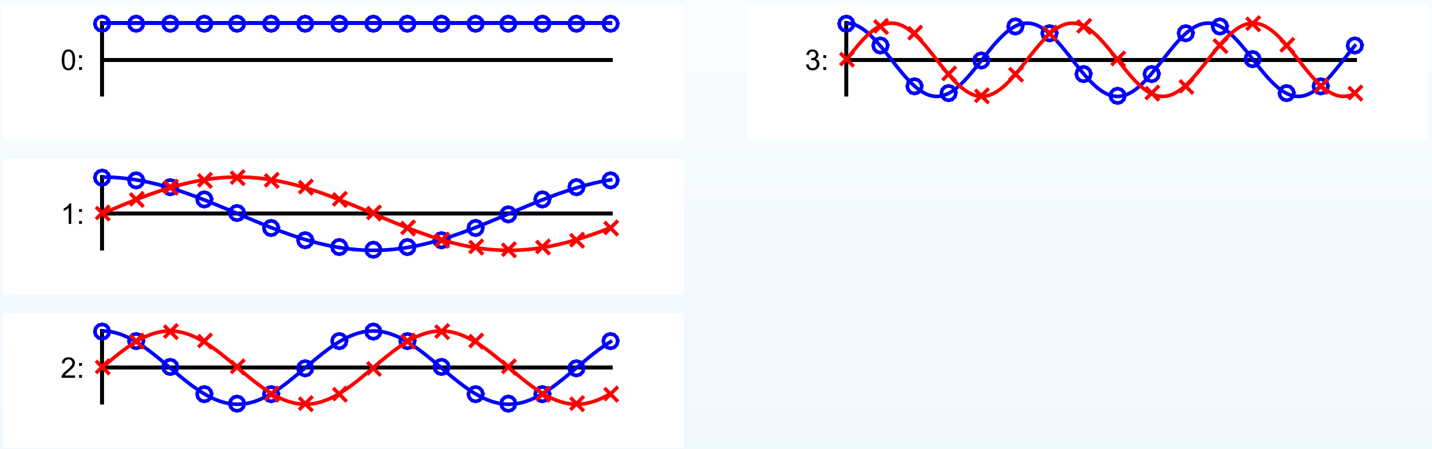

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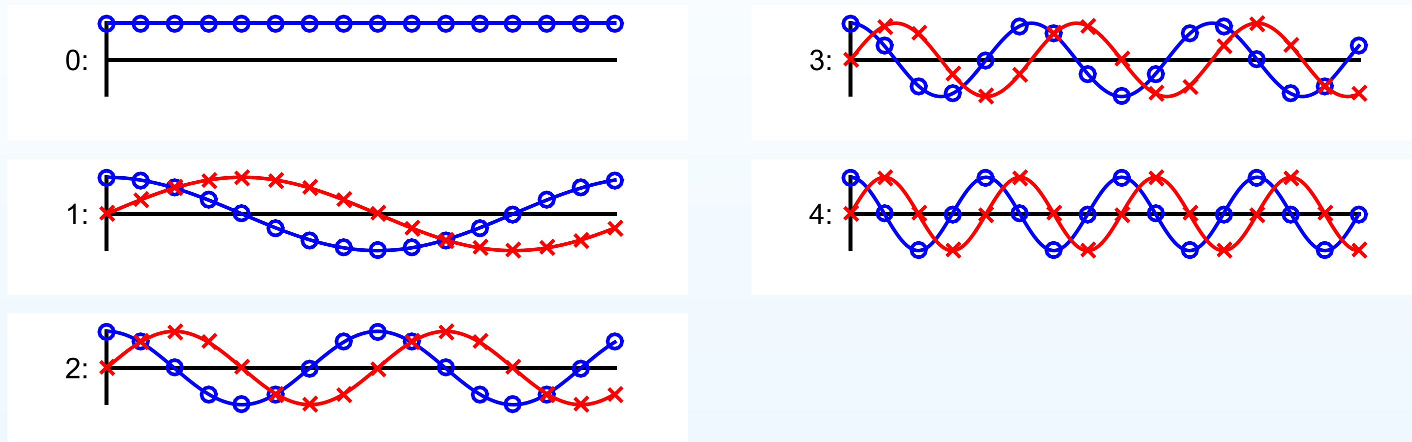

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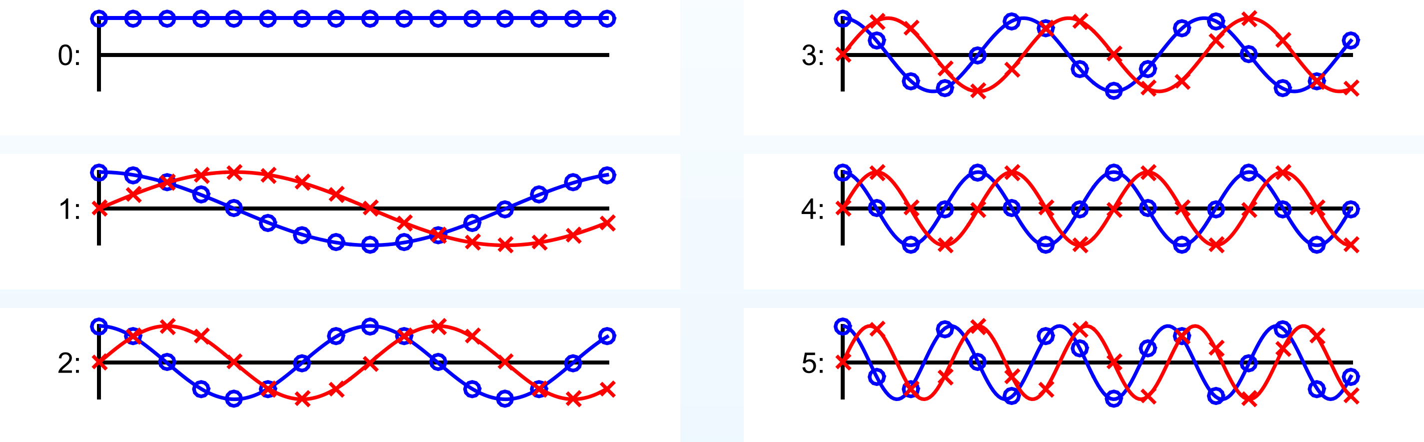

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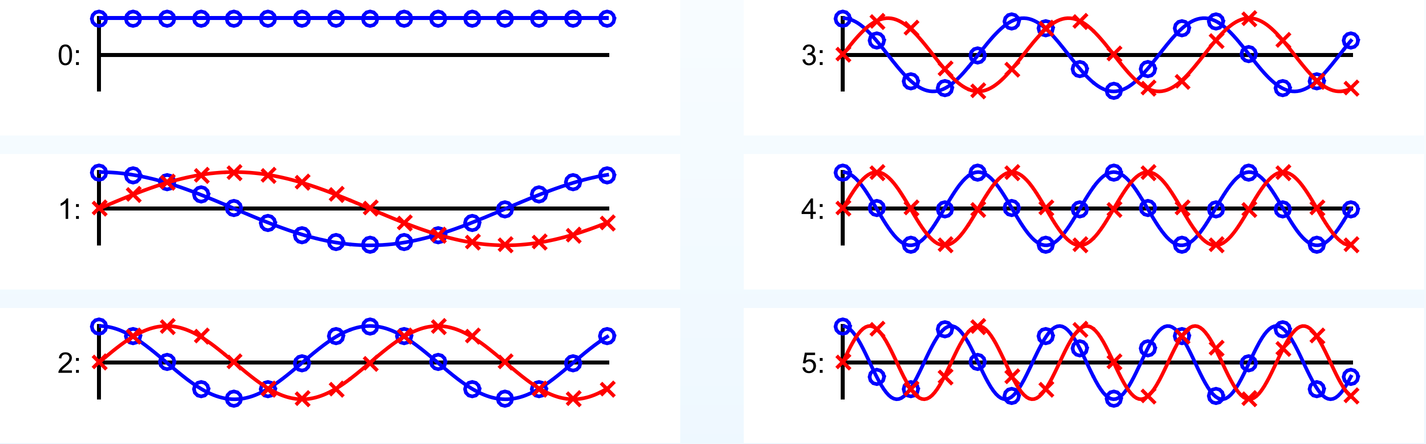

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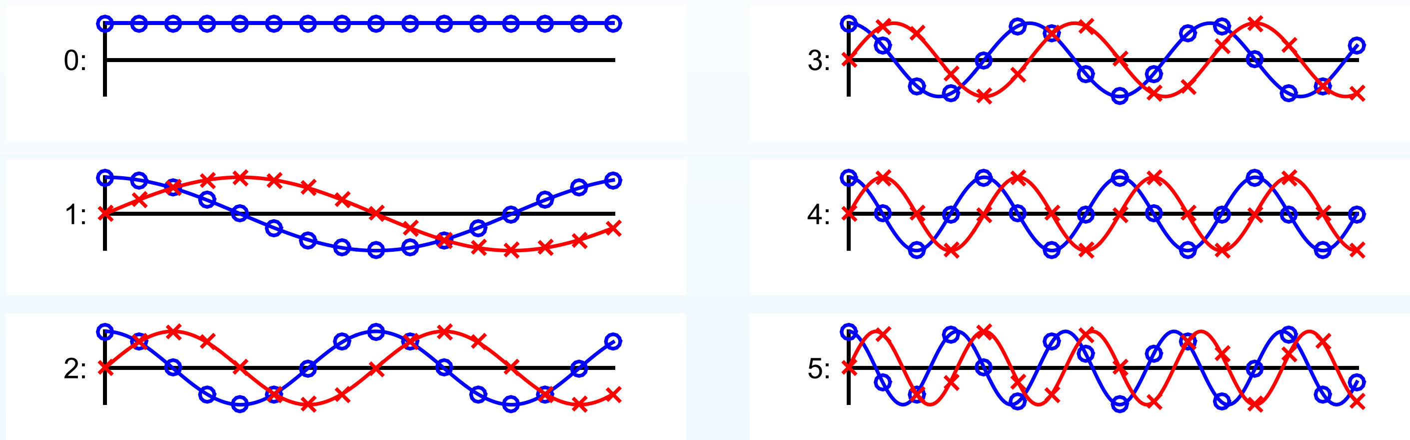

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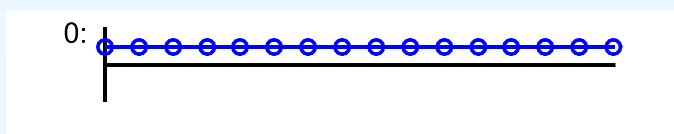

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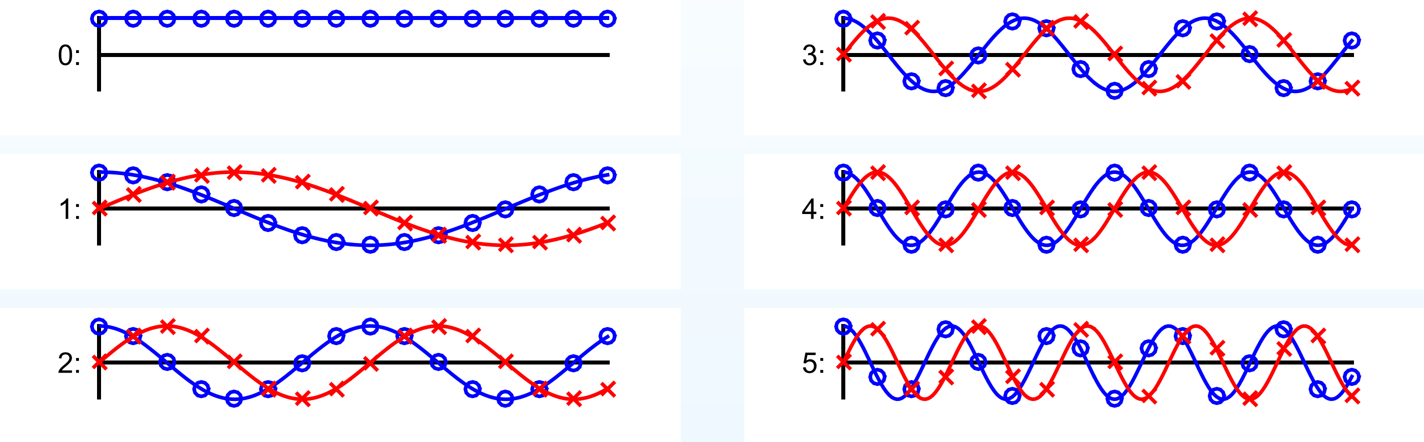

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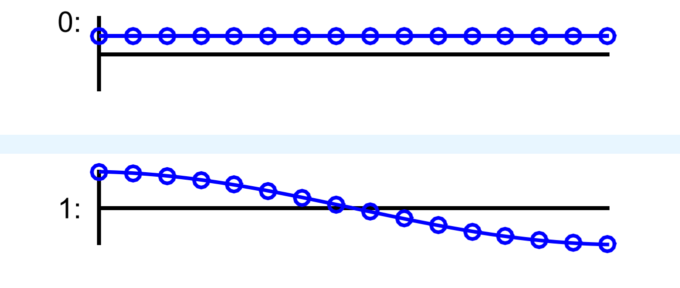

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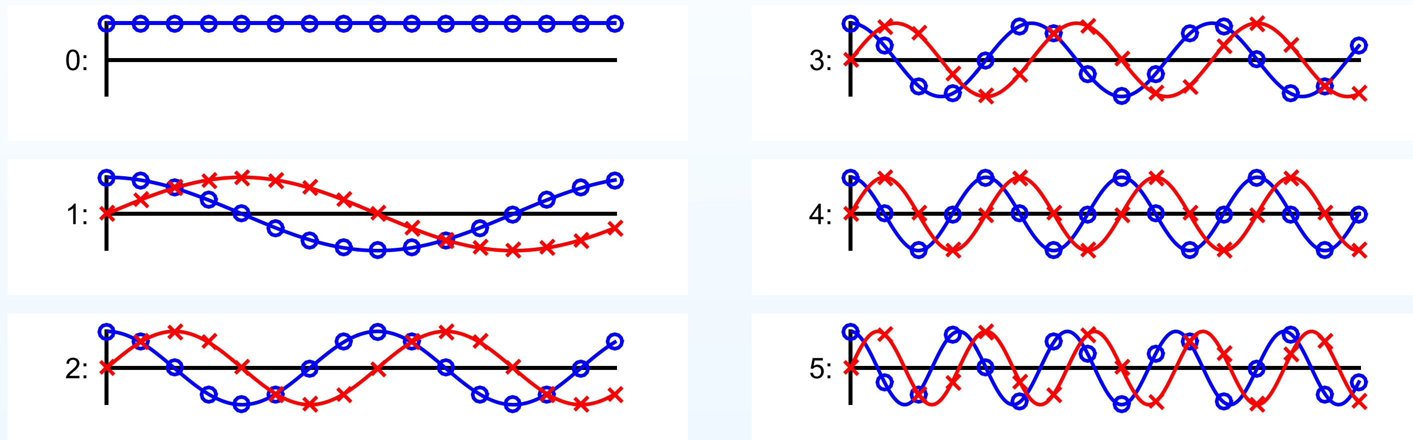

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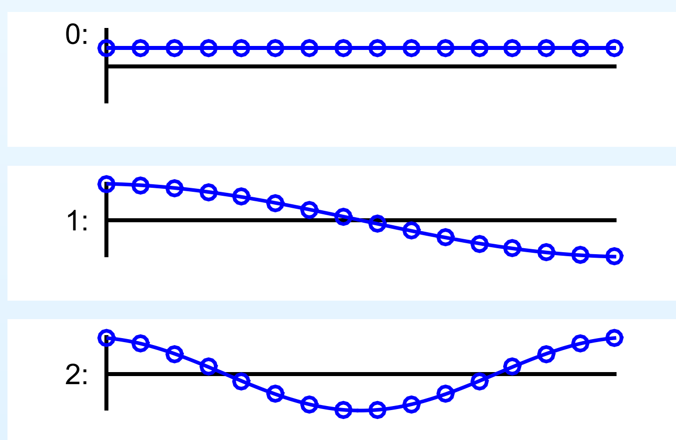

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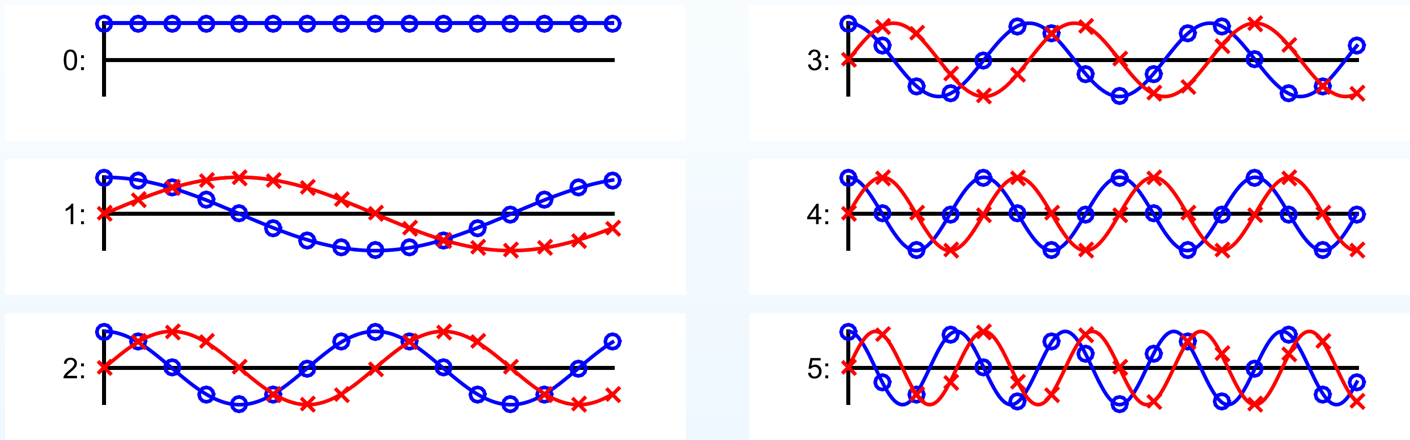

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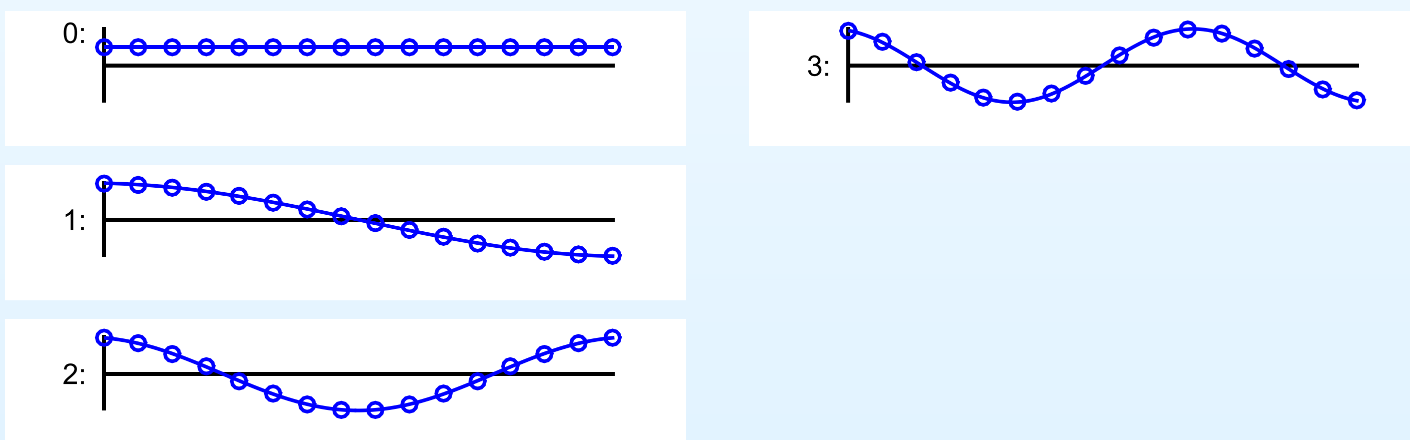

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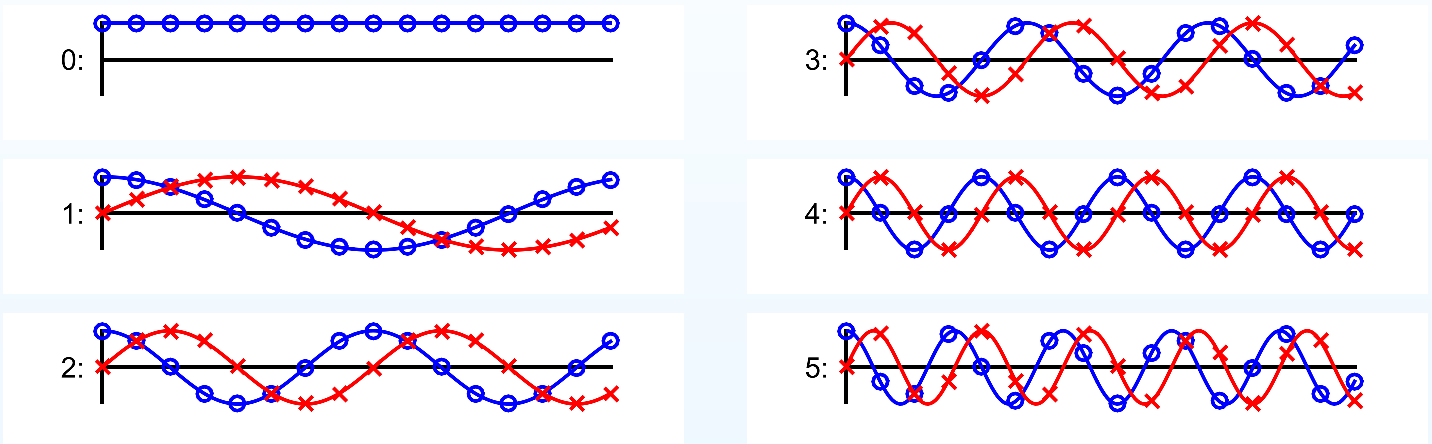

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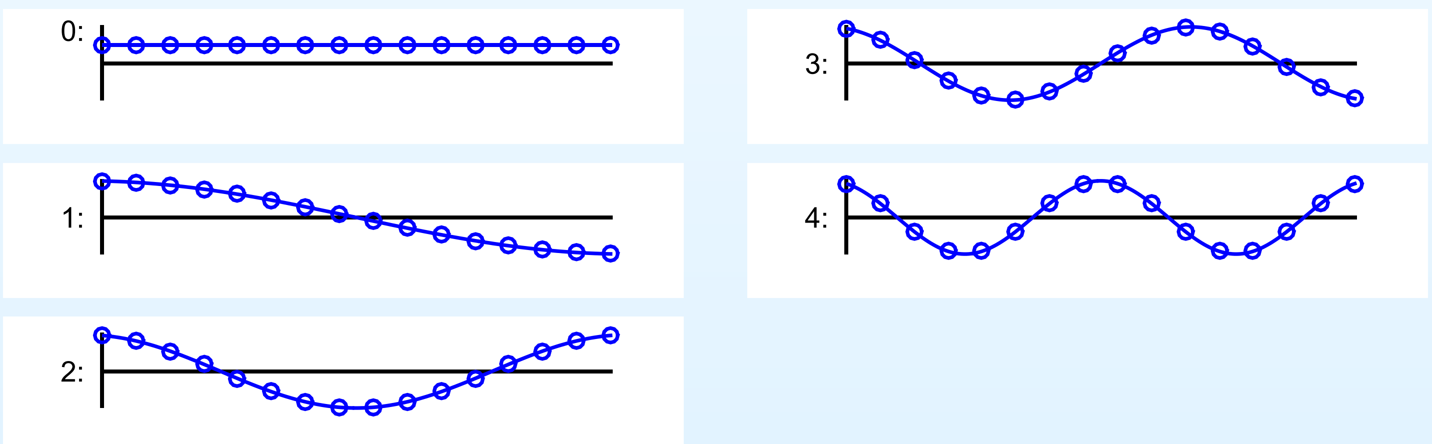

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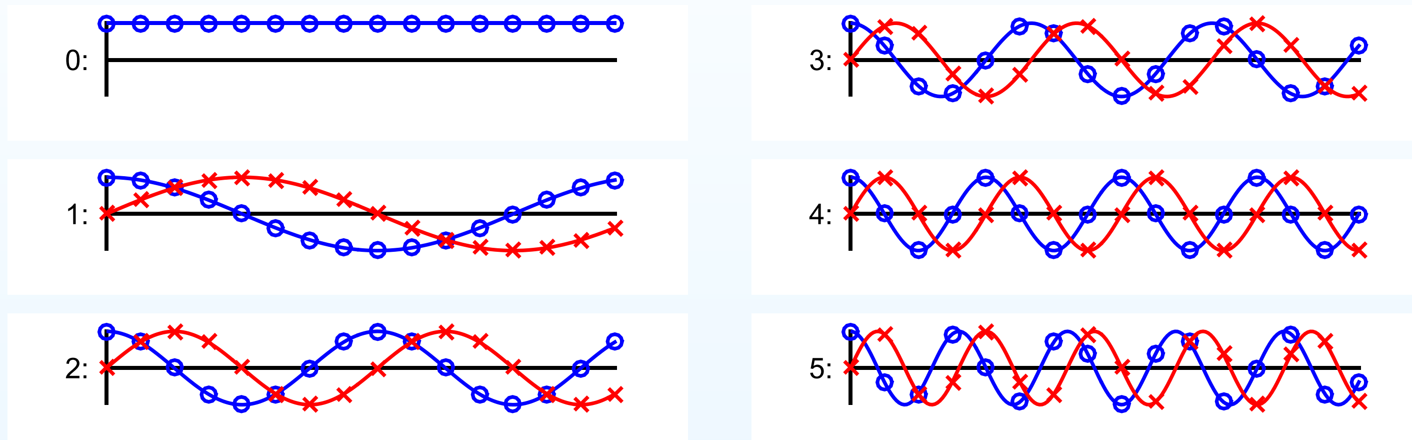

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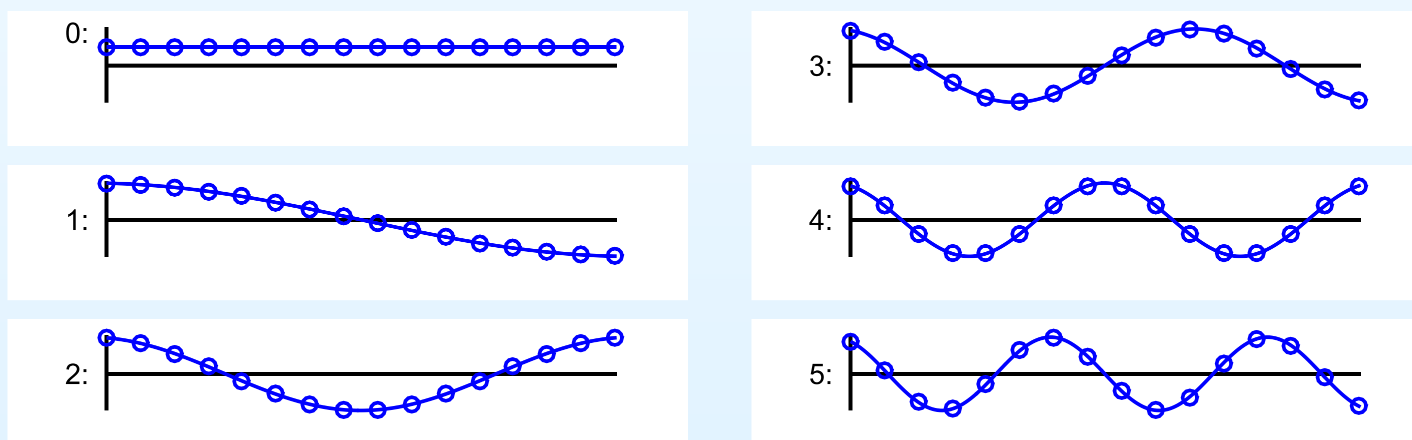

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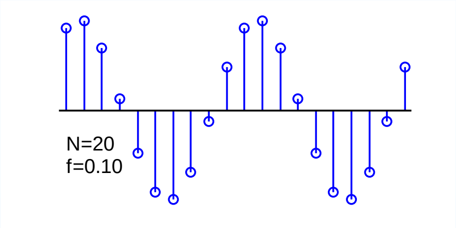

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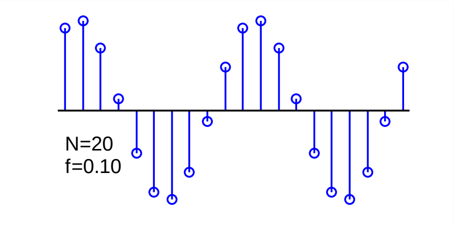

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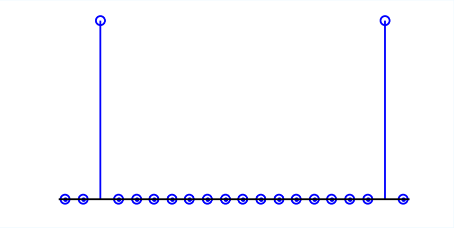

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$f = \frac{m}{N}$ $\qquad$ $f \neq \frac{m}{N}$

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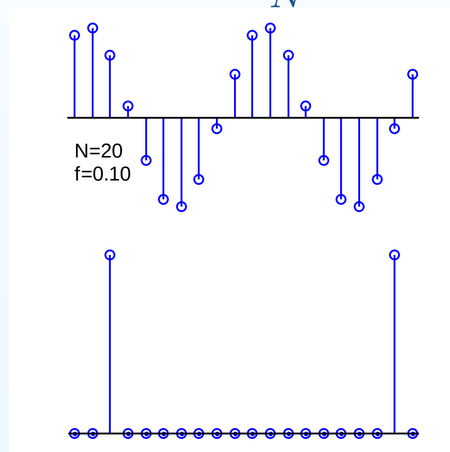
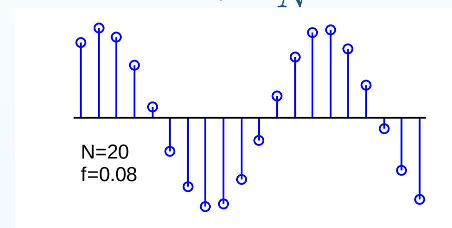
$|X_F[k]|$





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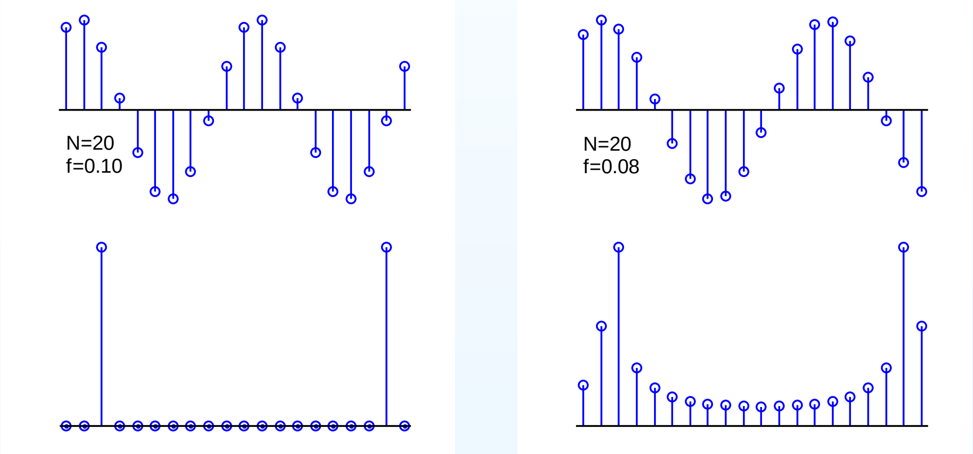

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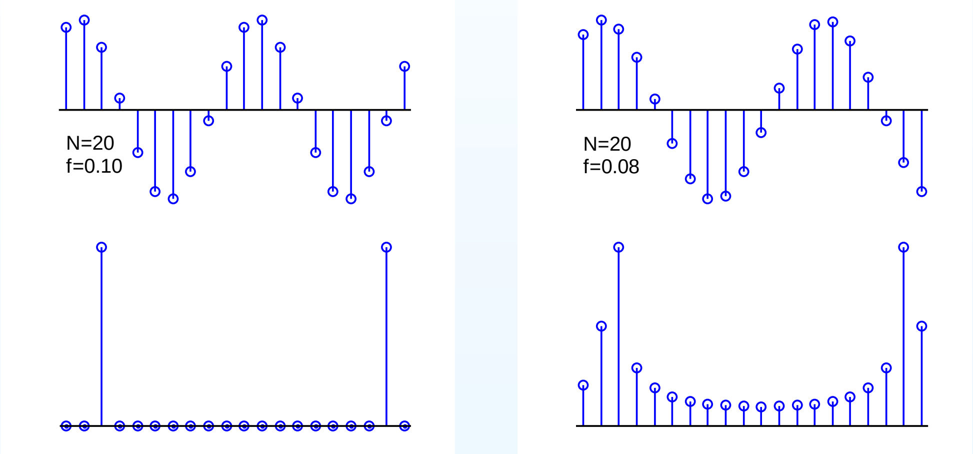

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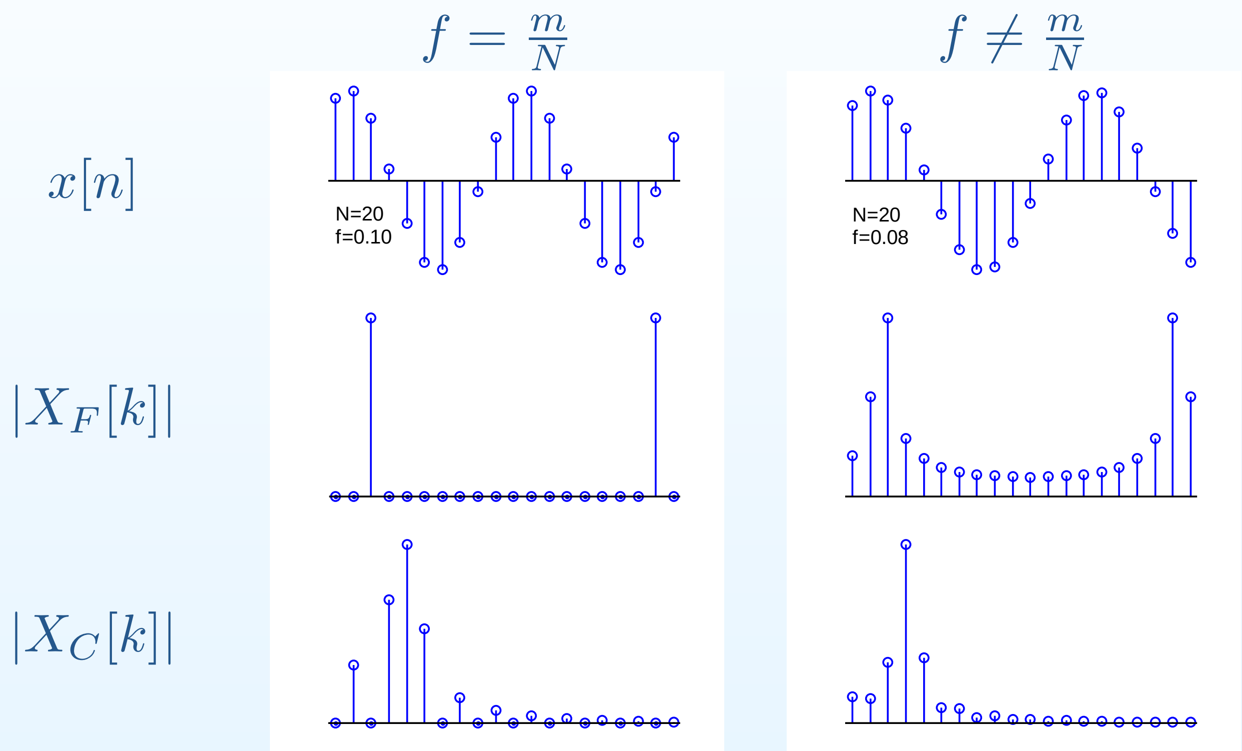

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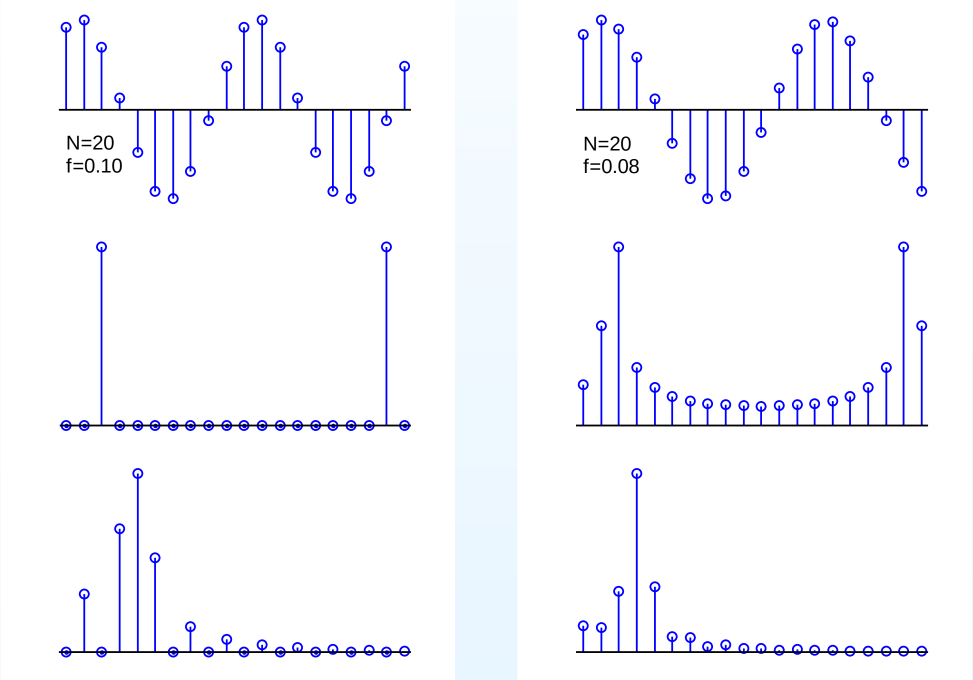

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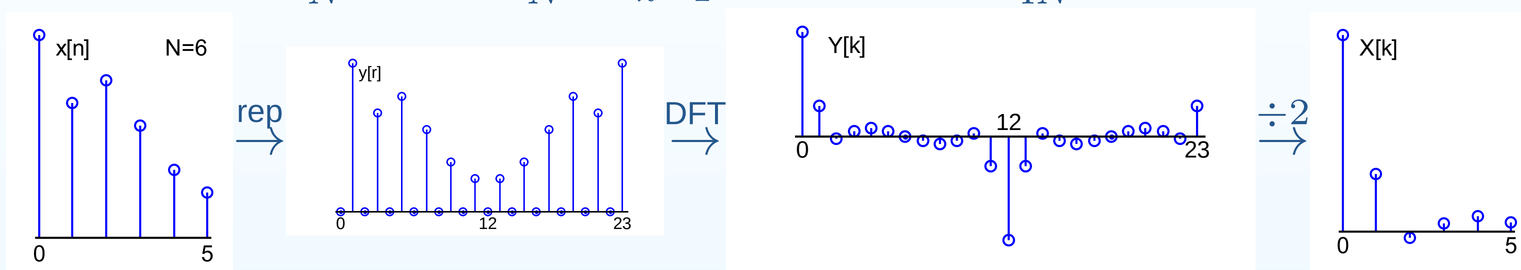

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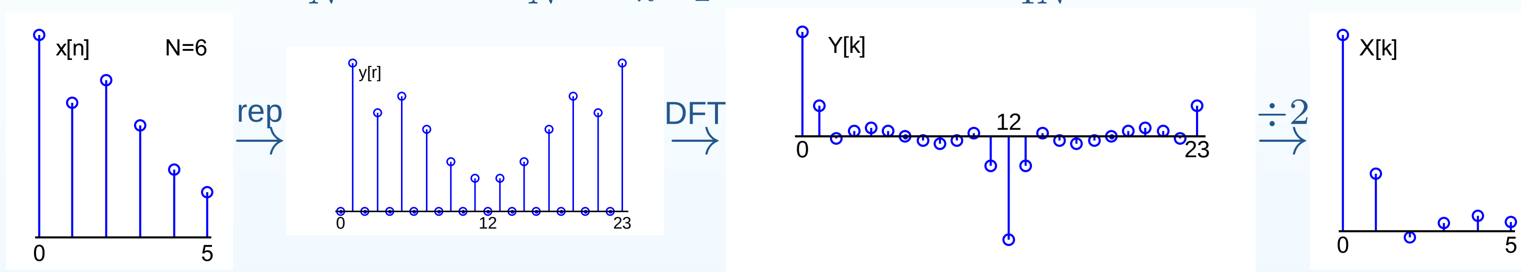

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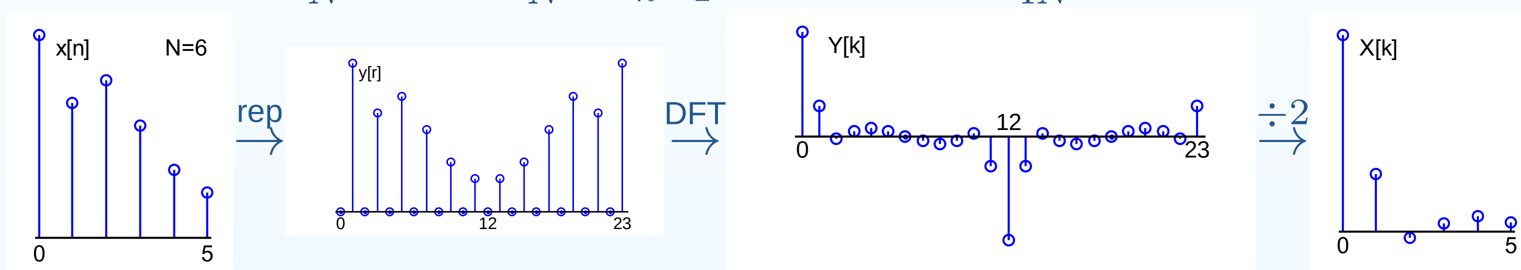

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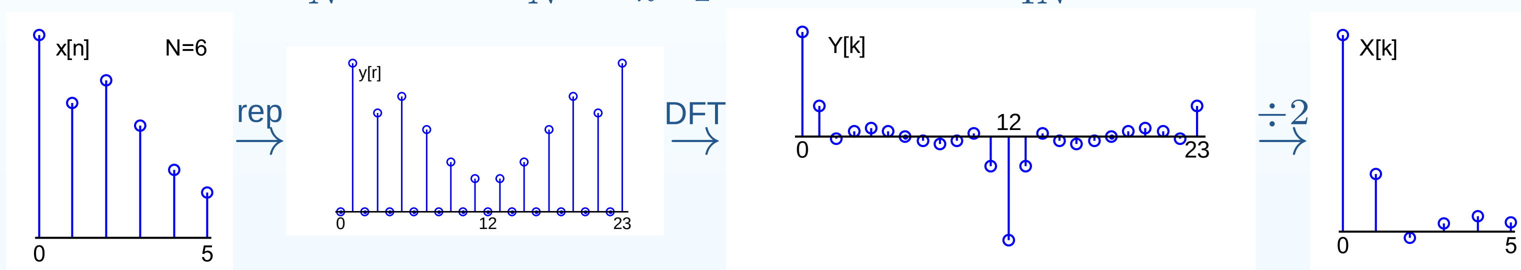

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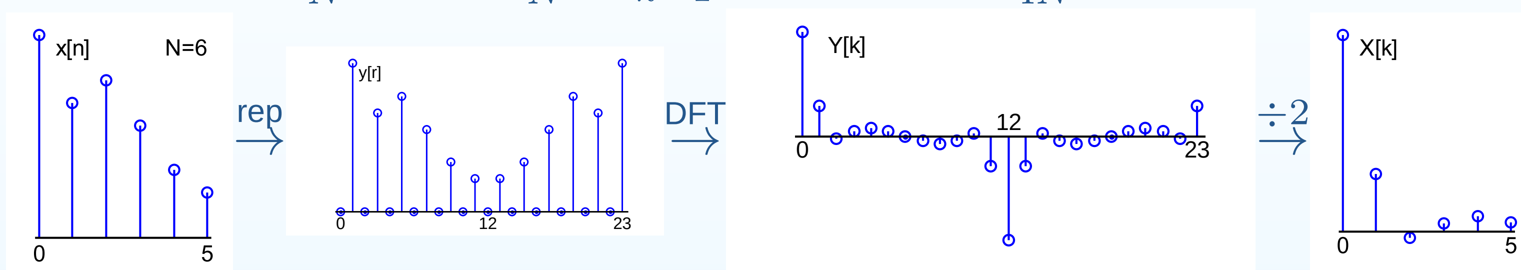

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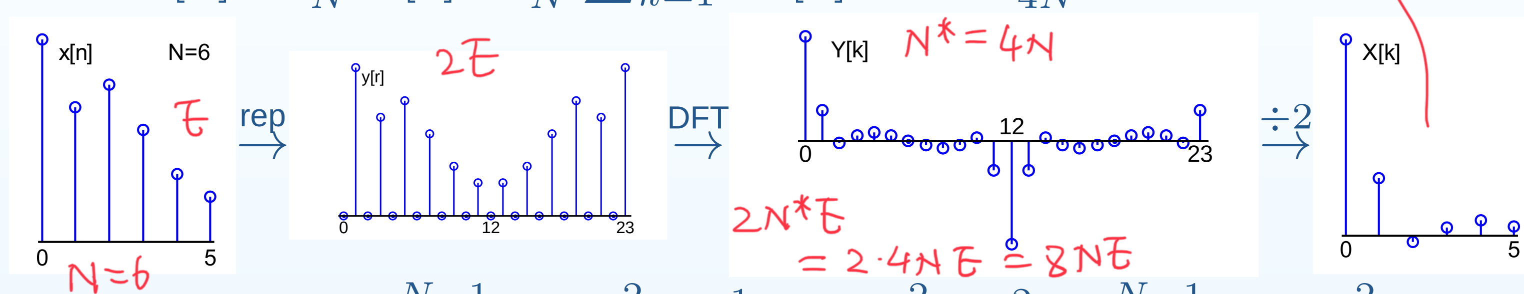

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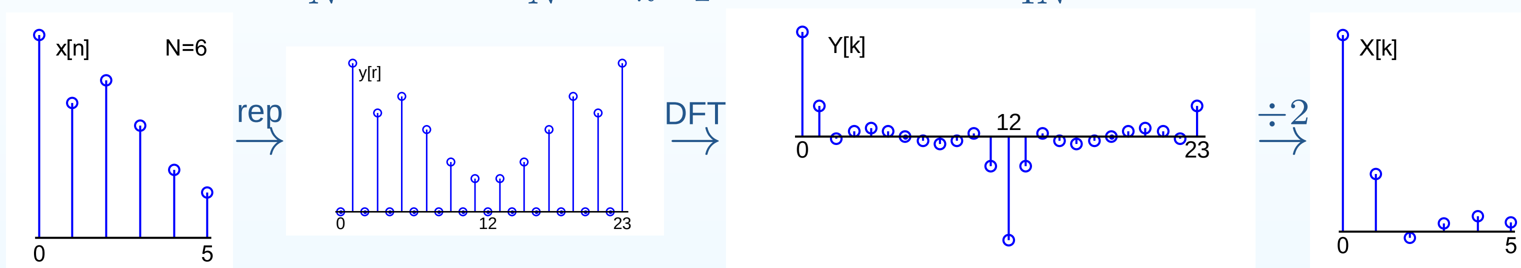

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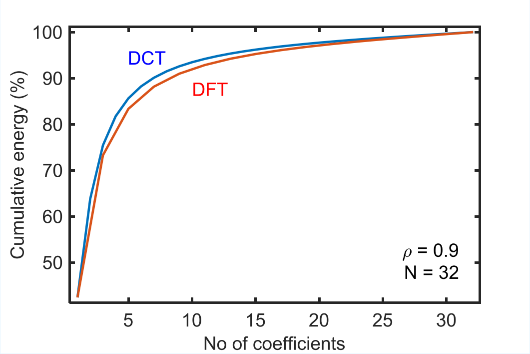

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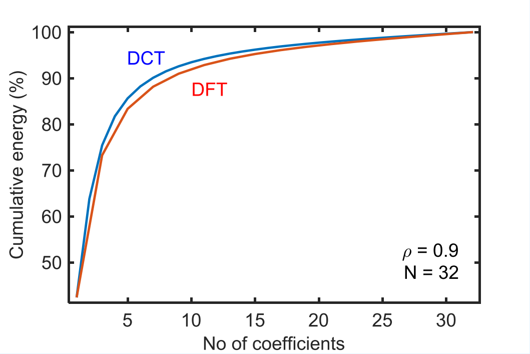

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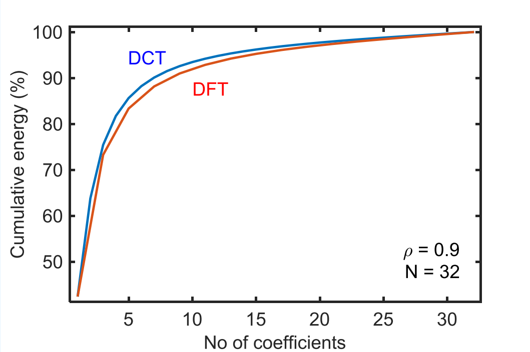

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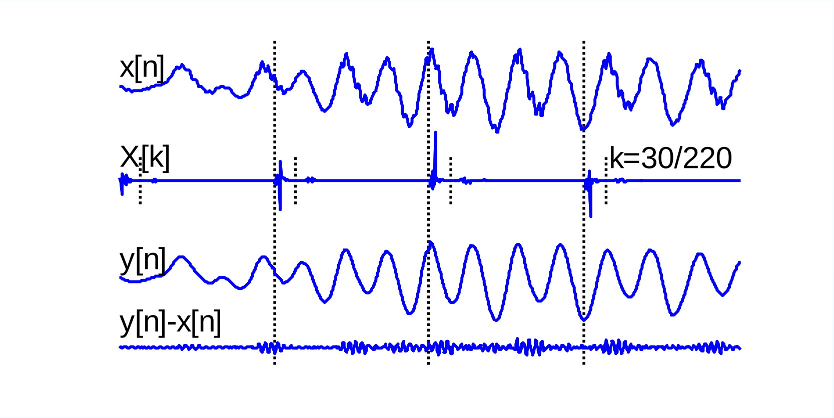
Energy compaction good for coding (low-valued coefficients can be set to 0)

Decorrelation good for coding and for probability modelling

Frame-based coding

- Divide continuous signal into frames



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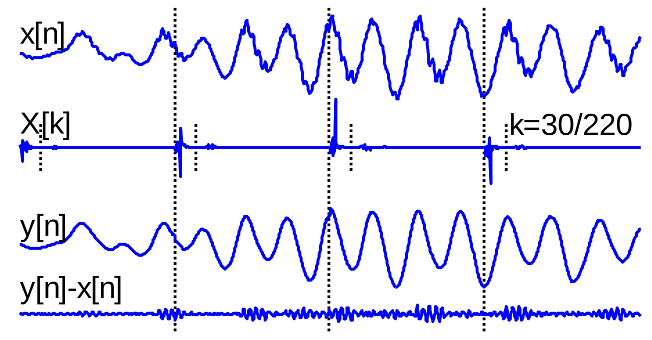

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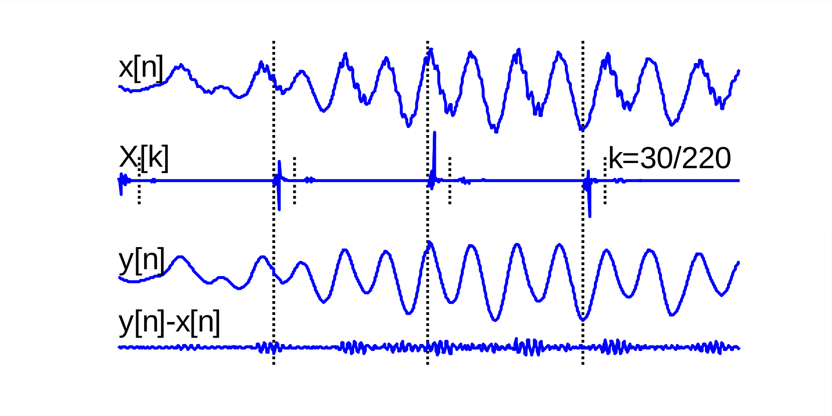
  - e.g. keep only 30 $X[k]$



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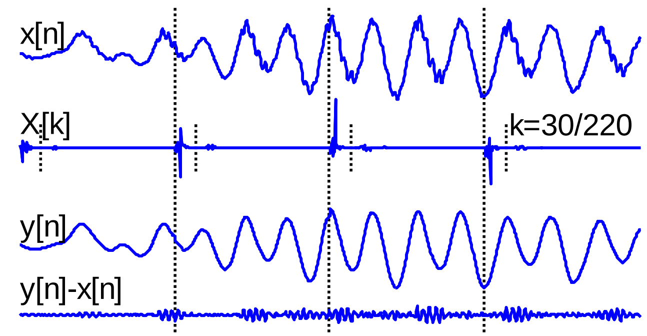

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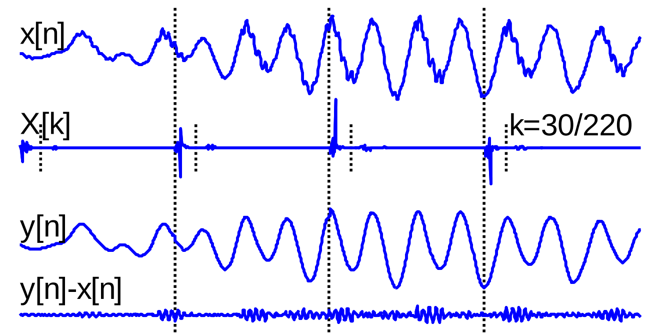

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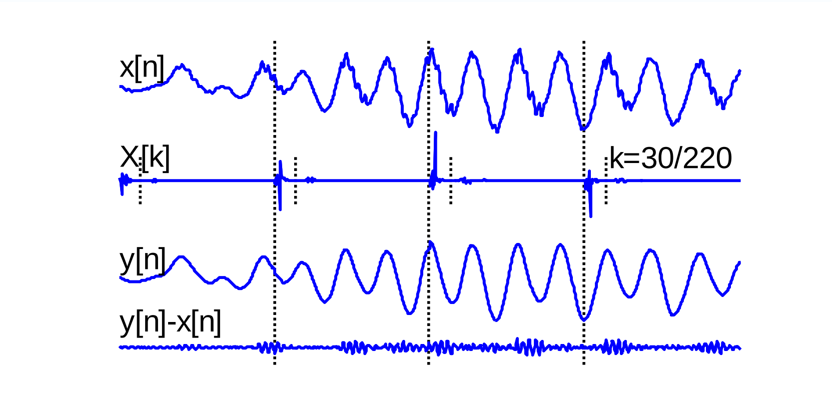
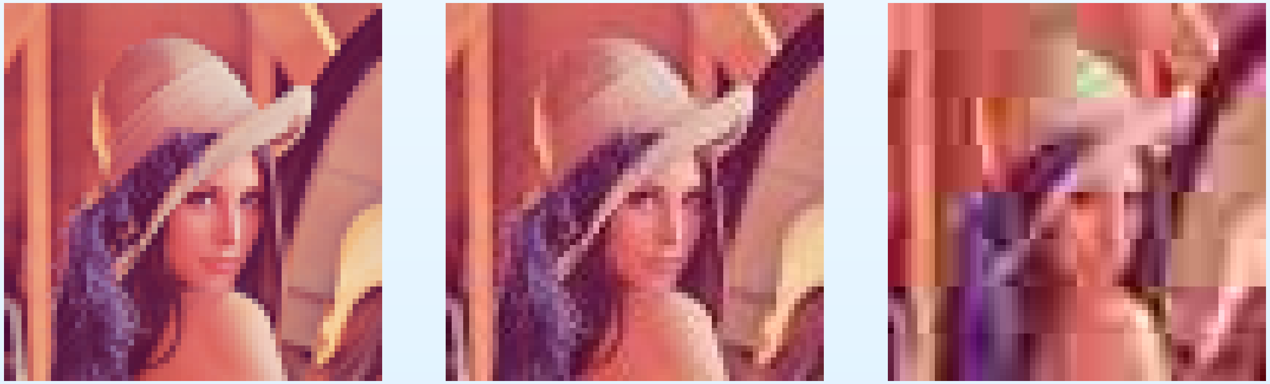


Problem: Coding may create discontinuities at frame boundaries
e.g. JPEG, MPEG use 8×8 pixel blocks

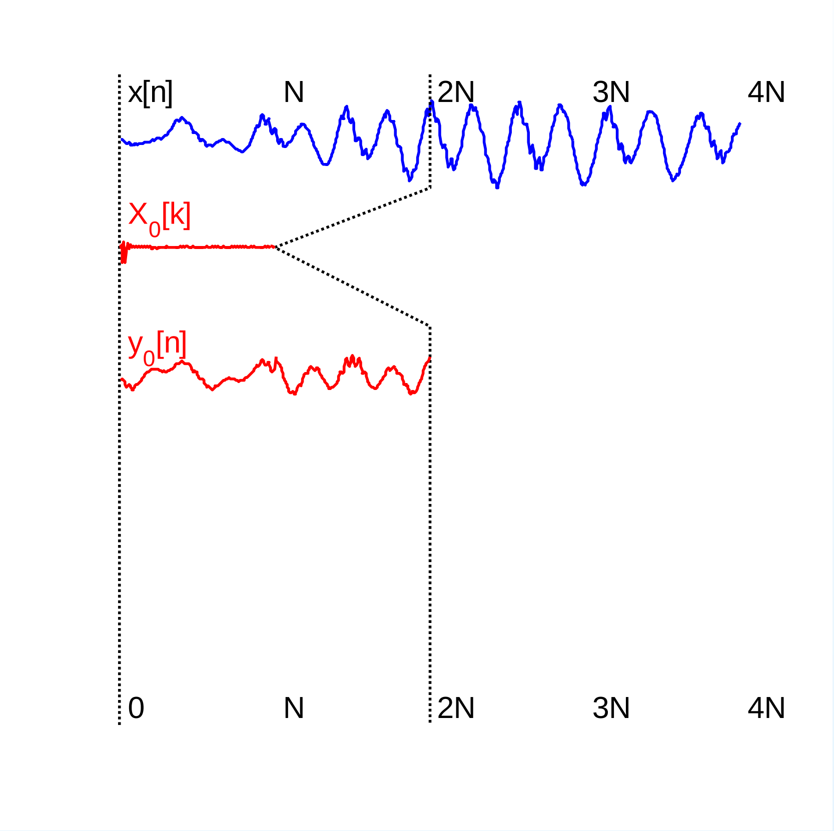


8.3 kB (PNG) 1.6 kB (JPEG) 0.5 kB (JPEG)

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$x[0 : 2N-1]$



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+

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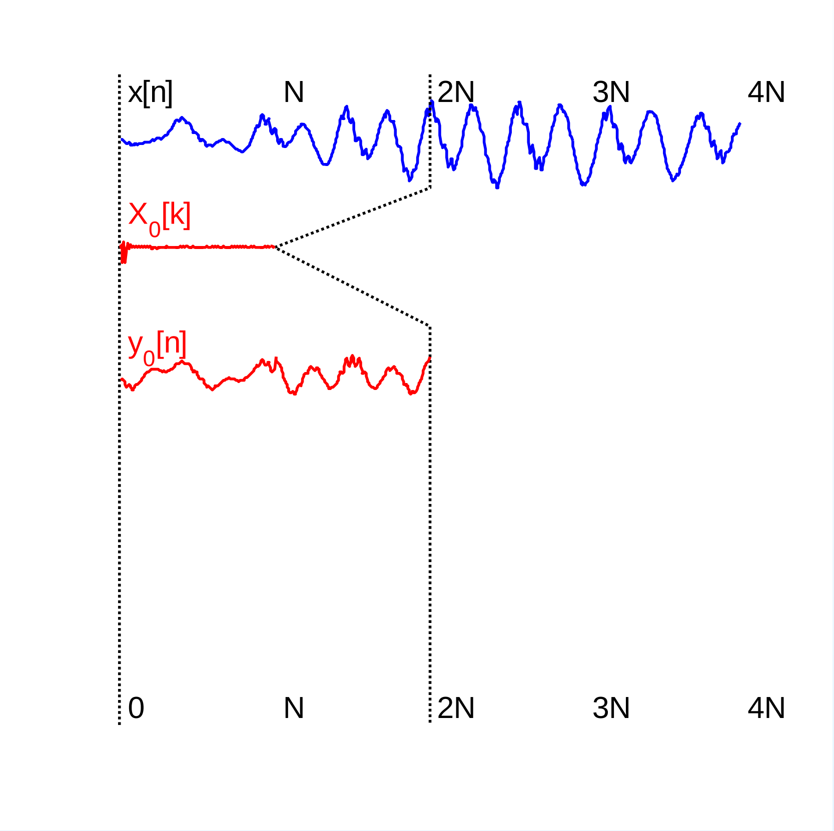

MDCT: $2N \rightarrow N$ coefficients

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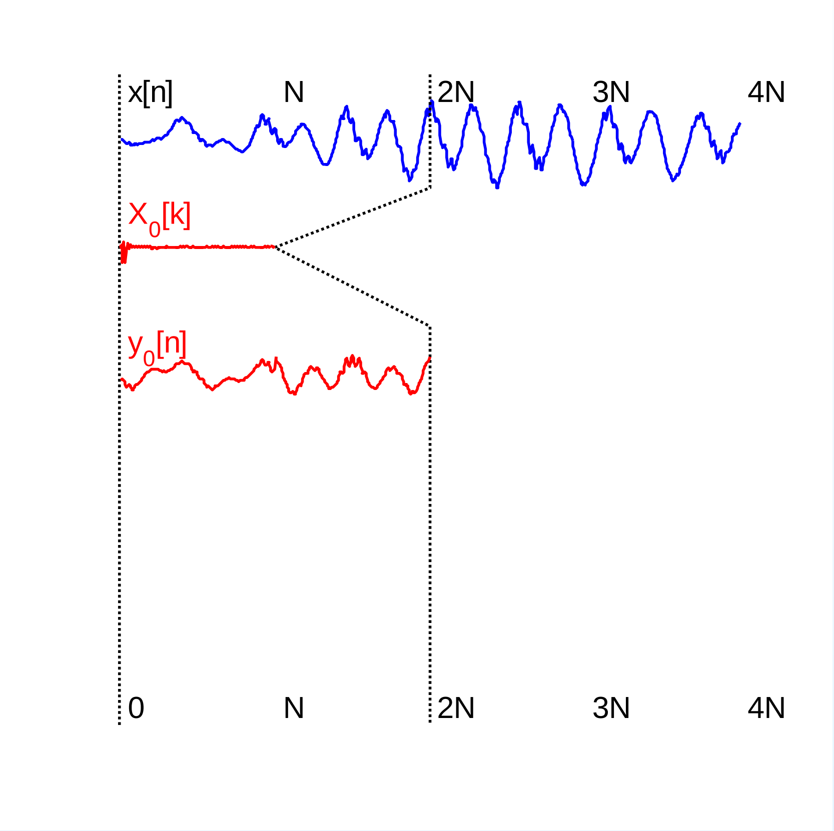
$$x[0:2N-1] \overset{\text{MDCT}}{\to} X_0[0:N-1] \overset{\text{IMDCT}}{\to} y_0[0:2N-1]$$



MDCT: $2N \to N$ coefficients, IMDCT: $N \to 2N$ samples

Lapped Transform

+

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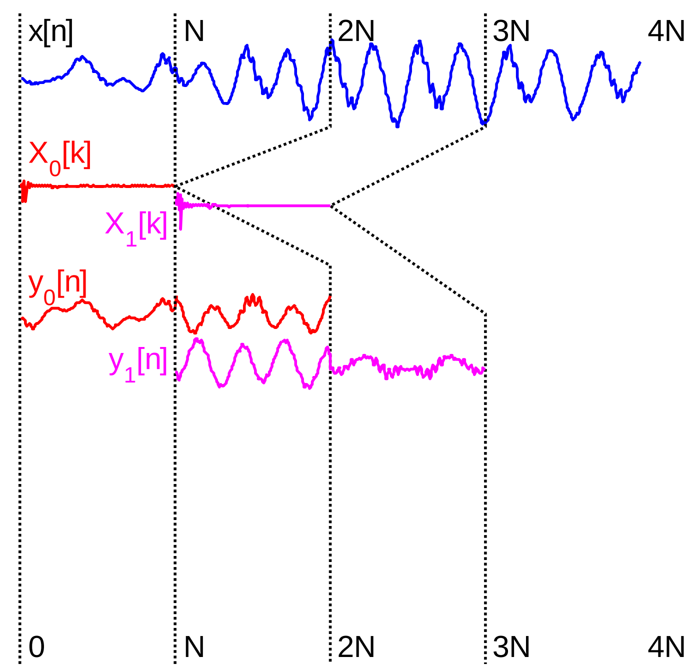

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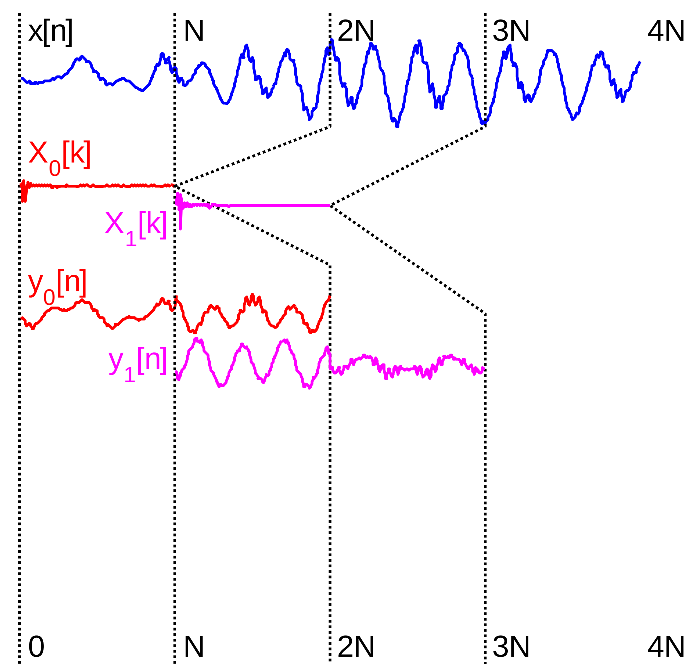

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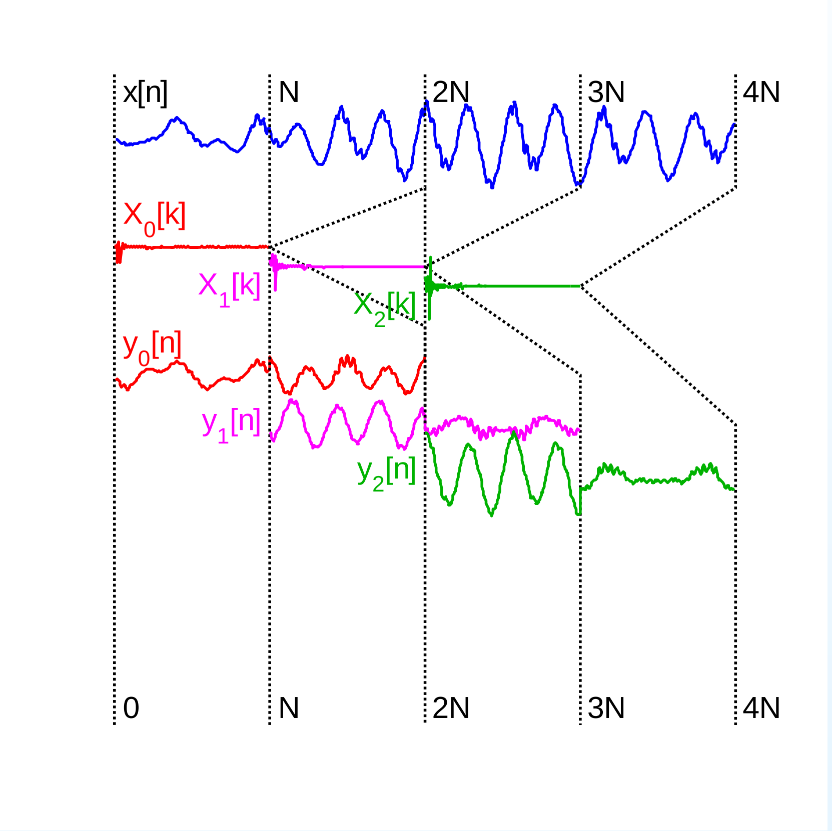

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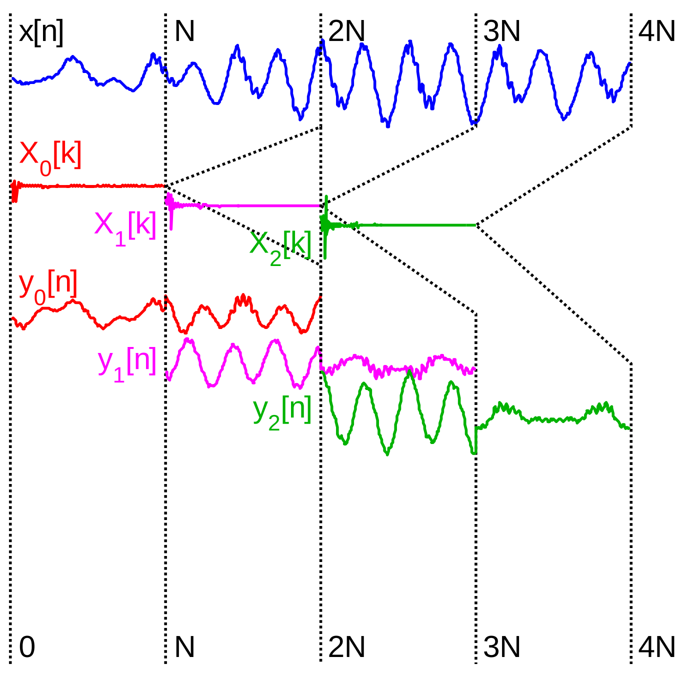

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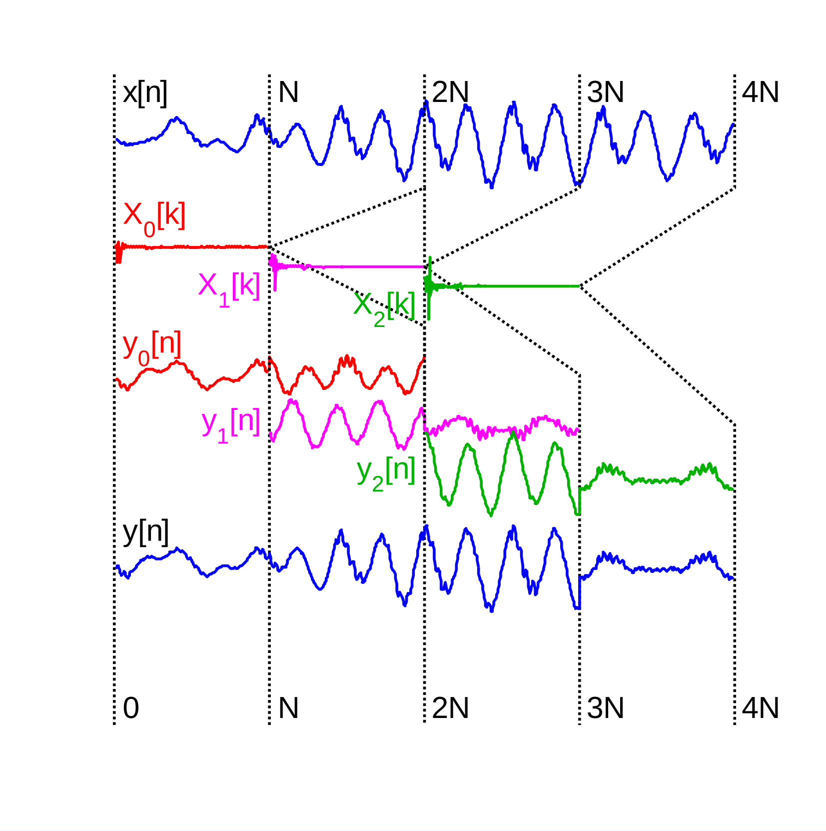

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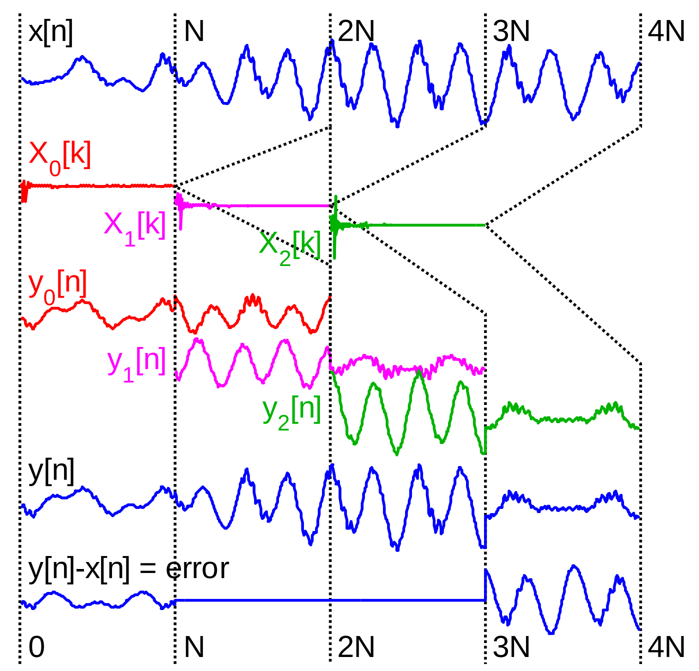

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$\frac{1}{N}\mathbf{M}^T\mathbf{M} = \frac{1}{2}\begin{bmatrix} \mathbf{I}-\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}+\mathbf{J} \end{bmatrix}$ with $\mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, $\mathbf{J} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$

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from first frame

from second frame

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Normally the $2N$-long $\mathbf{x}$ and $\mathbf{y}$ frames are windowed before the MDCT and again after the IMDCT to avoid any discontinuities; if the window is symmetric and satisfies $w^2[i] + w^2[i+N] = 2$ the perfect reconstruction property is still true.

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$$\mathbf{M} = \begin{bmatrix} 0.56 & 0.20 & -0.20 & -0.56 & -0.83 & -0.98 & -0.98 & -0.83 \\ -0.98 & -0.56 & 0.56 & 0.98 & 0.20 & -0.83 & -0.83 & 0.20 \\ 0.20 & 0.83 & -0.83 & -0.20 & 0.98 & -0.56 & -0.56 & 0.98 \\ 0.83 & -0.98 & 0.98 & -0.83 & 0.56 & -0.20 & -0.20 & 0.56 \end{bmatrix}$$

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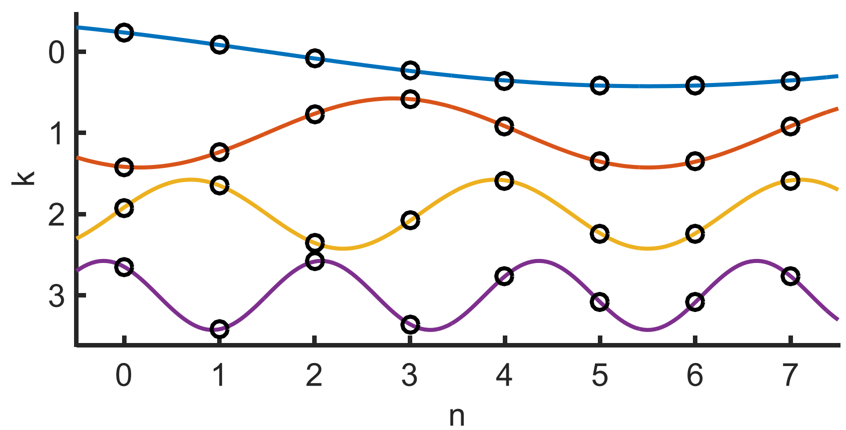

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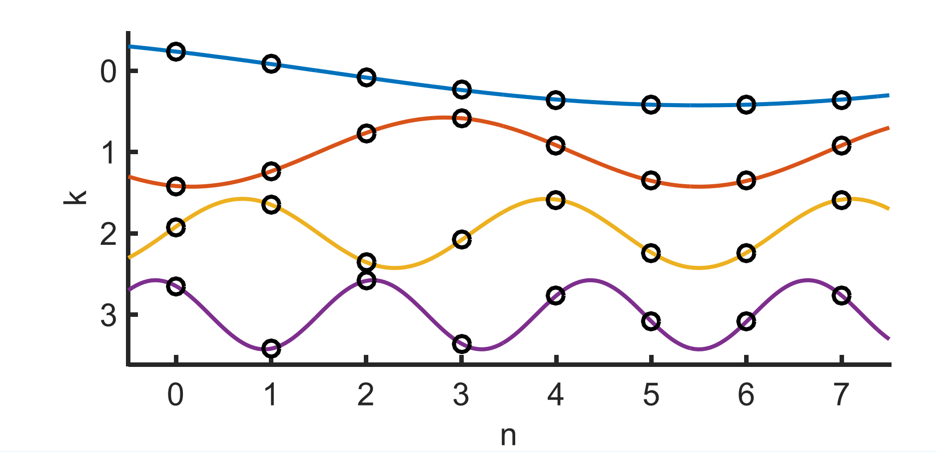

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The basis frequencies are $\{0.5, 1.5, 2.5, 3.5\}$ times the fundamental.

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- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \to N \to 2N$
- Aliasing errors cancel out when overlapping output frames are added
- Similar to DCT for energy compaction and decorrelation ☺
- Overlapping windowed frames can avoid edge discontinuities ☺
- Used in audio coding: MP3, WMA, AC-3, AAC, Vorbis, ATRAC

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\ \mathbf{x}\quad \overleftarrow{\mathbf{x}}\ \right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \to N \to 2N$
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For further details see Mitra: 5.

MATLAB routines

dct, idct	ODCT with optional zero-padding