IT 2014 Solutions

A — New Application
B — Bookwork
E — New Example
T — New Theory

1. a) Joint distribution

$x \backslash y$	0	1
0	0	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{3}$

$p(x=0)=\frac{1}{3}$ $\quad p(x=1)=\frac{2}{3}$

$p(y=0)=\frac{1}{3}$ $\quad p(y=1)=\frac{2}{3}$

$H(X) = H(Y) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918$ [2]E

$H(X|Y) = H(Y|X) = \frac{1}{3}H(0) + \frac{2}{3}H(\frac{1}{2}) = \frac{2}{3} = 0.667$ [2]E

average row entropy, $H(p)$: entropy function

$H(X,Y) = H(X) + H(Y|X) = 0.918 + \frac{2}{3} = 1.58$ [1]E

$I(X;Y) = H(X) - H(X|Y) = 0.918 - \frac{2}{3} = 0.251$ [1]E

b) $I(X_1;Y_1) = I(X_1;X_2) = 0$ since X_1 and X_2 are i.i.d. [1]E

$I(X_2;Y_2) = 0$ [1]E

$I(X_{1:2};Y_{1:2}) = I(X_{1:2};X_{2:1}) = H(X_{1:2}) = 2$ [3]E

$$I(X_1;X_2|Y_3) = \tfrac{1}{2}I(X_1;X_2|Y_3=0) + \tfrac{1}{2}I(X_1;X_2|Y_3=1)$$
$$= \tfrac{1}{2}I(X_1;X_1|Y_3=0) + \tfrac{1}{2}I(X_1;X_1\oplus 1|Y_3=1) \quad [3]E$$
$$= \tfrac{1}{2}H(X_1) + \tfrac{1}{2}H(X_1)$$
$$= H(X_1) = 1$$

c) i) Let $(\pi_0 \ \ \pi_1)$ be the stationary distribution

Then,

$$(\pi_0 \ \ \pi_1)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = (\pi_0 \ \ \pi_1)$$

We have

$$\pi_0(1-p) + \pi_1 q = \pi_0 \qquad [2]A$$

$$\pi_0 p = \pi_1 q$$

$$\pi_0 = \frac{q}{p}\pi_1$$

Since $\pi_0 + \pi_1 = 1$, we have

$$\left(1 + \frac{q}{p}\right)\pi_1 = 1 \qquad [2]A$$

Thus,

$$\pi_1 = \frac{p}{p+q} \qquad \pi_0 = \frac{q}{p+q}$$

ii) $$H(X) = \lim_{n\to\infty} H(X_n | X_{n-1}) \qquad [2]A$$

$$= \pi_0 H(p) + \pi_1 H(q) \quad \text{average row entropy}$$

$$= \frac{q}{p+q} H(p) + \frac{p}{p+q} H(q) \qquad (*) \qquad [2]A$$

iii) We know $H(X_n | X_{n-1}) \le H(X_n) \le 1$, so

$$H(X) \le 1 \qquad [3]A$$

Examining Eq. $(*)$, we find $H(X)=1$ if

$p = q = 0.5$.

2. a) i) (1) chain rule [1B]

(2) chain rule in another way [1B]

(3) $H(e|X,Y) \geq 0$ entropy is non-negative [1B]

$H(e|Y) \leq H(e)$ conditioning reduces entropy [1B]

(4) total probability theorem

(5) $H(e) = H(p_e)$ [2B]

Given Y and $e=0$, $X=Y$, so entropy $= 0$.

Given Y and $e=1$, $X \neq Y$ but can take any of the $|X|-1$ values, so entropy $\leq \log(|X|-1)$

(6) algebra [1B]

(7) $H(p_e) \leq 1$ [1B]

ii) The optimum detection is given by

$$\hat{X} = \begin{cases} 1 & y = a \\ 2 & y = b \\ 3 & y = c \end{cases}$$ [2E]

Thus, P_e is equal to the sum of the off-diagonal elements, i.e.,

$$P_e = \frac{1}{2}$$ [1E]

To evaluate Fano's inequality, we need

$$H(X|Y) = \frac{1}{3}H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4}) + \frac{1}{3}H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4}) + \frac{1}{3}H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4})$$

$$= H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4})$$ [2E]

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$$

$$= 1.5$$

Hence,

$$P_e > \frac{H(X|Y) - 1}{\log(|X| - 1)}$$

$$= \frac{1.5 - 1}{\log 2} \qquad [2E]$$

$$= 0.5$$

which is exactly the same as the actual P_e.

b) By definition

$$R(D) = \min I(X; \hat{X})$$

Such that $E[(X-\hat{X})^2] = D$ and $p(x) = \int p(x, \hat{x})\, d\hat{x}$.

The second condition is obviously true from Fig 2.2. [2T]

Now check the first condition:

$$E[(X-\hat{X})^2] = E\left[\left(\frac{D}{\sigma^2}X - \frac{\sigma^2 - D}{\sigma^2} Z\right)^2\right]$$

$$= \frac{D^2}{\sigma^4}\sigma^2 + \frac{(\sigma^2-D)^2}{\sigma^4}\frac{D\sigma^2}{\sigma^2 - D^2} \qquad [2T]$$

$$= \frac{D^2}{\sigma^2} + \frac{(\sigma^2-D)D}{\sigma^2}$$

$$= D.$$

The mutual information

$$I(X; \hat{X}) = h(\hat{X}) - h(\hat{X}|X)$$

$$= h(\hat{X}) - h\left(\frac{\sigma^2-D}{\sigma^2} Z\right) \qquad [2T]$$

Note that $\hat{X}$ has zero mean and variance

$$E[\hat{X}^2] = \left(\frac{\sigma^2-D}{\sigma^2}\right)^2\left(\sigma^2 + \frac{\sigma^2 D}{\sigma^2 - D}\right) \qquad [2T]$$

$$= \sigma^2 - D$$

So $h(\hat{X}) \le \frac{1}{2}\log[2\pi e(\sigma^2 - D)]$ Gaussian has maximum entropy

Hence,

$$I(X;\hat{X}) \le \frac{1}{2}\log[2\pi e(\sigma^2-D)] - h(Z) - \log\frac{\sigma^2-D}{\sigma^2}$$

$$= \frac{1}{2}\log[2\pi e(\sigma^2-D)] - \frac{1}{2}\log\left[2\pi e\frac{D\sigma^2}{\sigma^2-D}\right] - \frac{1}{2}\log\left(\frac{\sigma^2-D}{\sigma^2}\right)^2$$

$$= \frac{1}{2}\log\frac{\sigma^2}{D} \qquad [2T]$$

Since $I(X;\hat{X}) \le \frac{1}{2}\log\frac{\sigma^2}{D}$ for this example, the minimum mutual information also $\le \frac{1}{2}\log\frac{\sigma^2}{D}$. QED.

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a) i) (1) Chain rule [1B] each

(2) Conditioning reduces entropy

(3) definition of mutual info

(4) X_i's are independent

(5) from (2)

(6) definition of mutual info

(7) definition of mutual info

(8) memoryless channel

(9) chain rule

(10) conditioning reduces entropy

ii) From (3)-(7), if the channel has memory, i.e., y_i's are correlated for independent X_i's, then

$$I(X_{1:n}; Y_{1:n}) \geq \sum I(X_i; Y_i)$$ [3A]

On the other hand, if the channel is memoryless, then

$$I(X_{1:n}; Y_{1:n}) \leq \sum I(X_i; Y_i)$$ [3A]

Therefore, a channel with memory has higher capacity.

b) i) Weakly symmetric

$$C = \log 3 - H\left(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{2}\right) \qquad [3E]$$

$$= 0.126$$

ii) symmetric

$$C = \log 3 - H\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right)$$

$$= 0.085 \qquad [3E]$$

iii) Symmetric

$$C = \log 4 - H\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \qquad [3E]$$

$$= 1$$

4. a) Under no interference, it is a Gaussian channel

$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right) \qquad (**) \qquad [2B]$$

Under very strong interference, Y_1 first decodes X_2, while treating X_1 as noise. It then subtracts X_2 and decodes X_1, which is a clean channel with the same capacity as $(**)$. [2B]

This is possible if the rate of X_2 is less than [2A]

$$\frac{1}{2}\log\left(1+\frac{a^2P}{P+N}\right).$$

Thus, we have the same capacity if

$$\frac{1}{2}\log\left(1+\frac{P}{N}\right) \le \frac{1}{2}\log\left(1+\frac{a^2P}{P+N}\right)$$

[4 A]

$$\frac{P}{N} \le \frac{a^2P}{P+N}$$

$$a^2 \le \frac{P+N}{N} = 1+\frac{P}{N}$$

b) From the joint distribution, we obtain the marginal distributions:

$$P_{U_1} = \left(\alpha+\beta, \frac{\gamma}{m-1}, \dots, \frac{\gamma}{m-1}\right) \qquad [1E]$$

$$P_{U_2} = \left(\alpha+\gamma, \frac{\beta}{m-1}, \dots, \frac{\beta}{m-1}\right) \qquad [1E]$$

Thus,

$$H(U_1) = -(\alpha+\beta)\log(\alpha+\beta) - (m-1)\frac{\gamma}{m-1}\log\left(\frac{\gamma}{m-1}\right)$$

$$= -(\alpha+\beta)\log(\alpha+\beta) - \gamma\log\left(\frac{\gamma}{m-1}\right) \qquad [2E]$$

$$= H(\alpha+\beta, \gamma) + \gamma\log(m-1)$$

Similarly, $H(U_2) = H(\alpha+\gamma, \beta) + \beta\log(m-1)$ [2E]

$$= -(\alpha+\gamma)\log(\alpha+\gamma) - \beta\log\frac{\beta}{m-1}$$

Also,

$$H(U_1,U_2) = -\alpha\log\alpha - (m-1)\frac{\beta}{m-1}\log\left(\frac{\beta}{m-1}\right) - (m-1)\frac{\gamma}{m-1}\log\left(\frac{\gamma}{m-1}\right)$$ [2E]

$$= -\alpha\log\alpha - \beta\log\frac{\beta}{m-1} - \gamma\log\frac{\gamma}{m-1}$$

$$= H(\alpha,\beta,\gamma) + \beta\log(m-1) + \gamma\log(m-1)$$

$$H(U_1|U_2) = H(U_1,U_2) - H(U_2)$$ [2E]

$$= H(\alpha,\beta,\gamma) - H(\alpha+\gamma,\beta) + \gamma\log(m-1)$$

$$H(U_2|U_1) = H(U_1,U_2) - H(U_1)$$ [2E]

$$= H(\alpha,\beta,\gamma) - H(\alpha+\beta,\gamma) + \beta\log(m-1)$$

Hence, the slepian-wolf region is

$$R_1 \geq H(U_1|U_2)$$

$$R_2 \geq H(U_2|U_1)$$ [3B]

$$R_1+R_2 \geq H(U_1,U_2)$$

[For this question, the expressions are not unique.]