two properties

4: Linear Time Invariant Systems

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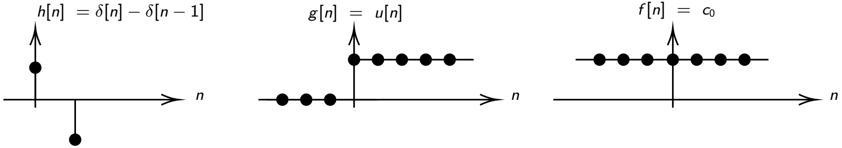

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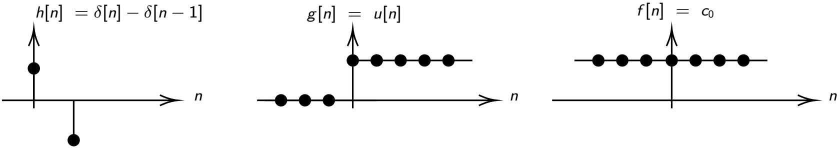

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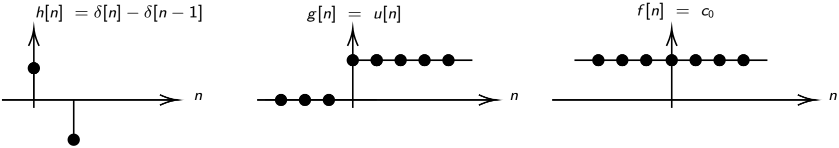

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3. From the figure above.

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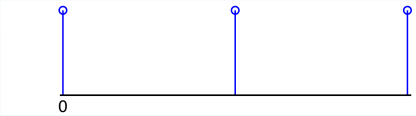

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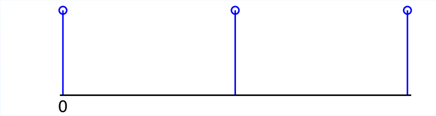

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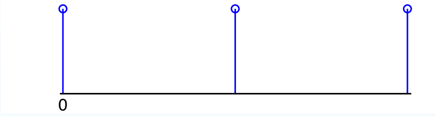
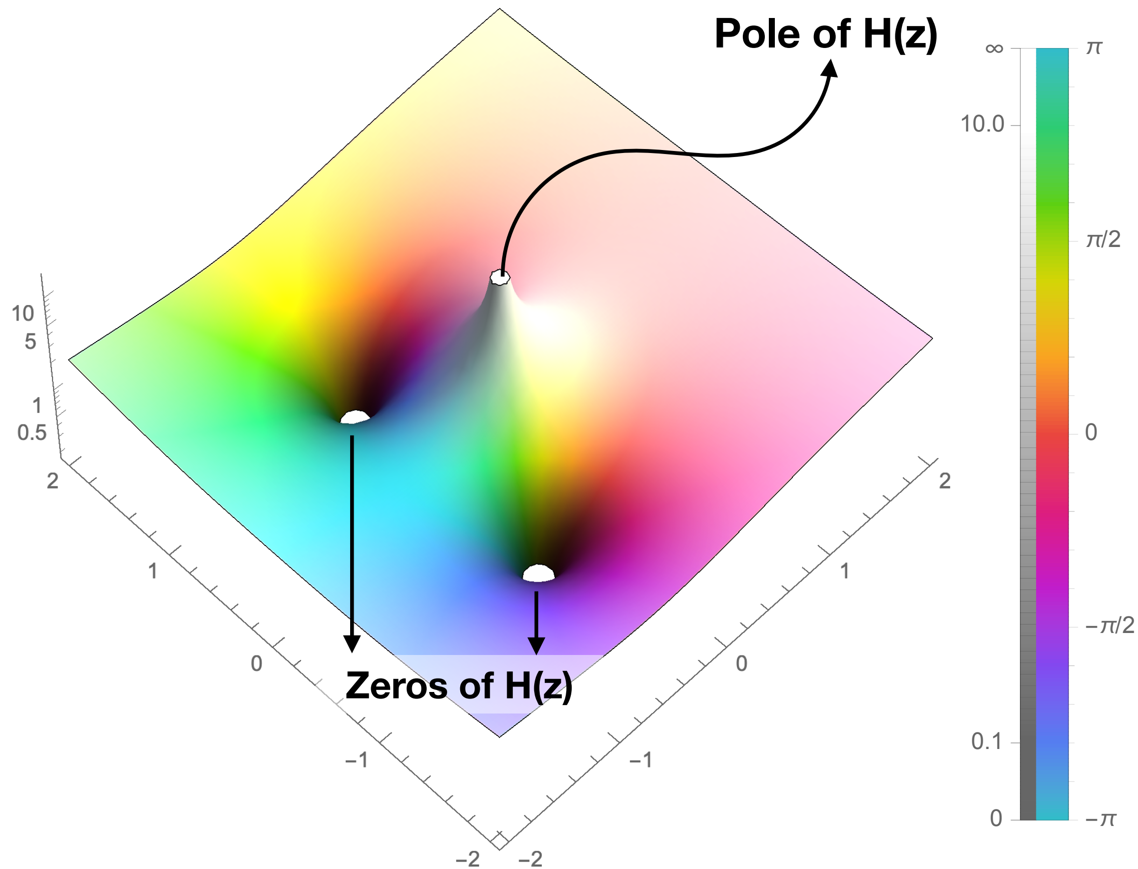
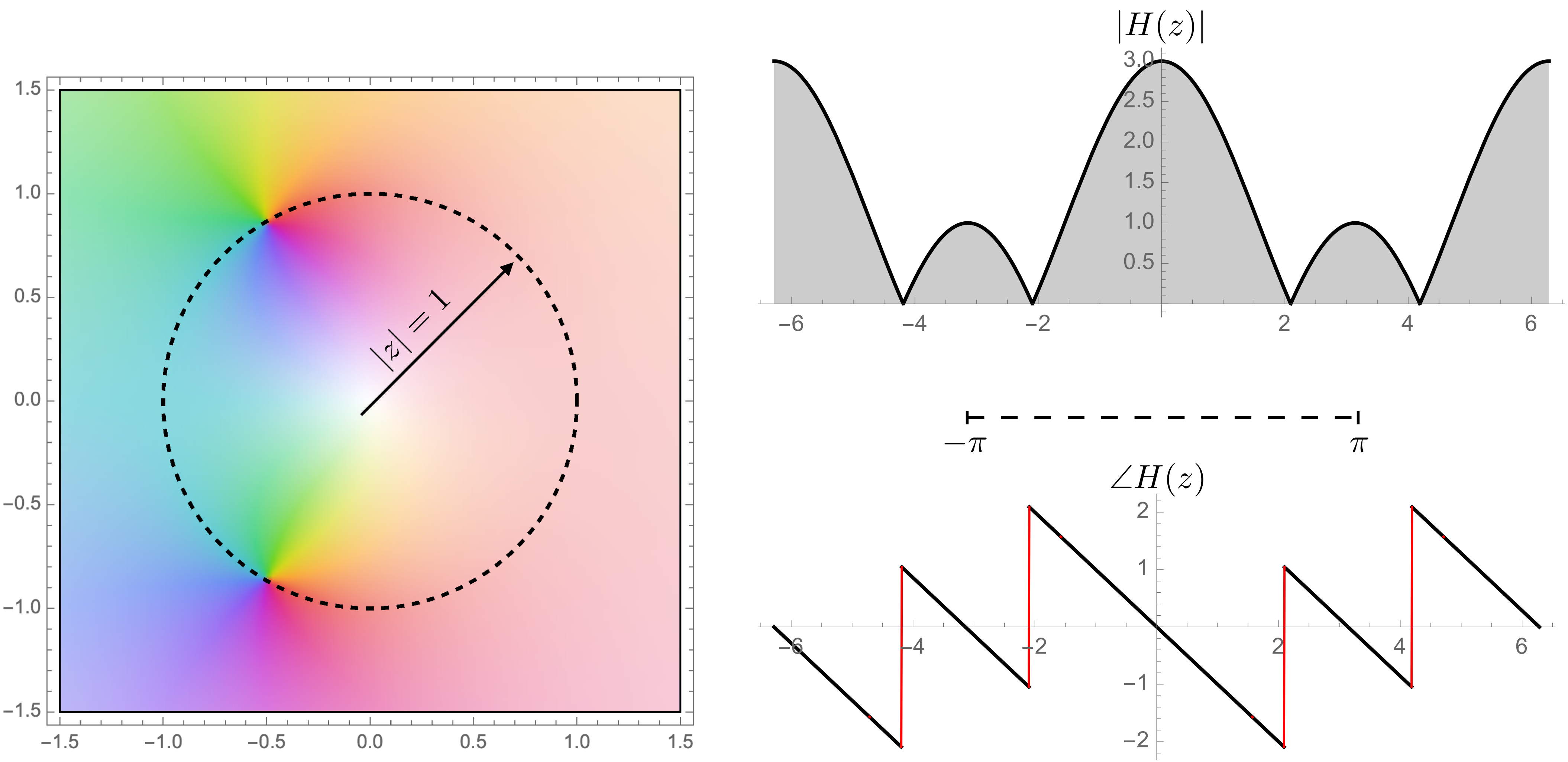


$$\begin{aligned} H(e^{j\omega}) &= 1 + e^{-j\omega} + e^{-j2\omega} \\ &= e^{-j\omega}\left(1 + 2\cos\omega\right) \end{aligned}$$

$$h[n] = \delta[n-1] + \delta[n] + \delta[n+1] \longleftrightarrow H(z) = z + 1 + z^{-1}$$





$|z| = 1$
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π
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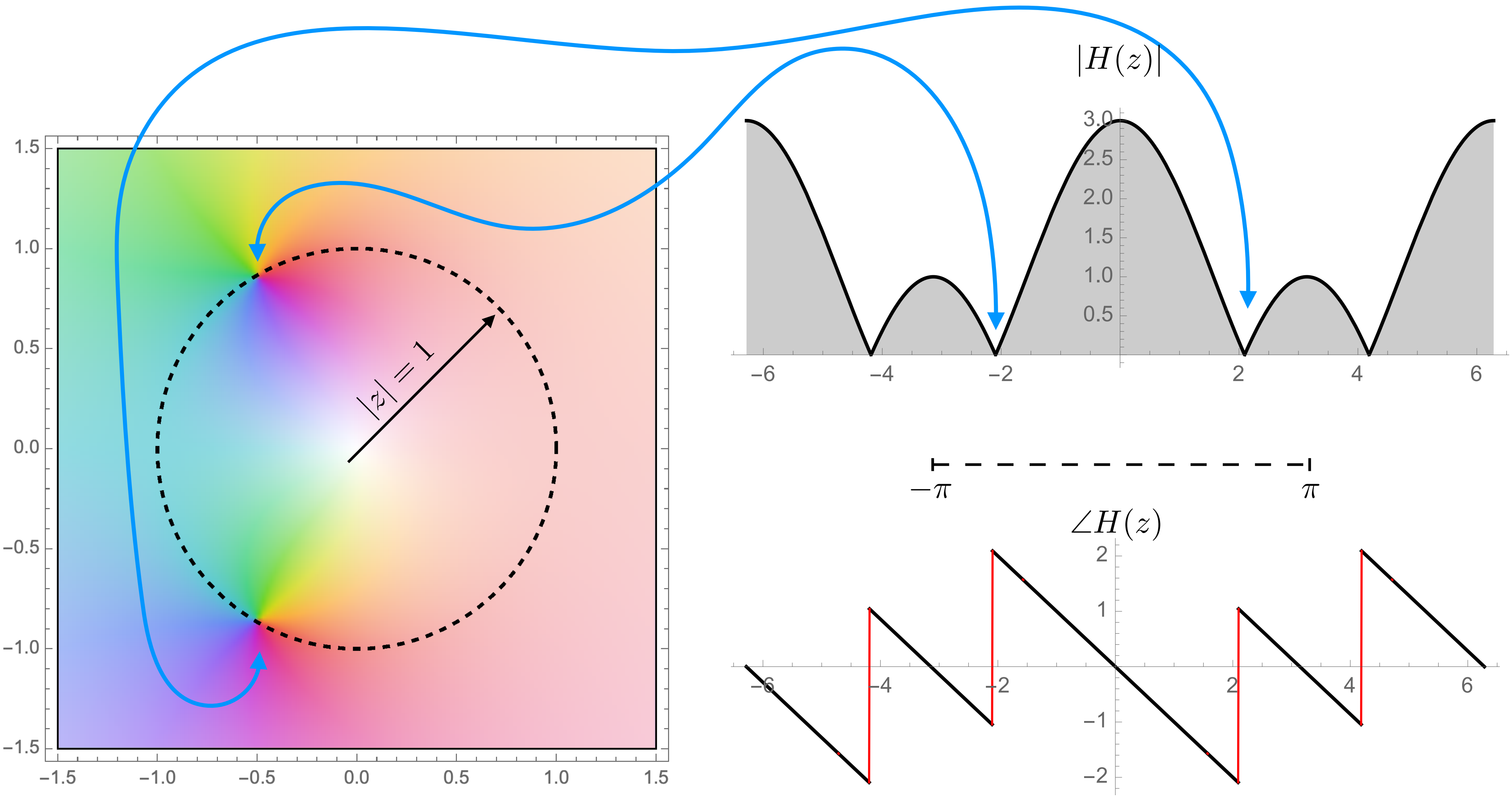

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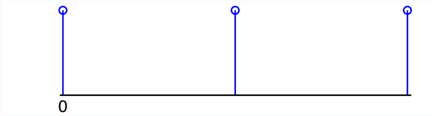

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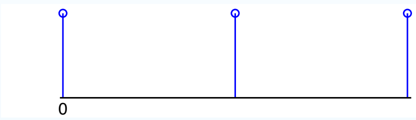

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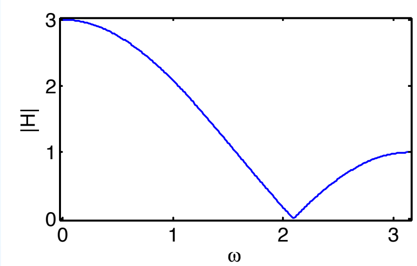

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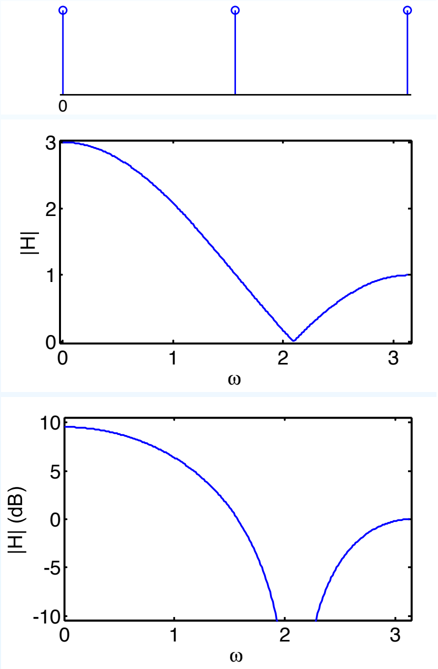

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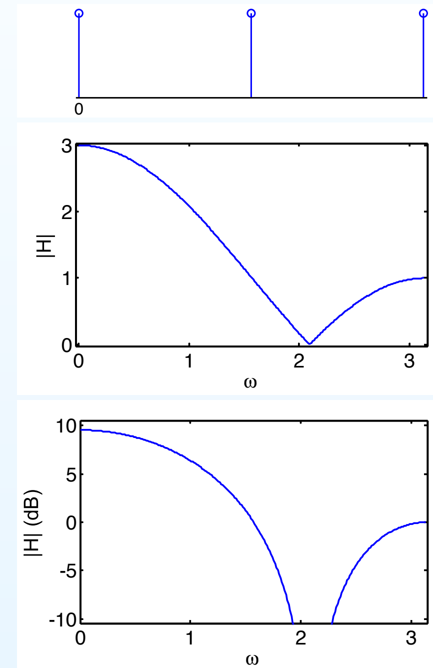

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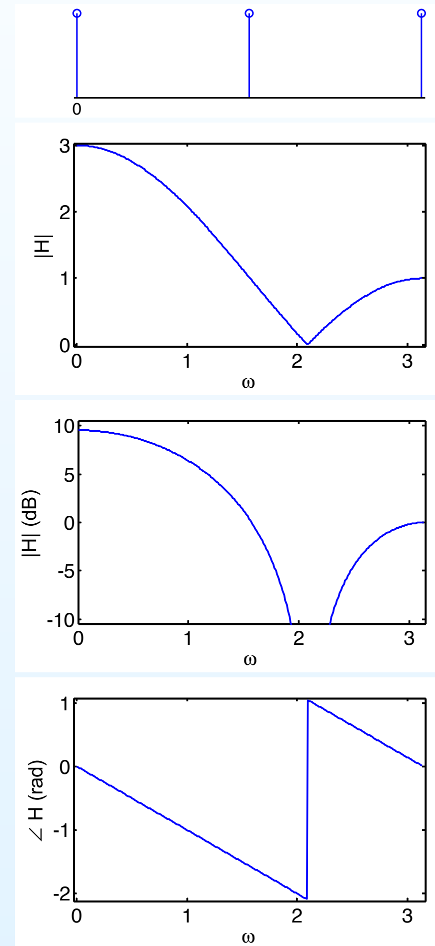

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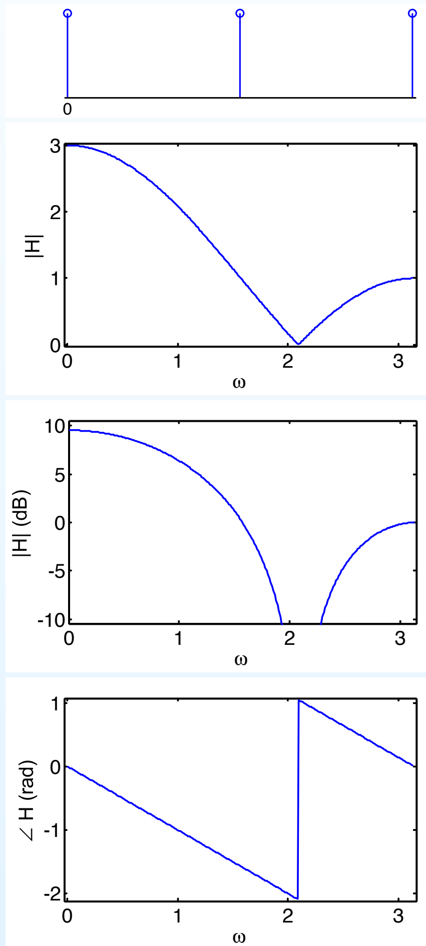

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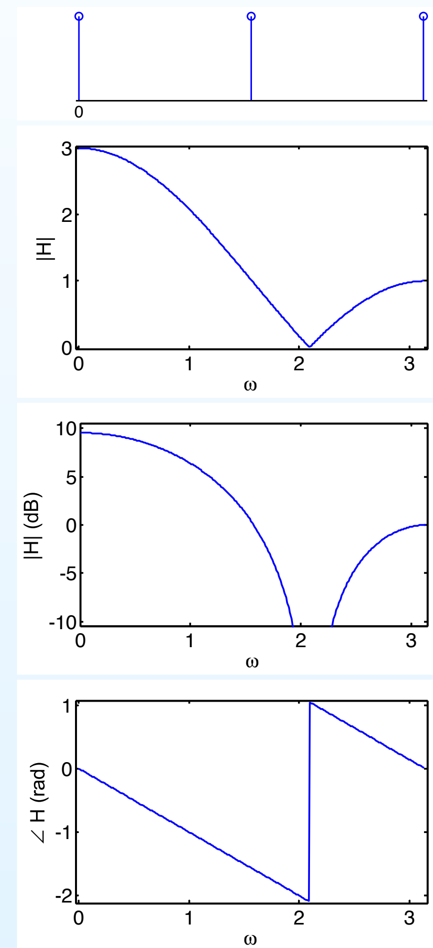

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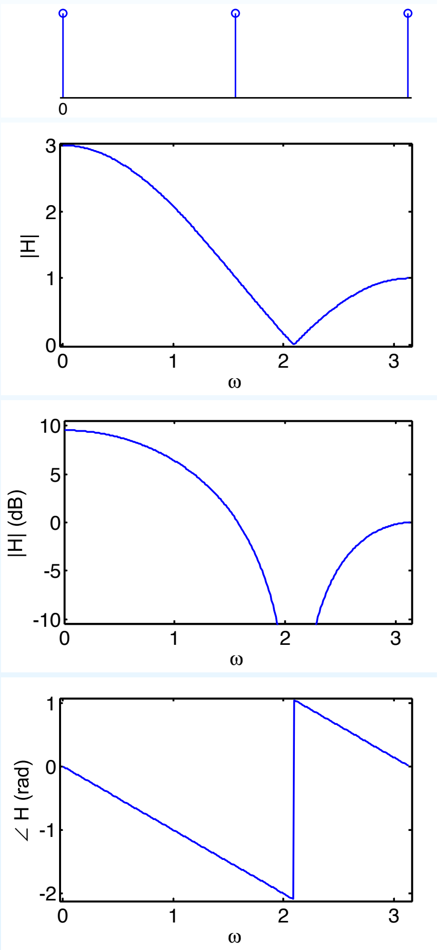

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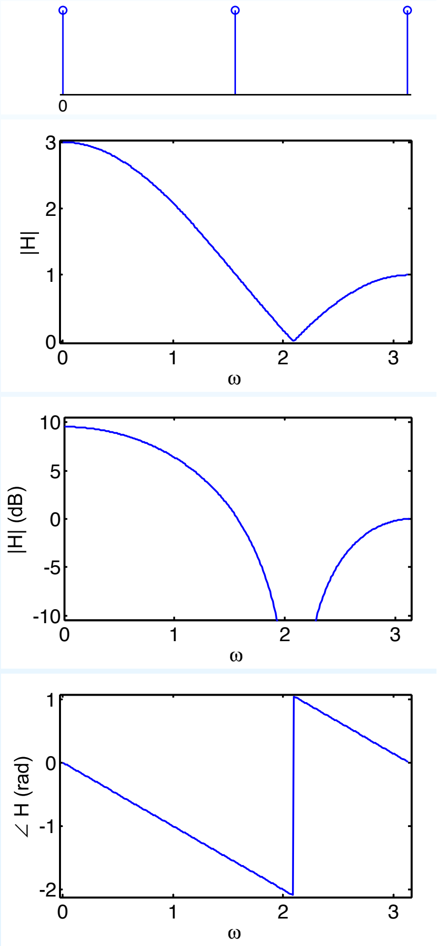

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i.e. output at time n depends only on inputs up to time n.

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All the systems we will deal with are causal.

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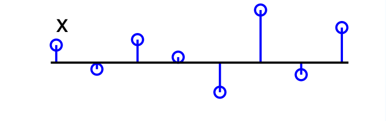
$y[n] = x[n] * h[n]$: convolve $x[0 : N-1]$ with $h[0 : M-1]$



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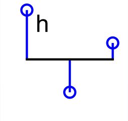

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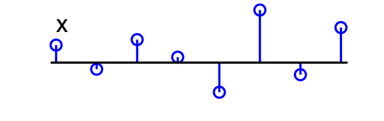

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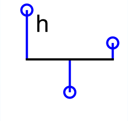

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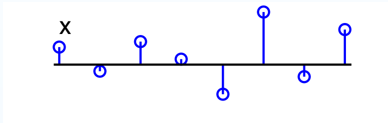

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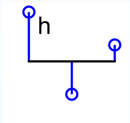

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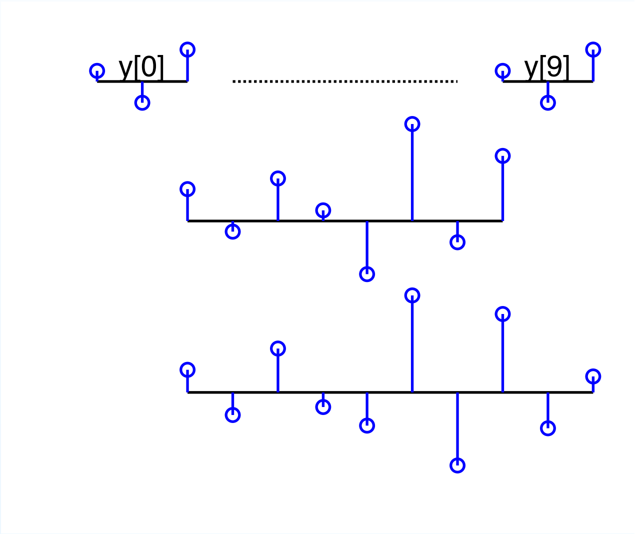

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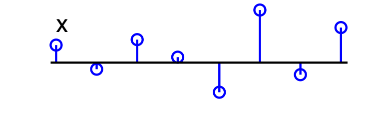

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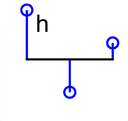

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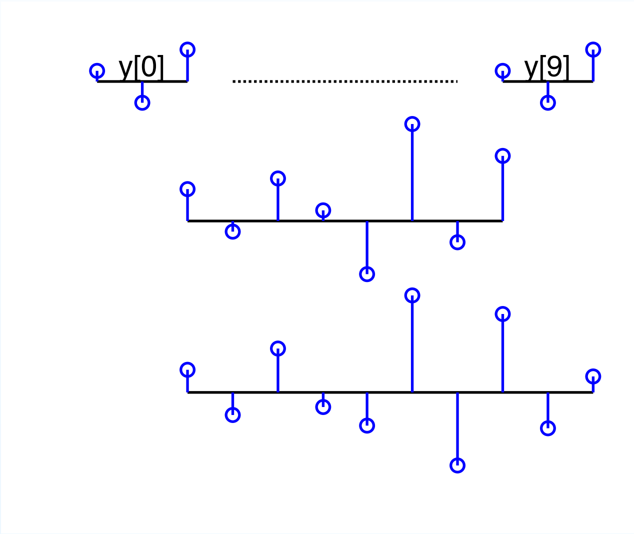

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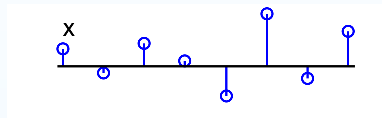

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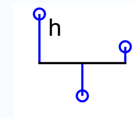

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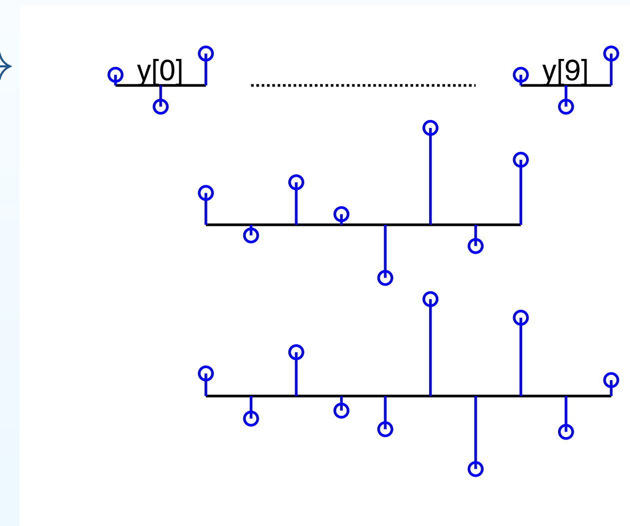

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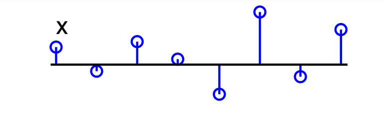

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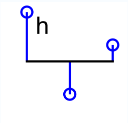

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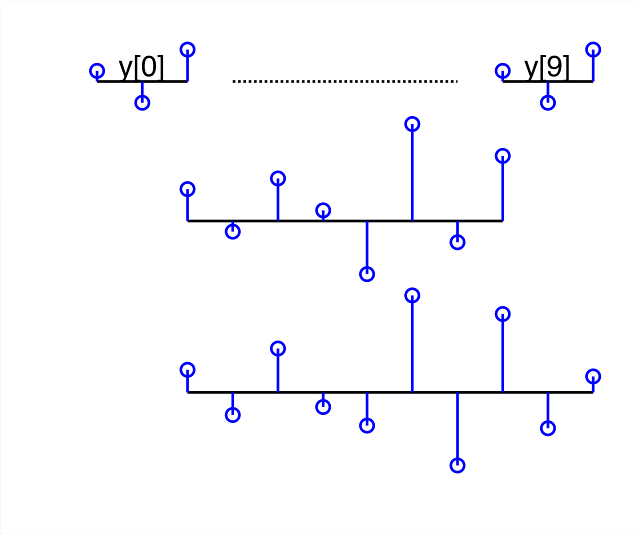

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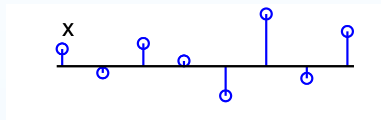

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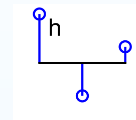

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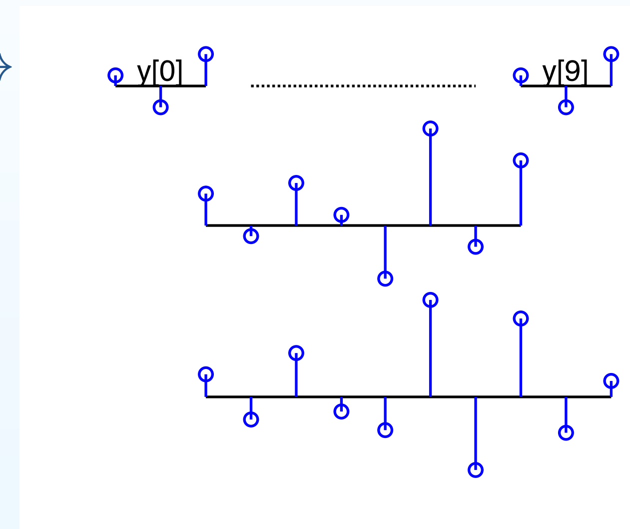

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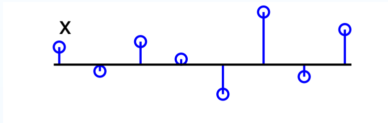

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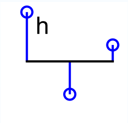

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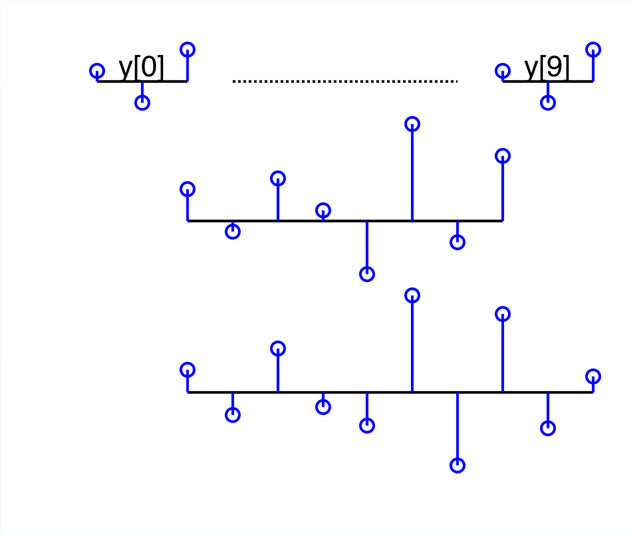

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Hence: $y[n] = \sum_{r=\max(0,n+1-N)}^{\min(M-1,n))} h[r]x[n-r]$

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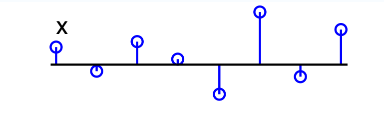

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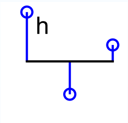

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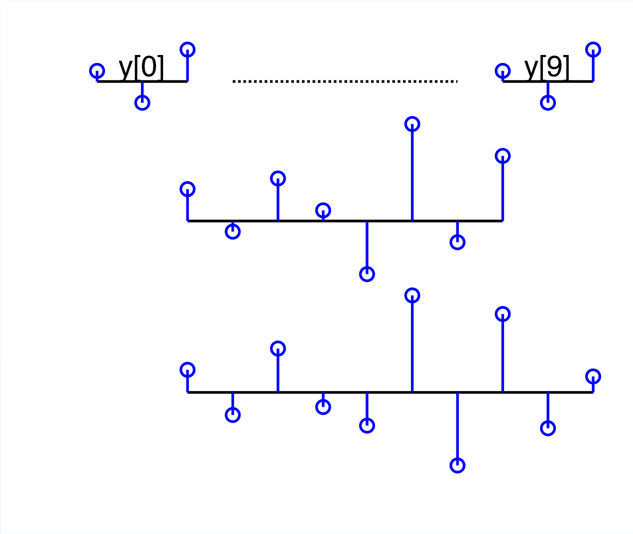

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$M + N - 1$ non-zero values

$N = 8$, $M = 3$
$M + N - 1 = 10$

Algebraically:

$$x[n-r] \neq 0 \Rightarrow 0 \le n - r \le N - 1$$
$$\Rightarrow n + 1 - N \le r \le n$$

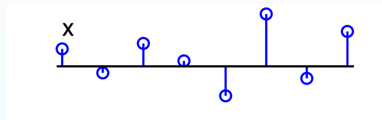
Hence: $y[n] = \sum_{r=\max(0,n+1-N)}^{\min(M-1,n))} h[r]x[n-r]$

We must multiply each $h[n]$ by each $x[n]$ and add them to a total

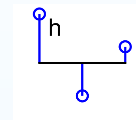
$\Rightarrow$ total arithmetic complexity ($\times$ or $+$ operations) $\approx 2MN$

Circular Convolution

$y_{\circledast}[n] = x[n] \circledast_N h[n]$: circ convolve $x[0:N-1]$ with $h[0:M-1]$



$\circledast_N$



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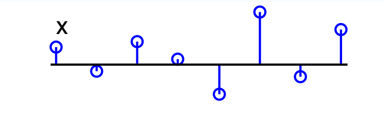

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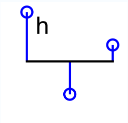

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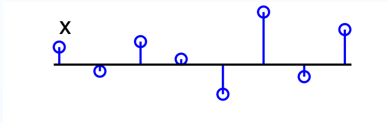

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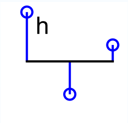

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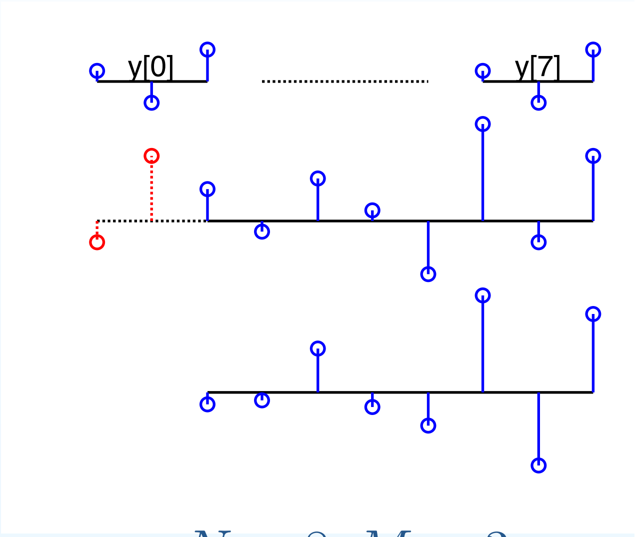

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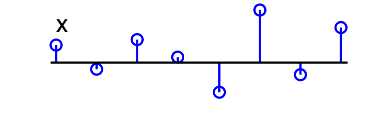

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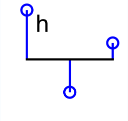

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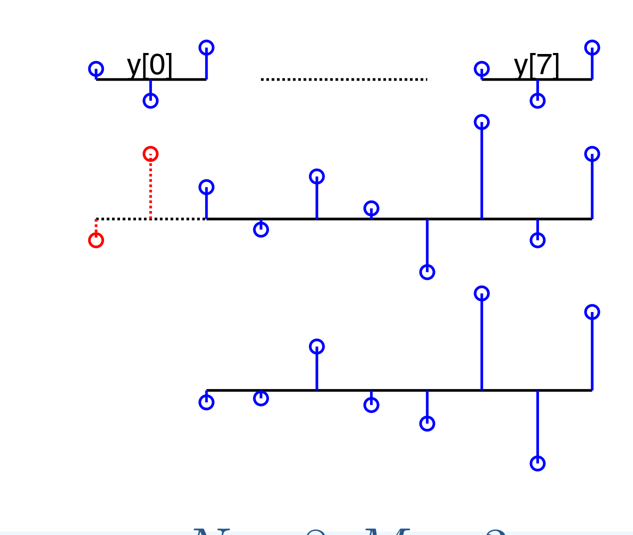

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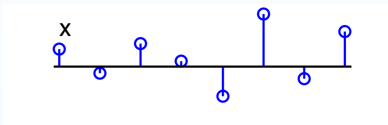

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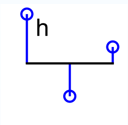

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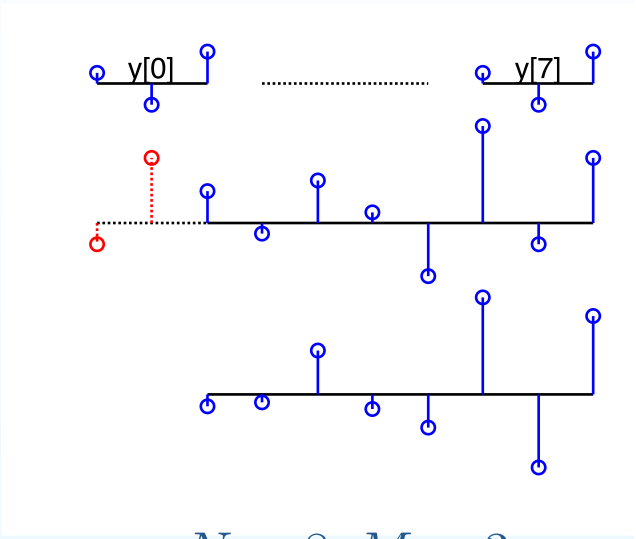

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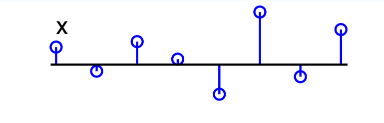

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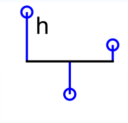

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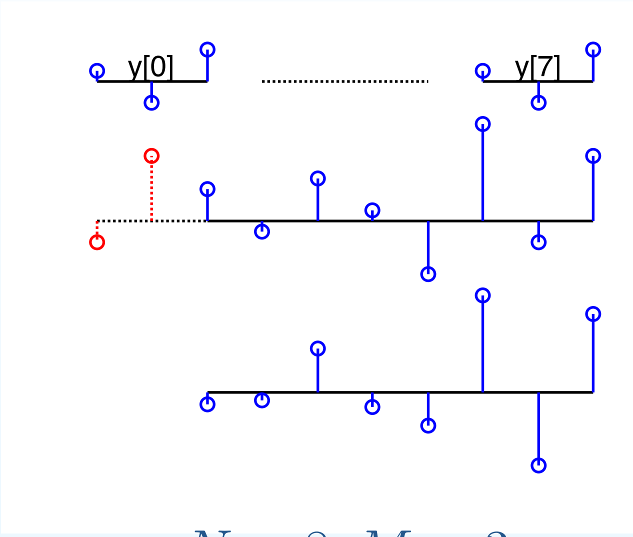

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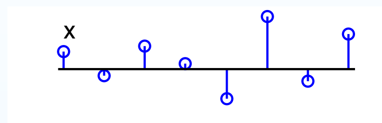

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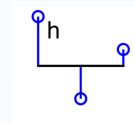

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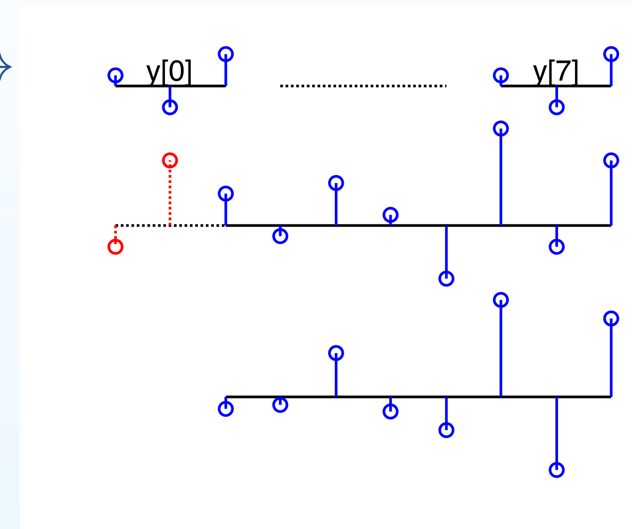

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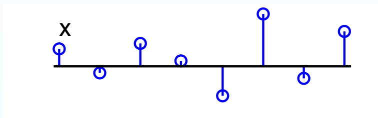

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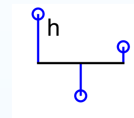

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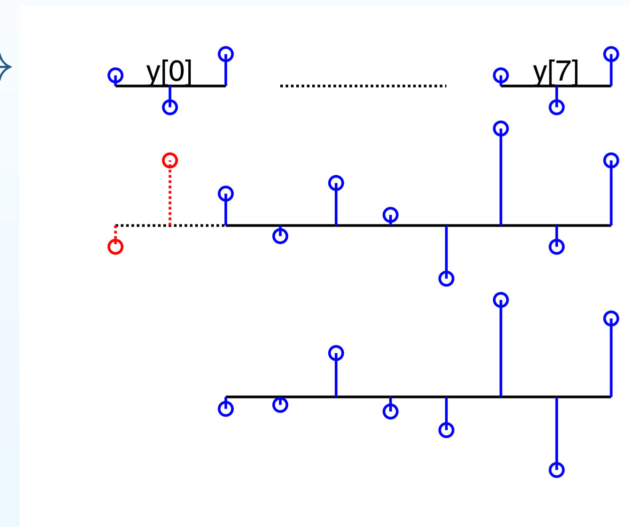

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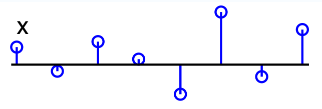

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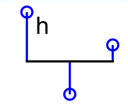

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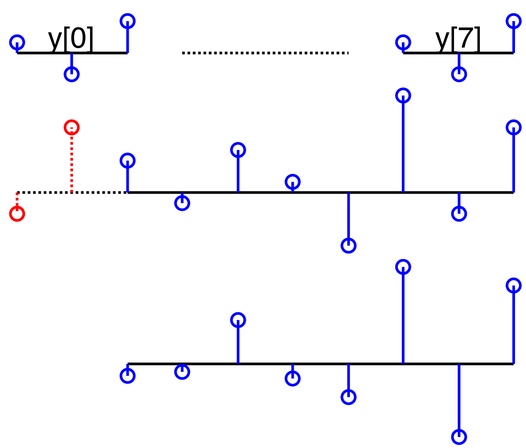

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Circular convolution is a necessary evil in exchange for using the DFT

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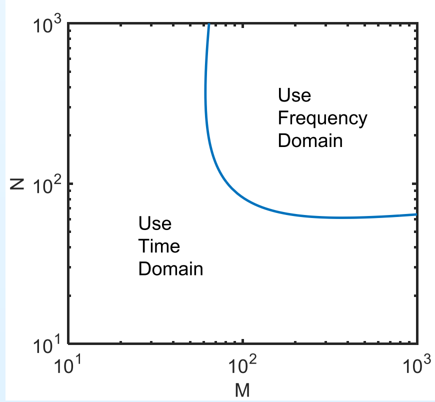

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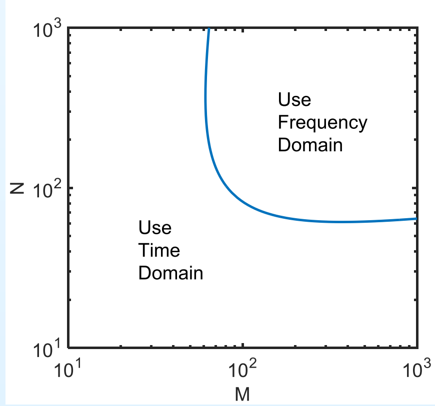

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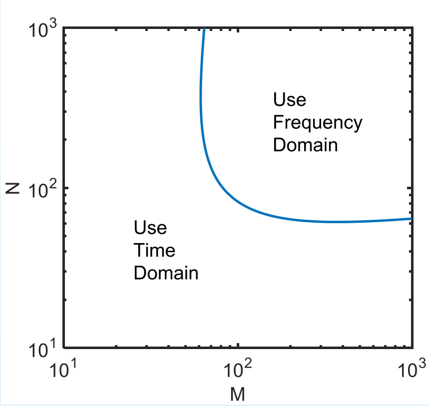

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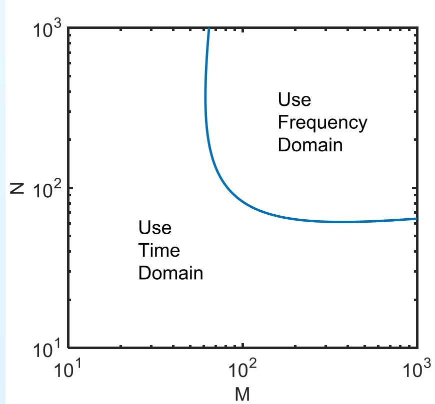

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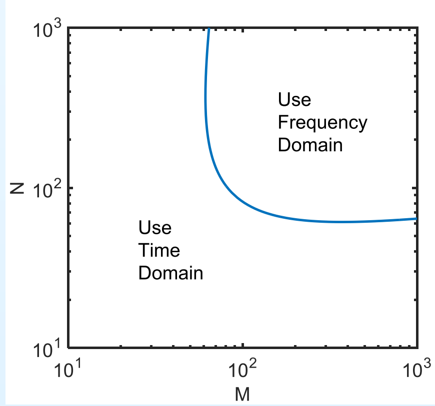

But: (a) DFT may be very long if N is large

(b) No outputs until all $x[n]$ has been input.

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For further details see Mitra: 4 & 5.

MATLAB routines

fftfilt	Convolution using overlap add
$x[n] \circledast y[n]$	real(ifft(fft(x).*fft(y)))