Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 10

Instructor: Dr. Ayush Bhandari

The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Let $A\left(e^{j\omega}\right) = 1 + e^{-j\omega}a$ and $B\left(e^{j\omega}\right) = b$. We try to approximate a filter given by

$$D(\omega) = \begin{cases} 1, & \omega < \frac{\pi}{6}, \\ 2, & \omega \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right], \\ 0, & \omega > \frac{\pi}{2}. \end{cases} \tag{1}$$

The weight function is $W(\omega) = 1, \forall \omega \in [0, \pi]$. We choose $\omega \in \left\{0, \frac{\pi}{4}, \pi\right\}$.

(a) Derive the numerical expressions of $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{d}$ such that

$$\left[\boldsymbol{U}^T, \boldsymbol{V}^T\right] \begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{d}. \tag{2}$$

We have that

$$\boldsymbol{U} = \left[-W(\omega_1) D(\omega_1) e^{-j\omega_1}, -W(\omega_2) D(\omega_2) e^{-j\omega_2}, -W(\omega_3) D(\omega_3) e^{-j\omega_3}\right], \tag{3}$$

$$\boldsymbol{V} = \left[W(\omega_1), W(\omega_2), W(\omega_3)\right]. \tag{4}$$

After substituting with the correct values of $\omega_1, \omega_2, \omega_3$, we have that

$$\boldsymbol{U} = \left[-1, \sqrt{2} + j\sqrt{2}, 0\right], \tag{5}$$

$$\boldsymbol{V} = [1, 1, 1]. \tag{6}$$

Furthermore

$$\boldsymbol{d} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}. \tag{7}$$

(b) Compute the real values a, b that minimize $\sum_k \left|E_E(\omega_k)\right|^2$.

Let $\boldsymbol{P} = \left[\boldsymbol{U}^T, \boldsymbol{V}^T\right]$. Currently, if we try to solve $\boldsymbol{P}\begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{d}$, the solution is complex. To ensure we get a real solution we compute

$$\begin{bmatrix} \mathrm{Real}(\boldsymbol{P}) \\ \mathrm{Imag}(\boldsymbol{P}) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \mathrm{Real}(\boldsymbol{d}) \\ \mathrm{Imag}(\boldsymbol{d}) \end{bmatrix}. \tag{8}$$

To find a, b that minimise $\sum_k \left|E_E(\omega_k)\right|^2$, we need to compute the pseudoinverse of $\bar{\boldsymbol{P}} = \begin{bmatrix} \mathrm{Real}(\boldsymbol{P}) \\ \mathrm{Imag}(\boldsymbol{P}) \end{bmatrix}$, which is amounts to $\bar{\boldsymbol{P}}^+ = \left(\bar{\boldsymbol{P}}^T \bar{\boldsymbol{P}}\right)^{-1} \bar{\boldsymbol{P}}^T$. We have that

$$\bar{\boldsymbol{P}} = \begin{bmatrix} -1 & 1 \\ -\sqrt{2} & 1 \\ 0 & 1 \\ 0 & 0 \\ \sqrt{2} & 0 \\ 0 & 0. \end{bmatrix}, \quad \bar{\boldsymbol{P}}^+ = \begin{bmatrix} \frac{\sqrt{2}}{17} - \frac{5}{34} & \frac{1}{34} - \frac{11\sqrt{2}}{68} & \frac{7\sqrt{2}}{68} + \frac{2}{17} & 0 & \frac{9\sqrt{2}}{34} + \frac{3}{34} & 0 \\ \frac{11-\sqrt{2}}{34} & \frac{4}{17} - \frac{3\sqrt{2}}{68} & \frac{5\sqrt{2}}{68} + \frac{15}{34} & 0 & \frac{2\sqrt{2}}{17} + \frac{7}{34} & 0 \end{bmatrix}. \tag{9}$$

Then the solution satisfies

$$\begin{bmatrix} a \\ b \end{bmatrix} = \bar{\boldsymbol{P}}^+ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{9\sqrt{2}}{34} - \frac{3}{34} \\ \frac{27}{34} - \frac{2\sqrt{2}}{17} \end{bmatrix} \approx \begin{bmatrix} -0.46 \\ 0.63 \end{bmatrix}. \tag{10}$$

2) In Optimal IIR filter design, we aim to approximate a target filter $D(\omega)$ by adjusting the parameters of a filter $H\left(e^{j\omega}\right) = \frac{B\left(e^{j\omega}\right)}{A\left(e^{j\omega}\right)} = \frac{b_0+b_1e^{-j\omega}+\cdots+b_Me^{-jM\omega}}{1+a_1e^{-j\omega}+\cdots+a_Ne^{-jN\omega}}$. Typically, the estimation is performed by minimising one of the two following equations

$$E_S(\omega) = W_S(\omega)\left[\frac{B\left(e^{j\omega}\right)}{A\left(e^{j\omega}\right)} - D(\omega)\right], \tag{11}$$

$$E_E(\omega) = W_E(\omega)\left[B\left(e^{j\omega}\right) - D(\omega)\,A\left(e^{j\omega}\right)\right]. \tag{12}$$

(a) Given that both equations involve nonlinear operations due to weighting functions $W_S(\omega), W_E(\omega)$, explain what is the challenge in optimising $E_S(\omega)$.

The unknowns in this case are a_k, b_k. Therefore $E_S(\omega)$ is a non-linear expression as a function of the unknowns. Even though $E_E(\omega)$ contains non-linear computations, the expression is linear in the unknown variables.

(b) The optimisation is performed at frequency samples $\{\omega_k\}_{k=1}^{K}$. Explain why we can estimate the filter when K is only half of the number of unknowns, which is $K = \frac{M+N+1}{2}$.

The expressions for $A\left(e^{j\omega}\right)$ and $B\left(e^{j\omega}\right)$ are complex. Given that we are looking for real coefficients a_k, b_k, we can double the number of equations by taken the real and imaginary parts

$$\begin{bmatrix} \mathrm{Real}\left(\boldsymbol{P}\right) \\ \mathrm{Imag}\left(\boldsymbol{P}\right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \mathrm{Real}\left(\boldsymbol{d}\right) \\ \mathrm{Imag}\left(\boldsymbol{d}\right) \end{bmatrix}, \tag{13}$$

where $\boldsymbol{P} = \left[\boldsymbol{U}^T, \boldsymbol{V}^T\right]$, $\boldsymbol{U} = \left[-W(\omega_1)\,D(\omega_1), \ldots, -W(\omega_K)\,D(\omega_K)\right]$ and $\boldsymbol{V} = \left[-W(\omega_1), \ldots, -W(\omega_K)\right]$. Therefore the final number of equations $2K$ is equal to the number of unknowns $2K = M + N + 1$.

(c) Explain what happens if K is very large, e.g. $K = 100 \cdot \frac{M+N+1}{2}$. Select one from the options below, and explain the choice.

i. The estimation would fail.

ii. The algorithm would estimate $D(\omega)$ with a very high degree of accuracy

iii. The algorithm works, but the accuracy is limited.

The least-squares algorithm does not fail. When the system is over-determined, it identifies the function $H(\omega)$ that is closest in norm to $D(\omega)$. However, because the number of parameters is the same, the solution cannot converge to the target function $D(\omega)$. Therefore the answer is iii.

(d) Explain what happens if $K < \frac{M+N+1}{2}$. Select one from the options below, and explain the choice.

i. The algorithm returns an error.

ii. The algorithm returns multiple solutions.

iii. The algorithm returns one correct solution.

In this case, the algorithm still looks for the solution that minimises $\sum_k |E_E(\omega_k)|^2$, so therefore it will return one solution. This represents the solution of (12) with minimum energy, and therefore it is a correct solution. Therefore the answer is iii.

(e) Let $W_E(\omega) = \cos^2\left(p\left(\omega - \omega_0\right)\right)$. Find p, ω_0 such that $E_E\left(k\frac{\pi}{20}\right) = 0, k = 1, \ldots, 20$, for any $H\left(e^{j\omega}\right)$ and $D(\omega)$.

To satisfy the requirement, we need $W_E\left(k\frac{\pi}{20}\right) = 0$. Given the definition of cos, we have that

$$\cos^2\left(p\left(k\frac{\pi}{20} - \omega_0\right)\right) = 0 \Leftrightarrow p\left(k\frac{\pi}{20} - \omega_0\right) = \frac{\pi}{2} + l_k\pi, l_k \in \mathbb{Z}, k = 1, \ldots, 20. \tag{14}$$

Therefore we can select $p = 20$, which gives us $k\pi - 20\omega_0 = \frac{\pi}{2} + l_k\pi$ which is satisfied for $l_k = k - 1$.