Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 1

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The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Which of the following statements is true and why?
(Multiple choices may be correct)

i) All discrete-time signals are digital signals.
ii) All digital signals are discrete-time signals.
iii) Some discrete-time signals are digital signals.
iv) Some digital signals are discrete-time signals.

By definition, digital signals are discrete in both amplitude and time. Hence, (ii) and (iii) are correct.

2) Let us define a sequence by,

$$x[n] = (r_1)^n u[n] - (r_2)^n u[-(n+1)] \tag{1}$$

where $r_1 = -1/3$ and $r_2 = 1/2$.

- On the z-plane, plot the poles and zeros together with the region-of-convergence.
- In the above sequence in (1), what happens if we exchange r_1 and r_2?

For the first part,

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-(n+1)].$$

Due to linearity of the z-transform, we evaluate the individual terms as follows,

$$\underbrace{\left(-\frac{1}{3}\right)^n u[n]}_{r_1^n u[n]} \xrightarrow{\text{Z}-\text{domain}} \frac{1}{1+\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$$

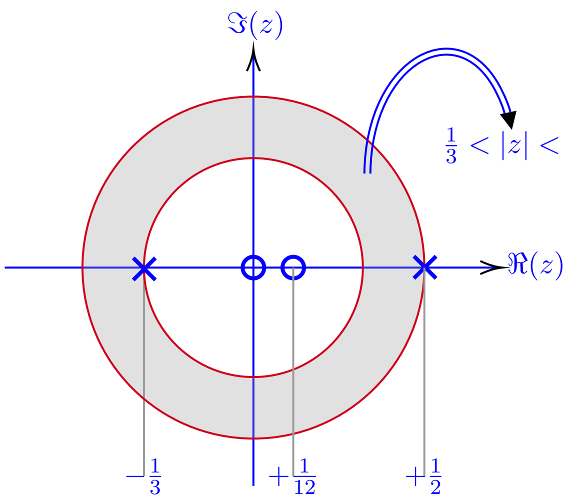
and,

$$\underbrace{-\left(\frac{1}{2}\right)^n u[-(n+1)]}_{-r_2^n u[-(n+1)]} \xrightarrow{\text{Z}-\text{domain}} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2}.$$

Hence,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z(z-1/12)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$

We see two zeros at $z = 0$ and $z = 1/12$ and two poles at $z = -1/3$ and $z = 1/2$. The RoC is marked as follows.



Inter-changing the roles of r_1 and r_2 results in,

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-(n+1)].$$

In the z-transform domain, this takes the form of,

$$X(z) = \underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{|z|<\frac{1}{3}} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}.$$

Since $[0, 1/3) \cap (1/2, \infty) = \emptyset$, the z-transform does not exist.

3) For some linear-time-invariant system, the transfer function is given by,

$$H(z) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}.$$

Suppose $x[n]$ is the input to the system and $y[n]$ is the output.

Derive the difference equation that is satisfied by $x[n]$ and $y[n]$.

The transfer function of a system $H(z)$ is the ratio of the output to the input. Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{2}{z}+\frac{1}{z^2}}{1+\frac{1}{4z}-\frac{3}{8z^2}} \Rightarrow X(z)\left(1+\frac{2}{z}+\frac{1}{z^2}\right) = Y(z)\left(1+\frac{1}{4z}-\frac{3}{8z^2}\right).$$

The above can be directly converted to the impulse-response format,

$$x[n] + 2x[n-1] + x[n-2] = y[n] + \tfrac{1}{4}y[n-1] - \tfrac{3}{8}y[n-2]$$
$$\Updownarrow$$
$$\sum_{k=0}^{2} a_k x[n-k] = \sum_{k=0}^{2} b_k y[n-k]$$

where,

	$k=0$	$k=1$	$k=2$
a_k	1	2	1
b_k	1	$\frac{1}{4}$	$-\frac{3}{8}$

.

4) **Filter Specification via Z-transform**

Suppose that a function is given by,

$$\phi(t) = \begin{cases} \frac{2}{3} - |t|^2 + \frac{|t|^3}{2} & 0 \leqslant |t| < 1 \\ \frac{(2-|t|)^3}{6} & 1 \leqslant |t| < 2 \\ 0 & 2 \leqslant |t| \end{cases}.$$

- Is $\phi(t)$ a symmetric function? Argue by plotting this function.
- Convert $\phi(t)$ into an FIR filter by sampling it at integer points, that is, $\phi(t), t = k$ where $k = 0, \pm 1, \pm 2, \ldots$.

  Let $\Phi(z)$ be the z-transform of this FIR filter sequence. Write the explicit form of $\Phi(z)$.

- Inverse filter design. Suppose that the FIR filter is defined by

$$p\left[k\right] = \phi\left(t\right)\big|_{t=k}, k = \mathbb{Z} \qquad \text{(that is, } k \text{ takes integer values).}$$

  Then, we say that $p_{\mathsf{inv}}[k]$ is an inverse-filter when,

$$p_{\mathsf{inv}}[k] * p[k] = \delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

  Identify the transfer function of $p_{\mathsf{inv}}[k]$ in terms of $\Phi\left(z\right)$.
  Write down the impulse-response of $p_{\mathsf{inv}}[k]$ given the definition of $\phi\left(t\right)$. Is $p_{\mathsf{inv}}[k]$ an FIR or IIR filter?
  Plot the impulse response of $p_{\mathsf{inv}}[k]$.

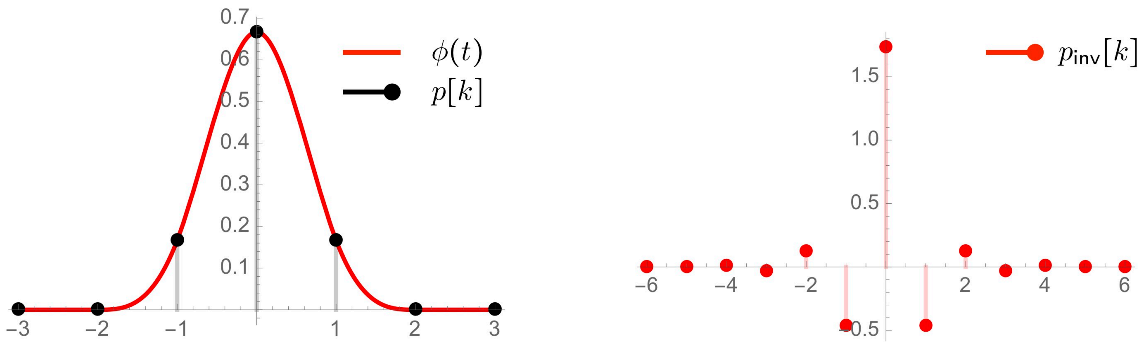


Fig. 1: Filter and Inverse Filter.

- This is a symmetric filter because in the definition of $\phi\left(t\right)$, the intervals are defined in terms of $|t|$ and hence, $\phi\left(t\right) = \phi\left(-t\right)$. The function is plotted in Fig. 1.
- By sampling on integer points, we obtain,

$$\phi\left(k\right)\big|_{k\in\mathbb{Z}} = \frac{1}{6}\left(\delta\left[k+1\right] + 4\delta\left[k\right] + \delta\left[k-1\right]\right) = p\left[k\right].$$

  Accordingly, its z-transform is given by,

$$\Phi\left(z\right) = \frac{1}{6}\left(z^{-1} + 4 + z\right).$$

- By the definition of the inverse filter, we have,

$$p\left[k\right] * p_{\mathsf{inv}}\left[k\right] = \delta\left[k\right] \xrightarrow{\text{z-domain}} \Phi\left(z\right)\Phi_{\mathsf{inv}}\left(z\right) = 1.$$

  And hence, we have,

$$\Phi_{\text{inv}}\left(z\right) = \frac{1}{\Phi\left(z\right)} = \frac{6}{z^{-1} + 4 + z}.$$

  where Φ_{inv} is the z-transform of p_{inv}.
  Let $b_0 = \sqrt{3} - 2$ be the smallest root of $z^{-1} + 4 + z$. Then, we have,

$$\Phi_{\text{inv}}\left(z\right) = \frac{-6b_0}{\left(1 - b_0 z^{-1}\right)\left(1 - b_0 z\right)} = \frac{-6b_0}{1 - b_0^2}\left(\frac{1}{\left(1 - b_0 z^{-1}\right)} + \frac{1}{\left(1 - b_0 z\right)} - 1\right).$$

This is a sum of first-order filters and its impulse response is given by,

$$p_{\text{inv}}[k] = \left(\frac{-6b_0}{1-b_0^2}\right) b_0^{|k|}.$$

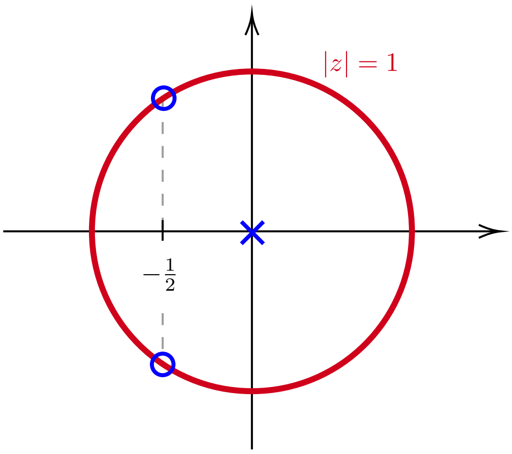
This is an IIR filter and is plotted in Fig. 1.

5) **Filter Identification**

The pole-zero plot of a discrete filter is given below.

When the input $x[n] = 1$ for all n, the output is exactly the same.

What is the impulse response of such a filter?

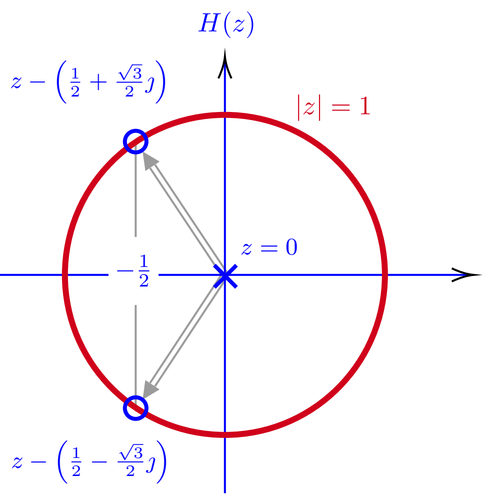


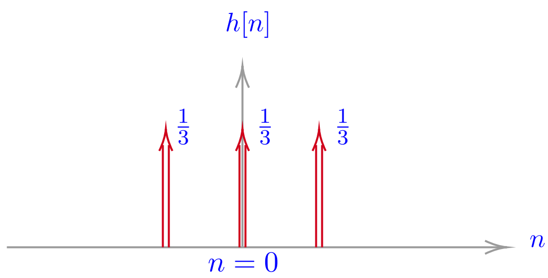
The filter is identified as,

$$h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right).$$

From the provided pole-zero plot, we observes the following poles and zeros,

Zero(s)	Pole(s)
$z_p = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$	$z_0 = 0$





Based on this information, we can construct the z-transform as follows,

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{\left(z - \frac{1}{2} + \jmath\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - \jmath\frac{\sqrt{3}}{2}\right)}{z}.$$

Also we are given that $\forall n, x[n] = 1 \Rightarrow y[n] = 1$.
This implies, $X(z)|_{z=1} = Y(z)|_{z=1} \Rightarrow H(1) = 1$.
Since $H(z) = z^{-1}K\left(z + 1 + z^{-1}\right)$, we obtain, $H(1) = 1 = 3K$ or $K = 1/3$ and,

$$H(z) = \frac{1}{3}\frac{z + 1 + z^{-1}}{z} \longleftrightarrow h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right).$$