Probability and Stochastic Processes

Exam 2019 solutions

B—bookwork, E—new example, T—new theory

1. Although similar (but not the same) questions have appeared before, students need some mathematical maturity to work out Q1.

a)

(i) Note that $0 < z < 1$.

$$F(Z) = P(Z \leq z) = P(XY \leq z)$$

2E

$$F(Z) = P\left(Y \leq \frac{Z}{X}\right) = \int_0^z \int_0^1 dy\, dx + \int_z^1 \int_0^{\frac{z}{x}} dy\, dx$$

2E

$$= z + \int_z^1 \frac{z}{x} dx = z + z\,[\ln(1) - \ln(z)] = z - z\ln(z)$$

$$f(z) = 1 - (\ln(z) + 1) = -\ln(z), \quad 0 < z < 1$$

2E

There is a common approach in 1(a): firstly derive F(z), then take derivative to get f(z)

(ii) $a = 1$ in the following.

$$F_Z(z) = P\left\{\frac{X}{Y} \leq z\right\} = P\{X \leq zY\}$$

$z < 1$

You need to be careful about the two cases; this was a common mistake

$$\begin{aligned} F_Z(z) &= P\{X \leq zY\} \\ &= \int_0^a \int_0^{zy} \frac{1}{a} \cdot \frac{1}{a}\, dx\, dy = \frac{z}{2}, \quad z \leq 1 \end{aligned}$$

2E

$z \geq 1$

$$\begin{aligned} F_Z(z) &= P\{X \leq zY\} \\ &= 1 - \int_0^a \int_0^{x/z} \frac{1}{a} \cdot \frac{1}{a}\, dy\, dx \\ &= 1 - \int_0^1 \frac{x}{z}\, dx = 1 - \frac{1}{2z} \quad z > 1 \end{aligned}$$

2E

$$f_Z(z) = \begin{cases} \frac{1}{2}, & z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$$

2E

(iii)

$$F_Z(z) = P(Z \le z) = P\left\{\frac{Y}{X+Y} \le z\right\}$$ [2E]

$$= P\left\{\frac{X}{Y} \ge \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \le \frac{1-z}{z}\right)$$

$$= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \le 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases}$$ [2E]

$$f_Z(z) = \begin{cases} \frac{1}{2(1-z)^2}, & 0 < z \le 1/2 \\ \frac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$ [2E]

b)

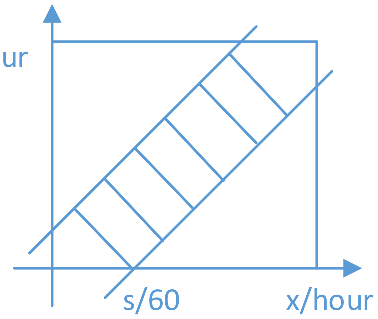
$$E[X] = \int_0^\infty x f(x) dx = \int_0^\infty \int_0^x 1 \cdot dy\, f(x) dx$$ [2T]

$$= \int_0^\infty \int_0^x f(x) dy\, dx = \int_0^\infty \int_y^\infty f(x) dx\, dy$$ [3T]

$$= \int_0^\infty P(X \ge y) dy = \int_0^\infty P(X \ge x) dx$$ [2T]

Question (b) is similar to a problem tested before (for a discrete random variable).

2.

a) 

[2E]

See the unit square in the figure, the origin is 5:00pm, x axis (unit: hour) represents the time Alice arrive and y axis represents the time Bob arrives. [3E]

So, “each is prepared to wait s/60 hours before leaving and they meet each other” can be mathematically described as

This is the key step

$\left\{|x-y| \le \frac{s}{60}\right\}$ (see the shadow area in the figure)

[3E]

$$P\left(|x-y| \le \frac{s}{60}\right) = \iint_{|x-y|\le\frac{s}{60}} f(x,y)\mathrm{d}x\mathrm{d}y = 1 - 4\times\left(\frac{1}{2}-\frac{s}{60}\right)^2$$

where (x,y) is a two-dimensional variable stands for the time Alice and Bob arrive, and its [2E]

density function is $f(x,y) = \begin{cases} 4, & 0 \le x \le 0.5, 0 \le y \le 0.5 \\ 0, & \text{otherwise} \end{cases}$

Finally, $1 - 4\times\left(\frac{1}{2}-\frac{s}{60}\right)^2 \ge \frac{3}{4} \Rightarrow s \ge 15$ min [3E]

b) The joint distribution of n observations (where $\bar{x} = 6$ is the empirical mean)

[3E]

$$f(X,\beta) = \frac{\beta^{n\alpha}(x_1 \dots x_n)^{(\alpha-1)}e^{-n\beta\bar{x}}}{\Gamma^{n}(\alpha)}, \quad x > 0, \alpha = 3, \beta > 0$$

Take partial derivative: [3E]

$$\frac{\partial f(X,\beta)}{\partial\beta} = \frac{n\alpha\beta^{n\alpha-1}(x_1 \dots x_n)^{(\alpha-1)}e^{-n\beta\bar{x}}}{\Gamma^{n}(\alpha)} - n\bar{x}\frac{\beta^{n\alpha}(x_1 \dots x_n)^{(\alpha-1)}e^{-n\beta\bar{x}}}{\Gamma^{n}(\alpha)}$$

[3E]

$$= \left(\frac{n\alpha}{\beta} - n\bar{x}\right)f(X,\beta)$$

[3E]

Thus, letting it be 0, we have $\beta = \frac{\alpha}{\bar{x}} = \frac{4}{6} = \frac{2}{3}$

Although it may look complicated, the final answer is really simple.

3. a) (i) $E[X(t)] = t + 0.5 \sin(\pi t)$ [4B]

(ii) $E[X(t_1)X(t_2)] = 2t_1 t_2 + 0.5 \sin \pi t_1 \sin \pi t_2$ [4B]

(iii) Not stationary [2B]

There is some ambiguity in Q(a). Here, the interpretation is that the coin is flipped once and for all, i.e., only flipped once in the beginning and doesn't change later. Some students interpreted it as flipping a coin at each time t, leading to different answers to (ii) and (iii). Both were accepted in marking.

b) Firstly note that the arrival rate $\lambda = 0.2$ student/minute. [1E]

i) The probability that k students are being serviced at a given (random) time, is exactly the probability that k students arrived within the last 10 minutes, but this probability is Poisson distributed, hence [2E]

students did well in (i). Note that there is no queue.

$$P(X = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \frac{2^k}{k!} e^{-2}, \quad k = 0,1,2,\ldots$$ [2E]

ii) The second case can be thought of random selection of a Poisson process, which gives rise to two Poisson processes, each with arrival rate $\lambda = 0.1$. This yields two queues with difference service times. The number of customers served is therefore the sum of the two queues, i.e., $k = k_1 + k_2$, where k_1 is Poisson distributed with parameter 1 and k_2 is Poisson distributed with parameter 2. this is the key step [3E] [2E]

We know that the sum of two Poisson distributions is again Poisson with parameter equal to the sum of the individual distributions' parameters, i.e., the number of students is Poisson distributed (with parameter 3) as [2E]

$$P(X = k) = \frac{3^k}{k!} e^{-3}, \quad k = 0,1,2,\ldots$$ [3E]

Part (ii) is a bit tricky.

4. routine question but requires some calculation

a) *The transition probability matrix is given by*

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$ [3E]

The limiting probabilities are obtained by solving the equations

$$\begin{aligned} \pi_0 &= \pi_1 + \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_1 &= \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_2 &= \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_3 &= \frac{1}{4}\pi_4 \\ \pi_4 &= \pi_0 \end{aligned}$$ [3E]

together with the normalization equation

[3E]

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

We obtain

$$\begin{aligned} \pi_4 &= \pi_0 \\ \pi_3 &= \frac{1}{4}\pi_4 \Rightarrow \pi_3 = \frac{1}{4}\pi_0 \\ \pi_2 &= \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \Rightarrow \pi_2 = \frac{1}{3}\left(\frac{1}{4}\pi_0\right) + \frac{1}{4}\pi_0 = \frac{1}{3}\pi_0 \\ \pi_1 &= \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \Rightarrow \pi_1 = \frac{1}{2}\left(\frac{1}{3}\pi_0\right) + \frac{1}{3}\left(\frac{1}{4}\pi_0\right) + \frac{1}{4}\pi_0 = \frac{1}{2}\pi_0 \end{aligned}$$ [3E]

therefore

$$\pi_0\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + 1\right) = 1 \Longrightarrow \pi_0 = \frac{12}{37}$$

and we get

$$\begin{aligned} \pi_0 &= \frac{12}{37} = 0.324 \\ \pi_1 &= \frac{6}{37} = 0.162 \\ \pi_2 &= \frac{4}{37} = 0.108 \\ \pi_3 &= \frac{3}{37} = 0.081 \\ \pi_4 &= \frac{12}{37} = 0.324. \end{aligned}$$ [3E]

Part (b) is similar to questions tested before (for 2 and 3 dimensions), but is easier.
In here, it's a one-dimensional random walk.

b)
i) Of the $2n$ steps, the chain must goes left for n steps, and right for n.
Therefore,

$P\{X_{2n} = 0\} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}$ [4E]

ii)

$$\sum_{n} P\{X_{2n} = 0\} = \sum_{n\geq 1} \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} = \sum_{n\geq 1} \left(\frac{1}{2}\right)^{2n} \frac{(2n)!}{n!\,n!}$$ [3E]

$$= \sum_{n\geq 1} \left(\frac{1}{2}\right)^{2n} 2^{2n} \frac{1}{\sqrt{\pi n}} = \sum_{n\geq 1} \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{\pi}} \zeta\left(\frac{1}{2}\right) = \infty$$

because the sum diverges. We therefore conclude that the origin is a recurrent state. [3T]

Here $\zeta(s)$ is the Riemann zeta function, which is well-known to converge for $s > 1$ but diverge otherwise.

PS. The famous Riemann hypothesis, considered one of the greatest unsolved problems in mathematics, asserts that any non-trivial zero s of $\zeta(s)$ on the complex plane has Re(s) = 1/2.