IMPERIAL COLLEGE LONDON

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EE9-FPN2-03
EE9-SC1

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2020

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Monday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions.

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Bhandari
Second Marker(s) : P.T. Stathaki

Question 1 [**40 Marks**]

a) Which of the following statements is true and why?

(Multiple choices may be correct)

i) All discrete-time signals are digital signals.
ii) All digital signals are discrete-time signals.
iii) Some discrete-time signals are digital signals.
iv) Some digital signals are discrete-time signals. [**2 Marks**]

b) What is the value of $f(t)$ below,

$$f(t) = \sum_{n=-\infty}^{n=+\infty} \mathrm{sinc}(t-n)$$

where t denotes continuous-time. (Hint: Use Nyquist–Shannon Sampling Theorem). [**2 Marks**]

c) It is typically assumed that the linear convolution operation is *associative* or,

$$f * (g * h) = f * g * h = (f * g) * h$$

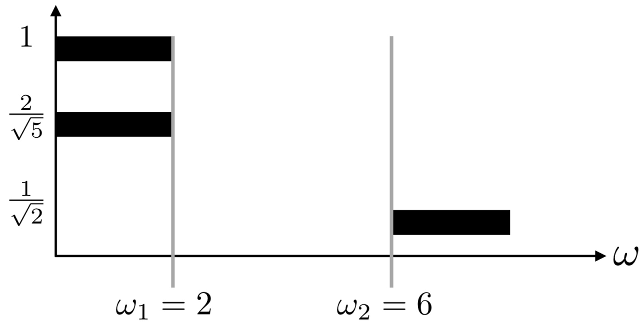
but it was shown in the class that this is only when the sums related to the convolution operations converge. Provide a counter-example when the convolution is <u>not</u> associative. (Hint: you can use one of the sequences as constant and another one as unit-step.) [**2 Marks**]

d) Suppose we are given two sequences of lengths L_1 and L_2. Under what condition are the linear and circular convolutions equivalent? (Just write the condition) [**2 Marks**]

e) Let $h[n] = \alpha^n u[n]$ be a given filter. What is the z–transform of M-times downsampled version of the same filter? [**2 Marks**]

f) Let $h(t) = \kappa e^{-\lambda t} u(t)$ be a continuous-time impulse response of some filter where κ is the gain, λ is the filter parameter, and $u(t)$ is the usual unit-step function. Find a range of values for κ and λ so that, the frequency-domain gain at zero frequency is unity and the following filter specifications are met.

[**4 Marks**]



g) Let us define a sequence by,

$$x[n] = (r_1)^n u[n] - (r_2)^n u[-(n+1)] \tag{1}$$

where $r_1 = -1/3$ and $r_2 = 1/2$. On the z-plane, plot the poles and zeros together with the region-of-convergence.

[**2 Marks**]

In the above sequence in (1), what happens if we exchange r_1 and r_2? [**2 Marks**]

h) As we have seen in the course, many linear-time-invariant systems are described by difference or differential equations. For instance, consider the filter described by differential equation,

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y_c(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x_c(t) \tag{2}$$

where $x_c(t)$ is the continuous-time input and $y_c(t)$ is the corresponding output. In order to work with discrete-time systems, a standard approach is to replace the derivative operator by finite differences. To this end, let us denote the central difference by, $\Delta^n x[k]$ where n is the order of difference and the difference operation is defined by the recursive operations,

$$\begin{cases} \Delta^0 x[k] = x[k] \\ \Delta^1 x[k] = \frac{1}{2}\left(x[k+1] - x[k-1]\right) \\ \Delta^n x[k] = \Delta^1\left(\Delta^{(n-1)} x[k]\right) \end{cases}.$$

Hence the continuous-time filter can be written as a discrete-time filter,

$$\sum_{n=0}^{N} a_n \left(\Delta^n y[k]\right) = \sum_{m=0}^{M} b_m \left(\Delta^n x[k]\right). \tag{3}$$

i) Let H_c and H_d be the transfer-functions of the continuous and discrete-time filters, respectively. How are the two transfer functions related? [**4 Marks**]

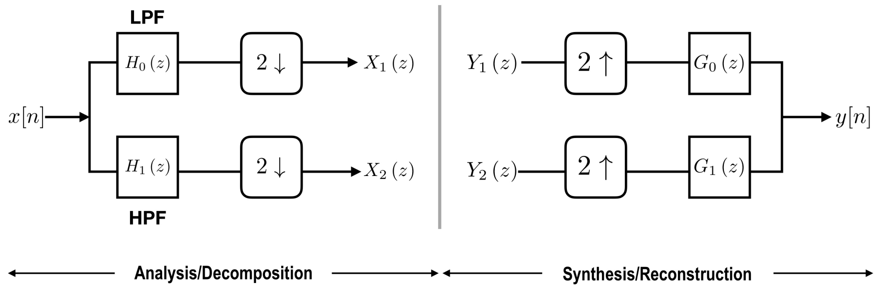
ii) What is the relation between continuous-filter frequency and digital-filter frequency? [**2 Marks**]

i) For some linear-time-invariant system, the transfer function is given by,

$$H(z) = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{3}{4}z^{-1}\right)}.$$

Suppose $x[n]$ is the input to the system and $y[n]$ is the output. Derive the difference equation that is satisfied by $x[n]$ and $y[n]$. **[4 Marks]**

j) Depicted below is a system that is describing what is known as the two-channel quadrature-mirror filterbank (QMF).



The input signal $x[n]$ undergoes filtering and decimation or downsampling ($\mathbf{2}\downarrow$). This is the signal analysis part. There on, given sequences or transfer functions, the synthesis part reconstructs the original signal by upsampling ($\mathbf{2}\uparrow$) followed by filtering. In the above H_0 and H_1 are low-pass and high-pass filters, respectively.

i) Identify outputs $X_1(z)$ and $X_2(z)$ given $x[n]$, $H_0(z)$ and $H_1(z)$. **[2 Marks]**

ii) How is $y[n]$ related to inputs transfer functions $Y_1(z)$ and $Y_2(z)$? **[2 Marks]**

iii) Suppose that one system's output becomes another system's input, that is,

$$Y_1(z) = X_1(z) \quad \text{and} \quad Y_2(z) = X_2(z).$$

What is the frequency response of $y[n]$? **[2 Marks]**

Under what conditions (in frequency domain), $y[n] = x[n]$? **[2 Marks]**

Let $h[n]$ be the impulse response of some filter $H(z)$. Furthermore, let $H_0(\omega) = H(\omega)$ and $H_1(\omega) = H(\omega - \pi)$. Identify the impulse response of $h_1[n]$, $g_0[n]$ and $g_1[n]$ in terms of $h[n]$. **[2 Marks]**

k) Identify parameters p and q in the filter, $h[n] = pq^n u[n]$ given that $h[1] = 1$ and the group delay at $\omega = \pi$ is $-1/3$. **[2 Marks]**

Question 2 [30 Marks]

a) **Properties of Filter**

Let $x[n], n \in \mathbb{Z}$ be a discrete-time sequence. The filter output is given by,

$$y[n] = \sum_{l=-K\left(\frac{M-1}{2}\right)}^{l=+K\left(\frac{M-1}{2}\right)} c_{M,K}[l]\, x[n+l] \tag{4}$$

where M (odd integer) and K are filter parameters and the filter coefficient $c_{M,K}[l]$ is given by,

$$c_{M,K}[l] = M^{-K} p_{M,K}[l] \tag{5}$$

where $p_{M,K}[l]$ is the solution to the equation,

$$\sum_{k=0}^{K(M-1)} z^k p_{M,K}\left[k - K\frac{M-1}{2}\right] = \left(\sum_{m=0}^{M-1} z^m\right)^K. \tag{6}$$

i) What is the time-domain impulse response of the filter? **[4 Marks]**

ii) What is $c_{M,1}[l]$? Plot the filter corresponding to $c_{M,1}[l]$. **[2 Marks]**

iii) When $K = 1$, what is the order of the filter? **[4 Marks]**

iv) What is the magnitude frequency response of the filter?
What is the effect of increasing K on the output? **[9+1 Marks]**

b) **Optimal FIR Filter Design**

Let f be a function with maximum frequency Ω_0 with derivative,

$$g(t) = \frac{d}{dt} f(t).$$

Suppose that we want to approximate g with linear combination of f, that is,

$$g(t) \approx \alpha_1 f(t) + \alpha_2 f(t - T)$$

where T is the sampling rate and $\{\alpha_1, \alpha_2\}$ are the coefficients of a 2-tap filter. As we have seen in one of the earlier questions, continuous derivative operators (e.g. (2)) are often replaced by finite differences (e.g. (3)) when working with FIR filters. In that case, $\{\alpha_1, \alpha_2\}$ are fixed and do not depend on signal's bandwidth. For a given bandwidth Ω_0, $\{\alpha_1, \alpha_2\}$ can be obtained by minimizing the least-squares error between the derivative operator and the 2-tap FIR-filter. Since we are working with least-squares error, the same operations can also be performed in the Fourier domain. Show that if the target filter is $\jmath\omega$ (in the Fourier domain), then, the optimal values of the filter taps are,

$$\alpha_2 = -\Omega_0 \frac{\sin\theta - \theta\cos\theta}{\theta^2 - \sin^2\theta} \quad \text{and} \quad \alpha_1 = -\alpha_2 \frac{\sin\theta}{\theta} \quad \text{where} \quad \theta = \Omega_0 T.$$

[10 Marks]

To ease the computations, we provide the following integrals that will be useful. In each case, $\theta = \Omega_0 T$.

$$\int_{-\Omega_0}^{\Omega_0} \cos(\omega T)\, d\omega = 2\tfrac{\sin\theta}{T} \qquad \int_{-\Omega_0}^{\Omega_0} \cos^2(\omega T)\, d\omega = +\tfrac{\sin(2\theta)}{2T} + \Omega_0$$

$$\int_{-\Omega_0}^{\Omega_0} \omega\sin(\omega T)\, d\omega = 2\tfrac{\sin\theta - \theta\cos\theta}{T^2} \qquad \int_{-\Omega_0}^{\Omega_0} \sin^2(\omega T)\, d\omega = -\tfrac{\sin(2\theta)}{2T} + \Omega_0$$

Question 3 [30 Marks]

Windowing, Low-Pass Filtering and Digital-to-Analog Reconstruction

a) Windowing is an important aspect of filter design. In this context, one typically multiplies a function or a sequence by a window to obtain a truncated function or sequence, respectively. Examples of window functions include, rectangular window, Hanning, Hamming, Blackman–Harris and Kaiser windows.

As we have seen, most of the windows have two common properties, (a) they are positive and (b) there are non-zero on a finite interval. Suppose $w(t) \geqslant 0$ is a window function that is non-zero only on the interval $[-L, L]$. Then prove that its Fourier Transform,

$$\widehat{w}(\omega) = \int w(t) e^{-\jmath\omega t} dt$$

is always maximum at the zero-frequency or mathematically, $|\widehat{w}(\omega)| \leqslant \widehat{w}(0)$. [4 **Marks**]

b) Suppose $f(t)$ is a function with maximum frequency Ω_0. Shannon's sampling formula allows us to write,

$$f(t) = \sum_{n=-\infty}^{n=\infty} f(nT) \operatorname{sinc}\left(\frac{t}{T} - n\right), \qquad T = \frac{\pi}{\Omega_0} \tag{7}$$

where T is the sampling rate. Suppose that the same function is sampled with sampling rate $T_0 \leqslant T$. We can interpret samples $f(nT_0)$ as Fourier Series coefficients, $c_n = f(nT_0)$ of some periodic function. Indeed for all n, c_n represent the coefficients of the periodic function,

$$\widehat{C}(\omega) = \frac{1}{T_0} \sum_{n=-\infty}^{n=\infty} c_n e^{-\jmath\omega n T_0}$$

and "windowing" this with a rectangular window leads to the digital-to-analog reconstruction in (7).

- Show that $\widehat{C}(\omega)$ is a periodic function. Identify its time-period. [2 **Marks**]
- Show the link between $\widehat{C}(\omega)$ and the Fourier transform of $f(t)$. [6 **Marks**]
- Show that "windowing" $\widehat{C}(\omega)$ results in (7) in the time domain. [6 **Marks**]
- Explain how the choice of rectangular window, in the above, leads to practical problems with implementation of (7), digital-to-analog reconstruction.

  [2 **Marks**]
- To be able to use more general window functions than the rectangular window, what conditions should be imposed on T_0? [4 **Marks**]
- Sketch an example of a windowing function that is suitable for reconstruction of $f(t)$ from its samples. Explain the reasons behind your choice. [4 **Marks**]
- What modifications should be made to formula (7) so that the new window function you have designed above can recover $f(t)$ from samples $f(nT_0)$? [2 **Marks**]

Question 4 [**30 Marks**]

Engineering and Implementation Aspects

a) **Linear Phase Filters**

Let $M = 8$ be the filter order. In the context of linear phase filter design,

- Write the equation for transfer function, $H(z)$ corresponding to a linear phase filter. [**2 Marks**]
- Show the diagram for **Direct Form** implementation. [**2 Marks**]
- Show the diagram for **Transpose Form** implementation. [**2 Marks**]
- The maximum sequential delay (MSD) is attributed to the number of additions and multiplications involved in the implementation of the filter. Compare the MSD for **Direct Form** and **Transpose Form** implementations. Argue which implementation is better. [**2 Marks**]

b) **Inverse Filter Design**

Suppose that a function is given by,

$$\phi(t) = \begin{cases} \frac{2}{3} - |t|^2 + \frac{|t|^3}{2} & 0 \leqslant |t| < 1 \\ \frac{(2-|t|)^3}{6} & 1 \leqslant |t| < 2 \\ 0 & 2 \leqslant |t| \end{cases} \cdot$$

- Is $\phi(t)$ a symmetric function? Argue by plotting this function. [**2 Marks**]
- Convert $\phi(t)$ into an FIR filter by sampling it at integer points.
  Let $\Phi(z)$ be the z-transform of this FIR filter. Write the explicit form of $\Phi(z)$. [**2 Marks**]
- Inverse filter design. Suppose that the FIR filter is defined by

$$p[k] = \phi(t)|_{t=k}, k = \mathbb{Z} \qquad \text{(that is, } k \text{ takes integer values).}$$

  Then, we say that $p_{\mathsf{inv}}[k]$ is an inverse-filter when,

$$p_{\mathsf{inv}}[k] * p[k] = \delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

  Identify the transfer function of $p_{\mathsf{inv}}[k]$ in terms of $\Phi(z)$. [**2 Marks**]
  Write down the impulse-response of $p_{\mathsf{inv}}[k]$ given the definition of $\phi(t)$. [**4 Marks**]
  Is $p_{\mathsf{inv}}[k]$ an FIR or IIR filter? [**1 Marks**]
  Plot the impulse response of $p_{\mathsf{inv}}[k]$. [**1 Marks**]

c) **Multirate System**

Consider the following system with input $x[n]$ and output $y[n]$.

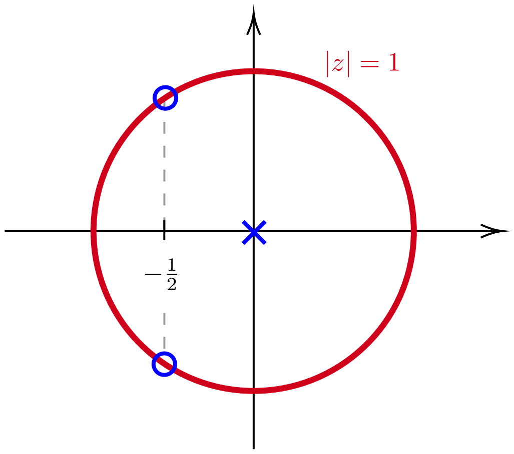
$$x[n] \rightarrow \boxed{\boxed{1:3}} \rightarrow \boxed{\boxed{2:1}} \rightarrow \boxed{\frac{z^{-6}}{\alpha - z^{-6} + \beta z^{-12}}} \rightarrow \boxed{\boxed{2:1}} \rightarrow \boxed{\boxed{1:3}} \rightarrow y[n]$$

Suppose that an equivalent representation of the above system is given by,

$$x[n] \rightarrow \boxed{\boxed{4:1}} \rightarrow \boxed{\frac{B(z)}{A(z)}} \rightarrow \boxed{\boxed{1:9}} \rightarrow y[n].$$

Identify α and β if $A(z)$ has roots at $1/4$ and $1/3$. [**5 Marks**] *d*) **Filter Identification**

The pole-zero plot of a discrete filter is given below. When the input $x[n] = 1$ for all n, the output is exactly the same. What is the impulse response of such a filter?



[5 Marks]