Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 9

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The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

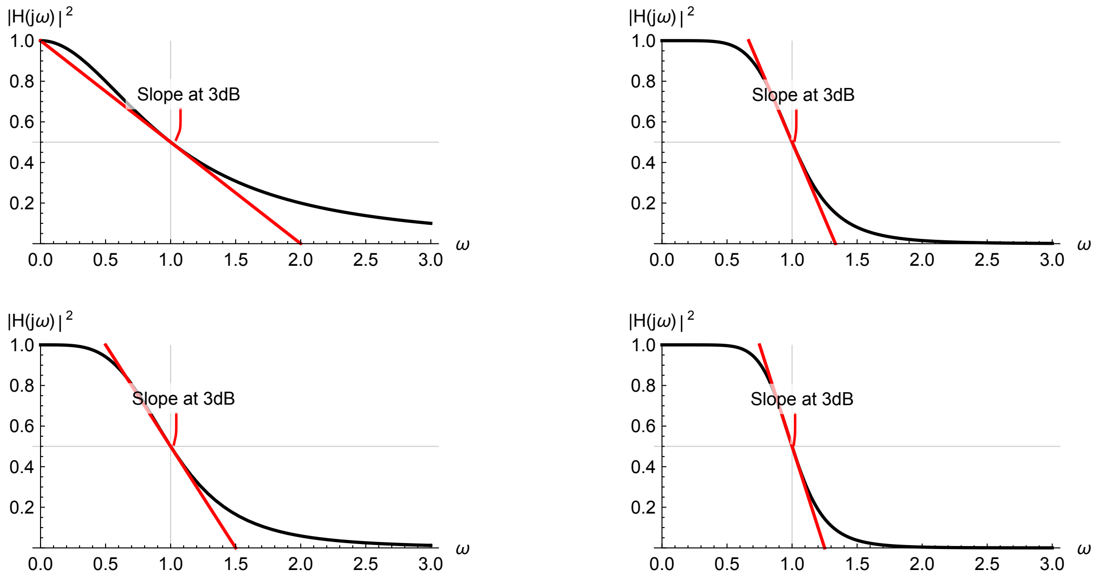

Fig. 1: Slope of Butterworth filter at the 3dB point. (Increasing filter orders, $N = 1, 2, 3, 4$).

1) Fig. 1 shows that as the order of an analog Butterworth filter is increased, the slope of $|H_a(j\omega)|^2$ at the 3 dB cutoff frequency ω_c also increases. Derive an expression for the slope of $|H_a(j\omega)|^2$ at ω_c as a function of the filter order N.

   The magnitude of the Butterworth filter's frequency response is

$$|H_a(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}. \tag{1}$$

   To evaluate the slope of $|H_a(j\omega)|^2$ at $\omega = \omega_c$, we may set $\omega_c = 1$ and evaluate the derivative at $\omega = 1$. Therefore, with

$$|H_a(j\omega)|^2 = \frac{1}{1 + \omega^{2N}} \tag{2}$$

   we have

$$\frac{d}{d\omega}|H_a(j\omega)|^2 = \frac{-2N\omega^{2N-1}}{(1 + \omega^{2N})^2}. \tag{3}$$

   Evaluating this at $\omega = 1$, we have

$$\left.\frac{d}{d\omega}|H_a(j\omega)|^2\right|_{\omega=1} = -\frac{N}{2}. \tag{4}$$

   Alternatively,

$$G(\omega) = |H_a(j\omega)|^2 = \frac{1}{(1 + \omega^{2N})}$$

   and hence,

$$G^{(1)}(\omega)\left(1 + \omega^{2N}\right) + \left(2N\omega^{2N-1}\right)G(\omega) = 0.$$

   Solving for the derivative,

$$G^{(1)}(\omega) = -\frac{\left(2N\omega^{2N-1}\right)G(\omega)}{(1 + \omega^{2N})} = -\frac{\left(2N\omega^{2N-1}\right)}{(1 + \omega^{2N})^2}.$$

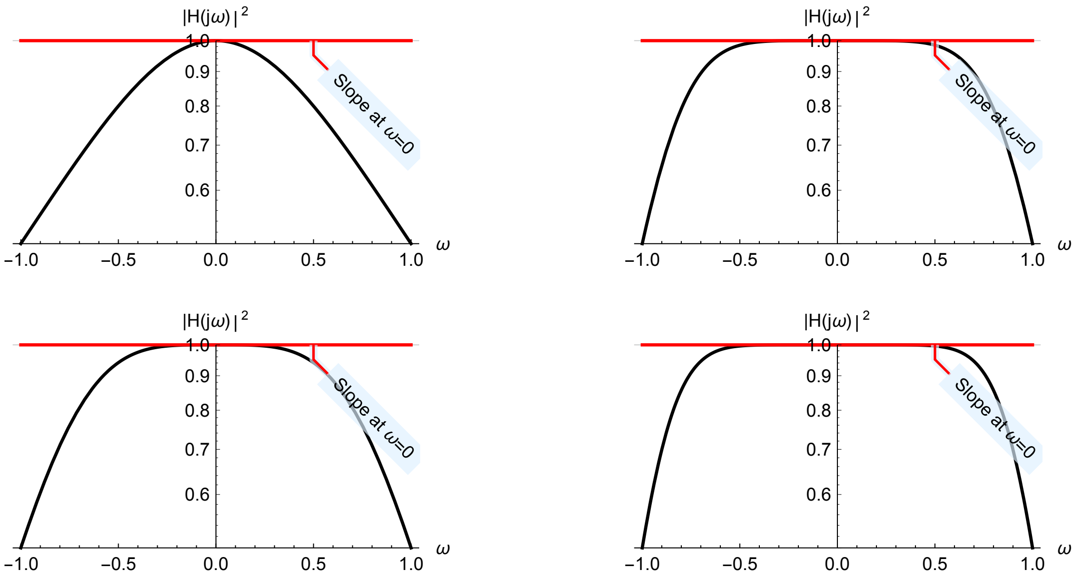

Fig. 2: Slope of Butterworth filter at $\omega = 0$. (Increasing filter orders, $N = 1, 2, 3, 4$).

2) Fig. 2 shows the frequency response of an N-th order low-pass Butterworth filter at $\omega = 0$. As we can see, the slope is zero at $\omega = 0$. Furthermore, the figure suggests that as the filter order increases, the higher order derivatives are also zero. This is known as the *maximally flat* property.

Mathematically, show that the Buttherworth filter is indeed *maximally flat* at $\omega = 0$, that is, the first $2N - 1$ derivatives of $|H_a(j\omega)|^2$ are equal to zero at $\omega = 0$.

A N-th order Butterworth filter has a magnitude squared frequency response fiven by

$$|H_a(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}. \tag{5}$$

Without any loss of generality, we may assume that $\omega_c = 1$ and evaluate the derivative of the function

$$G(\omega) = \frac{1}{1 + \omega^{2N}} \tag{6}$$

at $\omega = 0$. Multiplying both sides of this equation by $\left(1 + \omega^{2N}\right)$, we have

$$G(\omega) \cdot \left[1 + \omega^{2N}\right] = 1. \tag{7}$$

Setting $\omega = 0$, we have

$$G'(\omega)\Big|_{\omega=0} = 0. \tag{8}$$

Differentiating a second time gives

$$G''(\omega)\left[1 + \omega^{2N}\right] + G'(\omega)\left[4N\omega^{2N-1}\right] + G(\omega)\left[2N(2N-1)\omega^{2N-2}\right] = 0. \tag{9}$$

Again setting $\omega = 0$ we have

$$G''(\omega)\Big|_{\omega=0} = 0. \tag{10}$$

If we continue to differentiate k times, where $k \leq 2N - 1$, we have an equation of the form

$$G^{(k)}(\omega)\left[1 + \omega^{2N}\right] + \sum_{i=1}^{k-1} G^{(i)}(\omega) F_i(\omega) + G(\omega)\left[2N(2N-1)\ldots(2N-k+1)\omega^{2N-k}\right] = 0. \tag{11}$$

where $G^{(i)}(\omega)$ is the ith derivative of $G(\omega)$ and $F(\omega)$ is a polynomial in ω. Given that $G^{(i)}(\omega)$ is equal to zero at $\omega = 0$ for $i = 1, \ldots, k-1$, it follows that

$$G^{(k)}(\omega)\Big|_{\omega=0} = 0. \tag{12}$$

Differentiating $2N$ times, however, we have

$$G^{(2N)}(\omega)\left[1+\omega^{2N}\right] + \sum_{i=1}^{2N-1} G^{(i)}(\omega) F_i(\omega) + G(\omega)\cdot(2N)! = 0. \tag{13}$$

Therefore,

$$G^{(2N)}(\omega)\Big|_{\omega=0} = -G(\omega)\Big|_{\omega=0}\cdot(2N)! = -(2N)!, \tag{14}$$

which is non-zero, and the maximally flat property is established.

3) Let $H_c(s)$ be a continuous system defined by

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2}. \tag{15}$$

(a) Using impulse invariance design a discrete filter from the continuous-time second-order filter

Expanding $H_c(s)$ in a partial-fraction expansion, we produce

$$H_c(s) = \frac{0.5}{s+a+jb} + \frac{0.5}{s+a-jb}. \tag{16}$$

Therefore, the impulse-invariant transformation yields a discrete-time design with the system function

$$H(z) = \frac{0.5}{1-e^{(-a+jb)T}z^{-1}} + \frac{0.5}{1-e^{(-a-jb)T}z^{-1}} \tag{17}$$

$$= \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}. \tag{18}$$

Note that although we mapped the poles via $p_k = e^{s_k T}$, the zero at $s=-a$ was transformed into a zero at $z = e^{-aT}(\cos bT)$. Note also that the radius r of the z-plane poles is determined solely by the real part of s_k as $r = e^{-aT}$, while the angle θ of the z-plane poles is determined solely by the imaginary part of s_k as $\theta = bT$.

(b) For $a = 0.1$ and $b = 4$, convert $H_c(s)$ into a digital IIR filter by means of the bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \frac{\pi}{2}$.

First, we note that the analog filter has a resonant frequency $\Omega_r = 4$.

This frequency is to be mapped into $\omega_r = \frac{\pi}{2}$ by selecting the value of the parameter T. Given that

$$\Omega_r = \frac{2}{T}\tan\frac{\omega_r}{2}, \tag{19}$$

we must select $T = \frac{1}{2}$ to have $\omega_r = \frac{\pi}{2}$. Thus the mapping is

$$s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right). \tag{20}$$

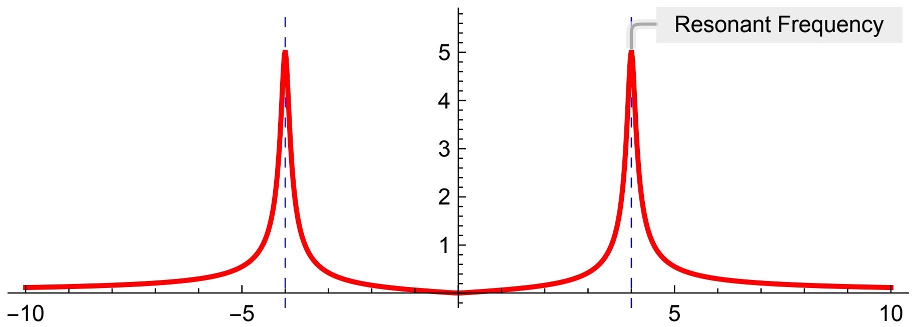

Fig. 3: Resonant frequency.

The resulting digital filter has the system function

$$H\left(z\right)=\frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}}. \tag{21}$$

We note that the coefficient of the z^{-1} term in the denominator of $H\left(z\right)$ is extremely small and can be approximated by zero. Thus we have the system function

$$H\left(z\right)=\frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}}. \tag{22}$$

This filter has poles at

$$p_{1,2}=0.987e^{\pm j\pi/2}, \tag{23}$$

and zeros at

$$z_{1,2}=-1,0.95. \tag{24}$$

Therefore, e have succeeded in designing a two-pole filter that resonates near $\omega=\frac{\pi}{2}$.