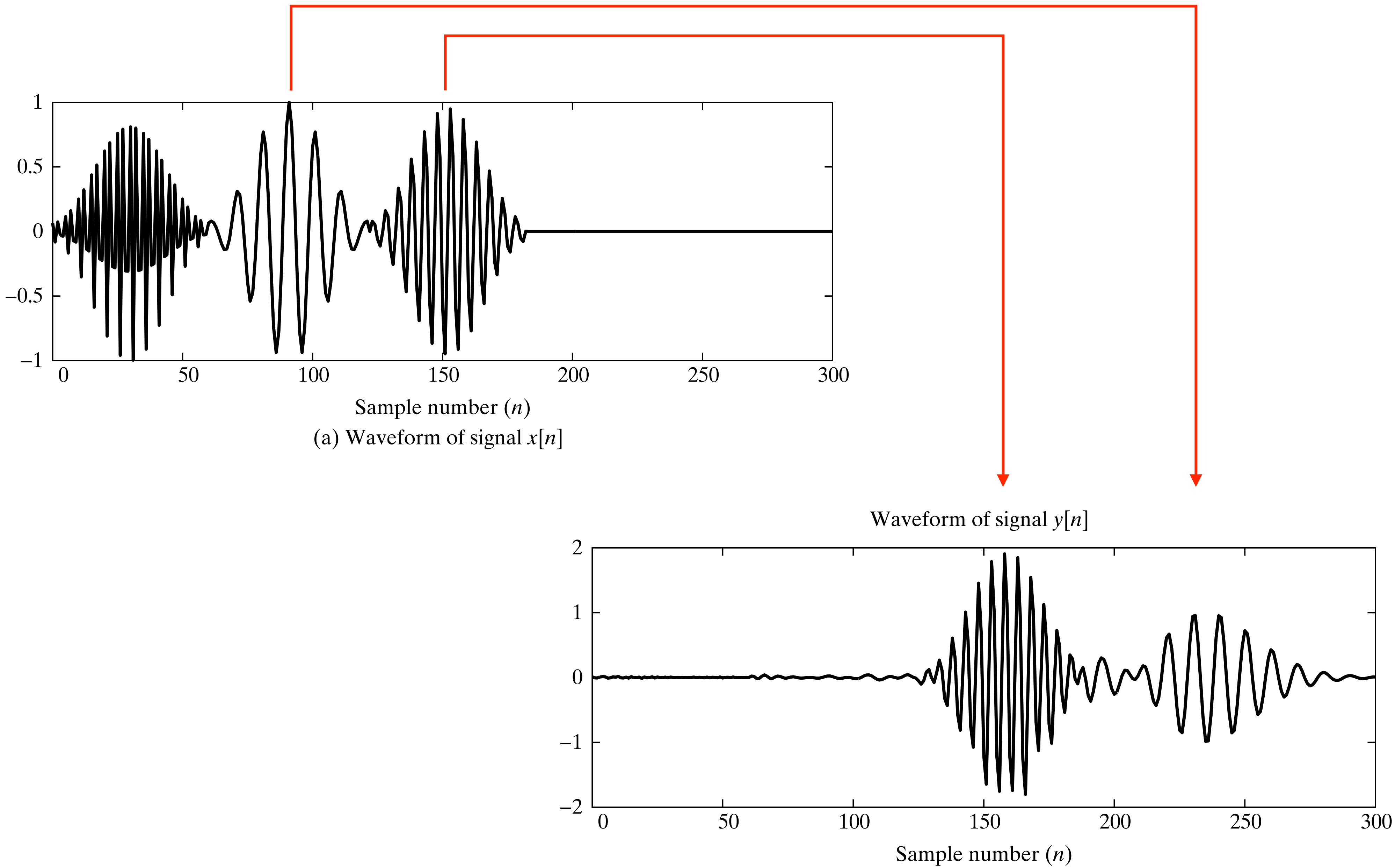
5: Filters



(a) Waveform of signal $x[n]$

Difference Equations

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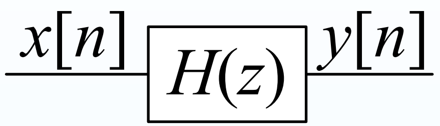

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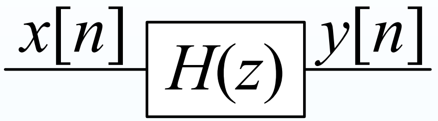

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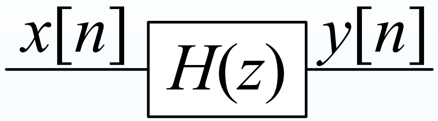

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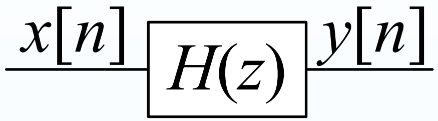

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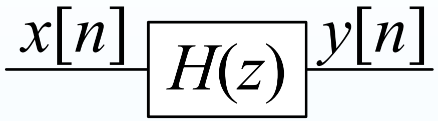

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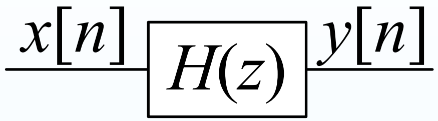

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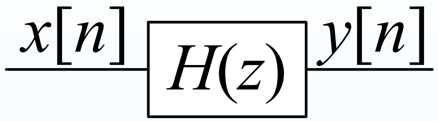

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Note negative sign in first equation.
Authors in some SP fields reverse the sign of the $a[n]$: BAD IDEA.

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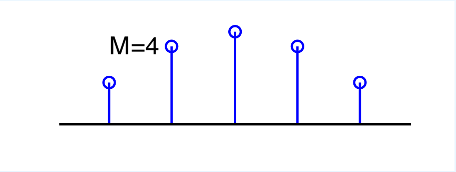

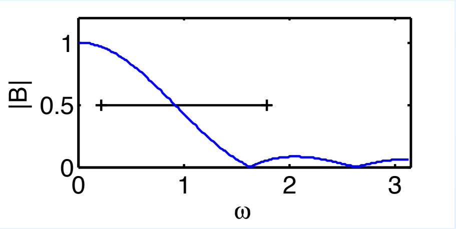

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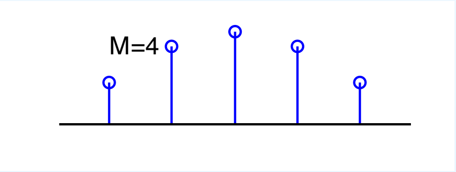

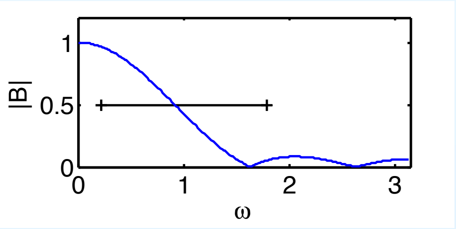

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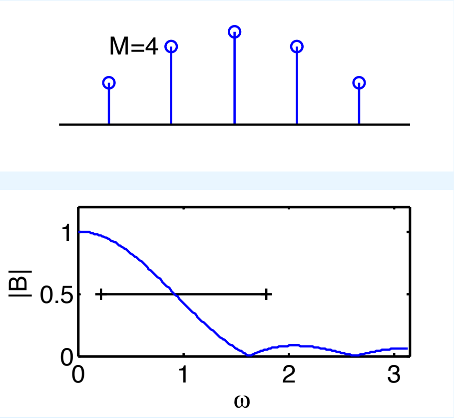

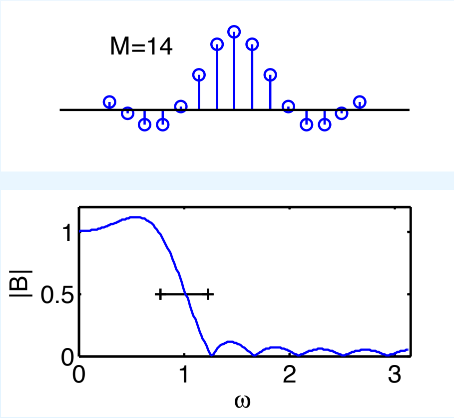

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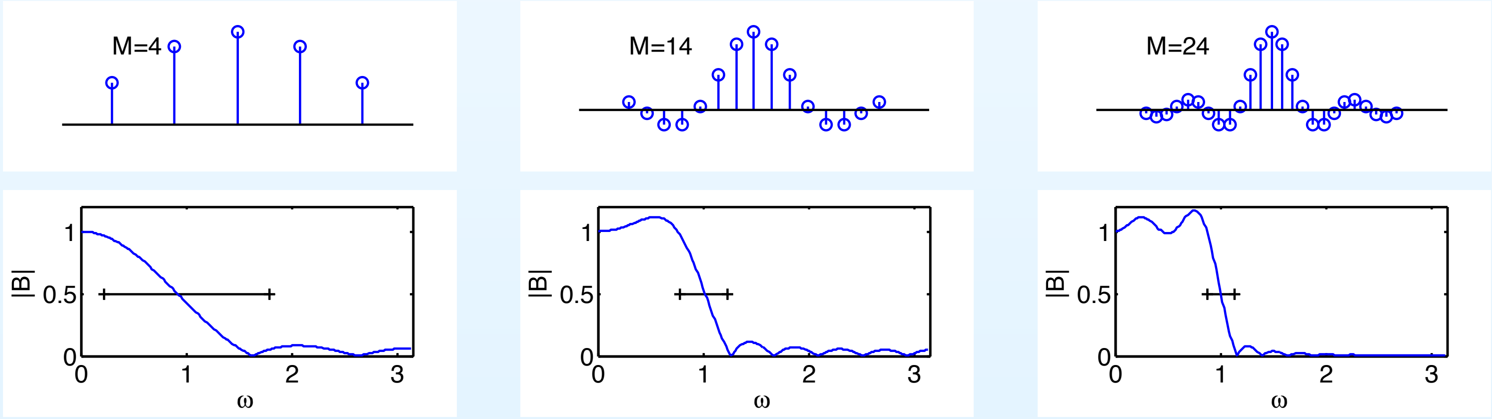

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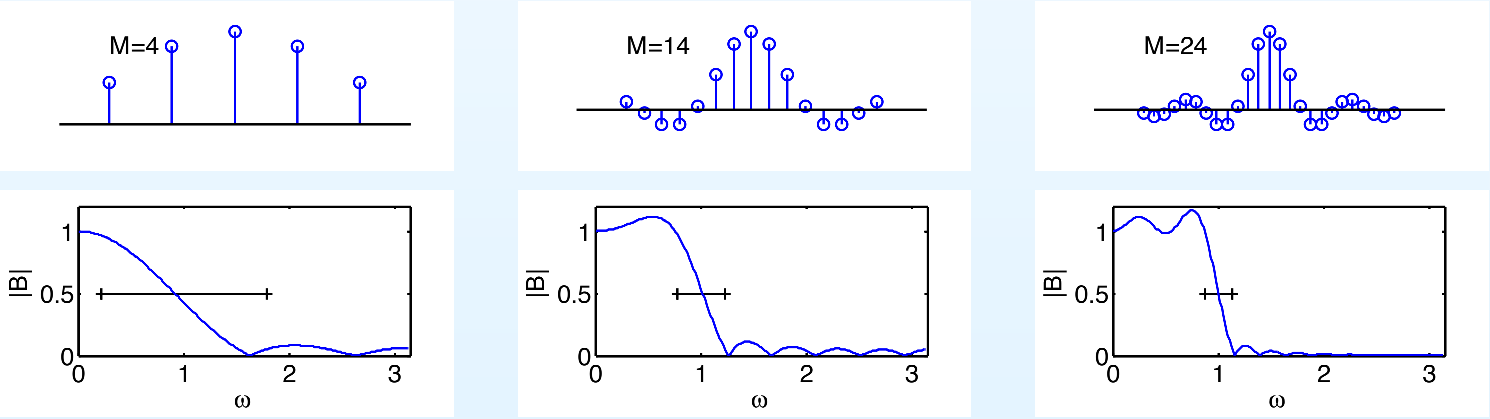

Rule of thumb: Fastest possible transition $\Delta\omega \geq \frac{2\pi}{M}$ (marked line)

FIR Symmetries

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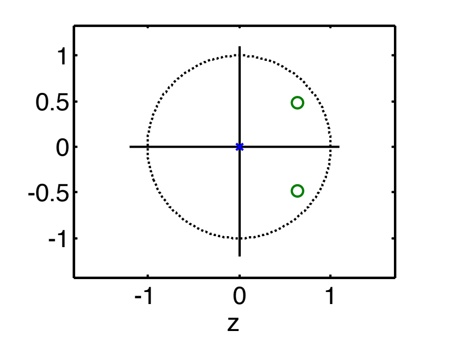
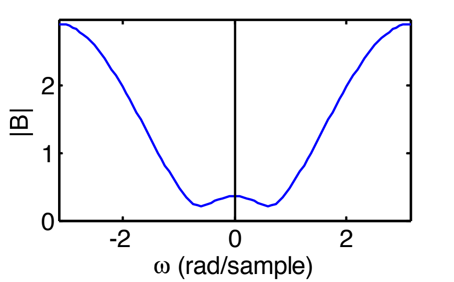
Real:

$[1,\ -1.28,\ 0.64]$





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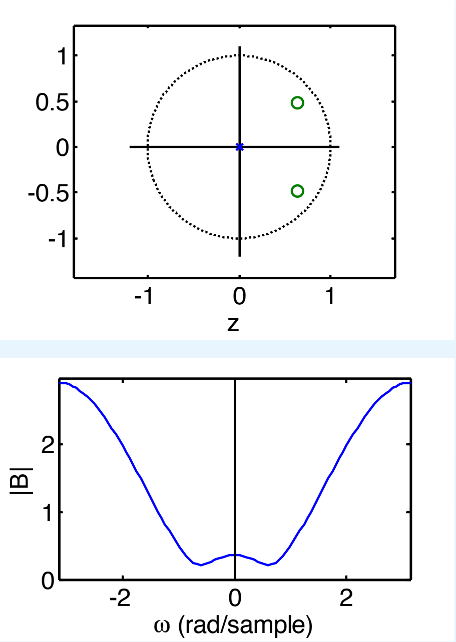

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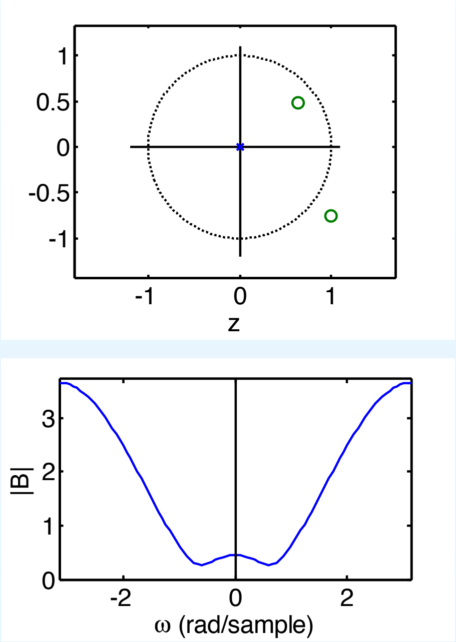

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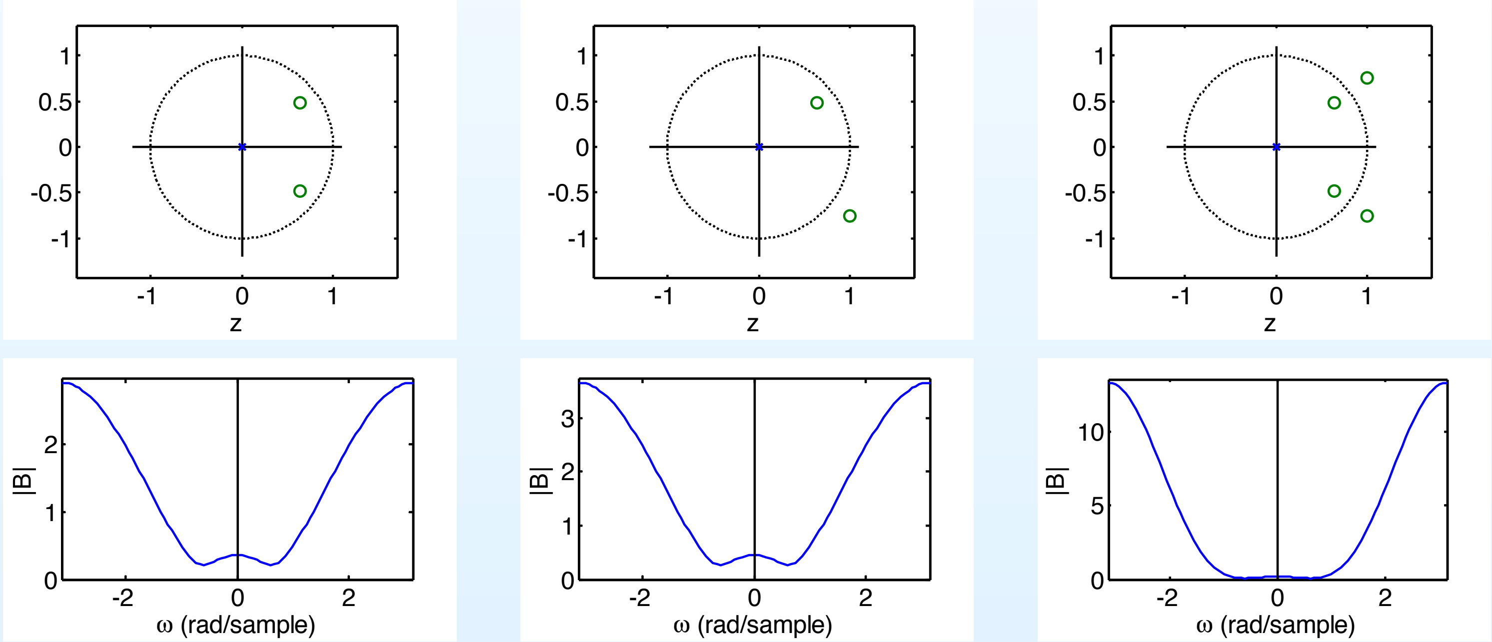
Real + Symmetric $b[n]$ $\Rightarrow$ conjugate+reciprocal groups of four or else pairs on the real axis

Real:	Symmetric:	Real + Symmetric:
$[1,\ -1.28,\ 0.64]$	$[1,\ -1.64+0.27j,\ 1]$	$[1,\ -3.28,\ 4.7625,\ -3.28,\ 1]$



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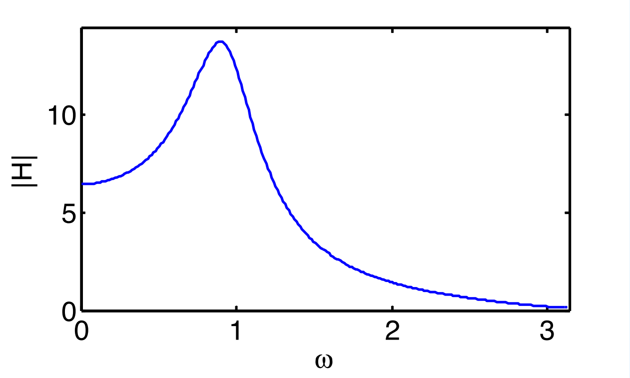
Example:

$H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$



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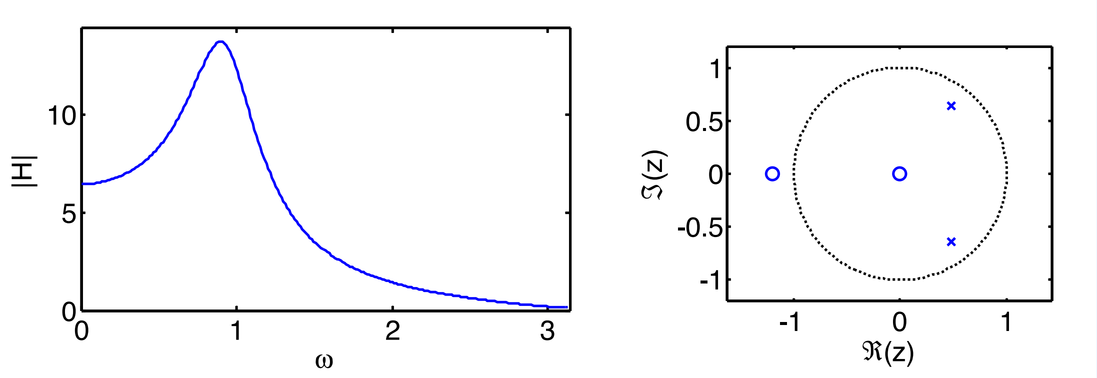

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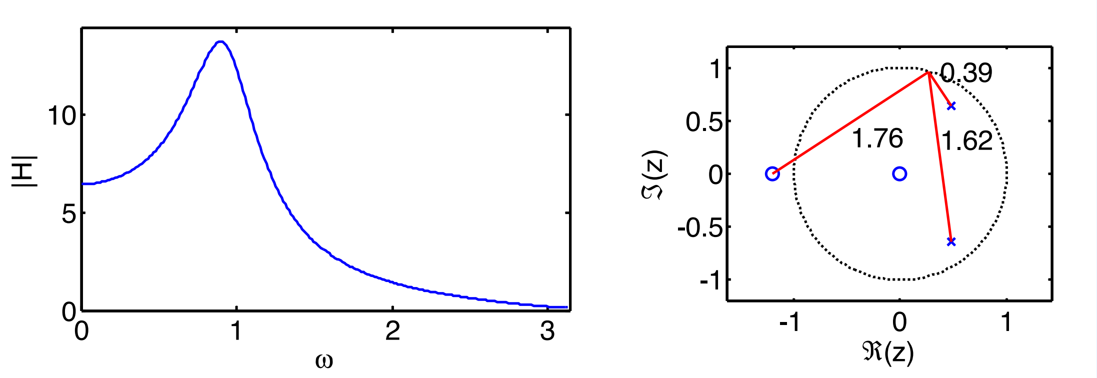
At $\omega = 1.3$: $\left|H(e^{j\omega})\right| = \frac{2\times 1.76}{1.62\times 0.39}$



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Factorize $H(z) = \frac{B(z)}{A(z)} = \frac{b[0]\prod_{i=1}^{M}\left(1-q_i z^{-1}\right)}{\prod_{i=1}^{N}\left(1-p_i z^{-1}\right)}$

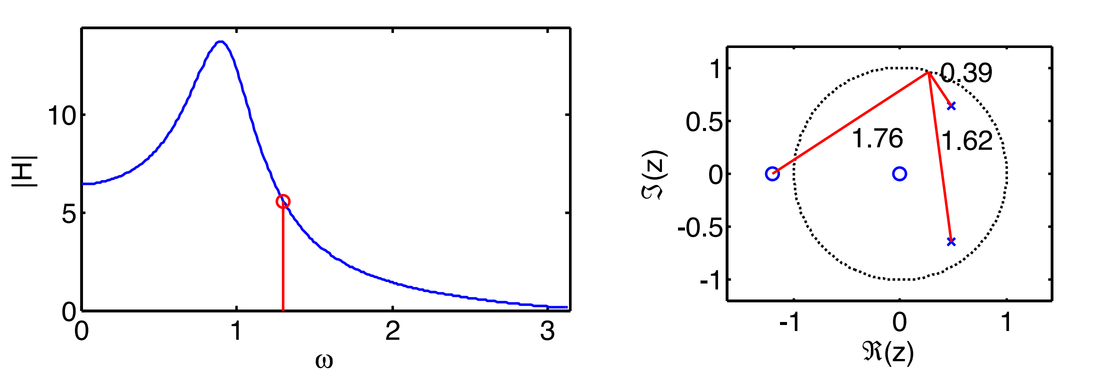
Roots of $A(z)$ and $B(z)$ are the "poles" $\{p_i\}$ and "zeros" $\{q_i\}$ of $H(z)$
Also an additional $N-M$ zeros at the origin (affect phase only)

$\left|H(e^{j\omega})\right| = \frac{|b[0]|\left|z^{-M}\right|\prod_{i=1}^{M}|z-q_i|}{|z^{-N}|\prod_{i=1}^{N}|z-p_i|}$ for $z = e^{j\omega}$

Example:

$H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}} = \frac{2\left(1+1.2z^{-1}\right)}{(1-(0.48-0.64j)z^{-1})(1-(0.48+0.64j)z^{-1})}$

At $\omega = 1.3$: $\left|H(e^{j\omega})\right| = \frac{2\times 1.76}{1.62\times 0.39} = 5.6$



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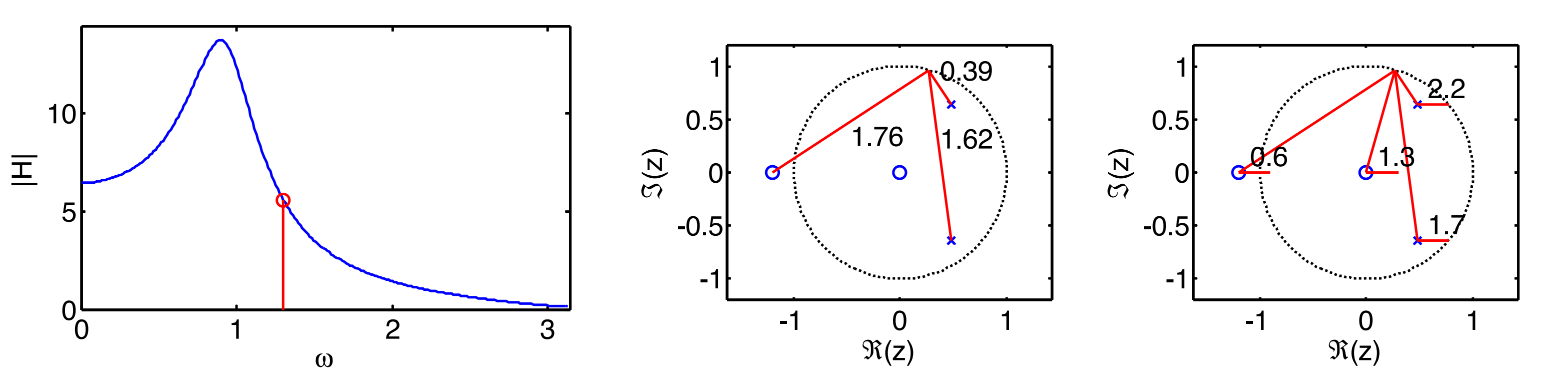
$\angle H(e^{j\omega}) = (0.6+1.3)-(1.7+2.2) = -1.97$



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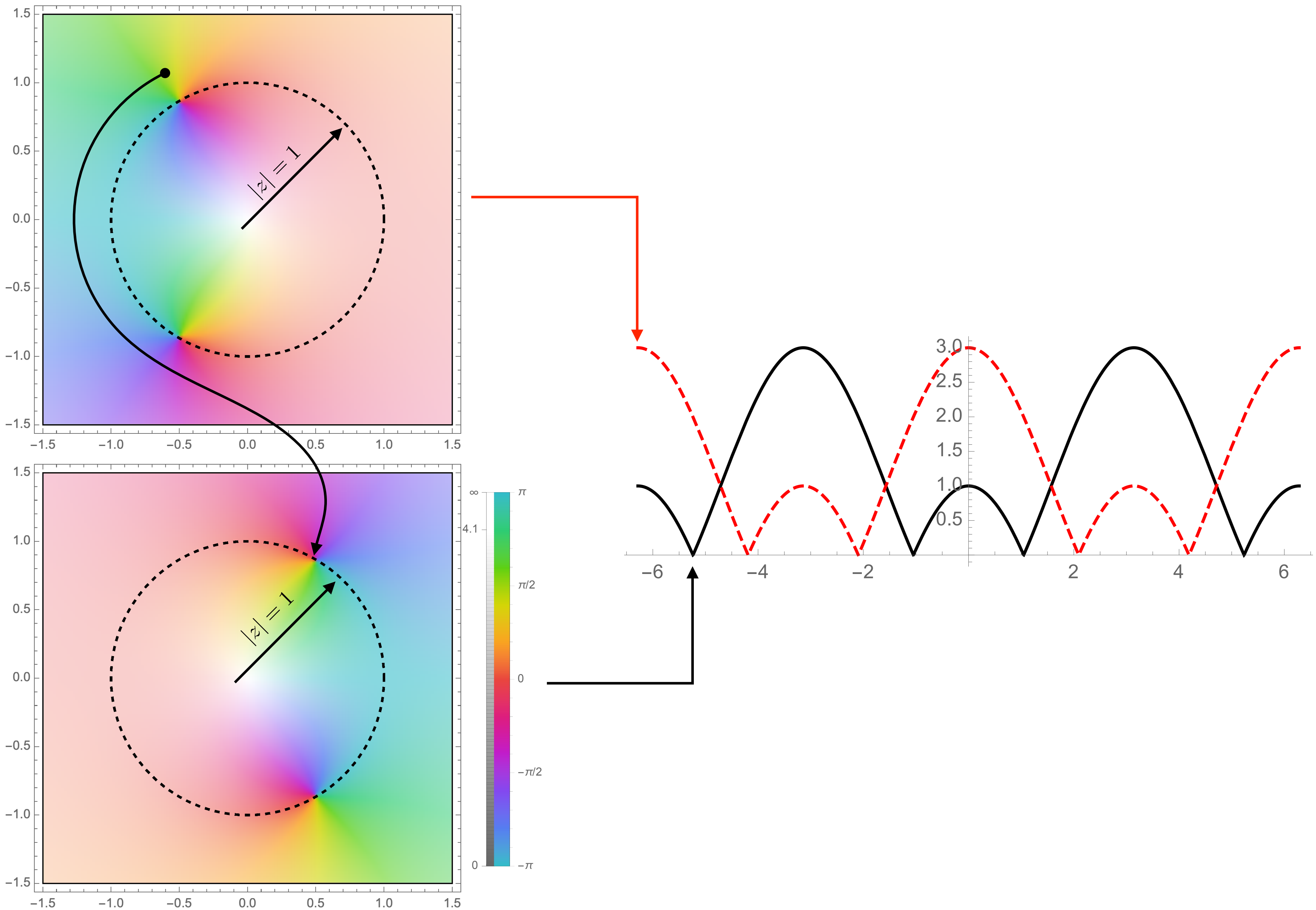
$|z| = 1$
$|z| = 1$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
∞
4.1
0
π
$\pi/2$
0
$-\pi/2$
$-\pi$
3.0
2.5
2.0
1.5
1.0
0.5
−6
−4
−2
2
4
6

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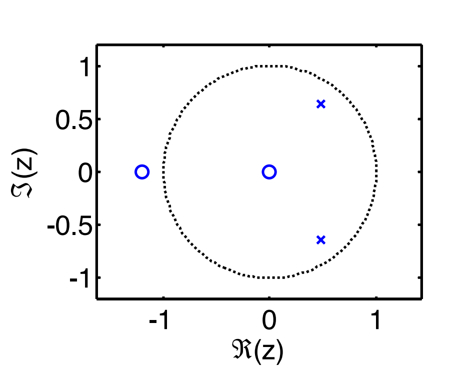

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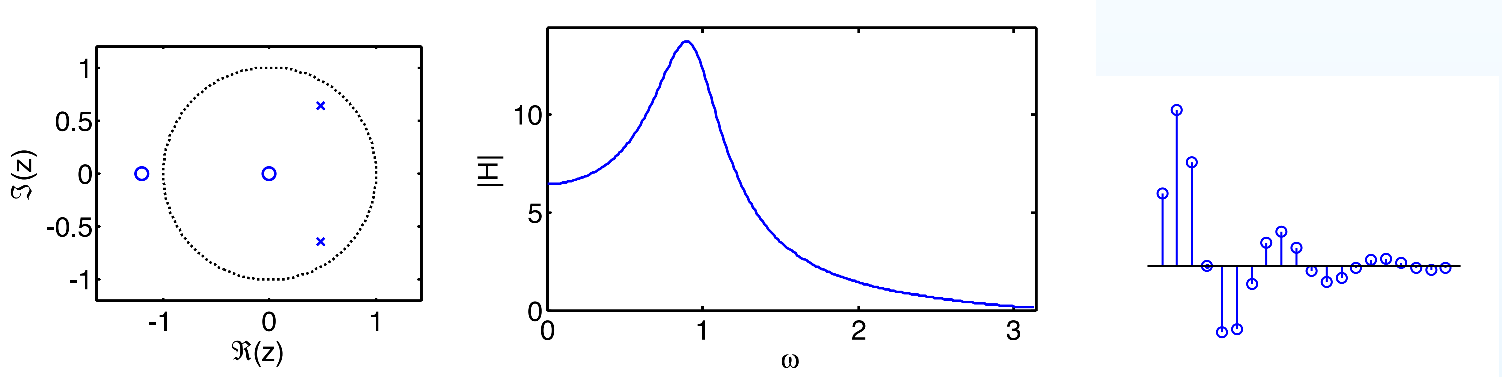

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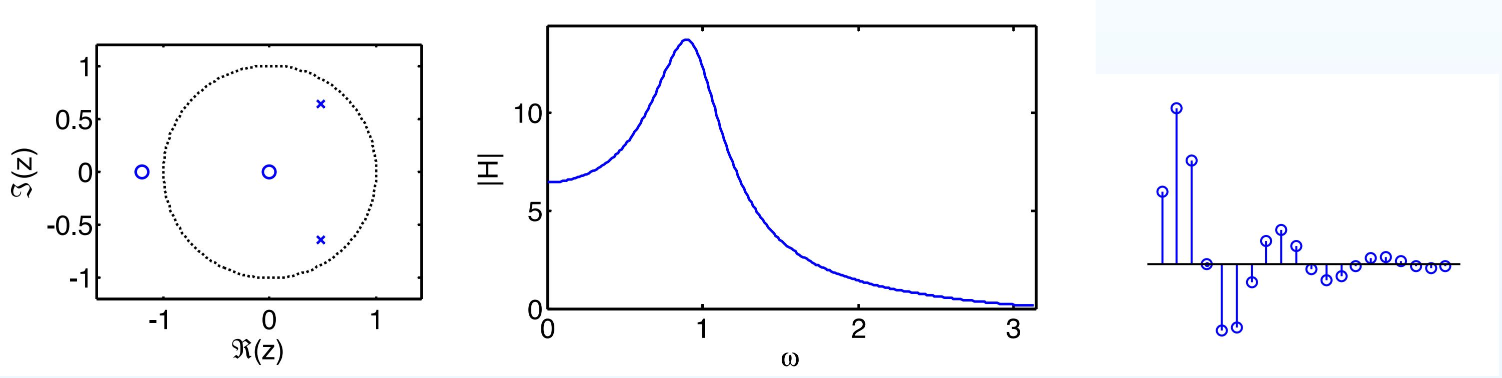

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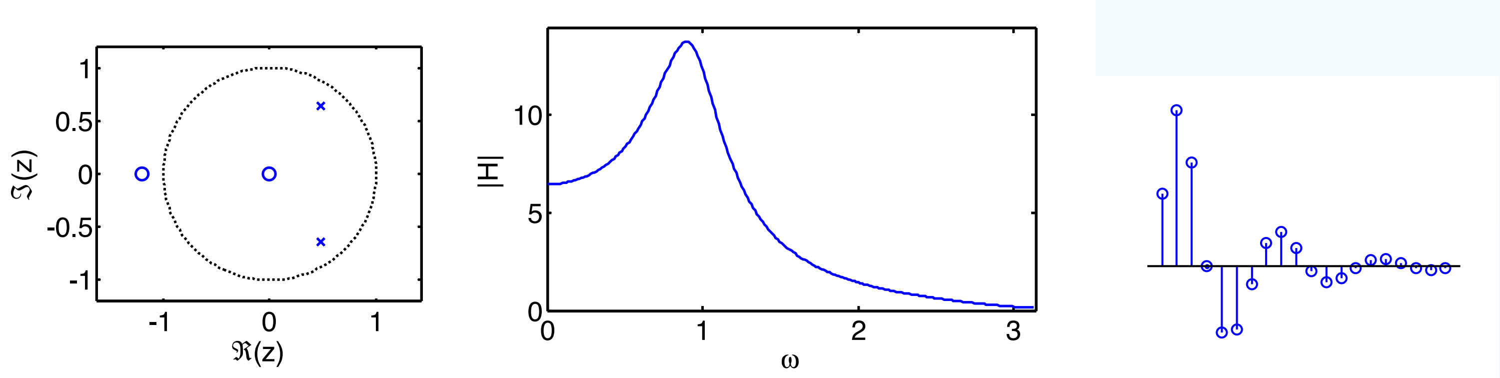

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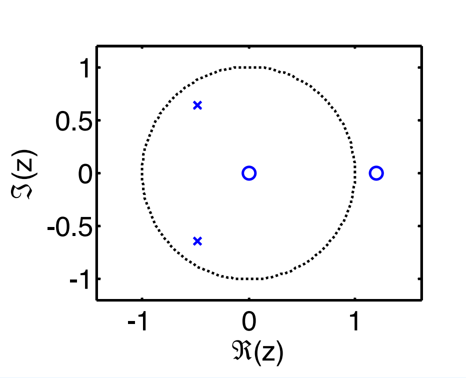

Pole and zero positions are negated

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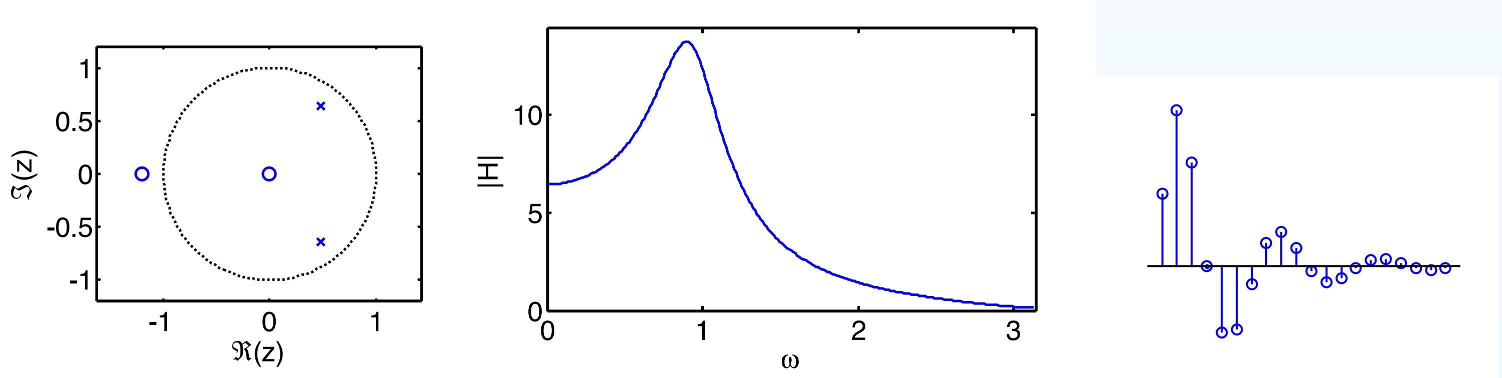

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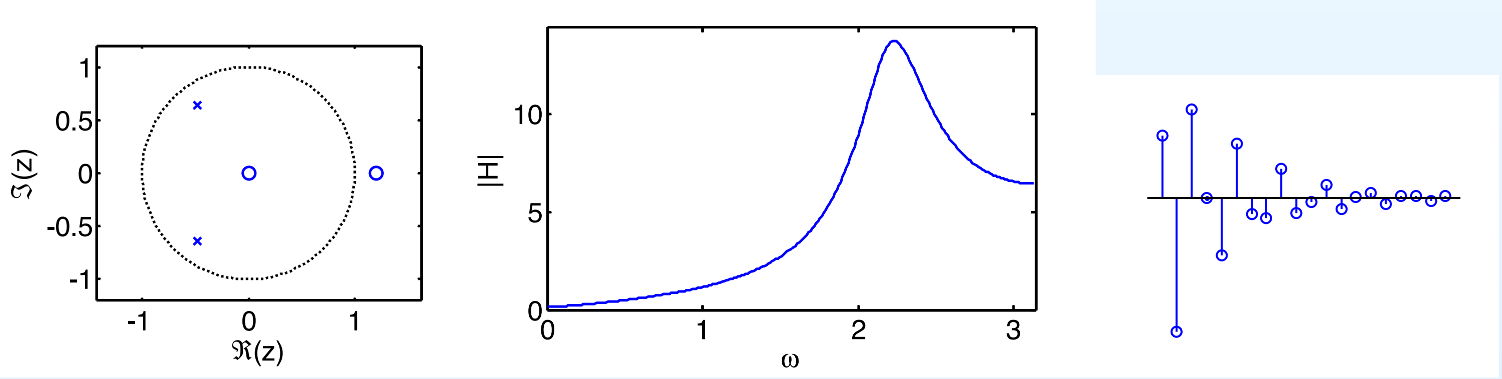

Pole and zero positions are negated, response is flipped and conjugated.

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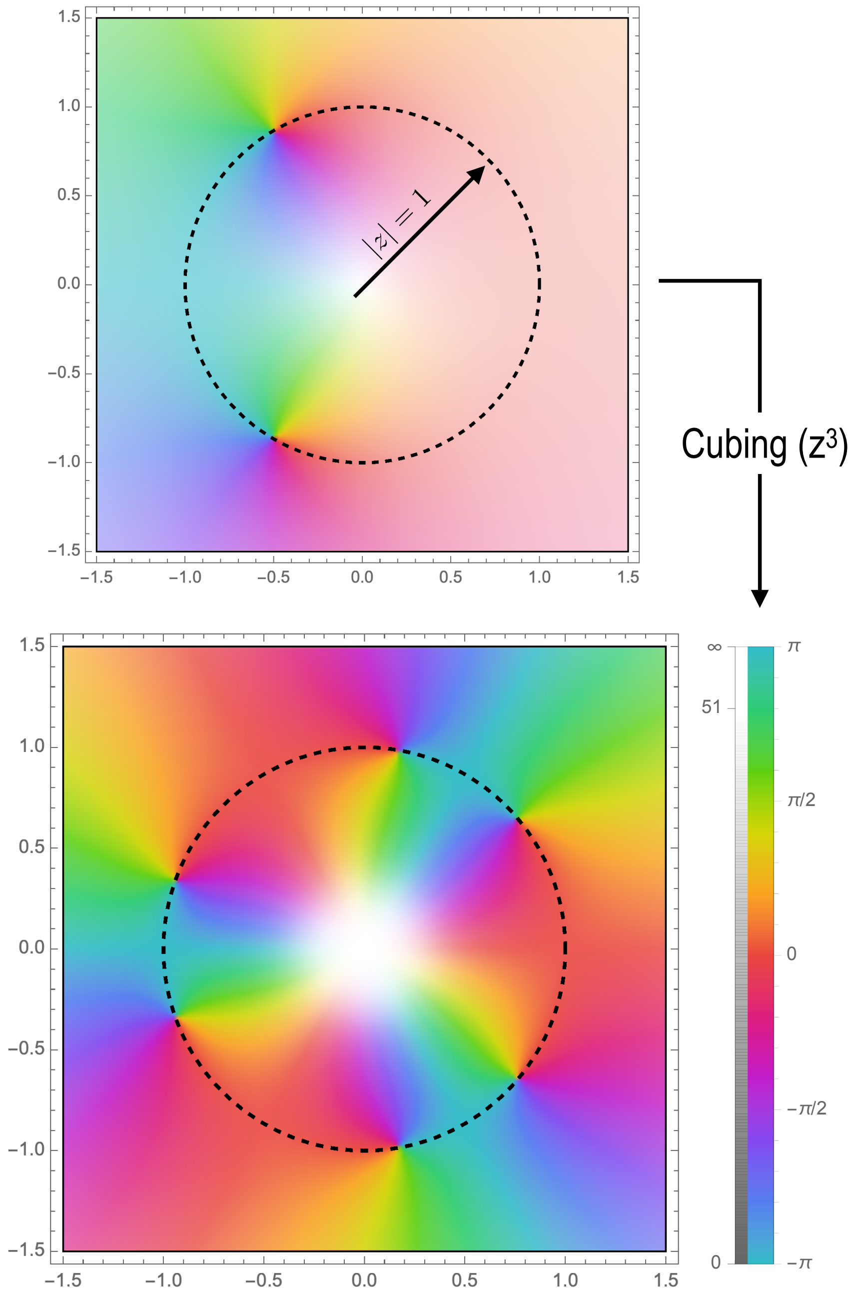
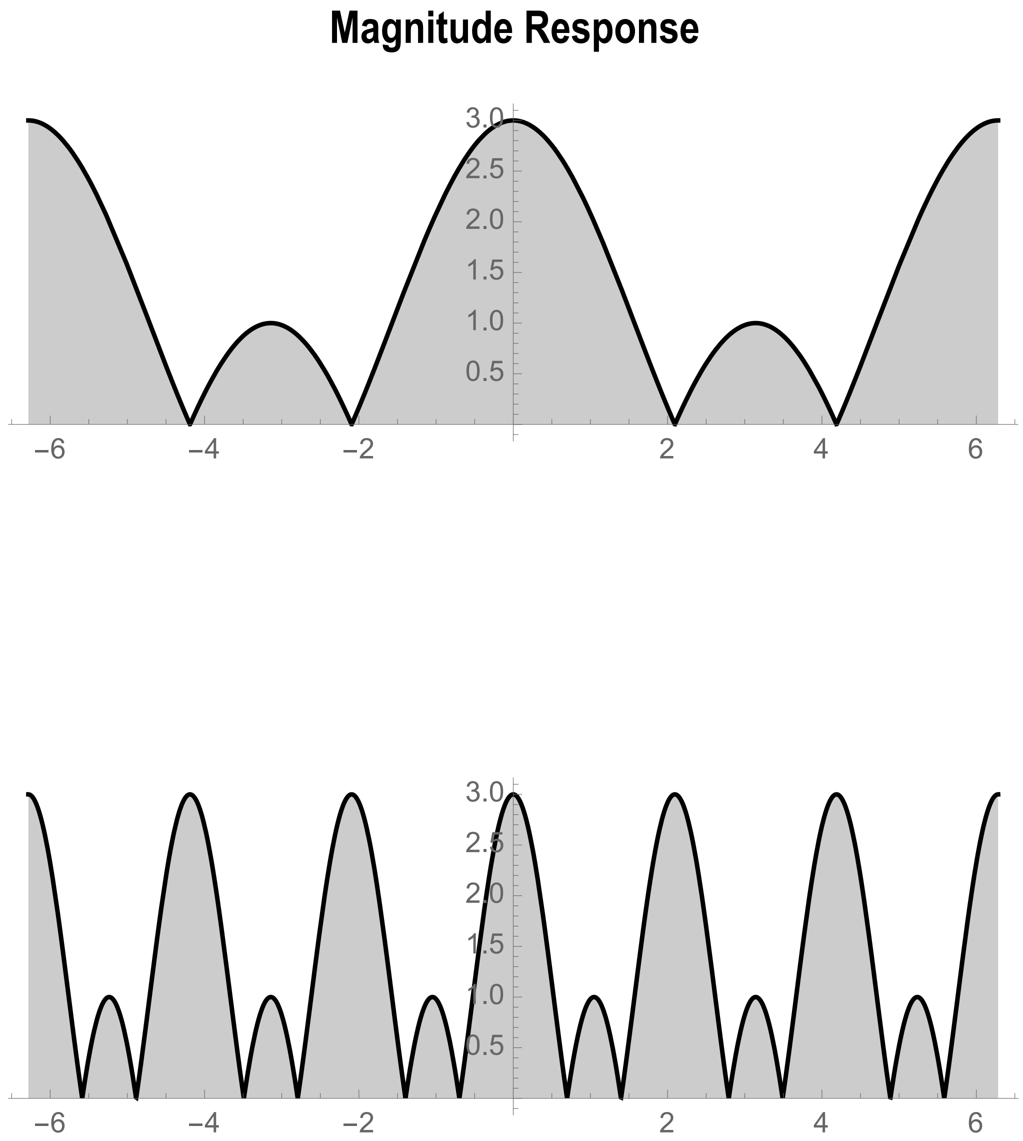

$|z| = 1$
Cubing (z^3)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
∞
51
0
π
$\pi/2$
0
$-\pi/2$
$-\pi$


Magnitude Response
3.0
2.5
2.0
1.5
1.0
0.5
−6
−4
−2
2
4
6

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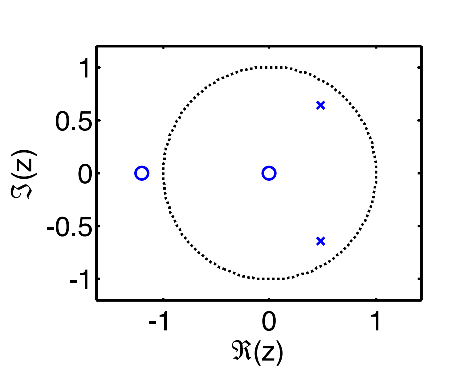

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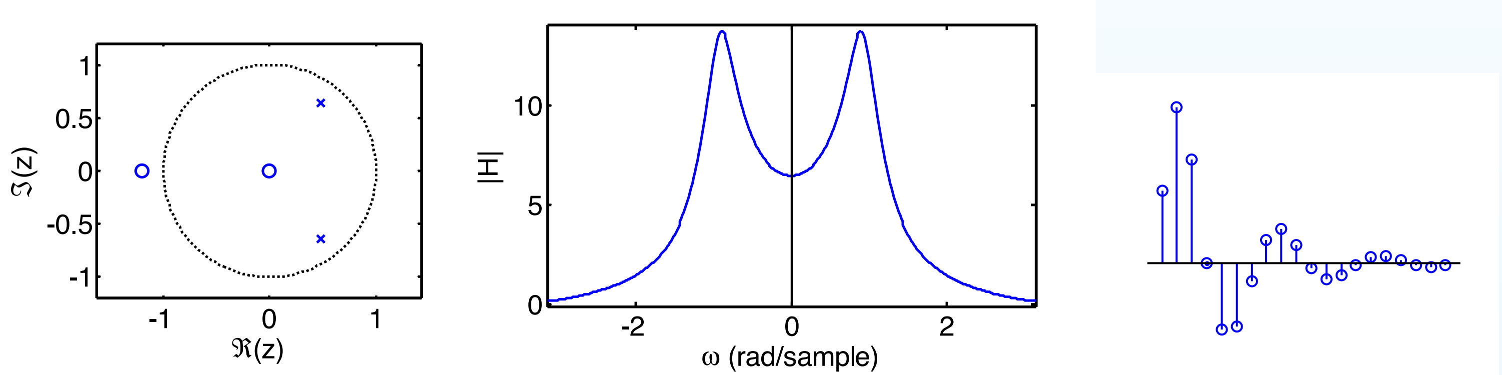

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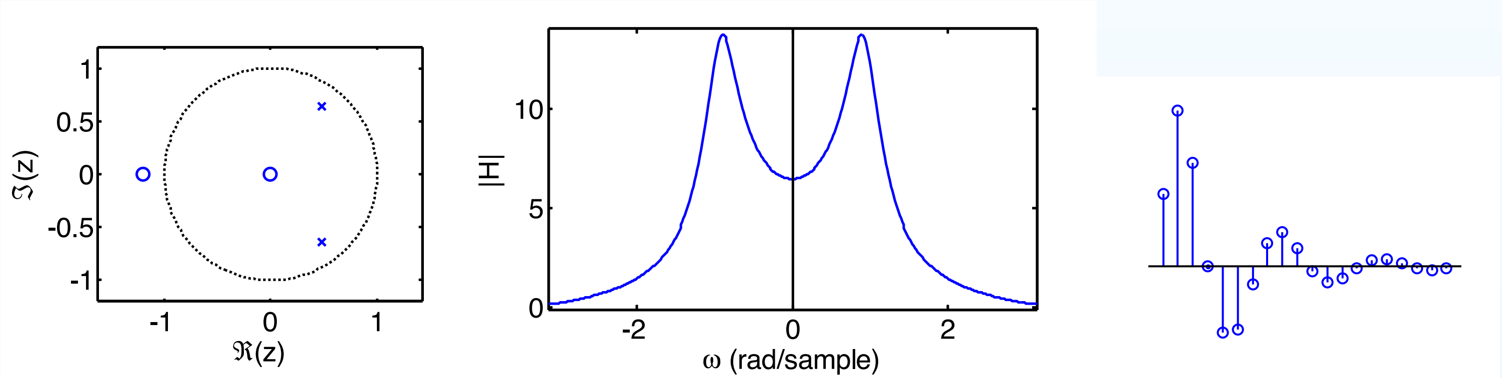

Cube z: $H_C(z) = \frac{2+2.4z^{-3}}{1-0.96z^{-3}+0.64z^{-6}}$ Insert 2 zeros between coefs

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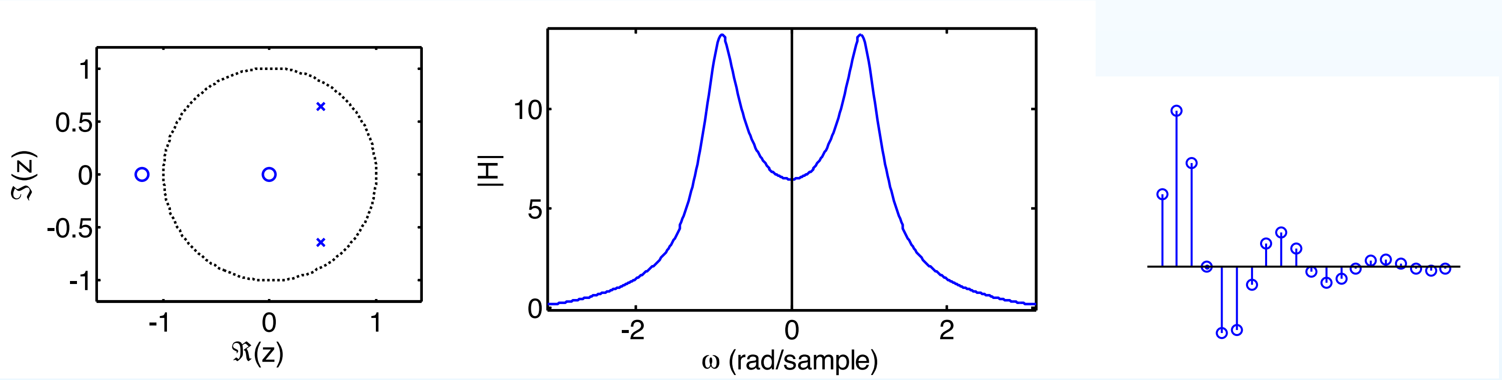

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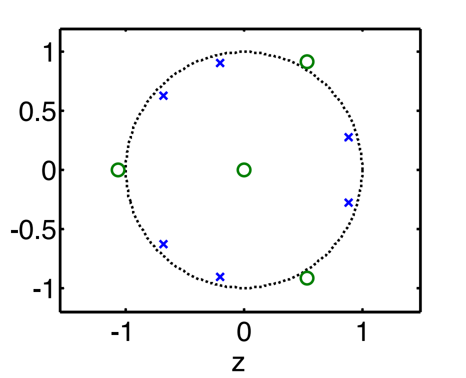

Pole and zero positions are replicated

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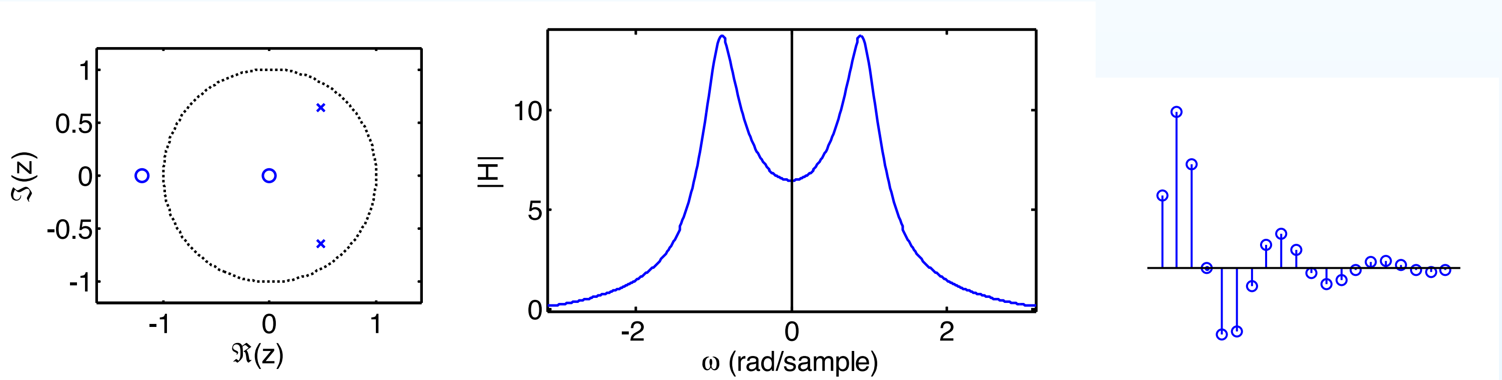

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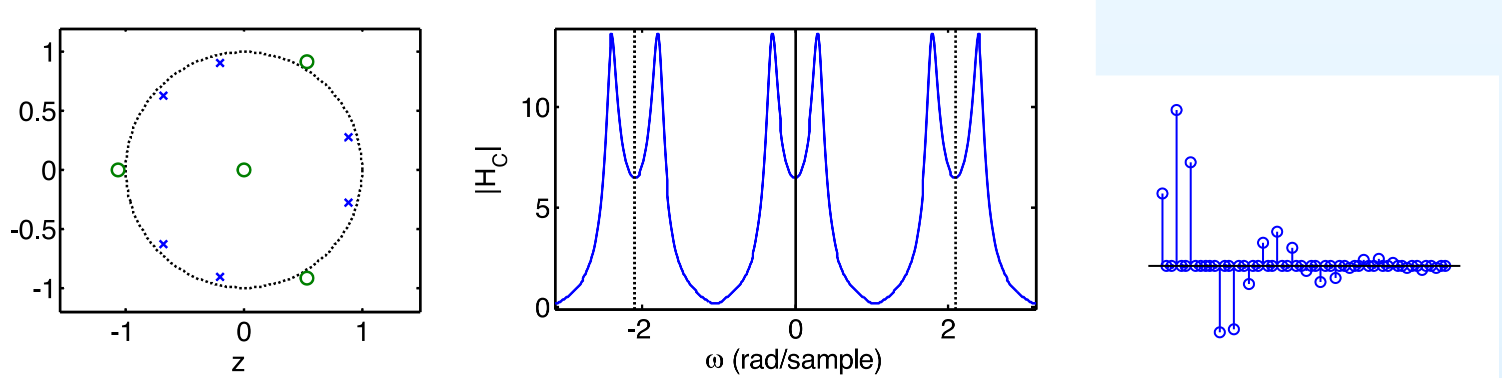

Pole and zero positions are replicated, magnitude response replicated.

Scaling z

Given a filter $H(z)$ we can form a new one $H_S(z) = H(\frac{z}{\alpha})$

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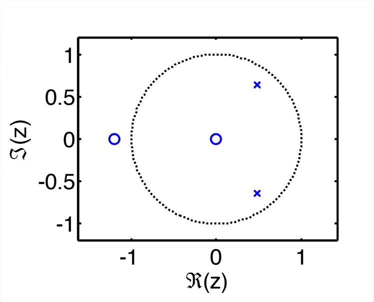

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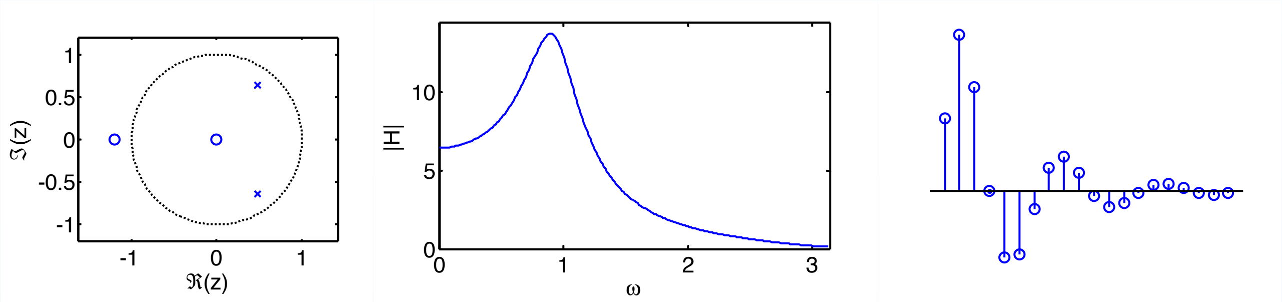

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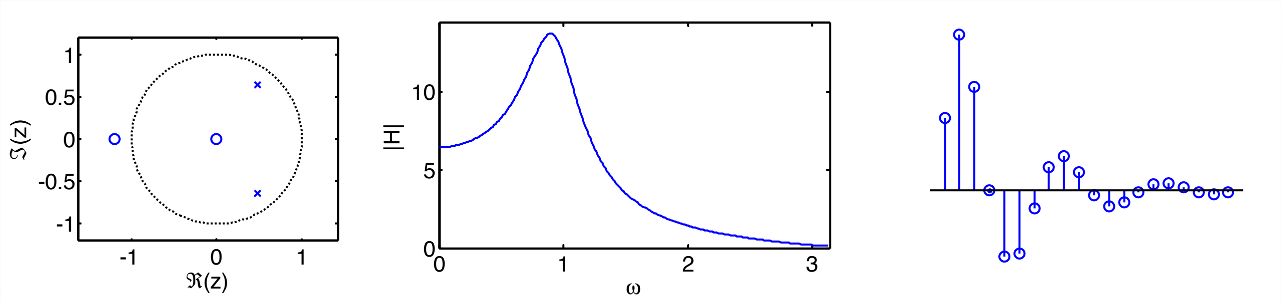

Scale z: $H_S(z) = H(\frac{z}{1.1}) = \frac{2+2.64z^{-1}}{1-1.056z^{-1}+0.7744z^{-2}}$

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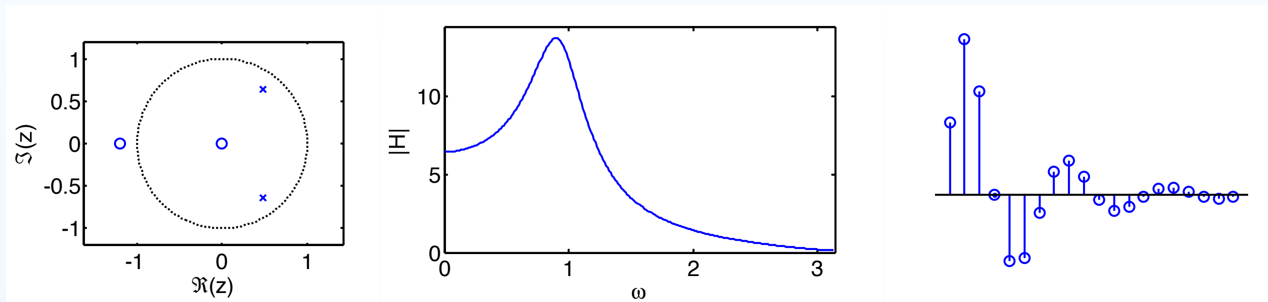

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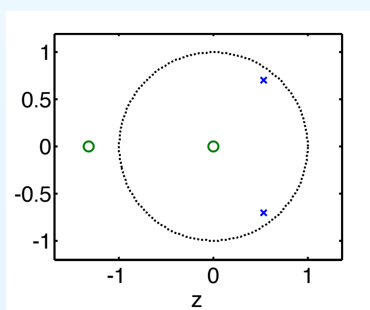

Pole and zero positions are multiplied by α

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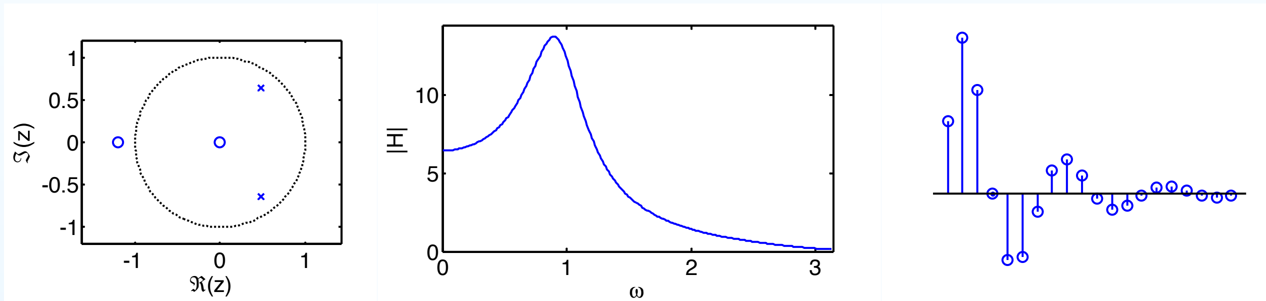

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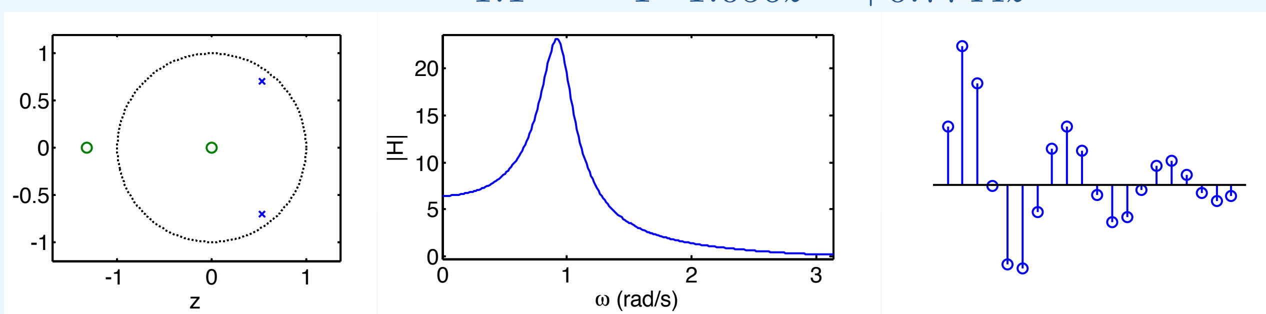

Pole and zero positions are multiplied by α, $\alpha > 1 \Rightarrow$peaks sharpened.

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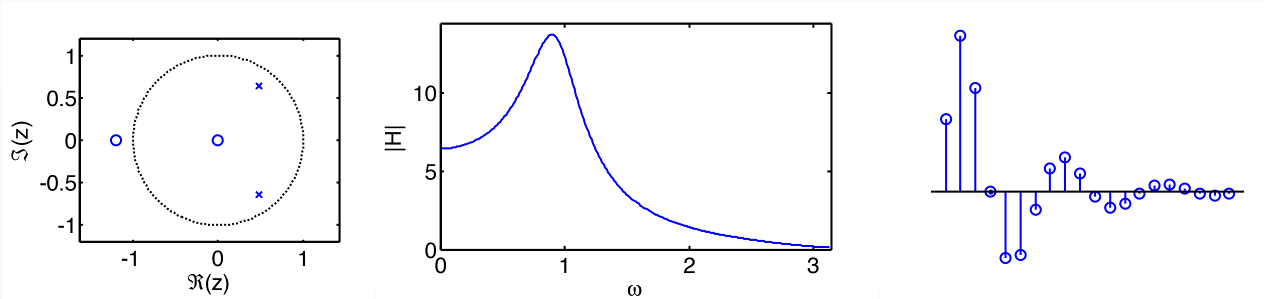

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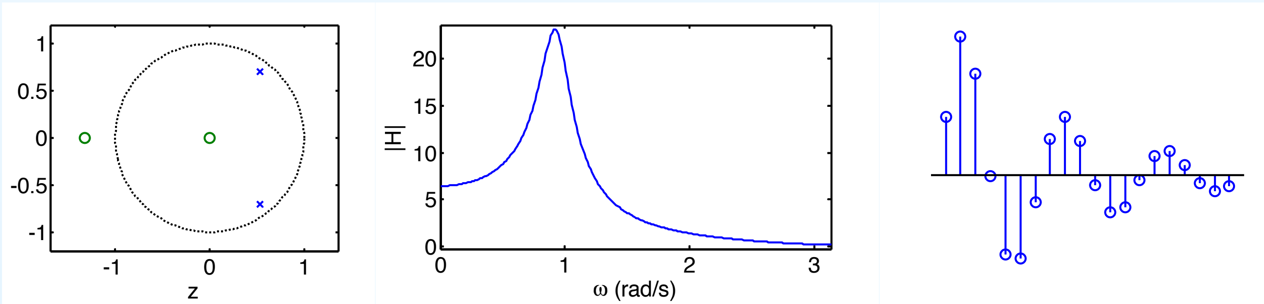

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Pole at $z = p$ gives peak bandwidth $\approx 2\,|\log|p|| \approx 2\,(1-|p|)$

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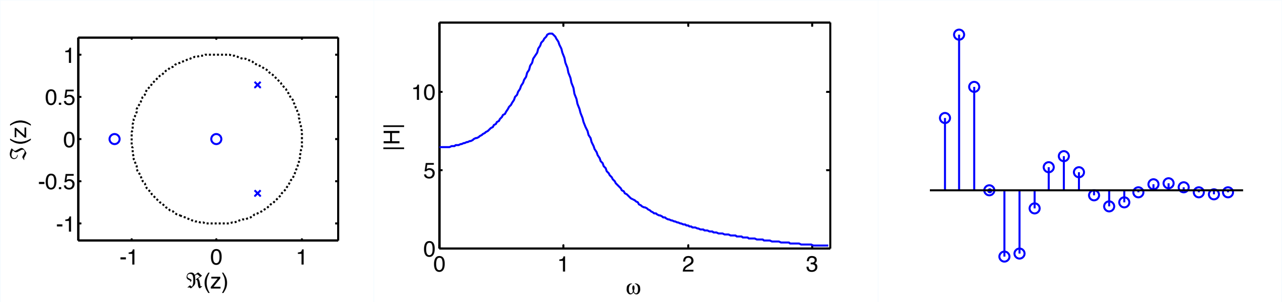

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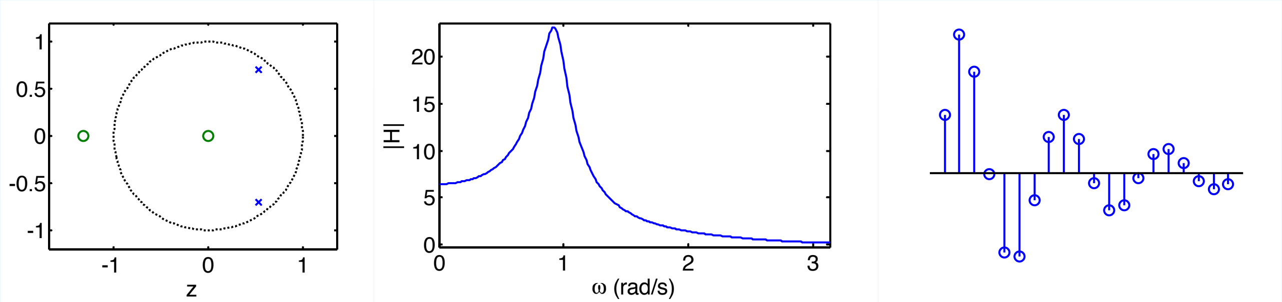

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Pole at $z = p$ gives peak bandwidth $\approx 2\left|\log|p|\right| \approx 2(1-|p|)$

For pole near unit circle, decrease bandwidth by $\approx 2\log\alpha$

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1st order low pass filter: extremely common

$$y[n] = (1-p)x[n] + py[n-1]$$

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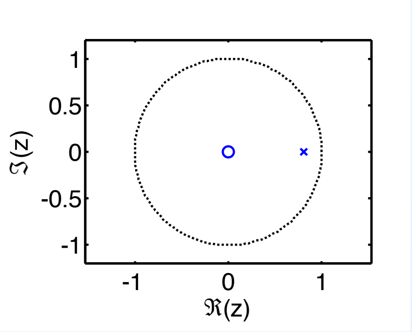
1st order low pass filter: extremely common

$$y[n] = (1-p)x[n] + py[n-1] \Rightarrow H(z) = \frac{1-p}{1-pz^{-1}}$$



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1st order low pass filter: extremely common

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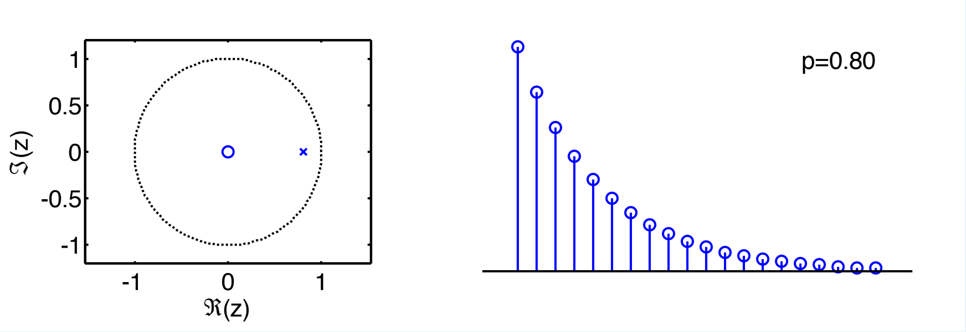
Impulse response:

$$h[n] = (1-p)p^n$$



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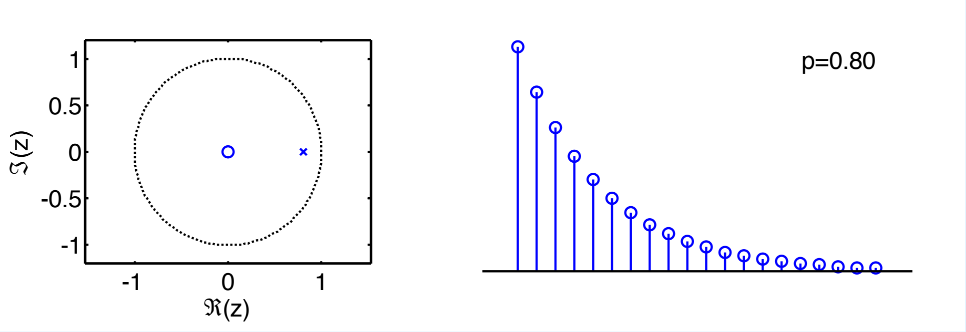
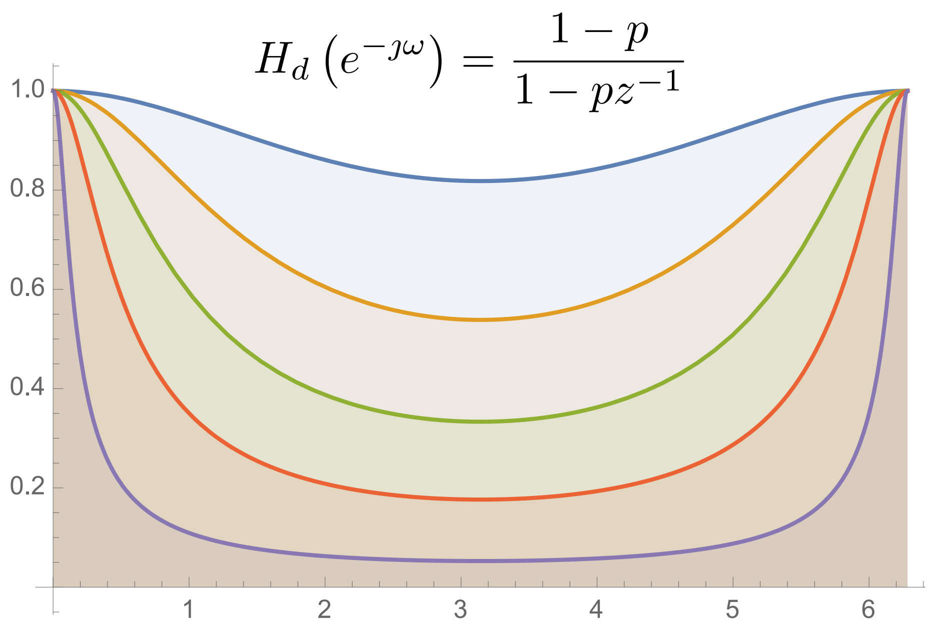
$$y[n] = (1-p)x[n] + py[n-1] \Rightarrow H(z) = \frac{1-p}{1-pz^{-1}}$$

Impulse response:

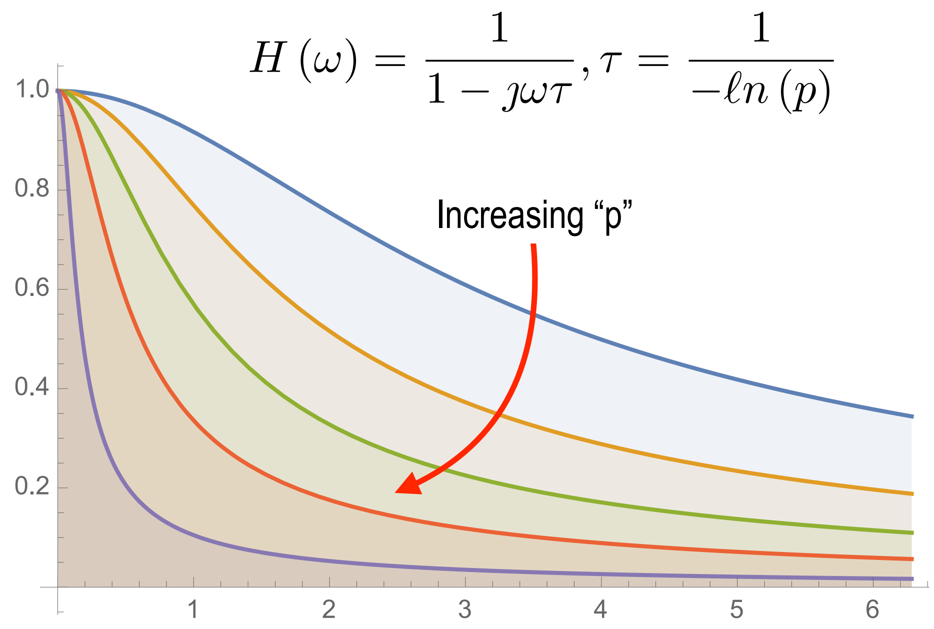
$$h[n] = (1-p)p^n = (1-p)e^{-\frac{n}{\tau}}$$

where $\tau = \frac{1}{-\ln p}$ is the time constant in samples.

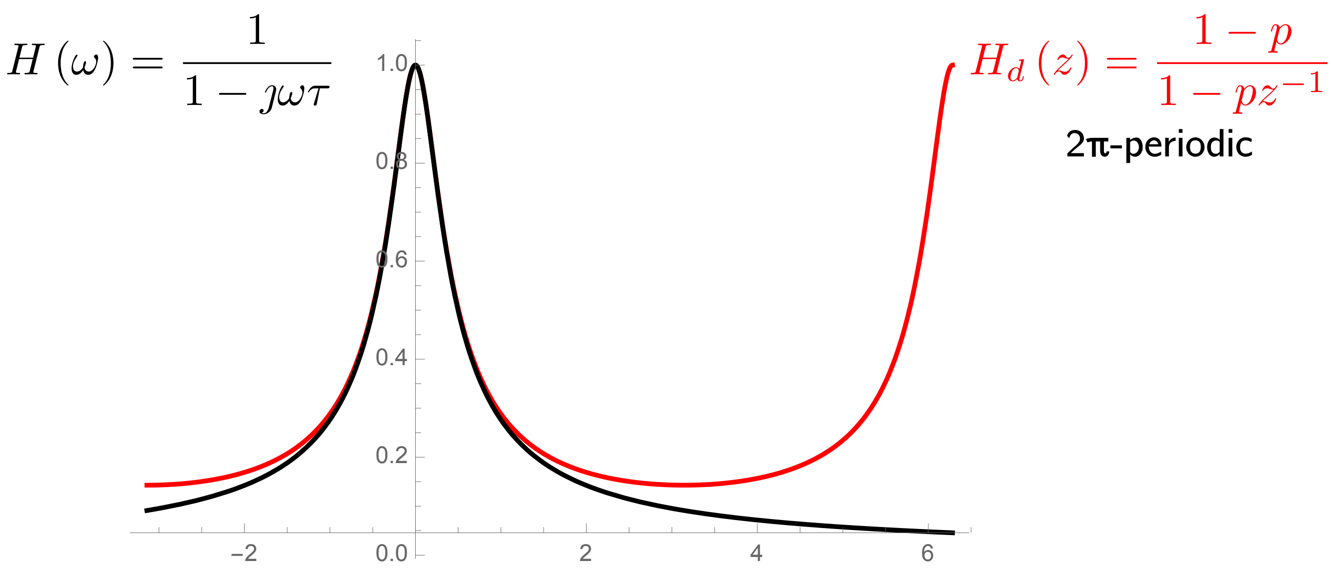




$H_d\left(e^{-\jmath\omega}\right) = \frac{1-p}{1-pz^{-1}}$
1.0
0.8
0.6
0.4
0.2
1
2
3
4
5
6



$H(\omega) = \frac{1}{1-\jmath\omega\tau}, \tau = \frac{1}{-\ell n(p)}$
Increasing "p"
1.0
0.8
0.6
0.4
0.2
1
2
3
4
5
6



$H(\omega) = \frac{1}{1-\jmath\omega\tau}$
$H_d(z) = \frac{1-p}{1-pz^{-1}}$
2π-periodic
1.0
0.8
0.6
0.4
0.2
0.0
−2
2
4
6

Low-pass filter

1st order low pass filter: extremely common

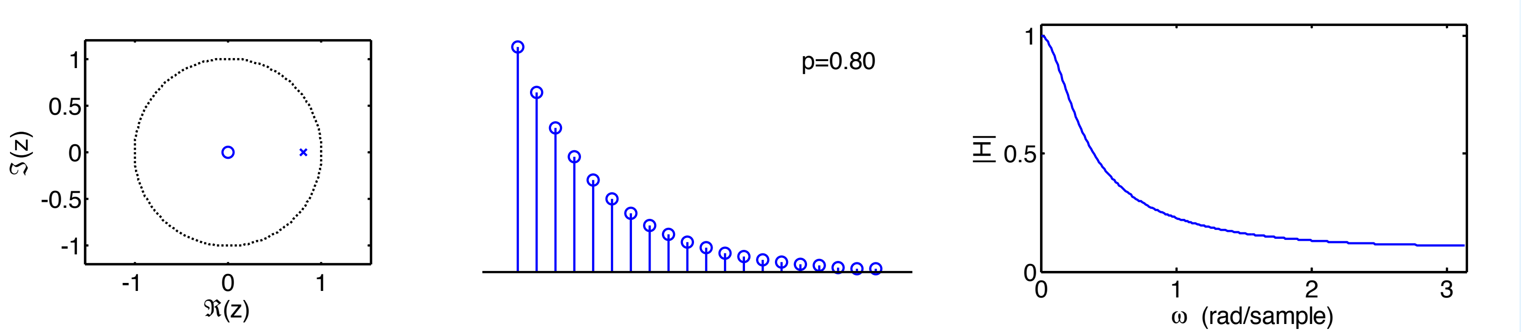
$$y[n] = (1-p)x[n] + py[n-1] \Rightarrow H(z) = \frac{1-p}{1-pz^{-1}}$$

Impulse response:

$$h[n] = (1-p)p^n = (1-p)e^{-\frac{n}{\tau}}$$

where $\tau = \frac{1}{-\ln p}$ is the time constant in samples.

Magnitude response: $\left|H(e^{j\omega})\right| = \frac{1-p}{\sqrt{1-2p\cos\omega+p^2}}$



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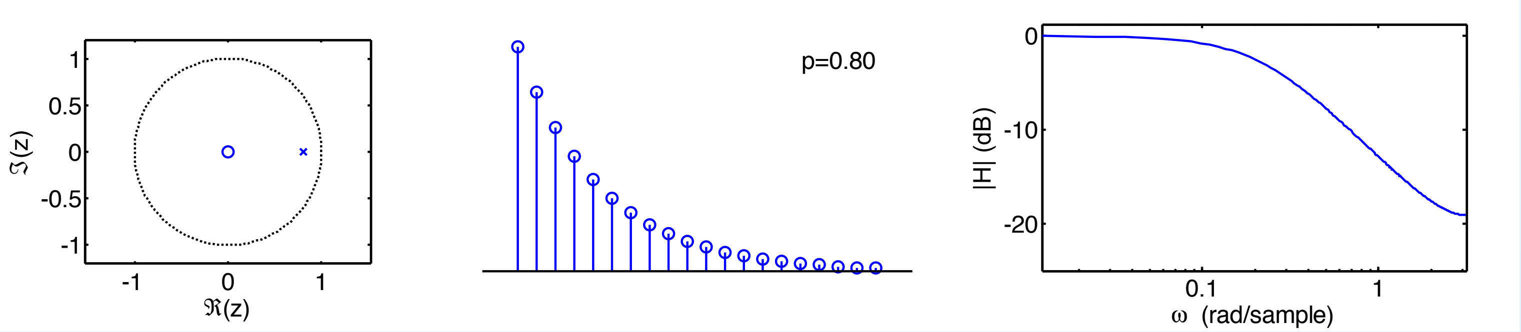

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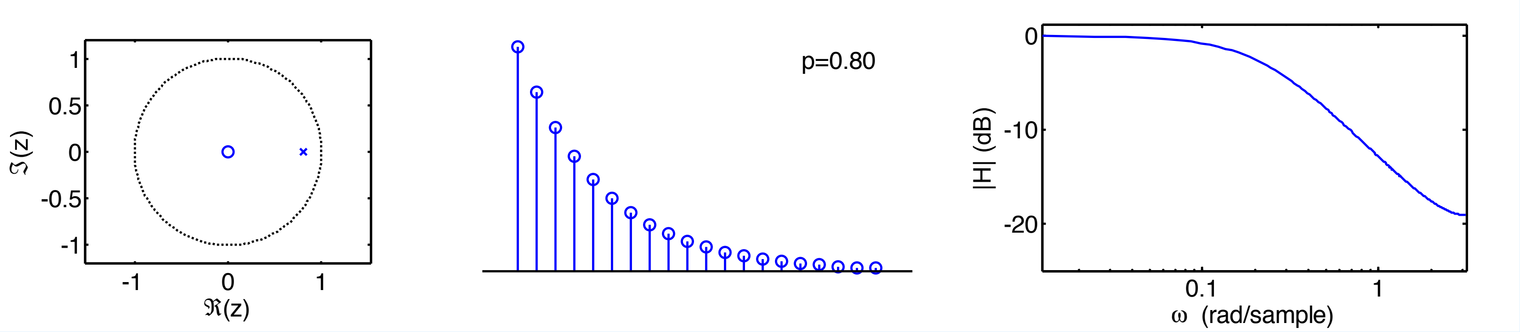
Low-pass filter with DC gain of unity.



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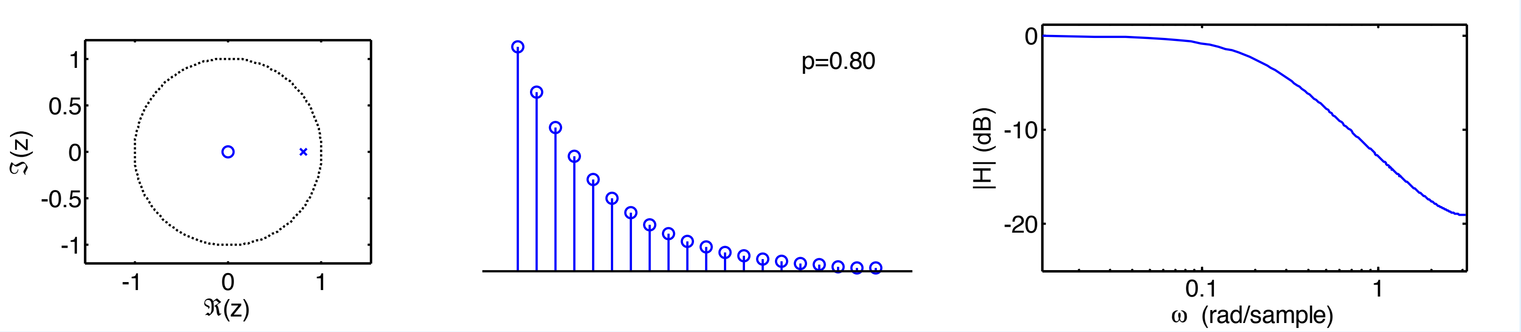

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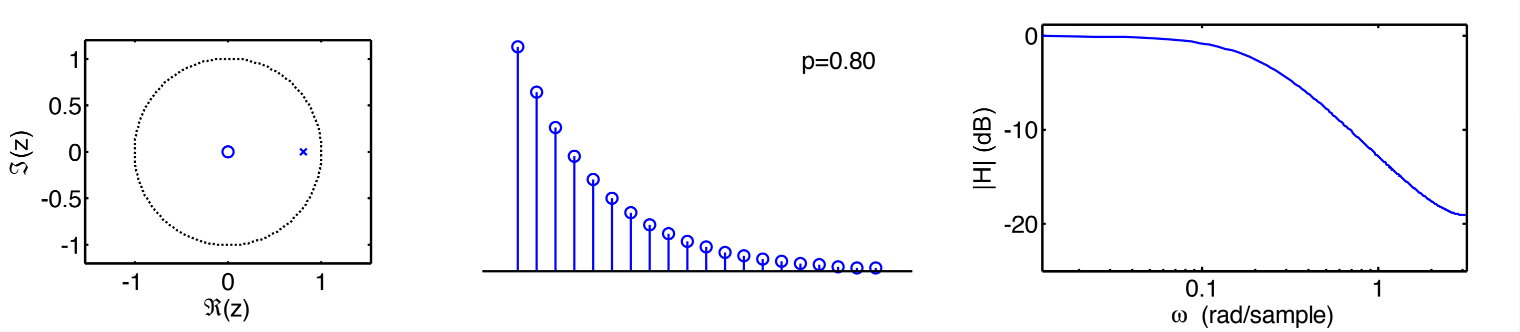

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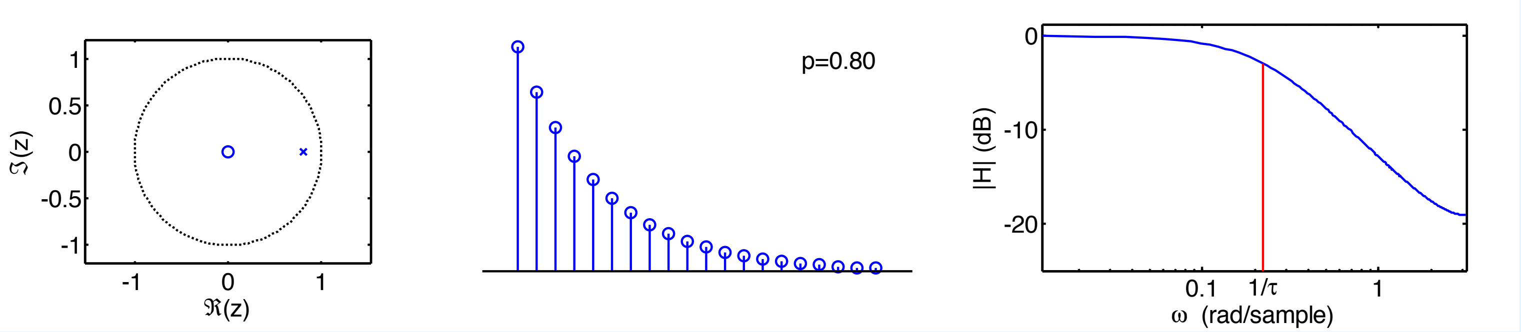

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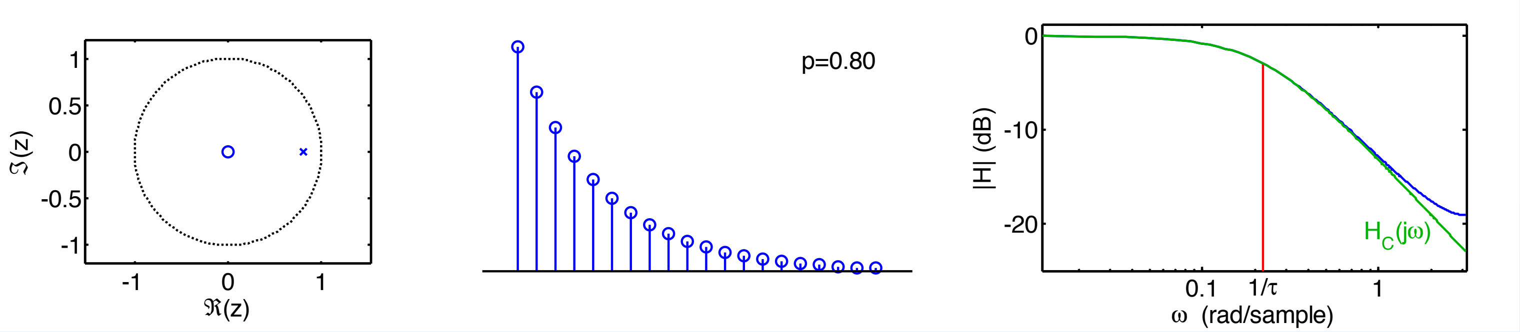

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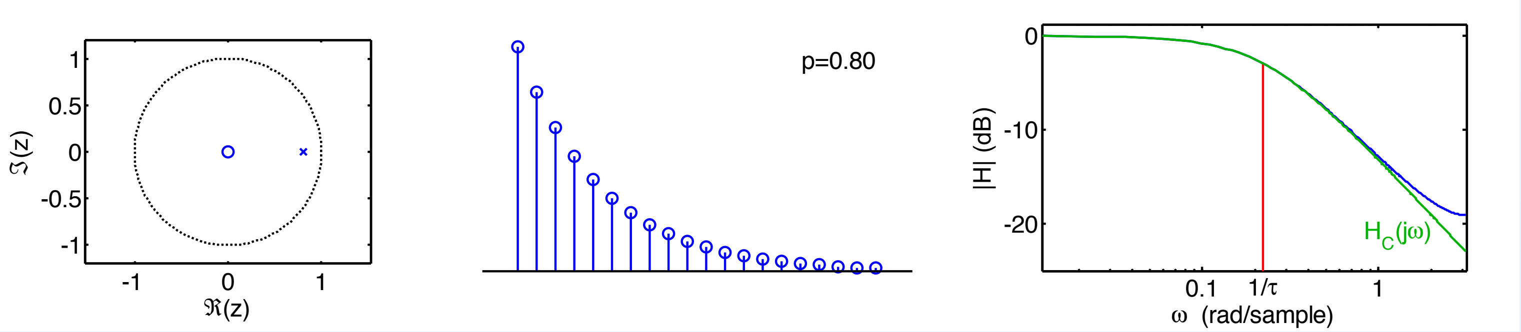

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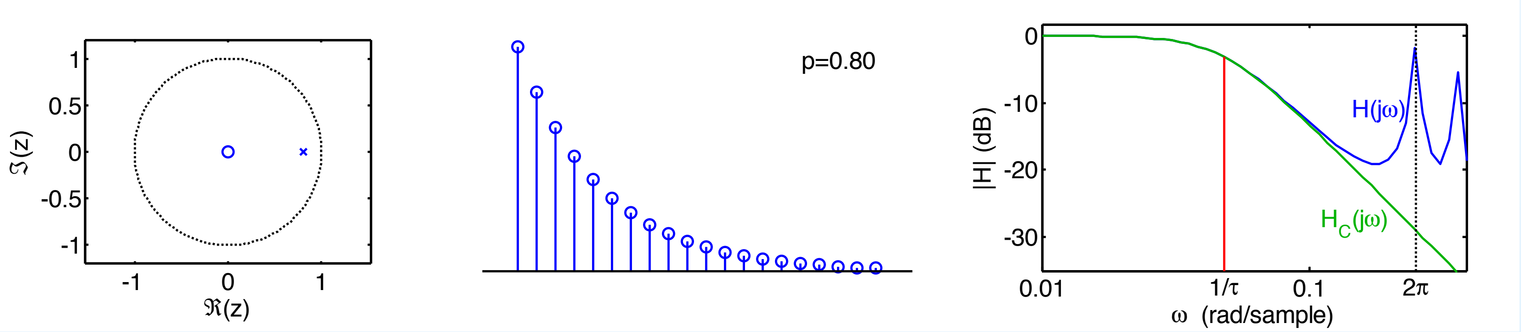

Allpass filters

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$$\Rightarrow H(e^{j\omega}) = \frac{\sum_{r=0}^{M} a^*[M-r]e^{-j\omega r}}{\sum_{r=0}^{M} a[r]e^{-j\omega r}} = e^{-j\omega M} \frac{\sum_{s=0}^{M} a^*[s]e^{j\omega s}}{\sum_{r=0}^{M} a[r]e^{-j\omega r}} \quad [s = M - r]$$

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Allpass filters

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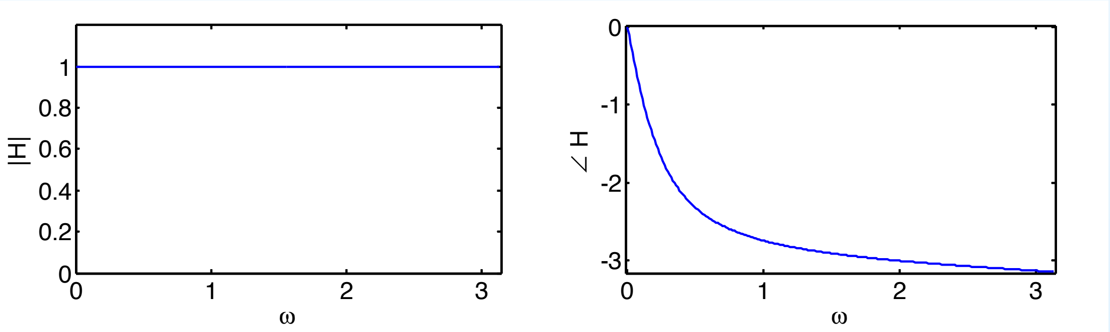

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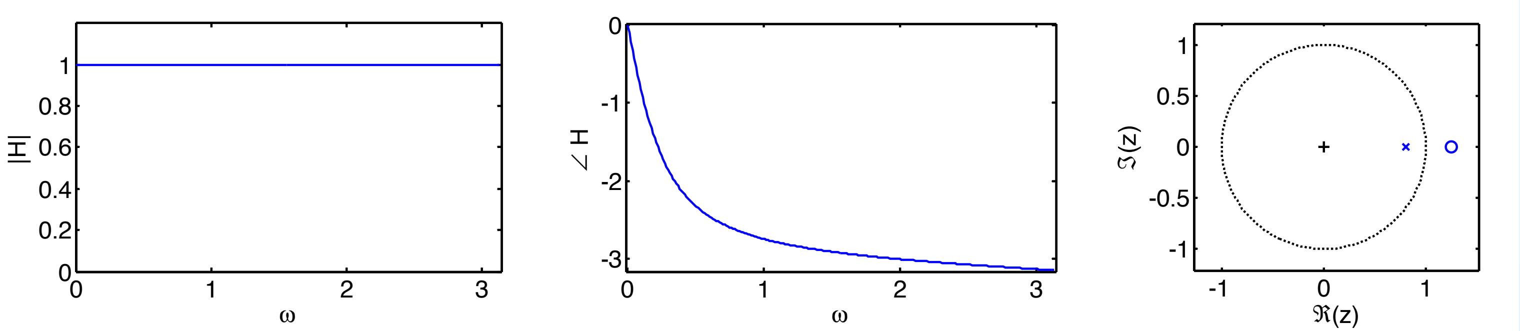

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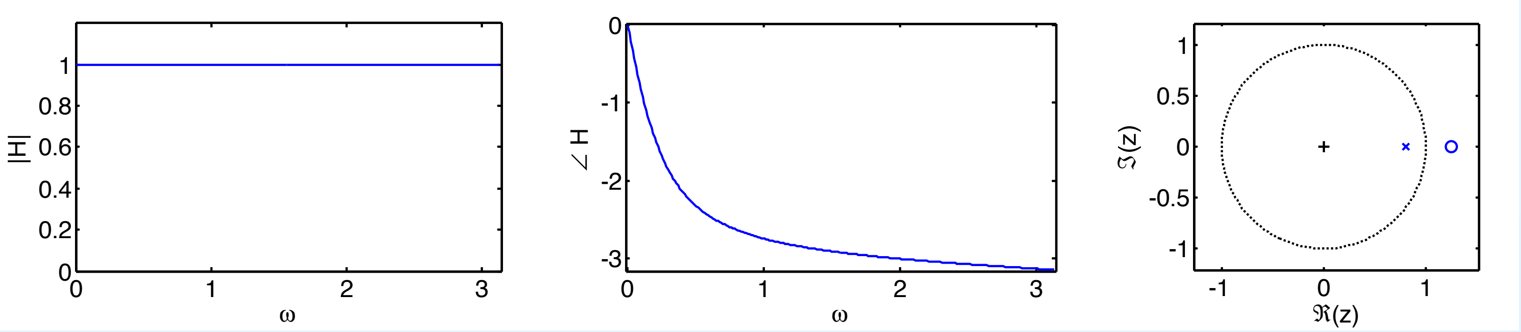

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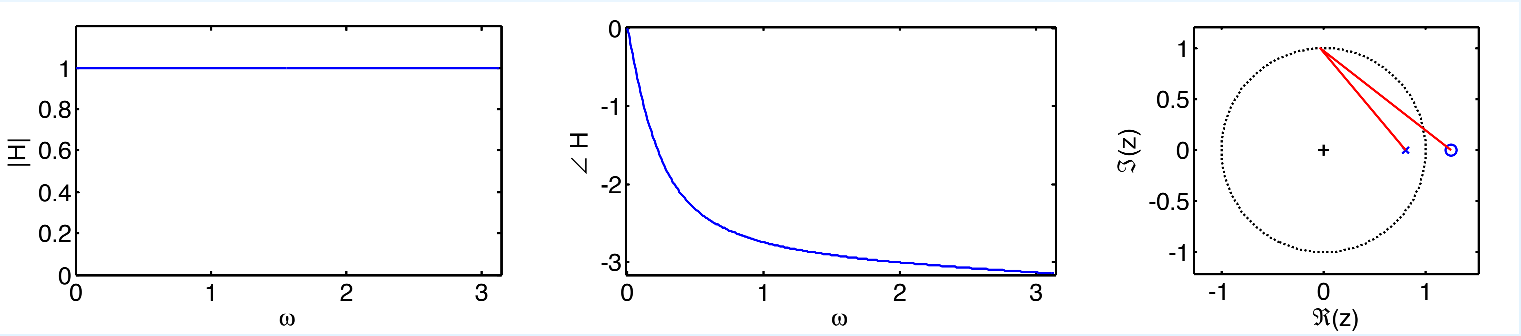
Constant distance ratio: $\left|e^{j\omega} - p\right| = |p|\left|e^{j\omega} - \frac{1}{p}\right| \forall\omega$



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Hence $\left|H(e^{j\omega})\right|=1\forall\omega \Leftrightarrow$ "allpass"

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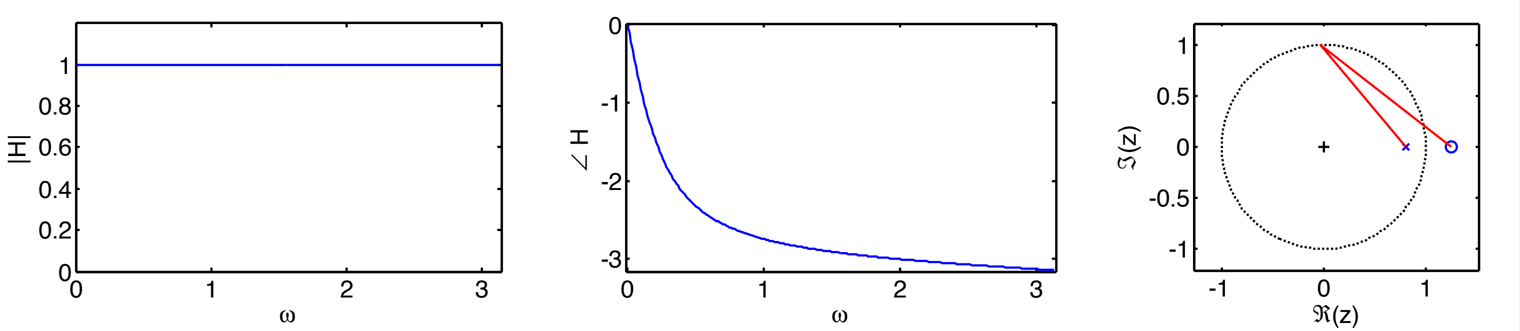
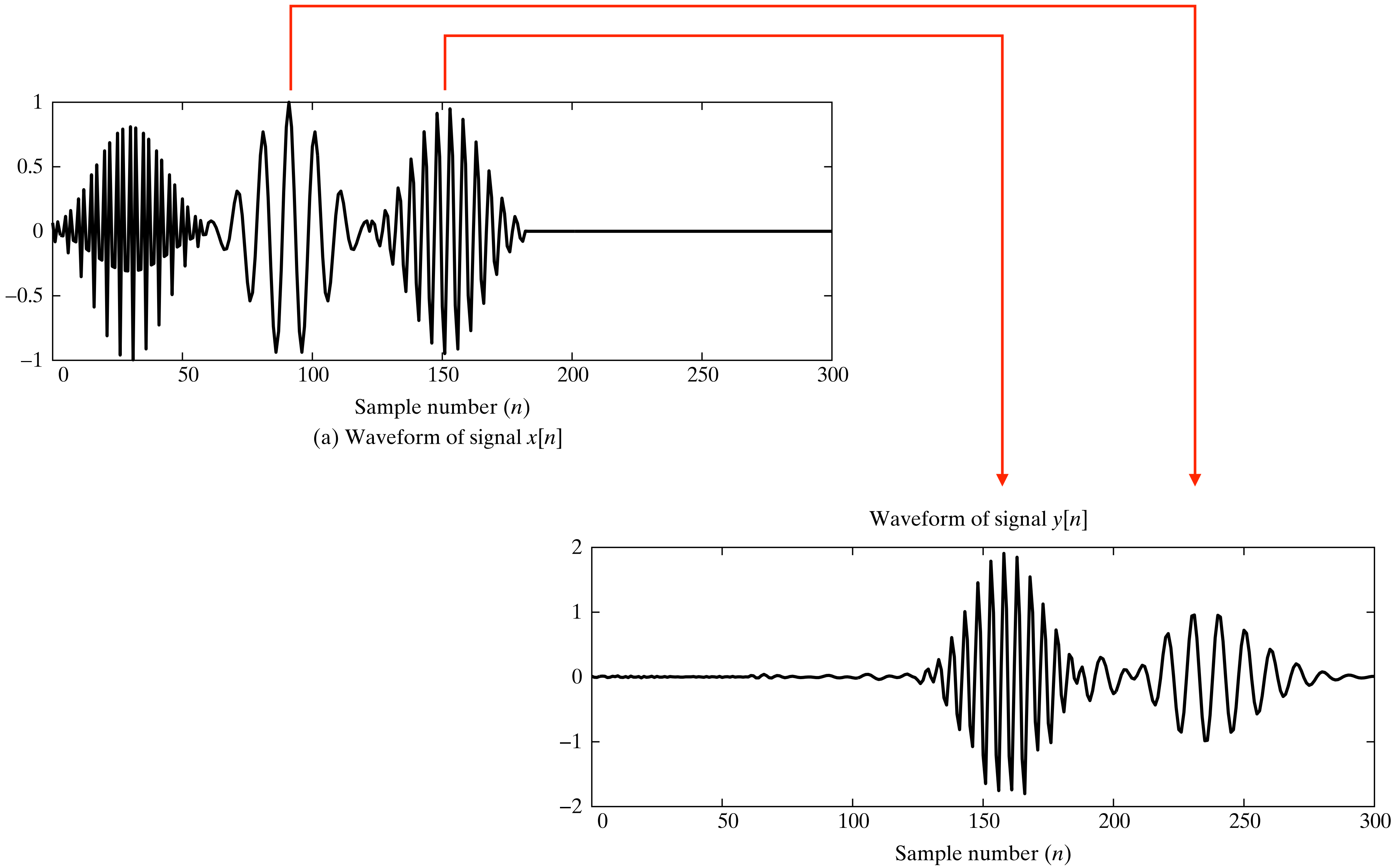


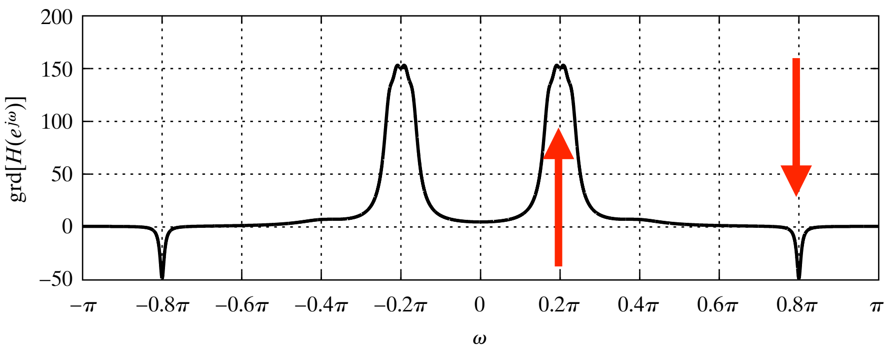
In an allpass filter, the zeros are the poles reflected in the unit circle.

Group Delay

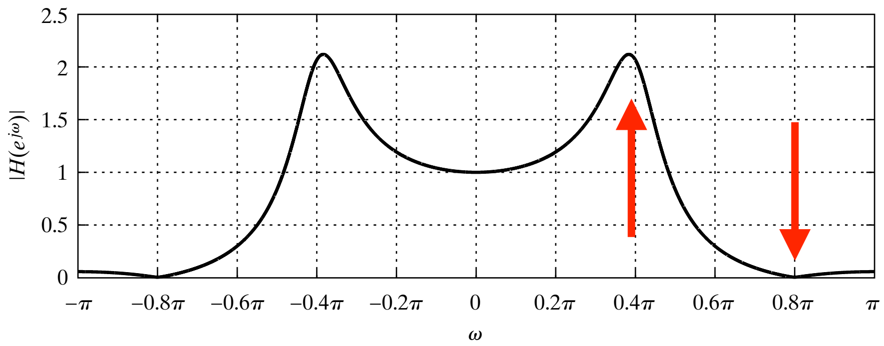
Group delay: $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$ = delay of the modulation envelope.



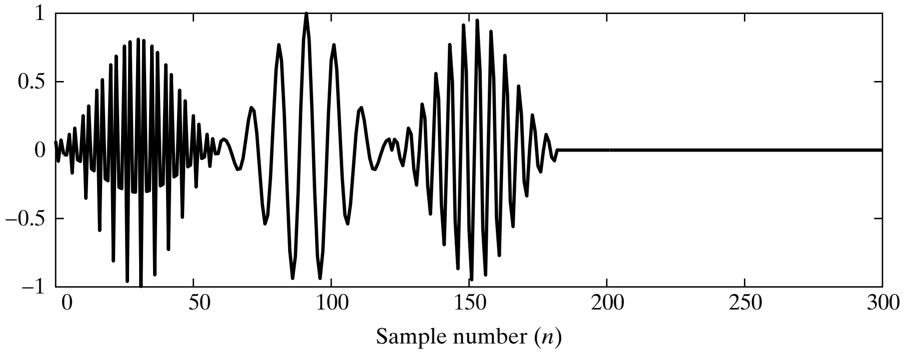
(a) Waveform of signal $x[n]$



(a) Group delay of $H(z)$



(b) Magnitude of Frequency Response



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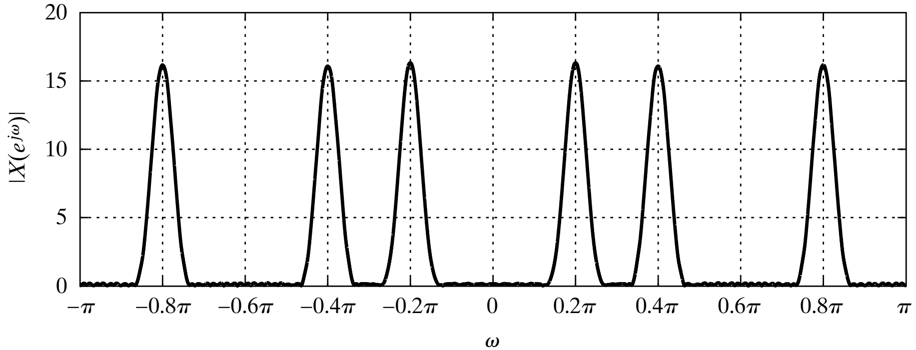
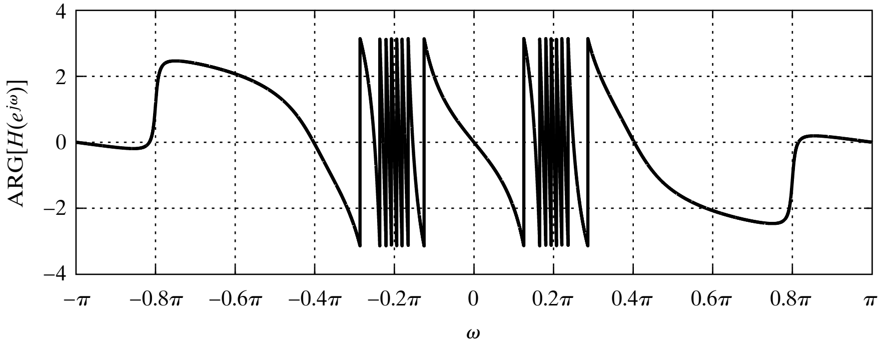
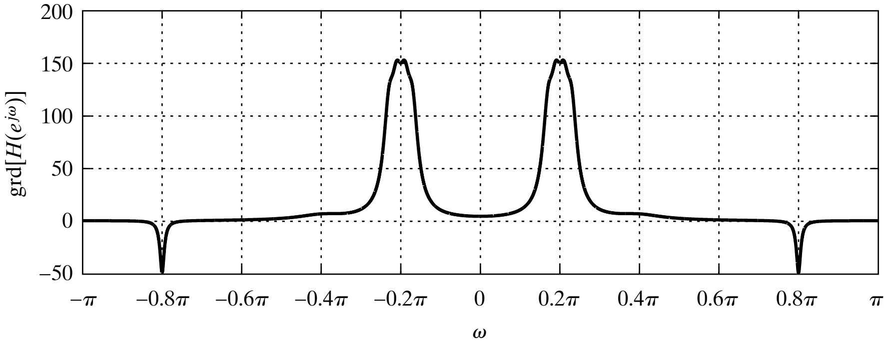


(b) Magnitude of DTFT of $x[n]$



(a) Principle Value of Phase Response



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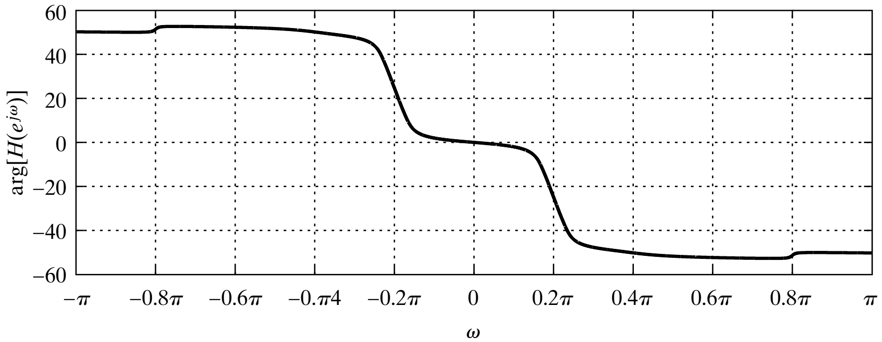
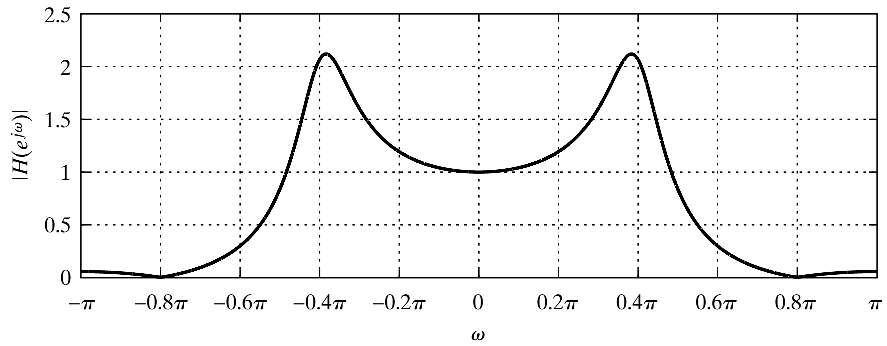
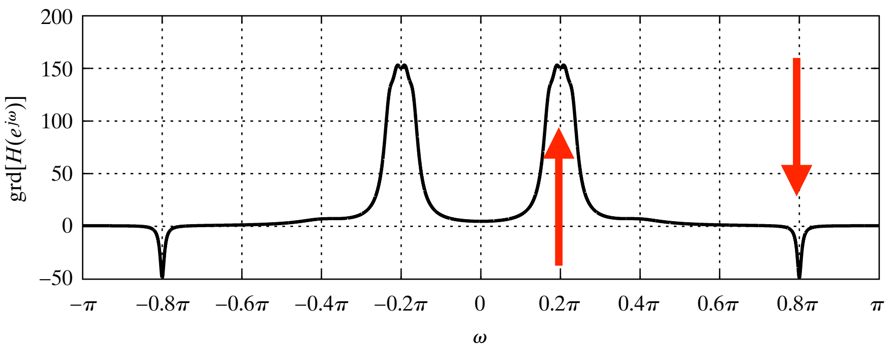


(b) Unwrapped Phase Response



(b) Magnitude of Frequency Response



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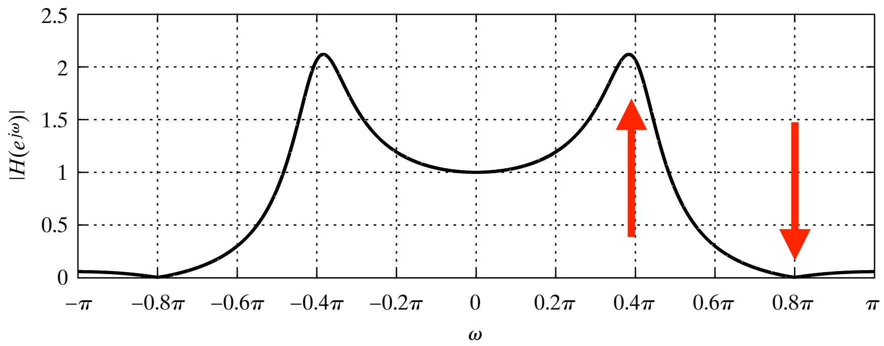
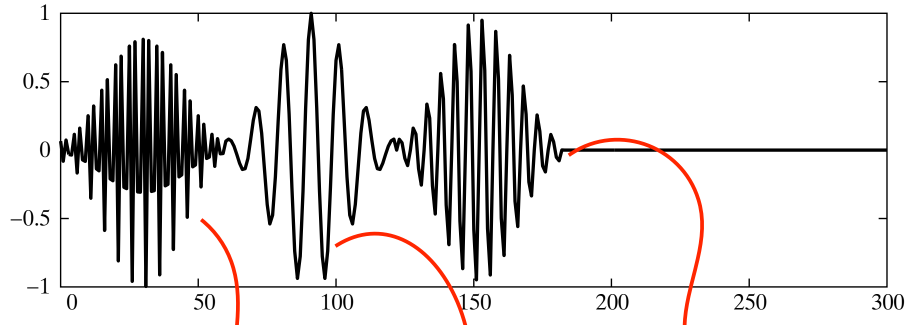


(b) Magnitude of Frequency Response



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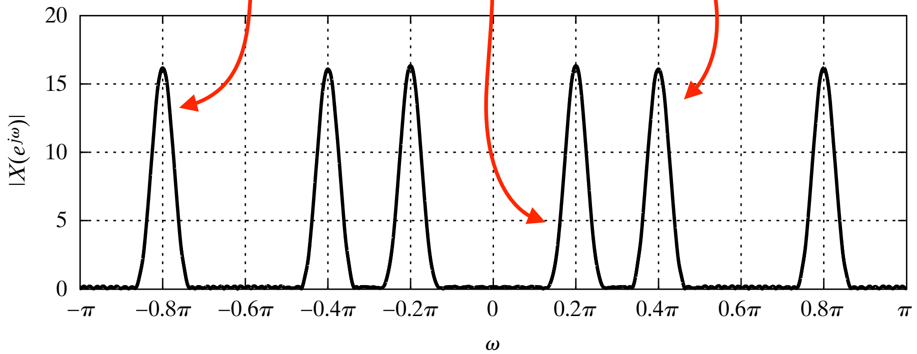
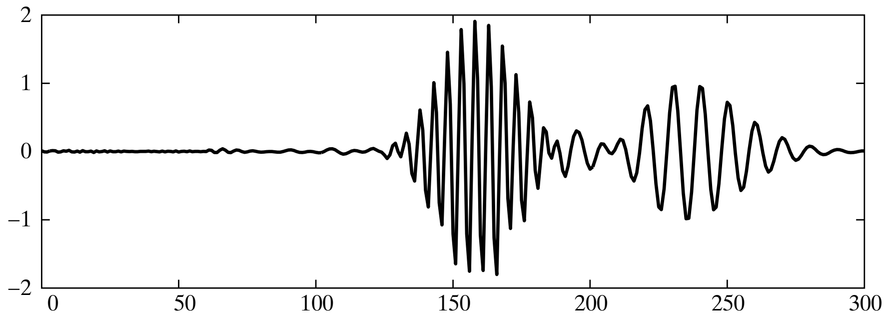


(b) Magnitude of DTFT of $x[n]$

Waveform of signal $y[n]$



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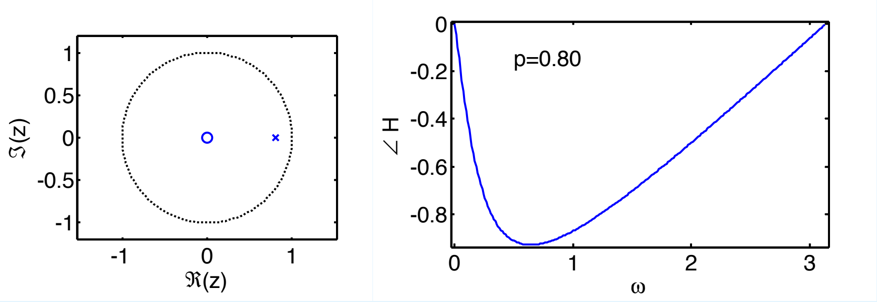

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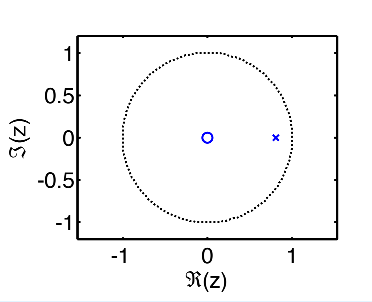

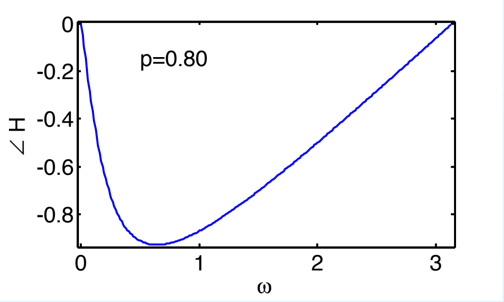

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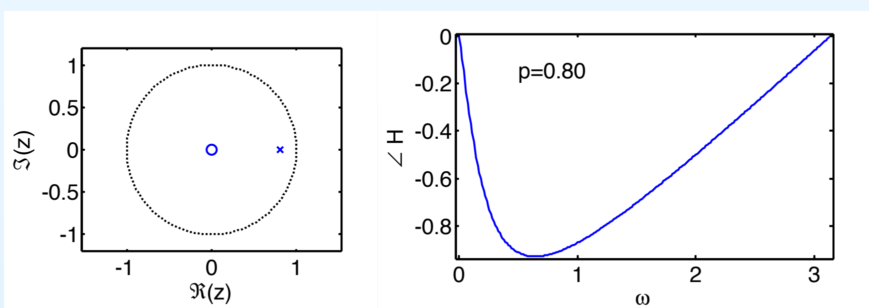

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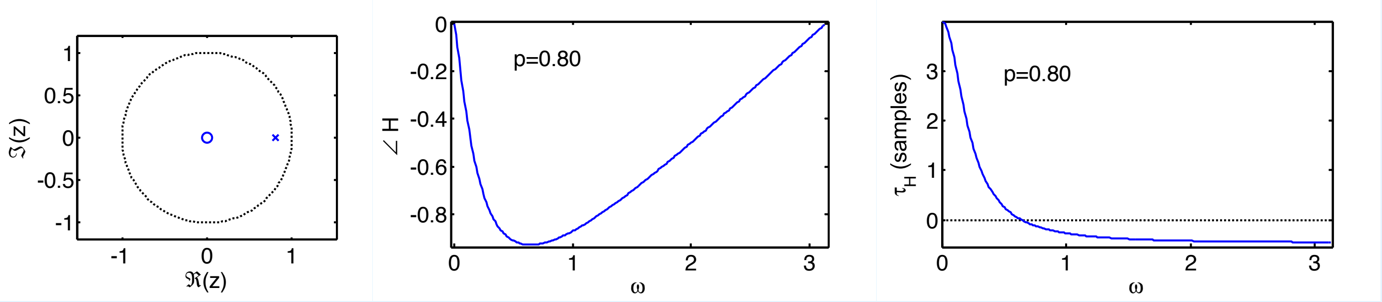

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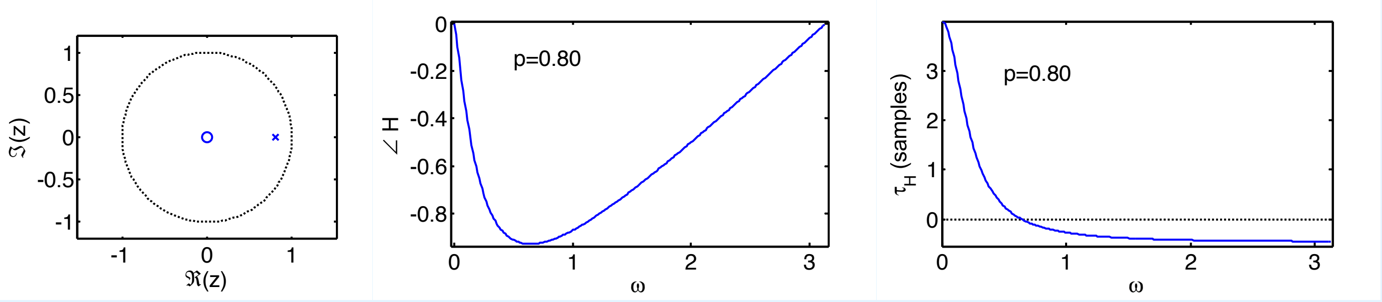

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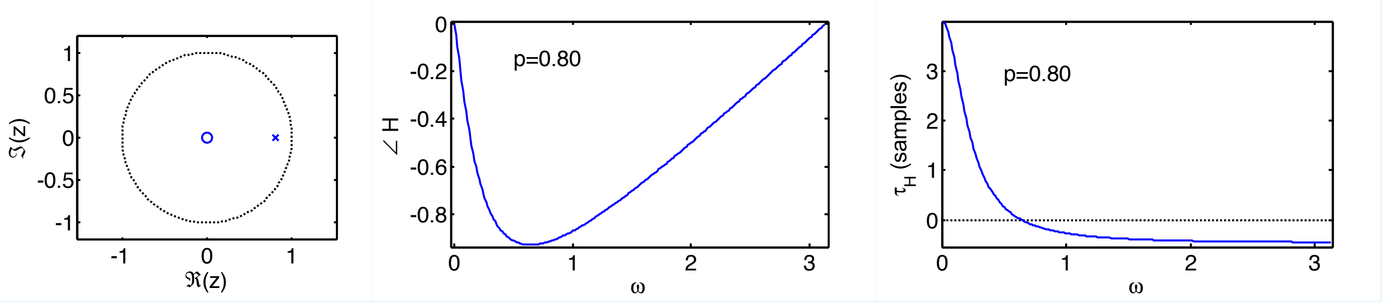

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Zeros on the unit circle count –½

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   Finally, the group delay simplifies to,

   $$\tau_H(\omega) = \mathrm{Re}\left(\frac{q}{e^{\jmath\omega} - q}\right) = \frac{q\cos(\omega) - q^2}{1 + q^2 - 2q\cos(\omega)}.$$

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$$\tau_H = \mathrm{Re}\left(-\frac{z}{H(z)}\partial_z H(z)\right) \text{ at } z = e^{\jmath\omega}$$

   and in the above (using derivative chain rule),

$$\partial_z H(z) = -\frac{pq}{\left(1 - qz^{-1}\right)^2 z^2} \xrightarrow{\text{Simplification}} -\frac{z}{H(z)}\partial_z H(z) = \frac{q}{z - q}.$$

   Finally, the group delay simplifies to,

$$\tau_H(\omega) = \mathrm{Re}\left(\frac{q}{e^{\jmath\omega} - q}\right) = \frac{q\cos(\omega) - q^2}{1 + q^2 - 2q\cos(\omega)}.$$

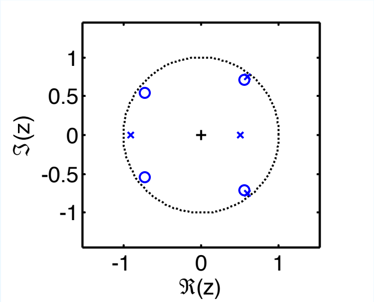
   We are given, $\tau_H(\pi) = -\frac{1}{3}$. We use this equation to solve for q,

$$\tau_H(\pi) = -\frac{q}{1 + q} = -\frac{1}{3} \Rightarrow q = \frac{1}{2}.$$

   Hence, $p = 2$.

Minimum Phase

Average group delay (over ω) = (# poles – # zeros) within the unit circle



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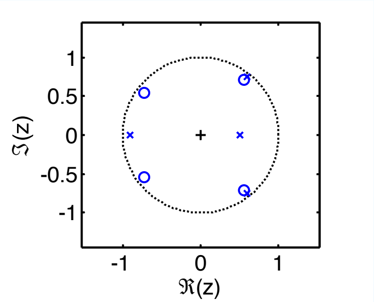

Minimum Phase

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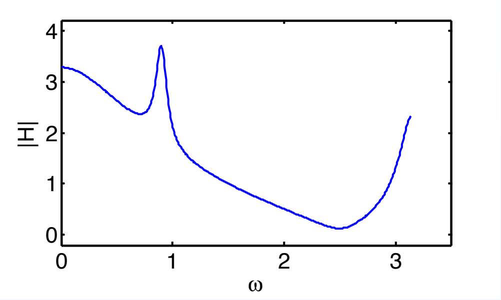

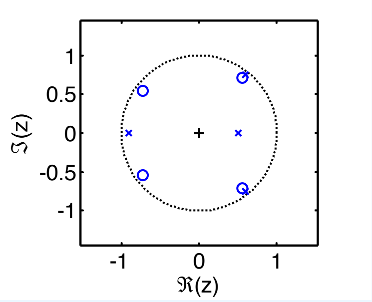

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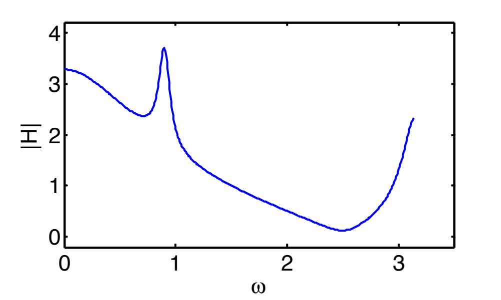

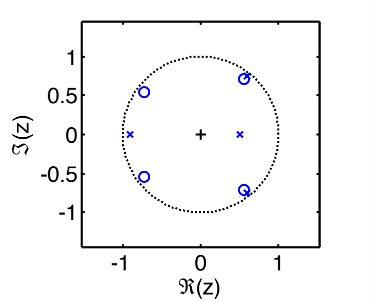

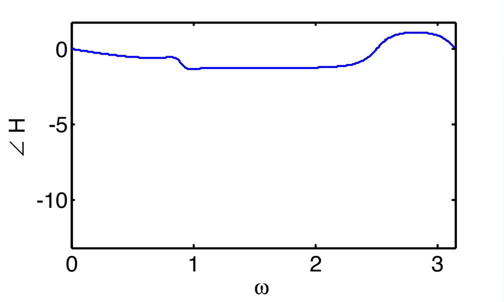

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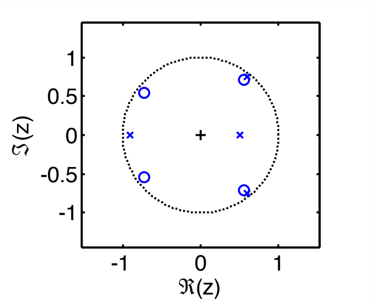

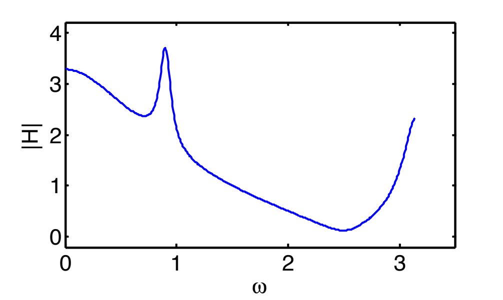

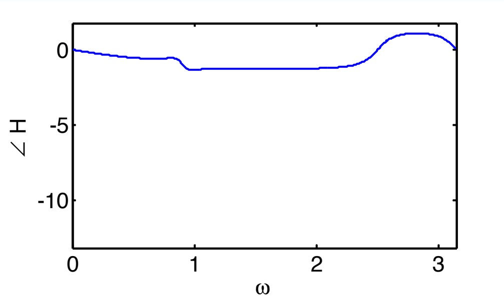

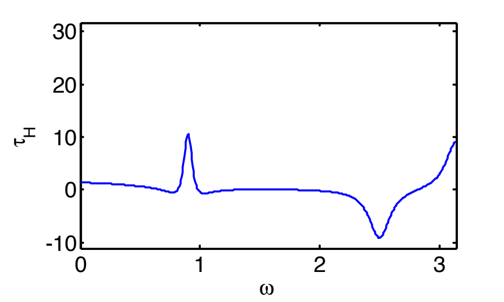

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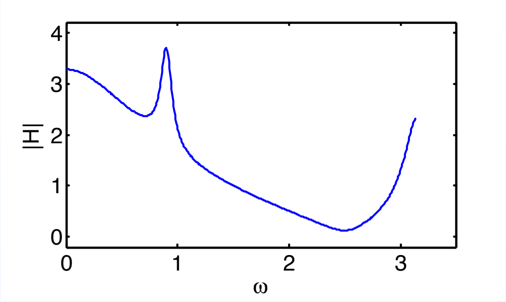

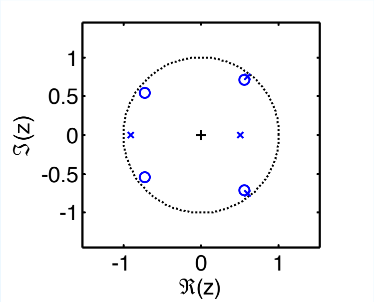

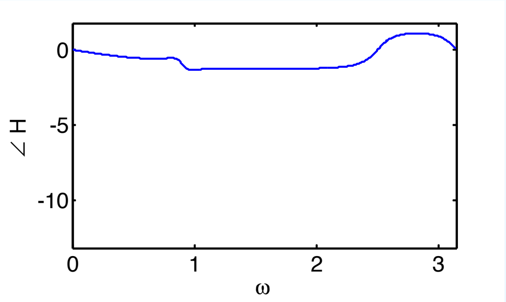

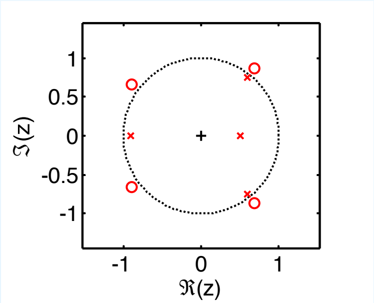

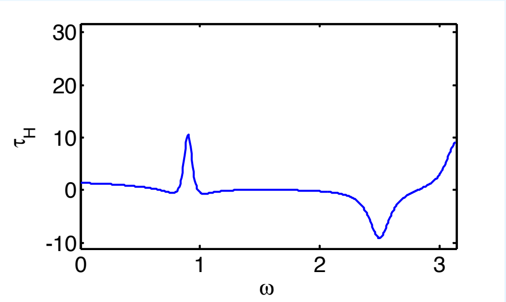

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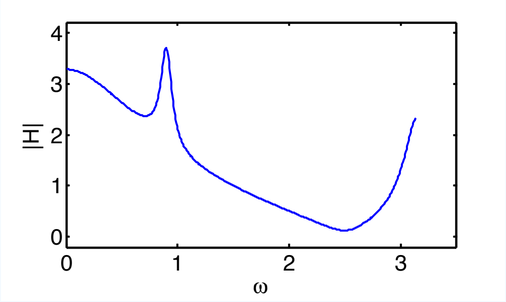

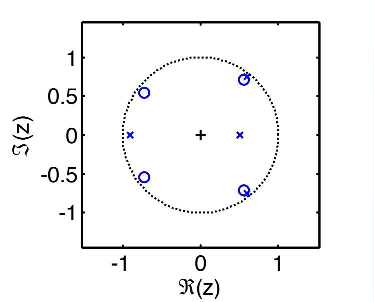

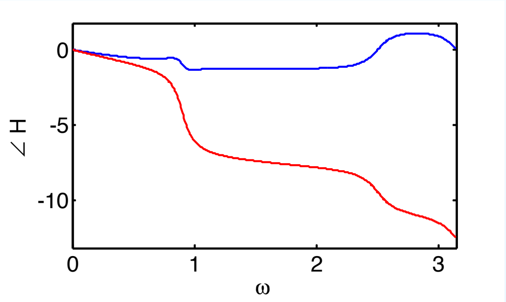

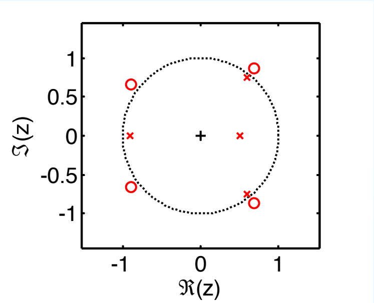

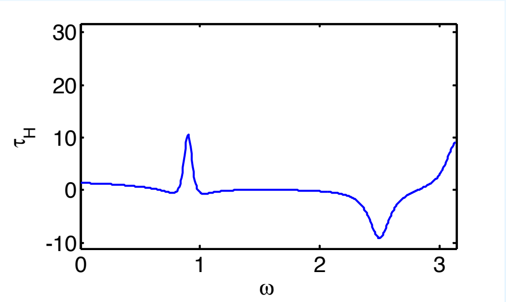

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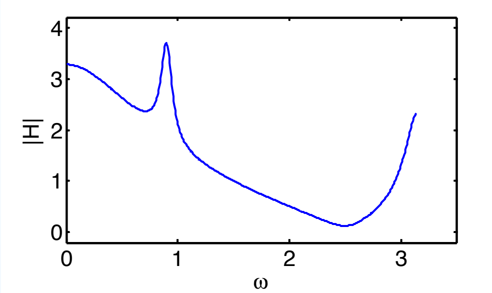

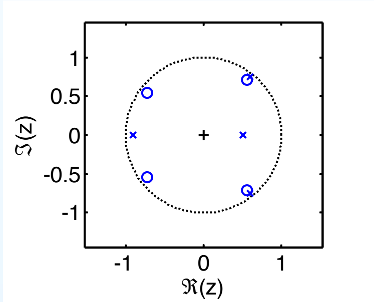

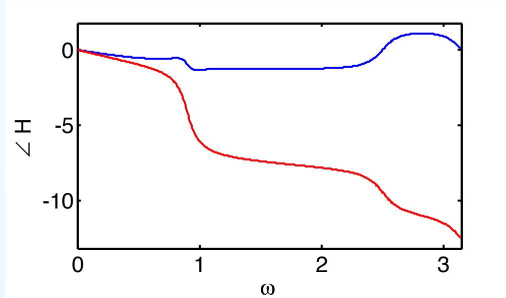

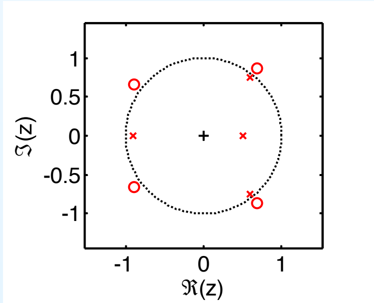

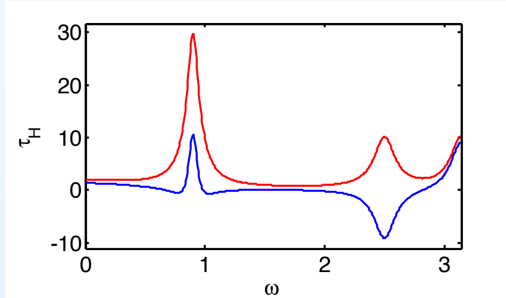

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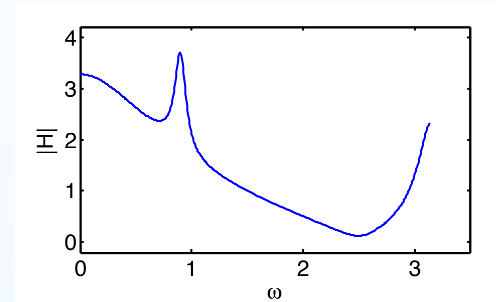
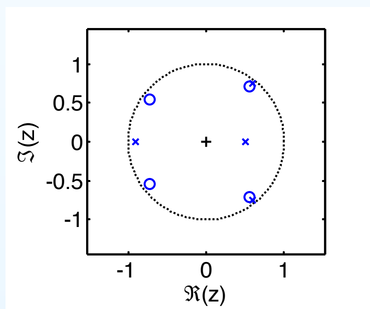
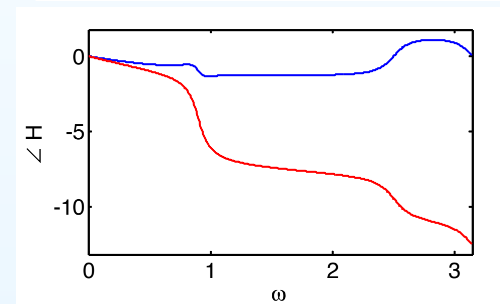
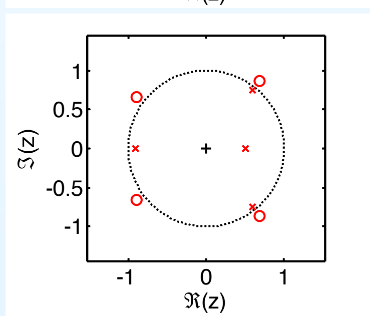
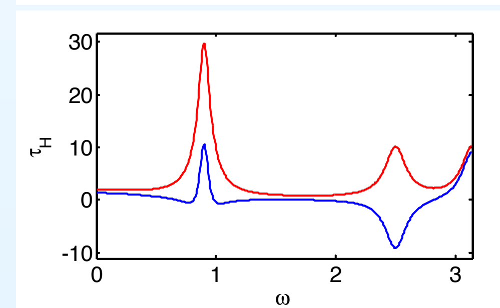
Reflecting an interior zero to the exterior multiplies $\left|H(e^{j\omega})\right|$ by a constant but increases average group delay by 1 sample.











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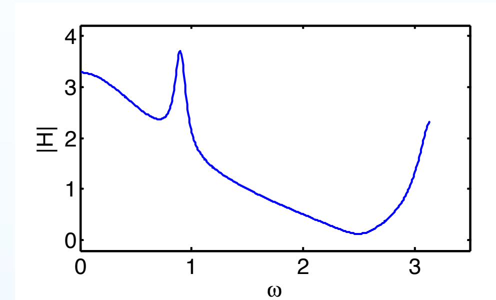

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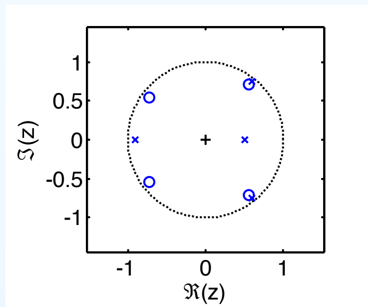

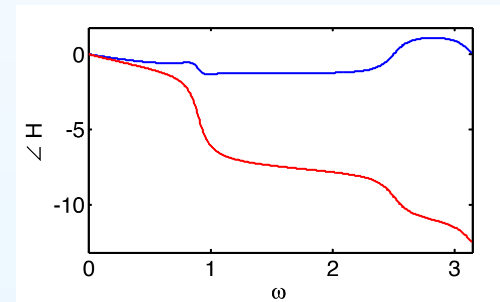

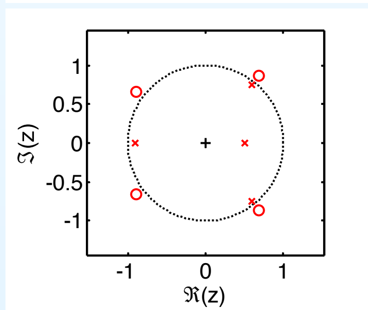

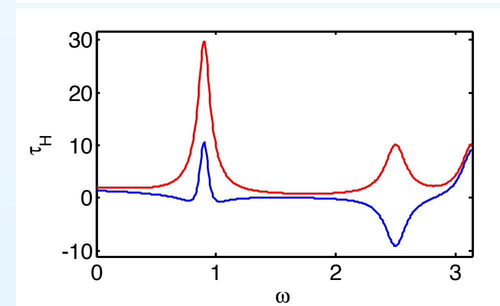

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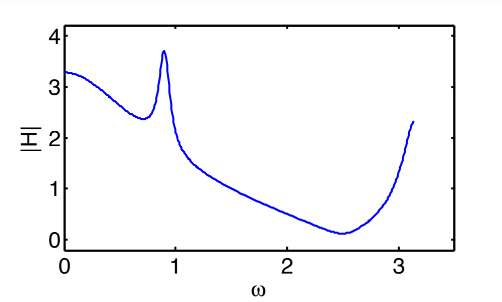

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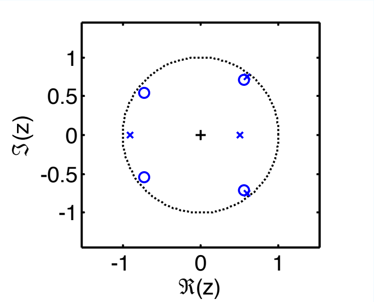

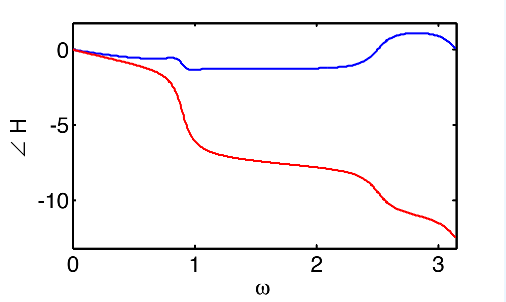

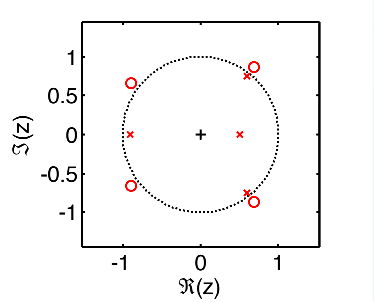

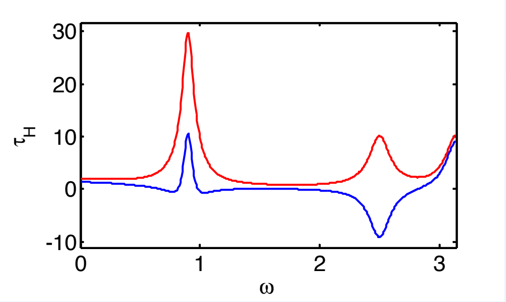

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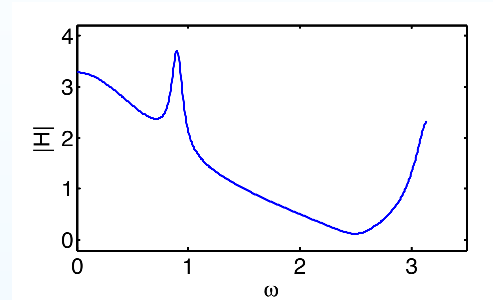

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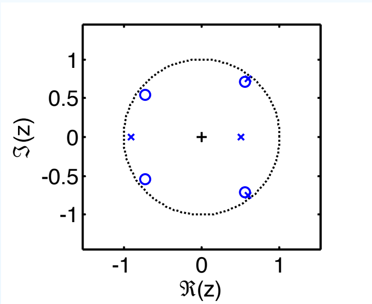

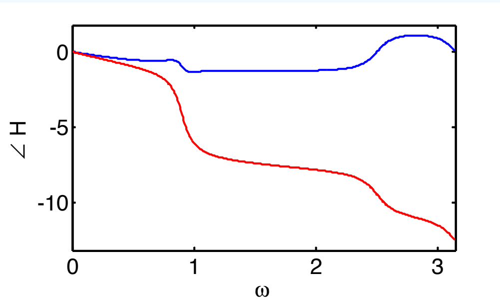

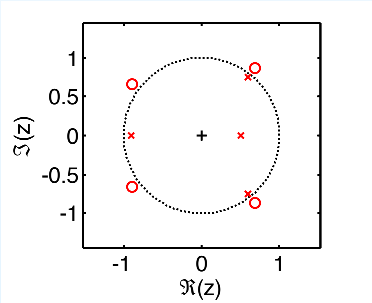

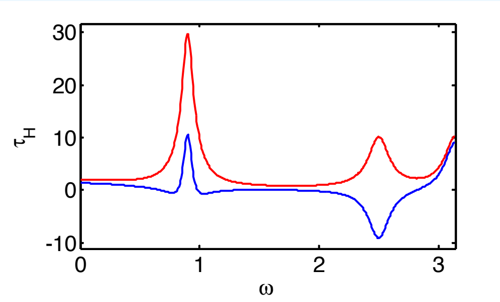

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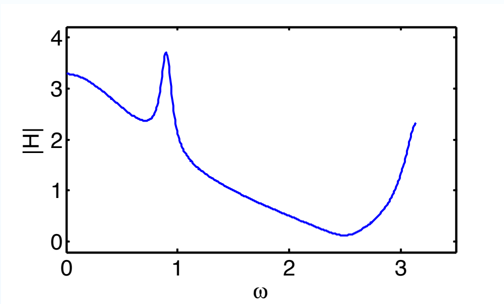

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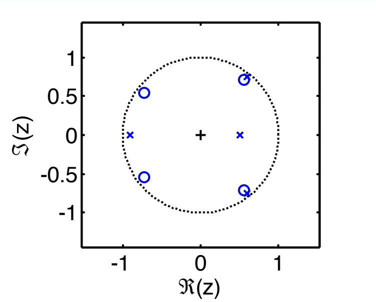

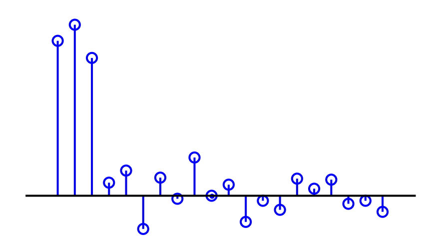

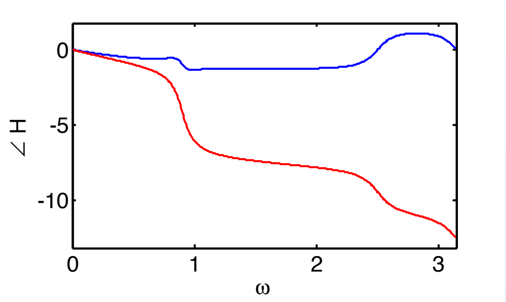

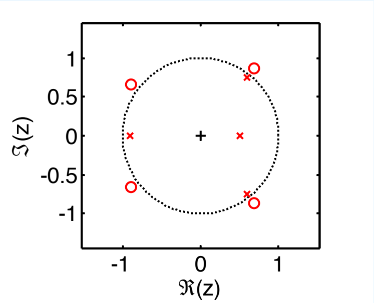

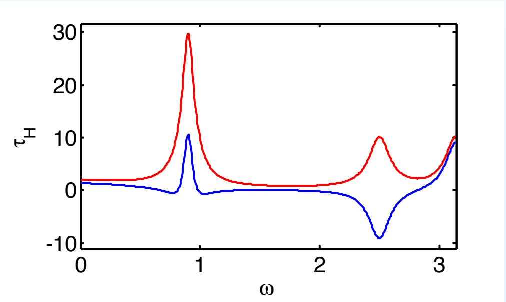

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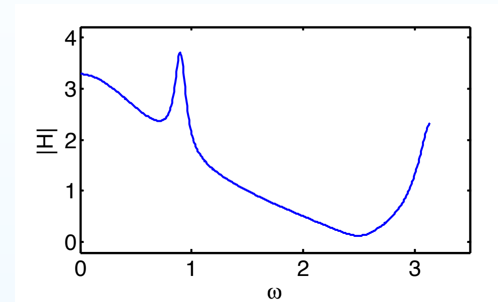

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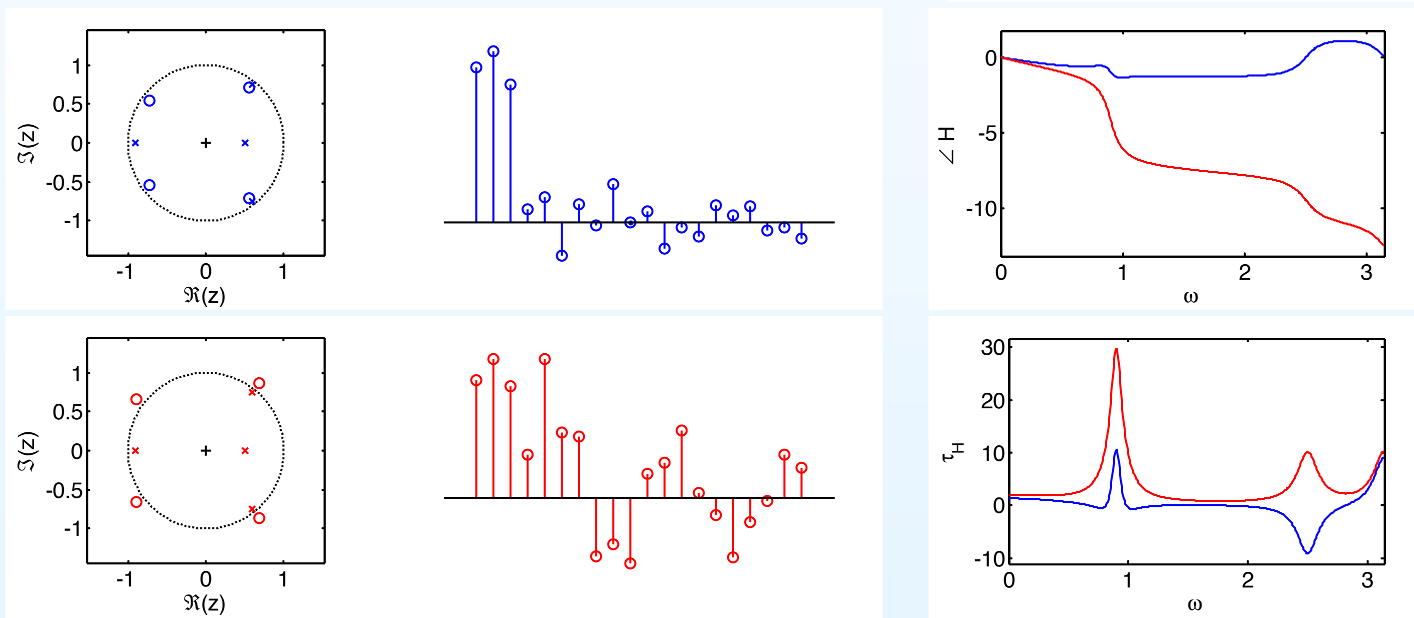

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Proof $\Leftarrow$:

$$2H(e^{j\omega}) = \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)}$$

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$$\begin{aligned} 2H(e^{j\omega}) &= \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)} \\ &= e^{-j\omega\frac{M}{2}} \sum_0^M h[n]e^{-j\omega\left(n-\frac{M}{2}\right)} + h[M-n]e^{j\omega\left(n-\frac{M}{2}\right)} \end{aligned}$$

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$$= e^{-j\omega\frac{M}{2}} \sum_0^M h[n]e^{-j\omega\left(n-\frac{M}{2}\right)} + h[M-n]e^{j\omega\left(n-\frac{M}{2}\right)}$$

$h[n]$ symmetric:

$$2H(e^{j\omega}) = 2e^{-j\omega\frac{M}{2}} \sum_0^M h[n]\cos\left(n-\frac{M}{2}\right)\omega$$

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Equivalently constant group delay: $\tau_H = -\frac{d\angle H(e^{j\omega})}{d\omega} = \alpha$

A filter has linear phase iff $h[n]$ is symmetric or antisymmetric:

$$h[n] = h[M-n] \quad \forall n \text{ or else } h[n] = -h[M-n] \quad \forall n$$

M can be even ($\Rightarrow \exists$ mid point) or odd ($\Rightarrow \nexists$ mid point)

Proof $\Leftarrow$:

$$\begin{aligned} 2H(e^{j\omega}) &= \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)} \\ &= e^{-j\omega\frac{M}{2}} \sum_0^M h[n]e^{-j\omega\left(n-\frac{M}{2}\right)} + h[M-n]e^{j\omega\left(n-\frac{M}{2}\right)} \end{aligned}$$

$h[n]$ symmetric:

$$2H(e^{j\omega}) = 2e^{-j\omega\frac{M}{2}} \sum_0^M h[n]\cos\left(n-\frac{M}{2}\right)\omega$$

$h[n]$ anti-symmetric:

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For further details see Mitra: 6, 7.

MATLAB routines

filter	filter a signal
impz	Impulse response
residuez	partial fraction expansion
grpdelay	Group Delay
freqz	Calculate filter frequency response