Information for students

This coursework is intended to be a sample exam paper.

It accounts for 15% of the mark for this course.

Deadline: 4:00pm, Friday 18 December, 2020. Pease submit a PDF copy to Blackboard.

Do not submit the MATLAB codes.

The Questions

1. Random variables.

a) The random variable X has a Gaussian distribution with mean 5 and standard deviation 2, and $Y = 2X + 4$. Find the mean, standard deviation and probability density function of Y.

[5]

b) Three-envelope puzzle. There are three sealed envelopes, one of which containing a check of £1m. You bet one envelope contains the check. Then I open another envelope, showing there is no check inside. Using Bayes' theorem, find the a posteriori probabilities, and determine whether you will stick to envelope you've chosen or switch to the third one. (Students are encouraged to run this experiment yourself.)

[5]

c) X and Y are independent, identically distributed (i.i.d.) random variables with common probability density function

$$f_X(x) = e^{-x}, \qquad x > 0$$

$$f_Y(y) = e^{-y}, \qquad y > 0$$

Find the probability density function of the following random variables:

i) $Z = X + Y$. [5]

ii) $Z = \min(X, Y)$. [5]

iii) $Z = \max(X, Y)$. [5]

2. Estimation and sequences of random variables.

a) The random variable X has the density $f(x) \sim c^4 x^3 e^{-cx}, \;\; x > 0$. We observe the i.i.d. samples x_i = 3.1, 3.4, 3.3. Find the maximum-likelihood estimate of parameter c.

[8]

b) The random variable X_i are i.i.d. and uniform in the interval (0, 1). Show that if $Y = \max X_i$ then the distribution function $F(y) = y^n$ for $0 \le y \le 1$.

[8]

c) If the autocorrelation function $R_S(\tau) = I \cdot e^{-|\tau|/T}$ where I is a constant and the linear MMSE estimate of $S(t - T/2)$ is given by $aS(t) + bS(t - T)$. Find the coefficients a and b and the corresponding mean-square error.

[9]

3. Random processes.

a) In the fair-coin experiment, we define the random process $X(t)$ as follows:

$X(t) = \sin \pi t$ if head shows;
$X(t) = 2t$ if tail shows.

i) Find the mean $E[X(t)]$. [2]

ii) Find the probability distribution function $F(x, t)$ for $t = 0.25$, $t = 0.5$, and $t = 1$. [3]

b) The random process $X(t)$ is real with autocorrelation $R(\tau)$.

i) Show that

$$P\{|X(t+\tau) - X(t)| \geq a\} \leq 2[R(0) - R(\tau)]/a^2.$$ [3]

ii) Express $P\{|X(t+\tau) - X(t)| \geq a\}$ in terms of the second order density $f(x_1, x_2; \tau)$ of $X(t)$. [2]

c) Suppose the wide-sense stationary random process $X(t)$ with $R_x(\tau) = 5\delta(\tau)$ passes through a linear system

$$Y'(t) + 2Y(t) = X(t)$$

Find $E[Y^2(t)], R_{xy}(t_1, t_2), R_{yy}(t_1, t_2)$ if the above equation holds for all t. [7]

d) Stochastic resonance. Suppose the input to the system with transfer function

$$H(s) = \frac{1}{s^2+2s+5}$$

is a wide-sense stationary process $X(t)$ with $E[X^2(t)] = 10$. Find the input power spectral density $S_x(\omega)$ such that the average power $E[Y^2(t)]$ of the resulting output $Y(t)$ is maximum.

[8]

4. Random processes and Markov chains.

a) Fix a parameter $\lambda \in (0, 1)$ and let $X_0, X_1, X_2, \dots$ be a sequence of independent random variables, whose distribution satisfies $P(X_j = -1) = P(X_j = 1) = 1/2$. Consider the following random sequence

$$Y_n = \sum_{i=0}^{n} X_i \lambda^i, \qquad n = 0,1,2,3,\dots$$

i) Show $\{Y_n\}$ is a martingale. [3]

ii) Derive the characteristic function of Y_n. [4]

iii) Show that for any set E

$$P(Y_{n+1} \in E) = \frac{1}{2} P(Y_n \in T_1^{-1}(E)) + \frac{1}{2} P(Y_n \in T_2^{-1}(E))$$

where $T_1(x) = \lambda x + 1$ and $T_2(x) = \lambda x - 1$. [4]

iv) What is the limiting distribution of Y_n as $n \to \infty$ if $\lambda = 1/2$? [4]

b) Consider the random walk with left barrier as in Lecture 9 with infinite state space $E = \{0,1,2,\dots\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & & & & 0 \\ q & 0 & p & & & \\ & q & 0 & p & & \\ & & . & . & . & \\ 0 & & . & . & . & \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. Write a computer program to simulate the random walk and show the realizations of $X(t)$ as a function of t, for

i) $p = 1/3$; [3]

ii) $p = 1/2$; [3]

iii) $p = 2/3$. [3]

iv) Discuss your findings. [3]

[Only show the simulation results as in Lecture 9 (not the program). Obviously, such a question cannot be tested in this way in the exam!]