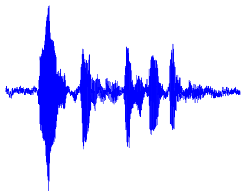
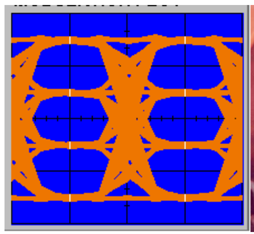
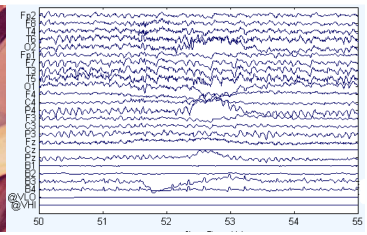
1: Introduction

Signals

- A signal is a numerical quantity that is a function of one or more independent variables such as time or position.
- Real-world signals are analog and vary continuously and take continuous values.
- Digital signals are sampled at discrete times and are quantized to a finite number of discrete values
- We will mostly consider one-dimensionsal real-valued signals with regular sample instants; except in a few places, we will ignore the quantization.
    - Extension to multiple dimensions and complex-valued signals is straighforward in many cases.

Examples:



Processing

- Aims to "improve" a signal in some way or extract some information from it
- Examples:
  - Modulation/demodulation
  - Coding and decoding
  - Interference rejection and noise suppression
  - Signal detection, feature extraction
- We are concerned with linear, time-invariant processing

Syllabus

Main topics:

- Introduction/Revision
- Transforms
- Discrete Time Systems
- Filter Design
  - FIR Filter Design
  - IIR Filter Design
- Multirate systems
  - Multirate Fundamentals
  - Multirate Filters
  - Subband processing

Sequences

We denote the n^{th} sample of a signal as $x[n]$ where $-\infty < n < +\infty$ and the entire sequence as $\{x[n]\}$ although we will often omit the braces.

Special sequences:

- Unit step: $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- Unit impulse: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$
- Condition: $\delta_{\text{condition}}[n] = \begin{cases} 1 & \text{condition is true} \\ 0 & \text{otherwise} \end{cases}$ (e.g. $u[n] = \delta_{n \geq 0}$)
- Right-sided: $x[n] = 0$ for $n < N_{min}$
- Left-sided: $x[n] = 0$ for $n > N_{max}$
- Finite length: $x[n] = 0$ for $n \notin [N_{min}, N_{max}]$
- Causal: $x[n] = 0$ for $n < 0$, Anticausal: $x[n] = 0$ for $n > 0$
- Finite Energy: $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ (e.g. $x[n] = n^{-1}u[n-1]$)
- Absolutely Summable: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow$ Finite energy

Time Scaling

For sampled signals, the n^{th} sample is at time $t = nT = \frac{n}{f_s}$ where $f_s = \frac{1}{T}$ is the sample frequency.

We usually scale time so that $f_s = 1$: divide all "real" frequencies and angular frequencies by f_s and divide all "real" times by T.

- To scale back to real-world values: multiply all *times* by T and all *frequencies* and *angular frequencies* by $T^{-1} = f_s$.
- We use Ω for "real" angular frequencies and ω for normalized angular frequency. The units of ω are "radians per sample".

Energy of sampled signal, $x[n]$, equals $\sum x^2[n]$

- Multiply by T to get energy of continuous signal, $\int x^2(t)dt$, provided there is no aliasing.

Power of $\{x[n]\}$ is the average of $x^2[n]$ in "energy per sample"

- same value as the power of $x(t)$ in "energy per second" provided there is no aliasing.

Warning: Several MATLAB routines scale time so that $f_s = 2$ Hz. Weird, non-standard and irritating.

z-Transform

The z-transform converts a sequence, $\{x[n]\}$, into a function, $X(z)$, of an arbitrary complex-valued variable z.

Why do it?

- Complex functions are easier to manipulate than sequences
- Useful operations on sequences correspond to simple operations on the z-transform:
  - addition, multiplication, scalar multiplication, time-shift, convolution
- Definition: $X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$

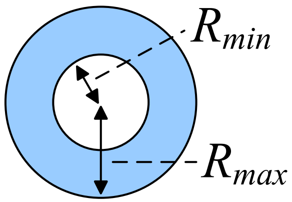
The set of z for which $X(z)$ converges is its *Region of Convergence* (ROC).

Complex analysis $\Rightarrow$: the ROC of a power series (if it exists at all) is always an annular region of the form $0 \leq R_{min} < |z| < R_{max} \leq \infty$.

$X(z)$ will always converge absolutely inside the ROC and may converge on some, all, or none of the boundary.

- "converge absolutely" $\Leftrightarrow \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| < \infty$

- finite length $\Leftrightarrow R_{min} = 0,\ R_{max} = \infty$
  - ROC may included either, both or none of 0 and ∞
- absolutely summable $\Leftrightarrow X(z)$ converges for $|z| = 1$.
- right-sided & $|x[n]| < A \times B^n \ \Rightarrow R_{max} = \infty$
  - $+$ causal $\Rightarrow X(\infty)$ converges
- left-sided & $|x[n]| < A \times B^{-n} \ \Rightarrow R_{min} = 0$
  - $+$ anticausal $\Rightarrow X(0)$ converges



[Convergence Properties]

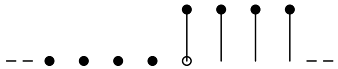
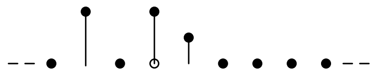
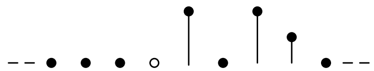
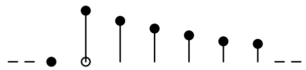
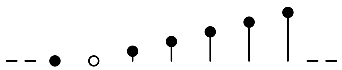
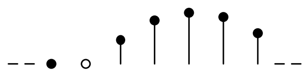
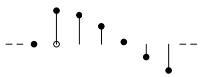
Null Region of Convergence:
It is possible to define a sequence, $x[n]$, whose z-transform never converges (i.e. the ROC is null). An example is $x[n] \equiv 1$. The z-transfom is $X(z) = \sum z^{-n}$ and it is clear that this fails to converge for any real value of z.

Convergence for $x[n]$ causal:
If $x[n]$ is causal with $|x[n]| < A \times B^n$ for some A and B, then $|X(z)| = \left|\sum_{n=0}^{\infty} x[n]z^{-n}\right| \leq \sum_{n=0}^{\infty} \left|x[n]z^{-n}\right|$ and so, for $|z| = R \geq B$, $|X(z)| \leq \sum_{n=0}^{\infty} AB^n R^{-n} = \frac{A}{1-BR^{-1}} < \infty$.

Convergence for $x[n]$ right-sided:
If $x[n]$ is right-sided with $|x[n]| < A \times B^n$ for some A and B and $x[n] = 0$ for $n < N$, then $y[n] = x[n-N]$ is causal with $|y[n]| < A \times B^{n+N} = AB^N \times B^n$. Hence, from the previous result, we known that $Y(z)$ converges for $|z| \geq B$. The z-transform, $X(z)$, is given by $X(z) = z^N Y(z)$ so $X(z)$ will converge for any $B \leq |z| < \infty$ since $\left|z^N\right| < \infty$ for $|z|$ in this range.

z-Transform examples

The sample at $n = 0$ is indicated by an open circle.

$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \leq \infty$
$-\alpha^n u[-n-1]$		$\frac{1}{1-\alpha z^{-1}}$	$0 \leq \lvert z \rvert < \alpha$
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \leq \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \leq \infty$
$\cos(\omega n)u[n]_{\omega=0.5}$		$\frac{1-z^{-1}\cos(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \leq \infty$

Note: Examples 4 and 5 have the same z-transform but different ROCs.

Geometric Progression: $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q} - \alpha^{r+1} z^{-r-1}}{1-\alpha z^{-1}}$

Rational z-Transforms

Most z-transforms that we will meet are rational polynomials with real coefficients, usually one polynomial in z^{-1} divided by another.

$$G(z) = g\frac{\prod_{m=1}^{M}\left(1-z_m z^{-1}\right)}{\prod_{k=1}^{K}\left(1-p_k z^{-1}\right)} = gz^{K-M}\frac{\prod_{m=1}^{M}(z-z_m)}{\prod_{k-1}^{K}(z-p_k)}$$

Completely defined by the poles, zeros and gain.

The absolute values of the poles define the ROCs:

$\exists R+1$ different ROCs

where R is the number of distinct pole magnitudes.

Note: There are $K-M$ zeros or $M-K$ poles at $z=0$ (easy to overlook)

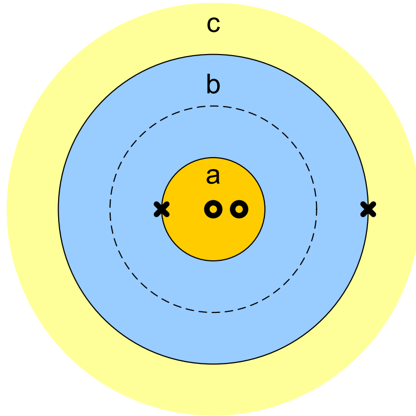
Rational example

$G(z) = \frac{8-2z^{-1}}{4-4z^{-1}-3z^{-2}}$

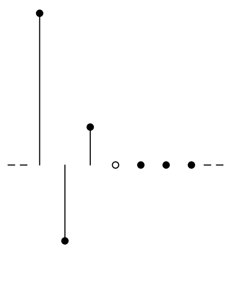
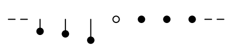
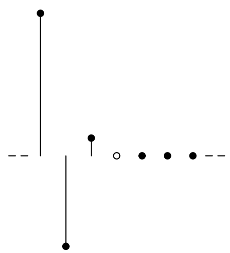
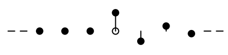
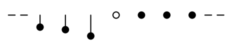
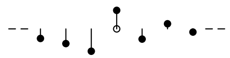
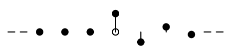
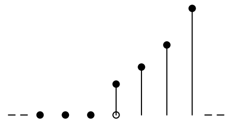
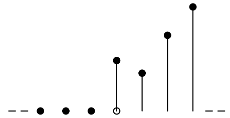
Poles/Zeros: $G(z) = \frac{2z(z-0.25))}{(z+0.5)(z-1.5)}$

$\Rightarrow$ Poles at $z = \{-0.5, +1.5)\}$,

Zeros at $z = \{0, +0.25\}$



Partial Fractions: $G(z) = \frac{0.75}{1+0.5z^{-1}} + \frac{1.25}{1-1.5z^{-1}}$

ROC	ROC	$\frac{0.75}{1+0.5z^{-1}}$	$\frac{1.25}{1-1.5z^{-1}}$	$G(z)$
a	$0 \le \lvert z \rvert < 0.5$			
b	$0.5 < \lvert z \rvert < 1.5$			
c	$1.5 < \lvert z \rvert \le \infty$			

$g[n] = \frac{1}{2\pi j} \oint G(z) z^{n-1} dz$ where the integral is anti-clockwise around a circle within the ROC, $z = Re^{j\theta}$.

Proof:

$$\begin{aligned}\frac{1}{2\pi j} \oint G(z) z^{n-1} dz &= \frac{1}{2\pi j} \oint \left(\sum_{m=-\infty}^{\infty} g[m] z^{-m}\right) z^{n-1} dz \\ &\overset{\text{(i)}}{=} \sum_{m=-\infty}^{\infty} g[m] \frac{1}{2\pi j} \oint z^{n-m-1} dz \\ &\overset{\text{(ii)}}{=} \sum_{m=-\infty}^{\infty} g[m] \delta[n-m] = g[n]\end{aligned}$$

(i) depends on the circle with radius R lying within the ROC

(ii) Cauchy's theorem: $\frac{1}{2\pi j} \oint z^{k-1} dz = \delta[k]$ for $z = Re^{j\theta}$ anti-clockwise.

$$\frac{dz}{d\theta} = jRe^{j\theta} \Rightarrow \frac{1}{2\pi j} \oint z^{k-1} dz = \frac{1}{2\pi j} \int_{\theta=0}^{2\pi} R^{k-1} e^{j(k-1)\theta} \times jRe^{j\theta} d\theta$$

$$= \frac{R^k}{2\pi} \int_{\theta=0}^{2\pi} e^{jk\theta} d\theta$$

$$= R^k \delta(k) = \delta(k) \qquad [R^0 = 1]$$

In practice use a combination of partial fractions and table of z-transforms.

tf2zp,zp2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \{z_m, p_k, g\}$
residuez	$\frac{b(z^{-1})}{a(z^{-1})} \rightarrow \sum_k \frac{r_k}{1-p_k z^{-1}}$
tf2sos,sos2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2sos,sos2zp	$\{z_m, p_k, g\} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{\in 1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2ss,ss2zp	$\{z_m, p_k, g\} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
tf2ss,ss2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$

Summary

- Time scaling: assume $f_s = 1$ so $-\pi < \omega \leq \pi$
- z-transform: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]^{-n}$
- ROC: $0 \leq R_{min} < |z| < R_{max} \leq \infty$
  - Causal: $\infty \in$ ROC
  - Absolutely summable: $|z| = 1 \in$ ROC
- Inverse z-transform: $g[n] = \frac{1}{2\pi j} \oint G(z) z^{n-1} dz$
  - Not unique unless ROC is specified
  - Use partial fractions and/or a table

For further details see Mitra:1 & 6.