Digital Signal Processing and Digital Filters

Imperial College London

Practice Sheet 6

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The purpose of the practice sheet is to enhance the understanding of the course materials. The practice sheet does not constitute towards the final grade. Students are welcome to discuss the problems amongst themselves. The questions will be discussed in the Q&A sessions with the instructor.

1) Let $h[n]$ be an ideal low pass filter given by

$$h[n] = \frac{\sin(0.4\pi(n-N))}{\pi(n-N)}. \tag{1}$$

(a) Compute its frequency response function $H(e^{j\omega})$.

The system $h[n]$ is a low-pass filter with bandwidth $\omega_0 = 0.4\pi$, and therefore

$$H(e^{j\omega}) = \begin{cases} e^{-jN\omega} & |\omega| \le 0.4\pi \\ 0 & 0.4\pi < |\omega| \le \pi. \end{cases} \tag{2}$$

(b) Calculate $|H(e^{j\omega_0})|$ and $\angle H(e^{j\omega_0})$ for $\omega_0 = 63.6$.

Clearly, ω_0 is outside $[-\pi, \pi]$. Using the periodicity of $H(e^{j\omega})$, we aim to solve

$$63.6 = 2N\pi + x, \quad \text{s.t. } x \in [-\pi, \pi], \tag{3}$$

which is satisfied for $N = 10$ and $x = 0.768$ rad/2. Given that $x < 0.4\pi$, we have that $H(e^{j\omega_0}) = e^{-jN \cdot 0.768}$. Then $|H(e^{j\omega_0})| = 1$ and $\angle H(e^{j\omega_0}) = -0.768N$.

(c) Design a window with a length of 20 sampling times (containing 21 samples) for $h[n]$ and choose N that would guarantee a linear phase response. Write the frequency response of the filter.

The impulse response has the form

$$h[n] = \frac{\sin[0.4\pi(n-N)]}{\pi(n-N)} \cdot w[n], \tag{4}$$

where $w[n] = 1_{[0,20]}[n]$. The rectangular window has length 20, and therefore we choose $N = 10$. The frequency response of $w[n]$ is $W(e^{j\omega}) = e^{-10j\omega}\frac{\sin(10.5\omega)}{\sin(0.5\omega)}$, which yields

$$H(e^{j\omega}) = 1_{[-0.4\pi, 0.4\pi]}(\omega) \cdot e^{-j20\omega} \circledast e^{-10j\omega}\frac{\sin(10.5\omega)}{\sin(0.5\omega)}, \omega \in [-\pi, \pi]. \tag{5}$$

2) A finite-length complex exponential signal is given by $x[n] = e^{j\omega n}, n \in [0, N-1]$. The DFT of $x[n]$ satisfies

$$|X[k]| = \left| \frac{\sin \frac{2\pi k - N\omega}{2}}{\sin \frac{2\pi k - N\omega}{2N}} \right|. \tag{6}$$

By using the approximation $\sin\theta \approx \theta, |\theta| < 0.2$ rad, show that $|X[k]|$ is approximately bounded by $2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$ for a suitable range of k. Give the range for k for which the bound applies and explain the significance of the term $\frac{2\pi k}{N} - \omega$.

The argument of sin in the denominator is "small" if $\frac{2\pi k - N\omega}{2N} < 0.2 \Leftrightarrow \left|k - \frac{\omega N}{2\pi}\right| < \frac{0.4N}{2\pi} \approx 0.0637N$. Within this range, $|X[n]| \approx \left|\sin \frac{2\pi k - N\omega}{2}\right| \cdot \frac{2N}{2\pi k - N\omega} = \left|\sin \frac{2\pi k - N\omega}{2}\right| \cdot 2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$. This proves the required result since the sin term is bounded by 1. Since $|X[k]|$ corresponds to a frequency of $\frac{2\pi k}{N}$, the term in parantheses gives the distance that a spectral component k is far away from the frequency ω in rad/sample. Thus, we have shown that, when using a rectangular window, the spectral leakage falls as $|k - k_0|^{-1}$ where $k_0 = \frac{\omega N}{2\pi}$ over the range $k \in (k_0 - 0.0637N, k_0 + 0.0637N)$.

3) Given a maximum transition width of $\Delta\omega = 0.2$ rad/s, compute the stop band gain for a filter with a number of $M = 50$ taps.

According to the formula for the number of taps, we have that

$$-20 \log_{10} \epsilon = M \cdot 2.2 \cdot \Delta\omega + 8 = 50 \cdot 2.2 \cdot 0.2 = 30 \Rightarrow \epsilon \approx 0.03. \tag{7}$$

4) Let $M_0 = \frac{-8-20\log_{10}\epsilon_0}{2.2\cdot\Delta\omega_0}$ denote the number of taps of a low pass filter, and let $\Delta\omega_1 = 2\Delta\omega_0$.

(a) Give general expressions of M_1 and ϵ_1 as function of M_0, ϵ_0 and $\Delta\omega_0$ that correspond to the number of taps and stop band gain corresponding to a filter with transition bandwidth $\Delta\omega_1 = 2\Delta\omega_0$.

$$M_0 = 2\frac{-8 - 20\log_{10}\epsilon_0}{2.2 \cdot 2\Delta\omega_0} = \frac{-16 - 40\log_{10}\epsilon_0}{2.2 \cdot \Delta\omega_1} = \frac{-8 - 20\log_{10}{\epsilon_0}^2}{2.2 \cdot \Delta\omega_1} - \frac{8}{2.2 \cdot \Delta\omega_1}. \tag{8}$$

Therefore we can choose $\epsilon_1 = {\epsilon_0}^2$. Given that $\frac{8}{2.2\cdot\Delta\omega_0}$ is not necessarily an integer, to guarantee that the requirements are satisfied, we choose

$$M_1 = \left\lceil M_0 + \frac{8}{4.4 \cdot \Delta\omega_0} \right\rceil, \tag{9}$$

where $\lceil \alpha \rceil$ is the smallest integer larger than α, also known as the *ceiling function.*

(b) What can be said about the change in number of taps and stop band gain? What are the conditions to guarantee this change?

If the original low pass filter is designed correctly, then $\epsilon_0 < 1$ and thus $\epsilon_1 = \epsilon_0^2 < \epsilon_0$. The number of taps is changed if and only if

$$\frac{8}{4.4 \cdot \Delta\omega_0} = \frac{4}{2.2 \cdot \Delta\omega_0} \geq 1 \Leftrightarrow \Delta\omega_0 \leq \frac{4}{2.2} \text{ rad/s} = 1.81 \text{ rad/s}. \tag{10}$$