DFT Sample Exam Problems with Solutions

1. Consider an $(2M+1)\times(2M+1)$ gray level real image $f(x,y)$ which is zero outside $-M \le x \le M$ and $-M \le y \le M$. Show that:
   (i) $F(-u,-v) = F^*(u,v)$ with $F(u,v)$ the two-dimensional Discrete Fourier Transform of $f(x,y)$.
   (ii) In order for the image to have the imaginary part of its two-dimensional Discrete Fourier Transform equal to zero, the image must be symmetric around the origin.
   (iii) In order for the image to have the real part of its two-dimensional Discrete Fourier Transform equal to zero, the image must be antisymmetric around the origin.

Solution

(i)

$$F(u,v) = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{-j\frac{2\pi}{(2M+1)}(ux+vy)}$$

$$F(-u,-v) = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$$

$F^*(u,v) = \sum_{-M}^{M}\sum_{-M}^{M} f^*(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)} = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$, since $f(x,y)$ is real.
We immediately observe that $F(-u,-v) = F^*(u,v)$.

(ii)

We can make the transformation $x \to -x$. In that case:

$F(u,v) = \sum_{M}^{-M}\sum_{M}^{-M} f(-x,-y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$.
If $f(x,y)$ is symmetric then $f(-x,-y) = f(x,y)$ and we write

$$F(u,v) = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$$

But $F^*(u,v) = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$
Therefore, if $f(x,y)$ is symmetric then $F(u,v) = F^*(u,v)$. Therefore, the imaginary part of $F(u,v)$ is zero.

(iii)

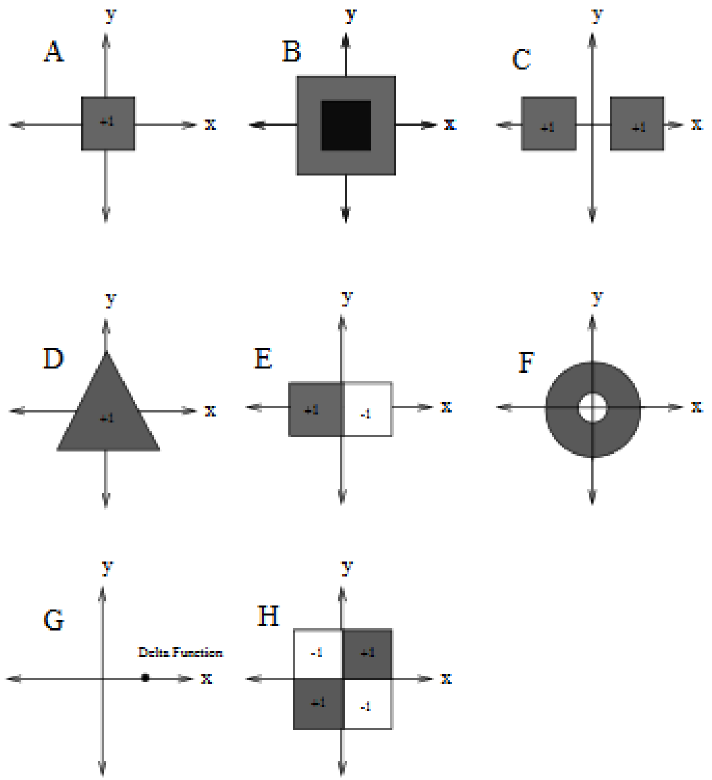
We can make the transformation $x \to -x$. In that case:

$F(u,v) = \sum_{M}^{-M}\sum_{M}^{-M} f(-x,-y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$.
If $f(x,y)$ is antisymmetric then $f(-x,-y) = -f(x,y)$ and we write

$$F(u,v) = -\sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$$

But $F^*(u,v) = \sum_{-M}^{M}\sum_{-M}^{M} f(x,y)e^{j\frac{2\pi}{(2M+1)}(ux+vy)}$
Therefore, if $f(x,y)$ is antisymmetric then $F(u,v) = -F^*(u,v)$. Therefore, the real part of $F(u,v)$ is zero.

2. Consider the images shown below (A to H). Using knowledge of properties of the two-dimensional Discrete Fourier Transform symmetries and not exact calculation of it, list which image(s) will have a two-dimensional Discrete Fourier Transform $F(u, v)$ with the following properties:
   (i) The imaginary part of $F(u, v)$ is zero for all u, v.
   (ii) $F(0,0) = 0$
   (iii) $F(u, v)$ has circular symmetry.
   (iv) The real part of $F(u, v)$ is zero for all u, v.



Solution

(i)

Having imaginary part equal to zero implies that the image is symmetric, as proven in the previous question. Therefore, the images which have a DFT with imaginary part equal to zero are A, B, C, F, H.

(ii)

$F(0,0) = \sum_{-M}^{M} \sum_{-M}^{M} f(x, y) = 0$. By observation we see that images which satisfy this property are E, H.

(iii)

$F(u, v)$ has circular symmetry if $f(x, y)$ has circular symmetry. This is image F.

(iv)

The real part of $F(u, v)$ is zero. This implies that the image is antisymmetric. The image which satisfies this condition is E.

3. Consider an $M \times M$ -pixel gray level image $f(x,y)$ which is zero outside $0 \leq x \leq M-1$ and $0 \leq y \leq M-1$. The image intensity is given by the following relationship

$$f(x,y) = \begin{cases} c, & x = x_1, x = x_2, 0 \leq y \leq M-1 \\ 0, & otherwise \end{cases}$$

where c is a constant value between 0 and 255 and x_1, x_2 $x_1 \neq x_2$ are constant values between 0 and $M-1$.

(i) Plot the image intensity.

(ii) Find the $M \times M$ -point Discrete Fourier Transform (DFT) of $f(x,y)$.

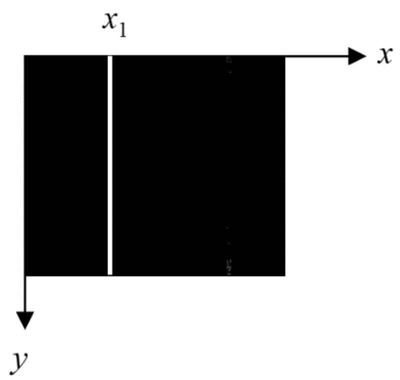
(iii) Compare the original image and its Fourier Transform.

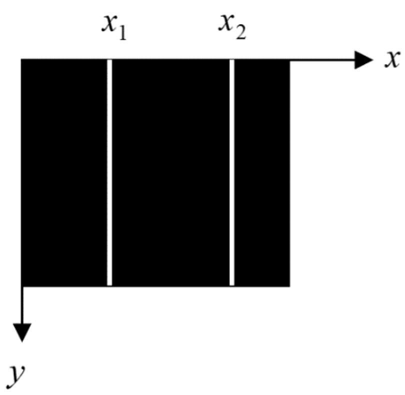
Hint: The following result holds: $\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}, |a| \leq 1$.

Solution

(i)

Plot the image intensity.





(ii)

For an image which contains only a single non-zero edge at $x = x_1$, the $M \times N$ -point Discrete Fourier Transform (DFT) of $f(x,y)$ is given as follows:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} = \sum_{y=0}^{N-1} f(x_1,y)e^{-j2\pi(ux_1/M+vy/N)}$$

$$= ce^{-j2\pi ux_1/M}\sum_{y=0}^{N-1} e^{-j2\pi vy/N} = ce^{-j2\pi ux_1/M}\sum_{y=0}^{N-1}(e^{-j2\pi v/N})^y = ce^{-j2\pi ux_1/M}\frac{1-(e^{-j2\pi v/N})^N}{1-e^{-j2\pi v/N}}$$

$$= ce^{-j2\pi ux_1/M}\frac{1-e^{-j2\pi v}}{1-e^{-j2\pi v/N}} \Rightarrow F(u,v) = \begin{cases} cNe^{-j2\pi ux_1/M}, & v = 0 \\ 0, & \text{otherwise} \end{cases}$$

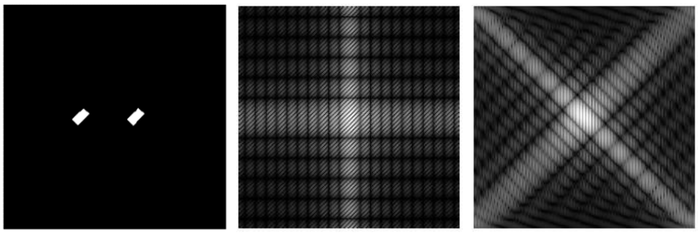
For the image with 2 non-zero edges

$$F(u,v) = \begin{cases} cN(e^{-j2\pi ux_1/M} + e^{-j2\pi ux_2/M}), & v = 0 \\ 0, & \text{otherwise} \end{cases}$$

(iii)

As seen a set of parallel straight lines in space implies a straight line perpendicular to the original one in frequency.

4. Consider the image shown in Figure 1.1(a) below. Two plots of magnitude of Two-Dimensional Discrete Fourier Transform (2D DFT) are shown in Figure 1.1(b) and 1.1(c) below. Discuss which one is the magnitude of the 2D DFT of the image of Figure 1.1(a). Justify your answer.



(a) **(b)** **(c)**

Figure 1.1

Solution

Figure (c) is the right answer since it contains edges which are perpendicular to the edges of the original image. As we know, each image in space produces a perpendicular image in the amplitude of the DFT.

5. Consider an $M \times N$ -pixel image $f(x,y)$ which is zero outside $0 \le x \le M-1$ and $0 \le y \le N-1$. In various image compression applications, we discard the transform coefficients with small magnitudes and use for coding only those with large magnitudes. Let $F(u,v)$ denote the $M \times N$ -point Discrete Fourier Transform (DFT) of $f(x,y)$. Let $G(u,v)$ denote $F(u,v)$ modified by

$$G(u,v) = \begin{cases} F(u,v), & \text{when } |F(u,v)| \text{ is large} \\ 0, & \text{otherwise} \end{cases}$$

Let

$$\frac{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |G(u,v)|^2}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |F(u,v)|^2} = \frac{9}{10}$$

We reconstruct an image $g(x,y)$ by computing the $M \times N$ -point inverse DFT of $G(u,v)$. Express $\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (f(x,y) - g(x,y))^2$ in terms of $\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)^2$.

Solution

The signal $f(x,y) - g(x,y)$ is obtained by the Inverse DFT of the signal $F(u,v) - G(u,v)$. Therefore, according to Parseval's theorem the energy of the signal $f(x,y) - g(x,y)$ is equal to the energy of the signal $F(u,v) - G(u,v)$. The signal $F(u,v) - G(u,v)$ consists of the DFT samples of $F(u,v)$ which were excluded in forming $G(u,v)$. Since, $G(u,v)$ captures 0.9 of the energy of $F(u,v)$, the signal $F(u,v) - G(u,v)$ will capture 0.1 of the energy of $F(u,v)$. Therefore,

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (f(x,y) - g(x,y))^2 = 0.1 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)^2$$