EE4-12
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EE9-FPN2-03

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2019

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected copy

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 30 April 10:00 am

Time allowed: 3:00 hours

Before the start of exam:
Q4 a) iii)

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions. Questions 1 & 2 should be answered in Booklet A and Questions 3 & 4 in Booklet B

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : P.A. Naylor, P.T. Stathaki

Digital Signal Processing and Digital Filters

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate, f, of a signal in Hz may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output	IIR	Infinite Impulse Response
CTFT	Continuous-Time Fourier Transform	LTI	Linear Time-Invariant
DCT	Discrete Cosine Transform	MDCT	Modified Discrete Cosine Transform
DFT	Discrete Fourier Transform	PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform	SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response		

A datasheet is included at the end of the examination paper.

1. (a) The discrete signals $x[n]$ and $y[n]$ are of length N and M respectively with $M \leq N$. $x[n]$ is zero outside the range $0 \leq n < N$ and $y[n]$ is zero outside the range $0 \leq n < M$. Consider the linear and circular convolution

$$v[n] = x[n] * y[n]$$
$$w[n] = x[n] \circledast_N y[n]$$

(i) State the values of n for which $v[n] = w[n]$ is necessarily true. [3]

(ii) If $x[n] = [5, 2, 3, 4, 7]$ and $y[n] = [1, 2, 3]$ determine the values of $v[1]$ and $w[1]$. [2]

(b) (i) Explain what is meant by saying that a linear time invariant system is "BIBO stable". [2]

(ii) The impulse response, $h[n]$, of a linear time-invariant system satisfies $\sum_{n=-\infty}^{\infty}|h[n]| = S$ where $S < \infty$. Prove that the system is BIBO-stable and also that $H(z)$ converges for $|z| = 1$. [3]

(c) The Discrete Time Fourier Transform (DTFT) of a normalized rectangular window of length $M+1$ samples is given by $W(e^{j\omega}) = \frac{\sin 0.5(M+1)\omega}{(M+1)\sin 0.5\omega}$.

(i) Determine the smallest positive value of ω for which $W(e^{j\omega}) = 0$. [1]

(ii) Determine the value of $20\log_{10}|W(e^{j\omega})|$ at $\omega = 0$. [2]

(iii) Determine the value of $20\log_{10}|W(e^{j\omega})|$ at $\omega = \frac{3\pi}{M+1}$ under the assumption that $\sin 0.5\omega \approx 0.5\omega$. [2]

(d) A filter impulse response $h[n]$ is of length $M+1$ and satisfies the symmetry condition $h[M-n] = h[n]$ for $0 \leq n \leq M$.

(i) Show that if M is odd, the frequency response $H(e^{j\omega})$ may be written as

$$H(e^{j\omega}) = Ae^{j\theta(\omega)}\sum_{n=0}^{\frac{M-1}{2}} h[n]\cos\left(n - \frac{M}{2}\right)\omega$$

and determine the expressions for A and $\theta(\omega)$. [4]

(ii) Determine an expression for the group delay of the filter defined as: $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. [1]

(e) **Figure 1** below shows the block diagram of a filter implementation comprising two delays, one multiplier with coefficient a and two adder/subtractor elements whose input polarities are as marked. All elements are drawn with their outputs on the right. Determine the transfer function $\frac{Y(z)}{X(z)}$. [5]

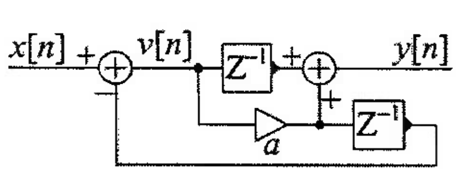

Figure 1

(f) (i) Show that, if $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ and a have the same is a complex-valued constant, then $x[n] = a^n u[n]$ and $y[n] = -a^n u[-n-1]$ have the same $z-$ transform but with different regions of convergence. You may use without proof the geometric progression formulae given in the datasheet. [2]

(ii) The z −transform $H(z)$ is given by

$$H(z) = \frac{2 + 17z^{-1}}{(2 - z^{-1})(1 + 4z^{-1})}$$

By expressing $H(z)$ in partial fraction form, determine the sequence, $h[n]$, whose z −transform is $H(z)$ and whose region of convergence includes $|z| = 1$. [3]

(g) A filter with input $x[n]$ and output $y[n]$ is defined by the difference equation

$$y[n] = ay[n-1] + (1-a)x[n]$$

(i) Determine the system function of the filter, $H(z)$, and the impulse response, $h[n]$, for $n = -1, 0, 1, 2$. [2]

(ii) State the values of z at which $H(z)$ has a pole or a zero. [2]

(iii) Determine the frequency at which the filter has a gain of -3dB. [3]

(iv) If the sample frequency is f_s fs, show that, for $n \geq 0$, the impulse response, $h[n]$, is equal to a sampled version of $g(t) = Ae^{-\frac{t}{\tau}}$ and determine the values of the constants A and τ. [3]

2. (a) A bilinear transformation of z is defined as $z = \frac{\hat{z}-\lambda}{1-\lambda\hat{z}}$, where λ is a constant parameter which satisfies the condition $|\lambda| < 1$.

(i) By proving that $|z|^2 = 1 + \frac{(|\hat{z}|^2-1)(1-\lambda^2)}{|1-\lambda\hat{z}|^2}$, show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [4]

(ii) Explain why the property stated in part (i) is important when using the bilinear transformation for filter design. [2]

(a) A lowpass filter of first order has the transfer function $G(z) = 1 + z^{-1}$.

(i) Determine the gain of the filter at $\omega = 0$. Find the factor by which the magnitude of the gain has decreased at the cutoff frequency $\omega_G = \frac{\pi}{2}$. [2]

(ii) Determine a trigonometrical expression for $|G(e^{j\omega})|$ and draw a dimensioned sketch of its value over the range $0 \le \omega \le \pi$. [4]

(iii) Using the appropriate z -plane transformation from the datasheet, transform $G(z)$ to a lowpass filter $H(z)$, with a cutoff frequency of $\omega_H = 0.2$. Calculate the numerical values of the filter coefficients when the transfer function of the filter is expressed in the standard form

$$g \times \frac{1 + b_1 z^{-1} + \cdots}{1 + a z^{-1} + \cdots}$$

[5]

(iv) Draw a dimensioned sketch of $|H(e^{j\omega})|$ over the range $0 \le \omega \le \pi$. [2]

(b) A quadratic transformation of z is defined as $z = -\hat{z}^2$.

(i) Show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [2]

(ii) If $z = e^{j\omega}$ and $\tilde{z} = e^{j\tilde{\omega}}$ sketch a graph of ω versus $\tilde{\omega}$ over the range $-\pi \le \tilde{\omega} \le \pi$. For all $\tilde{\omega}$, the value of ω should be chosen to lie in the range $-\pi \le \omega \le \pi$. [2]

(iii) A new filter is defined by $P(\tilde{z}) = H(z)$. Determine the numerical values of the coefficients of $P(\tilde{z})$ when expressed in the standard form given in (a)-(iii) above. [3]

(iv) Draw a dimensioned sketch of $|P(e^{j\omega})|$ over the range $0 \le \omega \le \pi$ and determine the values of ω within this range for which $|P(e^{j\omega})| = \sqrt{2}$.

(v) Explain the relationship between the bandwidth of the filter $P(e^{j\omega})$ and the cutoff frequency of the filter $H(e^{j\omega})$. [4]

3. a) Consider the multirate signal processing system shown in the block diagram of Fig. 3.1. This system multiplies the input sample rate by $\frac{P}{Q}$. In the following, let P and Q be coprime with $P < Q$.

i) Explain why the cutoff frequency of the lowpass filter $H(z)$ should be placed at the Nyquist rate of the output signal, $y[m]$ and give the normalized cutoff frequency, ω_0, in rad/sample in terms of P and/or Q.

Determine the required filter order M in terms of P and/or Q if the stop-band attenuation in dB is $a = 60$ and the normalized transition bandwidth is $\Delta\omega = 0.1\omega_0$. You may use the approximation formula $M \approx \frac{a}{3.5\Delta\omega}$, in which a and $\Delta\omega$ denote the stopband attenuation and the normalized transition bandwidth respectively.

[4]

ii) Estimate the average number of multiplications per input sample, $x[n]$, needed to implement the system in the form of Fig. 3.1 using the value of M from part a) i). [2]

iii) The filter $H(z)$ has a symmetrical impulse response

$$h[r] = g[r]w[r]$$

for $0 \le r \le M$ where $g[r]$ is the impulse response of an ideal lowpass filter with cutoff frequency ω_0 and $w[r]$ is a symmetrical window function.

Derive an expression for the ideal response, $g[r]$, in terms of ω_0, M and r.

[4]

b) It is intended to implement the filter $H(z)$ as a polyphase filter with commutated coefficients using the structure shown in Fig. 3.2.

i) Determine, in samples, the length of the filter impulse response $h_0[n]$ in terms of M, P and/or Q. Give an expression for the coefficients $h_0[n]$ in terms of the coefficients $h[r]$. [2]

ii) If $x[n] = 0$ for $n < 0$, give expressions for $v[0]$, $v[1]$, $v[2P+1]$ in terms of the input $x[n]$ and the coefficients $h_p[n]$. [2]

iii) Consider a particular example with $P = 5$ and $Q = 7$. Describe how the output decimator can be eliminated by changing both the sequence and rate at which the coefficient sets, $h_p[n]$ are accessed, and determine the new coefficient set order for this case. [3]

iv) Determine the number of multiplications per input sample for the system of part b) iii). You may assume that $M+1$ is a multiple of P. [2]

c) Consider an input signal, $x[n]$, with sampling rate 22 kHz. Consider also that the system is implemented as in part b) iii) with the values of a and $\Delta\omega$ as given in part a) i).

Determine the values of P, Q and M when the sample rate of the output, $y[m]$, is:
(i) 10 kHz and (ii) 10.1 kHz.

For each of these cases estimate the number of multiplications per input sample and the number of distinct coefficients that must be stored. [5]

d) Consider a Farrow filter employing a low-order polynomial $f_n(t)$ where

$$t = \frac{p}{P} \text{ for } 0 \le p \le P-1.$$

i) Using any relevant diagrams, give a brief explanation of the operation of the Farrow filter and explain clearly how the coefficients, $h_p[n]$, are obtained.
[2]

ii) Give an expression for the target value of $f_0(t)$ in terms of t, M, P and Q for the case that a rectangular window, $w[r] \equiv 1$, is used in the design of $H(z)$ and that $\omega_0 = \frac{\pi}{P}$.
[2]

iii) If the polynomials, $f_n(v)$, are of order $K = 5$, determine the number of coefficients that must be stored for each of the cases defined in part c).
[2]

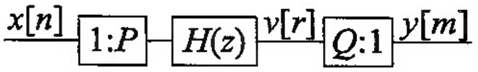

Figure 3.1

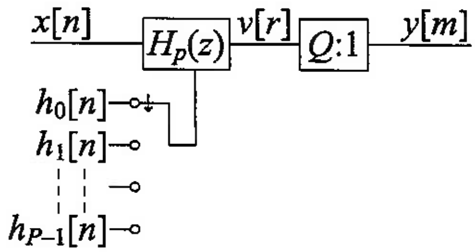

Figure 3.2

4. a) Let

$$x(t) = a(t)e^{j\phi(t)}$$

denote a complex-valued frequency-modulated signal with carrier frequency of 0 Hz and for which the peak frequency deviation is $d = 75\,\text{kHz}$. The amplitude of $a(t)$ is approximately constant with $a(t) \approx 1$. The phase of $a(t)$ is given by

$$\phi(t) = k\int_0^t m(\tau)d\tau$$

where k is a constant and $m(t)$ is a baseband audio signal with bandwidth $b = 15\,\text{kHz}$.

Let $x[n]$ denote the discrete-time signal formed from $x(t)$ sampled with a sampling frequency of 400 kHz.

i) The bandwidth of a double-sideband FM signal B is given by Carson's rule as

$$B = 2(d+b).$$

Starting from Carson's rule, determine the single-sided bandwidth, ω_0, of the discrete-time signal $x[n]$ in radians/sample.

[2]

ii) Show that $m(t) = k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right)$ where $\Im\,()$ denotes the imaginary part.

[4]

iii) The discrete-time system shown in the block diagram of Fig. ~~4.2~~ 4.1 is designed to implement the equation of part ii). Complex-valued signals are shown as bold connections. The block labelled "Conj" outputs the complex conjugate of its input. The block labelled $D(z)$ performs differentiation and is designed as an FIR filter using the window method with a target response

$$\overline{D}(e^{j\omega}) = \begin{cases} jc\omega & \text{for } |\omega| \le \omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a scaling constant.

Starting with the inverse DTFT, determine a simplified expression for the impulse response $\overline{d}[n]$ of $\overline{D}(z)$.

[4]

Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\overline{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$.

[4]

Let $s[n]$ denote the output of the differentiation block, $D(z)$, in Fig. 4.1. Consider the case of a practical implementation in which it is necessary to satisfy the constraint that $|s[n]| \le \frac{1}{\sqrt{2}}$. What is the maximum value of c that will ensure this constraint is satisfied. Clearly state all assumptions made.

[4]

b) The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies $\pm 100\,\text{kHz}$ around a centre frequency of $c \times 100\,\text{kHz}$ where the channel index, c, is an integer in the range $876 \le c \le 1079$. Figure 4.2 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

i) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of

$u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing. [3]

ii) In Fig. 4.2, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [3]

Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range -700 to $+700$ kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

Explain the purpose of the lowpass FIR filter, $H(z)$ in Fig. 4.2. [3]

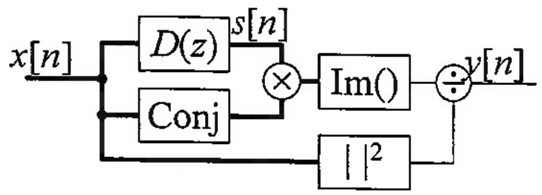

Figure 4.1

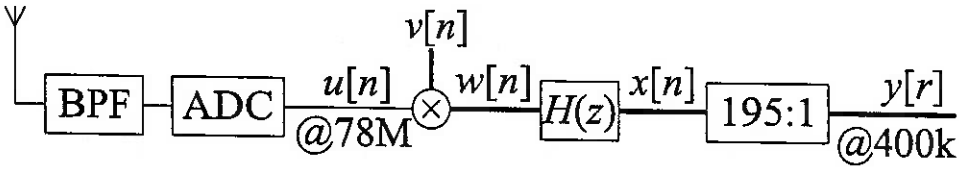

Figure 4.2

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = -\frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

$$-\boxed{H(z)}-\boxed{1{:}Q}- \;=\; -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample $v[n]$ by Q: $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator