Announcement

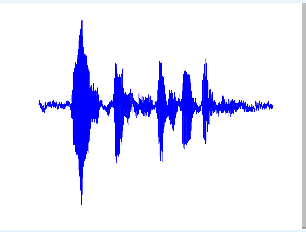
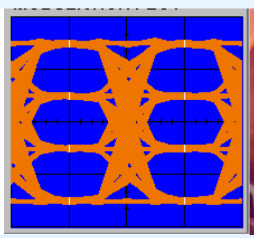
Updated lecture notes, handouts and a practice sheet with problems has been uploaded to the course website. http://alumni.media.MIT.edu/~ayush/course

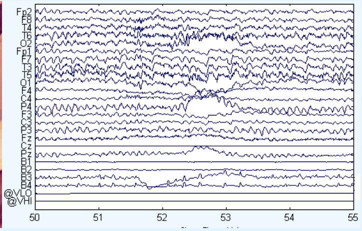
1: Introduction

Signals

- A signal is a numerical quantity that is a function of one or more independent variables such as time or position.

Examples:



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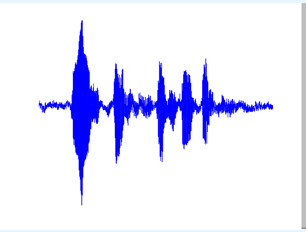

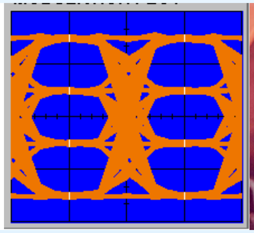

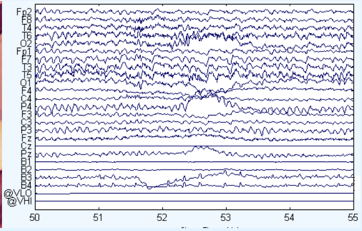

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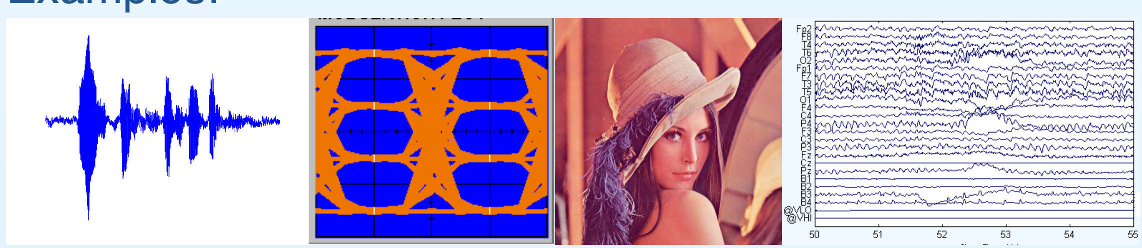

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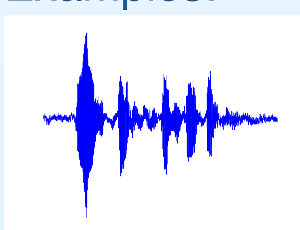
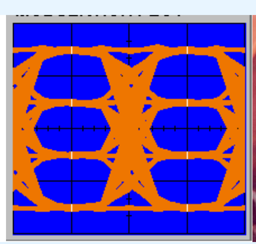
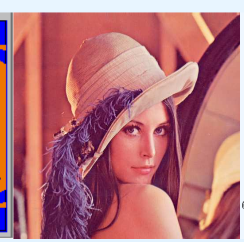
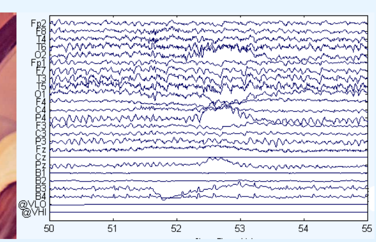

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- We will mostly consider one-dimensionsal real-valued signals with regular sample instants; except in a few places, we will ignore the quantization.
  - Extension to multiple dimensions and complex-valued signals is straighforward in many cases.

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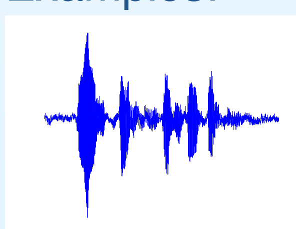
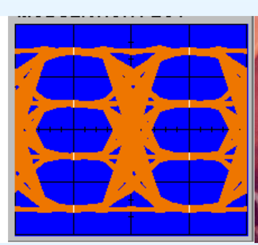
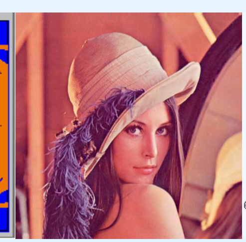
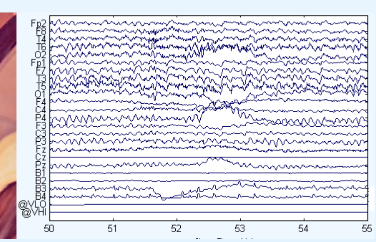

Processing

- Aims to “improve” a signal in some way or extract some information from it
- Examples:
  - Modulation/demodulation
  - Coding and decoding
  - Interference rejection and noise suppression
  - Signal detection, feature extraction
- We are concerned with linear, time-invariant processing

Syllabus

Main topics:

- Introduction/Revision
- Transforms
- Discrete Time Systems
- Filter Design
  - FIR Filter Design
  - IIR Filter Design
- Multirate systems
  - Multirate Fundamentals
  - Multirate Filters
  - Subband processing

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Warning: Several MATLAB routines scale time so that $f_s = 2$ Hz. Weird, non-standard and irritating.

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- Complex functions are easier to manipulate than sequences
- Useful operations on sequences correspond to simple operations on the z-transform:
  - addition, multiplication, scalar multiplication, time-shift, convolution
- Definition: $X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$

Region of Convergence

The set of z for which $X(z)$ converges is its *Region of Convergence* (ROC).

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3. Example: Power Series, $x[n] = r^n$.
$$r^0 + r + r^2 + r^3 + \cdots + r^N = \sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$
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   Series converges for $|r| < 1$ because,
$$|r| < 1, \quad \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r} = \frac{1}{1 - r}$$

Convergence: Example

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$$\left|2 - \sum_{n \leqslant N} \left(\frac{1}{2}\right)^n\right| = \left|2 - \frac{1 - \left(\frac{1}{2}\right)^{N+1}}{1 - \frac{1}{2}}\right| = \left(\frac{1}{2}\right)^N, \qquad \lim_{N \to \infty} \left(\frac{1}{2}\right)^N = 0.$$

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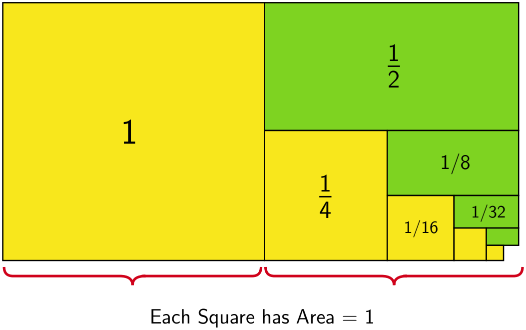
Visual Proof: Slicing "Squares".



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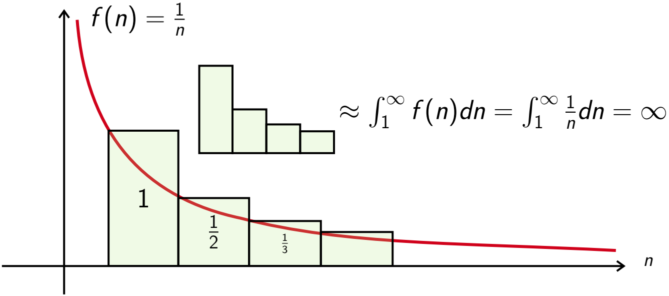
However, this series converges at ∞.



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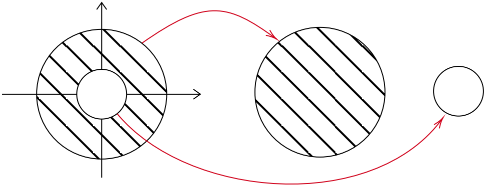
Visually:



ROC and its Shape

On the z-plane, the ROC is always an annulus, that is,

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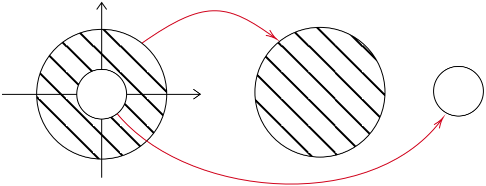

Why?

ROC and its Shape

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Why? Because... Let $z = re^{\jmath\theta}$ (polar co-ordinates).

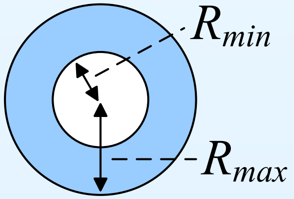
$$\begin{aligned} |X(z)| &= \left|\sum_n x[n]\, z^{-n}\right| \leqslant \left|\sum_n |x[n]| r^{-n}\right| \\ &= \underbrace{\sum_{n\geqslant 1} |x[-n]|\, r^n}_{\text{Anti-causal Part}} + \underbrace{\sum_{n\geqslant 0} \frac{|x[n]|}{r^n}}_{\text{Causal Part}} \end{aligned}$$

- On the z-plane, ROC is the region where both parts of the sum are finite.
- When the causal part is finite, the series converges on the exterior of some circle.
- When the anti-causal part is finite, the series converges on the interior of some circle.

Region of Convergence

The set of z for which $X(z)$ converges is its *Region of Convergence* (ROC).

Complex analysis $\Rightarrow$: the ROC of a power series (if it exists at all) is always an annular region of the form $0 \leq R_{min} < |z| < R_{max} \leq \infty$.



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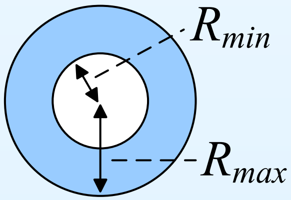

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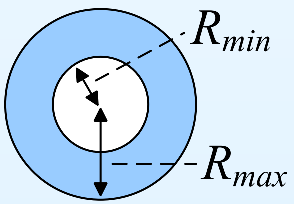

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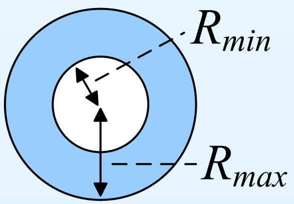

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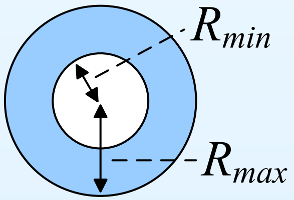

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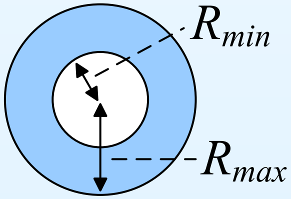

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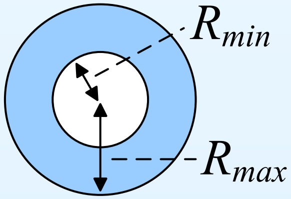

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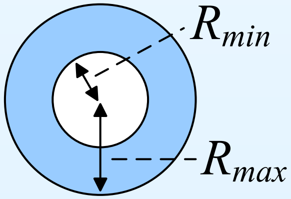

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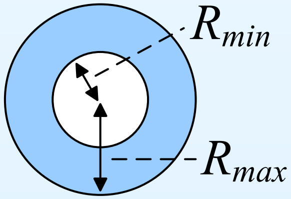

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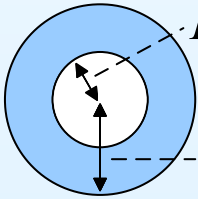
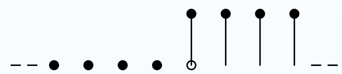


z-Transform examples

The sample at $n = 0$ is indicated by an open circle.

$u[n]$



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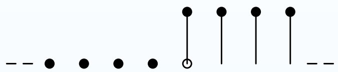
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$

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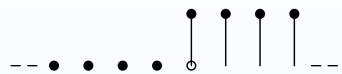
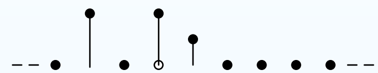
$u[n]$  $\frac{1}{1-z^{-1}}$ $1 < |z| \le \infty$

$x[n]$ 

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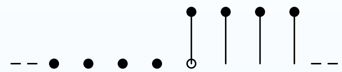
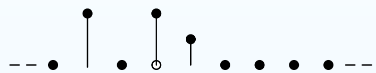
$u[n]$  $\frac{1}{1-z^{-1}}$ $1 < |z| \le \infty$

$x[n]$  $2z^2 + 2 + z^{-1}$ $0 < |z| < \infty$

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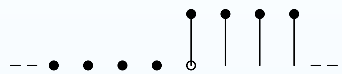
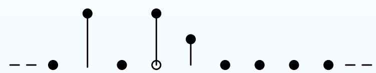
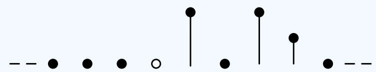
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$			

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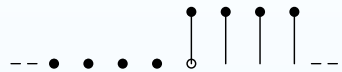
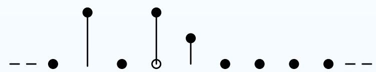
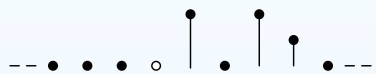
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$

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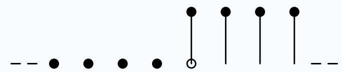
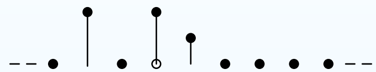
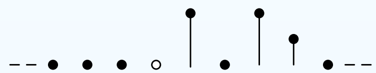
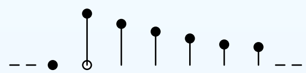
$u[n]$		$\frac{1}{1-z^{-1}}$	$1<\lvert z\rvert \le \infty$
$x[n]$		$2z^2+2+z^{-1}$	$0<\lvert z\rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2+2+z^{-1}\right)$	$0<\lvert z\rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$			

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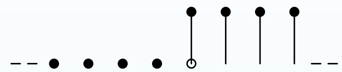
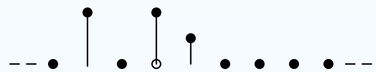
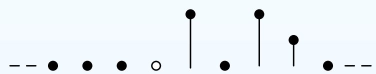
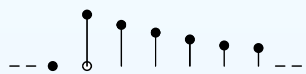
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$

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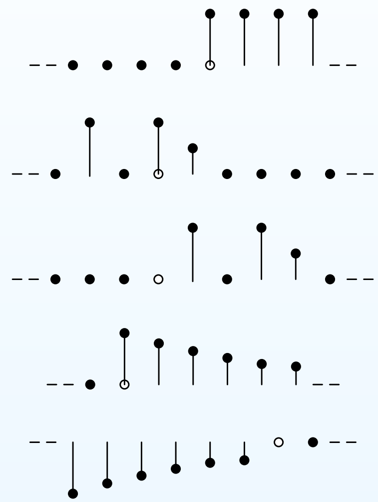

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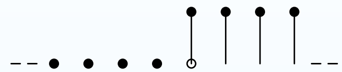
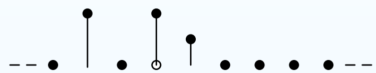
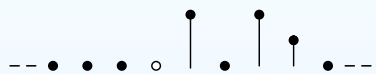
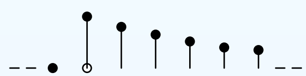
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
$-\alpha^n u[-n-1]$		$\frac{1}{1-\alpha z^{-1}}$	$0 \le \lvert z \rvert < \alpha$

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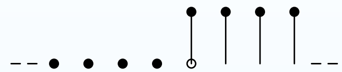
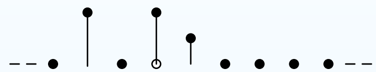
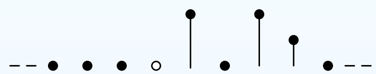
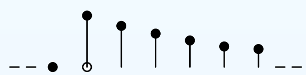
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z\rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z\rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z\rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z\rvert \le \infty$
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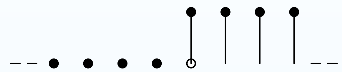
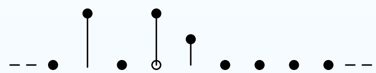
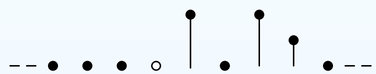
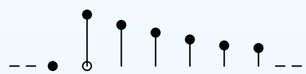
z-Transform examples

The sample at $n = 0$ is indicated by an open circle.

$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \leq \infty$
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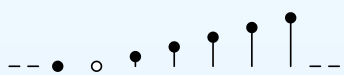
$nu[n]$			

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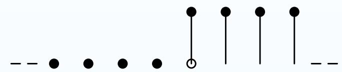
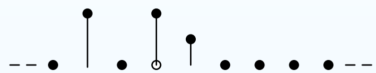
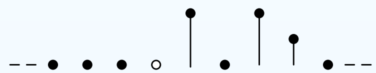
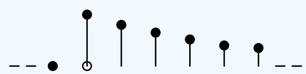
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
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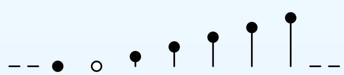
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \le \infty$

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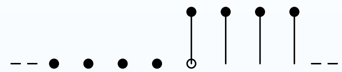
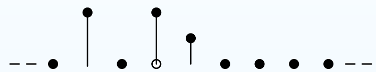
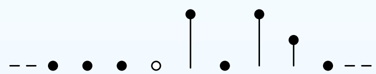
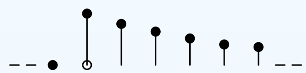
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z\rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z\rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z\rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z\rvert \le \infty$
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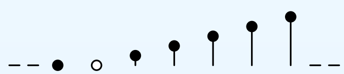
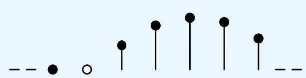
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z\rvert \le \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$			

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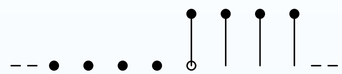
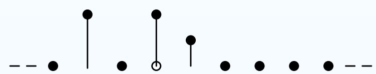
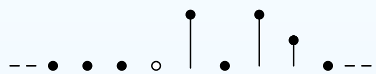
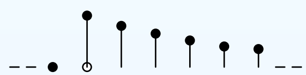
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \leq \infty$
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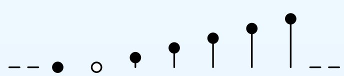
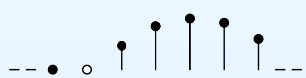
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \leq \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \leq \infty$

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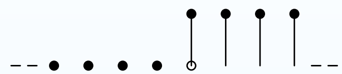
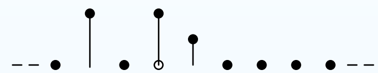
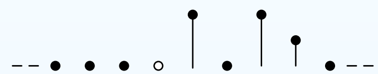
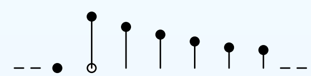
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z\rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z\rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z\rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z\rvert \le \infty$
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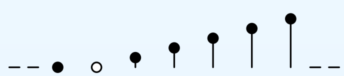
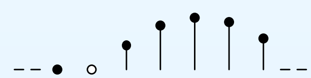
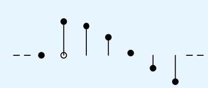
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z\rvert \le \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z\rvert \le \infty$
$\cos(\omega n)u[n]_{\omega=0.5}$			

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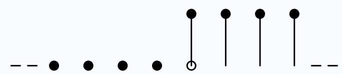
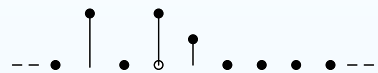
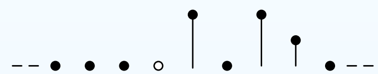
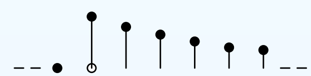
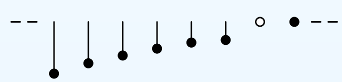
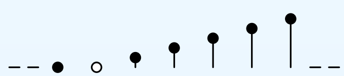
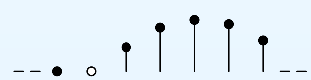
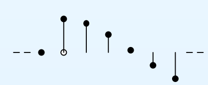
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \|z\| < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \|z\| \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \|z\| \le \infty$
$-\alpha^n u[-n-1]$		$\frac{1}{1-\alpha z^{-1}}$	$0 \le \|z\| < \alpha$
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \|z\| \le \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \|z\| \le \infty$
$\cos(\omega n)u[n]_{\omega=0.5}$		$\frac{1-z^{-1}\cos(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \|z\| \le \infty$

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Examples of Z-transform

1. Causal Sinusoid $x[n] = \cos(\omega_0 n + \xi)\, u[n]$.

   Start with,

$$x[n] = \frac{1}{2}\left(\exp(\jmath(\omega_0 n + \xi)) + \exp(-\jmath(\omega_0 n + \xi))\right) = \frac{\alpha}{2}\beta^n u[n] + \frac{\alpha^*}{2}(\beta^*)^n u[n].$$

   Clearly, $\left(\frac{\alpha}{2}\right)\beta^n u[n] \xrightarrow{\text{Z-transform}} \left(\frac{\alpha}{2}\right)\frac{1}{1-\beta z^{-1}}$ and likewise for the conjugate term.
   Finally,

$$X(z) = \left(\frac{\alpha}{2}\right)\frac{1}{1-\beta z^{-1}} + \left(\frac{\alpha^*}{2}\right)\frac{1}{1-\beta^* z^{-1}} = \frac{\cos\xi - z^{-1}\cos(\omega_0 - \xi)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}.$$

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2. Suppose $x[n] \to X(z)$. Find the z-transform of $nx[n]$. We have,

   $$X(z) = \sum_n x[n]\, z^{-n} \Rightarrow \frac{d}{dz}X(z) = -\sum_n \left(nx[n]\right) z^{-n-1} = -\frac{1}{z}\underbrace{\sum_n \left(nx[n]\right) z^{-n}}_{Z\{nx[n]\}}.$$

   Hence,

   $$Z\{nx[n]\} = -z\left(\frac{d}{dz}X(z)\right).$$

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Note: There are $K-M$ zeros or $M-K$ poles at $z=0$ (easy to overlook)

Rational example

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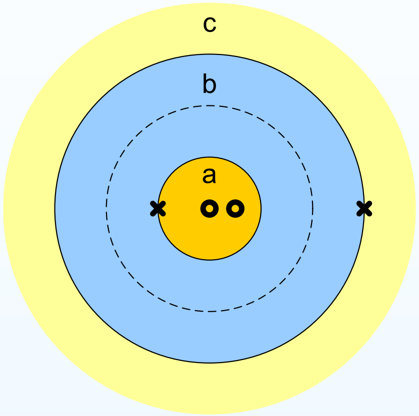

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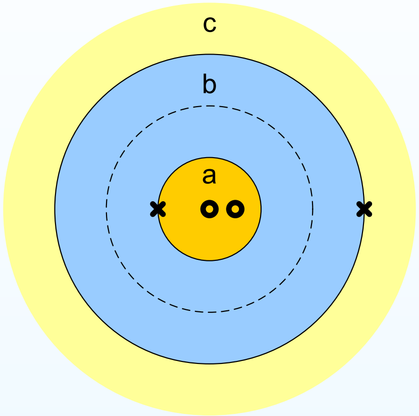

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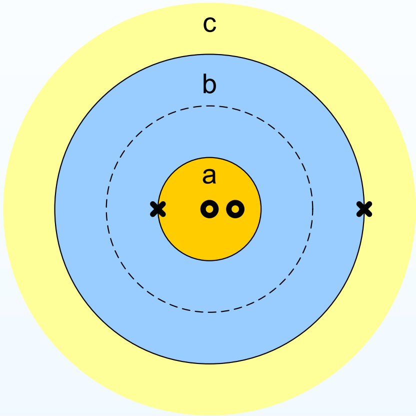

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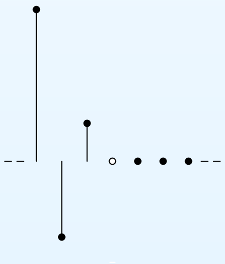
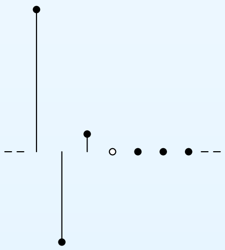
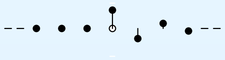
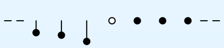
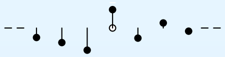
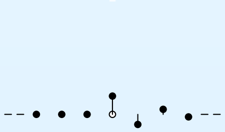
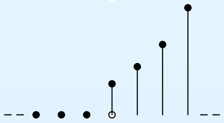
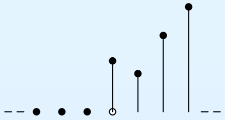
ROC	ROC	$\frac{0.75}{1+0.5z^{-1}}$	$\frac{1.25}{1-1.5z^{-1}}$	$G(z)$
a	$0 \le \lvert z \rvert < 0.5$			
b	$0.5 < \lvert z \rvert < 1.5$			
c	$1.5 < \lvert z \rvert \le \infty$			

Inverse z-Transform

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In practice use a combination of partial fractions and table of z-transforms.

MATLAB routines

tf2zp,zp2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \{z_m, p_k, g\}$
residuez	$\frac{b(z^{-1})}{a(z^{-1})} \rightarrow \sum_k \frac{r_k}{1-p_k z^{-1}}$
tf2sos,sos2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2sos,sos2zp	$\{z_m, p_k, g\} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{\in 1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2ss,ss2zp	$\{z_m, p_k, g\} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
tf2ss,ss2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$

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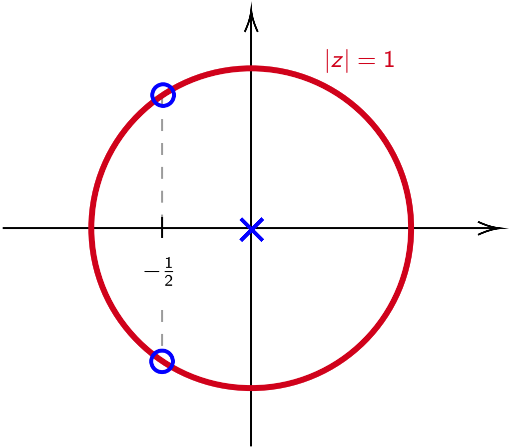
  - Not unique unless ROC is specified
  - Use partial fractions and/or a table

Exam 2020

The pole-zero plot of a discrete filter is given below.
When the input $x[n] = 1$ for all n, the output is exactly the same.
What is the impulse response of such a filter?



Further questions on practice sheet announced on course website.
`http://alumni.media.MIT.edu/~ayush/course`