

Probability and Stochastic Processes

Exam 2018 Solutions

1. part a) is book work; surprisingly, quite a number of students couldn't derive it.

a) It can be derived that

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})) , & y > 0, \\ 0, & \text{otherwise.} \end{cases} \quad [3]$$

Since the Gaussian pdf is even,

$$f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) , \quad y > 0. \quad [3]$$

Substituting $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$, we obtain

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}\sigma} e^{-y/2\sigma^2} \quad y > 0. \quad [4]$$

b) students did well in part b), but some didn't know the difference between these bounds

i) Markov

$$P(|X| > a) \leq \frac{E(|X|)}{a} = \frac{\sqrt{\frac{2}{\pi}}\sigma}{a} = \frac{\sqrt{\frac{2}{\pi}}\sigma}{3\sigma} = 0.265 \quad [4]$$

ii) Chebyshev

$$P(|X| > a) \leq \frac{\sigma^2}{a^2} = \frac{1}{9} = 0.111 \quad [4]$$

iii) Chernoff

$$P(|X| > a) = 2P(X > a) \leq 2e^{-\frac{a^2}{2\sigma^2}} = 2e^{-4.5} = 0.022 \quad [4]$$

Discussion: The accuracy of estimation improves since more moments are used from (i) to (iii), because more moments are used. [3]

students did well in Question 2, probably because of similar questions in previous years

2.

a)

For n samples we have

$$f(\underline{x}, c) = c^{4n} (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)} \quad [2]$$

$$\frac{\partial f(\underline{x}, c)}{\partial c} = 4n \cdot c^{4n-1} \cdot (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)} \quad [2]$$

$$- (x_1 + \dots + x_n) c^{4n} (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)}$$

$$= \left[\frac{4n}{c} - (x_1 + \dots + x_n) \right] f(\underline{x}, c) \quad [2]$$

$$= 0$$

$$c = \frac{4n}{x_1 + \dots + x_n} \quad [2]$$

In this problem, $n = 4$, so

$$c = \frac{4 \times 4}{16} = 1. \quad [2]$$

b) Recall the Wiener-Hopf equation standard question of linear prediction

$$\mathbf{c} = \mathbf{R}^{-1} \mathbf{r}$$

$$\sigma^2 = r_0 - \mathbf{r}^T \mathbf{R}^{-1} \mathbf{r}$$

i) If $n=1$, the Wiener-Hopf equation trivially reads

$$R_Y(0)c_1 = R_Y(1)$$

[2]

Therefore,

$$c_1 = \frac{R_Y(1)}{R_Y(0)} = \frac{2}{3}$$

[3]

Mean-square error

$$\sigma^2 = 3 - 4/3 = 5/3$$

ii) When $n = 2$, we have

$$\mathbf{R} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

[3]

Thus the coefficient vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{R}^{-1} \mathbf{r} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 4/5 \end{bmatrix} \quad [4]$$

Mean-square error

$$\sigma^2 = 3 - [1 \quad 2] \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 8/5 \quad [3]$$

3. a) 1 mark each except those marked otherwise

(1) linearity

(2),(3), convolution

(4) definition of power

(5) due to definition of power spectral density

(6) spectrum of $y_s(t)$ is $S(\omega)H(\omega)$ from (2); power spectrum of $n(t)$ is $S_{ww}(\omega) |H(\omega)|^2$ from (3) [2]

(7) $w(t)$ is white noise with power spectral density N_0

(8) because of Cauchy-Schwarz inequality **some student didn't know Cauchy-Schwarz**

$$\left| \int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0} d\omega \right|^2 \leq \int_{-\infty}^{+\infty} |S(\omega)e^{j\omega t_0}|^2 d\omega \int_{-\infty}^{+\infty} |H(\omega)|^2 d\omega \quad [2]$$

(9) Parseval's theorem

$$(10) \int_0^{+\infty} s(t)^2 dt = E_s$$

b) **part b) is a common question for Poisson processes.**

i) This is the same as the average time when the fourth student arrives.

$$E[T_4] = \frac{4}{\lambda} = \frac{4}{0.2} = 20 \text{ minutes} \quad [3]$$

ii) This means that the number of students arriving in the first half an hour is less than 4.

Let $t = 30$ minutes. Recall $N(t)$ has Poisson distribution: [2]

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots$$

Therefore,

$$\begin{aligned} P(N(t) < 4) &= P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) + P(N(t) = 3) \quad [3] \\ &= e^{-6} + 6e^{-6} + 18e^{-6} + 36e^{-6} = 0.151 \end{aligned}$$

iii) Let $t_1 = 10$ minutes, $t_2 = 20$ minutes. Since a Poisson process is memoryless,

$$P(N(t_1) \geq 1)P(N(t_2) - N(t_1) \leq 2) \quad [1]$$

$$= P(N(t_1) \geq 1)P(N(t_2 - t_1) \leq 2)$$

$$= [1 - P(N(t_1) = 0)][P(N(t_2 - t_1) = 0) + P(N(t_2 - t_1) = 1) + P(N(t_2 - t_1) = 2)] \quad [2]$$

$$= [1 - e^{-2}][e^{-2} + 2e^{-2} + 2e^{-2}] = 0.585$$

[2]

Students did less well in the last question.

A small number of student didn't know what is a martingale.

4. a) i) Using the identity $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$, we have

$$\begin{aligned} E[X_{n+1}] &= \frac{\cos\left\{\lambda\left[S_n + 1 - \frac{1}{2}(b-a)\right]\right\} + \cos\left\{\lambda\left[S_n - 1 - \frac{1}{2}(b-a)\right]\right\}}{2(\cos \lambda)^{n+1}} \\ &= \frac{\cos\left\{\lambda\left[S_n - \frac{1}{2}(b-a)\right]\right\} \cos\{\lambda\}}{(\cos \lambda)^{n+1}} \\ &= \frac{\cos\{\lambda[S_n - \frac{1}{2}(b-a)]\}}{(\cos \lambda)^n} = X_n \end{aligned} \quad [3]$$

remember, we need check the definition $E[X_{n+1}] = X_n$

Thus, X_n forms a martingale.

[2]

ii) Since X_n forms a martingale, we know

$$E[X_T] = X_0$$

Obviously

$$X_0 = \cos\left\{\frac{1}{2}\lambda(b-a)\right\} \quad [2]$$

Now

$$E[X_T] = E_T E_{S_T}[X_T|T] \quad [2]$$

$$= E_T \frac{E_{S_T} \cos\left\{\lambda\left[S_T - \frac{1}{2}(b-a)\right]\right\}}{(\cos \lambda)^T} \quad [2]$$

$$= E_T \frac{P(S_T = -a) \cos\left\{\lambda\left[-a - \frac{1}{2}(b-a)\right]\right\} + P(S_T = b) \cos\left\{\lambda\left[b - \frac{1}{2}(b-a)\right]\right\}}{(\cos \lambda)^T} \quad [2]$$

$$= E_T \frac{P(S_T = -a) \cos\left\{\lambda\left[\frac{1}{2}(b+a)\right]\right\} + P(S_T = b) \cos\left\{\lambda\left[\frac{1}{2}(b+a)\right]\right\}}{(\cos \lambda)^T} \quad [2]$$

$$= E_T \frac{\cos\left\{\lambda\left[\frac{1}{2}(a+b)\right]\right\}}{(\cos \lambda)^T} \quad [2]$$

$$= \frac{\cos\left\{\lambda\left[\frac{1}{2}(a+b)\right]\right\}}{E[(\cos \lambda)^T]} \quad [2]$$

part ii) is a bit tricky

Note that $P(S_T = -a) + P(S_T = b) = 1$.

(b) The stationary distribution satisfies

$$\pi_1 = \pi_0, \pi_2 = \frac{1}{2}\pi_1, \pi_3 = \frac{1}{3}\pi_2, \pi_4 = \frac{1}{4}\pi_3, \dots \quad [2]$$

from which we get

[2]

$$\pi_i = \frac{1}{i!} \pi_0$$

Since

$$\sum_{i=0}^{\infty} \pi_i = \sum_{i=0}^{\infty} \frac{1}{i!} \pi_0 = e \pi_0 = 1, \quad [3]$$

we find $\pi_0 = e^{-1}$ and

$$\pi_i = \frac{e^{-1}}{i!}, \quad i = 1, 2, 3, \dots \quad [3]$$

Common mistakes in part (b) are:

5

Some students treat it as a finite state space Markov chain;

Some couldn't calculate the infinite sum if $1/i!$.