Fundamentals of Wireless Communication

Capacity of Wireless Channels

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Contents

Introduction

Problem of communication over wireless fading channels

- Optimal performance achievable on a given channel
- Techniques to achieve such optimal performance

- The basic measure of performance is the ***capacity*** of a channel.

AWGN Channel Capacity and Channel Resources

Consider a continuous-time AWGN Channel with bandwidth W Hz, power constraint $\bar{P}$ watts, AWGN with psd ${}^{N_0}/_2$. For sampling at ${}^{1}/_{W}$, the discrete-time complex baseband channel:

$$y[m] = x[m] + w[m], where\ w[m] \text{ is } \mathcal{CN}(0, N_0) \text{ and is } i.i.d \text{ over time}$$

The capacity of the channel is:

$$C_{awgn}(\bar{P}, W) = W log\left(1 + \frac{\bar{P}}{N_0 W}\right) \dots\dots\dots\dots\dots\dots\dots (1) \text{ bps}$$

$Given\ SNR \coloneqq \frac{\bar{P}}{N_0 W}$

$$C_{awgn} = log(1 + SNR) \dots\dots\dots\dots\dots\dots\dots (2) \quad \text{bps/Hz}$$

(1) suggests that capacity of the channel depends on the basic resources:

- Received power $\bar{\boldsymbol{P}}$
- Bandwidth $\boldsymbol{W}$

Bandwidth limited regime SNR $\gg$ 1: Capacity logarithmic in received power but approximately linear in bandwidth.

$$Wlog\left(1+\frac{\bar{P}}{\mathrm{N}_0 W}\right) \approx W\left(\frac{\bar{P}}{\mathrm{N}_0 W}\right) log_2 e = \frac{\bar{P}}{N_0} log_2 e \ldots\ldots\ldots\ldots\ldots..(3)$$

Power limited regime SNR $\ll$ 1: Capacity linear in received power but insensitive to bandwidth.

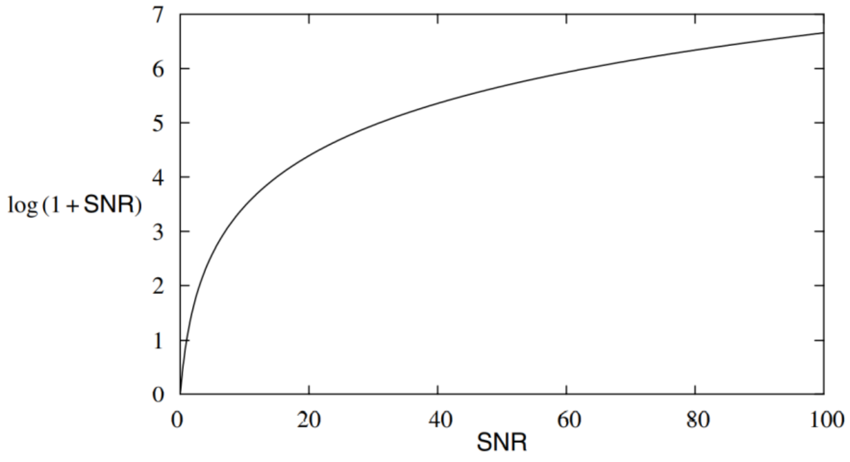
$$C_\infty = \frac{\bar{P}}{N_0} log_2 e \ldots\ldots\ldots\ldots\ldots(4)$$ bits/s

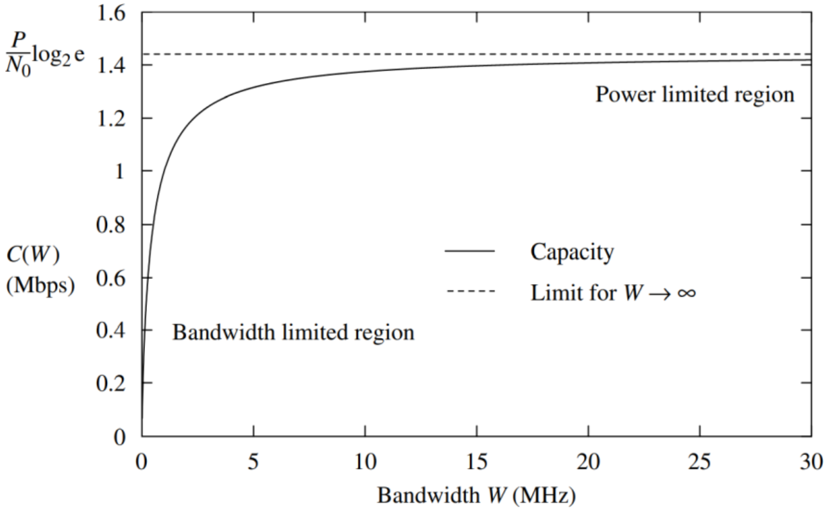
- From (4), the capacity is finite even if there is no bandwidth constraint.

The main objective is to minimize the required energy per bit ε_b rather than maximizing spectral efficiency in some communication applications.

The minimum ε_b, $\dfrac{\bar{P}}{C_{awgn}(\bar{P},W)}$ is achieved when $\bar{P} \rightarrow 0$

$$\Rightarrow \left(\frac{\varepsilon_b}{N_0}\right)_{min} = \lim_{\bar{P}\rightarrow 0} \frac{\bar{P}}{C_{awgn}(\bar{P},W)} = \frac{\bar{P}}{W log\left(1+\frac{\bar{P}}{N_0 W}\right)} = \frac{1}{log_2 e} = -1.59dB$$





Frequency – Selective AWGN Channel

Consider $a\ time - invariant\ L - tap$ frequency-selective AWGN channel:

$$y[m] = \sum_{l=0}^{L-1} h_l x[m-l] + w[m]$$

- OFDM converts it into a parallel channel:

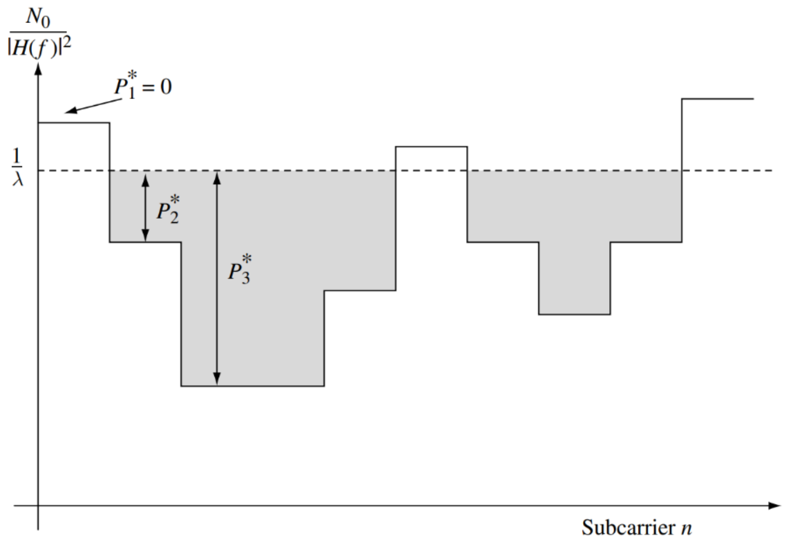
$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \widetilde{w}_n, \qquad n = 0,1, \dots, N_c - 1$$

The maximum rate of reliable communication with this scheme is:

$$C_{N_c} = \sum_{l=0}^{L-1} log\left(1 + \frac{{p_n}^* |\tilde{h}_n|^2}{N_0}\right) \dots \dots \dots \dots \dots \dots \dots (5)$$

Where, ${p_n}^*$ is the waterfilling power allocation, defined as: $\left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+$

$\lambda, the\ Lagrange\ multiplier$ is chosen such that the power constraint is met.



- The values $\dfrac{N_0}{|\tilde{h}_n|^2}$ plotted as a function of the sub-carrier index $n = 0,1, \dots, N_c - 1$
- P units of water per sub-carrier are filled into the vessel, the depth of the water at sub-carrier n is the ***power allocated to that sub-carrier.***

- $\dfrac{1}{\lambda}$ is the height of the water surface.

Capacity of Fading Channels

Slow Fading Channels

Considering a channel where the channel gain is random but remains constant for all time:

$$y[m] = hx[m] + w[m]$$

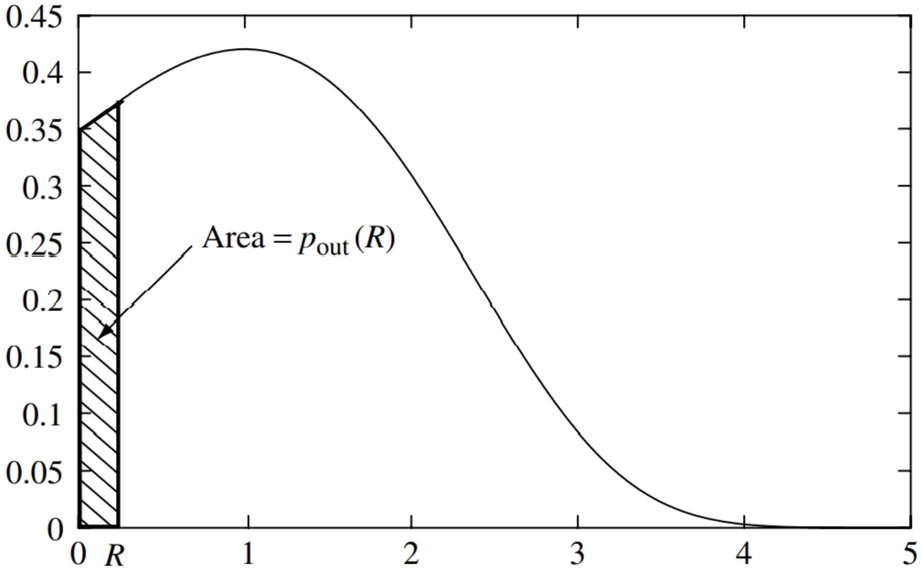
- **For slow fading situation**: $coherence\ time \gg latency\ requirement$
- It can be observed that there's no definite capacity. The performance is a function of h. Good $h \Rightarrow$ Good performance and vice versa.
- In this situation, we can define the notion of outage probability:

$$P_{out}(R) = \mathbb{P}\{\log(1 + |h|^2 \text{SNR} < R)\} \text{..................}(6)$$

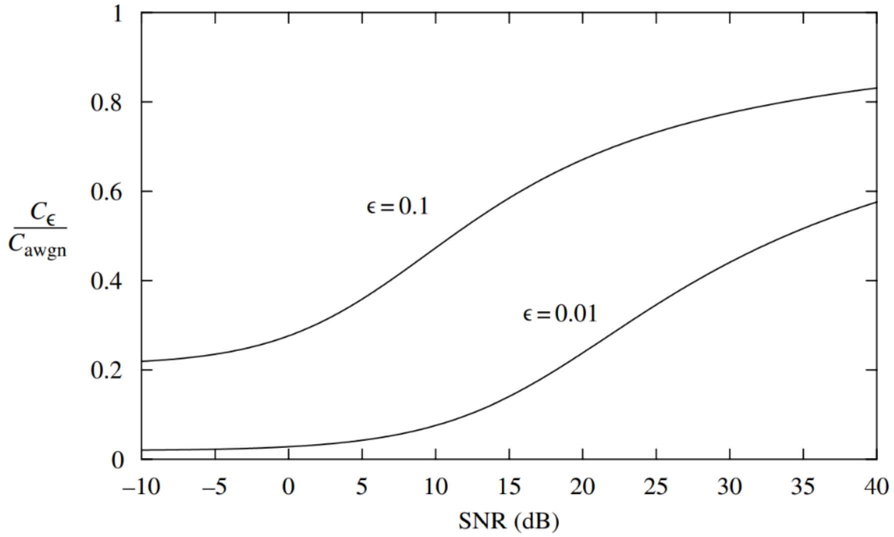
- A good performance measure is: $\epsilon - outage\ capacity$
- This is the largest rate of transmission R such that $P_{out}(R)$ is less than ϵ.

- $C_\epsilon = P_{out}{}^{-1}(\epsilon)$ ……………………………………………………..(7)

PDF: $\log(1+|h|^2\mathrm{SNR})$



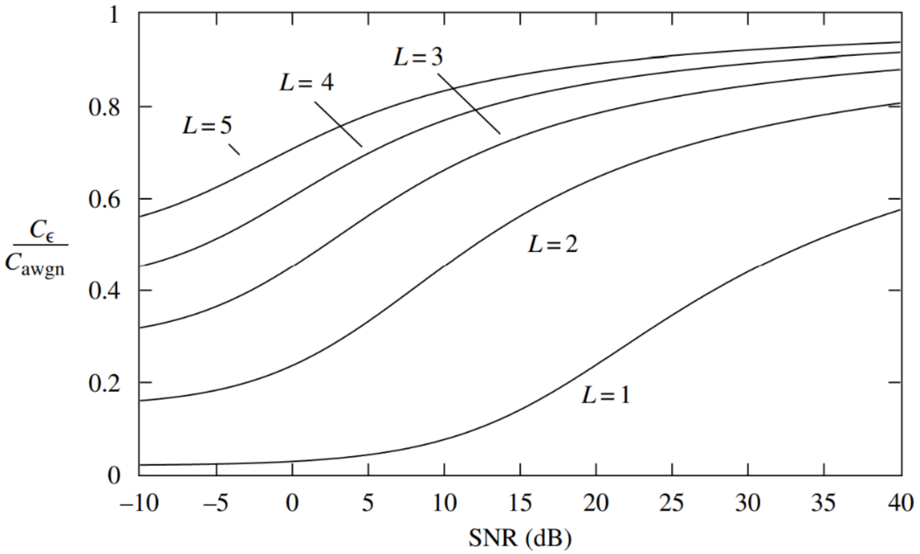
$$P_{out}(R) \approx \frac{2^R - 1}{SNR}$$



To assess the impact of fading, the ϵ−outage capacity is plotted as a fraction of the AWGN capacity at the same SNR

- If we increase the diversity of the channel by using L receive antennas

- Outage occurs whenever: $P_{out}{}^{rx}(R) = \mathbb{P}\left\{\log\left(1 + \left||h|\right|^2 \mathrm{SNR} < R\right)\right\} \dots .(8)$



- This causes dramatic improvement in the gain

- At ϵ = 0.01 and L = 2, the outage capacity is increased to 14% of the AWGN capacity (as opposed to 1% for L = 1).

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Fast Fading Channel

Model: $y[m] = h[m]x[m] + w[m]$

- When the codeword length spans many coherence periods, we are in the $so - called$ fast fading regime.

Block fading model: $h[m] = h_l$ remains constant over l^{th} coherence period of T_c Symbols and is $i.i.d$ across different coherence period.

Suppose coding is done over L coherence periods. If $T_c \gg 1$, L parallel sub-channels fade independently.

$$P_{out}(R) = \mathbb{P}\left\{\frac{1}{L}\sum_{l=1}^{L} log(1 + |h_l|^2 \text{SNR} < R)\right\} \qquad \text{.................}(9)$$

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- For finite L, $P_{out}(R) = \frac{1}{L}\sum_{l=1}^{L} log(1 + |h_l|^2\mathrm{SNR})$

As $L \to \infty$, and by the law of large numbers:

$$\frac{1}{L}\sum_{l=1}^{L} log(1 + |h_l|^2\mathrm{SNR}) \to \mathbb{E}[log(1 + |h|^2\mathrm{SNR})] \dots \dots \dots \dots (10)$$

$$\therefore C = \mathbb{E}[log(1 + |h|^2\mathrm{SNR})] \quad \mathrm{bps/Hz} \dots \dots \dots \dots \dots \dots \dots \dots (11)$$

- In the case of ideal interleaving, we assume $h[m]s$ are i.i.d.

For a large block length N and a given realization of fading gains $h[1], \ldots, h[N]$,

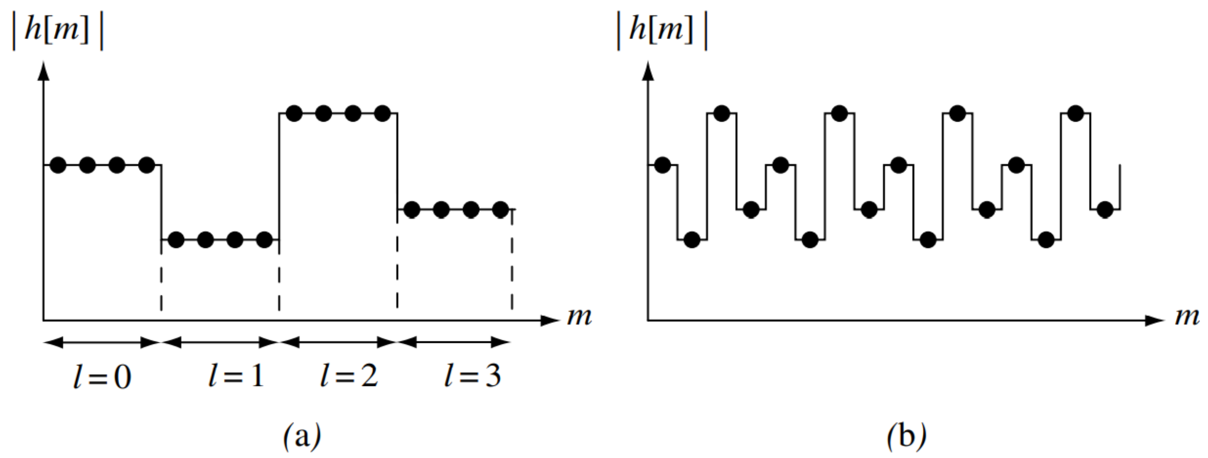
The maximum achievable rate through this interleaved channel is:

$$\frac{1}{N}\sum_{m=1}^{N} log(1 + |h[m]|^2 SNR) \quad bps/Hz$$

As $N \to \infty$, and by the law of large numbers:

$$\frac{1}{N}\sum_{m=1}^{N} log(1 + |h[m]|^2 SNR) \to \mathbb{E}[log(1 + |h|^2 SNR)]$$

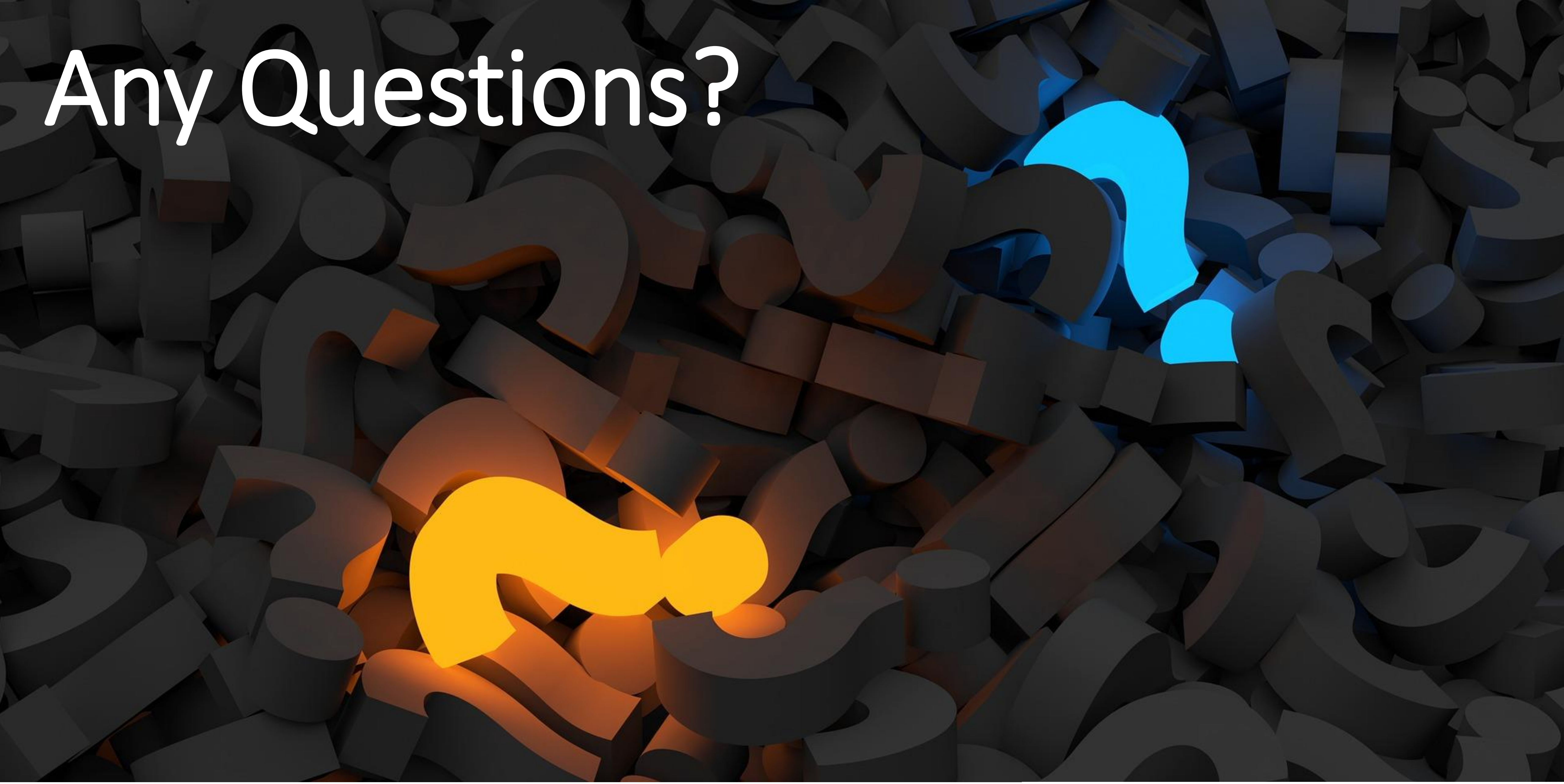
- Trajectory of the channel strength as a function of symbol time under **block fading model** and after **interleaving**



- Even with interleaving the capacity of the fast-fading channel can be achieved.

- **The benefit however is**: Capacity is achieved with a much shorter block length

Any Questions?



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