Fundamentals of Wireless Communication

MIMO I-Spatial Multiplexing and Channel Modeling

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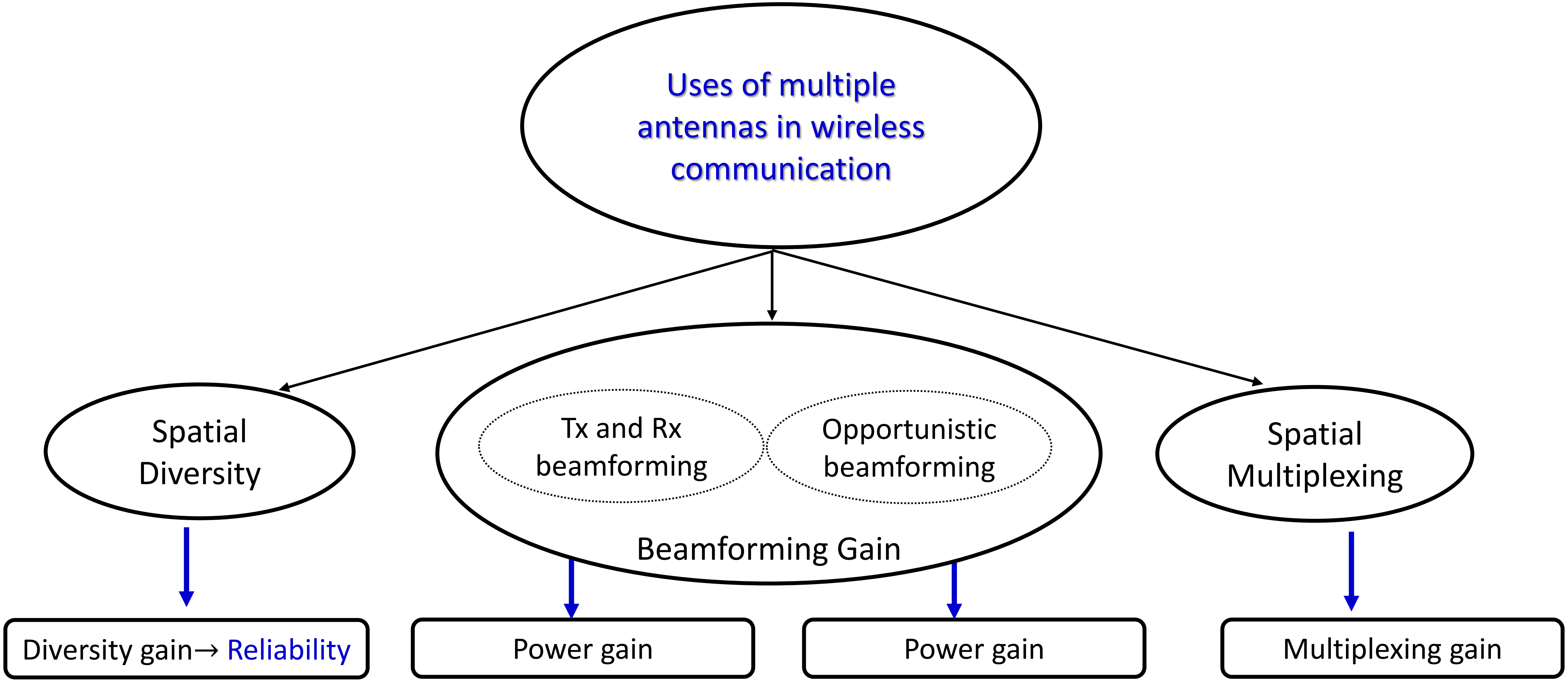
Introduction



Multiplexing Capability of Deterministic MIMO Channels

Time-invariant channel:

$$\boldsymbol{y} = \mathbf{H}\mathbf{x} + \boldsymbol{w}$$

$$\mathbf{X} \in \mathrm{C}^{n_t},\ \boldsymbol{y} \in \mathrm{C}^{n_r},\ \boldsymbol{w} \sim \mathcal{CN}\left(o, N_o \mathbf{I}_{n_r}\right)$$

$\mathbf{H} \in \mathrm{C}^{n_r x n_t}$ $\rightarrow$ deterministic and known to both transmitter and receiver

To find the capacity of **H**:

$$\mathbf{H} \longrightarrow \boxed{\text{SVD}} \longrightarrow \mathrm{U}\boldsymbol{\Lambda}\mathrm{V}^*$$

Decompose H into parallel, independent scalar Gaussian sub-channels

$\mathbf{U} \in \mathrm{C}^{n_r x n_r}$, $\mathbf{V} \in \mathrm{C}^{n_t x n_t}$ are unitary matrices

$\boldsymbol{\Lambda} \in \mathfrak{R}^{n_r x n_t}$, is a rectangular matrix with non-negative singular-valued diagonals ordered:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq, \lambda_{n_{min}} \text{ where } n_{min} \coloneqq \min(n_t, n_r)$$

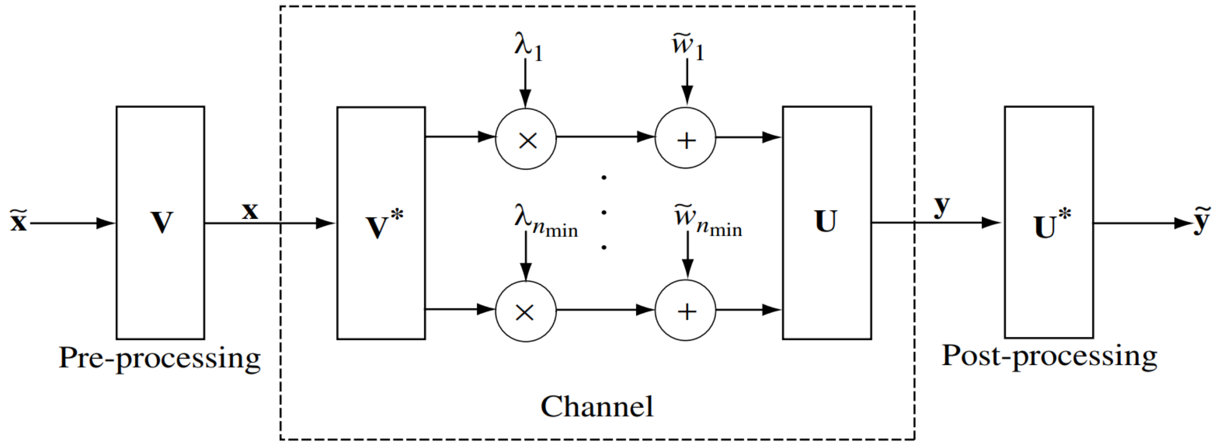
Since : $\mathbf{HH}^* = \mathbf{U}\Lambda\Lambda^t\mathbf{U}^*$,

The squared singular values λ_i^2 are eigenvalues of $\mathbf{HH}^*$ and $\mathbf{H}^*\mathbf{H}$

$$\Rightarrow \mathbf{H} = \sum_{i=1}^{n_{min}} \lambda_i \mathbf{u}_i \mathbf{v}_i^*$$

We define: $\tilde{\mathbf{x}} \coloneqq \mathbf{V}^*\mathbf{x}$, $\tilde{\mathbf{y}} \coloneqq \mathbf{U}^*\mathbf{y}$, $\tilde{\mathbf{w}} \coloneqq \mathbf{U}^*\mathbf{w}$ *then*

$$\boxed{\tilde{\mathbf{y}} = \Lambda\tilde{\mathbf{x}} + \tilde{\mathbf{w}}}$$



$$C = \sum_{i=1}^{n_{min}} log\left(1 + \frac{P_i^* {\lambda_i}^2}{N_0}\right)$$

Water filling power allocations:

$$P_i^* = \left(\mu - \frac{N_0}{{\lambda_i}^2}\right)^+$$

Rank and Condition Number

MIMO channel at high SNR:

Key determiners of performance:
- Rank
- Condition number

- Equal power allocated to the non-zero eigenmodes is asymptotically optimal.

$$C \approx \frac{1}{k}\sum_{i=1}^{k} log\left(1+\frac{P}{kN_0}{\lambda_i}^2\right) \approx klogSNR + \sum_{i=1}^{k} log\left(\frac{{\lambda_i}^2}{k}\right) \dots\dots\dots\dots\dots (1)$$

$$Where,\ k = number\ of\ non-zero\ {\lambda_i}^2$$

Multiplexing gain: $\boxed{k \approx rank(\mathbf{H}) \leq \min(n_t, n_r)}$

$$\sum_{i=1}^{k} {\lambda_i}^2 = \mathrm{Tr}[\mathbf{H}\mathbf{H}^*] = \sum_{i,j} |h_{ij}|^2$$

Condition number: $\frac{max_i \lambda_i}{min_i \lambda_i} \begin{cases} well-conditioned\ if\ \approx 1 \\ bad\ for\ spatial\ multiplexing\ if \gg 1 \end{cases}$

MIMO channel at low SNR:

- Rank and condition number → less relevant

The optimal policy is to allocate power only to the strongest eigenmode.

Capacity:

$$C \approx \frac{P}{N_0}\left(\max_i \lambda_i{}^2\right)\log_2 e \quad \text{bps/Hz}$$

No Multiplexing Gain

The MIMO channel provides a power gain of: $\max_i \lambda_i{}^2$

LOS SIMO and MISO Channel

Impulse response in passband: $h_i(\tau) = \alpha\delta\left(\tau - {}^{d_i}/_{c}\right),\ i = 1, \dots, n_r$

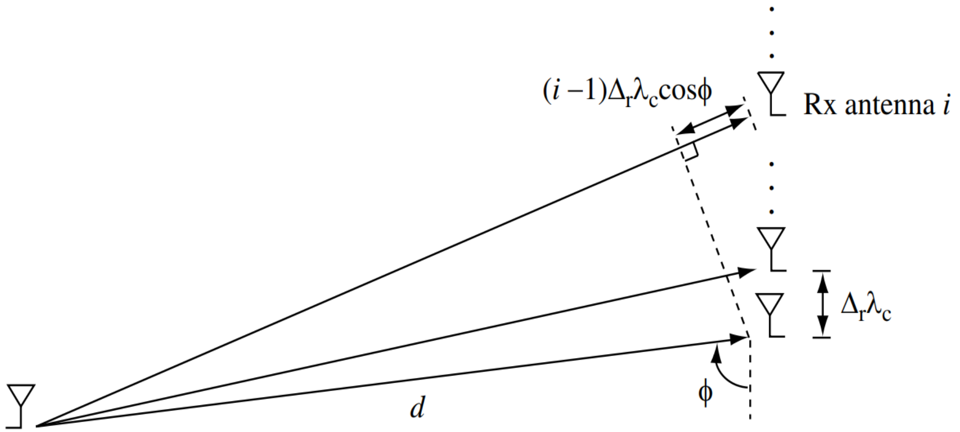
Equivalent complex baseband: $h_i = \alpha \exp\left(-\frac{j2\pi d_i}{\lambda_c}\right)$

- SIMO Channel:

$$\mathbf{y} = \mathbf{h}x + \mathbf{w},\ \textit{Where}\ \mathbf{h} = \left[h_1, \dots, h_{n_r}\right]^t \text{ is the } \textit{spatial signature.}$$

$$\mathbf{e_r}(\Omega) := \frac{1}{\sqrt{n_r}}\begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r\Omega) \\ \vdots \\ \exp(-j2\pi(n_r-1)\Delta_r\Omega) \end{bmatrix}$$

$$C = log\left(1 + \frac{P\|\mathbf{h}\|^2}{N_0}\right) = log\left(1 + \frac{P\alpha^2 n_r}{N_0}\right)$$

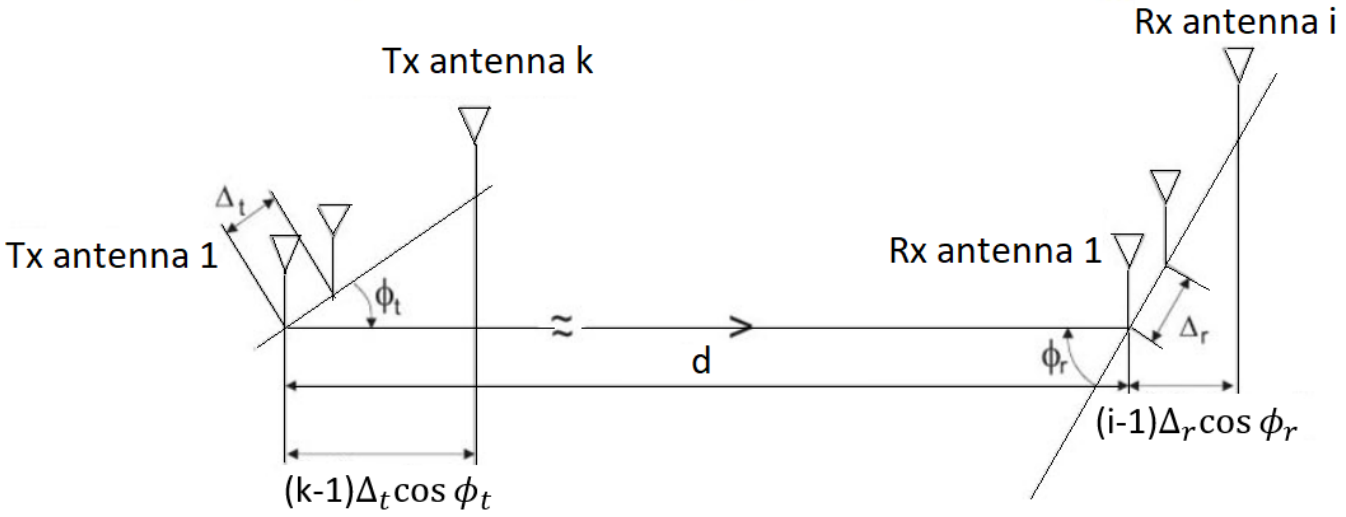


Power Gain but No degree of freedom Gain

MIMO With Only an LOS Path

MIMO channel with only direct line-of-sight paths between the antennas:

Unique nonzero singular value:
$\lambda_1 = \alpha\sqrt{n_t n_r}$



- Channel matrix:

$$\mathbf{H} = \alpha\sqrt{n_t n_r}\exp\left(-\frac{j2\pi d}{\lambda_c}\right)\mathbf{e}_\mathbf{r}(\Omega_\mathrm{r})\mathbf{e}_\mathbf{t}(\Omega_\mathrm{t})^* \;\ldots.\;(2)$$

- Capacity:

$$C = log\left(1 + \frac{P\alpha^2 n_t n_r}{N_0}\right) \ldots\ldots\ldots(3)$$

Power Gain but again, No degree of freedom Gain

Separated MIMO Transmit Antennas

Spatial signature transmit antenna k impinges on the receive antenna array:

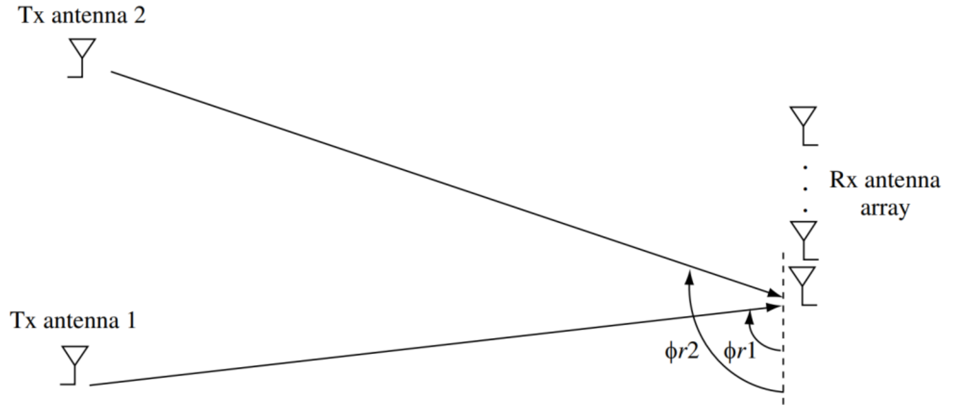
$$\mathbf{h}_k = \boldsymbol{\alpha}_k \sqrt{n_r} \exp\left(-\frac{j2\pi d_{1k}}{\lambda_c}\right) \mathbf{e}_{\mathbf{r}}(\Omega_{\mathrm{rk}}), k = 1, 2$$

Channel matrix, $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$ has distinct and linearly independent columns if separation in the directional cosine:

$$\boxed{\Omega_{\mathrm{r}} := \Omega_{r2} - \Omega_{r1} \neq 0 \mod \frac{1}{\Delta_r}}$$

Consequence: Two non-zero singular values $\lambda_1{}^2, \lambda_2{}^2 \Rightarrow$ **Two degrees of freedom**

NB: At this point **H** can still be ill-conditioned



How large can the angular separation be so that :

- **H** is well-conditioned?
- Two degrees of freedom can be utilized to achieve high capacity?

- Conditioning of **H** : determined by alignment of the spatial signatures of the two transmit antennas. The less aligned, the better the conditioning of **H**

$$f_r(\Omega_{\mathrm{r2}} - \Omega_{\mathrm{r1}}) \coloneqq \mathbf{e_r}(\Omega_{\mathrm{r1}})^* \mathbf{e_r}(\Omega_{\mathrm{r2}}) \ldots \ldots \ldots (4)$$

$$f_r(\Omega_{\mathrm{r}}) = \frac{1}{n_r} \exp\big(j\pi\Delta_r\Omega_r(\mathrm{n}_r - 1)\big) \frac{\sin(\pi L_r \Omega_r)}{\sin\left(\frac{\pi L_r \Omega_r}{\mathrm{n}_r}\right)}$$

Where, $L_{\mathrm{r}} \coloneqq \mathrm{n}_r\Delta_r$ is normalized length of receive antenna array.

Hence, $f_r(\Omega_{\mathrm{r}}) = |cos\theta| = \left| \frac{\sin(\pi L_r \Omega_r)}{\mathrm{n}_r \sin\left(\frac{\pi L_r \Omega_r}{\mathrm{n}_r}\right)} \right| \ldots \ldots \ldots \ldots (5)$

Assuming gains are the same,

$$\lambda_1{}^2 = \alpha^2 n_r(1 + |cos\theta|), \qquad \lambda_2{}^2 = \alpha^2 n_r(1 - |cos\theta|)$$

∴ condition of **H**:

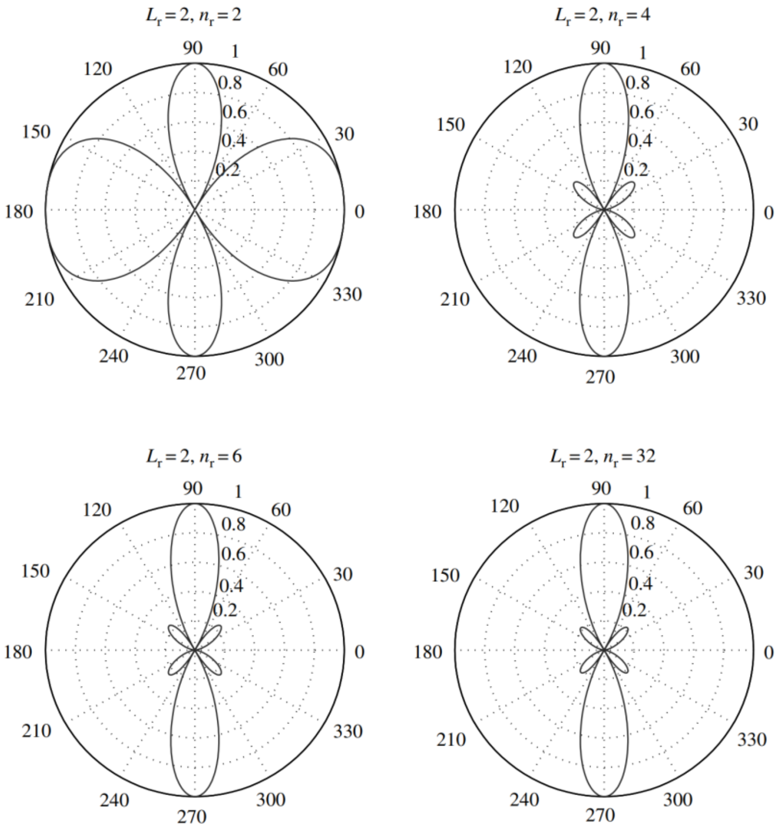
$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{1 + |cos\theta|}{1 - |cos\theta|}} \begin{cases} \text{ill} - \text{conditioned}, cos\theta \approx 1, \\ \text{well} - \text{conditioned}, \text{otherwise} \end{cases}$$

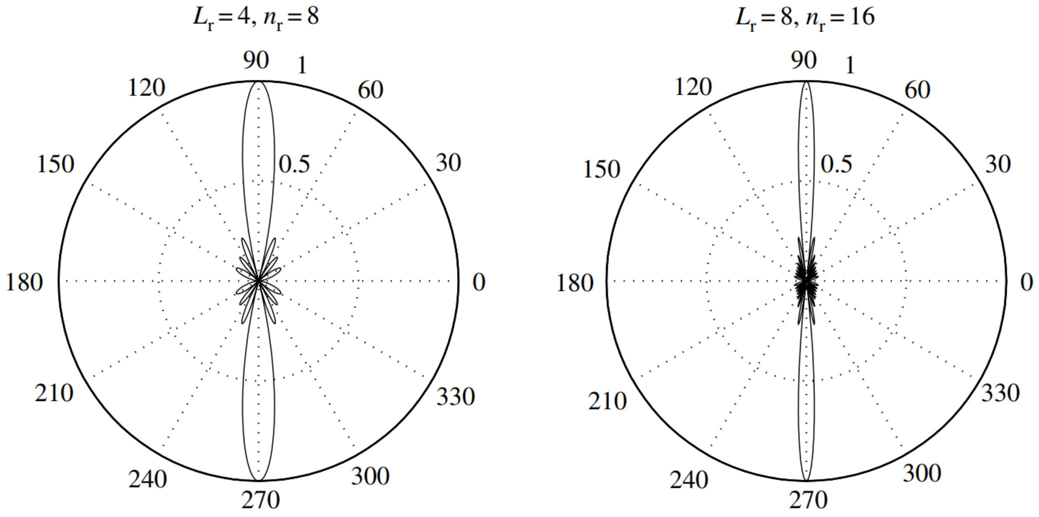
- $f_r(\Omega_\mathrm{r})$ is periodic with period: $\frac{n_r}{L_r} = \frac{1}{\Delta_r}$
- $f_r(\Omega_\mathrm{r})$ peaks at $\Omega_\mathrm{r} = 0; f_r(0) = 1$
- $f_r(\Omega_\mathrm{r}) = 0$ at $\Omega_\mathrm{r} = {}^{k}/_{L_r}\ , k = 1, .., n_r - 1$

Resolvability in angular domain:

- $|\Omega_\mathrm{r}| \ll \frac{1}{L_r}$
- $\Delta_r \leq {}^{1}/_{2}$

Angular Resolution





Main lobe-1 symmetric: $\Delta_r \leq {}^{1}/_{2}$

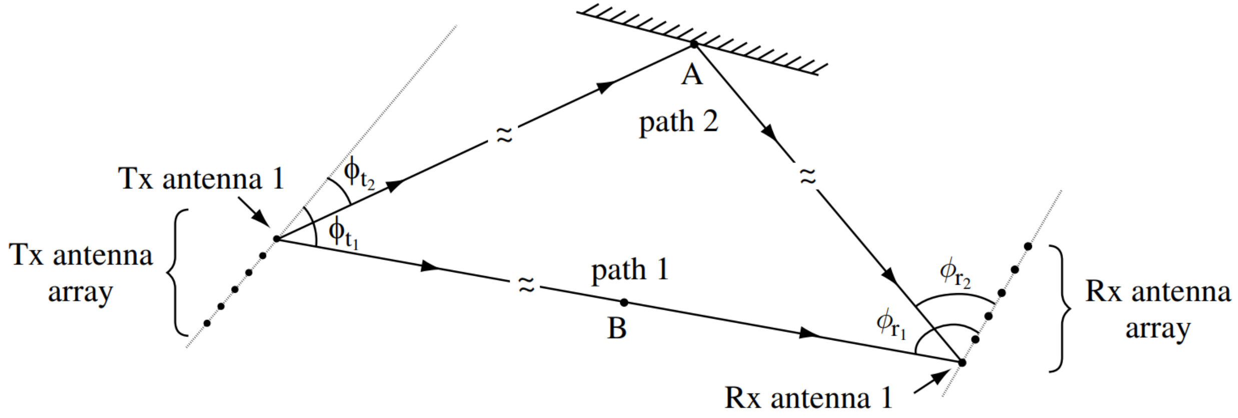
Main lobe-2 different symmetric: $\Delta_r \geq {}^{1}/_{2}$

Beamwidth: $\frac{2}{L_r}$

If the signal arrives from a single direction ϕ_0, then the optimal receiver projects the received signal onto the vector $\mathbf{e_r}(\cos\phi_0)$

MIMO With Two Transmission Paths

By the principle of superposition:



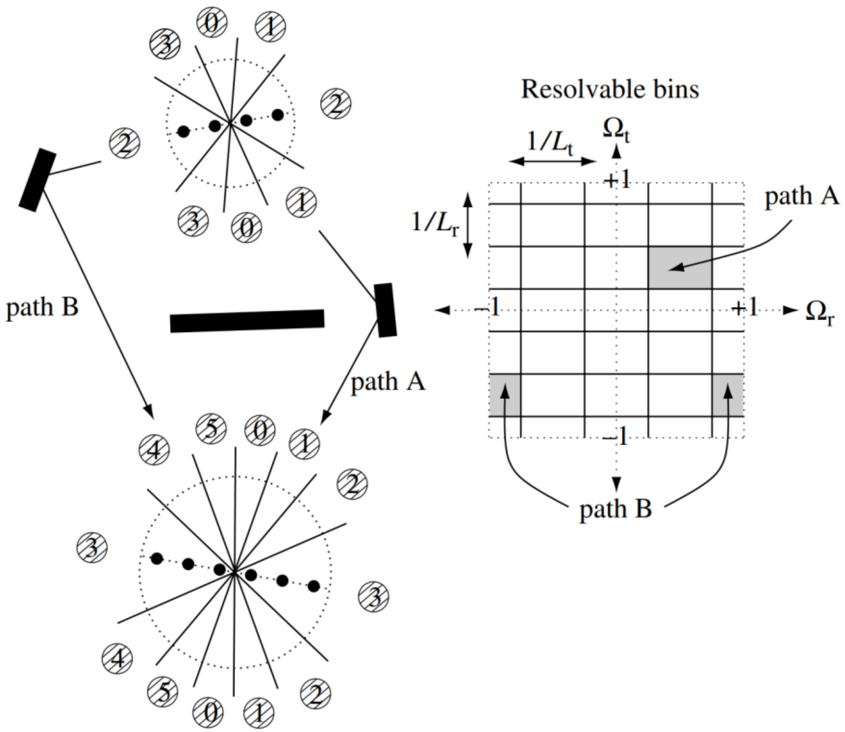
$$\mathbf{H} = \alpha_1^b \mathbf{e_r}(\Omega_{r1})\mathbf{e_t}(\Omega_{t1})^* + \alpha_2^b \mathbf{e_r}(\Omega_{r2})\mathbf{e_t}(\Omega_{t2})^* \quad \ldots\ldots\ldots (7)$$

- Beamforming pattern gives us antenna gain in different directions.
- Also importantly, gives us information on **angular resolvability**.

Multipath environments increase the rank of the channel matrix.

Modeling of MIMO Fading Channels

- L_r and L_t dictate degree of resolvability in angular domain.
- Paths with $|\Omega_\mathrm{r}| \ll \frac{1}{L_r}$ and $|\Omega_\mathrm{t}| \ll \frac{1}{L_t}$ are not resolvable by the arrays.



We do: Sample angular domain at fixed spacings.

For outgoing and incoming paths

Physical paths grouped into resolvable bins of angular width $\frac{1}{L_r}$ by $\frac{1}{L_t}$

Four receive antennas $(L_r = 2)$ and six transmit antennas $(L_t = 3)$

MIMO Multipath Channel

Consider MIMO multipath channel:

$$\boldsymbol{y} = \mathbf{Hx} + \boldsymbol{w} \ldots\ldots\ldots.. (7)$$

Let n_t and n_r be ULA of normalized lengths L_t and L_r respectively

Normalized separations: $\Delta_t = \frac{n_t}{L_t}$, $\Delta_r = \frac{n_r}{L_r}$

For an arbitrary number of physical paths between the transmitter and the receiver:

ith path has:

Attenuation: α_i

angle with transmitter array: $\phi_{ti}(\Omega_{ti} = \cos\phi_{ti})$

angle with receiver array: $\phi_{ri}(\Omega_{ri} = \cos\phi_{ri})$

Channel matrix **H**: $\sum_i \alpha_i^b \mathbf{e_r}(\Omega_{ri})\mathbf{e_t}(\Omega_{ti})^*$

$$\alpha_i^{\mathrm{b}} \coloneqq \alpha_i \sqrt{n_t n_r} \exp\left(-\frac{j2\pi d^{(i)}}{\lambda_c}\right)$$

$$\mathbf{e_r}(\Omega) \coloneqq \frac{1}{\sqrt{n_r}} \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r\Omega_r) \\ \vdots \\ \exp(-j2\pi(n_r-1)\Delta_r\Omega_r) \end{bmatrix}, \mathbf{e}_{\boldsymbol{t}}(\Omega) \coloneqq \frac{1}{\sqrt{n_t}} \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_t\Omega_t) \\ \vdots \\ \exp(-j2\pi(n_t-1)\Delta_t\Omega_t) \end{bmatrix}$$

$d^{(i)}$: distance between Tx1 and Rx1 on path i

Angular domain MIMO:

$$\boldsymbol{y^a} = \mathbf{U}_\mathrm{r}^*\mathbf{H}\mathbf{U}_\mathrm{t}\mathbf{x}^{\boldsymbol{a}} + \mathbf{U}_\mathrm{r}^*\mathbf{w} = \mathbf{H}^{\boldsymbol{a}}\mathbf{x}^{\boldsymbol{a}} + \mathbf{w}^{\boldsymbol{a}}$$

$\mathbf{U}_\mathrm{t}$ and $\mathbf{U}_\mathrm{r}$:are n_tx n_t and n_rx n_r unitary matrices respectively

$$h_{kl}^a = \mathbf{e}_\mathbf{r}\left({}^{k}/_{L_r}\right)^* \mathbf{H}\mathbf{e}_{\boldsymbol{t}}\left({}^{l}/_{L_t}\right)$$

$$\sum_i \alpha_i^b \left[\mathbf{e}_{\boldsymbol{r}}\left({}^{k}/_{L_r}\right)^* \mathbf{e}_\mathbf{r}(\Omega_{ri})\right]\left[\mathbf{e}_{\boldsymbol{t}}(\Omega_{ti})^*\mathbf{e}_{\boldsymbol{t}}\left({}^{l}/_{L_r}\right)\right]$$

$\mathbf{e}_{\boldsymbol{r}}\left({}^{k}/_{L_r}\right) \rightarrow$ main lobe around ${}^{k}/_{L_r}$

Bin

$\mathcal{R}_{\boldsymbol{k}}$ {all physical paths with most energy along $\mathbf{e}_{\boldsymbol{r}}\left({}^{k}/_{L_r}\right)$}

Bin $\mathfrak{I}_{\boldsymbol{l}}$ {all physical paths with most energy along $\mathbf{e}_{\boldsymbol{t}}\left({}^{l}/_{L_r}\right)$}

$$\mathbf{e}_{\boldsymbol{r}}\left({}^{k}/_{L_r}\right)^* \mathbf{e}_\mathbf{r}(\Omega_{ri})$$

Is significant for the ith path if:

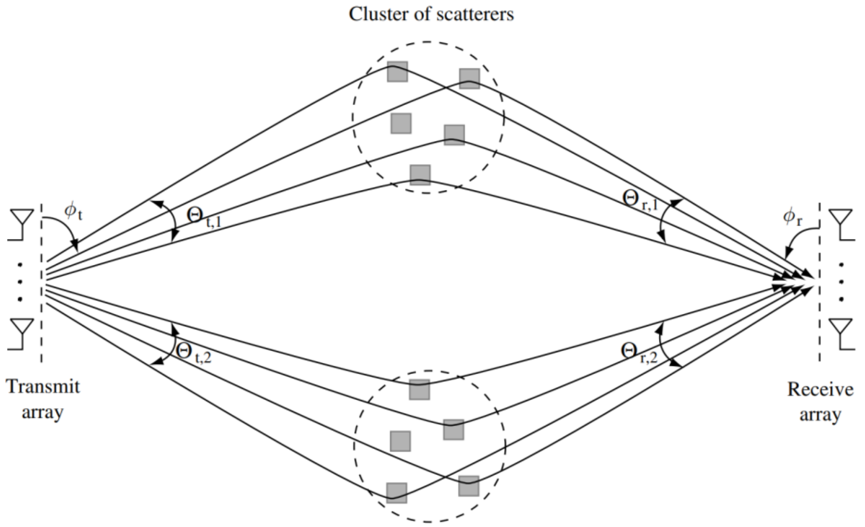
$$\left|\Omega_{ri} - \frac{k}{L_r}\right| < \frac{1}{L_r}$$

Clustered Response Model

Scatters and reflectors not located at

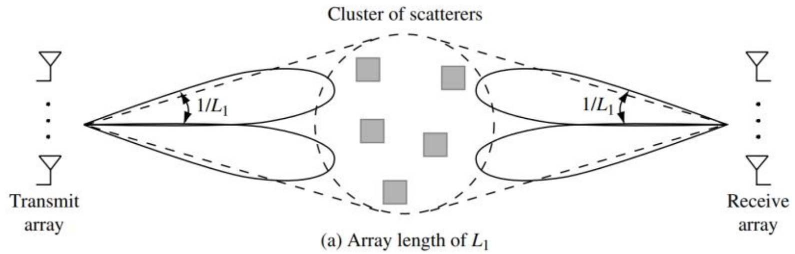
All directions from transmitter or receiver

- But are grouped into several clusters.

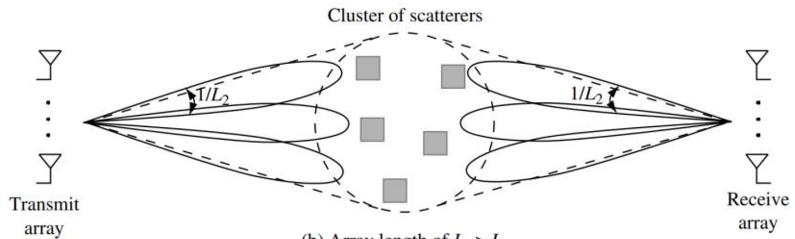


For L_r and L_t ,no. of degrees of freedom depends on:

- $\min(L_r\Omega_{r\ total}, L_t\Omega_{t\ total})$



(a) Array length of L_1



(b) Array length of $L_2 > L_1$

I.I.D Rayleigh Fading Model

Entries of channel gain matrix:

$$\mathbf{H}^{a}[m] \coloneqq \mathbf{U}_{\mathrm{r}}^{*}\mathbf{H}[m]\mathbf{U}_{\mathrm{t}} \rightarrow i.i.d\ CSCG$$

Physical basis of $i.i.d$ Rayleigh fading model:

richly scattered environment

- Significant number of multipaths in each of the resolvable angular bins.
- The energy should be equally spread out across these bins.

Antennas should be either ***critically*** or ***sparsely*** spaced.

- Sparser spacing → $easy\ to\ satisfy\ i.i.d\ CSCG$
- Entries of **H** → *less dependent*

If physical environment already provides scattering in all directions.

- Critical spacing → $i.i.d\ CSCG$
- Entries of **H** → *less dependent*

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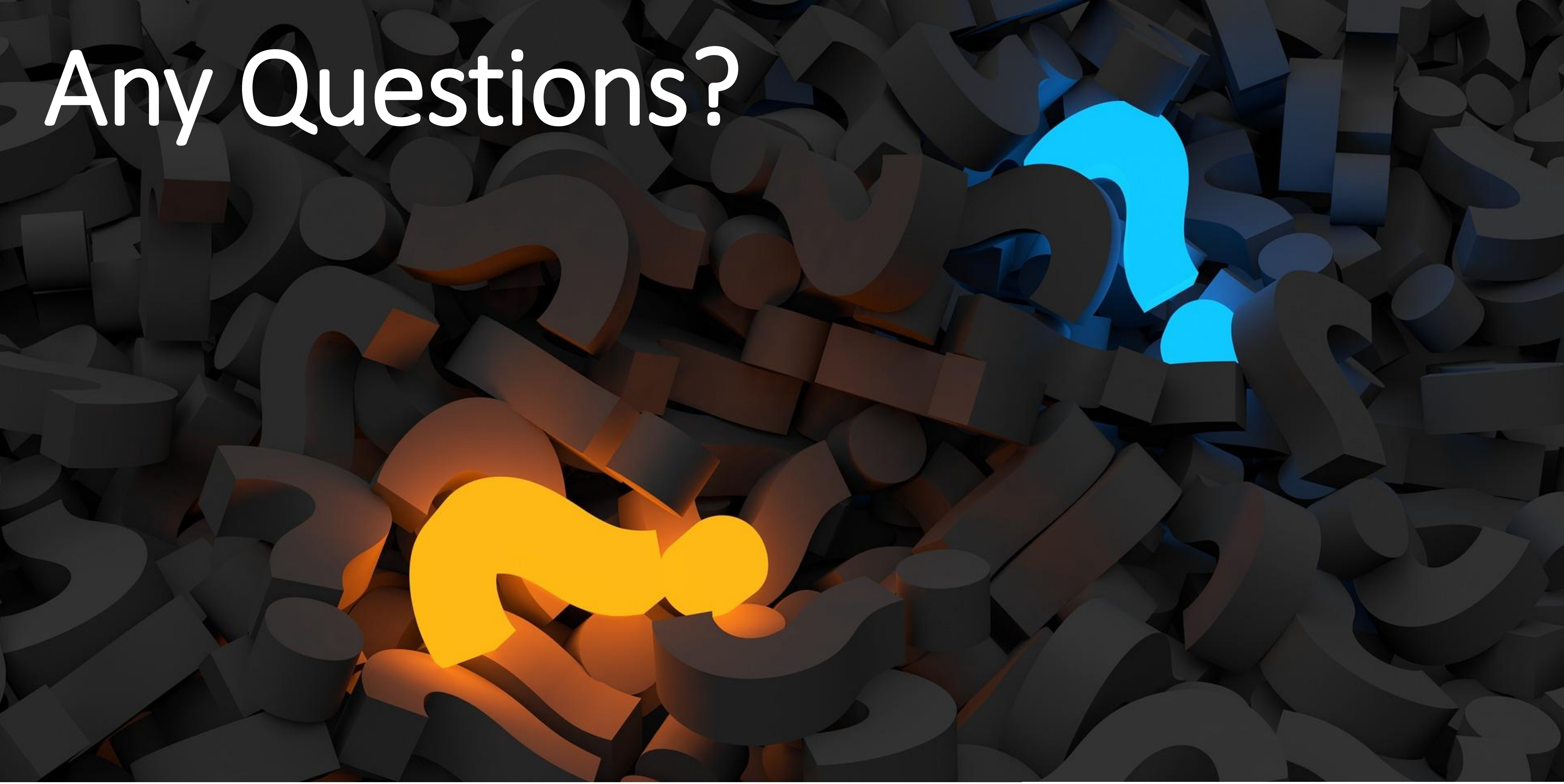
Any Questions?



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