Excercise 3.2. < 인공지능 대학원 2020 2691 강민국 >

To calculate $\lim(\sqrt{n^2+n}-n)$, we first show the given sequence converges, and then identify the sequence has $\sup(\sqrt{n^2+n}-n)=\alpha$.

① boundness.

Let S_n be the sequence $\sqrt{n^2+n}-n$ in $\mathbb{R}^1$.

If we assume S_n is bounded above and $1/2$ is a upper bound of S_n, then the following inequality should be satisfied.

$$\sqrt{n^2+n}-n \le \frac{1}{2}$$
$$\Rightarrow \sqrt{n^2+n} \le n+\frac{1}{2}$$
$$\Rightarrow n^2+n \le n^2+n+\frac{1}{4}$$
$$\Rightarrow 0 \le \frac{1}{4}$$

$\therefore$ $1/2$ is a upper bound of S_n ▫

② monotonic

Suppose $S_n \le S_{n+1}$, then the following inequality should be satisfied and necessary and sufficient condition.

$$\sqrt{n^2+n}-n \le \sqrt{(n+1)^2+n+1}-(n+1)$$
$$\Rightarrow \sqrt{n^2+n}+1 \le \sqrt{(n+1)^2+n+1}$$
$$\Rightarrow n^2+n+1+2\sqrt{n^2+n} \le n^2+3n+2$$
$$\Rightarrow 2\sqrt{n^2+n} \le 2n+1$$
$$\Rightarrow 4n^2+4n \le 4n^2+4n+1$$
$$\Rightarrow 0 \le 1$$ ▫

③ If $x < \frac{1}{2}$ and x is the least upper bound of S_n, then the following statement should be satisfied.

$$\sqrt{n^2+n} - n \le x$$

$$\Rightarrow \sqrt{n^2+n} \le x+n$$

$$\Rightarrow n^2+n \le x^2+2nx+n^2$$

$$\Rightarrow n(1-2x) \le x^2$$

$$\Rightarrow n \le \frac{x^2}{1-2x}$$

Since n is all possible positive integers and $x < \frac{1}{2}$,

The above inequality has contradicts (i.e. $x = \frac{1}{4}$).

$\therefore$ The monotonic and bounded sequence S_n converges and

$\lim \sqrt{n^2+n} - n = \frac{1}{2}$ □.

Excercise 3.5

Let $\limsup_{n\to\infty} a_n$, $\limsup_{n\to\infty} b_n$, $\limsup_{n\to\infty} (a_n+b_n)$ be a A, B, C.

If we assume $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ is not the form $\infty - \infty$,

we can make subsequences of a_n+b_n as follow:

$$\{a_n+b_n\} = \{a_1+b_1, a_2+b_2, a_3+b_3, \cdots\}$$

Select terms both term are positive.

$$\{a_j+b_j\}$$

Since all subsequences of a convergent sequence converge the same point, $\{a_j+b_j\}$ converges to C.

However, we can make subsequence of $\{a_{nk}\}, \{b_{nk}\}$ s.t $C = \lim_{k\to\infty}\{a_{n_k}\} + \lim_{k\to\infty}\{b_{n_k}\}$.

By the definition of upper limit, we can rewrite above equation as following inequality:

$$C = \limsup_{n\to\infty} (a_n+b_n) = \lim_{k\to\infty}\{a_{n_k}\} + \lim_{k\to\infty}\{b_{n_k}\} \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n .$$

Excercise 3.7

let t_n be $\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$ and n is positive integer, then t_n is a partial sum of $\sum \frac{\sqrt{a_n}}{n}$.

By the Cauchy-Schwarz inequality, we can make a new inequality as follow:

$$t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \leq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{k^2}\right) \leq \infty .$$

Since all partial sums of $\sum \frac{\sqrt{a_n}}{n}$ are bounded and $\frac{\sqrt{a_k}}{k}$ is non-negative, by the theorem 3.24, It is self-evident that $\sum \frac{\sqrt{a_n}}{n}$ Converges when $\sum a_n$ Converges. □

Excercise 3.8

we know two facts that $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded.

① For all $\varepsilon > 0$, there exists $m \geq n \geq N$ s.t $\left|\sum_{k=n}^{m} a_k\right| \leq \varepsilon$.

② $\{b_n\}$ is bounded, we can say that for any $\alpha \in \mathbb{R}$ and every $n \in \mathbb{Z}$, $\{|b_n|\} \leq \alpha$.

Since statement ① is always true for all $\varepsilon > 0$, we can rewrite as follow:

$$\forall \varepsilon > 0, \ \exists \ m \geq n \geq N, \ m, n \in \mathbb{Z} \quad \text{s.t} \ \left|\sum_{k=n}^{m} a_k\right| \leq \frac{\varepsilon}{\alpha}$$

$$\Rightarrow \left|\sum_{k=n}^{m} \alpha a_k\right| \leq \varepsilon$$

$$\Rightarrow \left|\sum_{k=n}^{m} b_k a_k\right| \leq \left|\sum_{k=n}^{m} \alpha a_k\right| \leq \varepsilon .$$

$\therefore \sum_{n=1}^{\infty} a_n b_n$ also Converges □

Excercise 3.9

(a) $\limsup_{n\to\infty} \sqrt[n]{n^3} = \limsup_{n\to\infty} n^{\frac{3}{n}} = \left(\limsup_{n\to\infty} n^{\frac{1}{n}}\right)^3 = \alpha.$

By the theorem 3.20 (c), above equation converges to 1. □.

$\therefore R = \frac{1}{\alpha} = 1$ □.

(b) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n!}\right|} = \limsup_{n\to\infty} \left(\frac{2^n}{n!}\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{2}{(n!)^{1/n}} = \alpha$.

Since I can't process above equation further, detour this using ratio test.

By the theorem 3.37, we can say following inequality is true.

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n!}} \le \limsup_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \le \limsup_{n\to\infty} \frac{2}{n+1} = +0$$

$\Rightarrow R = \frac{1}{\alpha} \ge \infty$ □.

(c) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^2}\right|} = \limsup_{n\to\infty} \frac{2}{n^{2/n}} = 2\left(\limsup_{n\to\infty} \sqrt[n]{n}\right)^2 = \alpha.$

By the Theorem 3.20 (c), $\underbrace{\limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}}}_{\alpha} = 2(1)$.

$\therefore R = \frac{1}{\alpha} = \frac{1}{2}$ □

(d) $\limsup_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|} = \limsup_{n\to\infty} \frac{n^{\frac{3}{n}}}{3} = \frac{1}{3} \limsup_{n\to\infty} n^{\frac{3}{n}} = \alpha$

By the excercise 3.9 (a), $\alpha = \frac{1}{3} \limsup_{n\to\infty} n^{\frac{3}{n}} = \frac{1}{3}$.

$\therefore R = \frac{1}{\alpha} = 3$ □.

Excercise 3.12.

Let r_n be the summation of $\sum_{m=n}^{\infty} a_m$, thus $a_k = -r_{k+1} + r_k > 0$.

Since $r_k > r_{k+1}$ is satisfied, we can form below inequality.

$$\sum_{k=m}^{n} \frac{a_k}{r_k} = \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \quad \text{for } n \geq m$$

$$\geq \frac{a_m + a_{m+1} + \cdots + a_n}{r_m}$$

$$= \frac{(r_m - r_{m+1}) + (r_{m+1} - r_m) + \cdots + (r_n - r_{n+1})}{r_m}$$

$$= \frac{r_m - r_{n+1}}{r_m}$$

$$> \frac{r_m - r_n}{r_m}$$

$$= 1 - \frac{r_n}{r_m}.$$

Also, by the condition $\sum a_n = \alpha$, we can calculate $\sum \frac{a_n}{r_n}$ iff and only iff r_n converges and does not converge to zero.

By the definition, below equation is satisfied.

$$r_n = \sum_{m=n}^{\infty} a_m$$

$$= \sum_{m=1}^{\infty} a_n - \sum_{m=1}^{n} a_m$$

$$= \alpha - \sum_{m=1}^{n} a_m$$

And its limit is $\lim_{n \to \infty} r_n = \alpha - \sum_{m=1}^{\infty} a_m = \alpha - \alpha = 0$.

$\therefore \sum \frac{a_n}{r_n}$ diverges.

Excercise 13.

Let $\sum_{k=0}^{n} a_k$, $\sum_{k=0}^{n} b_k$ be absolute Convergent series and Converge to α and β.

Cauchy product of $\sum a_k$, $\sum b_k$ is as follow:

$$\sum_{k=0}^{n} C_k = \sum_{k=0}^{n}\left(\sum_{l=0}^{k} a_l b_{k-l}\right).$$

To identify absolute Convergence of $\sum_{k=1}^{n} C_k$, we add absolute operations on the left and right side of above equation.

$$\sum_{k=0}^{n}|C_k| = \sum_{k=0}^{n}\left|\sum_{l=0}^{k} a_l b_{k-l}\right|$$

$$\Rightarrow \quad \le \sum_{k=0}^{n}\sum_{l=0}^{k}|a_l b_{k-l}|$$

$$\Rightarrow \quad = \sum_{k=0}^{n}\sum_{l=0}^{k}|a_l|\,|b_{k-l}|$$

$$\Rightarrow \quad = |a_0||b_0| + |a_0||b_1| + |a_1||b_0| + |a_0||b_2| + |a_1||b_1| + |a_2||b_0| + \cdots + |a_0||b_n| + |a_1||b_{n-1}| + \cdots |a_n||b_0|$$

$$\Rightarrow \quad = |a_0|(|b_0| + |b_1| + \cdots |b_n|) + |a_1|(|b_0| + |b_1| + \cdots |b_{n-1}|) + \cdots + |a_n||b_0|$$

$$\Rightarrow \quad \le |a_0|(|b_0| + |b_1| + \cdots + |b_n|) + |a_1|(|b_0| + |b_1| + \cdots |b_n|) + \cdots + |a_n|(|b_0| + |b_1| + \cdots |b_n|)$$

$$\Rightarrow \quad = (|a_0| + |a_1| + \cdots + |a_n|)(|b_0| + |b_1| + \cdots + |b_n|)$$

$$= \alpha\beta$$

$\therefore$ $\sum_{k=0}^{n}|C_k| \le \alpha\beta$ and this says Cauchy product of two absolutely Convergent Series Converges absolutely.

Excercise 3.14.

(a) Let s_n Converge to s, then there exists positive integers $n \geq N$ and ε is all positive real numbers s.t $d(s_n, s) < \varepsilon$.

Using above property, we can rewrite $|\sigma_n - s|$ as follow:

$$|\sigma_n - s| = \left| \frac{s_0 - s + s_1 - s + \cdots + s_n - s}{n+1} \right|$$

$$\leq \frac{|s_0 - s| + |s_1 - s| + \cdots |s_N - s|}{n+1} + \frac{|s_{N+1} - s| + \cdots + |s_n - s|}{n+1}$$

$$\leq \frac{(N+1) \max_{0 \leq j \leq N} (s_j - s)}{n+1} + \frac{(N+1) \max_{N+1 \leq j \leq n} (s_j - s)}{n+1}$$

$$\leq \frac{(N+1) \max_{0 \leq j \leq N} (s_j - s)}{n+1} + \frac{(N+1) \max_{N+1 \leq j \leq n} (\varepsilon)}{n+1}$$

$$= \underbrace{\frac{(N+1) \max_{0 \leq j \leq N} (s_j - s)}{n+1}}_{①} + \underbrace{\varepsilon \cdot \frac{N+1}{n+1}}_{②}$$

Since the term ① has fixed numerator, we can summerize above inequality as follow:

$$\frac{(N+1) \max_{0 \leq j \leq N} (s_j - s)}{n+1} + \varepsilon \cdot \frac{N+1}{n+1} \leq \frac{(N+1) \max_{0 \leq j \leq N} (s_j - s)}{n+1} + \varepsilon$$

$$\leq \varepsilon + \varepsilon$$

$$= 2\varepsilon$$

$\therefore \lim_{n \to \infty} \sigma_n = s$ □

(b) the sequence $\{s_n\} = (-1)^n$ does not converge, although $\lim_{n\to\infty} \sigma_n = 0$ □

(c) yes, Let $s_n = \begin{cases} 0 & n=0 \\ \log n & n \geq 1, \quad n \text{ is a perfect square.} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$

Since s_n has terms that increase and unbounded, $\limsup_{n\to\infty} s_n$ is infinity.

However, $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{0+\log 1+\frac{1}{2}+\frac{1}{3}+\log 4+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\log 9+\cdots}{n+1}$

$$\leq \lim_{n\to\infty} \frac{1}{n+1}\left(0+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)+\lim_{n\to\infty}\frac{1}{n+1}\left(1+2+3+\cdots+\lceil \log n \rceil\right).$$

$$\cancel{\lim_{n\to\infty}\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+\lim_{n\to\infty}\frac{1}{n+1}\left(\log 1+\log 2+\cdots+\log n\right)}$$

$$\cancel{\lim_{n\to\infty}\frac{1}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)+}$$

For $(m-1)^2 \leq n \leq m^2$, we can reformulate above inequality as follow:

$$\lim_{n\to\infty}\sigma_n \leq \lim_{n\to\infty}\frac{1}{n+1}\left(0+1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+\lim_{m\to\infty}\frac{1}{(m-1)^2}\left(\log 1+\log 2+\cdots+\log m\right)$$

$$\leq \lim_{n\to\infty}\frac{1}{n+1}\left(0+1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+\lim_{m\to\infty}\frac{1}{m+\frac{1}{m}-2}\cdot\log m$$

$$= \lim_{n\to\infty}\frac{1}{n+1}\left(0+1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+\lim_{m\to\infty}\frac{m\log m}{m^2-2m+1}$$

By the exercise 3.14 (a) and the fact $\lim_{m\to\infty}\frac{m\log m}{m^2-2m+1}=0$,

$$\lim_{n\to\infty}\sigma_n \leq 0+0$$

$$= 0$$ □

(d) Let $a_n = s_n - s_{n-1}$ and $n \geq 1$, then followings are satisfied.

$$s_n - \sigma_n = s_n - \left(\frac{s_0 + s_1 + \cdots + s_n}{n+1} \right)$$

$$= \frac{s_n - s_0 + s_n - s_1 + \cdots \ s_n - s_n}{n+1}$$

$$= \frac{(s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots \ s_1 - s_0) + (s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots \ s_2 - s_1) + \cdots}{n+1}$$

$$= \frac{(a_n + a_{n-1} + \cdots a_1) + (a_n + a_{n-1} + \cdots + a_2) + (a_n + a_{n-1} + \cdots + a_3)}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=1}^{n} k a_k$$

By the 3.14 (a), If $\lim_{n\to\infty} n a_n = 0$, then $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n} k a_k$ is also 0.

Above Condition says that $\lim_{n\to\infty} s_n - \sigma_n$ is 0 and the Condition $\lim_{n\to\infty} \sigma_n = \sigma$ says that $\lim_{n\to\infty} s_n$ is also σ.

☆(e)

Excercise 3.17

(a) $$x_{n+2} = \frac{\alpha + x_{n+1}}{1 + x_{n+1}} = \frac{\alpha + \frac{\alpha + x_n}{1 + x_n}}{1 + \frac{\alpha + x_n}{1 + x_n}} = \frac{\alpha + \alpha x_n + \alpha + x_n}{2x_n + \alpha + 1} = \frac{2\alpha + (\alpha+1)x_n}{2x_n + \alpha + 1}$$

$$= x_n + 2\left(\frac{\alpha - x_n^2}{\alpha + 1 + 2x_n} \right)$$

If $x_n > \sqrt{\alpha}$, then $x_{n+2} < x_n$.

Since $\alpha > 1$ be satisfied, above recurrence relation satisfies following inequality.

$$x_{n+2} = x_n + 2\left(\frac{(\sqrt{\alpha} - x_n)(\sqrt{\alpha} + x_n)}{\alpha + 1 + 2x_n} \right)$$

$$\geq x_n + \frac{2(\sqrt{\alpha} - x_n)(\sqrt{\alpha} + x_n)}{2(\sqrt{\alpha} + x_n)}$$

$$= x_n + \sqrt{\alpha} - x_n$$

$$= \sqrt{\alpha}$$

$\therefore$ For $x_1 > \sqrt{\alpha}$, $x_1 > x_3 > x_5 > \cdots$ is satisfied $\square$.

(b) let $x_1 > \sqrt{\alpha}$ and $x_2 = \frac{\alpha + x_1}{1 + x_1} = x_1 + \frac{\alpha - x_1^2}{1 + x_1}$.

Since $x_1 > \sqrt{\alpha}$, x_2 should be less than x_1 thus $\sqrt{\alpha}$.

By the results of (a), $x_{n+2} = x_n + 2 \frac{\alpha - x_n^2}{\alpha + 1 + 2x_n}$ should be satisfied.

Using same way in (a), we can demonstrate $x_2 < x_4 < x_6 \cdots$ $\square$

(c)

By the result (a),(b), we can say that the upper and lower bound of $\{x_{2n}\}$, $\{x_{2n+1}\}$ is $\sqrt{2}$.

let L_1, L_2 be the $\inf\{x_{2n+1}\}$ and $\sup\{x_{2n}\}$, then following should be satisfied.

$$L_1 = L_1 + 2\left(\frac{\alpha - L_1^2}{\alpha + 1 + 2L_1} \right) \Rightarrow \frac{\alpha - L_1^2}{\alpha + 1 + 2L_1} = 0$$

$$L_2 = L_2 + 2\left(\frac{\alpha - L_2^2}{\alpha + 1 + 2L_2} \right) \Rightarrow \frac{\alpha - L_2^2}{\alpha + 1 + 2L_2} = 0$$

$\therefore L_1 = L_2 = \sqrt{\alpha}$ and we can conclude that $\lim_{n \to \infty} x_n = \sqrt{\alpha}$ $\square$.

(d) Let $\varepsilon_n = |x_n - \sqrt{\alpha}|$, $\beta = \dfrac{\sqrt{\alpha}-1}{1+x_2}$ be an error and real number.

Since for all positive integer n, $x_2 < x_{2n-1}$ and $x_2 \leq x_{2n}$,

the following inequality is always satisfied.

$$1 + x_2 \leq 1 + x_i \quad \text{for all positive integer } i$$

$$\begin{aligned}
\varepsilon_{n+1} &= |x_{n+1} - \sqrt{\alpha}| \\
&= \left| \frac{\alpha + x_n}{1+x_n} - \sqrt{\alpha} \right| \\
&= \left| \frac{x_n - \sqrt{\alpha} - \sqrt{\alpha}(x_n - \sqrt{\alpha})}{1 + x_n} \right| \\
&= \left| \frac{x_n(1-\sqrt{\alpha}) + \sqrt{\alpha}(-1+\sqrt{\alpha})}{1+x_n} \right| \\
&= \left| \frac{1-\sqrt{\alpha}}{1+x_n} \right| \cdot \varepsilon_n \\
&\leq \beta\, \varepsilon_n.
\end{aligned}$$

Therefore $\varepsilon_n \leq \beta \varepsilon_{n-1} \leq \beta^2 \varepsilon_{n-2} \leq \cdots \leq \beta^{n-1} \cdot \varepsilon$.

we cannot compare the speed of convergence due to the fact that

I do not calculate the speed of 3.16.

Excercise 3.19

To insist $\chi(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$ is not Cantor set, we should show that

$\chi(a) \in \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ where $m \in \mathbb{N}$ and $k \in \{0\} \cup \mathbb{N}$.

we can summerize above statement as follow:

$$\chi(a) \in \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} \in \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\alpha_n}{3^{n-m}} \in (3k+1, 3k+2)$$

$$\Rightarrow \sum_{n=1}^{m-1} \frac{\alpha_n}{3^{n-m}} + \sum_{n=m}^{m} \frac{\alpha_n}{3^{n-m}} + \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}} \in (3k+1, 3k+2)$$

$$\Rightarrow 3A + \alpha_m + \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}} \in (3k+1, 3k+2), \quad A \in \mathbb{N}.$$

Since the lower bound of $\sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}}$ is zero and upper bound is one,

for some $B = \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}}$ and $0 \le B \le 1$, $3A + \alpha_m + B$ should be

a term of interval $(3k+1, 3k+2)$.

By the fact that $\alpha_m = 0$ or 2, there is no circumstance that

$3A + \alpha_m + B$ is an element of the interval. $(3k+1, 3k+2)$.

$\therefore \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$ is Cantor set. □

Excercise 3.20

Let ε be all positive Real number, then the convergent subsequence $\{p_{n_i}\}$ meets following condition.

$$d(p_{n_i}, p) \le \frac{\varepsilon}{2}, \quad n_i \ge N, \text{ and } N \in \mathbb{N}.$$

Also for all $\varepsilon > 0$, there exist positive integer M, n, m s.t $d(p_n, p_m) < \frac{\varepsilon}{2}$ and $n, m \ge M$.

By the triangular inequality, the following inequality is obvious.

$$d(p, p_m) \le d(p, p_{n_i}) + d(p_{n_i}, p_m) = \varepsilon.$$

$\therefore$ $\{p_n\}$ converges to p □

Excercise 3.21

Let $\{E_n\}$ be a closed and bounded subset of complete metric space X

S.t $\lim_{n\to\infty} \operatorname{diam} E_n$ is zero.

By the theorem 3.9 and 3.12, The sequence E_n is Cauchy and Convergent.

Since E_n is a closed sequence, the all possible intersections of E_n $(\bigcap E_n)$ are closed and every limit points of $\bigcap E_n$ is a point of $\bigcap E_n$.

Thus above statements say $\bigcap_{n=1}^{\infty} E_n$ has at least one point.

Suppose $E = \bigcap_{n=1}^{\infty} E_n$ has at least two points $p, q, r, \cdots$ s.t $d(p, q) > 0$.

Because $\operatorname{diam} E$ is the supremum of the distances among all points in E, the following inequality should be satisfied.

$$0 < d(p, q) \le \operatorname{diam} E \le \operatorname{diam} E_n$$

But $\lim_{n\to\infty} \operatorname{diam} E_n$ should be zero, $\bigcap_{n=1}^{\infty} E_n$ has an only one point. □

Excercise 3.23

For all arbitrary $\varepsilon > 0$, theve exists $N \in \mathbb{N}$ s.t $m, n \geq N$ and

$d(p_m, p_n) \leq \frac{\varepsilon}{2}$.

Same way, For all arbitrary $\varepsilon > 0$, there exists $M \in \mathbb{N}$ s.t $m, n \geq M$

and $d(q_m, q_n) \leq \frac{\varepsilon}{2}$.

By triangular inequality, for $n, m \geq \max(N, M)$, $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$

$$\leq d(p_n, p_m) + d(q_n, q_m) + d(q_m, p_m)$$

$$\leq \varepsilon + d(p_m, q_m)$$

$\therefore \; d(p_n, q_n) - d(p_m, q_m) \leq \varepsilon$ $\square$.