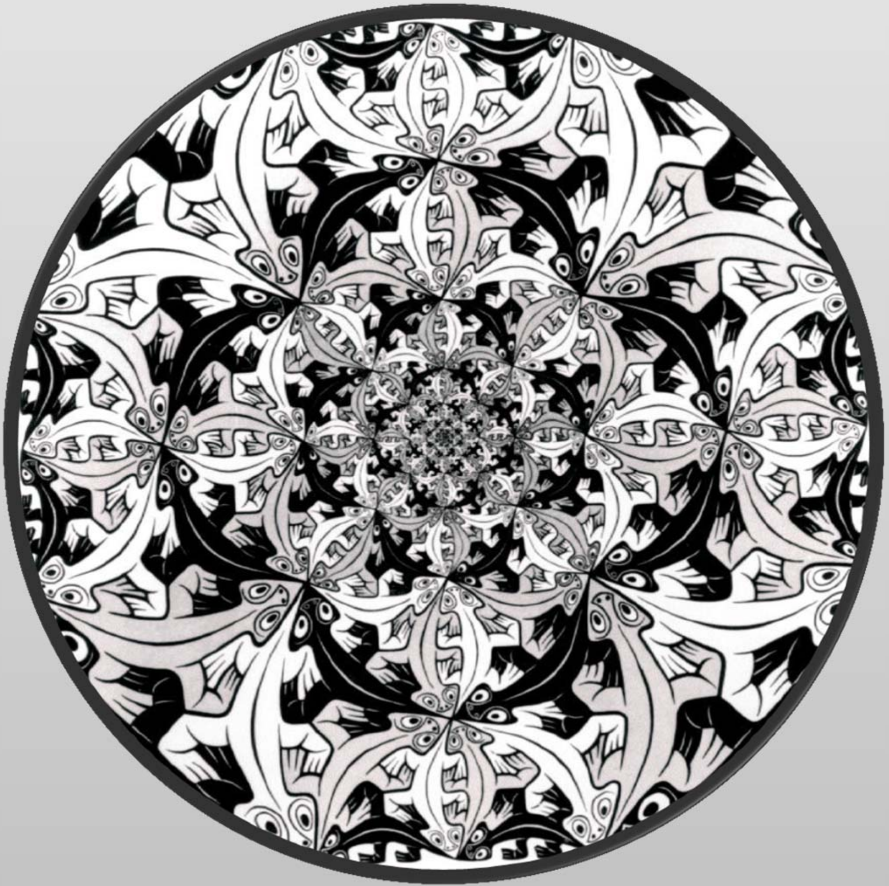
Computer Graphics



Lecture4 Transformation

Instructor: Dr. MAO Aihua

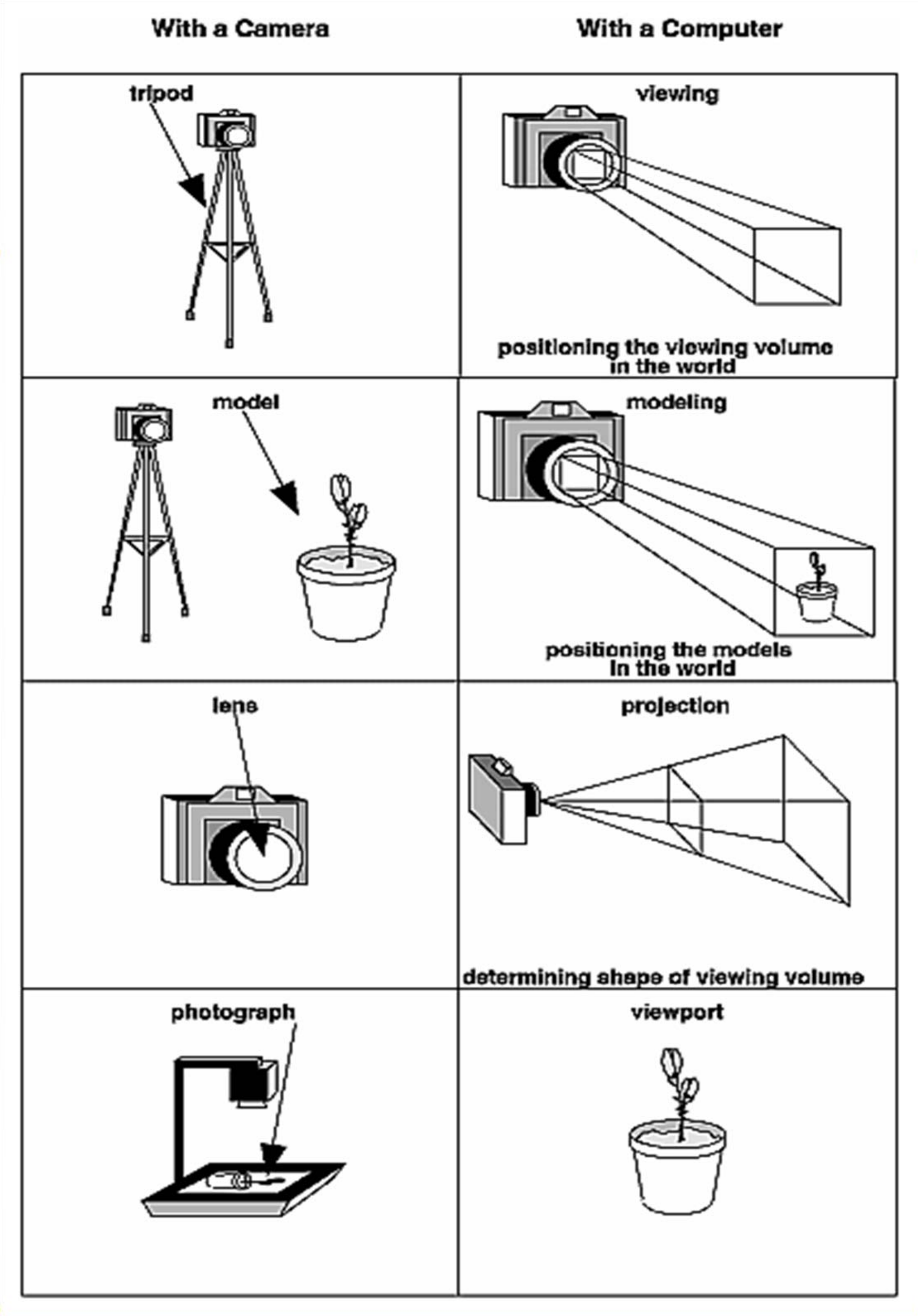
ahmao@scut.edu.cn

OpenGL: Coordinate system

A metaphor for transformation

Coornidate system

- the world coordinate
- the camera coordinate
- the local coordinate

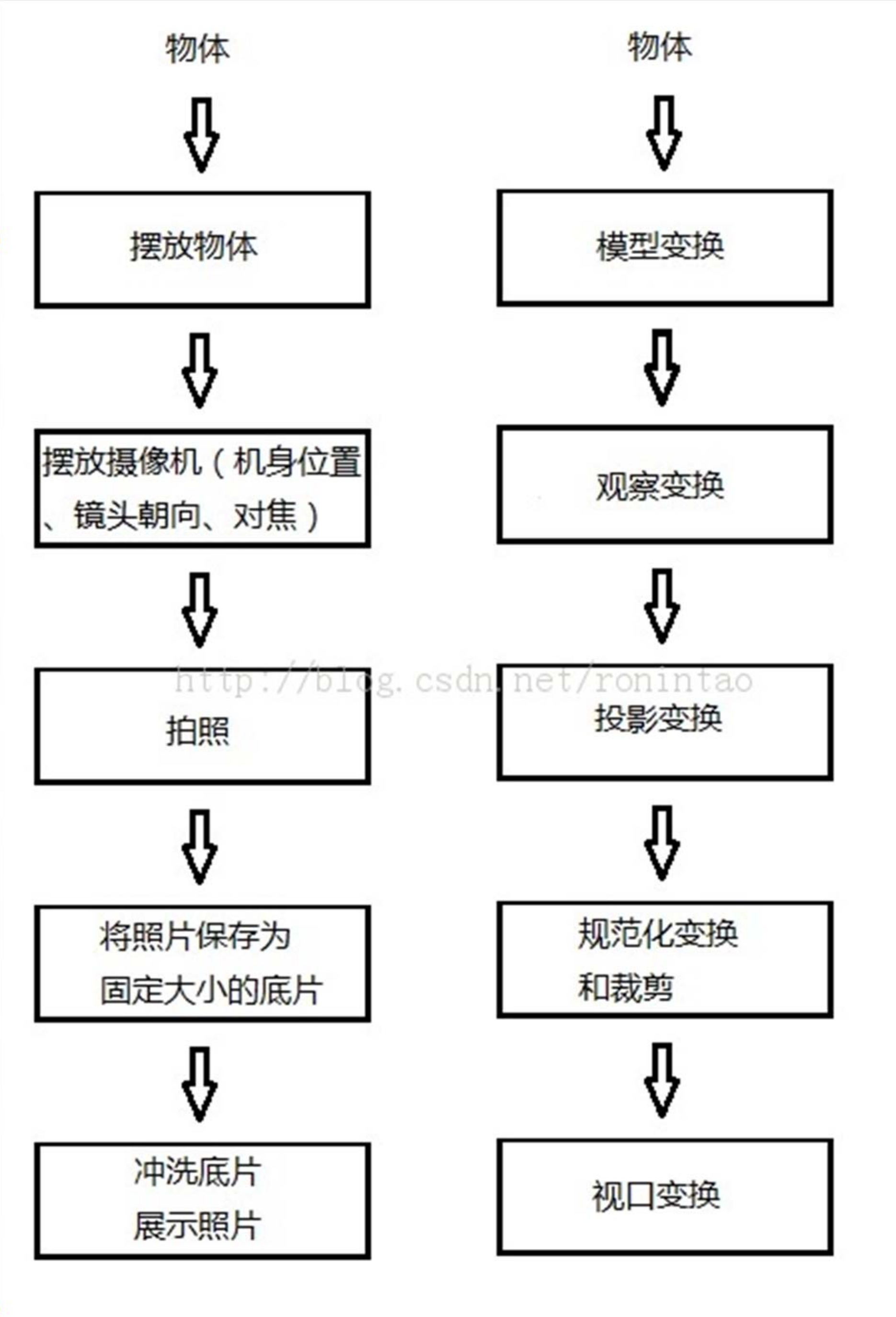


OpenGL: Coordinate system

A metaphor for transformation

Coornidate system

- the world coordinate
- the camera coordinate
- the local coordinate



Why we need modeling Transformations?

Specify transformations for objects

- Allows definitions of objects in own coordinate systems
- Allows use of object definition multiple times in a scene
  - *Remember how OpenGL provides a transformation stack because they are so frequently reused*

Chapter 5 from Hearn and Baker

H&B Figure 109

Overview

2D Transformations

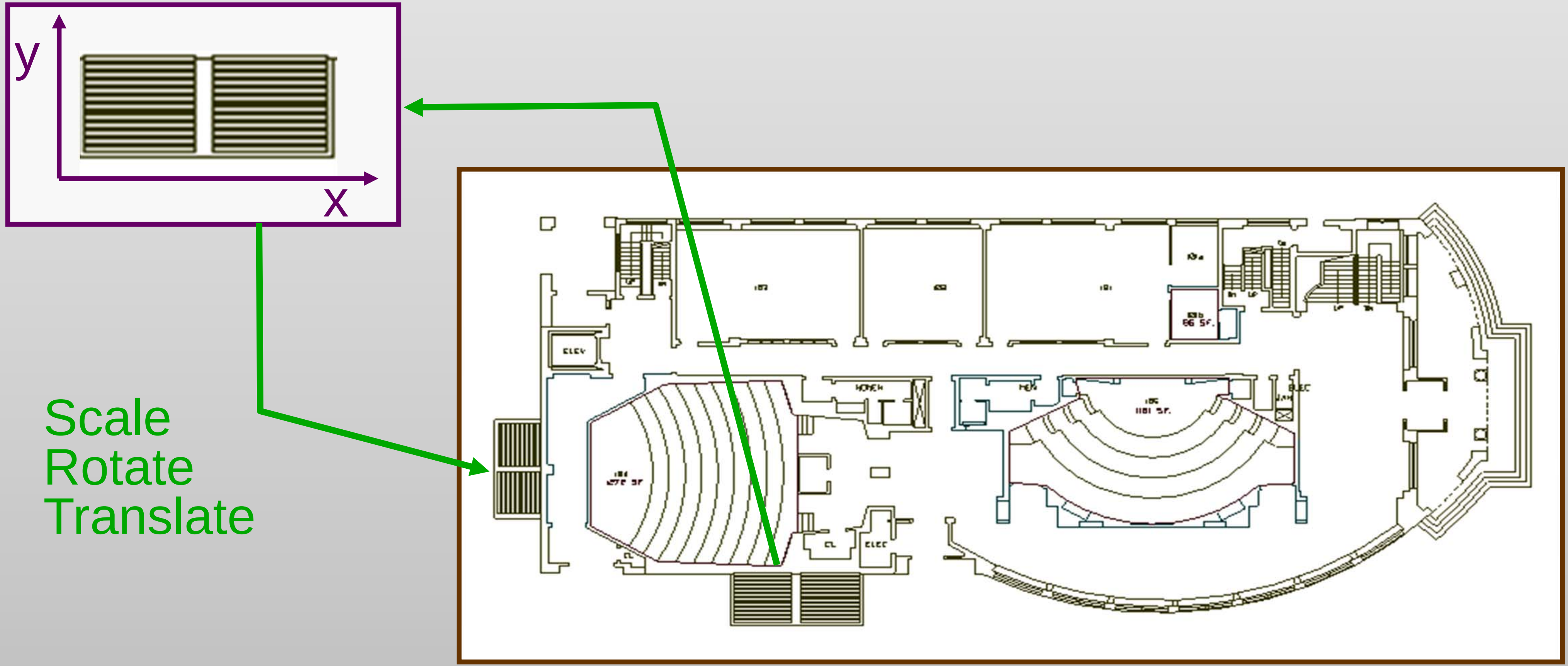
- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

2D Modeling Transformations

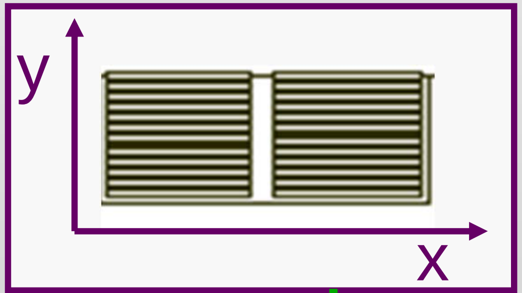
Modeling Coordinates



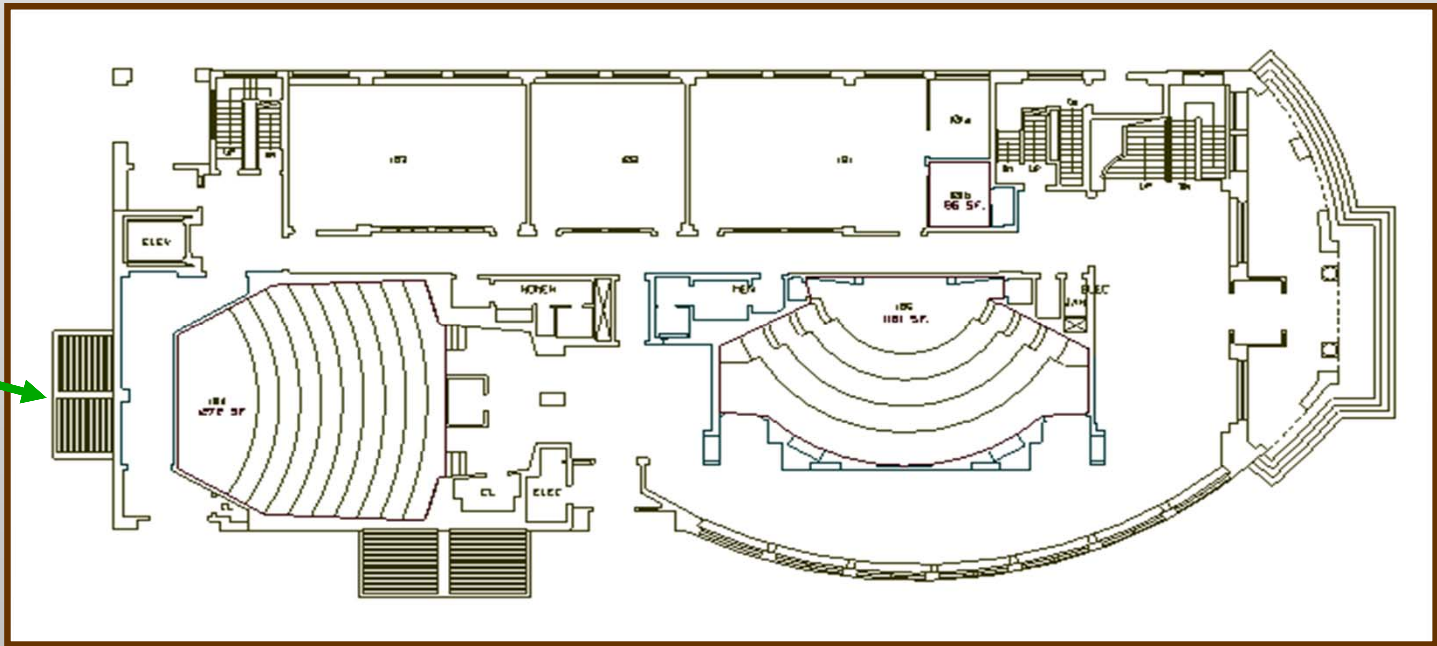
World Coordinates

2D Modeling Transformations

Modeling Coordinates



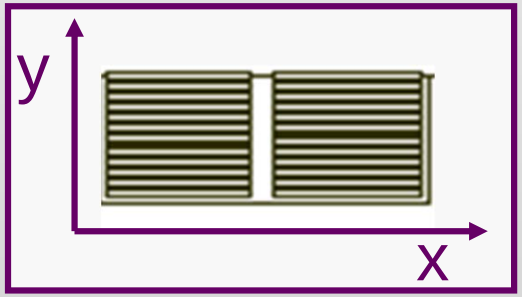
Let’s look at this in detail…

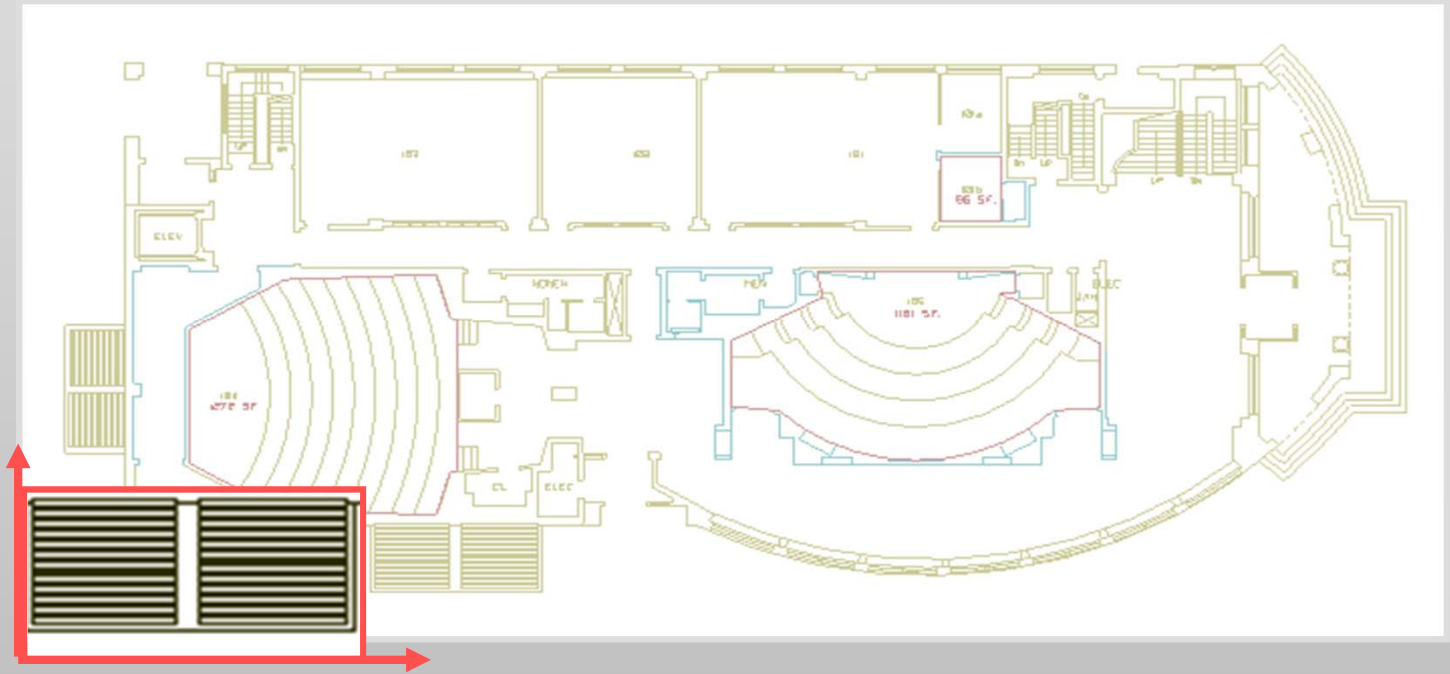


World Coordinates

2D Modeling Transformations

Modeling Coordinates

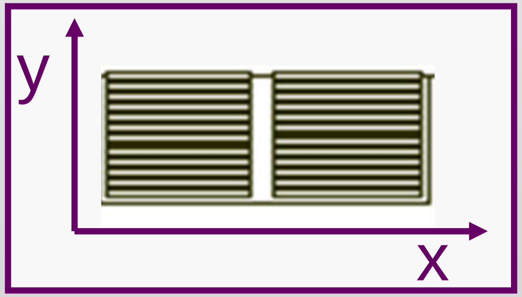




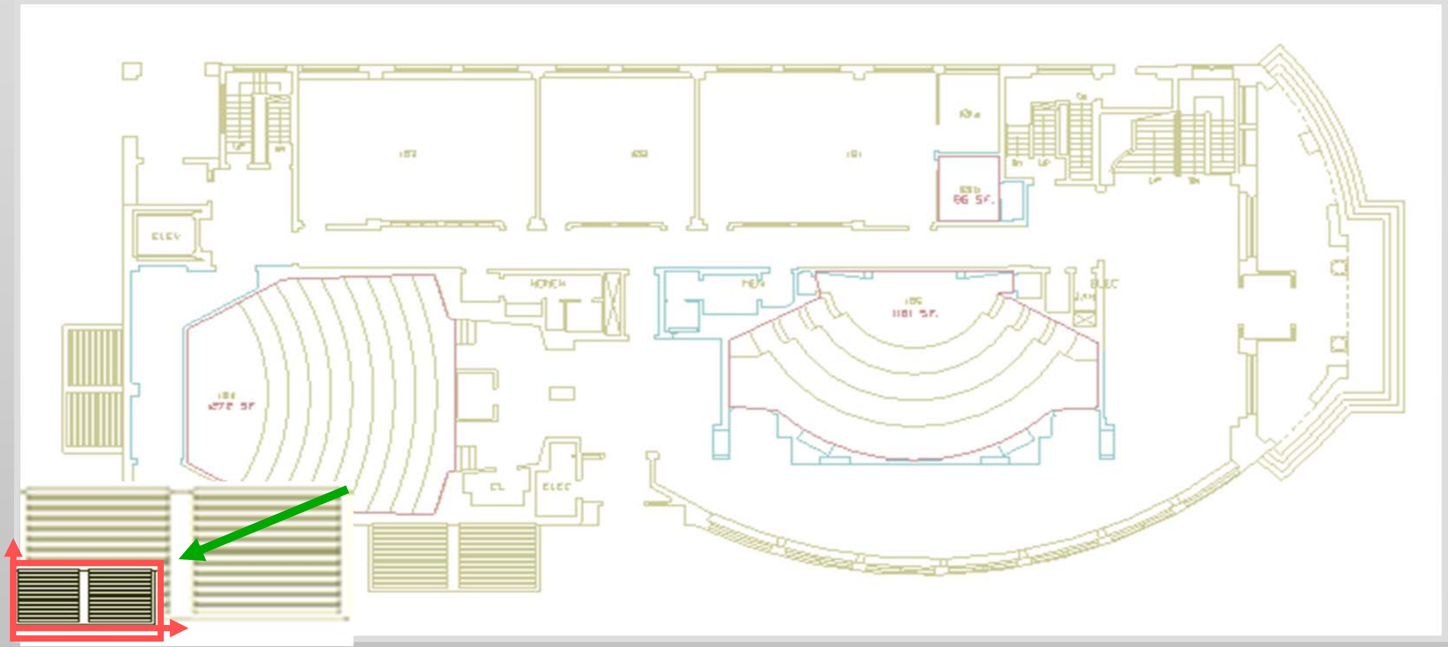
Initial location at (0, 0) with x- and y-axes aligned

2D Modeling Transformations

Modeling Coordinates

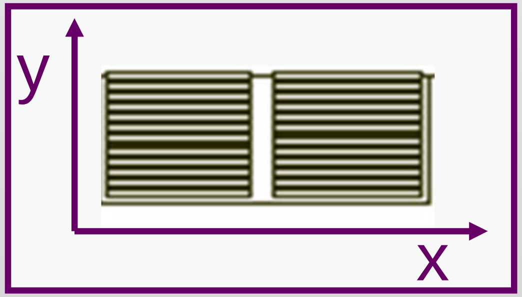


Scale 0.3, 0.3
Rotate -90
Translate 5, 3

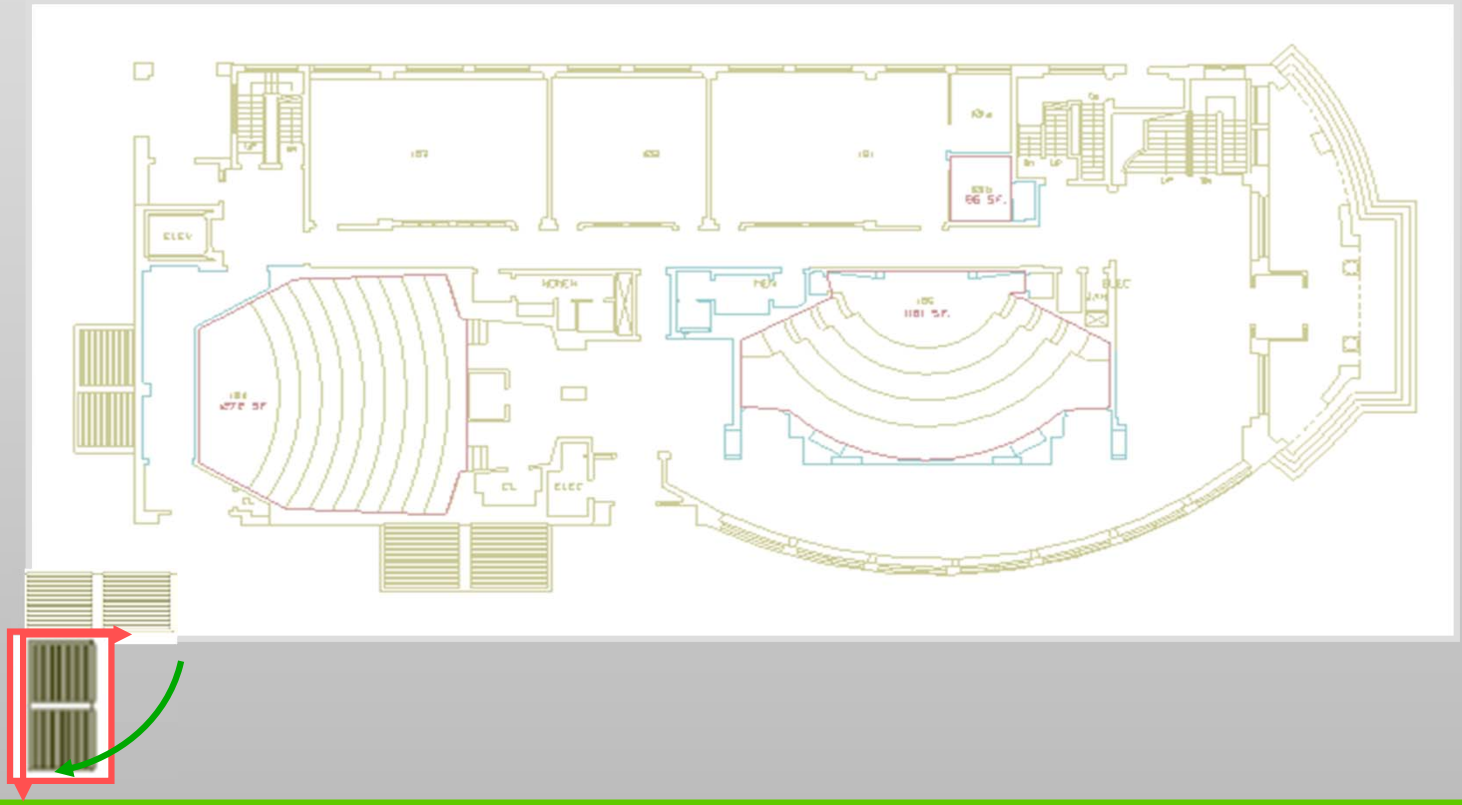


2D Modeling Transformations

Modeling Coordinates

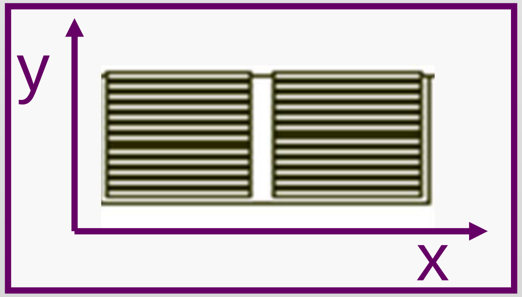


Scale .3, .3
Rotate -90
Translate 5, 3

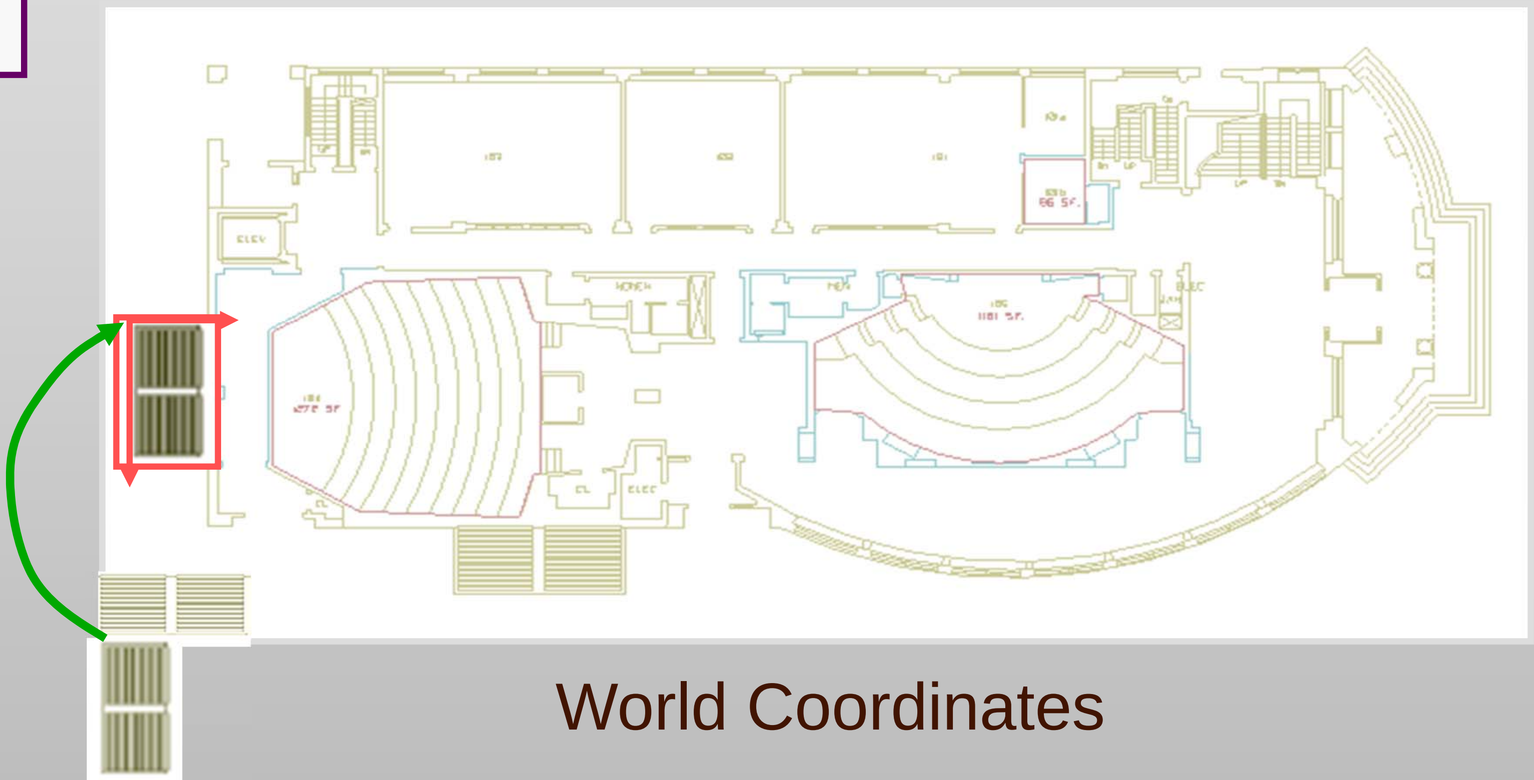


2D Modeling Transformations

Modeling Coordinates



Scale .3, .3
Rotate -90
Translate 5, 3

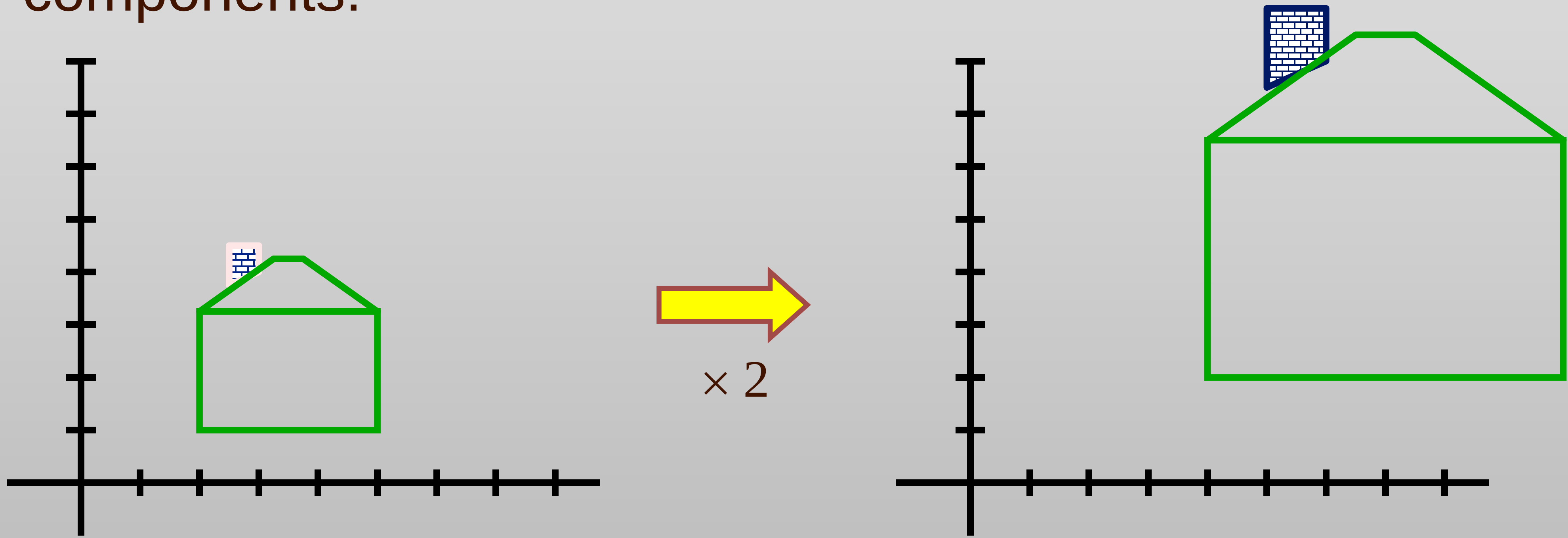


World Coordinates

Scaling

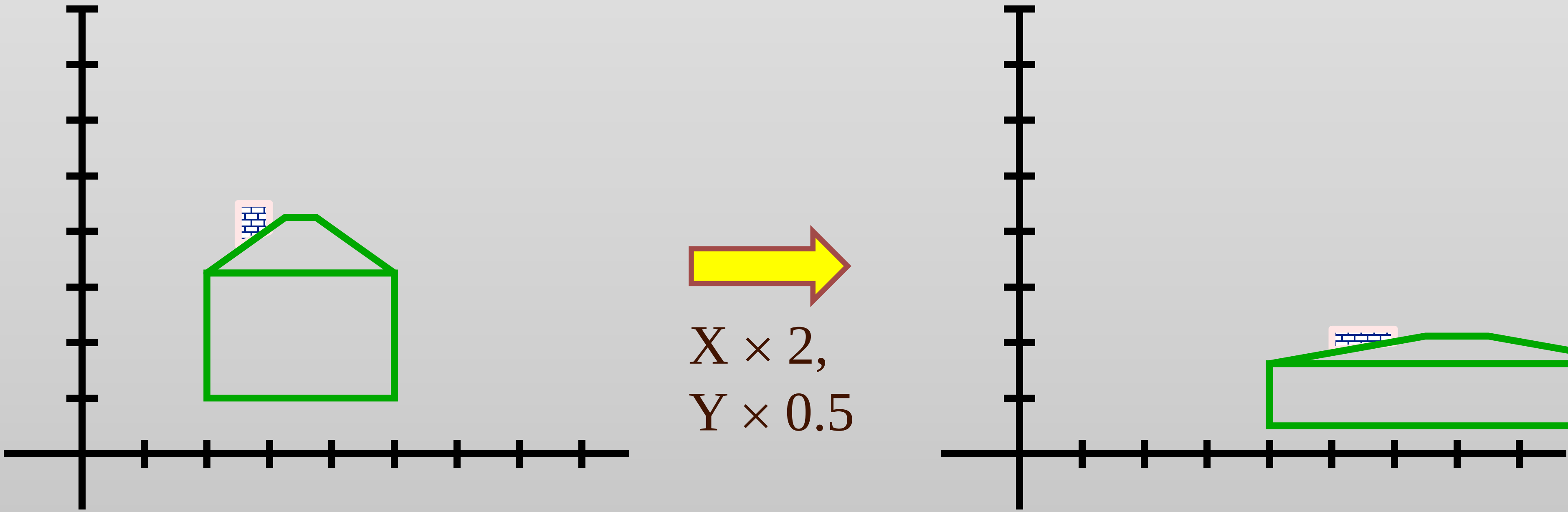
Scaling a coordinate means multiplying each of its components by a scalar

Uniform scaling means this scalar is the same for all components:



Scaling

Non-uniform scaling: different scalars per component:



How can we represent this in matrix form?

Scaling

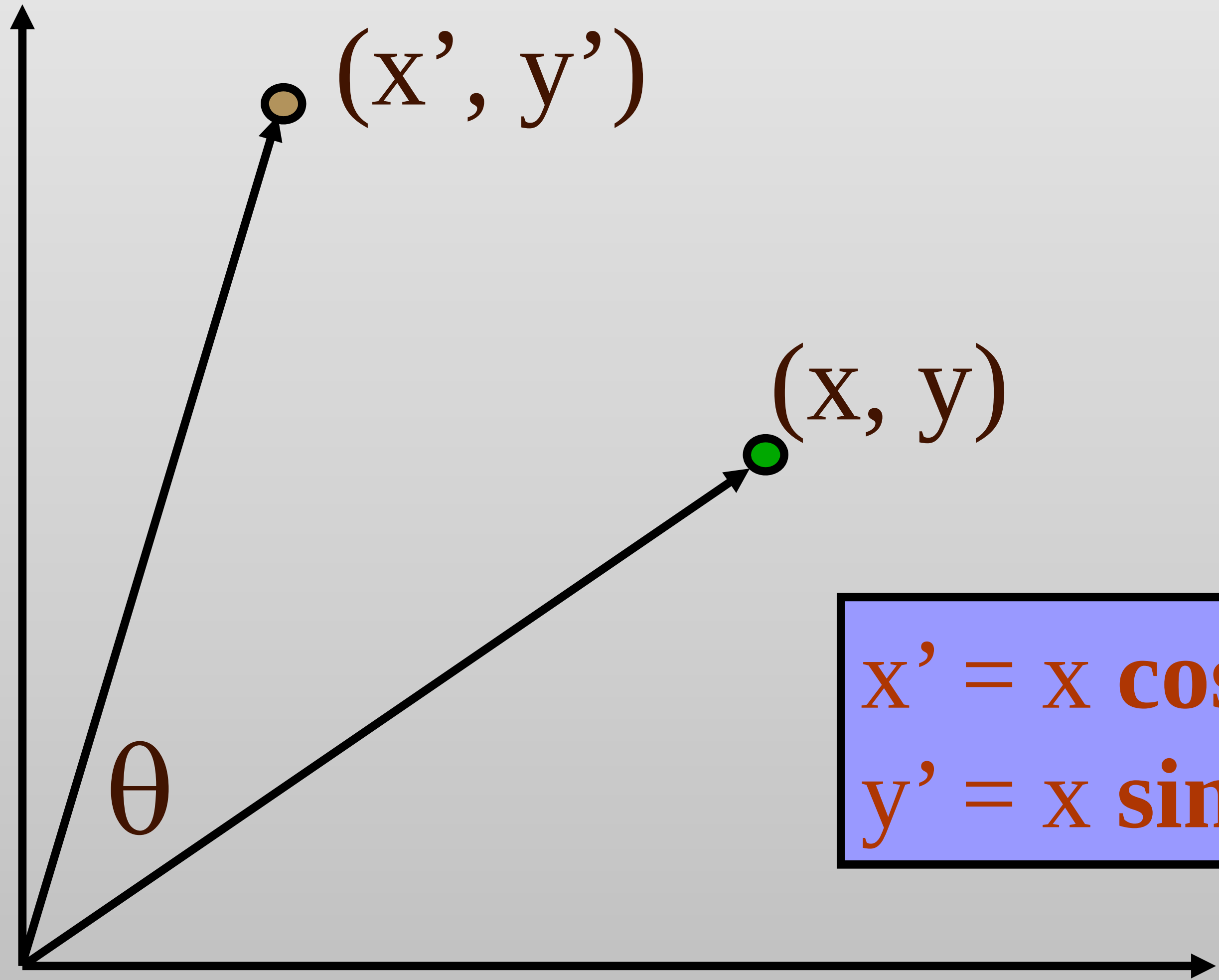
Scaling operation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

Or, in matrix form:

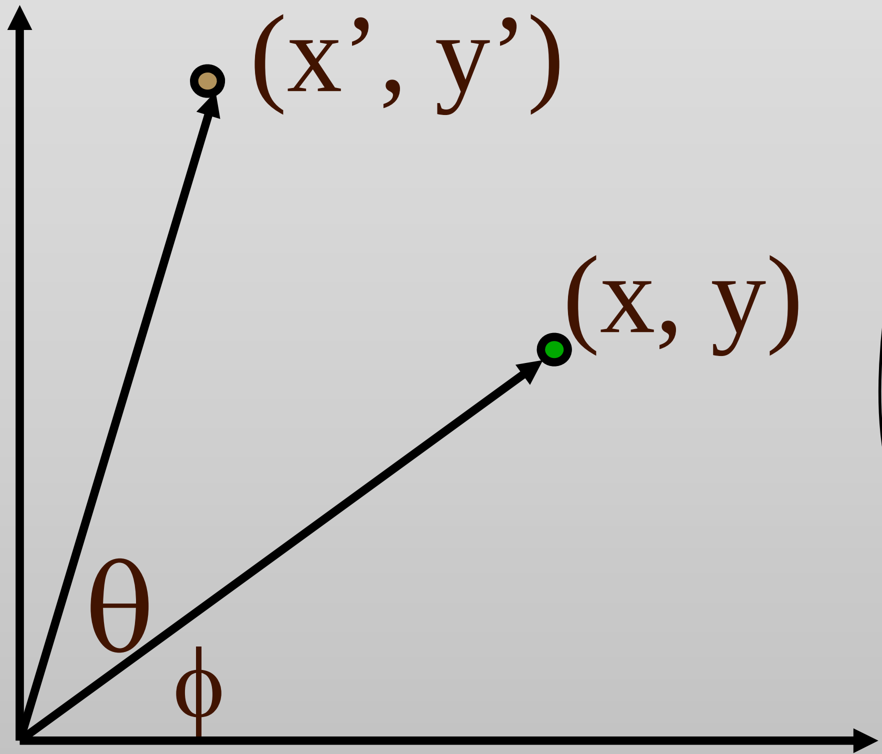
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation



$$x' = x\,\mathbf{cos}(\theta) - y\,\mathbf{sin}(\theta)$$
$$y' = x\,\mathbf{sin}(\theta) + y\,\mathbf{cos}(\theta)$$

2-D Rotation



$x = r \cos(\phi)$
$y = r \sin(\phi)$
$x' = r \cos(\phi + \theta)$
$y' = r \sin(\phi + \theta)$

Trig Identity…
$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$
$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…
$x' = x \mathbf{\cos}(\theta) - y \mathbf{\sin}(\theta)$
$y' = x \mathbf{\sin}(\theta) + y \mathbf{\cos}(\theta)$

2-D Rotation

This is easy to capture in matrix form:

x’ = x **cos**(θ) - y **sin**(θ)
y’ = x **sin**(θ) + y **cos**(θ)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- ***x’ is a linear combination of x and y***
- ***y’ is a linear combination of x and y***

Basic 2D Transformations

Translation:

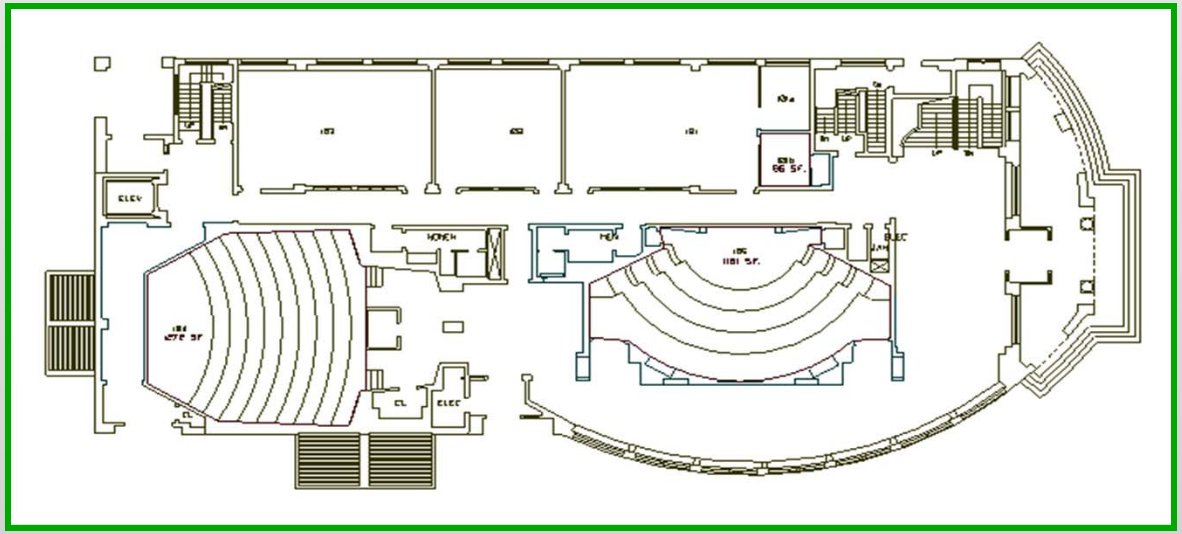
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



Transformations can be combined (with simple algebra)

Basic 2D Transformations

Translation:

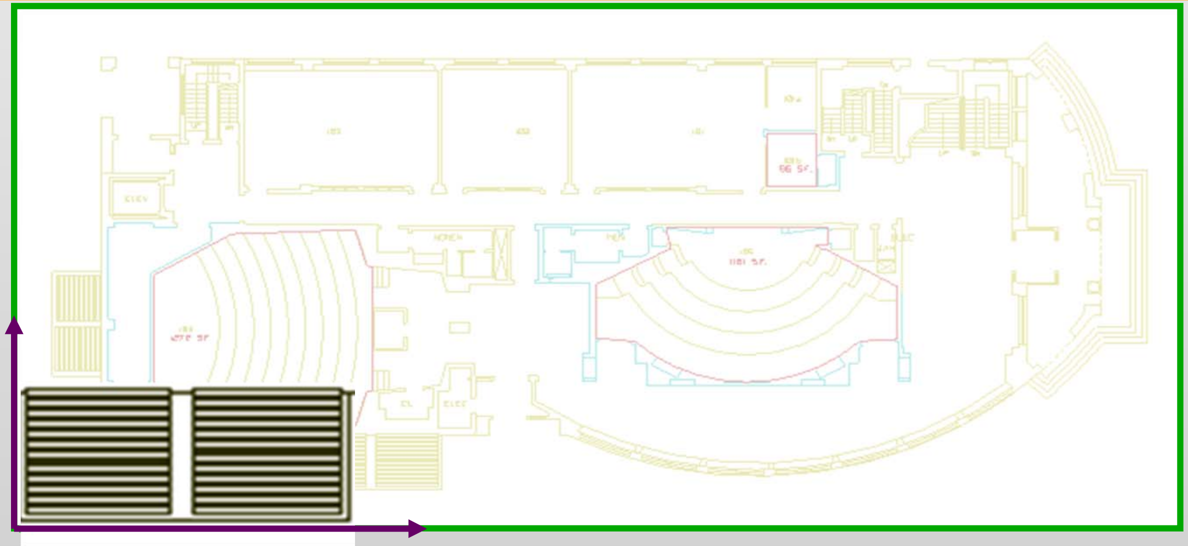
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



Basic 2D Transformations

Translation:

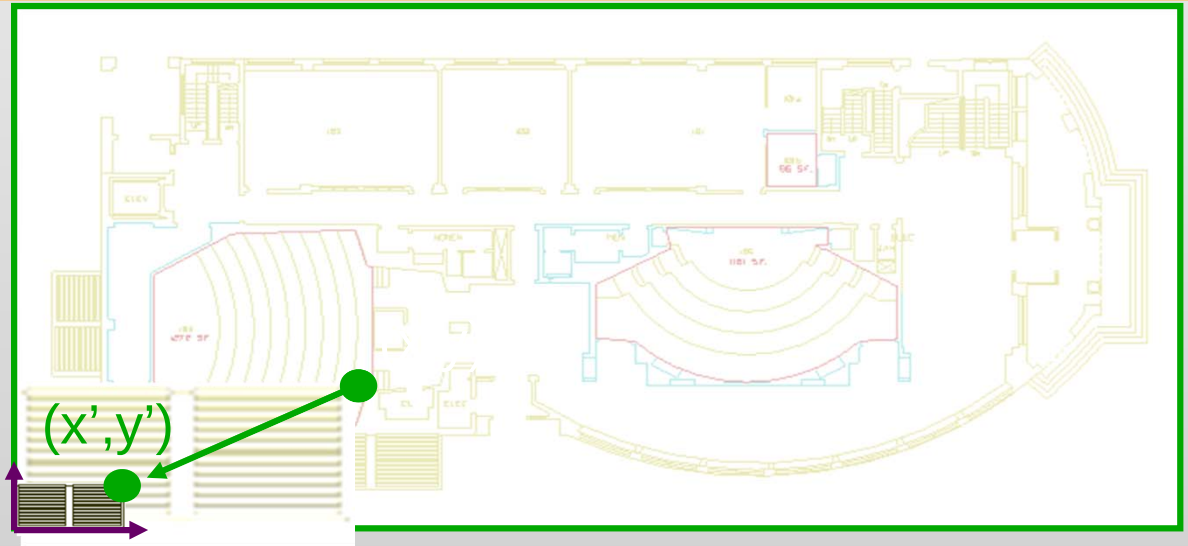
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



$x' = x*s_x$

$y' = y*s_y$

Basic 2D Transformations

Translation:

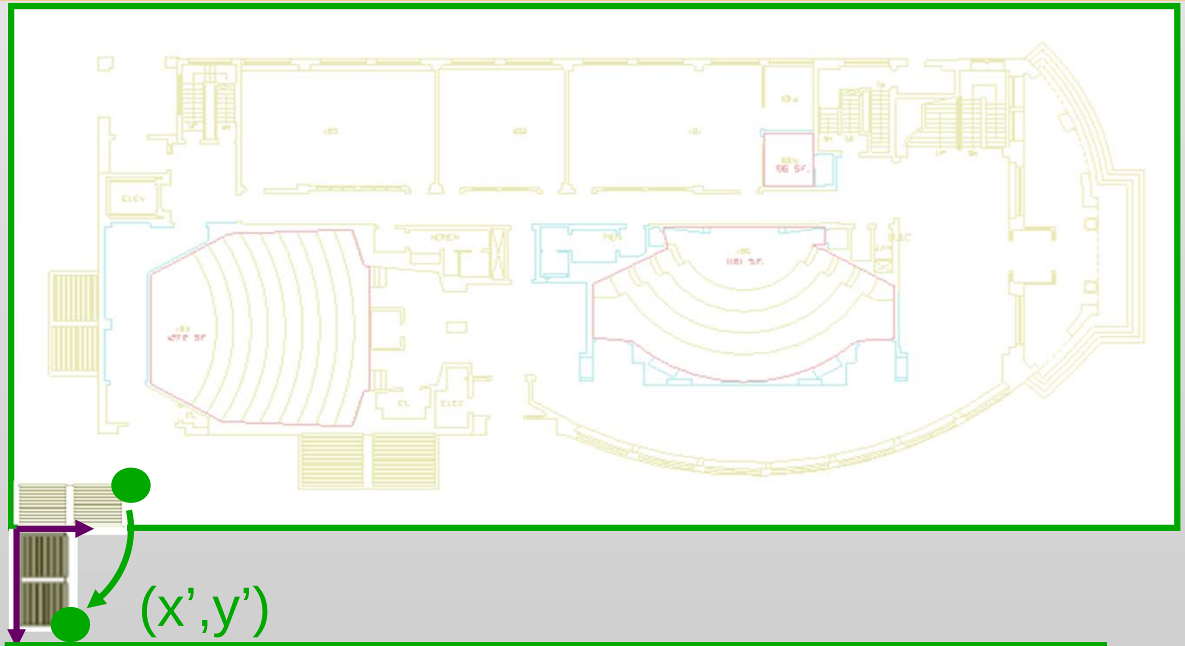
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



(x',y')

$x' = (x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta$

$y' = (x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta$

Basic 2D Transformations

Translation:

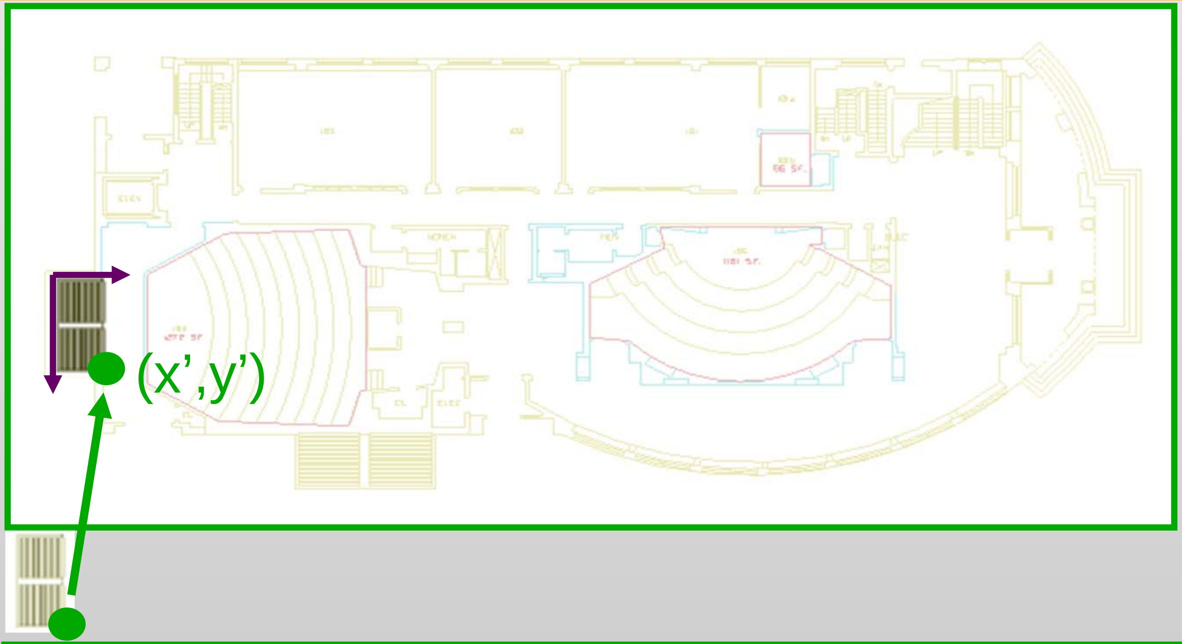
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



$x' = ((x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta) + t_x$

$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$

Basic 2D Transformations

Translation:

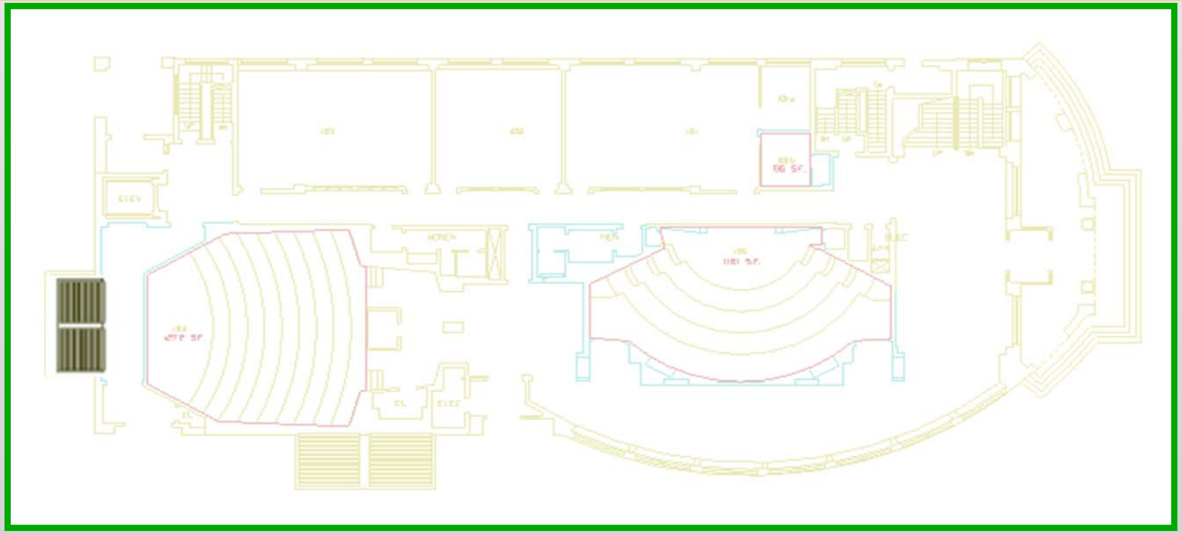
- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



$$x' = ((x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta) + t_x$$
$$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$$

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

Matrix Representation

Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiply matrix by column vector

⇔ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$

$$y' = cx + dy$$

Matrix Representation

Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations!

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Identity

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)

$$\boldsymbol{x}' = \boldsymbol{s}_x * \boldsymbol{x}$$
$$\boldsymbol{y}' = \boldsymbol{s}_y * \boldsymbol{y}$$

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 \\ 0 & \boldsymbol{s}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear

$$x' = x + sh_x * y$$
$$y' = sh_y * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

Has a 2D matrix? NO!

Only linear 2D transformations can be represented with a 2x2 matrix

Linear Transformations

Linear transformations are combinations of …

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear Transformations

Properties of linear transformations:

- Satisfies: $T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1T(\mathbf{p}_1) + s_2T(\mathbf{p}_2)$
- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

Homogeneous Coordinates

$$x' = x + t_x$$
$$y' = y + t_y$$

Q: Since it has no 2D matrix, how can we represent translation as a 3x3 matrix?

Homogeneous Coordinates

Homogeneous coordinates

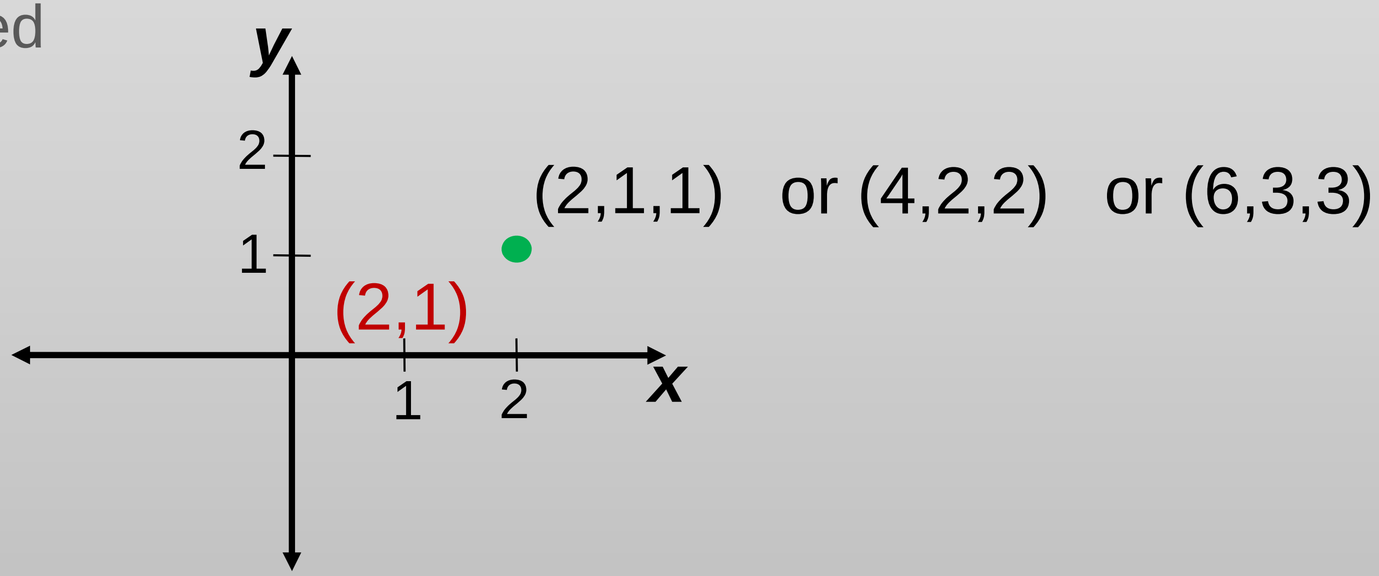
- represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Add a 3rd coordinate to every 2D point

- (x, y, w) represents a point at location (x/w, y/w)
- (x, y, 0) represents a point at infinity
- (0, 0, 0) is not allowed



Convenient coordinate system to represent many useful transformations

Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$
$$y' = y + t_y$$

A: Using the rightmost column:

$$\mathbf{T}ranslation = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

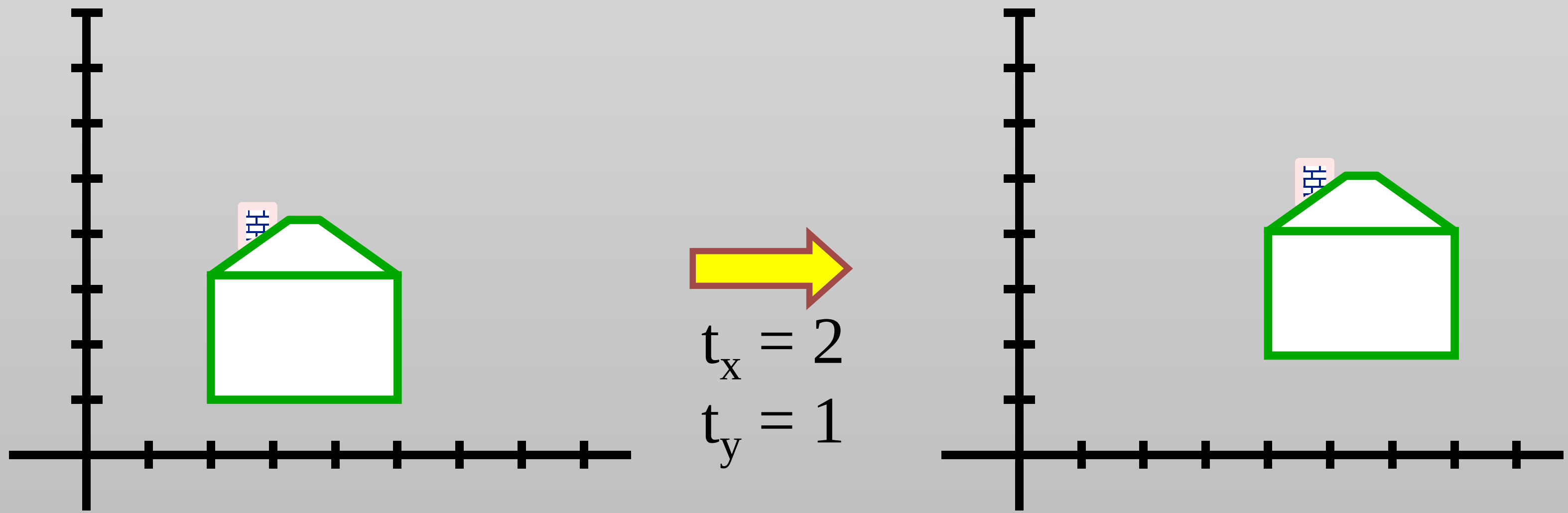
Translation

Example of translation

α

Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Basic 2D Transformations

Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \boldsymbol{t}_x \\ 0 & 1 & \boldsymbol{t}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 & 0 \\ 0 & \boldsymbol{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{sh}_x & 0 \\ \boldsymbol{sh}_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

Matrix Composition

Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = $T(t_x,t_y)$ $R(\Theta)$ $S(s_x,s_y)$ **p**

Matrix Composition

Matrices are a convenient and efficient way to represent a sequence of transformations

- General purpose representation

$$\mathbf{p}' = (T * (R * (S*\mathbf{p})\ )\ )$$

- Hardware matrix multiply

$$\mathbf{p}' = (T*R*S) * \mathbf{p}$$

Matrix Composition

Be aware: order of transformations matters

– *Matrix multiplication is not commutative*

$$\mathbf{p}' = T * R * S * \mathbf{p}$$

“Global” “Local”

Matrix Composition

After correctly ordering the matrices,

Multiply matrices together

What results is one matrix – store it (on stack)!

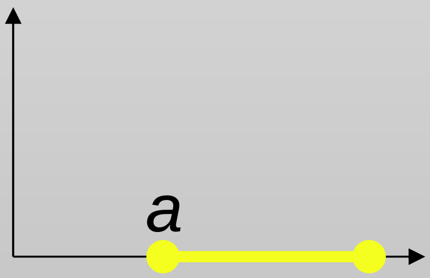
Multiply this matrix by the vector of each vertex

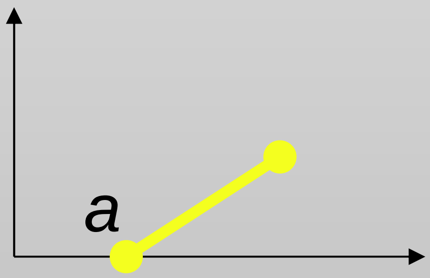
All vertices easily transformed with one matrix multiply

Matrix Composition

What if we want to rotate on a Arbitrary Center?

- Ex: Rotate line segment by 45 degrees about endpoint *a*

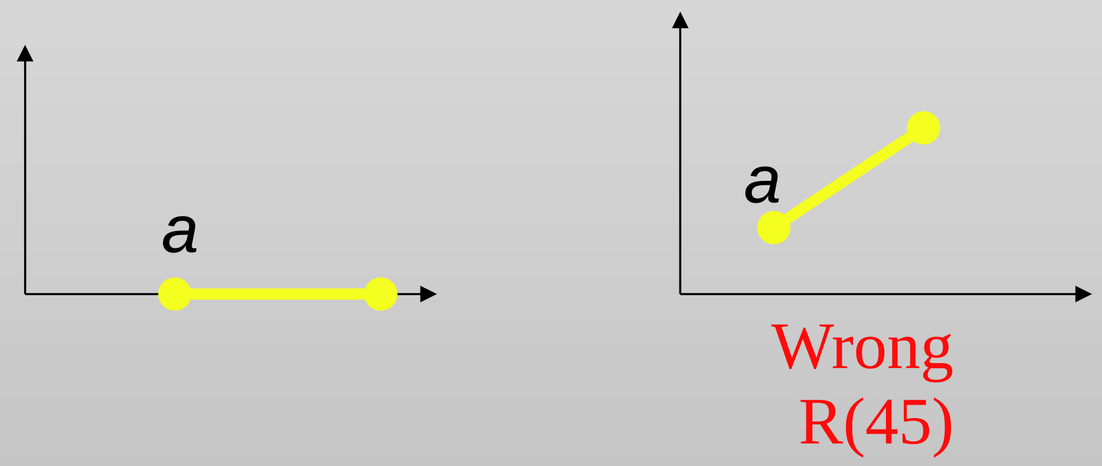




Arbitrary Rotation Center– Wrong Way

Our line is defined by two endpoints, the rotation center is not the origin (0,0)

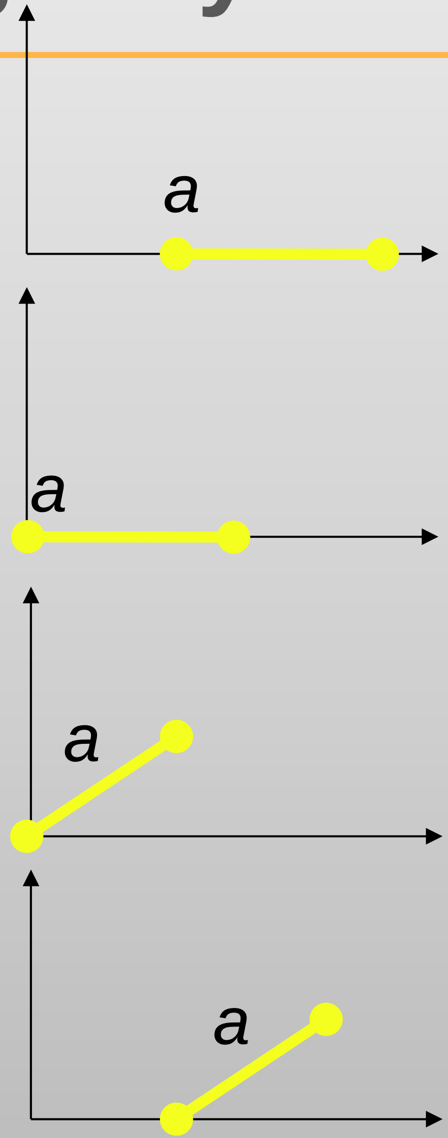
- Applying a rotation of 45 degrees, R(45), affects both points



- We could try to translate both endpoints to return endpoint *a* to its original position.

Arbitrary Rotation Center– Wrong Way

Isolate endpoint a from rotation effects



- First translate line so *a* is at origin: T (-3)
- Then rotate line 45 degrees: R(45)
- Then translate back so *a* is where it was: T(3)

Correct:
T(-3) R(45) T(3)

Matrix Composition

Will this sequence of operations work?

$$\begin{bmatrix}1 & 0 & -3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(45) & -\sin(45) & 0\\\sin(45) & \cos(45) & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_x\\a_y\\1\end{bmatrix}=\begin{bmatrix}a'_x\\a'_y\\1\end{bmatrix}$$

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

3D Transformations

Same idea as 2D transformations

- Homogeneous coordinates: (x,y,z,w)
- 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} \mathbf{x'} \\ \mathbf{y'} \\ \mathbf{z'} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 & 0 \\ 0 & \mathbf{s}_y & 0 & 0 \\ 0 & 0 & \mathbf{s}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix}$$

Scale

$$\begin{bmatrix} \mathbf{x'} \\ \mathbf{y'} \\ \mathbf{z'} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{t}_x \\ 0 & 1 & 0 & \mathbf{t}_y \\ 0 & 0 & 1 & \mathbf{t}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror about Y/Z plane

Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Reverse Rotations

Q: How do you **undo** a rotation of θ, $R(\theta)$?

A: Apply the inverse of the rotation… $R^{-1}(\theta) = R(-\theta)$

How to construct $R\text{-}1(\theta) = R(-\theta)$

- Inside the rotation matrix: $\cos(\theta) = \cos(-\theta)$
  - *The cosine elements of the inverse rotation matrix are unchanged*
- The sign of the sine elements will flip

Therefore… $R^{-1}(\theta) = R(-\theta) = R^T(\theta)$