- Hopfield network （HNN）

1.1982 proposed by Hopfield
2.HNN is an auto-associative memory network
3.Architecture：

Input：$\forall X_i \in \{-1,1\}$

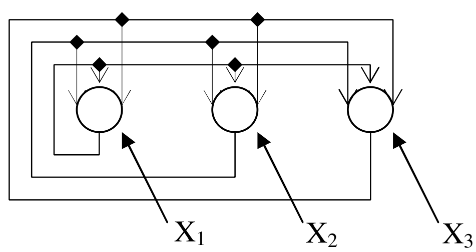
Output：Same as input single layer network

Transfer function：X_i ： $\begin{cases} +1 & net_j > 0 \\ X_j \quad \text{if} & net_j = 0 \\ +1 & net_j < 0 \end{cases}$

Weights： $W_{ij} = \sum_p X_i^p X_j^p$

$$W_{ii} = 0$$

Connections：



Method：

1. Learning：

a. Setup the network （nodes, patterns）

b. Setup weight matrix

$$W_{ij} = \sum_p X_i^p X_j^p \qquad W_{ii} = \phi$$

2. Recall

a. Read the weight matrix

b. Input the pattern X

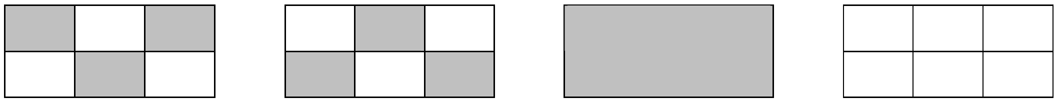
c. Compute new input （i.e. output） X^{new} $\longrightarrow$

$$net_j = \sum_i W_{ij} \ldots . X_i \text{ （or net = W • } X_i \text{）}$$

$$X_j \ : \begin{cases} +1 & net_j > 0 \\ X_j \quad \text{if} & net_j = 0 \\ +1 & net_j < 0 \end{cases}$$

d. Repeat ⓒ until converge （i.e. the net value is not changed or error very small）

Example（Use HNN）：Memorize the pattern as follow



P	X_1	X_2	X_3	X_4	X_5	X_6
X^1	1	-1	1	-1	1	-1
X^2	-1	1	-1	1	-1	1
X^3	1	1	1	1	1	1
X^4	-1	-1	-1	-1	-1	-1

Ans： $W_{12} = (1,-1,1,-1)\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 0 \qquad W_{12} = W_{21}$

$$W_{13} = (1,-1,1,-1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 4 \qquad W_{13} = W_{31}$$

$$W_{14} = (1,-1,1,-1)\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 0 \qquad W_{14} = W_{41}$$

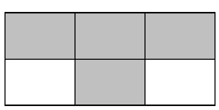
$$W_{15} = (1,-1,1,-1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 4 \qquad W_{15} = W_{51}$$

$$W_{16}\,(1,-1,1,-1)\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix} = 0 \qquad W_{16} = W_{61}$$

$$\begin{array}{ll}
W_{21} = 0 & W_{34} = 0 \\
W_{22} = 0 & W_{35} = 4 \\
W_{23} = 0 & W_{36} = 0 \\
W_{24} = 4 & W_{45} = 0 \\
W_{25} = 0 & W_{46} = 4 \\
W_{26} = 4 & W_{56} = 0
\end{array}$$

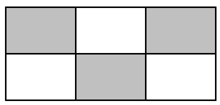
$$W = \begin{bmatrix} 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \end{bmatrix}$$

$net = W \cdot X^t$

Recall  $X^t = (1,1,1,-1,1,-1)$

$$W = \begin{bmatrix} 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \end{bmatrix} \bullet X^t \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 8 \\ 0 \\ 8 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$
\begin{bmatrix} 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 8 \\ -8 \\ 8 \\ -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}
$$



Liapunov function

This is an energy function that has been utilized to construct the method of HNN.

$$
E = \left(-\frac{1}{2}\right)\sum_i \left(\sum_j X_i W_{ij} X_j\right) + \sum_j \theta_j X_j
$$

Where $X_i \in \{-1,+1\}$, $W_{ij} = W_{ji}$, $W_{ii} = \phi$

$\theta_j = constant$

In order to let the function converge, we need to achieve $E_k^{new} - E_k <= \phi$

We let $X_j^{n+1} = \begin{cases} +1 & \sum_j W_{ij} X_j^n - \theta_j > 0 \\ X_j^n & if \quad \sum_j W_{ij} X_j^n - \theta_j = 0 \\ -1 & \sum_j W_{ij} X_j^n - \theta_j < 0 \end{cases}$

Pf： $E_k = \left(-\frac{1}{2}\right)\left\{ X_k \left(\sum_j W_{kj} X_j + \sum_i X_i W_{ik} \right) + \left(X_k^2\right) W_{kk} + \sum_{i+k} \sum_{j+k} X_i W_{ij} X_j \right\}$

$$
+ \theta_k X_k + \sum_{j+k} \theta_j X_i
$$

$$E_k^{new} = \left(-\frac{1}{2}\right)\left\{X_k^{new}\left(\sum_j W_{kj} + \sum_i X_i W_{ik}\right) + \left(X_k^{new}\right)^2 W_{kk} + \sum_{i\neq k}\sum_{j\neq k} X_i W_{ij} X_j\right\}$$

$$+\theta_k X_k^{new} + \sum_{j\neq k}\theta_j X_j$$

$$E_k^{new} - E_k = \left(-\frac{1}{2}\right)\left\{\left(X_k^{new} - X_k\right)\left(\sum_j W_{kj} X_j + \sum_i X_i W_{ik}\right)\right\} + \theta_k\left(X_k^{new} - X_k\right)$$

$$\sum_j W_{kj} \bullet X_j = \sum_j X_i \bullet W_{ik}$$

$$k \neq j \qquad\qquad i = k$$

$$\Delta E_k = E_k^{new} - E_k$$

$$= \left(-\frac{1}{2}\right)\left\{\left(X_k^{new} - X_k\right)\left(2\sum_j W_{kj} X_j\right)\right\} + \theta_k\left(X_k^{new} - X_k\right)$$

$$= \underbrace{-\left(X_k^{new} - X_k\right)}_{Ⓐ}\underbrace{\left(\sum_j W_{kj} X_j - \theta_k\right)}_{Ⓑ} \leq 0$$

討論：Case1 當 $X_k^{new} = -1 \rightarrow$ Ⓑ < 0

$$\therefore X_k = \begin{cases} +1 \Rightarrow < 0 \\ -1 \Rightarrow 0 \end{cases}$$

$\therefore$ Ⓐ ≤ 0 & Ⓑ <0 $\quad\therefore\ \Delta E_k \leq 0$

Case2 當 $X_k^{new} = +1$ ，Ⓑ > 0

當 $X_k = \begin{cases} +1 \Rightarrow 0 \\ -1 \Rightarrow < 0 \end{cases}$

$\Delta E_k \leq 0$

Liapunov → Auto associative

① 令 $W_{ij} = \sum_p X_i^{\ p} X_j^{\ p}$

$W_{ii} = \phi \qquad \theta_j = \phi$

$$E = \left(-\frac{1}{2}\right)\left\{\sum_i \sum_j X_i \left[\sum_p X_i^{\ p} X_j^{\ p}\right] X_j\right\}$$

② $W = \sum_p W^p = \sum_p \left(X_i^{\ p}\right)^t \cdot X_j^{\ p}$

$$E = \left(-\frac{1}{2}\right) X^t W^x \qquad X = \sum_i \sum_j X_i = \sum_i X_i \quad or = \sum_i \sum_j X_j = \sum_j X_j$$

③ $E^p = \left(-\frac{1}{2}\right) X^t \left(X^p X^p\right)^t \cdot X = \left(-\frac{1}{2}\right)\left(X^p \cdot X\right)^2$

當 $X^p = X$ 最大 $\therefore E^p$ 最小