PHY407 Lab-08 Report

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[1] *Q2 - Simulating the Shallow Water System*
[2] *Q1 - Electrostatics and Laplace's equation*

1. Q1 - ELECTROSTATICS AND LAPLACE'S EQUATION

For the pseudo-code of all the sub-question please refer to the python script titled `Lab08_Q1.py`

1.1. *Q1a - Gauss-Seidel Method without over-relaxation*

In this exercise, we calculate the electric potential everywhere in the box enclosing an electric capacitor, where the potential ϕ satisfies Laplace Equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

and the boundary conditions of $\phi(x = 0\text{cm}, y) = \phi(x = 10\text{cm}, y) = \phi(x, y = 0\text{cm}) = \phi(x, y = 10\text{cm}) = 0V$ (i.e. 0 voltage on the wall of the box) and $\phi(x = 2\text{cm}, y \in [2cm, 8cm]) = 1V$, $\phi(x = 8\text{cm}, y \in [2cm, 8cm]) = -1V$ (i.e. corresponds to the capacitor plates). We numerically found the solution by setting 101×101 grid points (separation between grid points $a = 1\text{mm}$) with the Gauss-Seidel method (without over-relaxation) and a target solution accuracy to $10^{-6}V$, where in the Gauss-Seidel method the potential value at each grid point at each step is updated by its adjacent grid points at the previous step given by:

$$\phi(x, y) \leftarrow \frac{1}{4}[\phi(x - a, y) + \phi(x + a, y) + \phi(x, y + a) + \phi(x, y - a)] \tag{2}$$

Using this approach, we counted that it took approximately 2158 iterations (of solving every points on the 101×101 grid) to reach a solution with the target accuracy, and we produced the following potential contour plot.

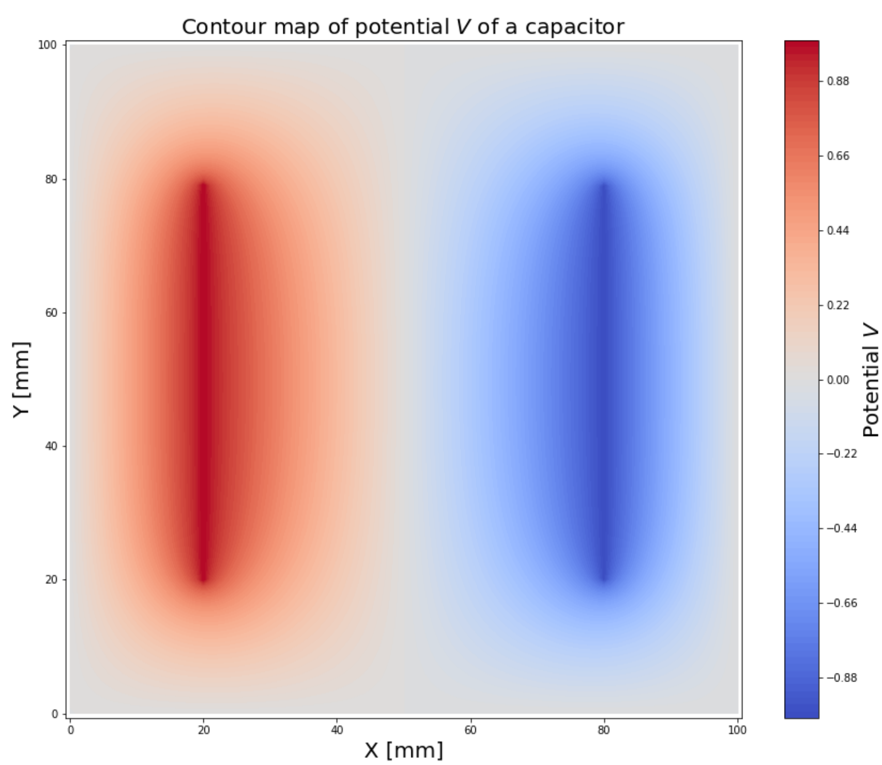

Figure 1: Contour plot of the potential inside the box. On the left is the positively charged capacitor plate located at $x = 20$ cm, and on the right is the negatively charged capacitor plate located at $x = 80$ cm.

1.2. *Q1b - Gauss-Seidel Method with over-relaxation*

For this part, we again used the Gauss-Seidel method but with over-relaxation this time, where the potential value at each for the next step is given by:

$$\phi(x,y) \leftarrow \frac{(1+\omega)}{4}[\phi(x-a,y)+\phi(x+a,y)+\phi(x,y+a)+\phi(x,y-a)] - \omega\phi(x,y) \tag{3}$$

where ω is the factor for over-relaxation

We repeated our calculations for $\omega = 0.1$ and $\omega = 0.5$, and we recorded the number of iterations it took to find the solution with the same target accuracy $10^{-6}V$ in the following table:

Over-relaxation factor	Iteration required
$\omega = 0$	2158
$\omega = 0.1$	1945
$\omega = 0.5$	1118

Table 1: Number of iterations required to find a solution in a box with target accuracy $10^{-6}V$. Note $\omega = 0$ is without over-relaxation as shown in part a.

We can see that for $\omega = 0.1$, the program is able to find the solution slightly faster, while for $\omega = 0.5$, it is able to find the solution almost twice as fast. Below is the contour plots for $\omega = 0.1$ and $\omega = 0.5$, but they looks essentially the same as in part a.

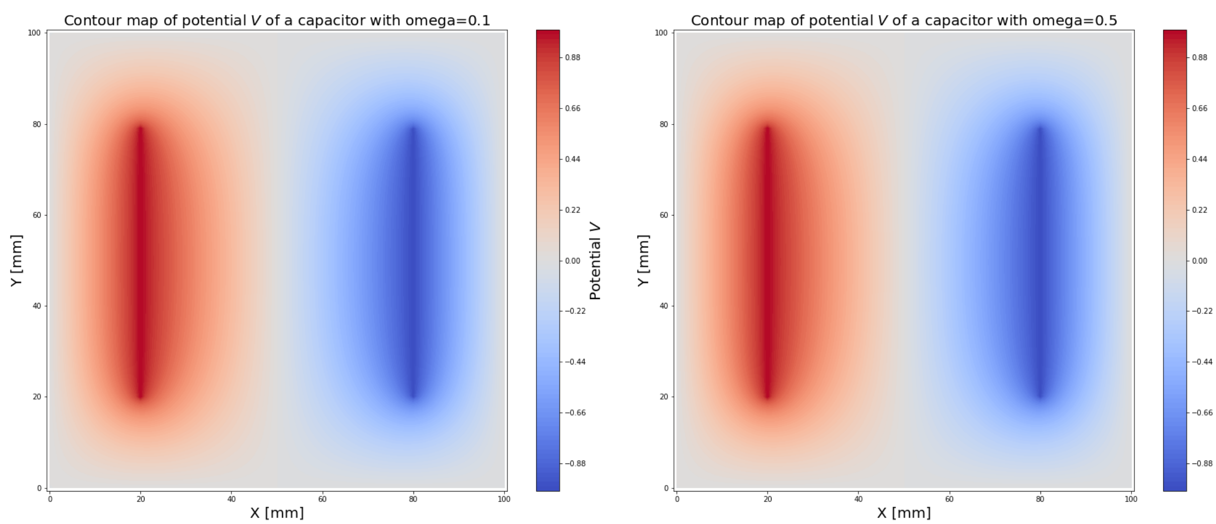

Figure 2: Contour plots of the potential when solved with over-relaxation

1.3. *Q1c - Electric field Stream Plot*

In addition we supplemented Figure 1 in part (a) with its electric field lines stream plot.

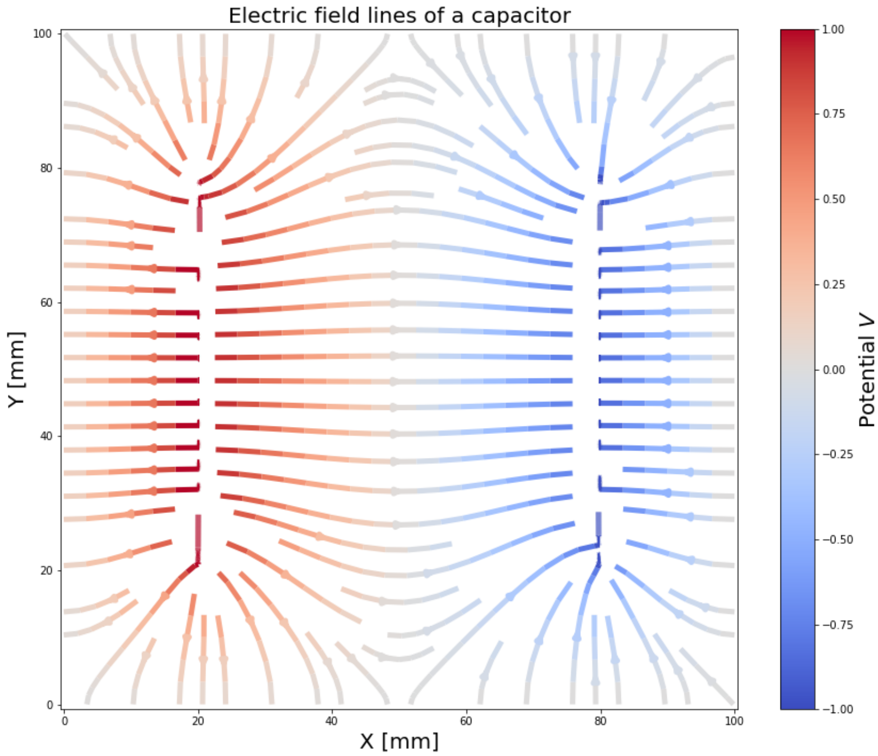

Figure 3: Stream plot of the electric field lines from capacitor in the box. The two empty regions corresponds to the capacitor plates, which is undefined by definition of the gradient.

2. Q2 - SIMULATING SHALLOW WATER SYSTEM

For the python code, plots and animation please refer to the script titled `Lab08_Q2.py`. Where we used some functions defined on the script titled `MyFunctions.py`.

2.1. *Q2a - 1D Shallow Water Equation in Flux-Conservative Form*

Using the 1-dimensional representation of the Navier-Stokes equations we can represent a shallow water system through a system of non-linear partial differential equations as follows.

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \mu \frac{\partial \mu}{\partial x} &= -g \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial t} + \frac{\partial (\mu h)}{\partial x} &= 0 \end{aligned} \tag{4}$$

Where μ is the velocity of the fluid, η is the free space altitude of the fluid with respect to the topographical surface η_b, $h = \eta - \eta_b$ is the water column height. We can rewrite this system of PDE's in flux conservative form as follows

$$\frac{\partial \mu}{\partial t} = -\mu \frac{\partial \mu}{\partial x} - g \frac{\partial \eta}{\partial x} \tag{5}$$

$$\frac{\partial \mu}{\partial t} = -\frac{\partial}{\partial x} \left(\frac{1}{2} \mu^2 + g\eta \right) \tag{6}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x} (\mu h) \tag{7}$$

$$\implies \frac{\partial \vec{\mu}}{\partial t} = -\frac{\partial \vec{F}(\vec{\mu})}{\partial x} \tag{8}$$

Where $\vec{\mu} = (\mu, \eta)$ and $\vec{F}(\vec{\mu}) = [1/2\mu^2 + g\eta, (\eta - \eta_b)\mu]$. We can discretize eq. 8 using the forward-time centered-space (FTCS) scheme by noting the following discretizations of the RHS and LHS.

$$\left. \frac{\partial \mu}{\partial t} \right|_j^n \approx \frac{1}{\Delta t} (\mu_j^{n+1} - \mu_j^n) \tag{9}$$

$$\left. \frac{\partial F}{\partial x} \right|_j^n \approx \frac{1}{2\Delta x} (F_{j+1}^n - F_{j-1}^n) \tag{10}$$

Where n is the time step index and j is the spatial step index. By noting eq. 9 and 10 and how we defined the vector function $\vec{F}$ we can write the FTCS discrete form for μ and η as follows.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} \left[\left(\frac{1}{2} u^2 + g\eta \right)_{j+1}^n - \left(\frac{1}{2} u^2 + g\eta \right)_{j-1}^n \right] \tag{11}$$

$$\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{2\Delta x} \left[((\eta - \eta_b)\mu)_{j+1}^n - ((\eta - \eta_b)\mu)_{j-1}^n \right] \tag{12}$$

2.2. *Q2b - Simulating the Shallow Water System*

Using eq. 11 and 12, which have been previously derived, we can simulate a shallow water system with the following properties: length $L = 1\,\mathrm{m}$, spatial step $\Delta x = 0.02\,\mathrm{m}$, temporal time step $\Delta t = 0.01\,\mathrm{s}$, flat bottom topography $\eta_b = 0$ and average water column height $H = 0.01\,\mathrm{m}$. Where the boundary conditions simulate rigid walls and the initial condition simulates some initial Gaussian-like perturbation in the fluid. Some plots of the simulation are provided below in figures 4 and 5, however, we encourage the reader to look at the animation of the simulation as this is far more interesting to watch than the figures.

2.3. *Q2c - Von Neumann Numerical Stability Analysis*

In the previous subsection we simulated the shallow water system described in 2.1. However in subsection 2.2 we saw the free surface altitude starts behaving nonphysically and the free surface altitude waves start diverging. This is not an issue with the system of partial differential equations we are solving for instead this is an issue with the

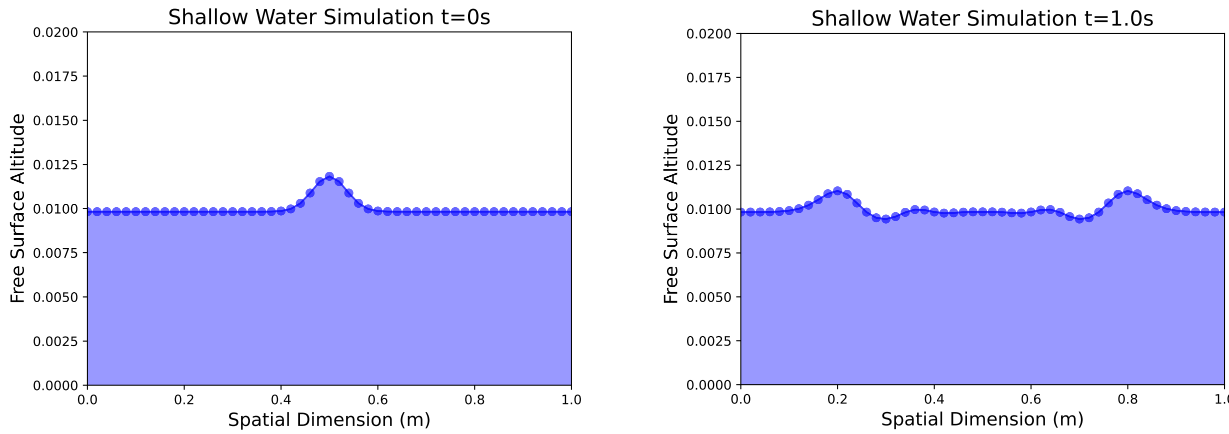

Figure 4: (Left) Shallow water simulation at initial state t_0 where initial condition is a Gaussian-like perturbation. (Right) Shallow water simulation after 1s, wave propagation is continuous and smooth.

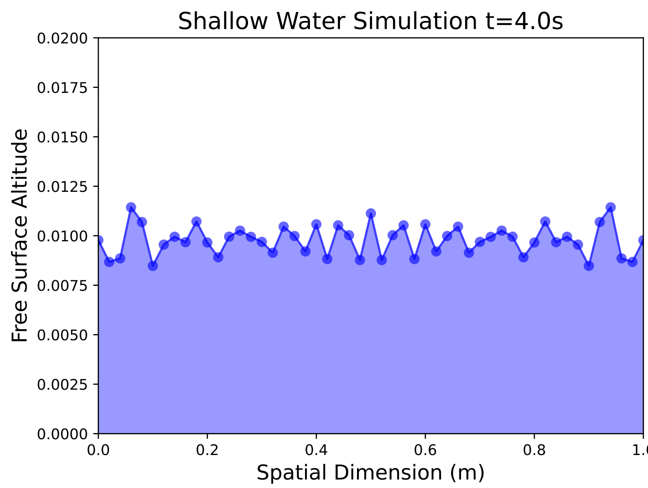

Figure 5: Shallow water simulation after 4s, we observe clear spikes in the fluid.

numerical method used to solve for μ and η. In order to study the numerical stability of the FTCS method we will implement a technique know as von Neumann stability analysis.

In order to perform this analysis we must first linearize the system of PDE's shown in eq. 4. In order to do so we must take a Taylor expansion of the right hand side where all the spatial partial derivatives are, and then we only consider the first order term of the Taylor expansion in order to only have linear terms and evaluate them at $(0, H)$.

$$\begin{aligned} \frac{\partial \mu}{\partial t} &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial t} &= -H\frac{\partial \mu}{\partial x} \end{aligned} \tag{13}$$

We then expand eq. 13 though the FTCS scheme

$$\mu(x, t + \Delta t) = \mu(x, t) - g\frac{\Delta t}{2\Delta x}(\eta(x + \Delta x, t) - \eta(x - \Delta x, t)) \tag{14}$$

$$\eta(x, t + \Delta t) = \eta(x, t) - H\frac{\Delta t}{2\Delta x}(\mu(x + \Delta x, t) - \mu(x - \Delta x, t)) \tag{15}$$

We then express the solutions of μ and η as Fourier coefficients i.e, $C_\mu(t)e^{ikx}$ to get the following relationships.

$$C_\mu(t+\Delta t) = C_\mu(t) - g\frac{\Delta t}{\Delta x}C_\eta(t)\sin(k\Delta x) \tag{16}$$

$$C_\eta(t+\Delta t) = C_\eta(t) - H\frac{\Delta t}{\Delta x}C_\mu(t)\sin(\Delta x) \tag{17}$$

We can rewrite this system of equations as a matrix representation

$$\vec{C}(t+\Delta t) = A\vec{C}(t) \tag{18}$$

$$\text{Where} \quad A = \begin{pmatrix} 1 & -g\frac{\Delta t}{\Delta x}\sin(k\Delta x) \\ -H\frac{\Delta t}{\Delta x}\sin(k\Delta x) & 1 \end{pmatrix} \tag{19}$$

We see here that for every time iteration the vector $\vec{C}(t)$ get multiplied by the matrix A. The von Neumann analysis states than if the eigenvalues of matrix A are greater than unity then the solution will diverge. We then continue to calculate the eigenvalues of A.

$$\begin{aligned} det(A-\lambda\mathbf{1}) &= (1-\lambda)^2 + gH\left(\frac{\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x) = 0 \\ \lambda &= 1 \pm i\sqrt{gH\left(\frac{\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x)} \\ |\lambda| &= \sqrt{1 + gH\left(\frac{\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x)} \end{aligned} \tag{20}$$

We clearly see that the magnitude of the eigenvalue will never be less than one therefore the solution will always diverge which leads us to conclude that the FTCS method on the shallow water equation is not stable.

$$\delta S = 0 \tag{21}$$