NEWCASTLE UNIVERSITY

Semester 2 2017/2018

Robust and Adaptive Control Systems

Time allowed – 2 hours

Instructions to candidates:

- Candidates must answer all THREE (3) questions.
- It is desirable to show the method of calculation and the steps taken to achieve the results.

[Turn Over]

Question 1 (30 marks):

A. Use $V(x,y)=x^2+y^2$ as a candidate Lyapunov Function to prove the stability of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -x+y-xy^2 \\ -2x-y-x^2y \end{bmatrix}$

[4 marks]

B. A system is given by $\dot{x}=y\left(13-x^2-y^2\right)$, $\dot{y}=12-x\left(13-x^2-y^2\right)$

a. Find its fixed points (it is given that $12-13x+x^3=(x-1)(x-3)(x+4)$).

[5 marks]

b. Determine the stability of the fixed points.

[7 marks]

c. Draw a sketch of the state space clearly indicating the fixed points, (real) eigenvectors and some trajectories.

[8 marks]

C. Given a system $\dot{x}=r+x^2$ with r being a constant

a. Create its bifurcation diagram.

[3 marks]

b. Sketch the responses (on the same graph) when r=-1 and $x(0)$=1.01 and 0.9 and -0.9 and -1.1 and -2.

[3 marks]

Question 2 (30 marks):

A. Prove that if $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ then $e^{At} = \begin{bmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{bmatrix}$.

[7 marks]

B. Write the exponential matrix of the normal form (i.e. $e^{\hat{A}t}$) of a state matrix A with eigenvalues:

a. -1, -2, -3, -4.

[2 marks]

b. -1, -1, -3, -4.

[3 marks]

C. Write the normal form (i.e. $\hat{A}$) of a state matrix A with eigenvalues:

a. -1±*i*, -1±*i*, -2±3*i*.

[3 marks]

b. -1, -1, -1, -2, -2, -3, -2±3*i*, -2±3*i*, -2±3*i* (the repeated eigenvalues do NOT give linear independent eigenvectors).

[5 marks]

D. Draw the state space of a homogeneous system $\dot{x} = Ax$, when:

a. $A = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix}$.

[2 marks]

b. $A = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}$.

[2 marks]

E. Complete the following table for a second order system with eigenvalues λ_1, λ_2. You must categorise the fixed point as focus, centre, node, saddle and its stability as stable, unstable and neutral:

Sign of eigenvalues	Type of fixed point	Stability
$\lambda_1 < \lambda_2 < 0$		
$\lambda_1 > \lambda_2 > 0$		
$\lambda_1 < 0 < \lambda_2$		
$\lambda = \mu + \nu j,\ \mu < 0$		
$\lambda = \mu + \nu j,\ \mu > 0$		
$\lambda = \mu + \nu j,\ \mu = 0$		

[6 marks]

Question 3 (40 marks):

A. Given the ODE $\ddddot{x} - 2\dddot{x} - 3\ddot{x} + 8\dot{x} - 4x = 0$

a. Find its characteristic equation and prove that 1, 2 and -2 are its solutions (1 is a double root).

[2 marks]

b. Write the Wronskian matrix.

[2 marks]

c. Find its general solution, how many initial conditions do we need in order to find its specific solution?

[2 marks]

B. A 15th order ODE has the following eigenvalues: $r_1 = -1, r_2 = -2, r_3 = -2, r_{4,5} = -3 \pm i, r_{6,7} = -4 \pm 2i, r_{8,9} = -4 \pm 2i,$ $r_{10,11} = -5 \pm 3i, r_{12,13} = -5 \pm 3i, r_{14,15} = -5 \pm 3i$. Find its 15 linear independent solutions.

[8 marks]

C. Prove that the 2nd generalised eigenvector of A can be found by solving $(A - rI)e^{(2)} = e^{(1)}$, where $e^{(1)}$ is the first generalised eigenvector.

[8 marks]

D. Given a nonlinear system $x^{(4)} = 3\ddot{x}^2 + \cos t\sqrt{x} - \frac{\dddot{x}x}{\sin t}u + \dot{x}^3 \frac{1}{\sqrt{t+1}}, t > 0$ find the correct control signal u such that we track the desired trajectory $x_d = e^{-t}$, by targeting the error dynamics to be given by $\tilde{x}^{(4)} + 7\tilde{x}^{(3)} + 17\ddot{\tilde{x}} + 17\dot{\tilde{x}} + 6\tilde{x} = 0$. Prove that the error dynamics are indeed defined by $\tilde{x}^{(4)} + 7\tilde{x}^{(3)} + 17\ddot{\tilde{x}} + 17\dot{\tilde{x}} + 6\tilde{x} = 0$.

[8 marks]

E. For a given 2nd order nonlinear system $\ddot{x} = f(x, \dot{x}, t) + g(x, \dot{x}, t)u$, find the control signal u such as $\frac{dV(s)}{dt} < 0$, where $V(s) = \frac{1}{2}s^2$ and $s = \dot{\tilde{x}} + \lambda \tilde{x}$, $\tilde{x}(t) = x(t) - x_d(t)$, with x_d being a desired smooth trajectory. It is essential that the system remains stable despite small perturbations in the system's parameters. Find the 2nd order ODE that describes the error $\tilde{x}(t)$.

[10 marks]

END