Deep Reinforcement Learning and its application to games

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- A video from Google DeepMind's Deep Q-learning playing Atari Breakout: `https://www.youtube.com/watch?v=TmPfTpjtdgg`
- *Mnih, Volodymyr, et al. "Playing atari with deep reinforcement learning." arXiv preprint arXiv:1312.5602 (2013).*
- code is also available `https://github.com/kuz/DeepMind-Atari-Deep-Q-Learner`

N.B.

- Apologies for those have seen it before

significance of this demo shows it's possible to use Neural Network to learn how to play a game, based on:

- sequences of screen images
- scores the game receives
- goal is to learn the best policy for **actions** to take

Surely you **don't need a menu** to learn how to play Atari. i.e., it's **model-free**!

Forget about the Neural network for a second, how is Reinforcement Learning (RL) different to conventional supervised learning?

- No data label like supervised learning, i.e., no *"best-action-for-that-screen"* label
- only **reward** signal
- feedback in **delayed**, not instantaneous
- data are not i.i.d., (consecutive frames are similar)
- agent's actions affects the subsequent data it receives.

Let's get started with some RL background.

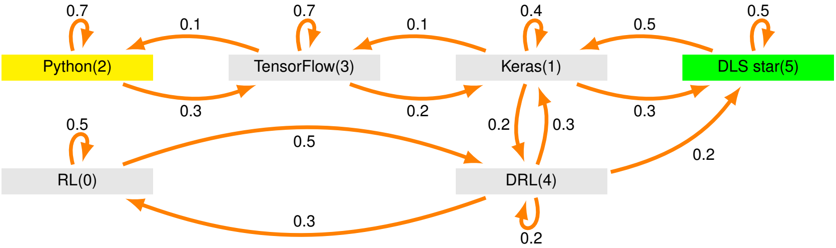
another way to look at it:

- RL uses training information that **evaluates the actions** taken rather than **instructs by giving correct actions**.
- a need for active exploration: explicit trial-and-error search for good behavior.
- **purely evaluative feedback** indicates how good the action taken is, but not whether it is the best or the worst action possible.
- **purely instructive feedback** indicates correct action to take, independently of the action actually taken. supervised learning

Application of RLs

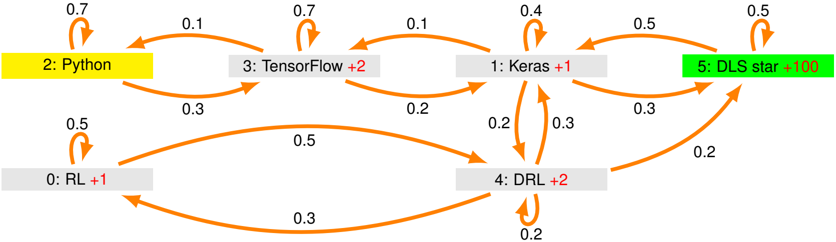
- **marketing**
  customer's attributes s, marketing actions a, customer signs up r
- **drone control**
  all avaiable sensor data a, controls s, not crashing r
- **chatbot**
  conversations to-date s, things that a robot will say a, customer satisfaction r

Markov Process



- one may start from **python** and generate sequences with transition probabilities to end up in **DLS star**. examples:
  - Python, Python, Python, TensorFlow, Keras, DLS star
  - Python, Python, Python, TensorFlow, TensorFlow, Keras, DRL, DRL RL, DLS star
  - Python, Python, TensorFlow, TensorFlow, Keras, DRL, DLS star
  - The question is: how we may able to measure “how good” each path? . . .

Let's add some rewards to being at each of the state:



What we care is the **total return** G_t: sum of **discounted** reward from time-step t

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \qquad \text{where } \gamma \in [0, 1]$$

note that G_t is a random variable
exercise what happens when $\gamma = 0$ and $\gamma = 1$

Markov Random Process: Bellman Equation (new)

- **state value function** $V(s)$ of MRP is expected total return starting from state s

$$
\begin{aligned}
V(s) &= \mathbb{E}_{s_{t+1}, s_{t+2}, \dots, r_{t+1}, r_{t+2}, \dots}[G_t | s_t = s] \\
&= \mathbb{E}_{s_{t+1}, s_{t+2}, \dots, r_{t+1}, r_{t+2}, \dots}\Big[R_{t+1} + \gamma \underbrace{\left(R_{t+2} + \gamma R_{t+3} + \dots\right)}_{G_{t+1}}\Big]
\end{aligned}
$$

- $\mathbb{E}[.]$ needs the integrate over $(s_1, s_2, \dots \in \mathcal{S}, r_1, r_2, \dots \in \mathcal{R})$:
- $s_1, s_2, \dots$ and $r_1, r_2, \dots$ are generated in the following fashion:

$$
s_0 \rightarrow (s_1, r_1) \qquad s_1 \rightarrow (s_2, r_2) \dots
$$

- for clarity, we let $s_t \rightarrow s_0$ and $s_{t+k} \rightarrow s_k$:

Markov Random Process: Bellman Equation (new)

- suppose we have a **universal state value function** $V(.)$:

$$V(.) = \sum_{s_0} \Pr(s_0) \sum_{s_1, r_1} \Pr(s_1, r_1 | s_0) \sum_{s_2, r_2} \Pr(s_2, r_2 | s_1) \sum_{s_3, r_3} \dots \left[r_1 + \gamma(r_2 + \gamma r_3 + \dots) \right]$$

- however, we usually specify value of $v_\pi(s_0)$ to evaluate:

$$V(s_0) = \underbrace{\sum_{s_1, r_1} \Pr(s_1, r_1 | s_0) \left(r_1 + \gamma \underbrace{\sum_{s_2, r_2} \Pr(s_2, r_2 | s_1) \sum_{s_3, r_3} \dots \left[r_2 + \gamma(r_3 + \gamma r_4 + \dots \right]}_{V(s_1) \triangleq \mathbb{E}[G_{t+1} | s_1]} \right)}_{V(s_0) \triangleq \mathbb{E}[G_t | s_0]}$$

$$= \mathbb{E}_{s_1, r_1} \left[r_1 + \gamma V(s_1) | s_0 \right]$$

$$= \mathbb{E}_{s_1} \left[R_1 + \gamma V(s_1) | s_0 \right] \text{ if } R_1 \text{ is deterministic}$$

Markov Random Process: Bellman Equation (new)

$$V(s_0) = \mathbb{E}_{s_1}\left[R_1 + \gamma V(s_1)|s_0\right]$$

- **Bellman equations**: value of the current state, $v(s)$ breaks up into (1) **immediate** and (2) **future** rewards.
- state value function $V(s)$ is written in a consecutive time steps
- difficult to estimate: because $V(s)$ also depends on various other $V(s')$ which occur at different times

Bellman Equation in matrix form

- to simplify, making R_t deterministic

$$V(s_0) = \mathbb{E}_{s_1}\left[R_1 + \gamma V(s_1) | s_0\right]$$

- say $s \in \{1, \dots, n\}$:

$$\underbrace{V(s_0 = 1)}_{v(1)} = \mathbb{E}_{s_1}\left[\underbrace{R_1(s_0 = 1)}_{R_1} + \gamma V(s_1) | s_0 = 1\right]$$
$$V(s_0 = 2) = \mathbb{E}_{s_1}\left[R_1(s_0 = 2) + \gamma v(s_1) | s_0 = 2\right]$$
$$\dots$$

take the first line,

$$\begin{aligned} v(1) &= \mathbb{E}_{s_1}\left[R_1 + \gamma V(s_1) | s_0 = 1\right] \\ &= R_1 + \gamma \mathbb{E}\left[V(s_1) | s_0 = 1\right] \\ &= R_1 + \gamma \left(\sum_{s_1=1}^{n} v(s_1) \Pr(1 \to s_1) \right) \\ &= R_1 + \gamma \left(\sum_{j=1}^{n} v(j) \Pr(1 \to j) \right) \\ &\dots \\ \implies v(n) &= R_n + \gamma \left(\sum_{j=1}^{n} v(j) \Pr(k \to j) \right) \end{aligned}$$

Bellman Equation in matrix form (2)

$$
\begin{aligned}
v(k) &= R_k + \gamma \left(\sum_{j=1}^{n} v(j) \Pr(k \to j) \right) \\
&= R_k + \gamma \mathcal{P}_{k,:}^{\top} \mathbf{v} \\
\implies \mathbf{v} &= \mathbf{R} + \gamma \mathcal{P} \mathbf{v}
\end{aligned}
$$

$$
\implies \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{1,1} & \dots & \mathcal{P}_{1,n} \\ \vdots & & \vdots \\ \mathcal{P}_{n,1} & \dots & \mathcal{P}_{n,n} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}
$$

the solution to **MRP** is straight forward:

$$
\begin{aligned}
\mathbf{v} &= \mathbf{R} + \gamma \mathcal{P} \mathbf{v} \\
(I - \gamma \mathcal{P}) v &= R \\
\mathbf{v} &= (I - \gamma \mathcal{P})^{-1} R
\end{aligned}
$$

Markov Decision Process (MDP)

- now agent has **actions**
- concept of **policy** π: take a state s_t as input and decides and action a_t

$$\pi(a|s) = \Pr(A_t = a|S_t = s)$$

- a policy is time-invariant (or stationary) and stochastic
- next state for an agent, now also depends on its action taken:

$$\mathcal{P}^a_{s \to s'} = \Pr(S_1 = s'|S_0 = s, A_0 = a)$$

- multiple transition matrix $\mathcal{P}$ each depends on the a taken
- once fixed π, MDP becomes MRP with transition probability $\mathcal{P}^\pi_{s \to s'}$:

$$\mathcal{P}^\pi_{s \to s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{s \to s'}$$

Markov Decision Process: Bellman Equation (new)

- given a policy π, **state value function** $v(s)$ is expected total return starting from state s

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi[G_t | s_t = s] \\ &= \mathbb{E}_\pi\Big[R_{t+1} + \gamma \underbrace{\left(R_{t+2} + \gamma R_{t+3} + \dots\right)}_{G_{t+1}}\Big] \end{aligned}$$

- $\mathbb{E}_\pi[.]$ needs the integrate over $(a_0, a_1, \dots \in \mathcal{A}, s_0, s_1, \dots \in \mathcal{S}, r_1, r_2, \dots \in \mathcal{R})$:

- then chain of changes are then:

$$s_0 \rightarrow a_0, \qquad (s_0, a_0) \rightarrow (s_1, r_1), \qquad s_1 \rightarrow a_1, \qquad (s_1, a_1) \rightarrow (s_2, r_2), \qquad \dots$$

- for clarity, we let $s_t \rightarrow s$ and $s_{t+1} \rightarrow s'$:

Markov Decision Process: Bellman Equation (new)

- suppose we have a **universal state value function** $V_\pi(.)$, i.e., no matter what the current state and action is:

$$
\begin{aligned}
& v_\pi(.) \\
& = \sum_{s_0} \Pr(s_0) \sum_{a_0} \pi(a_0|s) \sum_{s_1, r_1} \Pr(s_1, r_1|s_0, a_0) \sum_{a_1} \pi(a_1|s_1) \sum_{s_2, r_2} \Pr(s_2, r_2|s_1, a_1) \sum_{a_2} \cdots \sum_{s_3, r_3} \ldots \\
& \qquad [r_1 + \gamma(r_2 + \gamma r_3 + \ldots)]
\end{aligned}
$$

- however, we do know the value $v_\pi(s_0)$:

$$
\begin{aligned}
& V_\pi(s_0) \\
& = \underbrace{\sum_{a_0} \pi(a_0|s) \sum_{s_1, r_1} \Pr(s_1, r_1|s_0, a_0) \Big(r_1 + \gamma \underbrace{\sum_{a_1} \pi(a_1|s_1) \sum_{s_2, r_2} \Pr(s_2, r_2|s_1, a_1) \sum_{a_2} \cdots \sum_{s_3, r_3} \ldots [r_2 + \gamma(r_3 + \gamma r_4 + \ldots]}_{V_\pi(s_1) \triangleq \mathbb{E}_\pi[G_{t+1}|s_1]} \Big)}_{V_\pi(s_0) \triangleq \mathbb{E}_\pi[G_t|s_0]} \\
& = \sum_{a_0} \pi(a_0|s) \sum_{s_1, r_1} \Pr(s_1, r_1|s_0, a_0) (r_1 + \gamma \mathbb{E}_\pi[G_{t+1}|s_1])
\end{aligned}
$$

Bellman equation extends to $Q(s, a)$

summarise slides from before:

$$
\begin{aligned}
V_\pi(s) &= \sum_a \pi(a|s) \sum_{s',r'} \Pr(s', r'|s, a)\left(r' + \gamma v_\pi(s')\right) \\
&= \sum_a \pi(a|s) \sum_{s',r'} \Pr(s', r'|s, a)\left(r' + \gamma \mathbb{E}_\pi[G_{t+1}|s']\right) \\
&= \sum_a \pi(a|s) \mathbb{E}_{(s',r')\sim}\left[r' + \gamma v_\pi(s')\right]
\end{aligned}
$$

insert a to obtain Q function:

$$
\begin{aligned}
Q_\pi(s, a) &= \sum_{s',r'} \Pr(s', r'|s, a)\left(r' + \gamma v_\pi(s')\right) \\
&= \sum_{s',r'} \Pr(s', r'|s, a)\left(r' + \gamma \mathbb{E}_\pi[G_{t+1}|s']\right) \\
&= \mathbb{E}_{(s',r')\sim}\left[r' + \gamma v_\pi(s')\right]
\end{aligned}
$$

- this version is called **Bellman Expectation equation**
- expectation equation is linear in V, so you can solve for V using simple linear algebra
- Bellman Expectation equation is usually used to evaluate a known policy π

Bellman equation extends to $Q(s, a)$

since any policy π works, then:

$$Q_{\pi_*}(s, a) = \mathbb{E}_{(s', r') \sim} \left[r' + \gamma v_{\pi_*}(s') \right]$$
$$\text{or } Q_*(s, a) = \mathbb{E}_{(s', r') \sim} \left[r' + \gamma v_*(s') \right]$$

- this version is called **Bellman Optimality Equation**
- nonlinear (due to the max operation) in V, so there is no closed-form solution
- Bellman Expectation equation is usually used to evaluate a known policy π
- Many algorithms need to find the optimal solution (Q-learning, value iteration, policy iteration etc)
- optimality equation is used to learn the optimal policy π_*

Bellman optimality

- we know best $V_*(s)$ must be the best action from an optimal (state, action) pair: $Q_*(s, a)$:

$$V_*(s) = \max_a Q_*(s, a)$$

- and from before:

$$\begin{aligned}
V_*(s) &= \max_a \underline{Q_*(s, a)} \\
&= \max_a \underline{\mathbb{E}_{(s', r') \sim} \left[r' + \gamma v_*(s') \right]} \qquad \text{from last page} \\
&= \max_a \sum_{s', r'} \Pr(s', r' | s, a) \left(r' + \gamma v_*(s') \right) \\
&= \max_a \sum_{s', r'} \Pr(s', r' | s, a) \left(r' + \gamma \max_{a'} Q_*(s', a') \right) \\
&= \max_a \mathbb{E}_{(s', r') \sim} \left[r' + \gamma \max_{a'} Q_*(s', a') \middle| s \right] \\
&= \max_a \mathbb{E}_{(s', r') \sim} \left[r' + \gamma \max_{a'} Q_*(s', a') \right] \qquad \text{removed } \Big| s \text{ for clarity} \\
\implies Q_*(s) &= \mathbb{E}_{(s', r') \sim} \left[r' + \gamma \max_{a'} Q_*(s', a') \right]
\end{aligned}$$

- also $r' \triangleq r'(s, \pi(s), s')$

Value Iteration

$$
\begin{aligned}
&\textbf{loop} \forall s \in \mathbb{S} \\
&\qquad \textbf{loop} \forall a \in \mathcal{A} \\
&\qquad\qquad Q(s, a) = \sum_{s' \in \mathbb{S}} \mathcal{P}^{a}_{s \to s'} \left[R(s, a, s') + \gamma V(s') \right] \\
&\qquad V(s) = \max_{a} Q(s, a) \\
&\pi_*(S) = \arg\max_{a} \sum_{s' \in \mathbb{S}} \mathcal{P}^{a}_{s \to s'} \left[R(s, a, s') + \gamma V(s') \right]
\end{aligned}
$$

- implicitly solve for the state values under an ideal policy
- no need to define an actual policy during the iterations (although “current” policy is computable each iteration)

Policy Iteration

- choose an arbitrary policy π'
- **while** before some stopping criteria:

  $\pi = \pi'$

  compute the value function $V_\pi(1), \dots V_\pi(n)$ using policy π:

$$V_\pi(s_0) = R(s, \pi(s_0)) + \gamma \sum_{s_1 \in \mathbb{S}} \mathcal{P}^{a_0}_{s_0 \to s_1} V_\pi(s_1)$$

  improve the policy at each state:

$$\pi'(s_0) = \arg\max_{a_0} \left[R(s, a_0) + \gamma \sum_{s_1 \in \mathbb{S}} \mathcal{P}^{a_0}_{s_0 \to s_1} V_\pi(s_1) \right]$$

  - start a policy π_0 and iterate towards π_*, by estimating state value associated with the policy, and making changes to action choices
  - after each policy iteration, re-calculate value function for that policy. means you also work with value functions measure actual policies

Policy Evaluation

- given π, let's work out how good is v^{π}:

$$\begin{aligned} V^{\pi}(s) &= \mathbb{E}_{\pi}\left[R_t | s_t = s\right] \\ &= \mathbb{E}_{\pi}\left[R_{t+1} + \gamma V^{\pi}(s_{t+1}) | s_t = s\right] \\ &= \sum_a \pi(s, a) \sum_{s'} \mathcal{P}^a_{ss'}\left[R^a_{ss'} + \gamma V^{\pi}(s')\right] \end{aligned}$$

Policy Improvement

- given you are in state s, instead of following $\pi(s)$, what if we choose **another** policy, such that $a \neq \pi(s)$:
- look at $Q^{\pi}(s, a)$, i.e., a value function for taking a particular action a, instead of average of all actions:

$$\begin{aligned} Q^{\pi}(s, a) &= \mathbb{E}_{\pi}\left[R_{t+1} + \gamma V^{\pi}(S_{t+1} | S_t = s, a_t = a\right] \\ &= \sum_{s'} \mathcal{P}^a_{ss'}\left[R^a_{ss'} + \gamma V^{\pi}(s_{t+1})\right] \end{aligned}$$

- a should be chosen iff $Q^{\pi}(s, a) > V^{\pi}(s)$

Policy Improvement thereom

- **if** choosing $a \neq \pi(s)$ implies $Q^{\pi}(s,a) \geq V^{\pi}(s)$ for s
- **then** we choose policy π' for state s, and π for other $s' \neq s$
- this policy π' is at least as good as π, i.e., $V^{\pi'}(s) \geq V^{\pi}(s)$, i.e:

$$Q^{\pi}(s,a) > V^{\pi}(s) \implies V^{\pi'}(s) > V^{\pi}(s)$$

- proof:

$$\begin{aligned}
V^{\pi}(s) &\leq Q^{\pi}(s, \pi'(s)) \qquad \text{replace } s' \leftarrow \pi'(s) \\
&= \sum_{s'} \mathcal{P}_{ss'}^{\pi'(s)} \left[R_{ss'}^{\pi'(s)} + \gamma V^{\pi}(s_{t+1}) \right] \qquad \text{obvioulsy } \pi' \text{ gives different } s' \text{ than } \pi \\
&\equiv \mathbb{E}_{\pi'(s)} \left[R_{t+1} + \gamma V^{\pi}(S_{t+1}) | S_t = s \right] \\
&\leq \mathbb{E}_{\pi'(s)} \left[R_{t+1} + \gamma Q^{\pi}(S_{t+1}, \pi'(S_{t+1})) | S_t = s \right] \qquad \text{apply recursively to } V^{\pi}(S_{t+1}) \\
&= \mathbb{E}_{\pi'(s)} \left[R_{t+1} + \gamma \mathbb{E}_{\pi'(s')} \left[R_{t+2} + \gamma V^{\pi}(S_{t+2}) \right] | S_t = s \right] \\
&\equiv \mathbb{E}_{\pi'} \left[R_{t+1} + \gamma \mathbb{E}_{\pi'} \left[R_{t+2} + \gamma V^{\pi}(S_{t+2}) \right] | S_t = s \right] \\
&\vdots \\
&\leq \mathbb{E}_{\pi'} \left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+1} + \ldots | S_t = s \right] \\
&= V^{\pi'}(s)
\end{aligned}$$

Policy Improvement thereom (2)

- if apply this strategy to all states to get a new greedy policy:

$$\pi'(s) = \arg\max_a \left[Q^{\pi}(s,a) \right] \implies V^{\pi'} \geq V^{\pi}$$

- when $V^{\pi'} = V^{\pi} \implies$, and we know,

$$V^{\pi}(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[R^a_{ss'} + \gamma V^{\pi}(s') \right]$$

  we know:

$$V^{\pi'(s)} = \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[R^a_{ss'} + \gamma V^{\pi}(s') \right]$$

- a form of Bellman optimality equation
- therefore, $V^{\pi} = V^{\pi'} = V^*$

Bellman's equation: Three ways of solving it

$$V_\pi(s_0) = \mathbb{E}_\pi\left[G_t | s_0\right]$$

– could be approximated by Monte-carlo, i.e., sample $s_1, s_2, \ldots$ and compute G_t

$$= \mathbb{E}_\pi\left[r(s_0, \pi(s_0), s_1) + \gamma V_\pi(s_1)\right]$$

– could be approximated by Temporal Difference

$$= \sum_{a_0} \pi(a_0|s_0) \sum_{s_1} \mathcal{P}^{a_0}_{s_0 \to s_1}\left[r(s_0, \pi(s_0), s_1) + \gamma V_\pi(s_1)\right]$$

– could be solved exactly by Dynamic programming

Solve Bellman's equation using Temporal Difference

$$V_\pi(s_0) = \sum_{a_0} \pi(a_0|s_0) \sum_{s_1, r_1} \Pr(s_1, r_1|s_0, a_0)\big(r_1 + \gamma v_\pi(s_1)\big)$$

- drop $|s$ again for clarity:

$$\begin{aligned}
V^\pi(s) &= \mathbb{E}_{s'}\Big[r(s, \pi(s), s') + \gamma V^\pi(s')\Big] \\
\implies V^\pi(s) + \eta V^\pi(s) &= V^\pi(s) + \eta\Big(\mathbb{E}_{s'}\Big[r(s, \pi(s), s') + \gamma V^\pi(s')\Big]\Big) \\
\implies V^\pi(s) &= V^\pi(s) + \eta\Big(\mathbb{E}_{s'}\Big[r(s, \pi(s), s') + \gamma V^\pi(s')\Big] - V^\pi(s)\Big)
\end{aligned}$$

- instead of compute this expectation, in **each iteration** t, we sample a new state $\tilde{s'} \sim \Pr(s'| \dots)$

$$V^\pi_{t+1}(s) = V^\pi_t(s) + \eta\Big(r(s, \pi(s), \tilde{s'}) + \gamma V^\pi_t(\tilde{s'}) - V^\pi_t(s)\Big)$$

- note that the last equation is called **temporal difference**

Action-value (Q) function

- action-valued function $Q^{\pi}(s, a) = \mathbb{E}\big[G_t | S_t = s, A_t = a, \pi\big]$:
- expected total return starting from state s, taking action a, and then follow policy π
- Stochastic policy π:

$$V^{\pi}(s) = \mathbb{E}_{a \sim \pi(s)}[Q^{\pi}(s, a)]$$

- deterministic policy:

$$v^*(s) = \max_{a'} Q^*(s, a')$$

- from before;

$$\begin{aligned} V^*(s) &= \max_a \left(\mathbb{E}_{s'} \left[r(s, a, s') + \gamma \underbrace{V^*({\color{red}s'})} \middle| s \right] \right) \\ &= \underbrace{\max_a \left(\mathbb{E}_{s'} \left[r(s, a, s') + \gamma \big(\max_{a'} Q^*({\color{red}s'}, a') \big) \middle| s \right] \right)}_{Q^*(s', a') \text{ by definition}} \end{aligned}$$

- therefore:

$$Q^*(s, a) = \mathbb{E}_{s'} \left[r(s, a, s') + \gamma \big(\max_{a'} Q^*(s', a') \big) \middle| s, a \right] \Big)$$

Action-value (Q) function

$$Q^*(s,a) = \mathbb{E}_{s'}\left[r(s,a,s') + \gamma\left(\max_{a'} Q^*(s',a')\right)\Big| s,a\right]\Big)$$

- drop $|s, a$, let's solve this by **temporal difference**:

$$Q^\pi(s,a) = \mathbb{E}_{s'}\left[r(s,\pi(s),s') + \gamma\left(\max_{a'} Q^\pi(s',a')\right)\right]$$

$$\Longrightarrow \quad Q^\pi(s,a) + \eta Q^\pi(s,a) = Q^\pi(s,a) + \eta\left(\mathbb{E}_{s'}\left[r(s,a,s') + \gamma\left(\max_{a'} Q^\pi(s',a')\right)\right]\right)$$

$$\Longrightarrow \quad Q^\pi(s,a) = Q^\pi(s,a) + \eta\left(\mathbb{E}_{s'}\left[r(s,a,s') + \gamma\left(\max_{a'} Q^\pi(s',a')\right)\right] - Q^\pi(s,a)\right)$$

- instead of compute this expectation, in **each iteration** t, we sample a new state $(\tilde{s'}, \tilde{a}) \sim \Pr(s', a| \dots)$.

  Q-Learning: recursively:

$$Q(s,\tilde{a}) = Q(s,\tilde{a}) + \eta\Big(\underbrace{r(s,\tilde{a},\tilde{s'}) + \gamma\left(\max_{a'} Q(\tilde{s'},a')\right)}_{y} - Q(s,\tilde{a})\Big)$$

- let $\eta = 1$:

$$Q(s,\tilde{a}) = r(s,\tilde{a},\tilde{s'}) + \gamma\left(\max_{a'} Q(\tilde{s'},a')\right)$$

Require: choice of γ Rewards matrix R

1: $Q \leftarrow \mathbf{0}$
2: **for** each episode **do**
3: randomise initiate state s_0
4: **while** goal state not reached **do**
5: select $(a, s') \sim \Pr(a, s'|.)$
6: compute $\max_{a'} Q(s', a')$
7: $Q(s, a) \leftarrow r(s, a, s') + \gamma\big(\max_{a'} Q(s', a')\big)$
8: $s_t \leftarrow s_{t+1}$
9: **end while**
10: **end for**

Q and V functions

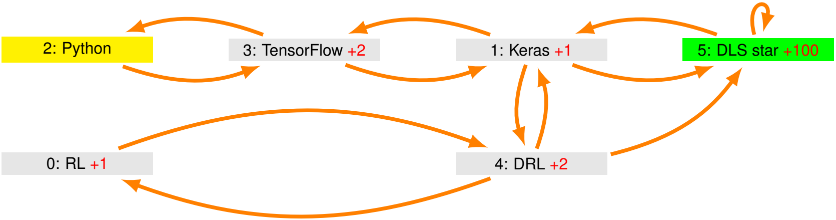
- if $a = \arg\max_a Q^\pi(s, a)$, then by setting $\pi'(a|s) = 1$, this policy is at least as good as π regardless of what π is
- if $Q^\pi(s, a) > V^\pi(s)$, then a is better than average since,

$$V^\pi(s) = \mathbb{E}_{a\sim\pi(a|s)}[Q^\pi(s, a)]$$

- obviously, we should increase $\pi(a|s)$ if $Q^\pi(s, a) > V^\pi(s)$

Q-Learning example

We took the example from the Markov Reward Process example earlier:



- there is small immediate rewards by going from one module to another
- you get a final large reward by becoming DLS star
- let $\gamma = 0.5$
- in this special example, $a = s'$, i.e., the action is to turn into the next state (module of studies).
- assume equal probabilities for all edges.

Q-Learning example: episode 1

$$R = \begin{array}{c} \\ \text{RL(0)} \\ \text{Ke(1)} \\ \text{Py(2)} \\ \text{TF(3)} \\ \text{DRL(4)} \\ \text{DLS*(5)} \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL(0)} \quad \text{Ke(1)} \quad \text{Py(2)} \quad \text{TF(3)} \quad \text{DRL(4)} \quad \text{DLS*(5)} \\ \begin{pmatrix} - & - & - & - & 2 & - \\ - & - & - & 2 & 2 & 100 \\ - & - & - & 2 & - & - \\ - & 1 & 0 & - & - & - \\ 1 & 1 & - & - & - & 100 \\ - & 1 & - & - & - & 100 \end{pmatrix} \end{array}$$

before

$$Q = \begin{array}{c} \\ \text{RL(0)} \\ \text{Ke(1)} \\ \text{Py(2)} \\ \text{TF(3)} \\ \text{DRL(4)} \\ \text{DLS*(5)} \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL(0)} \quad \text{Ke(1)} \quad \text{Py(2)} \quad \text{TF(3)} \quad \text{DRL(4)} \quad \text{DLS*(5)} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

after

$$Q = \begin{array}{c} \\ \text{RL(0)} \\ \text{Ke(1)} \\ \text{Py(2)} \\ \text{TF(3)} \\ \text{DRL(4)} \\ \text{DLS*(5)} \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL(0)} \quad \text{Ke(1)} \quad \text{Py(2)} \quad \text{TF(3)} \quad \text{DRL(4)} \quad \text{DLS*(5)} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- $s \sim \Pr(s|.) = 1$, i.e, Keras
- at $s = 1$, it has **allowable actions**: go to state $\{3, 4, 5\}$, i.e., $a \in \{3, 4, 5\}$
- $(a, s') \sim \Pr(a, s'|.) = (5, 5)$
- at $s' = 5$, it has **allowable actions**: $a' \in \{1, 5\}$:

$$\begin{aligned} Q(s, a) &= r(s, a, s') + \gamma(\max_{a'} Q(s', a')) \\ &= R(1, s' = 5) + 0.5 \max[Q(s' = 5, 1), Q(s' = 5, 5)] \\ &= 100 + 0.5 \times 0 = 100 \end{aligned}$$

- set $s \leftarrow s' \implies s = 5$, i.e., goal state, end

Q-Learning example: episode 2, Iteration 1

$$R = \begin{array}{c|cccccc} \text{S} \downarrow, \text{A} \rightarrow & \text{RL(0)} & \text{Ke(1)} & \text{Py(2)} & \text{TF(3)} & \text{DRL(4)} & \text{DLS*(5)} \\ \hline \text{RL(0)} & - & - & - & - & 2 & - \\ \text{Ke(1)} & - & - & - & 2 & 2 & 100 \\ \text{Py(2)} & - & - & - & 2 & - & - \\ \text{TF(3)} & - & 1 & 0 & - & - & - \\ \text{DRL(4)} & 1 & 1 & - & - & - & 100 \\ \text{DLS*(5)} & - & 1 & - & - & - & 100 \end{array}$$

before

$$Q = \begin{array}{c|cccccc} \text{S} \downarrow, \text{A} \rightarrow & \text{RL(0)} & \text{Ke(1)} & \text{Py(2)} & \text{TF(3)} & \text{DRL(4)} & \text{DLS*(5)} \\ \hline \text{RL(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{Ke(1)} & 0 & 0 & 0 & 0 & 0 & 100 \\ \text{Py(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{TF(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{DRL(4)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{DLS*(5)} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

after

$$Q = \begin{array}{c|cccccc} \text{S} \downarrow, \text{A} \rightarrow & \text{RL(0)} & \text{Ke(1)} & \text{Py(2)} & \text{TF(3)} & \text{DRL(4)} & \text{DLS*(5)} \\ \hline \text{RL(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{Ke(1)} & 0 & 0 & 0 & 0 & 0 & 100 \\ \text{Py(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{TF(3)} & 0 & 51 & 0 & 0 & 0 & 0 \\ \text{DRL(4)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{DLS*(5)} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

- $s \sim \Pr(s|.) = 3$
- at $s = 3$, it has **allowable actions**: go to state $\{1, 2\}$, i.e., $a \in \{1, 2\}$
- $(a, s') \sim \Pr(a, s'|.) = (1, 1)$
- at $s' = 1$, it has **allowable actions**: $a' \in \{3, 4, 5\}$:

$$\begin{aligned} Q(s, a) &= r(s, a, s') + \gamma(\max_{a'} Q(s', a')) \\ &= R(3, 1) + 0.5 \max[Q(1, 3), Q(1, 4), Q(1, 5)] \\ &= 1 + 0.5 \times 100 = 51 \end{aligned}$$

- set $s \leftarrow s' \implies s = 1$, i.e., **not** a goal state, keep on going

Q-Learning example: episode 2, Iteration 2

$$R = \begin{array}{c} \\ \text{RL}(0) \\ \text{Ke}(1) \\ \text{Py}(2) \\ \text{TF}(3) \\ \text{DRL}(4) \\ \text{DLS*}(5) \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL}(0) \quad \text{Ke}(1) \quad \text{Py}(2) \quad \text{TF}(3) \quad \text{DRL}(4) \quad \text{DLS*}(5) \\ \begin{pmatrix} - & - & - & - & 2 & - \\ - & - & - & 2 & 2 & 100 \\ - & - & - & 2 & - & - \\ - & 1 & 0 & - & - & - \\ 1 & 1 & - & - & - & 100 \\ - & 1 & - & - & - & 100 \end{pmatrix} \end{array}$$

before

$$Q = \begin{array}{c} \\ \text{RL}(0) \\ \text{Ke}(1) \\ \text{Py}(2) \\ \text{TF}(3) \\ \text{DRL}(4) \\ \text{DLS*}(5) \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL}(0) \quad \text{Ke}(1) \quad \text{Py}(2) \quad \text{TF}(3) \quad \text{DRL}(4) \quad \text{DLS*}(5) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 51 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

after

$$Q = \begin{array}{c} \\ \text{RL}(0) \\ \text{Ke}(1) \\ \text{Py}(2) \\ \text{TF}(3) \\ \text{DRL}(4) \\ \text{DLS*}(5) \end{array} \begin{array}{c} \text{S} \downarrow, \text{A} \rightarrow \quad \text{RL}(0) \quad \text{Ke}(1) \quad \text{Py}(2) \quad \text{TF}(3) \quad \text{DRL}(4) \quad \text{DLS*}(5) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 51 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- $s = 1$ from previous iteration
- at $s = 1$, it has **allowable actions**: go to state $\{3, 4, 5\}$, i.e., $a \in \{3, 4, 5\}$
- $(a, s') \sim \Pr(a, s' | .) = (5, 5)$
- at $s' = 5$, it has **allowable actions**: $a' \in \{1, 5\}$:

$$Q(s, a) = r(s, a, s') + \gamma(\max_{a'} Q(s', a'))$$

$$Q(1, 5) = R(1, 5) + 0.5 \max[Q(5, 1), Q(5, 5)]$$

$$= 100 + 0.5 \times 0 = 100$$

- set $s \leftarrow s' \implies s = 5$, i.e., goal state, end

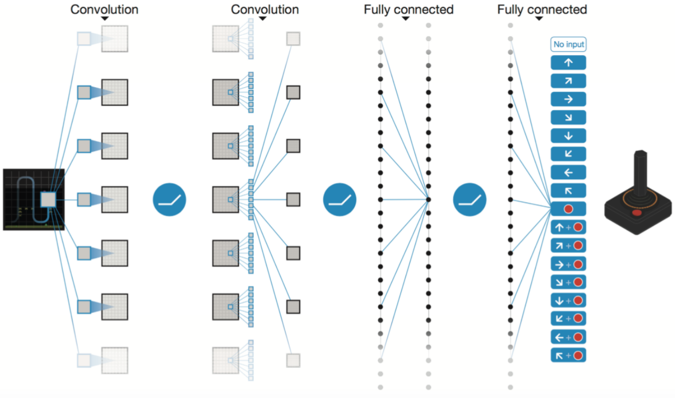
the state-action table gets updated until convergence.

On the setting of Atari

- the states are far too many!
- need a **function approximator** to estimate the action-value function, $Q(s, a|\theta) \approx Q^*(s, a)$
- guess what? Deep Neural Network helps!

Represent $Q(s, a)$ using neural networks

- The figure below represents a row of the Q function table earlier:



Conv [16] → ReLU → Conv [32] → ReLU → FC [256] → ReLU → FC [|A|]

- these are **not** softmax functions.

abstracted algorithm for Deep Q-Learning

Require: Initialize an empty replay memory
Require: Initialize the DQN weights θ

for each episode **do**
  for $t = 1, \dots T$ **do**
    with probability ϵ select $\tilde{a}$ random action
    otherwise, select:

$$\tilde{a} = \max_a \left(Q^*(s, a|\theta) \right)$$

    perform $\tilde{a}$ and receive rewards r_t and state s'.
    add tuple $(s, \tilde{a}, r_t, s')$ into replay memory
    Sample a mini-batch of tuples (s_j, a_j, r_j, s'_j) from the replay memory
    and perform stochastic gradient descent on the DQN, based on the loss function:

$$\Bigg(\underbrace{r_j + \gamma \big(\max_{a'} Q(s'_j, a'|\theta^-) \big)}_{y_j} - Q(s_j, a_j|\theta) \Bigg)^2$$

  end for
end for

innovation

- freeze parameters of target network $Q(s'_j, a'|\theta^-)$ for fixed number of iterations
- while updating the online network $Q(s; a; \theta_i)$ by gradient descent

double Deep Q-Leanring

- same values θ both to select and to evaluate an action:

$$
\begin{aligned}
y_j &= r_j + \gamma\big(\max_{a'} Q(s'_j, a' | \theta)\big) \\
&= r_j + \gamma\big(Q(s'_j, \arg\max_a Q(s'_j, a, \theta) | \theta)\big)
\end{aligned}
$$

- more likely to select overestimated values
- resulting in overoptimistic value estimates
- the solution is:

$$
y_j = r_j + \gamma\big(\max_{a'} Q(s'_j, \arg\max_a Q(s'_j, a, \theta) | \theta')\big)
$$

- still estimating value of policy according to current values defined by θ
- use second set of weights θ' to **fairly** evaluate value of this policy

In summary

- CNN and RNN are two of the building blocks in Deep Learning
- People have been putting them into many existing machine learning frameworks, and have generated many interesting stuff
- but there is plenty still needs to be explored!