A Quick Tutorial on Lagrangian Duality and Application to SVM

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## 1 Optimization with inequality constraints

A constrained optimization is of the following form (ignore the equality constraints for now):

$$\begin{aligned} &\min f(\mathbf{x}) \\ &\text{s.t. } g_i(\mathbf{x}) \leq 0 \ \forall i \in 1, \dots, m \end{aligned} \tag{1}$$

After defining $\mathbf{I}(u) = \begin{cases} 0, & \text{if } u \leq 0 \\ \infty, & \text{otherwise} \end{cases}$, we can turn a constrained equation into **unconstrained** equation:

$$J(x) = f(x) + \sum_i \mathbf{I}[g_i(x)] \tag{2}$$

it words, it makes infeasible region to have prohibitively large value, i.e., $\infty$ making it impossible to find a **minimization** solution in infeasible region

Similarly, in **maximization**, infeasible region are assigned value of $-\infty$ making it impossible to find a maximum solution in infeasible region

$$J(x) = f(x) - \sum_i \mathbf{I}[g_i(x)] \tag{3}$$

## 2 Looking at the lower Bound constraints

Replace $\mathbf{I}[g_i(x)]$ by its lower bound $\lambda_i g_i(\mathbf{x})$, with $\lambda_i \geq 0$. Therefore $J(x) \to \mathcal{L}(x, \lambda)$:

$$\mathcal{L}(x, \lambda) = f(\mathbf{x}) + \sum_i \lambda_i g_i(\mathbf{x}) \tag{4}$$

since $\lambda_i g_i(\mathbf{x})$ is lower bound of $\mathbf{I}[g_i(x)]$:

$$\begin{aligned} &\mathcal{L}(x, \lambda) \leq J(\mathbf{x}) \\ \text{i.e.,} \quad &\max_\lambda \mathcal{L}(\mathbf{x}, \lambda) = J(\mathbf{x}) \end{aligned} \tag{5}$$

if we were to minimize $\mathbf{x}$ on both sides:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\lambda} \mathcal{L}(\mathbf{x}, \lambda) &= \min_{\mathbf{x}} J(\mathbf{x}) \\ &= p^* \end{aligned} \tag{6}$$

In words, it means for $\mathcal{L}(\mathbf{x}, \lambda)$ we maximize $\lambda$ first, then minimize $\mathbf{x}$ and we obtain $J(\mathbf{x}^*)$. However, it is point-less to do so in that optimization order

## 3 swap the optimization order: $\min_x$ first, then $\max_\lambda$

from Eq(6)

$$\begin{aligned} \min_{\mathbf{x}} \max_{\lambda} \mathcal{L}(\mathbf{x}, \lambda) &= \min_{\mathbf{x}} J(\mathbf{x}) \\ \Longrightarrow \max_{\lambda} \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda) \leq \min_{\mathbf{x}} \max_{\lambda} \mathcal{L}(\mathbf{x}, \lambda) &= \min_{\mathbf{x}} J(\mathbf{x}) \\ \Longrightarrow \Big(d^* \equiv \max_{\lambda} \min_{x} \mathcal{L}(\mathbf{x}, \lambda)\Big) \leq \Big(p^* \equiv \min_{\mathbf{x}} \max_{\lambda} \mathcal{L}(\mathbf{x}, \lambda) &= \min_{\mathbf{x}} J(\mathbf{x})\Big) \end{aligned} \tag{7}$$

this relationship can be understood by **max-min inequality**

$$\sup_{\lambda} \inf_{x} f(\lambda, x) \leq \inf_{x} \sup_{\lambda} f(\lambda, x) \tag{8}$$

"the greatest of all minima" is less or equal to "the least of all maxima", **proof**:

$$\begin{aligned} & \inf_{x} f(\lambda, x) \leq f(\lambda, x), \forall \lambda \, \forall x \\ \Longrightarrow & \sup_{\lambda} \inf_{x} f(\lambda, x) \leq \sup_{\lambda} f(\lambda, x), \forall x \qquad \sup_{\lambda} \text{ both sides} \\ \Longrightarrow & \sup_{\lambda} \inf_{x} f(\lambda, x) \leq \inf_{x} \sup_{\lambda} f(\lambda, x) \qquad \text{on RHS: } \because \inf_{x} \in \forall x \end{aligned} \tag{9}$$

if strong duality holds:

$$d^* = p^* \tag{10}$$

## 4 advantage of dual function

in summary, the duality procedure is to find $\lambda^*$

$$\lambda^* = \arg\max_{\lambda} \big( \min_{x} \mathcal{L}(\mathbf{x}, \lambda) \big) \tag{11}$$

dual function $\min_x \mathcal{L}(\mathbf{x}, \lambda)$ is concave, even when the initial problem is not convex. Because it is a point-wise (in $\mathbf{x}$) infimum of affine functions:

$$\min_{x} \mathcal{L}(\mathbf{x}, \lambda) \triangleq \min_{x} \Big( f(\mathbf{x}) + \sum_{i} \lambda_i g_i(\mathbf{x}) \Big) \tag{12}$$

## 4.1 convex-concave theorem

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be compact convex sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function that is convex-concave:

$$\begin{aligned} &f(\cdot, y) : X \to \mathbb{R} \text{ is convex for fixed } y \\ &f(x, \cdot) : Y \to \mathbb{R} \text{ is concave for fixed } x \end{aligned} \tag{13}$$

then:

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y) \tag{14}$$

# 5 complementary slackness

## 5.1 when constraints are all satisfied: i.e., $g_i(\mathbf{x}^*) \leq 0 \ \forall i$

best $\lambda_i$ occurs when:

$$\lambda_i^* = \arg\max_{\lambda_i} \mathcal{L}(x, \lambda_i) = 0 \tag{15}$$

this is because we need $\lambda_i \geq 0$, and in the case:

$$g_i(\mathbf{x}) \leq 0 \text{ and } \lambda_i > 0 \implies \lambda_i g_i(\mathbf{x}) \leq 0 \tag{16}$$

so **max** occur when $\lambda_i = 0$

## 5.2 When constraints are not all satisfied: $\exists_i \ g_i(\mathbf{x}^*) > 0$

we can **maximize** $\mathcal{L}(\mathbf{x}, \lambda)$ by taking $\lambda_i \to +\infty$. We can see the way to prevent $\mathcal{L}(\mathbf{x}, \lambda)$ going to infinity is to locate new $\mathbf{x}*$ to be a "sub-optimal" solution of the unconstrained solution, a the contour where:

$$g_i(\mathbf{x}^*) = 0 \tag{17}$$

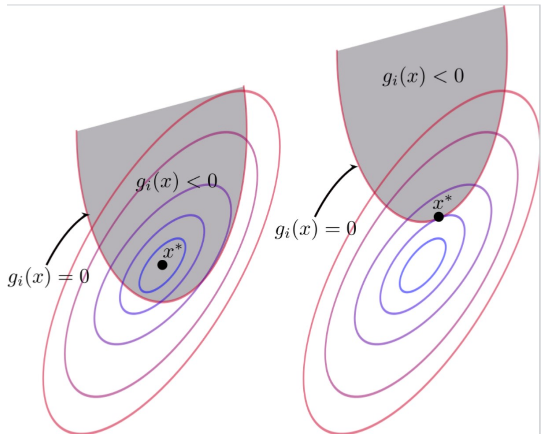
instead of original $\mathbf{x}^*$, i.e., optimal unconstrained solution $f(\mathbf{x}) = 0$

## 5.3 combine the two

Combine the above two cases, we found either $\lambda_i = 0$ or $g_i(\mathbf{x}) = 0$. We can specify it in a single equation:

$$\lambda_i g_i(\mathbf{x}) = 0 \tag{18}$$

This is called **complimentary slackness**. Diagrammatically, this is illustrated from a diagram from Wikipedia:




## 6 summary of KKT condition

**optimization problem** with both equality and inequality constraints:

$$
\begin{aligned}
\mathbf{x}^* &= \underset{\mathbf{x}}{\text{argmin}} f(\mathbf{x}) \\
&\quad \text{subject to } h_i(\mathbf{x}) = 0 \qquad \text{added for completeness} \\
&\quad \text{subject to } g_i(\mathbf{x}) \leq 0
\end{aligned} \tag{19}
$$

so how does duality procedure $\lambda^* = \arg\max_\lambda \min_x \mathcal{L}(\mathbf{x}, \lambda)$ being carried out in practice, also since we have additional equality constraint, we now have $\mathcal{L}(\mathbf{x}, \mu, \lambda)$ instead

1. obtain $\mathcal{L}_\lambda(\lambda) = \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mu, \lambda)$ by:

   (a) solve $\mathbf{x}'$, such that:

$$
\begin{aligned}
&\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}', \mu, \lambda) = 0 \\
\Longrightarrow\ &\nabla_{\mathbf{x}} \Big( f(\mathbf{x}') + \sum_{i=1}^{m} \mu_i h_i(\mathbf{x}') + \sum_{i=1}^{n} \lambda_i g_i(\mathbf{x}') \Big) = 0 \\
\Longrightarrow\ &\nabla_{\mathbf{x}} f(\mathbf{x}') + \sum_{i=1}^{m} \mu_i \nabla_{\mathbf{x}'} h_i(\mathbf{x}') + \sum_{i=1}^{n} \lambda_i \nabla_{\mathbf{x}} g_i(\mathbf{x}') = 0
\end{aligned} \tag{20}
$$

   (b) write $\mathbf{x}'$ in terms of $\lambda$ and substitute back into $\mathcal{L}(\mathbf{x}', \mu, \lambda)$ and obtain:

$$
\mathcal{L}_\lambda(\lambda) = \min_x \mathcal{L}(\mathbf{x}, \mu, \lambda) \tag{21}
$$

   note $\mathcal{L}_\lambda(\lambda)$ should contain no $\mathbf{x}$

   now we can $\max_\lambda \mathcal{L}_\lambda(\lambda)$ together with the complementary slackness conditions

2. to ensure **equality constraints**, we need to solve:

$$\begin{aligned}
&\nabla_\mu \mathcal{L}(\mathbf{x}', \mu, \lambda) = 0 \\
\Longrightarrow\ &\nabla_\mu f(\mathbf{x}') + \sum_{i=1}^{m} \mu_i \nabla_\mu h_i(\mathbf{x}') + \sum_{i=1}^{n} \lambda_i \nabla_\mu g_i(\mathbf{x}') = 0 \\
\Longrightarrow\ &\sum_{i=1}^{m} \mu_i \nabla_\mu h_i(\mathbf{x}') = 0
\end{aligned} \tag{22}$$

3. to ensure **Inequality constraints a.k.a. complementary slackness condition**

$$\begin{aligned}
\lambda_i g_i(\mathbf{x}) = 0, &\quad \forall i \\
\lambda_i \geq 0, &\quad \forall i \\
g_i(\mathbf{x}) \leq 0, &\quad \forall i
\end{aligned} \tag{23}$$

the final solution for dual $\lambda^*$ needs to be take account of all above equations, and let's see the classical example of solution for Support Vector Machine

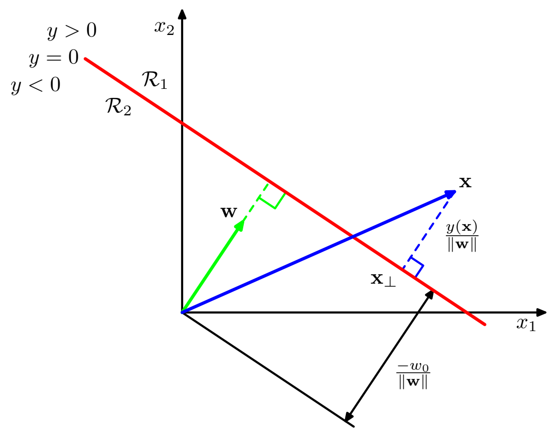
# 7 example through Support Vector Machine

## 7.1 Linear Discriminant Function (geometry)

$$y(\mathbf{x}) = \mathbf{x}^T \mathbf{w} + w_0 \tag{24}$$

if we let $r$ to be perpendicular distance between arbitrary point $\mathbf{x}$ from the decision surface, then, expression for $r$ can be solved as:

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_\perp + r\frac{\mathbf{w}}{\|\mathbf{w}\|} \qquad \text{sum of these two vectors} \\
\Longrightarrow\ \underbrace{\mathbf{w}^\top \mathbf{x} + w_0}_{y(\mathbf{x})} &= \mathbf{w}^\top \left(\mathbf{x}_\perp + r\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) + w_0 \qquad \text{apply } (\mathbf{w}^\top \times \quad + w_0) \text{ to both sides} \\
\Longrightarrow\ y(\mathbf{x}) &= \underbrace{\mathbf{w}^\top \mathbf{x}_\perp + w_0}_{=0} + \mathbf{w}^\top r \frac{\mathbf{w}}{\|\mathbf{w}\|} \\
\Longrightarrow\ y(\mathbf{x}) &= r\frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} = r\frac{\|\mathbf{w}\|^2}{\|\mathbf{w}\|} \\
\Longrightarrow\ r &= \frac{y(\mathbf{x})}{\|\mathbf{w}\|}
\end{aligned} \tag{25}$$




Our goal is to maximize margin $r$, making positive-labeled data to have $\hat{y} \geq 1$, and negative-label data to have $\hat{y} \leq 1$:

$$\max(\text{margin})_{\mathbf{w}, w_0} \implies \max\left(\frac{2}{\|\mathbf{w}\|}\right)$$
$$\text{subject to: } \begin{cases} \min(\mathbf{w}^T x_i + w_0) = 1 & i : y_i = +1 \\ \max(\mathbf{w}^T x_i + w_0) = -1 & i : y_i = -1 \end{cases}$$

resulting classifier $y = \text{sign}(\mathbf{w}^T + w_0)$ can be re-written as the **primal optimization**, and also combine the two constraints into a single equation:

$$\begin{aligned} &\min\left(\frac{1}{2}\|\mathbf{w}\|^2\right) \\ &\text{subject to: } \underbrace{y_i(\mathbf{w}^T x_i + w_0)}_{\text{both need to be SAME sign}} \geq 1 \\ &\implies 1 - y_i(\mathbf{w}^T x_i + w_0) \leq 0 \end{aligned} \tag{26}$$

### 7.2 Lagrangian Dual for SVM

in primal form, there is no kernel trick to exploit. So people are motivated to solve this in its **Lagrange dual**. there is no equality constraint in this case:

$$\mathcal{L}(\underbrace{w, b}_{\mathbf{x}}, \underbrace{\lambda}_{\text{there is no } \mu}) = \underbrace{\frac{1}{2}\|\mathbf{w}\|^2}_{f(\mathbf{x})} + \underbrace{\sum_{i=1}^{p} \mu_i h_i(\mathbf{x})}_{=0} + \sum_{i=1}^{N} \lambda_i [\underbrace{1 - y_i(w^T x_i + w_0)}_{g_i(\mathbf{x})}] \tag{27}$$

to solve $\mathbf{x}'$ for $\min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mu, \lambda)$, i.e., $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}', \mu, \lambda) = 0$

$$
\begin{aligned}
\frac{\partial \mathcal{L}(w,b,\lambda)}{\partial w} &= w - \sum_{i=1}^{N} \lambda_i y_i x_i = 0 \implies w' = \sum_{i=1}^{N} \lambda_i y_i x_i \\
\frac{\partial \mathcal{L}(w,b,\lambda)}{\partial b} &= \underbrace{\sum_{i=1}^{N} \lambda_i y_i}_{\text{not a function of } b} = 0
\end{aligned} \tag{28}
$$

### 7.3 write expression for $\mathcal{L}_\lambda(\lambda)$

substitute $\mathbf{x}'$ (in terms of $\lambda$), i.e.,:

$$
\begin{cases} w' & = \sum_{i=1}^{n} \lambda_i y_i x_i \\ \sum_{i=1}^{n} \lambda_i y_i & = 0 \end{cases}
$$

to $\quad \mathcal{L}(w,b,\lambda) = \frac{1}{2}\|\mathbf{w}\|^2 + \sum_{i=1}^{n} \lambda_i[1 - y_i(w^\top x_i + w_0)]$

$$
\begin{aligned}
\implies \mathcal{L}_\lambda(\lambda) &= \inf_x \mathcal{L}(w,b,\lambda) \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n} \lambda_i y_i x_i\Big)^\top \Big(\sum_{i=1}^{n} \lambda_i y_i x_i\Big) + \sum_{i=1}^{n} \lambda_i \Big[1 - y_i\Big(\Big(\sum_{i=1}^{n} \lambda_i y_i x_i\Big)^\top x_i + w_0\Big)\Big] \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{N} \lambda_i \lambda_j y_i y_j x_i^\top x_j - \sum_{i=1}^{n} \lambda_i y_i \Big(\sum_{j=1}^{n} \lambda_j y_j x_j^\top\Big) x_i - \underbrace{w_0 \sum_{i=1}^{n} \lambda_i y_i}_{=0} + \sum_{i=1}^{n} \lambda_i \\
&= \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j y_i y_j x_i^\top x_j \\
&\text{subject to: } \sum_{i=1}^{N} \lambda_i y_i = 0 \text{ and } \lambda_i \geq 0
\end{aligned} \tag{29}
$$

### 7.4 The dual problem

$$
\begin{gathered}
\underset{\lambda_1,\ldots\lambda_n}{\arg\max}\, \mathcal{L}_\lambda(\lambda) = \underset{\lambda_1,\ldots\lambda_n}{\arg\max} \left( \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j y_i y_j x_i^\top x_j \right) \\
\text{subject to: } \sum_{i=1}^{n} \lambda_i y_i = 0 \text{ and } \lambda_i \geq 0
\end{gathered} \tag{30}
$$

since $x_i^\top x_j$ can be replaced by kernel $\mathcal{K}(x_i, x_j)$

Use **complementary slackness:**

$$
\begin{aligned}
\lambda_i^* > 0 &\implies g_i(w^*, b^*) = 0 \\
&\implies 1 - y_i({w^*}^\top x_i + {w_0}^*) = 0 \\
&\implies y_i({w^*}^\top x_i + {w_0}^*) = 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{i.e., } x_i \text{ is support vector points} \\
\lambda_i^* = 0 &\implies g_i(w^*, b^*) < 0 \\
&\implies 1 - y_i({w^*}^\top x_i + {w_0}^*) < 0 \\
&\implies y_i({w^*}^\top x_i + {w_0}^*) > 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{i.e., } x_i \text{ is non support vector points}
\end{aligned} \tag{31}
$$

Since there is only a few $\lambda_i > 0$, dual inference is **efficient**!