algorithms in K12 math standards

Michael Garrett

20181016

Analysis of *Tennessee Math Standards*

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Michael Garrett

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Introduction

We need to know the lingo for the hierarchy and time-development of math learning. For example, it seems "standards" can refer to:

- the entire set of expectations, though it is not clear in practice whether these are thought of as expectations on the student or on the teachers;
- the entire, hierarchical set of learning progressions;
- a single statements of "what a student should know" (content);
- a single statement of "how a student should work" (practices).

There are two sets of statements about students: "Content standards" and "Standards for mathematical practice". In addition, there is a set guidelines, the "Literacy skills for mathematical proficiency", that are expectations about students competence in using language and mathematical symbols.

My take right now is that each statement of a content standard could be prefaced with the phrase, "By the end of this grade/course, a student should be able to…". For example:

- By the end of third grade, a student should be able to tell and write time to the nearest minute and measure time intervals in minutes (third grade content standard).
- By the end of third grade, a student should be able to make sense of problems [appropriate to their grade level] and persevere in solving them (mathematical practice standard 1, applied to third grade).
- By the end of the course "Integrated Math 1", a student should be able to interpret parts of an expression, such as terms, factors, and coefficients (a high school course standard).

The statements of the mathematical practice standards do not evolve between grade levels. The expectations at the end of each grade remains the same. Perhaps they could be prefaced with, "As always, a student should be able to…". For example,

- As always, a student should be able to make sense of problems and persevere in solving them (MP1);
- As always, a student should be able to construct viable arguments and critique the reasoning of others (MP3).

Content standards

It seems these are about about "mathematical fluency, procedural skills, and conceptual understanding" (as they are described in the section *The Standards for Mathematical Practice*).

Learning progressions

All math content is seen as following 4 main progressions that extend from K through 12th grade. These seem not to be named, but are largely the development of numbers, of operations, of geometry, and of data analysis.

K-8. Content as domains > clusters > standards

In K-8, the learning progressions track student's development through a three-tiered hierarchy. The content of mathematics learning is organized into 12 discrete areas of mathematics and several big ideas within each area with specific competencies that flesh out each big idea. These are called "domains", "clusters", and "content standards" respectively. The table of "standards" for each grade organizes the content standards into clusters, and the clusters into the 12 domains, but gives no information on when or how the teacher is to get the students to meet the grade-level standards.

For example, the content standard identified as `3.MD.A.1` describes how students should work with time. It is the 1st standard in cluster A of domain MD in 3rd grade. Some or all students may know this content when they enter 3rd grade, or they may be learning it in a science class, but it is the teacher's responsibility to make sure each child can meet this standard by the time they move to 4th grade. When and how is not addressed.

High school (9-12). Domains fall under conceptual categories

In 9th grade, curricula branch out and become built around "courses", and students are typically guided to take courses based on their academic background, aptitudes, and ambitions. The courses become how the learning progressions are delivered and they are responsible for targeting the content for the type of student that takes the course. However, the courses are not assigned to a specific grade level.

Perhaps due to the complexity of school work given to students in these years, an additional layer is added to the content hierarchy to describe the learning for high school. At this point, the many domains of high school math content are subsumed under 5 "conceptual categories". These conceptual categories carry forth the 4 learning progressions from K-8.

For example, the content standard `M1.A.SSA.A.1` is about finding meaning in algebraic expressions. It is the 1st standard in cluster A in the domain SSE of the conceptual category A as addressed in the course M1. As in the earlier standards, how or when to get a student to meet this standard is not described.

Summary

In summary, until 9th grade, domains are tracked within the learning progressions. Each domain presents a few big ideas that are operationalized into specific content standards. After that, domains are subsumed under conceptual categories which are tracked in the learning progressions. The specific content under each category is delineated within courses as domains, clusters, and content standards. Therefore, the content mastery expected of students will differ depending upon the courses they take, but they are expected to develop in each of conceptual categories taken on by high school education.

Standards for mathematical practice (MP)

Standards for practice are about "approaches, practices, and habits of mind". (Content standards are about "fluency, procedural skill, and conceptual understanding".) These practices are meant to bring students toward the actual work of practicing mathematicians, who are said to have mastered high-level "processes" and to have gained "proficiencies" through repeated practice.

Process "A particular course of action intended to achieve a result... pertains to problem solving, reasoning and proof, communication, representation, and connections".

Proficiency (noun) "[A skill] in the command of fundamentals derived from practice and familiarity, such as adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, [and having] productive dispositions toward the work at hand".

Processes are what mathematicians do as part of math work. Proficiencies are what they become good at by being immersed regularly in their particular kinds of work.

These standards are different from the content standards. They do not evolve from grade to grade, but are repeated verbatim for each year, and so appear more as a dimension along which each student is expected to progress. However, no benchmarks are offered to gauge a student's progress. Perhaps the framers envision that each child's competence along these dimensions is challenged again each year, or throughout the year.

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As a proficiency, "strategic competence" relates to part of Knuth's algorithmic thinking


Practices pertinent to algorithmic thinking:
* MP1 Make sense of problems and persevere in solving them
* MP3 Construct viable arguments and critique the reasoning of others
* MP5 Use appropriate tools strategically
* MP8 Look for and express regularity in repeated reasoning.

Do we want to look for instances of demonstrating competence in practice
standards in the work?
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Tasks that lead students to meet practice standards

The text envisions the kinds of schoolwork students should do to maintain professional "approaches, practices, and habits of mind" with work at the current grade level:

> To develop the processes and proficiencies addressed in the practice standards, students must be engaged in rich, high-level mathematical tasks... [These] demand a greater level of cognitive effort to solve than routine practice problems...

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This may help us differentiate algorithmic thinking from following procedures. Also, it may help clarify th
mathematics work.

Can we help teachers see the difference between the low-level
thinking of procedural work and the higher-level algorithmic thinking?
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Literacy skills

Close readings, practice technical vocabulary, appropriate writing and use of symbols.

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The work toward building algorithmic thinking in students can address these.
Perhaps we should call this "scaffolded algorithmic thinking", as it is in the
scaffolding that we transmit the cultural tools of language and symbol.
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Thinking about thinking

In 4th grade students are expected to

- "begin to interpret and represent multiplication as a comparison. They multiply and divide to solve contextual problems involving multiplicative situations, distinguishing their solutions from additive comparisons situations".

  Not sure what it means here to "distinguish solutions".

- "solve multi-step whole number contextual problems using the four operations representing the unknown as a variable within an equation"

  This is working toward formal algorithms. During 4th grade, they are expected to manage multiple steps and combine operations and to represent the intermediate results abstractly. Are they learning to see a pattern in the wording of a problem and apply a systematic procedure or are they ad hoc putting operations together.

Using strategies

First grade

Grade 1 standards use the word "strategies", but K do not.

By the end of first grade, students should be able to:

1.OA.C.5 Add and subtract within 20 using strategies such as counting on, counting back, making 10…

1.OA.C.6 Fluently add and subtract within 20 using mental strategies….

1.NBT.C.4 … Use concrete models, drawings, strategies based on place value,… to explain the reasoning used.

1.NBT.C.6 Subtract multiples of 10 from multiples of 10… using concrete models, drawings, strategies based on place value, properties of operations…

Second grade

In grade two, "strategies" appears in the descriptions of domains: "Students should solve a variety of problem types in order to make connections among contexts, equations, and strategies." (Operations and Algebraic Thinking), and "They should be able to explain why these strategies work" (Numbers and Operations in Base Ten), though it isn't clear to me what strategies they are referring to, perhaps "add and subtract within 1,000" or "add up to four two-digit numbers"

By the end of first grade, students should be able to:

2.NBT.B.5 "Fluently add and subtract within 100… using strategies based on place value…"

2.NBT.B.6 "Add up to four two-digit numbers using properties of operations and strategies based on place value"

2.NBT.B.7 "Add and subtract within 1000 using concrete models, drawings, strategies based on place value, …"

2.NBT.B.9 "Explain why addition and subtraction strategies work using properties of operations and place value…"

`It seems the "strategies" word appears around place value, mostly, at this level.`

Third Grade

In grade three, "strategies" is again used in the domain descriptions: "Students use increasingly sophisticated strategies based on properties of operations to fluently solve multiplication and division problems within 100" and "Students use… estimation strategies to assess the reasonableness of solutions" (Operations and Algebraic Thinking); "Students fluently add and subtract within 1000 using strategies and algorithms" (Number and Operations in Base Ten).

3.OA.B.5 "Apply properties of operations as strategies to multiple and divide."

3.OA.C.7 "Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 x 5 = 40, one knows 40 ÷ 5 = 8)"

3.OA.D.8 "Assess the reasonableness of answers using… estimation strategies including rounding"

3.NBT.A.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

3.NBT.A.3 Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9x80, 5x60) using strategies based on place value and properties of operations.

Fourth grade

In grade 4, "strategies" appears in the domain descriptions: "Students use strategies based on place value and the properties of operations to multiple a whole number up to four-digits by a one-digit number, and multiply two two-digit numbers", "They use these strategies and the relationship between multiplication and division to find whole number quotients and remainders up to four-digit dividends and one-digit divisors" (Number and Operations in Base Ten)

Operations and algebraic thinking

4.OA.A.3 "... Assess the reasonableness of answers [to contextual problems] using mental computation and estimatin strategies, including rounding"

Numbers and operations in base ten

4.NBT.B.4 "Fluently add and subtract within 1,000,000 using appropriate strategies and algorithms"

4.NBT.B.5 "Multiply a whole number of up to four digits by a one-digit whole number and multiple two two-digit numbers, using strategies based on place value and the properties of operations."

4.NBT.B.6 "Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division"

Fifth grade

Strategies appear in the domain descriptions: "Students should solve a variety of problem types in order to make connections among contexts, equations, and strategies" (Number and operations in fractions).

Number and operations in base ten

5.NBT.B.5 "Fluently multiply multi-digit whole numbers (up to three-digit by four-digit factors) using appropriate strategies and algorithms"

5.NBT.B.6 "Find whole-number quotients and remainders of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division".

5.NBT.B.7 "Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, the properties of operations, and/or the relationship between multiplication and division".

Identifying arithmetic patterns

Third grade

3.0A.D.9 "Identify arithmetic patterns... and explain them using properties of operations."

Forth grade

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Fifth grade

Arithmetic patterns appear in domains: "Students build on their understanding of patterns to generate two numerical patterns using given rules and identity relationships between the patterns" (Operations and algebraic thinking). Not clear what this means.

Operations and algebraic thinking

5.OA.A.3 "generate two numerical patterns using two given rules. *For example, given the rule 'Add 3' and the starting number 0, and given the rule 'Add 6' and the starting number 0, generate terms in the resulting sequences.* **a**. Identify relationships between corresponding terms in two numerical patterns. *For example, observe that the terms in one sequence are twice the corresponding terms in the other sequence.*"

Number and operations in base ten

5.NBT.A.2 "Explain patterns in the number of zeros of the product when multiplying a number by powers of 10. . . "

Using algorithms

Third grade

In grade three, "algorithms" appears in the domain descriptions: "Students fluently add and subtract within 1000 using strategies and algorithms" (Number and Operations in Base Ten)

Forth grade

4.NBT.B.4 "Fluently add and subtract within 1,000,000 using appropriate strategies and algorithms"

Fifth grade

Number and operations in base ten

5.NBT.B.5 "Fluently multiply multi-digit whole numbers (up to three-digit by four-digit factors) using appropriate strategies and algorithms"

Working with factors and remainders

Remainders, factors, multiples, primes, composites

These first appear in grade three, but are contextualized in grade 4.

Third grade

"They use these strategies and the relationship between multiplication and division to find whole number quotients and remainders up to four-digit dividends and one-digit divisors" (Number and Operations in Base Ten

Fourth grade

"This is the first time students find and interpret remainders in context. Students fin factors and multiples, and they identify prime and composite numbers" (Operations and algebraic thinking).

Combining fractions symbolically

See fourth grade Number and operations-fractions