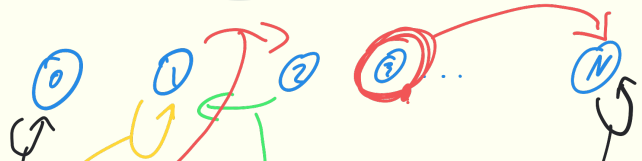
Birth Death Process



$p_{i,i+1} + p_{i,i-1} \le 1$

$p_{ii} = 1 - p_{i,i+1} - p_{i,i-1}$

$p_{00} = p_{NN} = 1$

Fixation x_i

$x_0 = 0$

$x_N = 1$

$x_i = p_{i,i-1}x_{i-1} + p_{ii}x_i + p_{i,i+1}x_{i+1}$ $\quad 0 < i < N$

Theorem

$$x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \gamma_k}{1 + \sum_{j=1}^{N} \prod_{k=1}^{j} \gamma_k} \qquad \gamma_k = \frac{p_{k,k-1}}{p_{k,k+1}}$$

proof:

$p_{i,i+1}x_{i+1} = -p_{i,i-1}x_{i-1} + x_i(1 - p_{ii})$

$1 = p_{ii} + p_{i,i-1} + p_{i,i+1}$

$p_{i,i+1}x_{i+1} = p_{i,i-1}(x_i - x_{i-1}) + x_i p_{i,i+1}$

$x_{i+1} - x_i = \frac{p_{i,i-1}}{p_{i,i+1}}(x_i - x_{i-1})$

$$\boxed{x_{i+1} - x_i = \gamma_i(x_i - x_{i-1})}$$

$x_2 - x_1 = \gamma_1(x_1 - \cancel{x_0}^{\,0}) = \gamma_1 x_1$

$x_3 - x_2 = \gamma_2(x_2 - x_1) = \gamma_2\gamma_1 x_1$

$x_4 - x_3 = \gamma_3(x_3 - x_2) = \gamma_3\gamma_2\gamma_1 x_1$

$\vdots$

$x_{i+1} - x_i = \prod_{k=1}^{i} \gamma_k x_1$

$x_N - x_{N-1} = \prod_{k=1}^{N-1} \gamma_k x_1$

$$x_i = \sum_{j=0}^{i-1}(x_{j+1} - x_j) = \left(1 + \sum_{j=1}^{i-1}\prod_{k=1}^{j}\gamma_k\right)x_1$$

$$x_N = 1 = \left(1 + \sum_{j=1}^{N}\prod_{k=1}^{j}\gamma_k\right)x_1$$

Neutral drift

$p_{i,i-1} = p_{i,i+1}$

$\gamma_i = 1$

$$x_i = \frac{1 + \sum_{j=1}^{i-1}\prod_{k=1}^{j} 1}{1 + \sum_{j=1}^{N-1}\prod_{k=1}^{j} 1} = \frac{1 + i - 1}{1 + N - 1} = \frac{i}{N}$$