

1. (a) Give the definition of Normal Form Game. [2]
 (b) For the rest of this question, consider the Normal Form Game with the following matrix representation:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Give the utilities to both players using the following strategy pairs:

- (i) $\sigma_r = (1, 0)$ $\sigma_c = (0, 1)$ [1]
 (ii) $\sigma_r = (1/2, 1/2)$ $\sigma_c = (1/3, 2/3)$ [1]
 (iii) $\sigma_r = (1/4, 3/4)$ $\sigma_c = (0, 1)$ [1]
 (c) Give the definition for the Lemke-Howson algorithm. [5]
 (d) Show that the vertices and their labels for the best response polytopes are given by:

For the row player best response player \mathcal{P} :

- $(0, 0)$ with labels: $\{0, 1\}$
- $(1/3, 0)$ with labels: $\{1, 2\}$
- $(0, 1/4)$ with labels: $\{0, 3\}$
- $(3/10, 1/10)$ with labels: $\{2, 3\}$

For the column player best response player \mathcal{Q} :

- $(0, 0)$ with labels: $\{2, 3\}$
- $(1/4, 0)$ with labels: $\{0, 3\}$
- $(0, 1/3)$ with labels: $\{1, 2\}$
- $(1/10, 3/10)$ with labels: $\{0, 1\}$

[4]

- (e) Draw the best response polytopes. [2]
 (f) Use the plots to carry out the Lemke-Howson algorithm with all possible initial dropped labels. [4]
 (g) Consider a modified Lemke-Howson algorithm that uses any pair of fully labeled vertices (and not necessarily $(0, 0)$). Use any pair of fully labeled vertices found in the previous question and find an initial dropped label that gives a different Nash equilibrium than the ones obtained in the previous question. [1]
 (h) Give a sketch of a proof, including potential assumptions that in a non degenerate game the number of Nash equilibria is odd. [4]

2. (a) Give the general definition of the Prisoner's Dilemma. [2]
 (b) What values of S, T give valid Prisoner's Dilemma games:

$$(i) \quad A = \begin{pmatrix} 3 & S \\ 5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & T \\ -1 & 1 \end{pmatrix} \quad [4]$$

$$(ii) \quad A = \begin{pmatrix} 2 & S \\ -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -2 \\ S & 1 \end{pmatrix} \quad [4]$$

- (c) Consider the following reactive players:

$$p = (3/5, 3/4) \quad q = (1/2, 1/4)$$

Obtain the Markov chain representation of a match between these two players. [4]

- (d) State a theoretic result giving the utility of two general reactive players in a Prisoner's dilemma match (as a function of (R, S, P, T)). [5]
 (e) Consider a reactive player $p = (x, x/2)$ and an opponent $q = (1/2, 1/4)$. Show that, for $(R, S, P, T) = (3, 0, 4, 1)$ the utility to the player p is given by:

$$u(x) = \frac{1}{(x-8)^2} (3x^2 - 38x + 112)$$

[5]

- (f) Using the above, show that:

$$\frac{du}{dx} = -\frac{10}{(x-8)^2}$$

and use this to identify the optimal value of x .

[4]

3. (a) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, obtain the following equation describing the corresponding evolutionary game:

$$\frac{dx}{dt} = x(f - \phi)$$

where $f = Ax$ and $\phi = fx$. [2]

- (b) Define a mutated population. [2]

- (c) Define an evolutionary stable strategy. [2]

- (d) State and prove a theorem giving a general condition for an Evolutionary stable strategy. [4]

- (e) Consider the following game $A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, obtain all evolutionary stable strategies. [5]

- (f) Consider the accompanying 2008 paper entitled “Studying the emergence of invasiveness in tumours using game theory” by Basanta et al.

- (i) Give a general summary of the paper. [3]

- (ii) There is a minor error in this paper in the game matrix, describe and suggest the fix. [2]

- (iii) How does the theorem in part 4 of this question relate to the findings of the paper? [2]

- (iv) Suggest an alternative area of game theory that could also be used. [3]

4. (a) Give the definition of a Moran process on a game. [4]
- (b) State and prove a theorem giving the fixation probabilities for a general birth death process. [6]
- (c) Consider the Markov process on the Prisoners Dilemma: $A = \begin{pmatrix} 4 & 1 \\ 6 & 2 \end{pmatrix}$ Use the above theorem to obtain the fixation probabilities for each strategy for $N = 6$. [4]
- (d) Consider the following two strategies for the Prisoners Dilemma:
- Tit For Tat: start by cooperating and then repeat the opponents previous message.
 - Alternator: start by cooperating and then alternate between defecting and cooperating.

Assuming a match lasting 5 turns show that the utility matrix between these two strategies corresponds to:

$$\begin{pmatrix} 20 & 18 \\ 18 & 16 \end{pmatrix}$$

- [5]
- (e) Obtain the fixation probabilities x_1 for each strategy for $N = 4$. [6]