Notebook

August 10, 2017

1 Best responses

1.1 Definition of a best response

In a two player game $(A, B) \in \mathbb{R}^{m \times n^2}$ a mixed strategy σ_r^* of the row player is a best response to a column players' strategy σ_c iff:

$$\sigma_r^* = \text{argmax}_{\sigma_r \in S_r} \sigma_r A \sigma_c^T$$

Similarly a mixed strategy σ_c^* of the column player is a best response to a row players' strategy σ_r iff:

$$\sigma_c^* = \text{argmax}_{\sigma_c \in S_c} \sigma_r B \sigma_c^T$$

In other words: a best response strategy maximise the utility of a player given a known strategy of the other player.

1.2 Best responses in the Prisoners dilemma

Consider the prisoners dilemma:

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

We can easily identify the pure strategy best responses by underlying the corresponding utilities. For the row player, we will underline the best utility in each column:

$$A = \begin{pmatrix} 3 & 0 \\ \underline{5} & \underline{1} \end{pmatrix}$$

For the column player we underling the best utility in each row:

$$B = \begin{pmatrix} 3 & \underline{5} \\ 0 & \underline{1} \end{pmatrix}$$

We see that both players' best responses are their second strategy.

1.3 Best responses in matching pennies

Consider matching pennies with the best responses underlined:

$$A = \begin{pmatrix} \underline{1} & -1 \\ -1 & \underline{1} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & \underline{1} \\ \underline{1} & -1 \end{pmatrix}$$

We see that the best response now depend on what the opponent does.
Let us consider the best responses against a mixed strategy (and apply the previous definition):

- Assume $\sigma_r = (x, 1-x)$
- Assume $\sigma_c = (y, 1-y)$

We have:

$$A\sigma_c^T = \begin{pmatrix} 2y-1 \\ 1-2y \end{pmatrix} \qquad \sigma_r B = \begin{pmatrix} 1-2x & 2x-1 \end{pmatrix}$$

```
In [1]: import sympy as sym
        import numpy as np
        sym.init_printing()

        x, y = sym.symbols('x, y')
        A = sym.Matrix([[1, -1], [-1, 1]])
        B = - A
        sigma_r = sym.Matrix([[x, 1-x]])
        sigma_c = sym.Matrix([y, 1-y])
        A * sigma_c, sigma_r * B
```

Out[1]:

$$\left(\begin{bmatrix} 2y-1 \\ -2y+1 \end{bmatrix}, \quad \begin{bmatrix} -2x+1 & 2x-1 \end{bmatrix} \right)$$

Those two vectors gives us the utilities to the row/column player when they play either of their pure strategies:

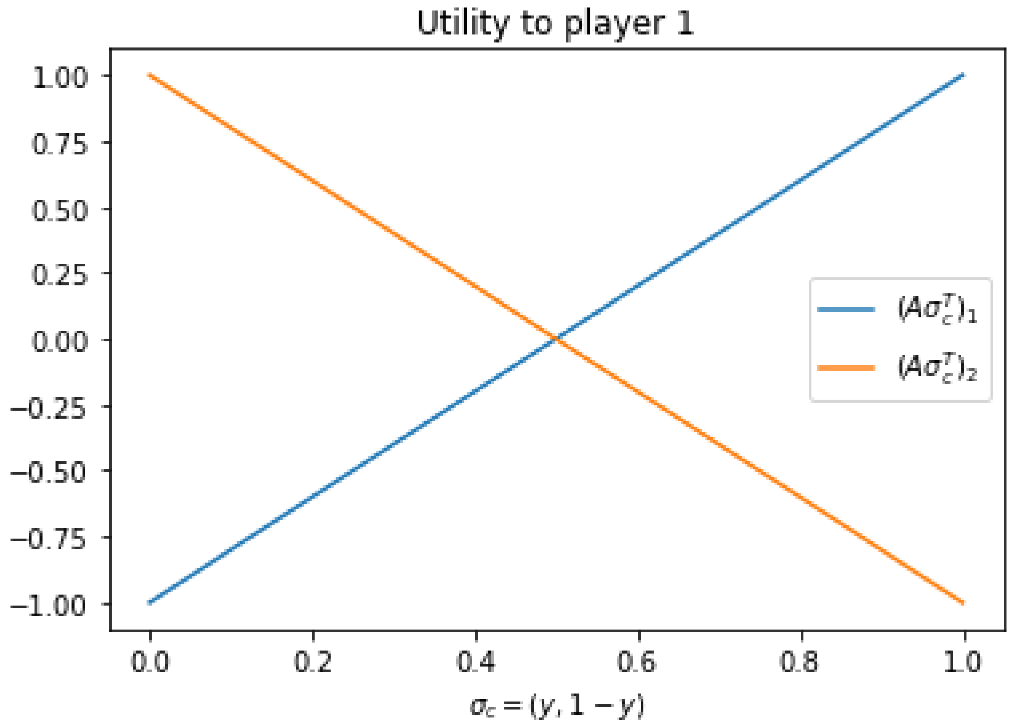
- $(A\sigma_c^T)_i$ is the utility of the row player when playing strategy i against $\sigma_c = (y, 1-y)$
- $(\sigma_r B)_j$ is the utility of the column player when playing strategy j against $\sigma_r = (x, 1-x)$

Let us plot these (using `matplotlib`):

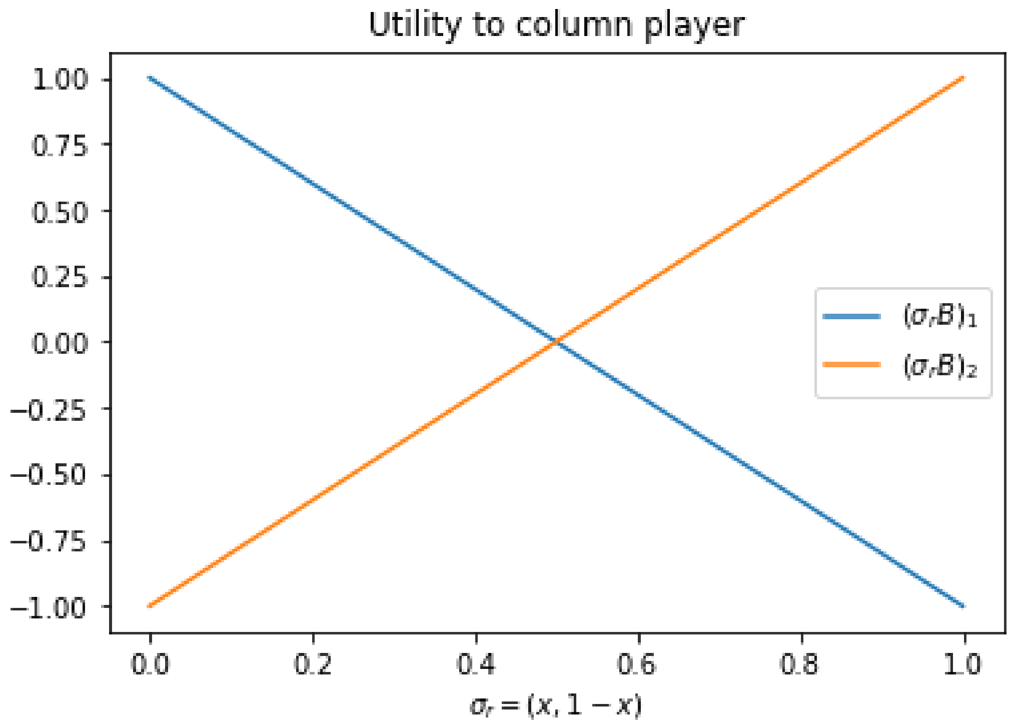
```
In [4]: import matplotlib
        import matplotlib.pyplot as plt
        %matplotlib inline
        matplotlib.rc("savefig", dpi=100)  # Increase the quality of the images (not needed)

        ys = [0, 1]
        row_us = [[(A * sigma_c)[i].subs({y: val}) for val in ys] for i in range(2)]
        plt.plot(ys, row_us[0], label="$(A\sigma_c^T)_1$")
        plt.plot(ys, row_us[1], label="$(A\sigma_c^T)_2$")
```

```
plt.xlabel("$\sigma_c=(y, 1-y)$")
plt.title("Utility to player 1")
plt.legend();
```




```
In [3]: xs = [0, 1]
        row_us = [[(sigma_r * B)[j].subs({x: val}) for val in xs] for j in range(2)]
        plt.plot(ys, row_us[0], label="$(\sigma_rB)_1$")
        plt.plot(ys, row_us[1], label="$(\sigma_rB)_2$")
        plt.xlabel("$\sigma_r=(x, 1-x)$")
        plt.title("Utility to column player")
        plt.legend();
```



We see that the best responses to the mixed strategies are given as:

$$\sigma_r^* = \begin{cases} (1,0), & \text{if } y > 1/2 \\ (0,1), & \text{if } y < 1/2 \\ \text{indifferent}, & \text{if } y = 1/2 \end{cases} \qquad \sigma_c^* = \begin{cases} (0,1), & \text{if } x > 1/2 \\ (1,0), & \text{if } x < 1/2 \\ \text{indifferent}, & \text{if } x = 1/2 \end{cases}$$

In this particular case we see that for any given strategy, the opponents' best response is either a pure strategy or a mixed strategy in which case they are indifferent between the pure strategies.

For example:

- If $\sigma_c = (1/4, 3/4)$ $(y = 1/4)$ then the best response is $\sigma_r^* = (0, 1)$
- If $\sigma_c = (1/2, 1/2)$ $(y = 1/2)$ then any mixed strategy is a best response **but** in fact both pure strategies would give the same utility (the lines intersect).

This observation generalises to our first theorem:

1.4 Best response condition

In a two player game $(A, B) \in \mathbb{R}^{m\times n^2}$ a mixed strategy σ_r^* of the row player is a best response to a column players' strategy σ_c iff:

$$\sigma_{r\,i}^* > 0 \Rightarrow (A\sigma_c^T)_i = \max_k (A\sigma_c^T)_k \text{ for all } 1 \leq i \leq m$$