Introduction

This thesis will lay out the literature on the price of anarchy bounds of single item, first price auctions and then construct a simulation to demonstrate these bounds for no-regret agents. Chapter one hopes to be a broad overview of the field to give context for this discussion, chapter two will go through the mathematical theory in a more rigorous way laying our what sort of behavior we expect and why, and chapter three will be the empirical results of a simulation which hopefully accord with what the theory says.

Note to the readers of this draft of chapter 1: the in-text citations generated by Latex are formatted incorrectly/in the wrong style for this draft and I am talking to CUS to try and get that fixed.

Chapter 1

Computational Economics and Auctions

Within the last three decades, computational economics has been on the rise. This branch of economic research encompasses two major ideas. One is that the increasing power of computers can help solve and understand classical economic problems through increasingly more complex simulations and numerical analysis. Two, that the mathematical methods developed in the field of theoretical computer science can be used to gain better understanding of existing models in terms of their algorithmic and computational complexity properties. We aim to take elements from both of these frameworks to better understand the social welfare of auctions at equilibrium.

1.1 Auctions, Equilibria, and Anarchy

It is perhaps obvious why economists would be inserted in studying auctions. Auctions are one of the most basic market structures that have roots going back to at least the ancient Greeks and still exist today in places such as art auctions and Ebay (Mochon & Saez (2015)). But in the past 15 years, economists have been joined by computer scientists who are increasingly interested in the strategic interactions of agents within this setting. Christos Papadimitriou said in a 2015 lecture at the Simons Institute that it was the advent of the internet, an artifact out of their control, that turned theoretical computer science into a "physical science." Now computer scientists had to "approach the internet with the same humility that economists approach the market..." He went on to say that "it also turned us (computer science) into a social science. It was obviously about people and incentives. Without understanding this, you cannot understand the internet" (Papadimitriou (2015)). It is within this framework that computer scientists first began to study auctions generally as they existed on the internet. At first this meant trying to understand the auctions present on the internet, Ebay and Googles sponsored search auctions. But now, more generally this can be seen as taking the mathematical tools of theoretical computer science and applying them as a lens to understand and explain the world beyond just the scope of the internet. Here specifically we are looking at the field algorithmic game theory, the study of the algorithms and complexity of strategic interactions. For economists, this can be thought of as a new toolbox for unpacking and understanding the models and structures that already dominate the field. For example, in the case of a Walsarian auctioneer who calculates the clearing price of an auction, their problem was shown to be NP-complete, a complexity class usually called "intractable" due to the time it takes to solve these problems (the best algorithms here are generally guess and check, which gets out of hand for large inputs)[1]. Using the lens computational complexity is one way of assessing what assumptions we are making about the computational power of our rational agents.

[1] NP is the class of problems that a given solution can be checked in polynomial time, i.e. $O(n^k)$ operations where n is the size of the input and k is any positive integer. NP-complete means that all problems in NP reduce to solving this problem. This is when you will usually here about "taking more time to compute then the age of the universe" etc...

1.1.1 The Price of Anarchy

Another idea that has come out of algorithmic game theory is the price of anarchy (POA), a mathematical way of showing the difference between the social welfare in the optimal case and in the worst case equilibrium for a game. More formally, the price of anarchy for a game is the ratio of the minimum equilibrium social welfare in a game over the best possible social welfare of the game. An illustrative example of the price of anarchy can be seen in selfish routing games such as the one pictured below in figure 1.1 that comes from Tim Roughgarden's "20 Lectures on Algorithmic Game Theory"Roughgarden (2016). In this game, each player chooses which edges

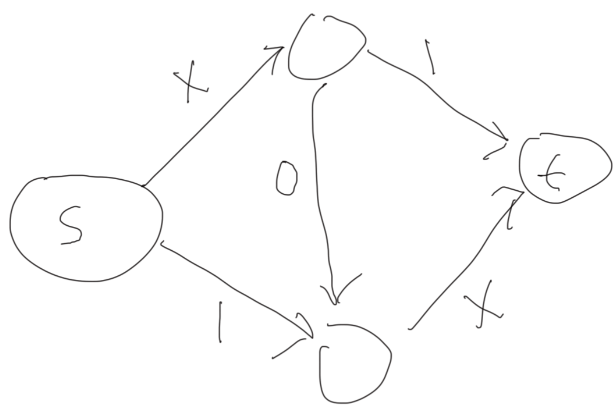

Figure 1.1: Routing game from s to t. Strategic interaction will make everyone worse off.

to take from the source s to reach the terminal t and try to do it in the least time, total weight of edges taken, as possible. For the edges labeled x, x is equal to the proportion of players who take the route. For example if 50% of the players take that edge then $x = 0.5$ [2]. This can be seen as analogous to traffic when driving a car, the more people take a road, the slower the traffic goes and the longer it takes to get somewhere. Knowing this, each player must choose which path to take to get to their destination as quickly as possible. As you can quickly verify, the best solution

[2]These figures are slightly off. The edges should have weights $f(x) = x$ or $f(x) = 1$ giving us the cost of using the edge as a function of the proportion of people who are using that edge. This will be fixed when I draw these graphs properly for the final draft and not in MS paint.

for society, minimizing the total driving time for all, is when half of the drivers take the top route, and half of the drivers take the bottom route taking in total 1.5 for each driver.

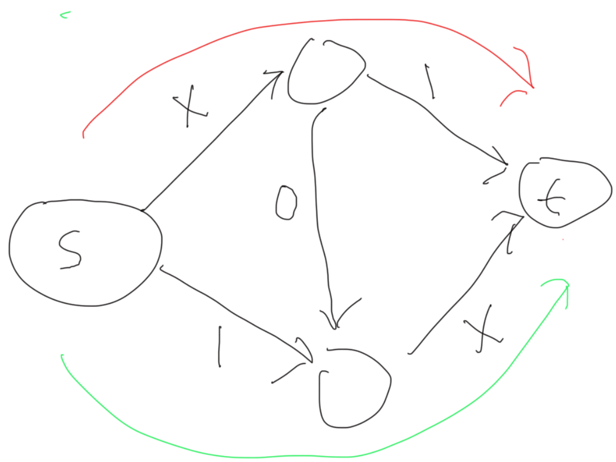

Figure 1.2: Socially optimal route: half taking top, half taking bottom

However, this is not an equilibrium. The drivers taking the top route have incentive to instead take the middle edge to try and lower their total time traveled. In fact, the Nash equilibrium will end up with all of the drivers taking the top x, going through the middle, and then the bottom x. Only then will no players have reason to deviate. This however, produces a worse outcome for all players and society as a whole (social welfare)[3].

Now, it take time 2 for each player to reach the terminal. Thus, under strategic interaction we see that the equilibrium is sub-optimal for society, and in this case all players as well. In this case, the price of anarchy can be computed as

$$POA = 3/4$$

. Interestingly bounds for the price of anarchy can be found for the class of game as a

[3]A careful observer will note that if the edge with cost 0 were not there, this would not be a problem. This is called Braess's Paradox where having this extra edge counter intuitively leads to worse outcomes. (Roughgarden (2016))

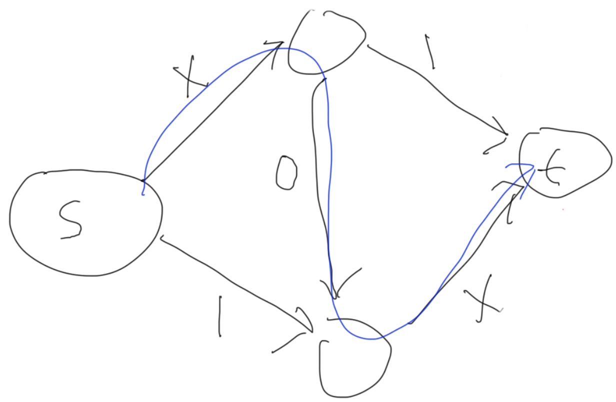

Figure 1.3: Route under strategic interaction

whole regardless of its individual construction. The price of anarchy bound for these selfish routing games is know to be 3/4 meaning that our example is as bad as the price of anarchy could get for any selfish routing game.

1.1.2 Auctions

This framework of the price of anarchy is now being applied to understand auctions by researchers in the algorithmic game theory field. Where for selfish routing games, the price of anarchy has been shown to have the tight bound above, the question for single price auctions only has approximate upper and lower bounds set on it, and is not known at all for double auctions. Before exploring this further, we should set up the mathematical framework of auctions that is necessary to compute such bounds.

An **auction** is a market mechanism, operating under specific rules that determines to whom one or more items will be awarded and at what price. For a bidder in an auction, the **value** v_i is how much the bidder values the item. This is sometimes called a private valuation as this value is generally unknown by the other participants in the auction. The **bid**, b_i, is the off that bidder i submits for for an item. **Sincere bidding** is when $v_i = b_i$, **underbidding** is when $v_i > b_i$, and **overbidding** is when

$v_i < b_i$. The highest bid made by any bidder is denoted b^*, and is the winning bid (if multiple bids equal the winning bid, then some tie-breaking rule must be used). The **selling price**, p^* is the final price that the bidder actually pays for the item (which depending on the auction type need not equal b^*). Under the **first-price** rule, the bid submitted by the winner is equal to the selling price. Before the auction begins, each bidder knows their personal or private value for the item. An auction consists of a set of bidders, $I = (1, 2, ..., N)$ and a seller. After the auction, the bidder i wins the item if their bid is higher than the bid placed by any other bidder k, $b_i > max_{k \neq b_k}$. In a single unit auction, the **income of the bidder** i is equal to their value of the item:

$$\Gamma_i^* = v_i$$

and the **surplus of the bidder** i is equal to the difference between income and price paid

$$\Pi_i^* = \Gamma_i^* - p^*.$$

If a bid placed by a bidder is less than the winning bid, they do not win anything and their income and surplus are both zero. The **seller's revenue** in a single unit auction is equal to the price paid by the winning bidder

$$R^* = p^*.$$

Pricing rules in an auction. **First-Price**: the winning bidder pays the amount of their bid, which is the highest bid of the auction: $p^* = b^*$. Also called **pay-what-you-bid** (PWYB). **Second-price**: the winning bidder pays an amount that is equal to the second highest bid for the awarded item Vickrey (1961) [4]. First price auctions will make people tend to bid lower than their private valuation since if they bid that

[4] The second price, sealed bid auction is also known as a Vickrey auction after the economist who invented it

value their profit is zero. Under this rule the item could be awarded to someone who value the item less. Second price auctions encourages bidder to bid their true values (as they will gain positive profits if they win no matter the second highest bid). This encourages an efficient allocation of items (Mochon & Saez (2015)). For the moment we will confine ourselves to first price auctions as this is where most of the strong results in POA analysis of auctions currently are.

Before discussing the known bounds on the price of anarchy for first price, single payer auctions it is worth understanding how the bids might lead to a non optimal outcome or what we mean by that. Because in a first price auction each bidder only knows how much they value the good and not the valuations of the other bidders they are acting under uncertainty as to how much they should bid. If they bid their exact private valuation then they will get a surplus of zero, or zero utility. In order for them to get some utility for the item they must be paying less then the exact amount they value the item. How much each bidder should "shade" their bid is determined by how much they think that the other party values the item. To capture this interaction auctions are represented as Bayesian games where each bidder is drawing their bids from distributions known to the other player (that need not be the same). If one player knows that the other is drawing from a distribution with a smaller mean than they are(i.e. probably doesn't value the item as much), the Bayes-Nash equilibrium will have them shade their bid less and this other person will shade their bid more. This can lead to the person who values the item less winning the auction and creating less social welfare.

Syrgkanis and Tardos proved in 2013 that the price of anarchy in first price, single item auctions is at least $1 - \frac{1}{e} \approx 0.63$. The exact upper bound on the price of anarchy for single payer, first price auctions remains unknown in the general case. This bound is true regardless of how many bidders there are or what distributions they are drawing their bids from.

This moves closer to answering the question for what is the price of anarchy at equilibrium, but these results do not pay attention to how the players arrive at these equilibria. In real world applications, we expect that players might play repeatedly in the same auction and learn as they play rather than come in with pre-computed strategies. This is especially true for when computing the equilibrium is computationally hard and the stakes of each individual auction is small. Given these observations, it is natural to ask questions about how the efficiency results carry over to adaptive game environments. The model for learning agents that is commonly used in the field is **no-regret learning**. An algorithm for a player satisfies the no-regret condition if, in the limit as the number of times the game is played goes to infinity, the average reward of the algorithm is at least as good as the average reward for the best fixed action in hindsight (assuming the sequence of actions for the other players remains unchanged)[5]. If each player incorporates this kind of learning algorithm, then it has been shown that these can converge to a larger class of equilibrium called correlated equilibrium where each player conditions their response on the expected action of the other player. Luckily, the previous theorem has been extended so that we know that the price of anarchy for the correlated equilibria of first price auctions are also at least 0.63 (Roughgarden et al. (2017))

With all of this set up, we now state our goal: to simulate no-regret learning algorithms for agents in a first price, single payer auction to see how well the price of anarchy holds. We also want to compare this with other learning algorithms that aren't no regret to see if this framework of using these algorithms is appropriate for making generalizations about equilibrium under learning.

[5]That is to say that the algorithm will converge to having a loss no worse than any fixed strategy we would have rather picked in hindsight as the limit goes to infinity. A more precise definition, example algorithms, and uses will be shown in chapter 2 to clarify what "regret" is and how this converges to zero

1.2 Simulated Agents and Simulated Economies

The use of computers in economics goes back all of the way to general purpose computers being invented in the 1940's. Wassily Leontif used a computer to invert a 39 x 39 matrix to help solve his input output model. Since then, computers use in economics has exploded. With computers, economists are able to solve bigger matrices, do Monte-Carlo simulations, create multinomial probit models, and use full information maximum likelihood estimation (Backhouse & Cherrier (2016)). While one branch of computational economics is focused on creating stronger and stronger calculators to facilitate empirical research, another branch has focused on creating simulated economies that allow economists to construct a blended version of theory and research within a computer program. Within these simulated economies, theories can be coded into the simulation which, when run can allow the researcher to conduct experiments that might not be practical to conduct in the real world.

One kind of a simulation that can be run is what is an agent-based model (ABM), a simulated system of autonomous decision makers called agents. These models are able to generate complex behavior even if only simple assumptions are made about the behavior of the coded agents. That is, these agents interacting with each other in complex ways are able to produce emergent phenomena in the macro structure of the system that are interesting

For example, in the 1970's Thomas Schilling created a computer simulation to try and understand how and why self segregated neighborhoods formed. Thus, he coded a virtual environment where agents were given a simple preference, they are only happy if they are not the minority in their neighborhood and will keep moving otherwise. This simple model can illustrate the main tenets of agent based modeling. First, we have agents who are representative of people in the real world. Their preferences to this respect are simple and easy to understand where they are only "happy" if half of their closest neighbors are the same as them. If they are not happy they will move somewhere else arbitrarily. These preferences can be represented as the short procedure, or algorithm, shown below where S is just the space they live in.

1. Draw a random location in S

2. If happy at new location, move there

3. Else, go to step 1

In this case, we get to choose what that environment looks like and like Shelling we can just say that it is a one by one unit square and we can say that their neighbors are the ten closest people to them (in Euclidean distance) on that square. When you run this simulation with green and orange dots representing the types of people you get the following behavior cycling through each of our 250 agents with the above procedure until every agent is happy.

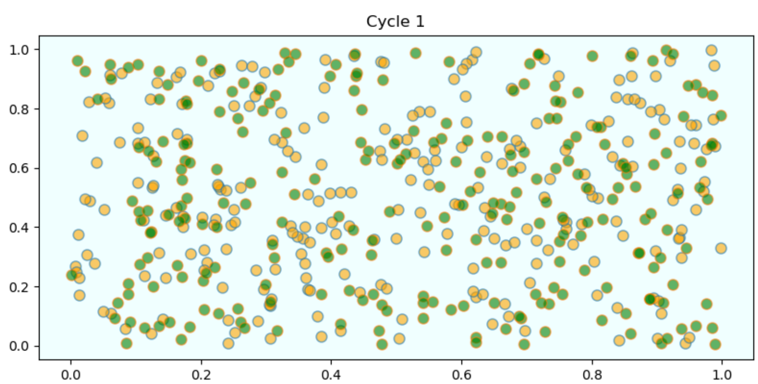

Figure 1.4: Schelling's Segregation Model: Cycle 1

As can be seen, in cycle 1 (Figure 1.4), the agents are well distributed among each other, but as they move in cycles 2-5, they become progressively more segregated. After 5 cycles all agents are happy and the simulation terminates. With very few assumptions about the agents' preferences, we can see the resulting emergent behavior of segregated neighborhoods in the system as a whole. Not only that, but the agents naturally move to an equilibrium as they adjust their behavior to what their neighbors are doing (Sargent & Stachurski (2019)).

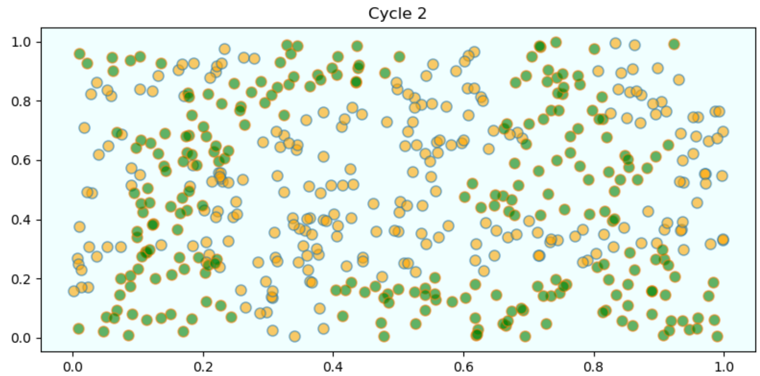

Figure 1.5: Schelling's Segregation Model: Cycle 2

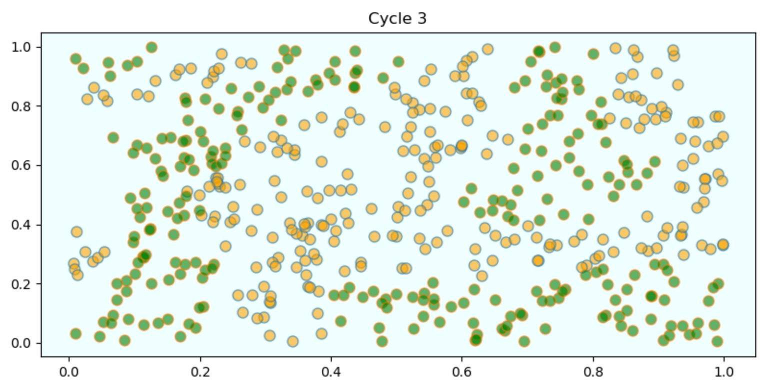

Figure 1.6: Schelling's Segregation Model: Cycle 3

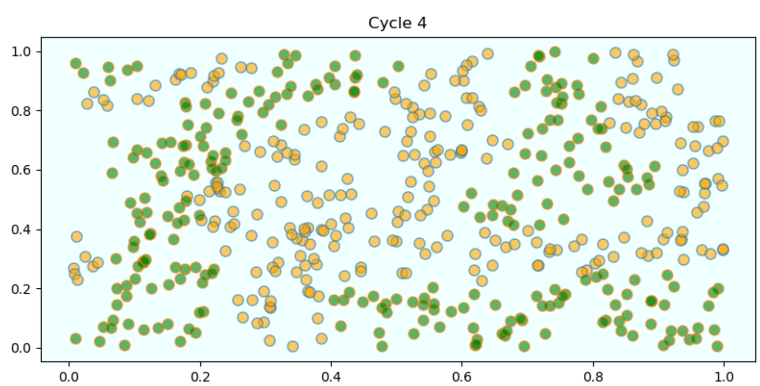

Figure 1.7: Schelling's Segregation Model: Cycle 4

Agent based modeling has been used to build and understand much more complicated systems then the example illustrated above. The Santa Fe Institute in New

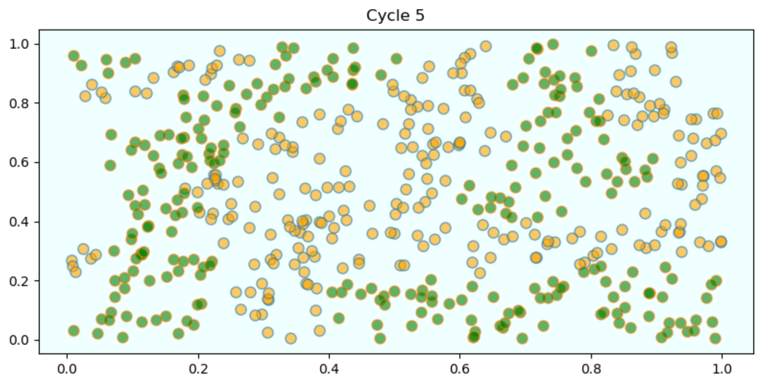

Figure 1.8: Schelling's Segregation Model: Cycle 5

Mexico is one of the main proponents of agent based modeling releasing a manifesto supporting using it to understand the "complexity" of economics from the ground up. They built the Santa Fe Artificial Stock Market in the 1990's to try and simulate the behavior of agents on the stock market and how they adapt their adapting trading strategies effects the outcome of the market. This is one of the first examples of agents learning and adapting to their environments as part of the model. In this model agents used a genetic algorithm to adapt their trading strategy at each period by modifying a string (for example 00011100) where each bit in the string told the agent to use a certain behavior or not. Those trading strategies that did well were coded to survive longer where the agents with worse strategies would randomly modify their own or take a more successful agents strategy (Arthur (1992)). This was supposed to be representative of the learning of traders on the stock market so that the insights taken from running these agents in simulation could be applied to learn something about the real world.

The authors of Santa Fe Stock Market paper at the time suggested that this was one of many algorithms that could be used to stand in for human behavior saying that reinforcement learning or deep learning could also be used to stand in for human intelligence[6] (CITATION NEEDED, This was in Shareen Joshi's thesis and in one of

[6]Reinforcement learning is when an algorithm plays a game repeatedly and updates its beliefs

their papers, but I am having trouble tracking down the quote. Will use her thesis to find original source or remove reference). Recent research in the field as well as in the field of multi-agent systems, a similar branch of computer science suggest that this is not the case. Because these simulations have agents competing in non-stationary environments that are changing from the perspective of any individual agent every time some other agent changes their behavior, the choice of algorithm dramatically changes how the system behaves (Rejeb & Guessoum (2005), Shoham & Leyton-Brown (2008)). Further, research that has compared the behavior of human agents in strategic settings to that of these algorithms have found that there is no general algorithm that best approximates human learning (Tesfatsion (2002)). From situation to situation, different algorithms more appropriately behave like humans. This is a problem only assuming that you want your agents to in some way represent human behavior, we might simply want our agents to represent the "rational choice" in any given situation. These algorithms aren't necessarily doing that either. As Holland and Miller say in their 1991 paper Artificial Adaptive Agents in Economic Theory, "Usually there is only one way to be fully rational, but there are many ways to be less rational." The way they suggest to get around this is to try and build models that have robust behavior across algorithm choice. This is probably not true for their own artificial stock market as it is not even robust across the choice of parameters to program the market. The literature of game theory provides a nice solution to this. The no-regret algorithms they use have the simple property of doing better in the long run than fixing their strategy randomly in the beginning. This is a simple learning requirement that means that this kind of algorithm should be more robust across representing human behavior. It also allows us to combine our simulated models and analytical models in a nice way as we can mathematically describe the processes of such algorithms behavior and we can code agents to use algorithms that have that

about what actions will lead to the best payoff. Deep learning uses deep neural networks to try and estimate the best outcomes in a fashion similar to regression.

behavior. This allows the learning process to be less of a black box and instead to be as simple and thus generalizable as possible.

Moving forward this paper aims to better understand the price of anarchy in auctions by simulating auctions using these no-regret learning algorithms[7] and seeing how they behave compared to the socially optimal equilibrium. These simulations will try to answer two main questions, what equilibrium do these algorithms converge to (if they converge at all) and what is the calculated price of anarchy in these systems compared to the bounds that theory tells us should exist. Using this framework to explore auctions we will start with a single-item first price auction as it is the best understood theoretically and is the easiest to code. We will then move to simultaneous first price auctions for complementary goods as this is also relatively well understood in terms of the price of anarchy. Finally, time permitting, we will discuss the open question of the price of anarchy in double auctions and discuss how one might go about proving bounds for it.

[7]these will be discussed in chapter 2, but no-regret learning is a type of reinforcement learning that have certain convergence properties

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