Chapter 2

Price of Anarchy in First-Price Auctions

This chapter lays out the mathematical framework for modeling auctions as Bayesian games of incomplete information, formally defines the price of anarchy in auctions and shows the bounds on the price of anarchy for first price, single payer auctions. It then follows up on this framework of analysis to show how these bounds can be extended to agents competing in auctions using learning algorithm's with a property called "no-regret" learning. This chapter primarily follows from the work of Tim Roughgarden, Vasilis Syrgkanis, and Éva Tardos who not only are all individually active in the field of algorithmic game theory and auctions, but also jointly authored the 2017 paper "The Price of Anarchy in Auctions" that is a survey of the entire topic (Roughgarden et al. (2017)). This chapter will be giving the important theorems and ideas that primarily come from this work and the individual work of these three authors, explaining them, demonstrating proofs where appropriate (I need to talk about where proof is appropriate with both advisors after you read this. I did not prove anything and am just citing/reconstructing results), and reconstruct the results that we will demonstrate with a simulation.

2.1 Auctions as Bayesian Game

Auctions are typically modeled as Bayesian games also known as games with incomplete information. While it is obvious why an auction consisting of the strategic interaction of bidders could be modeled as a game, how it should be modeled requires some thought. After all, while each bidder knows their own valuation of the item being sold, unless for some reason (and against their own interest) the other players announced their valuation of the item before the auction began our bidder will not know how the other players value the item. In fact, it is precisely this lack of information in sealed bid auctions that makes them interesting! If all players came into an auction knowing the valuation of other players, for first-price, single-item auctions assuming there was no tie, the bidder with the highest value would always win by bidding the valuation of the next highest bidder. Rather it is the uncertainty that players face about the valuation of other players that make things interesting as players must guess how much they should shade their bid (as again no player will ever bid above their valuation) based off of what they know about other players.

What can we say that bidders know about the valuations of other bidders? Certainly they do not know nothing as we all have reasonable expectations about what some item is worth to others. No one will value a candy bar at a million dollars. But a collector of candy bars might value it at a higher value than an ordinary person who just want to eat the candy bar. For each bidder, we could say that this bidder has a probability distribution from which they are drawing their valuation from that is know to the other bidders. As an example we might expect normal people's valuations of candy bars to be a normal distribution centered at \$1, but the collectors value might be a leplacian distribution (allowing for more black swan events) centered at \$5. This sort of strategic interaction where the players know the distributions that parameters of the game are drawn from are typically called games of incomplete information, or Bayesian games.

2.1.1 Bayesian Games

Bayesian games of incomplete information are games in which one or more of the players don't have "full knowledge" of the game that is being played. Introduced by John C. Harsanyi in 1967, rather than players knowing every parameter of the game situation such as utility functions, possible strategies, and information held by other players, each player knows a probability distribution from which these will be drawn. In his paper, Harsanyi says that this type of game can be thought of as a normal game, where "nature" goes first drawing from these probability distributions and assigning values before play begins without the players knowing which specific variation of the game they are playing (TODO CITE Harsanyi). Importantly, each player does know the probability distributions from which each value is selected. Formally such games are defined as follows,

Definition 1. A game with incomplete information, also known as a Bayesian game where there are n players, $G = (\mathcal{F}, S, P, u)$ consists of:

1. A set $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$, where $\mathcal{F}_i$ is the finite set of possible types for player i.
2. A set $S = S_1 \times \ldots \times S_n$, where S_i is the set of possible strategies for player i
3. A joint probability distribution $p(\mathcal{F}_1, \ldots, \mathcal{F}_i)$ over types. For finite type space, assume that $p(\mathcal{F}_i) > 0$ for all $\mathcal{F}_i \in \mathcal{F}$
4. Payoff functions $u_i : S \times \mathcal{F} \to \mathbb{R}$

(TODO CITE Levin)

Using this definition to model our auction the types of players will consist of the publicly known distribution from which they are drawing their valuation. That is, our auction will consist of bidders who know their own valuation of the item being bidded on, and the distribution from which each of the other players is drawing their own valuations. In a first-price, single item auction if player i wins with bid b_i, we define their payoff the winner to be $u_i = v_i - b_i$, the difference between their valuation and their bid and the losers all get $u_i = 0$ since they did not receive the item. Now, a strategy for a player is a function $s_i \in S_i$ that maps a valuation v_i in support of $\mathcal{F}$ to a bid $s_i(v_i)$ (i.e. taking into account the probability distribution for the other players).

Now, we move to the idea of equilibrium in this system. In games of complete information the central equilibrium concept is usually a Nash equilibrium, the set of strategies for all players in which each individual player cannot increase their utility by deviating from their strategy fixing the strategy of all the other players. That is, for every player if no one else changes strategy, their best option is to stay where they are, hence an equilibrium. This concept is now updated to give us a Bayes-Nash equilibrium where we must also incorporate the distributions for which players are drawing from.

Definition 2. A strategy profile constitutes a Bayes-Nash equilibrium if for every player i and every valuation v_i that the player might have, the player chooses a bid $s_i(v_i)$ that maximizes her conditional expected utility where the expectation is over the valuations of the other players, conditioned on a bidder i's valuation being v_i.

2.1.2 Formally Defining First-Price Auctions

NOTE: This section needs cleaning up since it is going to give a lot of duplicate information or redefine things slightly differently from previous sources.

To analyze an first-price auction as a game, we give the notation we will be using. This notation comes from Roughgarden et. al 's 2017 survey of the subject of the price of anarchy in auctions. For a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$, we let

$$x_i(\mathbf{b}) = \begin{cases} 1 & \text{if player } i \text{ is the winner} \\ 0 & \text{otherwise} \end{cases}$$

.

We let $p(\mathbf{b}) = \max_{i \in \{1,\ldots,n\}} b_i$ denote the selling price. The utility that a player i receives when their valuation is v_i is

$$u_i(\mathbf{b}; v_i) = (v_i - b_i) \cdot x_i(\mathbf{b})$$

.

Now, for a strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$, where each s_i is a function for player i's valuation v_i to their bid. We let $\mathbf{s}(\mathbf{v})$ denote the strategy vector resulting from the vector of valuations $\mathbf{v}$. For any given vector $\mathbf{x}$, we use $\mathbf{x}_{-i}$ to denote the vector $\mathbf{x}$ with the ith element removed.

Now, for a first-price auction we can say that a strategy profile $\mathbf{s} = (s_1, \ldots s_n)$ is a Bayes-Nash equilibrium if and only if

$$\mathbb{E}_{\mathbf{v}_{-i}}[u_i(\mathbf{s}(\mathbf{v}); v_i) \mid v_i] \geq \mathbb{E}_{\mathbf{v}_{-i}}[u_i(b_i^{'}, \mathbf{s}_{-i}(\mathbf{v}_{-i}); v_i) \mid v_i]$$

.

(Roughgarden et al. (2017))

2.1.3 Example Auction

We now turn to an example auction to clarify what has just been laid out. In this example we analyze the an auction between two players, Alice and Bob who are bidding on a candy bar where each select their valuations from the uniform distribution $[0, 1]$. This is a first-price, sealed bid auction where they each submit a bid for the candy bar simultaneously. How are Bob and Alice supposed to decide what to bid on this auction?

Propisition 1. *In the first-price, sealed bid auction with valuation distributed on*

$[0, 1]$, *the unique Bayesian-Nash equilibrium is* $\boldsymbol{s} = (s_1(v_1) = v_1/2, s_2(v_2) = v_2/2)$.

To show this first, we show that each player is using a best response. First, we note that We calculate that the expected value for player one is,

$$\begin{aligned}\mathbb{E}_{\mathbf{v}_{-1}}[u_1(\mathbf{s}(\mathbf{v}); v_1 \mid v_1)] &= \mathbb{E}_{v_2}[u_1(\ s_1(v_1), s_2(v_2);\ v_1 \mid v_1)] \\ &= (v_1 - b_1)\Pr[s_2(v_2) < b_1] + \frac{1}{2}(v_1 - b_1)\Pr[s_2(v_2) = b_1].\end{aligned}$$

since player expect to get their full utility with the probability that they outbid the other player, and expect to get one half that if they tie with the other bidder (since ties are broken with a coin toss). Now, assuming that player one bids, $b_1 \in [0, \frac{1}{2}]$.

... I'm not sure if I need this, leaving it out to write up more important parts.

2.1.4 Efficiency of First-Price Auctions

Examples of the Bayes-Nash equilibrium have been solved for various combinations of the number of players and distributions from which they draw their valuations. With n bidders it has been shown that the Bayes-Nash equilibrium strategy vector is composed of $s_i(v_i) = \frac{n-1}{n}v$ for all players i (TODO: cite, find source). Here the equilibrium is easy to calculate and efficient (meaning that the item will always be allocated to the player with the highest valuation). Neither efficiency nor ease of calculation are guaranteed for Bayes-Nash equilibria in this auction format. For example if we conduct an auction with two bidders, one choosing from the uniform distribution $[0, 1]$ and the other from the uniform distribution $[0, 2]$ it has been shown that the Bayes-Nash equilibrium for this auction is:

$$s_1(v_1) = \frac{4}{3v_1}\left(1 - \sqrt{1 - \frac{3v_1^2}{4}}\right)$$
$$s_2(v_2) = \frac{4}{3v_2}\left(\sqrt{1 + \frac{3v_2^2}{4}} - 1\right)$$

(CITE Krishna, 2002):

Here bidder one with the smaller valuation distribution knows that bidder two is more likely to to have a higher valuation than them. Thus player one must bid higher relative to their given valuation if they expect to win and so they shade their bid less than bidder two. This can lead to bidder one drawing a lower valuation than bidder two, but outbidding them regardless and winning the item. This is inefficient. More over, it has been shown that solving many of these asymmetric Bayes-Nash equilibrium requires finding a solution to a system of partial differential equations many of which have no closed-form solution (Roughgarden et al. (2017)). This means that if we expect bidders to do their homework before an auction and choosing their bidding strategy they might not know what to do. Given this, it is extremely hard to characterize or say things about what solutions to this format of auctions look like in general. However, just because we are not able to give a closed form for all of these equilibria, that does not mean we aren't able to say anything about them.

2.1.5 Price of Anarchy in First-Price Auctions

To try and get a sense of how inefficient these auctions can be, computer scientists have been applying a concept known as the price of anarchy to analyze these systems[1]. The price of anarchy is a way to compare the social welfare of a system or a game at

[1] Not that economists were uninterested in these questions of efficiency and welfare analysis in auctions before the computer scientists started studying them.

its best possible value to that of its worst possible equilibrium under strategic play. We must first define these concepts and then move to the point at hand. In the case of an auction the social welfare is the sum of the utilities of the players plus the revenue of the auctioneer.

Definition 3. The *social welfare* of a bid profile $\mathbf{b}$ when the valuation profile is $\mathbf{v} = (v_1, \ldots, v_n)$ is

$$SW(\mathbf{b}; \mathbf{v}) = \sum_{i=1}^{n} v_i \cdot x_i(\mathbf{b})$$

The price the winning bidder pays does not appear in this equation since the winning bidder is paying exactly as much as the auctioneer is getting and this term cancels out. This means that welfare is maximized when the bidder with the highest valuation is winner. Thus, if we let $x_i^*(\mathbf{v})$ be an indicator variable for whether or not a player i is the player with the highest valuation (ties broken arbitrarily), the maximum possible social welfare in a single-item auction is

$$\mathrm{OPT}(\mathbf{v}) = \sum_{i=1}^{n} v_i \cdot x_i^*(\mathbf{b})$$

.

Now, that we mathematically describe the social welfare in our system, we can define the price of anarchy.

Definition 4. The *price of anarchy* of an auction, with a valuation distribution $\mathcal{F}$, is the smallest value of the ratio

$$\frac{\mathbb{E}_{\mathbf{v}}[SW(\mathbf{s}(\mathbf{v}); \mathbf{v})]}{\mathbb{E}_{\mathbf{v}}[\mathrm{OPT}(\mathbf{v})]},$$

ranging over all Bayes-Nash equilibrium $\mathbf{s}$ of the auction.

The above definition applies to individual auctions dependent on choice of $\mathcal{F}$ and n. We generally only discuss the price of anarchy for the format of the auction, in our case any possible first-price auction. The price of anarchy for the first-price auction format is then the worst possible price of anarchy for any choice of the number of players n or valuation distributions $\mathcal{F}$. Note that the price of anarchy (for either an individual auction or the format of auction) is a number between 0 and 1, and that the closer it is to one, the "better" we can guarantee the system's social welfare will be[2].

Incredibly, bounds on the price of anarchy for the format of first-price auctions have been found. Again, this allows us to characterize how much worse the social welfare for the system could be at (Bayes-Nash) equilibrium no matter how many players we have or what distributions they are choosing their valuations from. This sort of guarantee is incredible, especially for systems where we may not want, or it may not be feasible to have a central authority pre-calculate the way to optimize social welfare in a system. Rather, we can trust that the system will perform at least so well under strategic interaction.

Theorem 1. *The price of anarchy in first-price single-item auctions format is at least* $1 - \frac{1}{e} \approx 0.63$

(CITE: Syrgkanis & Tardos 2013)(Possibly include: no bound better than 0.87 is possible (CITE Hartline and Taggart 2014))

(Possibly include below thm, might be referenced by later proofs)

Theorem 2. *Every Bayes-Nash equilibrium of a first-price auction with correlated valuation distributions has expected social welfare at least* $1 - \frac{1}{e}$ *times the optimal welfare.*

(CITE: Syrgkanis, 2014) (Tight bound for above)

Theorem 1 tells us that no matter how many players we have or what weird

[2]Much of the literature for POA (including the paper introducing the idea) defines it as the opposite ratio, Optimal/Worst-EQ where smaller values indicate better systems. For some reason the auction literature defines it as Worst-EQ/Optimal, so I will remain consistent with them.

distributions we try and give them, we cannot construct a first-price auction that will achieve less than 0.63% of the optimal social welfare at Bayes-Nash equilibrium.

2.2 Extending Results to No-Regret Agents

These results hold for simple first-price auctions, but it is natural to ask questions about how robust these results are. First of all, how do people arrive at a Bayes-Nash equilibria if there doesn't exist a closed form way to express it? Secondly, do these results hold for mixed Bayes-Nash equilibria (randomizing between bidding strategies) or other larger, more realistic equilibrium concepts? To address these questions, Roughgarden et al use a set of extension theorems that take us through a general mechanism design setting and allow the class of equilibria our bounds hold for to be expanded. This extension will take us to a concept of no-regret learning.

2.2.1 General Auction Mechanisms

In order to understand (or even state) the proofs and theorems required to take us to our expanded set of equilibria, we must introduce more notation and the idea of a general mechanism design setting and a general auction setting (this is unfortunate for us due to an expansion of scope, but actually makes these theorems quite powerful!). In a general mechanism design setting, the auctioneer solicits an action a_i from all of the players, i, from some action space $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Given an action profile $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}$, the auctioneer decides an outcome $o(\mathbf{a})$ among the set of possible outcomes $\mathcal{O}$. This outcome includes a payment $p_i(o)$ that each player must give to the auctioneer. Denote the revenue of the auctioneer $\mathcal{R}(O) = \sum_i p_i(o)$. Players receive some utility as a function of their valuation, v_i and the outcome which we write $u_i(o; v_i)$. Let $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$. Now with this new notation out of the way, we move to the concept of a smooth auction.

2.2.2 Smooth Auctions

Definition 5. For parameters $\lambda \geq 0$ and $\mu \geq 1$, an auction is (λ, μ)-smooth if for every valuation profile $\mathbf{v} \in \mathcal{V}$ there exist action distribution $D_1^*(\mathbf{v}), \ldots, D_n^*(\mathbf{v})$ over $\mathcal{A}_1, \cdots, \mathcal{A}_n$ such that for every action profile $\mathbf{a}$,

$$\sum_i \mathbb{E}_{a_i^* \sim D_i^*(\mathbf{v})}[u_i(a^*, \mathbf{a}_{-i}; v_i)] \geq \lambda \mathrm{OPT}(\mathbf{v}) - \mu \mathcal{R}(\mathbf{a})$$

Theorem 3. *If an auction is (λ, μ)-smooth, then for every profile $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of independent valuation distributions over $\mathcal{V}_1, \ldots, \mathcal{V}_n$, every Bayes-Nash equilibrium of the auction has expected welfare at least $\frac{\lambda}{\mu} \cdot \mathbb{E}_{\boldsymbol{v} \sim \mathcal{F}}[OPT(\boldsymbol{v})]$.*

(Roughgarden et al. (2017))

2.2.3 No-Regret Learning

Consider an auction with n players that is repeated for T time steps. At each iteration t, bidder i draws a valuation v_i from $\mathcal{F}_i$ and chooses an action a_i^t which can depend on the history of play. After each iteration, the players observes the actions taken by the other players (TODO: potentially can be relaxed to only need to see utility for their own action, which makes simulating easier. Need to follow up on source). A player i is said to use a no-regret learning algorithm if, in hindsight their average regret (difference between average utility of strategy vs algorithm) for any alternative strategy $a_i^{'}$ goes to zero or becomes negative as $T \to \infty$. When all players use this algorithm it results in a vanishing regret sequence.

Definition 6. A sequence of action profiles $\mathbf{a}^1, \mathbf{a}^2, \ldots, a^T$ is a *vanishing regret sequence* if for every player i and action $a_i^{'} \in \mathcal{A}_i$,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (u_i(a_i^{'}, \mathbf{a}_{-i}^t; v_i) - u_i(\mathbf{a}^t; v_i)) \leq 0$$

Theorem 4. *If an auction is (λ, μ)-smooth, then for every valuation profile $\boldsymbol{v}$, every vanishing regret sequence of the auction has expected welfare at least $\frac{\lambda}{\mu} \cdot OPT(\boldsymbol{v})$ as $T \to \infty$.*

(Roughgarden et al. (2017)) (TODO: add second source)

The end result is that since single-payer first-price auctions are $(1 - \frac{1}{e}, 1)$ smooth (TODO, demonstrate that), we know that this bound will hold for auctions with no regret learning agents. Thus, we will try and build a simulation that uses one of the algorithms that fulfills this property and demonstrate that for any arbitrary number of players and distributions from which they pick their valuations from, they converge to an equilibrium social welfare greater than $1 - \frac{1}{e}$.