Simulating the Price of Anarchy in Auctions

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Robert S. Irvin

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Jeffery Parker   David Perkinson

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Contents

Abstract

The purpose of this thesis is to demonstrate via simulation the efficiency guarantees of the price of anarchy in first-price, single-item auctions. An agent based model is used where each agents strategy can be updated to demonstrate the efficiency properties of the system under this strategy.

The simulations find that agents using known Bayes-Nash equilibria strategies converge to a price of anarchy greater then the lower bound given by theory. Next, we see that agents who use an arbitrary bidding strategy still achieve high efficiency in both symmetric and asymmetric auctions despite their lack of intelligence, and that as the number of bidders increases these auctions approach full efficiency. Finally, we see that no-regret learning agents converge to outcomes both better than the efficiency guarantees that theory would suggest, and better than the arbitrary bidders.

The results from this experiment both demonstrate the correctness of the price of anarchy theory and give high confidence in efficiency of first-price single-item auctions. While even arbitrary agents perform quite well, we see that agents who are capable of learning and reflecting human behavior converge to highly efficient outcomes.

Introduction

When people act rationally, society can suffer. Economists often talk about the tragedy of the commons, a scenario where people follow their incentives to over-utilize a public good and ruin it for society at large. In the tragedy of the commons, each agent's rational behavior is what leads the system to converge to overuse, wasted resources, and utility lower in the long run than if they had been able to preserve the resource. While the idea of the tragedy of the commons is about how incentives can lead to the overuse of public resources, there is a larger problem at the heart of it: how and why is it that when people behave strategically it often leads to worse outcomes for both the individual and society as a whole?

The fact that selfish behavior leads to a worse outcome in the tragedy of the commons is a product of the system it takes place in. Because the incentives of this game line up with individualistic choices that are bad for society, rational agents will always choose to over-utilize the public good. It would only be if some outside force came in and made people behave correctly that these agents could maintain behaviors that would lead to the best outcome for society in the long run. Socially optimal choice for all agents is not an equilibrium when agents behave rationally in this case, but we can still imagine what it would be like if agents were forced to make socially responsible choices by some external force. This is an economic dream,

to have some *deus economica* control the actions of every agent to maximize some empirically derived social welfare function. Every action preplanned and every util maximized for the good of us all. In lieu of an economic god ruling us, maybe some benevolent dictator could be set up instead?

Even if a benevolent dictator can be found, who is to say that they will be able to calculate the exact optimal solutions for large scale problems? Imagine a nation's government trying to calculate the socially optimal number of shoes to manufacture each year for millions of people each with different shoe sizes and preferences? It seems doubtful that any government would be capable of doing this as there is too much data to process and too many changing circumstances to react quickly to societal need. Rather, most countries leave shoe manufacturing to a free market which has firms that can pop up and respond to the demands of customers at will. That is not to say that the free market is perfect for shoes: we have no way of knowing if its allocation of resources end up being the best possible. In fact, it is highly doubtful that it is. There is only one socially optimal outcome (which presumably our omnipotent god of economics could calculate) but there exist an infinite number of outcomes which are worse than optimal. Our question is one of trying to understand what makes some markets or systems behave in a way that is good for society and some terrible for society when everyone is behaving selfishly, or, how can we describe the effect of selfish behavior on a system. One way of approaching an answer to this question is instead of trying to understand how agents behave in the worst case (from a social welfare perspective) for a given system or game. That is, we want to bound the systems worst possible social cost when people are behaving rationally. This is called the *price of anarchy*, the worst possible price paid in terms of social welfare at the

equilibria for rational agents within that system.

The idea of equilibria in a system is also problematic. How are we to presume that agents converge to an equilibria? Are they able to easily calculate it and decide to go there to begin with? No. We should be concerned with the set of outcomes or equilibria that happens not only for fully rational agents with infinite computational power, but also with agents who learn how to behave as they participate, who dynamically react to the changes in strategy by the other players.

This thesis is about simulating the price to society for anarchic behavior within a system (game) who learn to play as they go. Computer scientists and economists have been proving the bounds of this price for various systems both at the fully rational equilibria, and at the equilibria that certain learning agents converge to. We create a simulation to demonstrate this bound in one of the most simple markets, a sealed bid first-price, single-item auction. This is an auction where everyone bids simultaneously, one-shot on a single item and the highest bidder gets the item. Thus, by constructing a simulation of an auction in this way, we can be more sure that when this auction format is used in real life that these auctions are producing outcomes that are at least as efficient as our lower bound, and we no longer have to worry about finding a benevolent dictator.

Chapter 1

Auctions and Computational Economics

Within the last three decades, computational economics has been on the rise. This branch of economic research encompasses two major ideas. One is that the increasing power of computers can help solve and understand classical economic problems through increasingly more complex simulations and numerical analysis. Two, that the mathematical methods developed in the field of theoretical computer science can be used to gain better understanding of existing models in terms of their algorithmic and computational complexity properties. We aim to take elements from both of these frameworks to better understand the social welfare of auctions at equilibrium.

1.1 Auctions, Equilibria, and Anarchy

It is perhaps obvious why economists would be interested in studying auctions. Auctions are one of the most basic market structures that have roots going back to at least

the ancient Greeks and still exist today in places such as art auctions and Ebay (Mochon and Saez, 2015). In the past 15 years, economists have been joined by computer scientists who are increasingly interested in the strategic interactions of agents within this setting. Christos Papadimitriou said in a 2015 lecture at the Simons Institute that it was the advent of the internet, an artifact out of computer scientists control, that turned theoretical computer science into a "physical science." Now computer scientists had to "approach the internet with the same humility that economists approach the market..." He went on to say that "it also turned us [computer science] into a social science. It was obviously about people and incentives. Without understanding this, you cannot understand the internet"(Papadimitriou, 2015). It is within this framework that he says computer scientists first began to study auctions as they existed on the internet such as Ebay and Google's sponsored search auctions. Now, computer scientists are moving beyond the internet, taking the mathematical tools of their discipline and applying them as a lens to understand and explain the world. This thesis is looking at the field of algorithmic game theory: the study of the algorithms and complexity of strategic interactions. For economists, this can be thought of as a new toolbox for unpacking and understanding the models and structures that already dominate the field. For example, in the case of a Walrasian auctioneer who calculates the clearing prices of a combinatorial auction, their problem was shown to be NP-complete, a complexity class usually called "intractable" due to the time it takes to solve these problems (the best algorithms here are generally guess and check, which gets out of hand for large inputs)(Papadimitriou, 2015)[1]. Using the lens of

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[1]NP is the class of problems that a given solution can be checked in polynomial time, i.e. $O(n^k)$ operations where n is the size of the input and k is any positive integer. NP-complete means that all problems in NP reduce to solving this problem. This is when you will usually hear "taking more time to compute than the age of the universe" etc...

computational complexity, analysis of how many operations or how much memory a problem must take to solve, is one way of assessing what assumptions we are making about the computational power of our rational agents.

1.1.1   The Price of Anarchy

One of the major ideas to come out of algorithmic game theory is the *price of anarchy* (POA), a mathematical way of showing the difference between the social welfare in the optimal case and in the worst case equilibrium for a game. This term was first coined by two computer scientists Elias Koutsoupias and Christos Papadimitriou, who were using the price of anarchy to understand network games and more generally were looking at how this concept could be used to understand behavior on the internet (Koutsoupias and Papadimitriou, 1999; Papadimitriou, 2001). More formally, the price of anarchy for a game is the ratio of the minimum equilibrium social welfare in a game over the best possible social welfare of the game (The actual formal definition is given in definition 2.1). An illustrative example of the price of anarchy from Roughgarden (2016, 15-16) are selfish routing games played on graphs such as the one pictured below in figure 1.1.

In this game, each player chooses which edges to take on the directed graph from the source s to reach the terminal t. Each edge has an associated cost, expressed as a function of the proportion of players who take that edge, x, and their total cost is sum of the costs for each of the edges they took to reach t. Player's objective is to minimize their cost of reaching the terminal by picking the best route they can. In the game depicted in 1.1, we see that routes either cost, 0, 1, or x, where x is the proportion of the players who take that route ($x \in [0, 1]$) For example if 50% of the
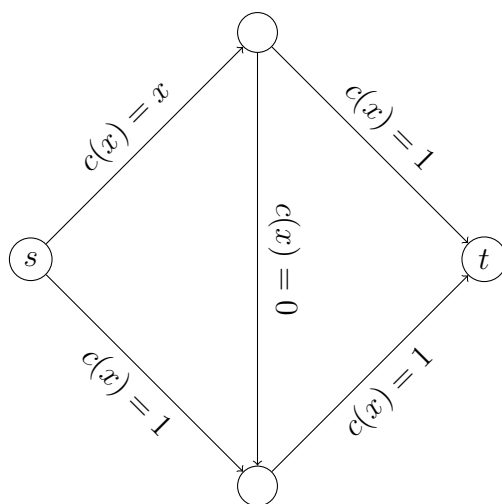
Figure 1.1: Routing game from s to t. Strategic interaction will make everyone worse off.

players take that edge then $x = 0.5$. This can be seen as analogous to traffic when driving a car, the more people take a road, the slower the traffic goes and the longer it takes to get somewhere. Knowing this, each player try to choose which path to take to get to their destination as quickly as possible. As you can quickly verify, the best solution for society, minimizing the total driving time for all, is when half of the drivers take the top route, and half of the drivers take the bottom route costing in total 1.5 for each driver.

However, this is not an equilibrium. The drivers taking the top route have incentive to instead take the middle edge to try and lower their total time traveled. In fact, the Nash equilibrium will end with all of the drivers taking the top x, going through the middle, and then the bottom x. Only then will no players have reason to deviate. These actions to individually and strategically try to decrease their cost end up producing a worse outcome for all players and society as a whole (the social
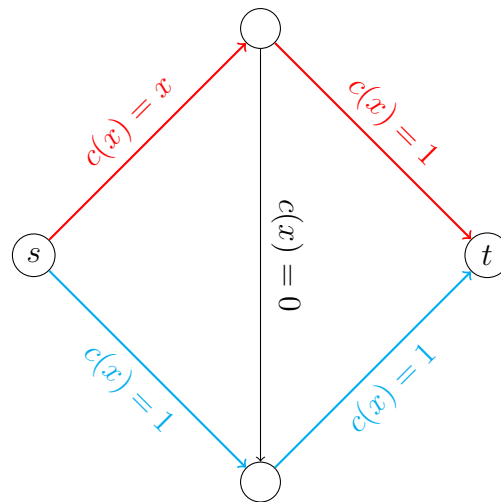
Figure 1.2: Socially optimal routing, not at equilibrium

welfare)[2].
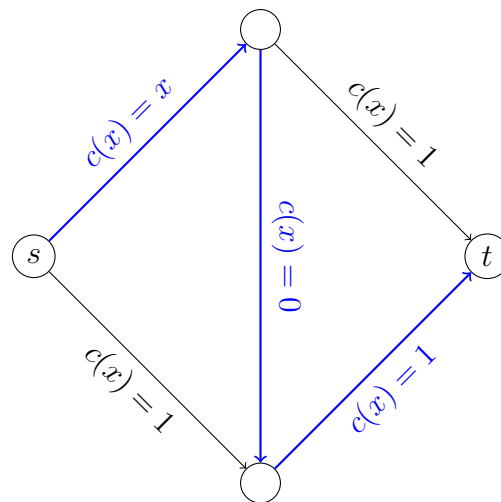
Figure 1.3: Route under strategic interaction, the only Nash equilibrium

Now, it will take time 2 for each player to reach the terminal. Thus, under strategic

interaction we see that the equilibrium is sub-optimal for society, and in this case all

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[2]A careful observer will note that if the edge with cost 0 were not there, this would not be a problem. This is called Braess's Paradox where having this extra edge counter intuitively leads to worse outcomes (Braess, 1968).

players as well. So, the ratio of the social welfare (total costs to all players) for the worst possible equilibria compared to the best possible (but not necessarily equilibria) constructed can be computed

$$\frac{\sum_{n=1}^{N} n \cdot 2}{\sum_{n=1}^{N} n \cdot 1.5} = \frac{4}{3}$$

where we sum the cost for our N players in each of the two scenarios described above depicted in figure 1.2, and figure 1.3 (Roughgarden et al., 2017). For this network, there is only one (Nash) equilibrium for selfish routing games, however, price of anarchy bounds can be found for the class of game as a whole regardless of its individual construction. These bounds tell us how much worse then optimal we can expect a system at equilibrium to behave in the worst case. The price of anarchy bound for these selfish routing games with affine cost functions (i.e. $c(x) = ax + b$) is know to be 4/3, meaning that our example is as bad as the ratio between the best possible social welfare and the worst possible equilibria could get for any network to play this game (Roughgarden, 2007).

1.1.2   Auctions

The lens of the price of anarchy is now being applied to understand auctions by researchers in the algorithmic game theory. Where for selfish routing games, the price of anarchy has been shown to have the tight bound above, the price of anarchy for single price auctions only has approximate upper and lower bounds set, and is not known at all for double auctions. Before exploring this further, we should set up the mathematical framework of auctions that is necessary to understand the social

welfare in auctions, as well as how strategic interaction arises.

An *auction* is a market mechanism, operating under specific rules that determines to whom one or more items will be awarded and at what price. To do this, an auctioneer solicits a bid, or a sequence of bids from each of the players for the items being auctioned and then uses some fixed rule (or mechanism) to determine who is awarded the items and how much each bidder must pay. While we may traditionally think about ascending auctions for a single item, called an *English auction*, where the auctioneer keeps soliciting bids until no one is willing to bid anymore, this is not the only kind of auction. There can be many different rules for determining who the winner is, or what price each of the bidders must pay (regardless of if they win or not) depending on what the objective of the auctioneer is (Mochon and Saez, 2015). For example a popular auction format for (precocious) e-commerce websites like QuiBids is where each player must pay 40 cents to bid, and are only allowed to bid one cent more then the precious bid. The winner will then be the last person to bid who must pay the final bid price plus shipping for the item on top of the price they paid for each of the bids (QuiBids, 2020). The QuiBids auction format is seemingly designed to maximize the revenue of the auctioneer.

Using the notation from Mochon and Saez (2015) for each bidder i in an auction, their *valuation* v_i is how much they value the item. This is sometimes called a private valuation as this value is generally unknown by the other participants in the auction. The *bid*, b_i, is the offer that bidder i submits for an item. *Sincere bidding* is when $v_i = b_i$, *underbidding* is when $v_i > b_i$, and *overbidding* is when $v_i < b_i$. The highest bid made by any bidder is denoted b^*, and under the highest price mechanism is also the winning bid (if multiple bids equal the winning bid, then some tie-breaking rule

must be used). The *selling price*, p^* is the final price that the bidder actually pays

for the item (which, depending on the auction type, need not equal b^*). Under the

first-price rule, the bid submitted by the winner is equal to the selling price. Before

the auction begins, each bidder knows their personal or private value for the item. An

auction consists of a set of bidders, $I = (1, 2, ..., N)$, and a seller. After the auction,

the bidder i wins the item if their bid is higher than the bid placed by any other

bidder, $b_i > \max_{i \neq k}\{b_k\}$. In a single unit auction, the *income of the bidder i* is equal

to their value of the item:

$$\Gamma_i^* = v_i$$

and the *surplus of the bidder i* is equal to the difference between income and price

paid

$$\Pi_i^* = \Gamma_i^* - p^*.$$

If a bid placed by a bidder is less than the winning bid, they do not win anything and

their income and surplus are both zero. The *seller's revenue* in a single unit auction

is equal to the price paid by the winning bidder

$$R^* = p^*.$$

There are multiple different pricing rules in an auction that determine who gets

allocated the item. In a *first-price* auction the winning bidder pays the amount of

their bid, which is the highest bid of the auction: $p^* = b^*$. Also called *pay-what-*

you-bid (PWYB). In a *second-price* auction, the winning bidder pays an amount that

is equal to the second highest bid for the awarded item (Vickrey, 1961) [3]. People

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[3]The second price, sealed bid auction is also known as a Vickrey auction after the economist who

tend to bid lower than their private valuation in first-price auctions since if they bid that value, their profit is zero regardless of whether they win. Thus, with the first-price rule the item could be awarded to someone who values the item less than other bidders. Second price auctions encourage bidders to bid their true valuations (as they will gain positive profits if they win no matter the second highest bid). This encourages an efficient allocation of items (Mochon and Saez, 2015). For the moment we will confine ourselves to first-price auctions as this is where most of the strong results in price of anarchy analysis of auctions currently are.

Before discussing the known bounds on the price of anarchy for first-price, single-item auctions, it is worth understanding how the bids might lead to a non-optimal outcome and what we mean by that. Due to only knowing their own valuation of the good, each bidder must act under uncertainty as to how much they should bid to beat out the unknown valuations of the other bidders. However, if they bid their exact private valuation then they will get a surplus of zero, or zero utility. In order for them to get some utility for the item they must be paying less then the exact amount they value the item. How much each bidder should bid less than their valuation, or *shade*, their bid is determined by how much they think that the other party values the item. To capture this interaction, auctions are represented as Bayesian games (see definition 2.1) where each bidder is drawing their valuation from distributions known to the other players (that need not be the same). If one player knows that the other is drawing from a distribution with a smaller mean than they are (i.e. probably doesn't value the item as much), the Bayes-Nash equilibrium will have them shade their bid less and this other person will shade their bid more. This can lead to the

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invented it.

person who values the item less winning the auction and creating less *social welfare* (see definition 2.4), the summed surplus of all bidders and the seller.

Syrgkanis and Tardos proved in 2013 that the lower bound of the price of anarchy in first-price, single-item auctions is at least $1 - \frac{1}{e} \approx 0.63$. The exact upper bound is unknown, but it has been proven that it can be no better than 0.87 (Hartline et al., 2015). This bound is true regardless of how many bidders there are or what distributions they are drawing their bids from. So, for any first-price single-item auction at equilibrium, we can say that the social welfare for any equilibrium must be at least 63% of the optimal social welfare (the maximum possible social welfare achievable in this system).

This moves us closer to understanding the social welfare produced by first-price single-item auctions, but these results do not pay attention to how the players arrive at these equilibria. Not to mention that in real world applications, we expect that players might play repeatedly in the same auction and learn as they play rather than come in with pre-computed strategies. This is especially true for when computing the equilibrium is computationally hard and the stakes of each individual auction is small. Given these observations, it is natural to ask questions about how the efficiency results carry over to adaptive game environments. One notion of learning commonly used in both theoretical computer science and game theory is *no-regret learning*. An algorithm for a player satisfies the no-regret condition if, in the limit as the number of times the game is played goes to infinity, the average reward of the algorithm is at least as good as the average reward for the best fixed action in hindsight (assuming the sequence of actions for the other players remains unchanged)

[4]. The no-regret condition is not in itself a technique for learning, but rather a characteristic of a learning algorithm in a fixed action environment. It says that each player using an algorithm shown to satisfy the no-regret condition will do at least on average as well as if they had picked the best possible fixed action. If each player incorporates this kind of learning algorithm, then it has been shown that these can converge to a larger class of equilibrium called *coarse-correlated equilibrium* where each player conditions their response on the expected action of the other player (Blum and Mansour, 2007). Remarkably, Roughgarden et al. (2017) produced a way to extend the price of anarchy bounds proven for Bayes-Nash equilibria to the larger set of coarse-correlated equilibria. Hence, that we know that the price of anarchy for the set of coarse-correlated equilibria for first-price, single-item auctions are also at least 0.63, and that if the bidders in the auction are no-regret learners, we can arrive there.

We now state our goal: to simulate no-regret learning algorithms for agents in a first-price, single-payer auction to demonstrate the price of anarchy for this auction format.

1.2   Simulated Agents and Simulated Economies

The history of computers in economics, as outlined in Backhouse and Cherrier (2016), goes all the way back to general purpose computers being invented in the 1940's. Wassily Leontif used a computer to invert a 39 x 39 matrix to help solve his input output model. Since then, computers use in economics has exploded. With comput-

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[4]That is to say that the algorithm will converge to having an average utility no worse than any fixed strategy we would have rather picked in hindsight after seeing the outcome for each of their actions (see definition 2.8).

ers, economists are able to invert bigger matrices, do Monte-Carlo simulations, create multinomial probit models, and use full information maximum likelihood estimation. While one branch of computational economics is focused on creating stronger and stronger calculators to facilitate empirical research, another branch has focused on creating simulated economies that allow economists to construct a blended version of theory and research within a computer program. Within these simulated economies, theories can be coded into the simulation which, when run, can allow the researcher to conduct experiments that might not be practical to conduct in the real world.

One kind of a simulation that can be run is called an agent-based model, a simulated system of autonomous decision makers (agents). These models are able to generate complex behavior even if only simple assumptions are made about the behavior of the coded agents. That is, these agents interacting with each other in complex ways are able to produce emergent phenomena in the macro structure of the system.

For example, in the late 1960's and early 1970's, Thomas Schilling created computer simulations to try and understand how and why self segregated neighborhoods formed. He coded a virtual environment where agents were given a simple preference, they are only happy if they are not the minority in their neighborhood and will keep moving otherwise (Schelling, 1969). This simple model can illustrate the main tenets of agent based modeling. The implementation we illustrate comes from Thomas Sargent and John Stachurski's 2019 lectures in quantitative economics [5]. First, we have agents who are representative of people in the real world. Their preferences to this respect are simple and easy to understand where they are only "happy" if half of their

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[5]These lectures are a treasure trove of information on how to use Python to construct economic models. The simulation in chapter 3 was built in part using them as a guide.

closest neighbors are the same as them. If they are not happy they will move some-
where else arbitrarily. These preferences can be represented as the short procedure,
or algorithm, shown below where S is just the space they live in:

1. draw a random location in S,

2. if happy at new location, move there,

3. else, go to step 1.

In this case, we get to choose what that environment looks like and like Shelling
we can just say that it is a one by one unit square and we can say that their neighbors
are the ten closest people to them (in Euclidean distance) on that square. When you
run this simulation with green and orange dots representing the types of people you
get the following behavior cycling through each of our 250 agents with the above
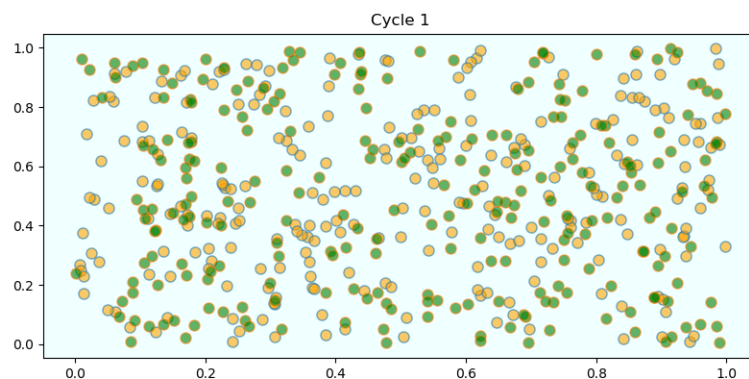procedure until every agent is happy.

Figure 1.4: Schelling's Segregation Model: Cycle 1

As can be seen, in cycle 1 (Figure 1.4), the agents are well distributed among each
other, but as they move in cycles 2-5, they become progressively more segregated.
After 5 cycles all agents are happy and the simulation terminates. With very few
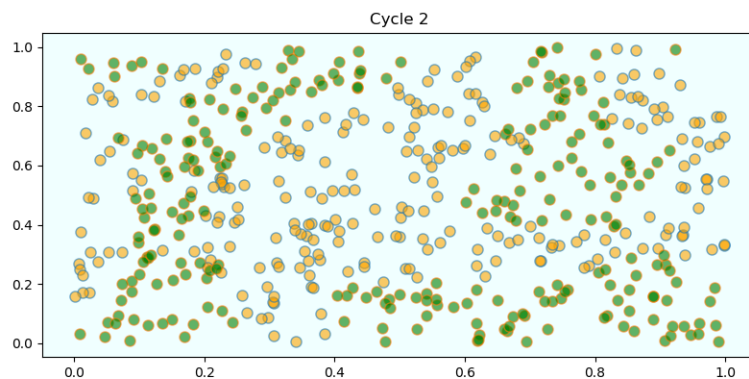
Figure 1.5: Schelling's Segregation Model: Cycle 2
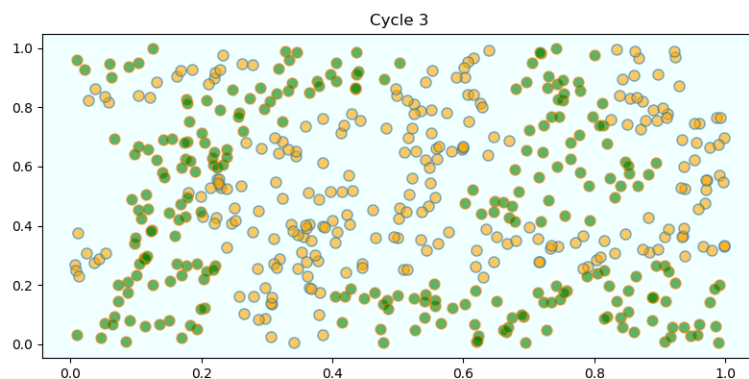
Figure 1.6: Schelling's Segregation Model: Cycle 3

assumptions about the agents' preferences, we can see the resulting emergent behavior of segregated neighborhoods in the system as a whole. Not only that, but the agents naturally move to an equilibrium as they adjust their behavior to what their neighbors are doing (Sargent and Stachurski, 2019).

Agent based modeling has been used to build and understand much more complicated systems then the example illustrated above. The Santa Fe Institute in New Mexico is one of the main proponents of agent based modeling releasing a manifesto
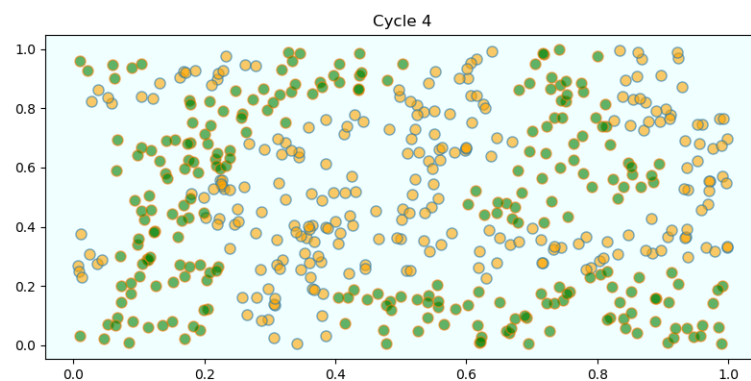
Figure 1.7: Schelling's Segregation Model: Cycle 4
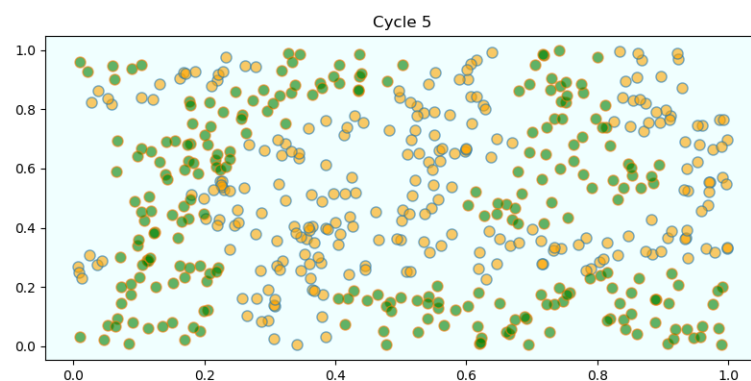
Figure 1.8: Schelling's Segregation Model: Cycle 5

supporting using it to understand the "complexity" of economics from the ground up (Backhouse and Cherrier, 2016). They built the Santa Fe Artificial Stock Market in the 1990's to try and simulate the behavior of agents on the stock market and how their adapting trading strategies effects the outcome of the market. The Santa Fe Artificial Stock Market is one of the first examples of agents learning and adapting to their environments as part of the simulation(LeBaron, 2002). In this simulated market, agents used a genetic algorithm to adapt their trading strategy at each period by

modifying a string (for example 00011100) where each bit in the string told the agent to use a certain behavior or not. Those trading strategies that did well were coded to survive longer where the agents with worse strategies would randomly modify their own or take a more successful agent's strategy (Arthur, 1992). This genetic algorithm was supposed to be representative of the learning of traders on the stock market so that the insights taken from running these agents in simulation could be applied to learn something about the real world.

The authors of Santa Fe Stock Market paper suggested that their genetic algorithm was one of many algorithms that could be used to stand in for human behavior saying that reinforcement learning or deep learning could also be used to stand in for human intelligence in a simulation[6]. More specifically they suggest that an appropriate algorithms can be chosen so long as they behave reasonably well like humans in that scenario, something like a Turing test (Arthur (1991)). Unfortunately, research that has compared the behavior of human agents in strategic settings to that of these algorithms have not found some general algorithm that best approximates human learning. From game to game, different algorithms more appropriately behave like humans who were experimentally playing the same games(Tesfatsion, 2002). Having no algorithm to approximate humans in a simulation is a problem only assuming that you want your agents to in some way represent human behavior; we might simply want our agents to represent the "rational choice" in any given situation. Unfortunately, the learning algorithms we have today aren't necessarily fully rational in the normal economics sense either. While the artificial agents may learn how to maximize their

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[6]Reinforcement learning is when an algorithm plays a game repeatedly and updates its beliefs about what actions will lead to the best payoff. Deep learning uses deep neural networks to try and estimate the best outcomes in a fashion similar to regression.

utility, in general learning algorithms are simply searching the space as best they can rather than computing outright how they should behave. As Holland and Miller say in their 1991 paper "Artificial Adaptive Agents in Economic Theory", "Usually there is only one way to be fully rational, but there are many ways to be less rational." Since they recognize the agents they are creating for their simulation probably fall in the "less rational" camp, they suggest to get around this is to try and build models that have robust behavior across agent learning algorithm choice. If most learning algorithms lead to similar results (for whatever you are trying to prove), we may be fairly confident that we are looking at a property of learning in the system that leads to the result you think you have demonstrated rather than the particular outcome reached by the specific algorithm you choose.

The literature of game theory provides another solution for trying to approximate human learning, no-regret learning. This is a simple learning requirement that means that this kind of algorithm should (hopefully) be more robust across representing human behavior. It also allows us to combine our simulated models and analytical models in a nice way. Because of the well described properties of no-regret algorithms, one can describe and prove the behavior of systems using this behavior in a stronger way than we ever could using a simulation. Since there are proven no-regret algorithms, we can also code this behavior into our simulated agents that might help us understand the system better than just looking at the analytical results.

Moving forward, this thesis aims to better understand the efficiency of auctions by simulating them using no-regret learning agents. The main aim of this simulation will be to demonstrate the price of anarchy bounds proven by Roughgarden et al. (2017), and then to beyond that see how efficient the auctions are. We will do this

with first-price, sealed-bid auction as they are the best understood theoretically and
simple to code (compared to the other auction formats price of anarchy bounds given
in Roughgarden (2007)).

Chapter 2

Price of Anarchy in First-Price Auctions

This chapter lays out the mathematical framework for modeling auctions as Bayesian games of incomplete information, formally defines the price of anarchy in auctions, and shows the bounds on the price of anarchy for first-price, single-item auctions. It then follows up on this framework of analysis to show how these bounds can be extended to agents competing in auctions using learning algorithms with a property called "no-regret" learning. This chapter primarily follows from the work of Tim Roughgarden, Vasilis Syrgkanis, and Éva Tardos who not only are all individually active in the field of algorithmic game theory and auctions, but also jointly authored the 2017 paper "The Price of Anarchy in Auctions" that is a survey of the entire topic. This chapter will be giving the important theorems and ideas that primarily come from this work and the individual work of these three authors, explaining them, demonstrating proofs where appropriate, and reconstructing the important ideas that

we will be using in our simulation.

2.1   Auctions as Bayesian Games

Auctions are typically modeled as Bayesian games, also known as games with incomplete information. As you would expect, this is simply a type of game where the players don't know everything. While it is obvious why an auction consisting of the strategic interaction of bidders could be modeled as a game, how it should be modeled requires some thought. After all, while each bidder knows their own valuation of the item being sold, unless for some reason (and against their own interest) the other players announced their valuation of the item before the auction began our bidder will not know how the other players value the item. In fact, it is precisely this lack of information in sealed-bid auctions that makes them interesting! If all players came into an auction knowing the valuation of other players, for first-price, single-item auctions assuming there was no tie, the bidder with the highest value would always win by bidding the valuation of the next highest bidder. Rather it is the uncertainty that players face about the valuation of other players that make things interesting as players must guess how much they should shade their bid (as again no player will ever bid above their valuation) based off of what they know about other players.

What can we say that bidders know about the valuations of other bidders? Certainly they do not know nothing as we all have reasonable expectations about what some item is worth to others. No one will value a candy bar at a million dollars. But a collector of candy bars might value it at a higher value than an ordinary person who just wants to eat the candy bar. For each bidder, we could say that this bid-

der has a probability distribution from which they are drawing their valuation from that is known to the other bidders. As an example we might expect normal people's valuations of candy bars to be a normal distribution centered at $1, but the collector's value might be a Laplacian distribution (allowing for more black swan events) centered at $5. This sort of strategic interaction where the players know the distributions which parameters of the game are drawn from are typically called *games of incomplete information*, or *Bayesian games*.

2.1.1 Bayesian Games

Bayesian games of incomplete information are games in which one or more of the players lack "full knowledge" of the game that is being played. Introduced by John C. Harsanyi in 1967, rather than players knowing every parameter of the game situation such as utility functions, possible strategies, and information held by other players, each player knows a probability distribution from which these will be drawn. In his paper, Harsanyi says that this type of game can be thought of as a normal game, where "nature" goes first drawing from these probability distributions and assigning values before play begins without the players knowing which specific variation of the game they are playing (Harsanyi, 1967). Importantly, each player does know the probability distributions from which each value is selected. Formally, slightly modifying the definition given in Nisan (2007), Bayesian games are defined as follows,

Definition 2.1 (Bayesian Game). A game with (independent private values and) *incomplete information* on a set of n players consists of:

(a) For each player i, a set of *strategies* S_i, letting $S = S_1 \times \ldots \times S_n$.

(b) For every player i, a set of *types* T_i, and a prior distribution F_i on T_i. A value $t_i \in T_i$ is the private information that i has, and $\mathcal{F}_i(t_i)$ is the a priori probability that i gets type t_i. $T = T_1 \times \ldots \times T_n$ and $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$.

(c) For every player i, a *utility function* $u_i : T_i \times S \to \mathbb{R}$, where $u_i(t_i, s_1, \ldots s_n)$ is the utility achieved by player i, if their type is t_i, and the profile of strategies played by all players is $s_1, \ldots s_n$.

Lets break this definition down before we apply it to auctions. For each player i, there is some set of strategies, or actions that they can take in the game, S_i. These strategy sets need not be finite, and they need not be the same for each player. In rock paper scissors of example $S_i = $ "*rock*","*paper*","*scissors*" for the two players $i = 1$ and $i = 2$. Now, we create a larger set S that is the Cartesian product of each individual players strategy sets. Elements of this set will then be the different possible strategy combinations. So, in the rock paper scissors example we get $S = $ "*rock*","*paper*","*scissors*","*rock*","*paper*","*scissors*", and we can sample $s \in S$ such as $s = ($"*rock*","*paper*"$)$ or $s = ($"*rock*","*rock*"$)$. We call these vectors s, *strategy profiles*, where $s = (s_1, \ldots, s_n)$. Often time game theorists want to understand how a player i would choose their strategy based off of the strategy of others. Hence, as is common in the literature, we let s_{-i} denote a strategy profile with the *ith* element removed, the ordered tuple of everyone else's strategies. In rock paper scissors if $s_{-i} = $ "*rock*", player i would probably would wished they had played paper.

The second part of definition 2.1, is what separates Bayesian games distinct. We have a set of types for each player and some probability distribution over this set that tells how likely each player is to be of a certain type. We say that the probability distribution is *a priori* on the set of types which means that the probability of each

type is known to the other players before the game starts. Here an illustrative example is are card games such as poker or five card draw. The type of each player would be their current hand of cards that they drew from the deck. Players only know their own cards, not the other players. They do however *a priori* know how likely each possible hand is for the other player. In a card game, the probability distribution for types are not independent. If one player draws the ace of spades, the probability that the other player has a hand containing that card is zero (supposing they are using only one deck). To simplify talking about this, we will generally speak about drawing some type t from $\mathcal{F}$.

The third part of definition 2.1 is the utility function, u_i for each player. This is a function that maps the strategies played by each of the players (which gives the outcome of the game), and the type of player they are to a real number, their "utility" for this outcome. We will often write $u_i(t_i, s)$ instead of $u_i(t_i, s_1, \ldots s_n)$ for convenience. u_i can be defined deterministically or probabilistically, for example as is common in the case that both players play strategies that leads to a tie one might let the tie be broken arbitrarily and have the winning player get the utility.

2.1.2 First-Price Auctions

We now move to using the formal structure of Bayesian games to model auctions. Here the types of players will be their valuation, v_i, how much they value the item being bidded on. There is also $\mathcal{F}_i$, the publicly known distribution from which they are drawing their valuation for the item being auctioned. To avoid confusion in the notation, we will generally just speak of players drawing their valuation v_i from their distribution $\mathcal{F}_i$. For example if each of the players was uniformly drawing their

valuation from $[0, 1]$, then their type t_i, would be $t_i = x | 0 \leq x \leq 1$ and $\mathcal{F}_\rangle$ is the uniform distribution over that interval. Hence, in Bayesian auctions bidders know their own valuation of the item being bidded on and the distribution from which each of the other players are drawing their valuations. In a sealed-bid first-price, single item auction if player i wins with bid b_i, we define the utility of the winner to be $u_i = v_i - b_i$, the difference between their valuation and their bid and the losers all get $u_i = 0$ since they did not receive the item. A strategy for a player is a function $s_i \in S_i$ that maps a valuation v_i in support of $\mathcal{F}$ to a bid, $b_i = s_i(v_i)$ (i.e. taking into account the probability distribution for the other players).

We now move on to the idea of equilibrium in this system. In games of complete information the central equilibrium concept is usually a Nash equilibrium, the set of strategies for all players in which each individual player cannot increase their utility by deviating from their strategy fixing the strategy of all the other players. That is, for every player if no one else changes strategy, their best option is to stay where they are, hence an equilibrium. This concept is now updated to give us a Bayes-Nash equilibrium where we must also incorporate the distributions which players are drawing from. For an auction, Bayesian-Nash equilibria are defined as follows by Roughgarden et al. (2017):

Definition 2.2 (Bayes Nash Equilibrium Auctions). a strategy profile, **s** constitutes a *Bayes-Nash equilibrium* if for every player i and every valuation v_i that the player might have, the player chooses a bid $s_i(v_i)$ that maximizes her conditional expected utility where the expectation is over the valuations of the other players, conditioned on a bidder i's valuation being v_i.

This is an equilibria because, for the given strategy profile **s**, every player i is

maximizing their expected utility with their current choice of s_i. Hence, no player has incentive to deviate from their chosen strategy with this profile.

Now, if for a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$, we let

$$
x_i(\mathbf{b}) = \begin{cases} 1 & \text{if player } i \text{ is the winner} \\ 0 & \text{otherwise.} \end{cases}
$$

,

then the utility a player i receives when their valuation is v_i is

$$
u_i(\mathbf{b}; v_i) = (v_i - b_i) \cdot x_i(\mathbf{b}),
$$

and we can say that a strategy profile $\mathbf{s} = (s_1, ldots, s_n)$ is a Bayes-Nash equilibrium if and only if

$$
\mathbb{E}_{\mathbf{v}_{-i}}[u_i(\mathbf{s}(\mathbf{v}); v_i) \mid v_i] \geq \mathbb{E}_{\mathbf{v}_{-i}}[u_i(b'_i, \mathbf{s}_{-i}(\mathbf{v}_{-i}); v_i) \mid v_i]
$$

(Roughgarden et al., 2017). The above equation is one way of symbolically writing definition 2.1 for auctions.

2.1.3 Example Auction

We now turn to an example auction to clarify what has just been laid out. In this example we analyze an auction between two players, Alice and Bob who are bidding on a candy bar where each draw their valuations from the uniform distribution on $[0, 1]$. This is a first-price, sealed-bid auction where they each submit a bid for a

(single) candy bar simultaneously. How are Alice and Bob supposed to decide what to bid on this auction? Supposing that Alice and Bob can calculate a Bayes-Nash equilibrium for this example, and that example is unique, they would know that their best strategy would be to bid their strategy at this equilibrium. But how would our bidders go about calculating the equilibrium? We recreate the proof of the existence of such an equilibrium as from Nisan (2007). An example of proving the uniqueness of this equilibria appears in Levin (2002).

Proposition 2.3. *In a first-price auction among two players with prior distributions distributions of their valuations v_1, v_2, uniform over the interval $[0, 1]$ the unique Bayesian-Nash equilibrium is $\boldsymbol{s} = (s_1(v_1) = v_1/2, s_2(v_2) = v_2/2)$.*

Proof. (Nisan, 2007) First we show that this is a Bayesian-Nash equilibrium. Let us consider which bid x is Alice's best response if Bob uses bidding strategy $s(v_2) = v_2/2$, where Alice's valuation is v_1 and Bob's is v_2. The utility of Alice if she wins is $v_1 - x$ if she wins and pays x, and 0 if she loses. Thus, her expected utility from a bid x is $\mathbb{E}[u_1] = \Pr[\text{Alice wins with bid } x] \cdot (v_1 - x)$, where the probability is over $\mathcal{F}_2$, the prior distribution of v_2. Now, Alice wins if she outbids Bob so, $x \geq s_2(v_2) = v_2/2$. Since v_2 is distributed uniformly in $[0, 1]$ we can calculate the probability she wins with a bid x:

$$\Pr[\text{Alice wins with bid } x] = \begin{cases} 0 & \text{if } x < 0, \\ 2x & \text{if } 0 \leq x \leq 1/2, \\ 1 & \text{if } x > 1/2. \end{cases}$$

We see that the optimal value of x is in range $0 \leq x \leq 1/2$ as $x = 1/2$ gives higher utility than any $x > 1/2$, and since any $x < 0$ will give utility 0. Thus, to optimize the value of x, we find the maximum of the function $2x(v_1 - x)$ over the interval $0 \leq x \leq 1/2$. Taking the derivative with respect to x and setting this equal to zero, we get $2v_1 - 4x = 0$ (noting that the second derivative is negative, and thus this is a maximum). This equation has solution $x = v_1/2$. □

2.1.4 Known Bayes-Nash Equilibria of First-Price Auctions

Examples of the Bayes-Nash equilibria for the single-item auction format have been solved for various choices of the number of players and prior valuation distributions. For example, *symmetric auctions*, where all of the bidders have the same distribution of valuations, with n players Chawla and Hartline (2013) showed that the Bayes-Nash equilibrium strategy profile is the vector is composed of $s_i(v_i) = \frac{n-1}{n}v_i$ for all players i [1]. That is, if each player bids $b_i = s_i(v_i) = \frac{n-1}{n}v_i$, then no one will have any reason to deviate. This is a generalization of what we showed above in the case of two players; if we plug $n = 2$ we get a resulting strategy for each player i ($i \in 1, 2$) of $s_i(v_i) = v_i/2$. So, as the number of players increases, each player must shade their bid less and less, bidding closer and closer to their true valuation. The Bayes-Nash equilibria for symmetric auctions are *efficient*, meaning that the item will always be allocated to the player with the highest valuation. Efficiency is what economists want out of their auctions. When the item is always awarded to the person who values the item the most, this is seems like a reasonably fair way to allocate the item, which is what economics is all about! However, in general efficiency is not guaranteed at

---

[1]This is a bit awkward to parse, but it is worth emphasizing that the Bayes-Nash equilibria is the strategy profile as a whole.

Bayes-Nash equilibria of first-price auctions. For example if we conduct an auction with two bidders, one choosing from the uniform distribution $(0, 1]$ and the other from the uniform distribution $(0, 2]$ as demonstrated by Krishna (2002), the Bayes-Nash equilibrium for this auction is:

$$s_1(v_1) = \frac{4}{3v_1} \left(1 - \sqrt{1 - \frac{3v_1^2}{4}} \right)$$
$$s_2(v_2) = \frac{4}{3v_2} \left(\sqrt{1 + \frac{3v_2^2}{4}} - 1 \right)$$

Here bidder one (who has the smaller prior valuation distribution) knows that bidder two is more likely to have a higher valuation than them. Thus, player one must bid higher relative to their given valuation if they expect to win, and so they shade their bid less than bidder two. This can lead to bidder one drawing a lower valuation than bidder two, but outbidding them regardless and winning the item. For example if $v_1 = 1$ and $v_2 = 1.1$, then $s_1(1) \approx 0.667$ and $s_2(1.1) \approx 0.462$. This is inefficient since bidder one wins the item even though they have a lower valuation. From an economists perspective this presents a problem, sometimes the structure of this market leads to inefficient outcomes, and sometimes it doesn't. How can we figure out how inefficient this market structure is, or happy optimal we expect the outcomes to be for this structure.

Unfortunately, it has been shown that solving many of these asymmetric Bayes-Nash equilibrium requires finding a solution to a system of partial differential equations which have no closed-form solution (Roughgarden et al., 2017). Thus, even with

infinite time we could not got through and compute for every possible combination of the number of players and valuation distributions if they lead to efficient outcomes. Having no closed form solution also suggests it may be unreasonable to expect bidders to "do their homework before an auction" so that everyone can show up and just start bidding their strategy for the Bayes-Nash equilibria. Thus, it would seem hard to characterize the efficiency outcomes of firs-price auctions even though it seems like such a simple format. However, the price of anarchy literature gives us a workaround. Instead of spending the time to characterize the efficiency of (solvable) Bayes-Nash equilibria, we can find a bound on the worst cast equilibrium outcome!

2.1.5   Price of Anarchy in First-Price Auctions

To try and get a sense of how inefficient first-price auctions can be, computer scientists have been applying a concept known as the price of anarchy (see definition **??** below), one way to characterize systems at equilibrium[2]. The price of anarchy is a way to compare the social welfare of a system or a game at its best possible value to that of its worst possible equilibrium under strategic play. Before moving on, we must mathematically define what we mean by "social welfare" if we are going to In the case of an auction, the social welfare is the sum of the utilities of the players, $u_i(\mathbf{s}, v_i) = (v_i - b_i) \cdot x_i(b_i)$, plus the revenue of the auctioneer, $\sum_i^n b_i \cdot x_i$.

Definition 2.4 (Social Welfare)**.** The *social welfare* of a bid profile **b** when the

---

[2]Not that economists were uninterested in these questions of efficiency and welfare analysis in auctions before the computer scientists started studying them.

valuation profile is $\mathbf{v} = (v_1, \ldots, v_n)$ is,

$$SW(\mathbf{b}; \mathbf{v}) = \sum_{i=1}^{n} v_i \cdot x_i(\mathbf{b}).$$

??

The price the winning bidder pays does not appear in this equation since the winning bidder is paying exactly as much as the auctioneer is getting, and this term cancels out. Hence, the social welfare is maximized when the bidder with the highest valuation wins the auction. If we let $x_i^*(\mathbf{v})$ be an indicator variable for whether or not a player i is the player with the highest valuation (ties broken arbitrarily), the maximum possible social welfare in a single-item auction is

$$\text{OPT}(\mathbf{v}) = \sum_{i=1}^{n} v_i \cdot x_i^*(\mathbf{v}).$$

Now that we have mathematically described the social welfare in our system, we can define the price of anarchy.

Definition 2.5 (Price of Anarchy). The *price of anarchy* of an auction, with a valuation distribution $\mathcal{F}$, is the smallest value of the ratio:

$$\frac{\mathbb{E}_{\mathbf{v}}[SW(\mathbf{s}(\mathbf{v}); \mathbf{v})]}{\mathbb{E}_{\mathbf{v}}[\text{OPT}(\mathbf{v})]}, \tag{2.1}$$

ranging over all Bayes-Nash equilibrium $\mathbf{s}$ of the auction.

The above definition applies to individual auctions which are dependent on the choice of $\mathcal{F}$ and n. We generally only discuss the price of anarchy for the format of

the auction which in our case is first-price single-item auctions. The price of anarchy for the first-price auction format is then the worst possible price of anarchy for any choice of the number of players n or valuation distributions $\mathcal{F}$. Note that the price of anarchy (for either an individual auction or the format of auction) is a number between 0 and 1, and that the closer it is to one, the "better" we can guarantee the system's social welfare will be[3].

Incredibly, bounds on the price of anarchy for the format of first-price auctions have been found. Again, this allows us to characterize how much worse the social welfare for the system could be at (Bayes-Nash) equilibrium no matter how many players we have or what distributions they are choosing their valuations from. This sort of guarantee is incredible, especially for systems where we may not want, or it may not be feasible to have a central authority pre-calculate the way to optimize social welfare in a system. Rather, we can trust that the system will perform at least so well under strategic interaction.

Theorem 2.6. *(Syrgkanis and Tardos, 2013) The price of anarchy in first-price single-item auctions format is at least* $1 - \frac{1}{e} \approx 0.63$

Theorem 2.6 tells us that no matter how many players we have or what weird distributions we try and give them, we cannot construct a first-price auction that will achieve less than 0.63% of the optimal social welfare at Bayes-Nash equilibrium (if it exists).

---

[3]Much of the literature for POA (including the paper introducing the idea) defines it as the opposite ratio, Optimal/Worst-EQ where smaller values indicate better systems (Koutsoupias and Papadimitriou, 1999). For some reason the auction literature defines it as Worst-EQ/Optimal, so I will remain consistent with them. This is confusing as *price* of anarchy makes you think it should be a number that gets bigger as it gets worse.

2.2 Extending Results to No-Regret Agents

These results hold for simple first-price auctions, but it is natural to ask questions about how robust these results are. First of all, how do people arrive at a Bayes-Nash equilibria if there doesn't exist a closed form way to express it? Secondly, do these results hold for mixed Bayes-Nash equilibria (randomizing between bidding strategies) or other larger, more realistic equilibrium concepts? To address these questions, Roughgarden et. al use a set of extension theorems that take us through a general mechanism design setting and allow the class of equilibria our bounds hold for to be expanded. This extension will take us to a concept of no-regret learning. We briefly sketch key theorems from this thesis that lead us to an equilibrium concept that applies to learning agents.

2.2.1 General Auction Mechanisms

In order to understand (or even state) the proofs and theorems required to take us to our expanded set of equilibria, we must introduce more notation and the idea of a general mechanism design setting and a general auction setting (this is unfortunate for us due to an expansion of scope, but actually makes these theorems quite powerful!). Mechanism design can be thought of as reverse game theory. Instead of thinking about how to play a game to maximize your utility, you are now thinking about how to design a game so that rational agents will behave in a certain way. This influence on players is generally exerted through payoffs and cost to play in the game. While we will not be spending much time with this math, it is useful to note that this framework reflects an ability for the auctioneer to fine tune how the game is

played. In a general mechanism design setting, the auctioneer solicits an action a_i from all of the players, i, from some action space $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Given an action profile $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}$, the auctioneer decides an outcome $o(\mathbf{a})$ among the set of possible outcomes $\mathcal{O}$. This outcome includes a payment $p_i(o)$ that each player must give to the auctioneer. Denote the revenue of the auctioneer $\mathcal{R}(O) = \sum_i p_i(o)$. Players receive some utility as a function of their valuation, v_i and the outcome which we write $u_i(o; v_i)$. Let $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$.

2.2.2 Smooth Auctions

Smooth auctions are a way of describing auctions in the general mechanism design framework that allow all possible deviations by players to be better accounted for in the math. The details of this are omitted from this paper, but for the complete details on this see Roughgarden et al. (2017). The definition of a smooth auction is as follows, where D_i^* is a new concept of an action distribution that is just a priori distribution over the belief about what actions the other players will take

Definition 2.7 (Smooth Auctions). For parameters $\lambda \geq 0$ and $\mu \geq 1$, an auction is (λ, μ)-smooth if for every valuation profile $\mathbf{v} \in \mathcal{V}$ there exist action distribution $D_1^*(\mathbf{v}), \ldots, D_n^*(\mathbf{v})$ over $\mathcal{A}_1, \cdots, \mathcal{A}_n$ such that for every action profile $\mathbf{a}$,

$$\sum_i \mathbb{E}_{a_i^* \sim D_i^*(\mathbf{v})}[u_i(a^*, \mathbf{a}_{-i}; v_i)] \geq \lambda \text{OPT}(\mathbf{v}) - \mu \mathcal{R}(\mathbf{a})$$

Now, following from the proof of theorem 2.6, Roughgarden et. al show that first-price, single-item auctions are $(1 - \frac{1}{e}, 1)$-smooth. This is done by using the last step of

their proof of theorem 2.6. They then go on to develop a series of extension theorems that allow us to characterize characterize the price of anarchy for equilibria of auctions when they are not just for pure-strategies. Specifically we will be looking at how these can be extended a set of outcomes for agents who learn to play the game as they go.

2.2.3   No-Regret Learning

Consider an auction with n players that is repeated for T time steps. At each iteration t, bidder i draws a valuation v_i from $\mathcal{F}_i$ and chooses an action a_i^t which can depend on the history of play. After each iteration, the players observes the actions taken by the other players. A player i is said to use a no-regret learning algorithm if, in hindsight their average regret (difference between average utility of strategy vs algorithm) for any alternative strategy $a_i^{'}$ goes to zero or becomes negative as $T \to \infty$. When all players use this algorithm it results in a vanishing regret sequence.

Definition 2.8 (Vanishing Regret)**.** A sequence of action profiles $\mathbf{a}^1, \mathbf{a}^2, \ldots, a^T$ is a *vanishing regret sequence* if for every player i and action $a_i^{'} \in \mathcal{A}_i$,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (u_i(a_i^{'}, \mathbf{a}_{-i}^t; v_i) - u_i(\mathbf{a}^t; v_i)) \leq 0$$

Theorem 2.9 (Roughgarden (2007))**.** *If an auction is (λ, μ)-smooth, then for every valuation profile $\boldsymbol{v}$, every vanishing regret sequence of the auction has expected welfare at least $\frac{\lambda}{\mu} \cdot OPT(\boldsymbol{v})$ as $T \to \infty$.*

The end result is that since single-payer first-price auctions are $(1 - \frac{1}{e}, 1)$ smooth, we know that this bound will hold for auctions with no regret learning agents. Thus,

we will try and build a simulation that uses one of the algorithms that fulfills this property and demonstrate that for any arbitrary number of players and distributions from which they pick their valuations from, they converge to an equilibrium social welfare greater than $1 - \frac{1}{e}$.

Chapter 3

Simulating the Price of Anarchy

This chapter constructs a novel simulation of first-price single-payer auctions to demonstrate that the ratio of actual social welfare to optimal social welfare is within the bounds given by the price of anarchy for this format. First, the construction of this simulation is discussed. Next, we demonstrate the results of running the simulation on bidders using known Bayes-Nash equilibrium strategies and an arbitrary bidding strategy. Finally, we demonstrate that these bounds also hold for no-regret learning agents in a fixed action variant of the first-price auction.

3.1   A Simulated Auction Environment

To simulate sequential first-price auctions, we construct a program in python that allows us to create an arbitrary number of bidders, each with valuation distributions of our choice who simultaneously bid on an item being auctioned for as many sequential auctions, or rounds, as we choose. Each round represents an auction for a new, but similar item where the valuation distributions for each bidder remains the

same. At the beginning of each round, every bidder draws a new valuation from their distribution. The bidders then each simultaneously submit a bid to the auction and the winner is determined by the highest bid (ties are broken with equal probability among those with the same bid). After the winner is selected, they are given utility $u(v, b) = v - b$, the difference between their valuation and bid that round. At each round the total social welfare is v_i, the valuation of the winning bidder as per definition 2.4. The optimal social welfare each round then is the highest valuation of any bidder. These values are summed across auctions to get the total social welfare for this sequence of auctions and to see what the optimal social welfare would have been. This is laid out in the pseudo-code version of the sequential auction below where we use the super script t to denote which round each variable is from[1].

---

[1]The "←" symbol used in the algorithm means assignment of value. For example $x \leftarrow 1$ is the variable x is assigned a value of 1. This reduces the ambiguity of using the "=" symbol which could be also be a statement or proposition in pseudo-code.

Algorithm 1: Sequential First-Price Single-Item Auction

Initialize SW, OW, and POA to zero

for $t = 1, \ldots, T$ **do**

  Each bidder draws their valuation v_i^t from their distribution F_i;

  Each bidder uses their strategy s_i^t to submit a bid;

  The highest bidder is assgined the object and they pay their bid, if tie, a

   winner is choosen randomly among them;

  Each player has their utilites updated according to if they won the object;

  if *player i wins the auction* **then**
  $\quad | \quad SW \leftarrow SW + v_i^t$

  if *player j has the highest valuation* **then**
  $\quad | \quad OW \leftarrow OW + v_j^t$
  $POA \leftarrow \frac{SW}{OW}$

3.1.1   Simulating Bayes-Nash Equilibria

Using the simulation outlined above, we first demonstrate that bidders using known Bayes-Nash equilibrium strategies have an average price of anarchy greater than 0.63. First, we simulate the case of two bidders each drawing their valuation from the uniform distribution $[0, 1]$. We are using the hard coded strategies that $s(v_i^t) = \frac{v_i^t}{2}$ for each bidder which was shown to be the unique Bayes-Nash Equilibrium in chapter 2. The results are shown in table 3.1 below, where the cumulative price of anarchy is given up to the specified round specified [2].

This example is obviously silly to simulate since this equilibrium is fully efficient. As each player bids half their valuation, the winner is always the player with the

---

[2]The POA is calculated as per algorithm 1.

Round	POA
1	1.0000
10	1.0000
100	1.0000
1,000	1.0000
10,000	1.0000
100,000	1.0000

Table 3.1: Price of anarchy in two player symmetric auction

highest valuation. Hence the actual social welfare is always the same as the optimal social welfare:

$$\frac{SW(\mathbf{s}(\mathbf{v}); \mathbf{v})}{\text{OPT}(\mathbf{v})} = 1.$$

This does however give us a simple check of correctness for our simulation. We now move to simulate the more interesting example of two bidders with asymmetric distributions. We let bidder one chose their valuation from the uniform distribution over the interval $[0, 1]$ and bidder two chooses their valuation from from the uniform distribution on the interval $[0, 2]$. As stated in chapter 2, the unique Bayes-Nash equilibrium for this auction is:

$$s_1(v_1) = \frac{4}{3v_1}\left(1 - \sqrt{1 - \frac{3v_1^2}{4}}\right)$$

$$s_2(v_2) = \frac{4}{3v_2}\left(\sqrt{1 + \frac{3v_2^2}{4}} - 1\right)$$

Again, this auction is not fully efficient as the bidder who is drawing their valuation

from the smaller distribution has to shade their bid less (bid higher relative to their valuation) in this equilibrium which can sometimes result in the person who values it less winning the item. We simulate this sequentially 100,000 times and get the following results as shown in table 3.2:

Round	POA
1	1.0000
10	1.0000
1,000	0.9919
10,000	0.9924
100,000	0.9935

Table 3.2: Price of anarchy in two player asymmetric auction

While this auction is not fully efficient, it is highly efficient. The price of anarchy in this auction never drops below 0.99. The main reason this happens is that while it is possible for the bidder drawing from the smaller distribution to win even if they have the smaller valuation, this only occurs when their two valuations are relatively close. This means that the social welfare lost in this case is not much, even if it is not fully efficient. In both of these auctions we see that they are well above the 0.63 lower bound guaranteed for all first-price single-item auction formats. The fact that they perform so well is not surprising given that this is such a simple system. e would expect people who are rationally maximizing their own utility to mostly win if they have the highest bid. If anything, this simulation highlights the need to find an exact lower bound on the price of anarchy for Bayes-Nash equilibria as it would be more surprising if we could find an instance of this auction that produces highly inefficient outcomes.

3.1.2   Minimal Intelligence Bidders

Before going on to simulate the auction using no-regret bidders, we first move to establish a baseline for how well each auction setting (based on number of bidders and distribution choice) performs with agents that are not learning. The simulations of Bayes-Nash equilibria in the previous sections indicate that we can generally expect this system to perform well, but how well? How well will this system perform with even arbitrary bidders?

To try and answer this question, we construct agents that bid randomly between zero and their (current) valuation every round i.e. each players bid is chosen uniformly from the distribution $[0, v^t]$. We call these agents minimally intelligent since they are not overbidding, but they are also clearly using a nonsensical strategy for an auction. One should note that while this is a bad strategy, it is possible to formulate much worse strategies for our bidders from both utility maximization and efficiency standpoints[3]. This in some way represents agents who are incapable of learning (or doing their homework before they arrive) and thus continually randomly select from all options. The results for simulating sequential first-price auctions with 2, 10, and 100 agents using this strategy are shown in the table 3.3 below.

Table 3.3 shows how the price of anarchy evolves through the rounds as the number of rounds goes to 1,000. Since the agents are not learning anything and the bids are randomly sampled each time, these efficiency results should approximately converge to the expected value as $T \to \infty$. To better illustrate the relationship between price of anarchy in each of these sequential auctions and the number of bidders, we graph

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[3]It's rather fun to think up such strategies! For example, if each bidder shaded little when they had a low draw and shaded a lot when they had a high valuation, this can lead to highly inefficient outcomes

Round	2 Agent POA	10 Agent POA	100 Agent POA
1	0.9000	0.8410	0.9859
10	0.9312	0.8987	0.9419
1,000	0.9279	0.8812	0.9480
10,000	0.9150	0.8880	0.9467
100,000	0.9164	0.8887	0.9470

Table 3.3: Price of anarchy in two player asymmetric auction

the above simulation now run on 2 to 100 bidders. Each of these simulations is run 100 times, and the min, max, and average POA are graphed below in figure 3.1.
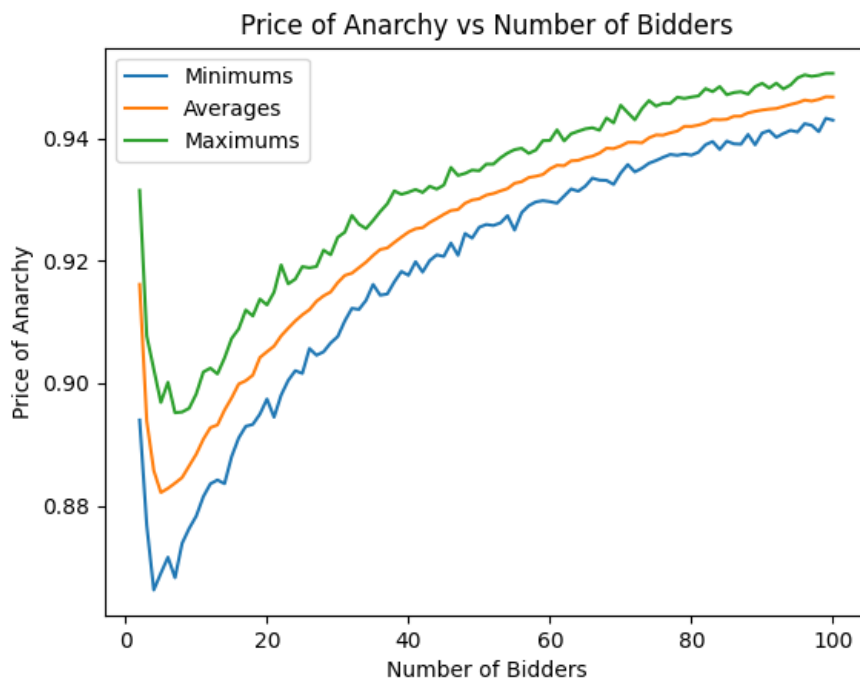

Figure 3.1: POA for 2 to 100 symmetric bidders

We see three things, one is that when all bidders are drawing from the same uniform distribution the market is incredibly efficient regardless of how smart the bidders are when choosing their strategy. Even with arbitrary strategies this auction

still performs remarkably well. The second thing to note is that in this setting, as the number of bidders increases the social welfare also increases. This makes sense as we would expect the probability of a reasonably high valuation winning to increase as there are more valuations per round. The third thing we notice, and most surprising of all, is that the increase in the price of anarchy as the number of players increases is not monotonic. At lower level of bidders, the efficiency actually decreases as we add more bidders. This seems to be because the probability that one player outbids another with a significantly higher valuation increases as you add players when you only have a few to begin with. Still though, the probability of someone with a high valuation winning is quite high and we see the price of anarchy approach 1 as n goes to infinity. This is exactly what we expect as if we had a near infinite number of bidders, one of them would almost certainly bid arbitrarily close to their own valuation.

We now conduct a similar simulation but with asymmetric bidders. In this case, half of the bidders are drawing uniformly from $[0, 1]$ and half from $[0, 2]$. The results are shown in table 3.4.

Round	2 Agent POA	10 Agent POA	100 Agent POA
1	0.1.000	1.0000	0.9726
10	0.9115	0.8178	0.9331
1,000	0.9054	0.8658	0.9310
10,000	0.9092	0.8630	0.9297
100,000	0.9112	0.8640	0.9304

Table 3.4: Price of anarchy in two player asymmetric auction

Again we see that the market is quite efficient in this case regardless of how smart the bidders are. We again also see that it seems to converge to optimal (i.e. to 1) as the number of bidders increases but non-monotonically. We simulate this for 2 to

100 bidders again 10 times at each number of bidders with $T = 1,000$, where on odd numbers we allow the extra bidder to have distribution $[0, 1]$. This is shown is figure 3.2.
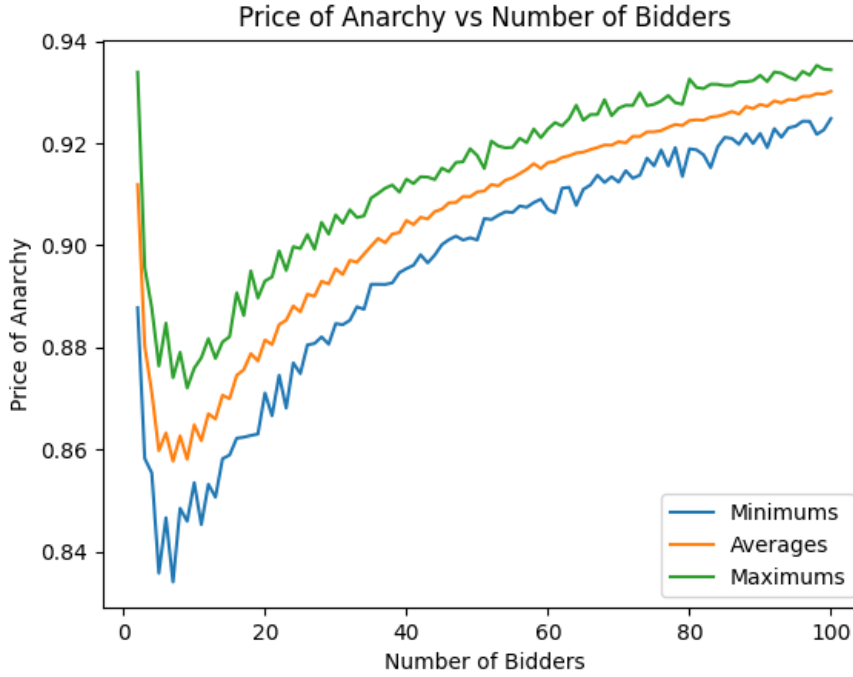

Figure 3.2: POA for 2 to 100 asymmetric bidders

Note that the baseline price of anarchy for minimal-intelligence agent auctions changes as we change the distributions. While the format seems quite efficient and well above the price of anarchy bounds (which again are only guaranteed for equilibria, which this is not) even with agents following these minimally intelligent strategies, we have no guarantee that there is not some way to construct this so that these agents wouldn't do much worse. Given that it seems as the number of agents increases, the efficiency also seems to increase, one would expect that this would need to be done by picking more interesting distributions for the bidders to choose from and only having

two bidders.

3.2 Simulating No-Regret Bidders

With the results from the Bayes-Nash equilibrium demonstrated and an efficiency baseline established, we move on to simulating the bounds of no-regret learning agents in first-price auctions. Again, this is the equilibria we expect auctions to converge to if each agent is using a no-regret learning algorithm. To use these algorithms we do however have to make a concession to the environment we are simulating: we must now simulate an auction where the bidders are only allowed a finite number of actions.

3.2.1 Multiplicative Weights Algorithm

The no-regret learning algorithm we will use to train our bidders is called the multiplicative weights algorithm. It has been shown to satisfy the no-regret property in papers such as Littlestone and Warmuth (1994) or Freund and Schapire (1999), but our implementation of it comes from Roughgarden (2016). If each of our bidders use the no-regret learning algorithm our auction should converge to an outcome greater than the 0.63 price of anarchy bound from chapter 2.

Before giving the algorithm, a few words are probably necessary to understand where it comes from and what it is trying to do. First, this algorithm is what is called an *online* algorithm. That is an algorithm that takes its inputs sequentially as it goes rather than getting all of its inputs up front. Next, this algorithm and many other algorithms for players learning in games are based around the player only

having a fixed number of actions they can choose from. For each round in the game, the player gets outside advice from "experts" who recommend to the player what to do at each round and the player picks among them to decide what to do. For us, these experts are our strategies that will map a valuation to a bid. At each time step t, the player picks the action to play and then after that, some adversary picks the utilities to assign for each action that could have been taken. This is a stronger condition than we will need as the adversary in a first-price auction is the cumulative action of the other players, where the highest bid determines which strategies (if any) the player could have taken and won. However, in the general case this algorithm has been shown to be no-regret in the face of an adversary directly picking the utilities the learning agent receives (see Lugosi (2006) for a broad overview of prediction with expert advice and learning in games).

Algorithm 2: Multiplicative Weights (MW) Algorithm

Initialize $w^1(a) = 1$ for every $a \in A$

for $t = 1, 2, \ldots, T$ **do**

    Use distribution $p^t = \frac{w^t}{\sum_{a \in A} w^t(a)}$ over actions to pick $a \in A$ and output a.

    Given the utility vector u^t, for every action $a \in A$ use the formula

    $w^{t+1}(a) = w^t(a) \cdot (1 - \eta u^t(a))$ to update its weight.

The logic of this algorithm is simple. At each time step we see how well each of the possible actions performed and increase the weight, or probability of selecting that action in the future. This increase is done proportionally to how well the action did as determined and *learning rate*, η, is chosen before starting the procedure. As demonstrated in Roughgarden (2016) and Blum and Mansour (2007), the MW algo-

rithm is no-regret if $\eta = \sqrt{(\ln n)/T}$ where n is the number of actions that this agent

can choose from, and T is the number of rounds that will be played (yes, this assumes

that the player knows that up front).

   This algorithm fulfills our purpose of simulating agents learning in auctions, but

in some sense it is unsatisfying that the learning algorithm requires a finite set of

actions that the bidder does not even get to choose. It would be more interesting

if we were able to give our agents some reasonable algorithm that allowed them to

formulate their own strategies or mappings between their valuation and bid rather

than choosing from a pre-made set. This would introduce a whole host of other

problems from the reasonableness of expecting agents to implement such algorithms

to the ability to prove that such algorithms converge to an equilibria. Part of the

beauty of using multiplicative weights is that it is simple and has nice mathematical

properties. Using more interesting learning techniques these properties might no

longer hold and then we would have to wonder what our simulation is really showing[4].

3.2.2   Uniform Distribution Simulations

The first simulation we run is a repeat of the symmetric auction setting, but now using

agents learning with the multiplicative weights algorithm. We create 101 strategies

that bidders can choose from to shade their bid, from bidding zero percent of their

valuation with one percent increases up to bidding 100 percent of their valuation,

$S = \{0, 0.01 \cdot v_i, 0.02 \cdot v_i, \ldots, 0.99 \cdot v_i, v_i\}$. First, we run the simulation with two

symmetric bidders each choosing their valuation from the uniform distribution over

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[4]This thesis grew out of an interest in doing just that, throwing "smarter" algorithms such as
neural networks and machine learning into existing multi-agent simulations. It becomes hard to tell
what the point of such simulations are when the dynamics might just be properties of the interaction
of the specific algorithms used and not of the system itself.

$[0, 1]$. The results are shown below in table 3.5.

Round	POA
1	1.0000
10	0.9517
1,000	0.9129
10,000	0.9578
100,000	0.9947

Table 3.5: Price of anarchy in two player asymmetric auction with no-regret learning

Here we can see that after 100,000 rounds the price of anarchy converges to 0.99 and near perfect efficiency as the agents learn how to play the game. It's important to point out here that the above table is not an average, but simply one run of the simulation. Since each time we run the simulation it is possible for the bidders to learn to converge to a new equilibrium, the POA values for each simulation can be different. While we are mainly concerned with the lowest possible POA that the system converges too as a demonstration of the proven bounds, it is also interesting to see what the average and best case simulations resulted in, as these will give us more confidence about the robust efficiency of the system. We now repeat this using 2 through 100 agents, each symmetric and drawing from a uniform $[0, 1]$ as above. We run 100 simulations with each number of agents, each for 1,000 rounds[5]. This gives us the following results as shown in figure 3.3

The efficiency in the minimum case with our learning agents is about as good as the best-case with the random guessing agents. It makes sense that the efficiency would increase as we would expect people with higher valuations to win more often as people learn to minimize their regret. It is theoretically possible in some games that

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[5]This is a relatively small number, but necessarily chosen for the sake of computation time with 100 agents who learn.
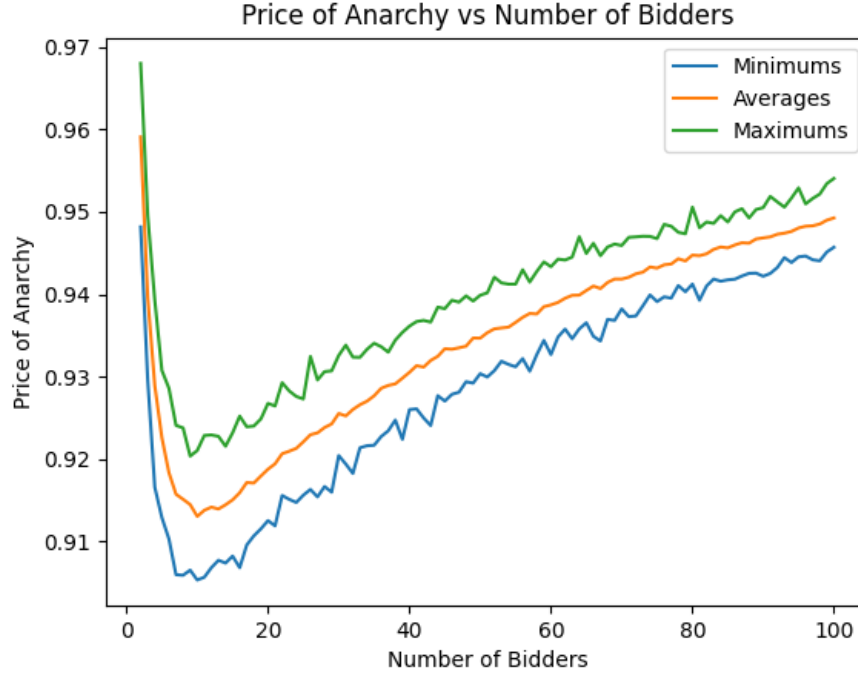
Figure 3.3: POA for 2 to 100 symmetric, no-regret bidders

social welfare could be better off with everyone choosing arbitrary strategies than under learning agents, but we see in this game, and with our learning agents, that is not the case.

Finally, we simulate the case of having two bidders with asymmetric valuation distributions where one draws uniformly from $[0, 1]$ and the other draws from $[0, 2]$. We can see the result of simulating this 100,000 times in table 3.6 below.

Here we see the agents converge to a very good efficiency similar to the behavior we saw when using the Bayes-Nash equilibrium for this setting and better than in the two bidder minimum-intelligence case. Now again we conduct this simulation with 2 to 100 asymmetric bidders. For each number of bidders we simulate this 100 times and when there are an odd number of bidders there is an extra $[0, 1]$ bidder. Each

Round	POA
1	1.0000
10	0.8901
1,000	0.9314
10,000	0.9770
100,000	0.9961

Table 3.6: Price of anarchy in two player asymmetric auction with no-regret learning

sequential auction consists of 100,000 rounds. The results are shown in figure 3.4.
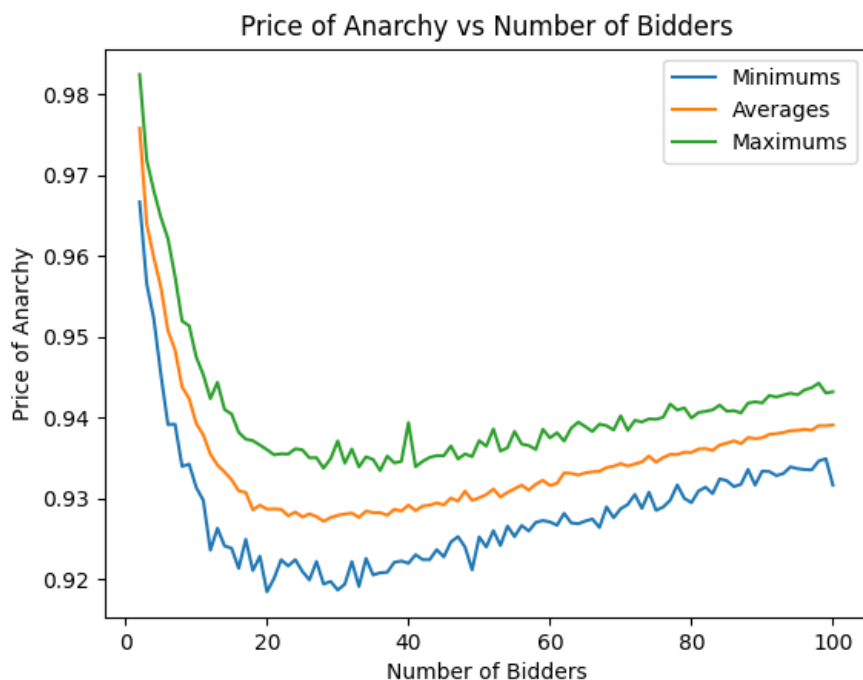

Figure 3.4: POA for 2 to 100 asymmetric, no-regret bidders

Figure 3.4 is similar to the figures we have seen for every other auction. We see very high efficiency for two or three bidders, and then see a dramatic decrease as it goes to 9. We then see the numbers slowly climb back towards being fully efficient in the best, worst, and average cases. Comparing the results of our simulations, we

see that while the asymmetric learners were more efficient for lower levels of bidders, they do not approach fully efficient in the same way symmetric no-regret bidders do as n grows. Rather, it starts to level off faster. This seems reasonable based off of our expectations and similar results with minimally intelligent bidders.

Conclusion

Our simulation demonstrated the efficiency greater than the price of anarchy bounds with no-regret bidders. The price of anarchy theorems that give the bounds for the efficiency of the system tell us that these should hold no matter what number of bidders and what distributions we give them. By all indications with both learning, and non-learning agents first-price single-item auctions are incredibly efficient. It makes one wonder how low one could get the simulated price of anarchy to be. It also begs for a better proof giving the exact lower bound on the price of anarchy for no-regret learners. Combining the results from the learning and non-learning agents we should feel incredibly confident in the efficiency of first-price auctions implemented in the real world. It seems like they behave remarkably well as a way to allocate resources to those who value them the most.

Moving forward using simulations to understand the price of anarchy in auctions, there are two obvious ways to extend this research. One is to build a simulation for simultaneous first-price auctions or all-pay auctions (where everyone must pay regardless of winning) where the price of anarchy bounds for no-regret agents have also been proven using smooth-auctions (definition 2.7) (Roughgarden et al., 2017). For simultaneous auctions, or any sort of auction where bidders must make many decisions

at once (in this case bids), simulating no-regret learning becomes much harder as the number of actions increases. The simulations presented here were already slow, its hard to imagine how long it would take to run a simulation of such a procedure. The other obvious extension is to try simulating this again using different algorithms. Specifically we could try different no-regret algorithms and see if they converge to similar efficiency outcomes, but it might also be interesting to try modern machine learning techniques. Again, if one uses techniques that don't allow us to bound the efficiency of their outcomes, it can be hard to know how sensitive the results are to particular parameters or what it might possibly do in every scenario. That is why no-regret learning is so nice as a stand in for humans. It is simple to implement, it is well behaved, and it is boundable. Hence, we should also expect human agents learning in auctions to make choices that lead to mostly efficient outcomes.

Appendix A

Simulation Code

The code for this simulation is available at the public GitHub repository:

https://github.com/15rsirvin/Computational-Economics

in the sequential auction folder. There are different python files for each of the simulations where the relevant parts are changed to make them minimally intelligent bidders or asymmetric. The simulation is coded in python and mostly consists of loops to sequentially run each auction, update each bidder, and then allow each bidder to learn. We run each simulation in parallel on a different processor to try and speed up the process, but is is still quite slow for large numbers of simulations, rounds, or bidders. That is because this process is essentially $O(T \cdot n \cdot m)$, where T is the number of rounds, n the number of bidders, and m is the number of simulations at that number of bidders.

We ran our code on a google cloud platform "compute engine" server to speed up the simulation process with a faster computer and to allow the simulation to run continuously. This enabled us to choose high values of T, m, and n, despite how long

the code takes to execute with such values.

If someone were to try and replicate parts of this thesis, I would recommend re-writing the code in a faster language like C or C++. Because this code scales linearly with any individual input, optimizing the speed at which the loop executes should have a massive effect on the total time it takes to compute. It might also be possible to algorithmically optimize this simulation so that it can scale sub-linearly. Ultimately we decided not to spend to much time optimizing the code as letting it run on a remote server was not a huge issue.

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