Combinatorial Optimization & Quantum Computing

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Abstract

In this paper, an introduction to combinatorial problems, optimization, and quantum computing are discussed as well as their overlap and relation to finance. Quantum speedups, which compare a quantum and classical computer's time to solve a given problem algorithmically are explained, in addition to how quantum computers have the ability to outperform its classical counterpart. The idea of Quantum Machine Learning is also discussed and how it can be leveraged to solve combinatorial problems.

1 Introduction

1.1 Combinatorial Optimization

Combinatorial optimization is a sub-field of mathematical optimization that is related to algorithm theory and computational complexity theory. It plays an important role in several fields including, artificial intelligence, machine learning, and applied mathematics.

Combinatorial optimization is the process of searching for maximums or minimums of an objective function (cost function) whose domain is a large and discrete configuration space [9]. In the domain of these problems, the space of possible solutions is typically too great to search using exhaustive search methods. In many cases, there does not exist an exact algorithm that is feasible to find the optimal solution. As a result, randomized search algorithms such as random-restart hill climbing, simulated annealing, and genetic algorithms must be employed [9]. The results of these algorithms cannot be confirmed to be optimal as they run into computational and time limitations. Some well known examples of combinatorial optimization problems include the knapsack problem, the traveling salesman problem (TSP), and the Job-shop scheduling problem. For instance, the travelling salesman problem is defined as, given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the original city? [9].

TSP belongs to the class of combinatorial optimization problems known as NP-complete. It has no “quick” solution and the complexity of calculating the best route will increase as more destinations are added to the problem [9]. TSP can be solved by analyzing every round-trip route to determine the shortest path. However, as the number of destinations increase, modern computers reach their computational limit. With 10 destinations, there can be more than 300,000 round-trip permutations and combinations. With 15 destinations, the number of possible routes could exceed 87 billion [9].

In portfolio asset selection and modern portfolio theory, the objective is to maximize expected returns based on a given level of market risk. The question of asset selection while minimizing risk is a combinatorial one [7]. The process of constructing a portfolio in modern portfolio theory involve calculating the desired asset allocation or asset mix. By using a combinatorial optimization algorithm, an optimal combination of investments that minimize risk while maximizing returns can be found.

1.2 Quantum Computing

Quantum computing (QC) works in a similar overall setup to a classical computer. However, due to the small scale it is built on, QC uses properties like superposition of states and quantum entanglement to perform computations. Classical computers use bits to store data and perform calculations while QC uses qubits (quantum bits). Qubits are essentially the binary units of quantum computing that are in a superposition of states as opposed to a bit which is in either one state or another. This relationship is shown in Figure 1 below.

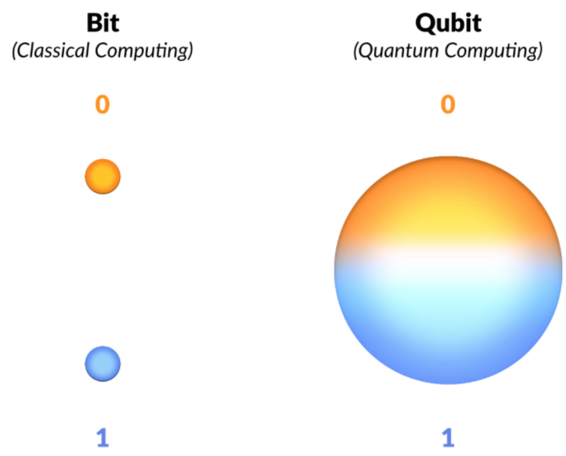

Figure 1: Difference between a bit and a qubit [4]

A qubit has a probability of being in one state or another while a bit is always in one state. A bit will either be a 0 or a 1 while a qubit can be represented as,

$$|\psi >= \alpha|0> +\beta|1>$$

where $|\psi>$ represents the current state and α and β are complex numbers that represent the probability of being in state 0 or state 1. The actual probabilities of being in state 0 or 1 can be obtained by taking the magnitude squared of α and β respectively.

In classical computers, logic gates are used to perform operations. Similarly, QCs have their own versions of gates that are used to perform matrix operations in a quantum circuit. An image of how these gates are represented as well as their descriptions can be found in Appendix A. Quantum entanglement is a property of quantum mechanics that causes the actions of one qubit to alter another. In terms of QC, it allows for multiple qubit states to be acted on simultaneously. Essentially there is a relationship between qubits that means when one is measured, the results will effect the probability of the other being in a certain state.

In today's world, there are three different types of quantum computers. Quantum Anealers are the smallest of the three and only perform one function using a process called quantum annealing. Next is an Analog Quantum Computer which can perform multiple calculations and contains anywhere from 50 to 100 qubits. The last and largest of the three is the Universal Quantum Computer which can perform any type of calculation and will contain around 100,000 qubits [3].

2 Quantum Computing in Finance

2.1 Quantum Machine Learning

Quantum machine learning (QML) will be one of the most significant uses of quantum computing in the financial industry. QML has two main categories, quantum-applied machine learning and quantum enhanced machine learning [14]. The advantage of QML is its ability to solve classical optimization problems in a shorter period of time compared to classical computers, as well as to recognize patterns in data that are too complex for classical computers [1]. QML works by mapping the system being looked at into a quantum system (states). It is able to perform supervised (classification), unsupervised (clustering), and semi-supervised machine learning [14]. QML tends to be integrated with classical systems where the training is done by the quantum computer and the results are checked against the classical systems' expectations [2]. A diagram of this setup is shown in below in Figure 2.

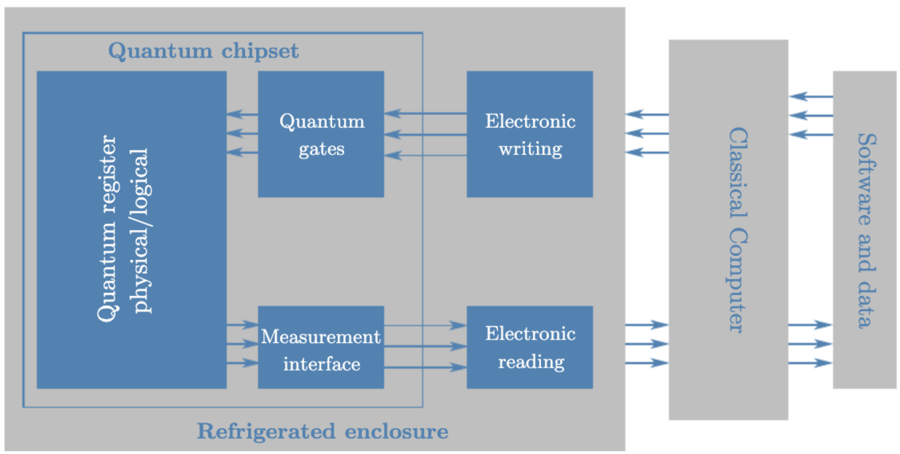

Figure 2: Quantum computer integrated with a classical computer [14]

One of the main advantages QML offers (especially to the financial industry) is quantum speedup. Quantum speedup refers to the number of evaluations a quantum algorithm must do to reach an end result compared to that of a classical computer [2]. In QML, quantum algorithms tend to be faster at training models using quantum circuits than classical algorithms due to the nature of qubits and quantum entanglement. Examples of QML subroutines and their speedups can be found in Figure 4 in Appendix A.

While speedups are beneficial, one of the major disadvantages of QML is the requirement of classical data (cdata) to be mapped to quantum data (qdata). QML requires the input to be qdata and outputs qdata. Therefore the system requires an encoder, such as a quantum ram (QRAM), to shift from cdata to qdata. Another issue that QML brings is that it is still mostly theoretical which means the exact number of gates and cost of integration is still unclear [14].

The speedup to machine learning that QML offers can be attributed to quantum parallelism, which is the ability of a quantum memory register to exist in a superposition of base states [8]. This implies that instead of processing each input one at a time, operations are performed on a superposition of all possible inputs at the same time [6]. Quantum parallelism allows operations to be performed significantly faster compared to a classical computer. For example Grover's algorithm performs a search over an unordered set of N = 2n items to find the unique element that satisfies some condition [13]. While the best classical algorithm for searching over an unordered data set requires O(N) time, Grover's algorithm performs the search on a quantum computer in only $O(\sqrt{N})$ operations [13]. The quadratic speed up would allow combinatorial problems like the TSP to become computationally feasible. Search algorithms imploring quantum parallelism can have a vast range of applications. For example, it can be applied to portfolio optimization where it can find an optimal combination of stocks given a finite set of stocks. Since this problem is essentially a combinatorial one, it can be solved in quadratic time using QML.

The main objective in quantum algorithm design is to construct an algorithm which requires less time and resources to find an optimal solution to a specific computational task. The exponential speed-up of Shor's factorization algorithm indicates that other problems in the NP class can be reduced to BQP on a quantum computer [5]. In computational complexity theory, bounded-error quantum polynomial time (BQP) is the class of problems solvable by a quantum computer in polynomial time. NP-complete is a class of decision problems that does not have an efficient algorithm that can solve it in polynomial time. Travelling salesman problem is NP-complete which means it cannot be solved in polynomial time. If it can be reduced to BQP on a quantum computer then it can be solved in polynomial time. However, it is not yet known whether NP is a subset of BQP, and the informal consensus is that this containment is in fact very unlikely [5].

3 Drawbacks

One of the main drawbacks of quantum computing is decoherence. Decoherence refers to an uncontrolled interaction of a system's (or state's) wave function with its environment leading to unintentional entanglement. This interaction leads to a breakdown in the wave function and a resulting loss in quantum behaviour removing any advantage a quantum computer may have over a classical one [10]. A common example is the double-slit experiment where the output changes if the results are measured. While an error correcting algorithm can be applied, to create an error-tolerant qubit requires thousands of physical qubits again removing the advantage of using a quantum system. Even so, there are algorithms built on processors that expect decoherence and are able output reliable results regardless of the drawback [10].

As mentioned in section 2.1, quantum computers require an encoder in order to work with a classical computer. The issue here is that fault-tolerant QRAMs would require quadrillions of qubits to be able to store sufficient data [11]. Another hardware issue comes from the environment needed to maintain qubit superposition states. In today's world, this includes near absolute zero temperatures, almost complete environment isolation (controlled environment), and radiation shielding [11].

4 Conclusion

In this paper quantum computing and combinatorial optimization was analyzed in terms of its applications in finance. Quantum computing, which relies on quantum nature through entanglement and qubits, is able to produce quantum speedups on classical problems. The superposition of qubits allows quantum computers to utilize quantum parallelism which is the mechanism that supports computational speedups. The processing speed of QCs varies from quadratic to logarithmic, which is a significant increase in efficiency compared to current classical methods. Commercial use of quantum computers in finance is currently not feasible due to the major drawbacks that come with their upkeep. Unintentional entanglement in addition to the required type of environment and number of qubits needed for hardware shows that quantum computing is still far away from being competitive in the financial field.

References

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Appendix A

The following table contains both image representations and descriptions of the different logic gates that are used in a quantum circuit in order to perform operations.

Gate	Transformation on Bloch sphere (defined for single qubit)
X	π-rotation around the X axis, Z→–Z. Also referred to as a bit-flip.
Z	π-rotation around the Z axis, X→–X. Also referred to as a phase-flip.
H	maps X→Z, and Z→X. This gate is required to make superpositions.
S	maps X→Y. This gate extends H to make complex superpositions. (π/2 rotation around Z axis).
$S^\dagger$	inverse of S. maps X→–Y. (-π/2 rotation around Z axis).
T	π/4 rotation around Z axis.
$T^\dagger$	-π/4 rotation around Z axis.

Figure 3: Gates used to perform computations in a quantum computer [12]

In the Box 1 Table, speedups are taken with respect to their classical counterpart(s)—hence, $O(\sqrt{N})$ means quadratic speedup and $O(\log(N))$ means exponential relative to their classical counterpart.

Box 1 Table | Speedup techniques for given quantum machine learning subroutines

Method	Speedup	Amplitude amplification	HHL	Adiabatic	qRAM
Bayesian inference[106,107]	$O(\sqrt{N})$	Yes	Yes	No	No
Online perceptron[108]	$O(\sqrt{N})$	Yes	No	No	Optional
Least-squares fitting[9]	$O(\log N)$*	Yes	Yes	No	Yes
Classical Boltzmann machine[20]	$O(\sqrt{N})$	Yes/No	Optional/No	No/Yes	Optional
Quantum Boltzmann machine[22,61]	$O(\log N)$*	Optional/No	No	No/Yes	No
Quantum PCA[11]	$O(\log N)$*	No	Yes	No	Optional
Quantum support vector machine[13]	$O(\log N)$*	No	Yes	No	Yes
Quantum reinforcement learning[30]	$O(\sqrt{N})$	Yes	No	No	No

*There exist important caveats that can limit the applicability of the method[51].

Figure 4: Quantum Machine Learning subroutines data [1]