An isothermal electromagnetic 3-field gyro-fluid model*

X. Q. Xu and P. W. Xi

Lawrence Livermore National Laboratory, Livermore, CA 94550 USA

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Abstract

An isothermal truncation of the electromagnetic gyro-fluid model of Snyder and Hammett [Phys. Plasmas 8, 3199 (2001)] is developed for ELM simulations. The ion gyrocenter density and electron density are combined to yield a gyro-kinetic vorticity density equation. The set of nonlinear electromagnetic gyro-fluid equations consists of gyro-kinetic vorticity density, ion gyro-center density, the generalized Ohms law and Ampere's law. The first-order Pade's approximation to $\Gamma_0(b) = 1/(1+b)$ is used to get potential by inverting the gyrokinetic vorticity density. In the limit of small ion gyro-radius length, $b = k_\perp^2 \rho_i^2 \ll 1$ (to first order finite Larmor radius approximation in b), this set of equations is shown to be the same as two-fluid model that include finite Larmor radius (FLR) effects. We demonstrate that the complicated nonlinear gyro-viscous tensor in the two-fluid model naturally appears in the isothermal gyro-fluid model as the FLR-corrected ExB convection for the ion gyro-center density in the gyro-kinetic vorticity density equation and the FLR-corrected gyro-kinetic vorticity density. This offers a simple, yet adequate description of ion dynamics that is relatively easy to implement in nonlinear simulation codes. We also show that the gyro-kinetic vorticity density is the charge density only in the cold-ion limit.

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I. DERIVATION OF GYRO-FLUID VORTICITY EQUATIONS

The isothermal gyro-fluid 1+0 model can be obtained from Snyder-Hammett model [1]:

$$\begin{aligned}
\frac{\partial n_{iG}}{\partial t} + \mathbf{v}_{EG} \cdot \nabla n_{iG} &= -\left(\frac{2}{eB}\right) \mathbf{b_0} \times \kappa \cdot \nabla p_{iG} & (1)\\
\frac{\partial n_e}{\partial t} + \mathbf{v}_E \cdot \nabla n_e &= \left(\frac{2}{eB}\right) \mathbf{b_0} \times \kappa \cdot \nabla p_e - \nabla_\parallel \left(n_e v_{\parallel e}\right), & (2)\\
\frac{\partial A_\parallel}{\partial t} &= -\nabla_{||}\phi + \frac{1}{n_e e}\nabla_{||} p_e + \frac{\eta}{\mu_0}\nabla_\perp^2 A_\parallel - \frac{\eta_H}{\mu_0}\nabla_\perp^4 A_\parallel & (3)\\
J_{||} &= -\frac{1}{\mu_0}\nabla_\perp^2 A_\parallel = -n_e e v_{\parallel e}, & (4)\\
n_e &= \bar{n}_i - n_i \left[1 - \Gamma_0(b)\right] \frac{Z_i e \phi}{T_0}, & (5)\\
\bar{n}_i &= \Gamma_0(b)^{1/2} n_{iG}. & (6)
\end{aligned}$$

where $\bar{n}_i$ is the gyro-phase independent part of the real space ion density. The notation n_{iG} is the ion gyro-center density and n_i is the particle density (equal to n_e in the limit of small Debye length, $k\lambda_D \ll 1$). For the various definitions of density, the relation between the particle and gyro-center representations is given by the gyro-kinetic Poisson equation, Eq. (5).

Definitions of various quantities associated with plasma physics are as follows:

$$\begin{aligned}
\mathbf{v}_{EG} &= \mathbf{b}_0 \times \nabla_\perp \Phi_G / B,\\
\mathbf{v}_E &= \mathbf{b}_0 \times \nabla_\perp \phi / B,\\
\tilde{\mathbf{B}} &= \nabla A_\parallel \times \mathbf{b}_0. & (7)
\end{aligned}$$

The notation $\Phi_G = \Phi = \Gamma^{1/2}(b)\phi$ has been introduced for gyro-averaged electric potential. Here $\nabla_\parallel F = B\partial_\parallel (F/B)$ for any F, $\partial_\parallel = \partial_\parallel^0 + \tilde{\mathbf{b}} \cdot \nabla, \tilde{\mathbf{b}} = \tilde{\mathbf{B}}/B, \partial_\parallel^0 = \mathbf{b_0} \cdot \nabla, \kappa = \mathbf{b_0} \cdot \nabla \mathbf{b_0}$. The symbol tilde represents the fluctuation quantities.

Since in the long wavelength regime of a quasi-neutral plasma $\bar{n}_i$ and n_e are two large numbers and are almost equal $\bar{n}_i \sim n_e$ and Eq. (5) can be rewritten as $1 - \bar{n}_i/n_e \simeq (k_\perp \rho_s)^2 e\phi/T_e$, where $(k_\perp\rho_s)^2 \ll 1$ and $e\phi/T_e \sim 1$, the desired solution of Poisson equation as written depends on the difference of two large and almost equal number. Therefore it is difficult to obtain accurate numerical solutions because the numerical errors in $(\bar{n}_i(\mathbf{x}, t) - n_e(\mathbf{x}, t))$ may be on the same order as the ion polarization density.

Here we propose an alternative formulation. We define two new variables: gyrokinetic vorticity density $\varpi_G = eB(n_e - n_{iG})$ and gyrokinetic total pressure $p_G = p_{iG} + p_e = n_{iG}T_{iG} +$

$n_eT_e = (n_{iG} + n_e)T_0$, assuming electron temperature T_e being equal to ion temperature T_{iG} $T_e = T_{iG} = T_0$. The gyrokinetic ion and electron density equation (1) and (2) can be rewritten as a new combination:

$$\frac{\partial \varpi_G}{\partial t} + \mathbf{v}_E \cdot \nabla \varpi_G - eB(\mathbf{v}_{EG} - \mathbf{v}_E) \cdot \nabla n_{iG} = 2\mathbf{b_0} \times \kappa \cdot \nabla p_G + B\nabla_{\|} j_{\|}, \tag{8}$$

$$\varpi_G = eB\left\{\Gamma_0^{1/2}(b)n_{iG} - n_{iG} - Z_i n_i e\left[1 - \Gamma_0(b)\right]\phi/T_0)\right\}, \tag{9}$$

$$\frac{\partial p_G}{\partial t} + \mathbf{v}_E \cdot \nabla p_G + T_0(\mathbf{v}_{EG} - \mathbf{v}_E) \cdot \nabla n_{iG} = 0. \tag{10}$$

Here the parallel current term has been neglected in pressure equation. This formulation naturally couples different domains (core, the SOL and the private flux region) together in the edge region through boundary conditions as in BOUT code. The conlisional stress tensors can easily be included in connection to fluid descriptions.

A. Gyro-fluid vorticity density equation in the limit of small ion gyro-radius length

In the long-wavelength limit,

$$\Gamma_0(b) = \frac{1}{1+b} \simeq 1 - b, \tag{11}$$

$$\Gamma_0^{1/2}(b) = \frac{1}{1+b/2} \simeq 1 - \frac{b}{2}, \tag{12}$$

$$b = -\rho_i^2 \nabla_\perp^2, \tag{13}$$

$$\varpi_G = \frac{eB}{T_0}\rho_i^2\left[n_0 Z_i e \nabla_\perp^2 \phi + \frac{1}{2}\nabla_\perp^2 p_i\right], \tag{14}$$

$$\Phi_G - \phi = \frac{1}{2}\rho_i^2 \nabla_\perp^2 \phi, \tag{15}$$

then the gyrokinetic vorticity density becomes

$$\frac{\partial \varpi_G}{\partial t} + \mathbf{v}_E \cdot \nabla \varpi_G - \frac{eB}{2}\rho_i^2\left[\frac{\mathbf{b}_0 \times \nabla_\perp\left(\nabla_\perp^2 \phi\right)}{B}\right] \cdot \nabla n_{iG} = 2\mathbf{b_0} \times \kappa \cdot \nabla p_G + B\nabla_{\|} j_{\|}. \tag{16}$$

Clearly, this is different from two-fluid model. For the isothermal model, which neglects all considerations of temperature dynamics, we can rewrite the gyrokinetic vorticity density as

$$\frac{\partial \varpi_G}{\partial t} + \mathbf{v}_E \cdot \nabla \varpi_G - \frac{eB}{2T_0}\rho_i^2\left[\frac{\mathbf{b}_0 \times \nabla_\perp\left(\nabla_\perp^2 \phi\right)}{B}\right] \cdot \nabla p_{iG} = 2\mathbf{b_0} \times \kappa \cdot \nabla p_G + B\nabla_{\|} j_{\|}. \tag{17}$$

The third term on the left-hand side is the finite Larmor radius (FLR) corrected ExB convection term for ion gyro-center density $eB(\mathbf{v}_{EG} - \mathbf{v}_E) \cdot \nabla n_{iG}$ in gyrokinetic vorticity density equation, which is part of two-fluid gyro-viscous tensor besides the additional corrections in gyrokinetic vorticity density due to ion dynamics.

B. Comparison of two-fluid model with gyro-fluid model in the limit of small ion gyro-radius length

In order to compare with two-fluid vorticity density $\varpi = \varpi_G + (eB/2T_0)\rho_i^2\nabla_\perp^2 p_i$, we will use Poisson bracket properties and the simplified density equation,

$$\frac{\partial p_{iG}}{\partial t} + \mathbf{v}_E \cdot \nabla p_{iG} \simeq 0. \tag{18}$$

we can then rewrite the finite Larmor radius (FLR) corrected ExB convection term for ion gyro-center density $eB(\mathbf{v}_{EG} - \mathbf{v}_E) \cdot \nabla p_{iG}$ in gyrokinetic vorticity density equation.
Define the Poisson bracket for variables f and g, $\mathbf{v}_f \cdot \nabla g = [f, g]$, the useful manipulations include

$$\nabla_\perp^2 [f, g] \;=\; \nabla \cdot [f, \nabla_\perp g] + \nabla \cdot [\nabla_\perp f, g], \tag{19}$$

$$\nabla \cdot [f, \nabla_\perp g] \;=\; [\nabla_\perp f, \nabla_\perp g] + [f, \nabla_\perp^2 g], \tag{20}$$

Therefore we obtain the relationship

$$[\nabla_\perp^2 f, g] \;=\; \nabla_\perp^2 [f, g] - 2[\nabla_\perp f, \nabla_\perp g] - [f, \nabla_\perp^2 g], \tag{21}$$

which also can be rewritten as

$$[\nabla_\perp f, \nabla_\perp g] \;=\; \frac{1}{2}\nabla_\perp^2 [f, g] - \frac{1}{2}[\nabla_\perp^2 f, g] - \frac{1}{2}[f, \nabla_\perp^2 g]. \tag{22}$$

Using Eqs. (14), (17) and (18), we have

$$\begin{aligned}
[\nabla_\perp^2 \phi, p_i] \;&=\; -\frac{\partial}{\partial t}\nabla_\perp^2 p_i - 2[\nabla_\perp \phi, \nabla_\perp p_i] - [\phi, \nabla_\perp^2 p_i], \\
&=\; \left\{-\frac{\partial}{\partial t}\nabla_\perp^2 p_i - [\phi, \nabla_\perp^2 p_i]\right\} - \nabla_\perp^2 [\phi, p_i] + [\nabla_\perp^2 \phi, p_i] + [\phi, \nabla_\perp^2 p_i], \\
&=\; \left\{-\frac{\partial}{\partial t}\nabla_\perp^2 p_i - (\mathbf{v}_E \cdot \nabla)\, \nabla_\perp^2 p_i\right\} \\
&-\; n_i Z_i e \left[\mathbf{v}_{p_i} \cdot \nabla(\nabla_\perp^2 \phi)\right] + (\mathbf{v}_E \cdot \nabla)\, \nabla_\perp^2 p_i - \nabla_\perp^2 \left(\mathbf{v}_E \cdot \nabla p_i\right)
\end{aligned} \tag{23}$$

Then we obtain the two-fluid vorticity density equation,

$$\begin{aligned}
\frac{\partial \varpi}{\partial t} \;&+\; \mathbf{v}_E \cdot \nabla \varpi = 2\mathbf{b_0} \times \kappa \cdot \nabla p_G + B\nabla_\parallel j_\parallel \\
&-\; \frac{1}{2\omega_{ci}} \left\{n_i Z_i e \mathbf{v}_{Pi} \cdot \nabla(\nabla_\perp^2 \phi)\right\} \\
&+\; \frac{1}{2\omega_{ci}} \left\{\mathbf{v}_E \cdot \nabla(\nabla_\perp^2 p_i) - \nabla_\perp^2 (\mathbf{v}_E \cdot \nabla p_i)\right\}
\end{aligned} \tag{24}$$

This equation is the same as two-fluid version of vorticity equation (2) given by Xu et al [2], excluding external momentum sources, ion viscosity, and the cross-term in vorticity definition, which is given here again for comparison:

$$
\begin{aligned}
\frac{\partial \varpi}{\partial t} + \left(\mathbf{v}_E + v_{\|i}\mathbf{b}_0\right)\cdot\nabla\varpi &= (2\omega_{ci})\,\mathbf{b_0}\times\kappa\cdot\nabla p \\
&+ n_i Z_i e\frac{4\pi v_A^2}{c^2}\nabla_{\|} j_{\|} \\
&- \frac{1}{2}\left\{n_i Z_i e\mathbf{v}_{Pi}\cdot\nabla(\nabla_\perp^2\phi)\right\} \\
&+ \frac{1}{2}\left\{\mathbf{v}_E\cdot\nabla(\nabla_\perp^2 p_i) - \nabla_\perp^2(\mathbf{v}_E\cdot\nabla p_i)\right\}
\end{aligned} \tag{25}
$$

This resolves the long-standing issue regarding the difference in vorticity equation derived from two-fluid and gyrokinetic framework. The gyro-viscous terms emerges naturally from the FLR nonlinearities in the ion gyrocenter density in the limit of small ion gyro-radius length. Furthermore, the gyro-fluid results show a simple physics picture and can be easily implemented in simulation codes.

II. A COMPLETE SET OF THE 3-FIELD ELECTROMAGNETIC GYRO-FLUID EQUATIONS

For the isothermal model, which neglects all considerations of temperature dynamics, we can rewrite the gyrokinetic vorticity density as

$$
\frac{\partial \varpi_G}{\partial t} + \mathbf{v}_E\cdot\nabla\varpi_G - eB(\mathbf{v}_{EG}-\mathbf{v}_E)\cdot\nabla n_{iG} = 2\mathbf{b_0}\times\kappa\cdot\nabla p_G + B\nabla_{\|} j_{\|}, \tag{26}
$$

$$
\frac{\partial p_G}{\partial t} + \mathbf{v}_E\cdot\nabla p_G + T_0(\mathbf{v}_{EG}-\mathbf{v}_E)\cdot\nabla n_{iG} = 0, \tag{27}
$$

$$
\frac{\partial A_{\|}}{\partial t} = -\nabla_{\|}\phi + \frac{1}{n_e e}\nabla_{\|} p_e + \frac{\eta}{\mu_0}\nabla_\perp^2 A_{\|} - \frac{\eta_H}{\mu_0}\nabla_\perp^4 A_{\|} \tag{28}
$$

$$
\varpi_G = eB\left\{\Gamma_0^{1/2}(b) n_{iG} - n_{iG} - Z_i n_i e\left[1-\Gamma_0(b)\right]\phi/T_0\right\}, \tag{29}
$$

$$
n_e = \frac{1}{2}\left(\frac{p_G}{T_0} - \frac{\varpi_G}{eB}\right), \tag{30}
$$

$$
n_{iG} = \frac{1}{2}\left(\frac{p_G}{T_0} + \frac{\varpi_G}{eB}\right), \tag{31}
$$

$$
p_e = n_e T_0, \tag{32}
$$

$$
J_{\|} = -\frac{1}{\mu_0}\nabla_\perp^2 A_{\|}. \tag{33}
$$

Definitions of various quantities associated with plasma physics are as follows:

$$\begin{aligned}
\mathbf{v}_{EG} &= \mathbf{b}_0 \times \nabla_\perp \Phi_G / B, \\
\mathbf{v}_E &= \mathbf{b}_0 \times \nabla_\perp \phi / B, \\
b &= -\rho_i^2 \nabla_\perp^2, && (34) \\
\tilde{\mathbf{B}} &= \nabla A_\parallel \times \mathbf{b}_0. && (35)
\end{aligned}$$

The notation $\Phi_G = \Phi = \Gamma^{1/2}(b)\phi$ has been introduced for gyro-averaged electric potential. $\varpi_G = eB(n_e - n_{iG})$ and $p = p_{iG} + p_e = (n_{iG} + n_e)T_0$. Here $\nabla_\parallel F = B\partial_\parallel(F/B)$ for any F, $\partial_\parallel = \partial_\parallel^0 + \tilde{\mathbf{b}} \cdot \nabla, \tilde{\mathbf{b}} = \tilde{\mathbf{B}}/B, \partial_\parallel^0 = \mathbf{b_0} \cdot \nabla, \kappa = \mathbf{b_0} \cdot \nabla \mathbf{b_0}$.

A. The 3-field gyro-fluid model in the limit of small ion gyro-radius length

$$\begin{aligned}
\frac{\partial \varpi_G}{\partial t} &+ \mathbf{v}_E \cdot \nabla \varpi_G - \frac{eB}{2T_0}\rho_i^2 \left[\frac{\mathbf{b}_0 \times \nabla_\perp \left(\nabla_\perp^2 \phi\right)}{B}\right] \cdot \nabla p_{iG} = 2\mathbf{b_0} \times \kappa \cdot \nabla p_G + B\nabla_\parallel j_\parallel. && (36) \\
\frac{\partial p_G}{\partial t} &+ \mathbf{v}_E \cdot \nabla p_G + \frac{1}{2}\rho_i^2 \left[\frac{\mathbf{b}_0 \times \nabla_\perp \left(\nabla_\perp^2 \phi\right)}{B}\right] \cdot \nabla p_{iG} = 0, && (37) \\
\frac{\partial A_\parallel}{\partial t} &= -\nabla_\parallel \phi + \frac{1}{n_e e}\nabla_\parallel p_e + \frac{\eta}{\mu_0}\nabla_\perp^2 A_\parallel - \frac{\eta_H}{\mu_0}\nabla_\perp^4 A_\parallel && (38) \\
\varpi_G &= \frac{eB}{T_0}\rho_i^2 \left[n_0 Z_i e \nabla_\perp^2 \phi + \frac{1}{2}\nabla_\perp^2 p_i\right], && (39) \\
n_e &= \frac{1}{2}\left(\frac{p_G}{T_0} - \frac{\varpi_G}{eB}\right) && (40) \\
n_{iG} &= \frac{1}{2}\left(\frac{p_G}{T_0} + \frac{\varpi_G}{eB}\right) && (41) \\
p_e &= n_e T_0, && (42) \\
J_\parallel &= -\frac{1}{\mu_0}\nabla_\perp^2 A_\parallel. && (43)
\end{aligned}$$

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[1] P. B. Snyder and G. W. Hammett, Phys. Plasmas 8, 3199 (2001).

[2] X. Q. Xu, R. H. Cohen, T. D. ROGNLIEN, and J. R. Myra, Phys. Plasma 7, 1951-1958 (2000).