Hamiltonian four-field model for nonlinear tokamak dynamics

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The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics, as well as other effects related to compressibility and electron adiabaticity. Much simpler than some previous descriptions of the same physics, it still preserves essential features of the underlying exact dynamics. In particular, because it is a Hamiltonian dynamical system it conserves the appropriate Casimir invariants, as well as avoiding implicit, unphysical dissipation. Here the model is derived and interpreted, its Hamiltonian nature is demonstrated, and its constants of motion are extracted.

I. INTRODUCTION

The four-field model is a system of coupled fluid equations describing magnetized plasma motion in an axisymmetric confinement device such as a tokamak. It is intended to model such phenomena as sawtooth oscillation and tokamak disruption, especially in their nonlinear stages.[1] While emphatically a simplified system, in which numerous geometrical and dynamical effects are neglected, the equations attempt to represent nonideal processes, including finite-ion-Larmor-radius (FLR) terms and electron adiabaticity, in a manner consistent with significant physical constraints. In particular, when explicit dissipation is omitted the model is shown to define a (generalized or noncanonical) Hamiltonian dynamical system. (See, for example, Ref. 2.)

The four-field model is an outgrowth of reduced magnetohydrodynamics[3] (RMHD), whose enlarged physics requires four, rather than three, independent field variables. It also has much in common with several earlier extensions of RMHD,[4–8] especially with regard to motivation. By far its closest antecedent is the approximate four-field model of Hazeltine, Kotschenreuther, and Morrison[4] (hereafter referred to as HKM); in fact, the present work is to be viewed primarily as an improvement upon the model of HKM.

The usefulness of reduced fluid models has been discussed repeatedly in the literature,[3–8] and the specific scientific intentions of the four-field model are discussed in some detail in HKM. Thus we are content here to remark that the four-field model is a generalization of RMHD that allows for slow evolution (frequencies comparable to the diamagnetic drift), long mean-free-path electron dynamics, and various effects of plasma compressibility in a simple, albeit nonrigorous, way. Like its predecessors[4,5] the four-field model reproduces such features of kinetic and FLR physics as the "semicollisional" conductivity; gyroviscosity-modified, nonlinear diamagnetic convection; curvature-modified drift-tearing instability; and diffusion in a stochastic magnetic field. Also, like its predecessors, the four-field model omits temperature gradients and kinetic effects of magnetic trapping. Finally, unlike the model of HKM (but in common with the underlying physics it attempts to represent), the four-field model's ideal version not only conserves energy, but is a Hamiltonian dynamical system.

Three equivalent versions of the model are presented in Sec. II. The derivation is given in Sec. III, while Sec. IV is devoted to a discussion of the system's dynamical invariants.

Because of the model's similarity to HKM a detailed discussion of each of its terms would largely repeat previous literature.[4] Therefore we confine our comments to those features that distinguish the present work from its predecessors.

It seems appropriate to comment first upon the general significance, in such approximate field theories as RMHD, of the generalized Hamiltonian property.

The Hamiltonian nature of a system need not be obvious from inspection, particularly when the system is described in terms of noncanonical variables. Yet consequences stronger than, for example, energy conservation are still present; an example is the energy conserving but non-Hamiltonian Boltzmann equation. In particular, noncanonical Hamiltonian systems typically possess special constants called Casimir invariants. The magnetic and cross helicities are examples [cf. Eqs. (7) and Sec. IV]. Also, noncanonical Hamiltonian systems have an underlying (although not necessarily convenient) description in terms of canonical variables. In terms of these variables the system conserves phase-space volume and other Poincaré invariants.

The simplest way to guarantee that some dynamical system is Hamiltonian is to demonstrate that it represents faithfully, at least in some asymptotic limit, the actual evolution of charged particles. Thus, for example, Vlasov theory and ideal magnetohydrodynamics (MHD) can be shown to have the (generalized) Hamiltonian property.[2] However, not all systems of interest to plasma physics can be derived systematically from exact microscopic dynamics. Especially in nonlinear regimes, progress frequently demands the use of simplified models in which the Hamiltonian property is problematic. A major concern in the application and interpretation of such models is the possibility of unphysical dissipation.

Physical dissipation enters exact formulations explicitly through such mechanisms as collision operators or resistive terms. In the case of nonrigorously derived models, however, dissipation can also enter implicitly and unintentionally because of uncontrolled approximation. No resistivity or collisional term occurs in this case—the system appears

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purely nondissipative—yet underlying phase-space conservation and other invariants are lost. Typically the magnitude and even the effective sign of this fake dissipation is uncontrolled.

It has been shown that RMHD is a Hamiltonian system.[9] Certain extensions of RMHD, discussed in Sec. III, similarly preserve the Hamiltonian property; furthermore, a Hamiltonian representation of 2-D FLR physics has been found.[10] Nonetheless it shall become clear that the Hamiltonian property of reduced fluid models must be considered extremely fragile. Among the myriad of physically plausible four-field models, each conserving energy and yielding correct, FLR-modified linear equations, only a tiny subset is Hamiltonian. The system described in this work is shown to be Hamiltonian; we believe it is the subset's simplest member.

II. DESCRIPTION OF THE MODEL

A. Four-field equations

We present here the dissipationless version of the four-field model, noting that dissipative terms (resistivity, diffusion, and viscosity) can be introduced straightforwardly *a posteriori*. Our notation is conventional and very similar to that of, for example, RMHD.[3] As noted in the Introduction we refer the reader to previous work[3–8] for more detailed discussion and interpretation.

The four normalized fields are W, ψ, p, and v; they have the following physical significance:

W measures the scalar parallel *vorticity*;
ψ measures the poloidal magnetic *flux*;
p measures the electron *pressure*; and
v measures the ion *parallel velocity*.

In addition to the above normalized variables, the model involves three constant parameters: the electron beta $\beta \equiv 8\pi n_c T_e / B_T^2$; the temperature ratio $\tau \equiv T_i/T_e$; and the FLR parameter $\delta \equiv c/(2\omega_{pi} a)$, where ω_{pi} is the ion plasma frequency and a is the plasma minor radius. The product

$$\tau\delta^2\beta = T_i/(m_i \Omega_i^2 a^2)$$

is evidently proportional to the ion Larmor radius, suggesting the convenient abbreviation

$$\rho^2 \equiv \tau\delta^2\beta.$$

Our normalizations of the field variables are also conventional, and follow HKM in detail. Thus $\psi = (\epsilon B_T a)^{-1} A_\zeta$, where ϵ is the inverse aspect ratio and A_ζ is the toroidal component of the vector potential; $\varphi = c\Phi/(\epsilon v_A B_T a)$, where Φ is the electrostatic potential and v_A is the Alfvén speed; $v = v_\parallel/(\epsilon v_A)^{-1}$, where $v_\parallel$ is the ion parallel velocity; and $p = (\beta/\epsilon)(n/n_c - 1)$, where n is the plasma density. We also introduce a velocity streamfunction F according to

$$(1 + \rho^2 \nabla_\perp^2) F = \varphi + \delta\tau p, \qquad (1)$$

where $\nabla_\perp$ is the 2-D gradient operator in the plane transverse to the magnetic field. The function F, which differs slightly from its counterpart in HKM, is a streamfunction in the sense that the normalized ion velocity tranverse to **B** is $\hat{\zeta} \times \nabla_\perp F$. As in HKM the right-hand side (rhs) of (1) yields the expected combination of electric and diamagnetic drifts, while the $O(\delta^2)$ term involving $\nabla_\perp^2$ on the left-hand side (lhs) gives a FLR correction. In terms of F, the normalized vorticity variable w is given by

$$W \equiv \nabla_\perp^2 F.$$

Similarly, the normalized parallel currenty density is related to ψ via

$$J \equiv \nabla_\perp^2 \psi.$$

Finally we define h, a normalized "horizontal" distance, by $h \equiv (R - R_0)/a$, where R is the major radius and R_0 the major radius of the magnetic axis. This quantity enters the equations only in the form $\nabla_\perp h$, which is the lowest-order field line curvature.

The four-field model can then be expressed as

$$\frac{\partial}{\partial t} W + [F, W] + \nabla_\parallel J + (1+\tau)(1 + \rho^2 \nabla_\perp^2)[h, p]$$
$$= \delta\tau\left[\nabla_\perp \cdot [p + 2\beta h, \nabla_\perp F] + \tfrac{1}{2}\rho^2 \nabla_\perp^2 [p + 2\beta h, W]\right]$$
$$- \rho^2 \nabla_\perp^2 \nabla_\parallel \left(J + \frac{v}{2\delta}\right), \qquad (2)$$

$$\frac{\partial}{\partial t}\psi + \nabla_\parallel \varphi - \delta \nabla_\parallel p = 0, \qquad (3)$$

$$\frac{\partial}{\partial t} p + [\varphi, p + 2\beta h] = 2\delta\beta\left[[p, h] - \nabla_\parallel\left(J + \frac{v}{2\delta}\right)\right], \qquad (4)$$

$$\frac{\partial}{\partial t} v + [\varphi, v] + \frac{1}{2}\nabla_\parallel [p + \tau(p - \delta\beta W)]$$
$$= \rho^2 [v, \nabla_\perp^2 (F - \delta\tau p)] + \frac{2\rho^2}{\delta}[v, h]. \qquad (5)$$

Here we use the conventional bracket symbol defined by

$$[f, g] \equiv \hat{\zeta} \cdot \nabla_\perp f \times \nabla_\perp g,$$

where $\hat{\zeta}$ is a unit vector in the toroidal direction. Also, the parallel gradient operator is defined by

$$\nabla_\parallel f \equiv \frac{\partial f}{\partial \zeta} + [f, \psi].$$

Equations (3) and (4) express the generalized (collisionless) Ohm's law and the particle conservation law precisely as in HKM. Equation (2), the shear-Alfvén law, differs from HKM in including several additional FLR and compressibility terms on the rhs. Similarly, the parallel acceleration law, Eq. (5), includes previously omitted physics. All the additional terms are numerically small, since δ and β are typically small in tokamak experiments. The significance of these correction terms is discussed in Sec. II C.

This system conserves the following energy (Hamiltonian) functional:

$$H \equiv \tfrac{1}{2}\langle |\nabla_\perp F|^2 + v^2 + |\nabla_\perp \psi|^2 + (1+\tau)p^2/(2\beta)\rangle, \qquad (6)$$

which differs from that of HKM. Here the angular brackets denote an integral over the system volume (effects of the volume boundary are ignored). This functional is easily understood to be the sum of the parallel and perpendicular fluid kinetic, poloidal magnetic field, and internal energies. In addition to the energy functional, the four-field model conserves the following four Casimir (or "helicity" type) invariants:

$$
\begin{aligned}
C_1 &= \langle A(\psi)\rangle, \\
C_2 &= \langle B(\psi)(p + 2\beta h)\rangle, \\
C_{3,4} &= \langle C_{\pm}\,[2\delta\beta v + \beta\psi \pm \sqrt{2}\rho(2\delta\beta W \\
&\qquad - \rho^2\nabla_\perp^2 p - p - 2\beta h)\,]\rangle.
\end{aligned} \tag{7}
$$

These constants are associated with the magnetic helicity, density, and generalizations of the cross helicity, respectively. When there are magnetic surfaces, such as in the case of axisymmetry or single helicity dynamics, the functions A, B, and $C_\pm$ are arbitrary. For general 3-D dynamics C_1 and C_2 remain conserved provided $A(\psi) = \psi$, $B(\psi) = \text{constant}$, and $C_\pm(x) = x$.

Equations (1)–(7) are the main results of this paper. We next rewrite the system in a form that makes manifest its Hamiltonian character.

B. Hamiltonian form

In order to display the Hamiltonian structure of the four-field model it is convenient to introduce the following set of variables:

$$
\begin{aligned}
&\xi^1 = \nabla_\perp^2(F - \delta\tau p/2), \quad \xi^2 = \psi, \\
&\xi^3 = p + 2\beta h, \quad \xi^4 = v.
\end{aligned} \tag{8}
$$

We shall refer to the ξ^i as "field variables" to distinguish them from the "physical variables" W, ψ, p, and v.

When the total system energy is expressed in terms of the ξ^i, it becomes

$$
\begin{aligned}
H[\xi] \equiv \tfrac{1}{2}\langle &|\nabla_\perp(\nabla_\perp^{-2}\xi^1) + (\delta\tau/2)\nabla_\perp(\xi^3 - 2\beta h)|^2 \\
&+ |\nabla_\perp\xi^2|^2 + (1+\tau)(\xi^3 - 2\beta h)^2/(2\beta) \\
&+ (\xi^4)^2\rangle,
\end{aligned} \tag{9}
$$

where $\nabla_\perp^{-2}$ represents the inverse Laplacian operator, whose occurrence in fluid Hamiltonians is conventional.

Now we can express the four-field model for evolution of the ξ^i in the following form:

$$
\frac{\partial}{\partial t}\xi^1 = [H_1,\xi^1] + \nabla_\parallel H_2 + [H_3,\xi^3] + [H_4,\xi^4], \tag{10}
$$

$$
\frac{\partial}{\partial t}\xi^2 = \nabla_\parallel(H_1 + 2\delta\beta H_3), \tag{11}
$$

$$
\frac{\partial}{\partial t}\xi^3 = [H_1 + 2\delta\beta H_3,\xi^3] - \beta\nabla_\parallel(H_4 - 2\delta H_2), \tag{12}
$$

$$
\frac{\partial}{\partial t}\xi^4 = [H_1,\xi^4] - \beta\nabla_\parallel H_3 + \delta\tau[\xi^3 - 2\delta\beta\xi^1,H_4]. \tag{13}
$$

Here functional derivatives of the Hamiltonian are indicated by subscripts, $H_i \equiv \delta H/\delta\xi^i$; they are given by

$$
\begin{aligned}
&H_1 = -F, \quad H_2 = -J, \\
&H_3 = [(1+\tau)/2\beta]p - (\delta\tau/2)W, \quad H_4 = v,
\end{aligned} \tag{14}
$$

and can easily be written in terms of the field variables by means of Eqs. (8). Note that Eqs. (10)–(13) are simpler in form than Eqs. (2)–(5), especially since the latter can only be used in conjunction with Eq. (1).

To express the four-field model in Hamiltonian form, first let F and G be arbitrary functionals of the fields ξ^i, with $F_i \equiv \delta F/\delta\xi^i$ as usual. Then, implicitly summing over paired indices, we define a Poisson bracket by

$$
\{F,G\} = \left\langle C^{ij}_k\xi_k[F_i,G_j] + C^{ij}_2\left(F_i\frac{\partial G_j}{\partial\xi}\right)\right\rangle, \tag{15}
$$

where the coefficient matrix C^{ij}_k, which is symmetric with respect to its upper indices, has the following nonzero components:

$$
\begin{aligned}
&C^{1j}_k = C^{j1}_k = \delta_{kj}, \\
&C^{23}_k = C^{32}_k = 2\delta\beta\delta_{k2}, \\
&C^{33}_k = 2\delta\beta\delta_{k3}, \\
&C^{34}_k = C^{43}_k = -\beta\delta_{k2}, \\
&C^{44}_k = -\delta\tau(\delta_{k3} - 2\delta\beta\delta_{k1}).
\end{aligned} \tag{16}
$$

We remark that Eqs. (15) and (16) define a true Poisson bracket: It is bilinear, antisymmetric, satisfies the Jacobi identity

$$
\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \tag{17}
$$

and acts as a derivation, i.e.,

$$
\{F,GH\} = \{F,G\}H + G\{F,H\}.
$$

We also remark that C^{ij}_k is a rather simple matrix, at least in the sense of being sparse.

The Hamiltonian version of Eqs. (2)–(5) is given by

$$
\frac{\partial}{\partial t}\xi^i = \{\xi^i,H\}. \tag{18}
$$

The invariance of the "Casimirs" defined by Eqs. (7) then follows from the identities $\{C_i,F\} = 0$ for $i = 1$–4 and F arbitrary.

C. Discussion

Here we consider the significance of the new FLR and compressibility terms appearing in the present model, basing our discussion on Eqs. (2)–(5) for convenience.

In the ion dynamics, Eqs. (2) and (5), FLR corrections are apparent; they are manifested by the familiar operator $\rho^2\nabla^2$ and have a well-known interpretation in terms of averages over the Larmor orbit. The FLR terms manifested on the rhs of Eq. (2) describe, in particular, nonlinear diamagnetic convection and ion gyroviscosity. In linear theory (where the perturbation is assumed to vary more sharply than the equilibrium) these terms reproduce the ion drift-frequency corrections found in linearized gyrokinetic analysis; a detailed discussion of their physical significance can be found elsewhere.[4–6]

Another type of FLR correction is most apparent in Eq. (5), although also present elsewhere: the $\delta\beta W$ correction to the ion pressure, $\tau p \to \tau(p - \delta\beta W)$. It can be identified with a well-known residue from the "gyroviscous cancellation"; thus gyroviscosity is known[11] to modify the ion scalar pressure p_i in a FLR plasma according to

$$
p_i \to p_i[1 - (2\Omega_i)^{-1}\mathbf{b}\cdot\nabla\times\mathbf{V}_i], \tag{19}
$$

where Ω_i is the ion gyrofrequency, $\mathbf{V}_i$ is the ion fluid velocity, and $\mathbf{b}$ is a unit vector in the direction of the magnetic field. When Eq. (19) is expressed in terms of the four-field normalized variables and reduced for large aspect ratio, it yields $p - \delta\beta W$.

All FLR terms in Eqs. (2) and (5) have been derived by systematic ordering procedures in previous work[5]; however, the rigorous ordering also produces a host of additional corrections of similar form. Thus the present model, which is extremely simple compared to the rigorous version, contains a *selection* of gyroradius corrections. We will presently discuss the grounds for this selectivity.

The remaining terms of interest involve the plasma compressibility, given by the rhs of Eq. (4). Equation (4) coincides with a previous conservation law and has been discussed in detail elsewhere[4]; we recall that the term involving h is the perpendicular compressibility, resulting from curvature of the magnetic field, while the term involving $\nabla_\parallel$ is the parallel compressibility of the electron flow $V_{\parallel e} \propto v + 2\delta J$. The new feature here is the appearance of explicit compressibility terms in Eq. (2), as seen, for example, in its last term. We point out that the contribution of compressibility to the shear-Alfvén law, although rarely taken into account, is easily understood. First, the vorticity associated with diamagnetic acceleration, $\hat{\xi}\cdot\nabla\times(d/dt)(\hat{\xi}\times\nabla p)$, evidently involves $\nabla^2(d/dt)p_i$ and therefore the Laplacian of the compressibility $p_i\nabla\cdot\mathbf{V}_i$. Second, gyroviscosity can be shown[4,5] to contribute terms of the same form. Equation (2) displays the sum of these two contributions, which together with the factor of $\frac{1}{2}$, also occur in the rigorous version.[5]

This comment helps explain the appearance of the modified vorticity $\xi^1 = \nabla_\perp^2(F - \delta\tau p/2)$ as a basic field in the system. The second term correctly accounts for plasma compressibility in the shear-Alfvén law. Perhaps fortuitously, it also contributes to a correct accounting of ion diamagnetic convection terms.

Thus the new terms are physically plausible in the sense that rigorous ordering arguments yield correction terms of the same form. However, because the rigorous analysis also reveals numerous other FLR effects, the new terms do not make Eqs. (2)–(5) more "exact" in any formal sense. Why then do these particular corrections appear?

The correction terms in Eqs. (2)–(5) are best characterized as being the *minimal* additions to a cold-ion theory which preserve the following essential physical properties.

(i) Reasonable cold-ion ($\tau\to 0$) limit; specifically we require that the $\tau = 0$ version agree with that of the previous four-field model, whose physical reasonableness was discussed in HKM.

(ii) Agreement in the linear regime with kinetic theory of ion diamagnetic effects; in particular we require that the ion diamagnetic frequency enter the linearized four-field model in the manner predicted by gyrokinetics.[6]

(iii) Hamiltonian structure; we insist upon a dynamical law of the form of Eq. (18), where the bracket is antisymmetric, satisifies Jacobi's identity, and acts as a derivation.

The four-field equations presented here satisfy these requirements, and they do so minimally in the sense that the model obtained by omission of any term does not.

III. DERIVATION

Because we seek a drastically simplified description of FLR physics—indeed, the simplest system that satisfies the requirements (i)–(iii) of Sec. II—our derivation of the four-field model cannot rely on simple ordering procedures. Instead, it is based on a mapping procedure that is motivated by asymptotically rigorous models.

A. The gyromap

A high-beta version of RMHD that includes both electron and ion drift corrections, but excludes compressibility, is obtained by a rigorous ordering procedure in Ref. 5. This three-field model is given by

$$\frac{\partial}{\partial t}\nabla_\perp^2\varphi + [\varphi,\nabla_\perp^2\varphi] + \nabla_\parallel J + (1+\tau)[h,p] + \delta\tau\nabla_\perp\cdot[p,\nabla_\perp\varphi] = 0, \tag{20}$$

$$\frac{\partial}{\partial t}\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \tag{21}$$

$$\frac{\partial}{\partial t}p + [\varphi,p] = 0, \tag{22}$$

and conserves the following energy:

$$H = \tfrac{1}{2}\langle|\nabla_\perp\varphi|^2 + |\nabla_\perp\psi|^2 + 2\delta p\nabla_\perp^2\varphi - \tau\delta^2|\nabla_\perp p|^2 - 2(1+\tau)hp\rangle; \tag{23}$$

it is also a Hamiltonian system.

For reasons of clarity we now specialize to the axisymmetric case. The generalization to three dimensions is straightforward, involving nothing more than the replacement

$$[f,\psi]\to\nabla_\parallel f. \tag{24}$$

If this replacement is made in a Poisson bracket then it can be shown in general that the Jacobi identity is maintained.

The axisymmetric version of Eqs. (20)–(22) has the following Poisson bracket:

$$\{F,G\} = \langle U[F_U,G_U] + \psi([F_U,G_\psi] + [F_\psi,G_U]) + p([F_U,G_p] + [F_p,G_U]) + \delta\tau p[\nabla_\perp F_U;\nabla_\perp G_U]\rangle\,. \tag{25}$$

Here, we have used $\delta F/\delta U \equiv F_U$, etc. and in the last term the semicolon notation is defined by

$$[\mathbf{A};\mathbf{B}] = \sum_i [A_i,B_i]\,.$$

Because of the last term, the form of the bracket of (25) differs from previous brackets in that it involves more derivatives. Yet one can prove directly that Eq. (25) satisfies the Jacobi identity.

Now consider the zero ion-temperature limit. Setting τ equal to zero we obtain

$$\frac{\partial}{\partial\tau}\nabla_\perp^2\varphi + [\varphi,\nabla_\perp^2\varphi] + \nabla_\parallel J + [h,p] = 0, \tag{26}$$

$$\frac{\partial}{\partial\tau}\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \tag{27}$$

$$\frac{\partial}{\partial\tau}p + [\varphi,p] = 0. \tag{28}$$

Apart from removing the ion pressure from Eq. (26), the only effect of taking this limit has been to remove ion gyroviscosity physics. Observe that the term involving the parameter δ in Eq. (27), unlike the gyroviscous effect in Eq. (20), reflects electron physics; it is the Hall term.

At zero τ the Hamiltonian becomes

$$H=\tfrac{1}{2}\langle|\nabla_\perp\varphi|^2+|\nabla_\perp\psi|^2+2\delta p\nabla_\perp^2\varphi-2hp\rangle \tag{29}$$

and the Poisson bracket reduces to

$$\{F,G\}=\langle U[F_U,G_U]+\psi([F_U,G_\psi]+[F_\psi,G_U]) + p([F_U,G_p]+[F_p,G_U])\rangle, \tag{30}$$

which differs from Eq. (25) only in that it lacks the gyro-term.

Now comes the crucial observation: *Poisson brackets for systems without ion gyroviscosity physics can be mapped into those with ion gyroviscosity physics by a simple linear transformation*. The transformation amounts to changing to a frame moving at one-half the magnetization velocity. The magnetization velocity is defined by $\mathbf{v}_M=(\nabla\times\mathbf{M})/ne$, where $\mathbf{M}$ is the magnetization. We call this transformation the *gyromap*.

The gyromap was first observed in Ref. 5 for a 2-D model with compressibility. We will demonstrate it here for the brackets of Eqs. (20)–(22).

Technically the mapping we are refering to is a Lie algebra isomorphism; the brackets of Eqs. (25) and (30) are isomorphic. [In Sec. IV we use this algebraic fact to simply obtain the complicated constants of motion of Eqs. (7).] Physically the transformation amounts to defining a new variable U' by

$$U'=U+(\delta\tau/2)\nabla_\perp^2 p, \tag{31}$$

which yields the following relation between the new and old streamfunctions:

$$\varphi'=\varphi+(\delta\tau/2)p. \tag{32}$$

Here, the second term evidently corresponds to the velocity of the moving frame. One can show that in reduced ordering, $(\delta\tau/2)\nabla_\perp^2 p=(\hat{\xi}\cdot\nabla\times\mathbf{v}_M)/2$, where $\mathbf{M}=p\mathbf{B}/B^2$.

By the chain rule for functional derivatives, the transformation on the field variables induces the following relations among the derivatives:

$$\left.\frac{\delta}{\delta U}\right|_{U,p,\psi}=\left.\frac{\delta}{\delta U'}\right|_{U',p,\psi},\quad \left.\frac{\delta}{\delta \psi}\right|_{U,p,\psi}=\left.\frac{\delta}{\delta \psi}\right|_{U',P,\psi},$$
$$\left.\frac{\delta}{\delta p}\right|_{U,p,\psi}=\left.\frac{\delta}{\delta p}\right|_{U',p,\psi}+\frac{\delta\tau}{2}\nabla_\perp^2\left.\frac{\delta}{\delta U'}\right|_{U',p,\psi}. \tag{33}$$

Inserting $U=U'-(\delta\tau/2)\nabla_\perp^2 p$ and Eqs. (33) into Eq. (25) gives

$$\{F,G\}=\langle U'[F_{U'},G_{U'}]+\psi([F_{U'},G_\psi]+[F_\psi,G_{U'}]) + p([F_{U'},G_p]+[F_p,G_{U'}])\rangle. \tag{34}$$

Equation (34) has precisely the same form as Eq. (30). Thus we see that the bracket for Eqs. (20)–(22) can be obtained from its $T_i=0$ limit by reversing the transformation that we have just performed. We obtain the bracket for the four-field model in a similar way.

B. Four-field derivation

As noted, our derivation of the new field equations begins with the cold-ion form of the previous four-field model.[4] This cold-ion model is asymptotically correct and easily obtained by straightforward ordering arguments. Setting $\tau=0$ in previous formulas (cf. Sec. II A), we obtain

$$\frac{\partial}{\partial t}\nabla_\perp^2 F'+[F',\nabla_\perp^2 F']+\nabla_\parallel J+[h,p]=0, \tag{35}$$

$$\frac{\partial}{\partial t}p+[F',p]+\beta\nabla_\parallel(v+2\delta J)-2\beta[h,F'-\delta p]=0, \tag{36}$$

$$\frac{\partial}{\partial t}\psi+\nabla_\parallel F'-\delta\nabla_\parallel p=0, \tag{37}$$

$$\frac{\partial}{\partial t}v+[F',v]+\frac{1}{2}\nabla_\parallel p=0. \tag{38}$$

Here, F' is the velocity streamfunction, which in the $\tau=0$ limit is equal to φ. The energy conserved by this system is

$$H=\tfrac{1}{2}\langle|\nabla_\perp F'|^2+|\nabla_\perp\psi|^2+v^2+p^2/(2\beta)\rangle. \tag{39}$$

We define the field variables by

$$(\xi^{1'},\xi^{2'},\xi^{3'},\xi^{4'})=(\nabla^2F',\psi,p+2\beta h,v). \tag{40}$$

Hence, using the notation $H_i=\delta H/\delta\xi^{i'}$,

$$H_1=-F',\quad H_2=-J,\quad H_3=p/2\beta,\quad H_4=v. \tag{41}$$

The axisymmetric versions of Eqs. (35)–(38) can be written as

$$\frac{\partial}{\partial t}\xi^{j'}=[H_i,C_k'^{ij}\xi_k'], \tag{42}$$

where the $C_k'^{ij}$ are given by the $\tau=0$ limit of Eq. (16):

$$C_k'^{1j}=C_k'^{j1}=\delta_{kj},\quad C_k'^{23}=C_k'^{32}=2\delta\beta\delta_{k2},$$
$$C_k'^{33}=2\delta\beta\delta_{k3},\quad C_k'^{34}C_k'^{43}=-\beta\delta_{k2}, \tag{43}$$

and

$$C_k'^{44}=0. \tag{44}$$

Now the axisymmetric equations of motion can be expressed in Hamiltonian form:

$$\frac{\partial}{\partial t}\xi^{j'}=\{\xi^{j'},H\}, \tag{45}$$

where the bracket is defined by

$$\{F,G\}=\langle C_k'^{ij}\xi_k'[F_i,G_j]\rangle \tag{46}$$

for arbitrary functionals F and G. We omit the straightforward demonstration that this bracket, satisfying Jacobi's identity, is a proper Poisson bracket.

In other words, the cold-ion limit of the previous four-field model is, like MHD, reduced MHD, and many other models, a Hamiltonian system. One obvious result is that energy of Eq. (39) is conserved since $\{H,H\}=0$.

For finite T_i the Hamiltonian of Eq. (39) is altered, without rigorous justification, in two ways. First, $F'=\varphi$ is replaced by F, the streamfunction of Eq. (1); this change is easily understood *a posteriori*, as shown below. Second, the internal energy is modified to include the ion contribution: $p^2/(2\beta)\to(1+\tau)p^2/(2\beta)$. These unsurprising changes yield the Hamiltonian of Eq. (6), whose physical plausibility was discussed in Sec. II.

Less straightforward are the finite-τ modifications of the Poisson bracket. In this regard, it is convenient to treat the parallel and perpendicular dynamics separately.

Consider first the parallel dynamics. It is clear that our task is to justify the replacement of Eq. (44) by Eq. (16). We do this in an *ad hoc* manner, using three constraints to construct the coefficient C_k^{44}. First note that at finite τ the streamfunction F differs, to leading order in δ, from the potential φ by $\delta\tau p$, a term that gives rise to the ion diamagnetic drift. On the other hand, as first shown by Mikhailovskii,[12] the parallel flow is advected only by the electrostatic drift, as indicated in Eq. (5). These two facts enforce the first term of Eq. (16). Finally, one finds that the resulting bracket satisfies Jacobi's identity only if the remaining term of Eq. (16) is also appended.

Similar "brute-force" procedures—inelegant but straightforward—have been attempted in the construction of perpendicular dynamics at finite τ, but without success. The perpendicular dynamics, involving gyroviscosity and perpendicular compressibility, are much more complicated and the physical constraints less clear than in the parallel case. Notice in particular that each proposed finite-τ modification must be checked for consistency with the Jacobi identity; the unwieldy form of typical FLR corrections [cf., for example, Eq. (25)] makes such checks extremely tedious.

Fortunately the gyromap permits a much simpler and more reliable implementation of FLR physics. To obtain the appropriate bracket for the above Hamiltonian we consider the reverse of the map defined by Eq. (31), setting

$$\begin{aligned} \xi^{1'} &= \xi^{1''} + (\delta\tau/2)\nabla_\perp^2(\xi^{3''} - 2\beta h), \\ \xi^{i'} &= \xi^{i''}, \quad i = 1,2,4, \end{aligned} \tag{47}$$

where

$$(\xi^{1''},\xi^{2''},\xi^{3''},\xi^{4''}) = (\nabla^2 F,\psi,p + 2\beta h,v).$$

The chain rule yields

$$\begin{aligned} \frac{\delta}{\delta\xi^{i'}} &= \frac{\delta}{\delta\xi^{i''}}, \quad i = 1,2,4, \\ \frac{\delta}{\delta\xi^{3'}} &= \frac{\delta}{\delta\xi^{3''}} - \frac{\delta\tau}{2}\nabla_\perp^2\frac{\delta}{\delta\xi^{1''}}. \end{aligned} \tag{48}$$

Inserting Eqs. (47) and (48) into the "parallel-corrected" $T_i = 0$ bracket defined by Eqs. (43), (16), and (46) produces the correct four-field bracket, which together with the Hamiltonian of Eq. (6), produces Eqs. (2)–(5).

In Sec. II B we chose to write the Hamiltonian equations in terms of the variables $\boldsymbol{\xi}$ defined by Eq. (8). Thus the Hamiltonian of Eq. (6) becomes that of Eq. (9) and the bracket obtained above in terms of $\boldsymbol{\xi}''$ becomes that given by Eqs. (15) and (16).

Notice that the electrostatic potential need not be defined for this closed system; the four-field variables ξ^i are advanced in time without knowledge of φ. It is nonetheless of interest to identify φ in terms of the four fields. There are two arguments leading to the correct answer, as given by Eq. (1).

First we can demand agreement between Eq. (3), involving φ, and Eq. (11) for the ξ^i. The point here is that Eq. (3) is free of FLR physics and derived easily from electron momentum conservation. Thus we use Eqs. (46)–(49) to find

$$\begin{aligned} \frac{\partial\xi^2}{\partial t} &= -\left[\psi,\frac{\delta H}{\delta\xi^{1'}}\right] - 2\delta\beta\left[\psi,\frac{\delta H}{\delta\xi^{3'}}\right] \\ &= -\left[\psi,\frac{\delta H}{\delta\xi^{1}}\right] - 2\delta\beta\left[\psi,\frac{\delta H}{\delta\xi^{3}} - \frac{\delta\tau}{2}\nabla_\perp^2\frac{\delta H}{\delta\xi^{1}}\right] \\ &= [\psi,(1+\rho^2\nabla_\perp^2)F - \delta\tau p] - \delta[\psi,p], \end{aligned}$$

which agrees with Eq. (3) only if

$$\varphi = (1+\rho^2\nabla_\perp^2)F - \delta\tau p,$$

as in Eq. (1).

The second argument proceeds by ordering directly the Braginskii gyroviscosity tensor as in Ref. 5. We express the ion velocity as

$$\mathbf{v} = \epsilon v_{\mathrm{A}}(\hat{\boldsymbol{\zeta}}\times\nabla_\perp F + v\hat{\boldsymbol{\zeta}}) + O(\epsilon^2),$$

and compute the $O(\epsilon)$ portion of the ion momentum balance equation; the result again is precisely Eq. (1).

IV. CASIMIR INVARIANTS

A. Derivation

Noncanonical field theories generally have a special class of constants of motion called Casimir invariants. These are entropylike or helicitylike constants, such as the magnetic and cross helicities of MHD. Since the four-field model, unlike ideal MHD, contains FLR physics and in addition is reduced, it is not obvious what these constants should be. [Direct calculation from Eqs. (2)–(5) leads to enormous and nearly hopeless labor.] We determine the Casimirs in this section using the bracket formalism.

By definition Casimir invariants are constants that commute with all functionals, i.e., C is a Casimir invariant if

$$\{C,F\} = 0, \quad \text{for all } F. \tag{49}$$

One can use Eq. (49) to obtain the constants. We begin with the 2-D, parallel corrected, cold-ion bracket of Eqs. (43), (16), and (46). Equation (49) can be manipulated, by partial integration, into the form

$$\{C,F\} = -\langle F_i[C_k^{ij}\xi^{k'},C_j]\rangle = 0. \tag{50}$$

Here, we have systematically set surface terms to zero. Independent of the boundary conditions necessary for the vanishing of these terms, the Casimirs so obtained will be constants of motion in the sense that their integrands will satisfy local conservation equations.

Now since Eq. (50) must be true for all functionals F, it follows that the coefficient of each F_i must vanish. This gives a system of four partial differental equations

$$C_k^{ij}[\xi^{k'},C_j] = 0, \quad i = 1,\dots,4, \tag{51}$$

which can be solved straightforwardly. We thus obtain the following four Casimir invariants:

$$C^{(1)} = \langle A(\xi^{2'})\rangle, \quad C^{(2)} = \langle \xi^{3'}B(\xi^{2'})\rangle, \tag{52}$$

$$C^{(3,4)} = \langle C_\pm[2\delta\beta\xi^{4'} + \beta\xi^{2'} \pm \sqrt{2}\rho(2\delta\beta\xi^{1'} - \xi^{3'})]\rangle, \tag{53}$$

where the $C_\pm$ are arbitrary functions.

Now in order to obtain the Casimirs for the four-field model it is necessary to map from the primed to the physical variables. We know that the quantities thus obtained will be Casimirs since the (parallel corrected) $T_i = 0$ bracket is isomorphic to the four-field bracket written in terms of the physical variables $W=\nabla_\perp^2 F$, ψ, p, and v. There is a one-to-one correspondence between Casimir invariants of isomorphic brackets. Thus we obtain the following Casimir invariants:

$$C^{(1)} = \langle A(\psi)\rangle,\quad C^{(2)} = \langle (p+2\beta h)\beta(\psi)\rangle,$$
$$C^{(3,4)} = \langle C_\pm\,[2\delta\beta v+\beta\psi \pm \sqrt{2}\times\rho(2\delta\beta W-\rho^2\nabla_\perp^2 p - p - 2\beta h)\,]\rangle\,. \tag{54}$$

These quantities are constants for the axisymmetric version of Eqs. (2)–(5), i.e., where $\nabla_\parallel$ is replaced by $-[\psi,\cdot\,]$. For three dimensions the functions A, B, and $C_\pm$ are restricted, as mentioned in Sec. II A. This restriction, among other things, is discussed in Sec. IV B.

B. Discussion

The restriction of axisymmetry for constancy of the Casimir invariants can be eased. In fact, the existence of the above Casimirs for arbitrary functions A, B, and $C_\pm$ in three dimensions is tantamount to the existence of a solution $\tilde\psi$ to[2]

$$\nabla_\parallel\tilde\psi = \frac{\partial\tilde\psi}{\partial\zeta} + [\tilde\psi,\psi] = 0. \tag{55}$$

The question of the existence of a global $\tilde\psi$ is the same as that of the existence of a constant of motion for the one degree-of-freedom Hamiltonian system, for which the poloidal plane is the phase space, ζ is the time, and ψ is the Hamiltonian. Stated yet another way, the existence of $\tilde\psi$ is equivalent to the existence of magnetic surfaces. In the general case it is unlikely that $\tilde\psi$ exists (recall that ζ is a periodic variable).

Nevertheless, let us assume that $\tilde\psi$ exists and change variables; we will use the field $\tilde\psi$ instead of ψ. We wish to transform our 3-D Poisson bracket, Eq. (15), into one written in terms of the variable $\tilde\psi$. To do this we relate ψ and $\tilde\psi$ variations of an arbitrary functional F. This yields

$$\nabla_\parallel\frac{\partial F}{\partial\psi} = \left[\frac{\delta F}{\partial\tilde\psi},\tilde\psi\right]. \tag{56}$$

Upon inserting Eq. (56) into Eq. (15) we see that the transformation $\psi\to\tilde\psi$ takes the 3-D four-field bracket into the axisymmetric bracket with $\tilde\psi$ replacing ψ. This bracket has the Casimir invariants of Eq. (54) for *arbitrary* functions A, B, and $C_\pm$, but with $\tilde\psi$ replacing ψ.

Thus we have shown that the existence of the general Casimir invariants is tantamount to the existence of magnetic surfaces. It follows that the degree to which one believes magnetic surfaces exist in a tokamak discharge should be the same as the degree to which one believes Casimir invariants with arbitrary functions A, B, and $C_\pm$ exist.

One case in which solutions to Eq. (55) do exist is that of helical symmetry. Then one has $\psi(r,\tilde\theta,t)$, where $\tilde\theta=\theta-\zeta/q_0$ and it can be shown by direct substitution that the following solves Eq. (55):

$$\tilde\psi(r,\tilde\theta,t) = \psi(r,\tilde\theta,t) = r^2/(2q_0). \tag{57}$$

Here, $\tilde\psi$ is the helical flux function.

Let us next consider the meaning of the Casimir invariants. We have mentioned that these invariants are related to the magnetic and cross helicities. Specifically, they are the remnants of the ideal MHD quantities that survive our ordering procedure. The cross helicity also survives our inclusion of FLR physics, which is manifest in the fact that $\mathbf{v}\cdot\mathbf{B}$ has an additional term $\mathbf{v}_M\cdot\mathbf{B}$ arising from the gyromap. Since all four of our Casimir invariants have one of the two forms

$$\tilde C^{(1)} = \langle f(\chi)\rangle,\quad \tilde C^{(2)} = \langle \Upsilon g(\chi)\rangle, \tag{58}$$

where f and g are arbitrary functions, we will discuss their meaning in general terms for the fields χ and Υ. If we divide our physical domain into cells, which we label by the value of χ at, say, the center, then the invariant $\tilde C^{(1)}$ determines the number of cells with a particular value of χ. This can be shown by choosing f to be the characteristic function. The same procedure can be used to show that the invariant $\tilde C^{(2)}$ determines the sum of the values of Υ on those cells with a particular value of the field χ. Neither of these invariants determine spatial correlation, i.e., the placement of the cells with a given value.

To conclude we take limits of the Casimir invariants, Eqs. (54), and show that they reduce to previously obtained Casimir invariants. To facilitate this we rewrite $\tilde C^{(3)}$ and $\tilde C^{(4)}$ as follows:

$$\tilde C^{(3)} \equiv \langle [C_-(D-\rho E)+C_-(D+\rho E)]/(4\delta)\rangle,$$
$$\tilde C^{(4)} \equiv \langle [C_+(D-\rho E)-C_+(D+\rho E)]/(4\delta\beta\sqrt{2}\rho)\rangle, \tag{59}$$

where $D\equiv\beta(\psi+2\delta v)$ and $E\equiv\sqrt{2}[(1+\rho^2\nabla_\perp^2)p - 2\delta\beta\nabla_\perp^2 F+2\beta h\,]$. In the cold-ion limit $\rho,\tau\to 0$ and $F\to\varphi$, and the Casimir invariants of Eq. (59) become

$$\tilde C^{(3)} = \langle C_-(\psi+2\delta v)/(2\delta)\rangle,$$
$$\tilde C^{(4)} = \langle [\nabla_\perp^2\varphi-(p+2\beta h)/(2\delta\beta)]C'_+(\psi+2\delta v)\rangle. \tag{60}$$

We can further take the limit $\delta\to 0$ and obtain the invariants for compressible reduced MHD:

$$\tilde C^{(3)} = \langle vC'_-(\psi)\rangle,$$
$$\tilde C^{(3)} = \langle \nabla_\perp^2\varphi C'_+(\psi)-(p+2\beta h)vC''_+(\psi)/\beta\rangle. \tag{61}$$

This model was introduced in HKM.

V. SUMMARY

The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics, as well as other effects related to compressibility and electron adiabaticity. Much simpler than a rigorous or even reduced description of the same physics, it still preserves essential features of the underlying exact dynamics.

The model is given by Eqs. (2)–(5) in terms of physical variables and by Eqs. (10)–(13) in terms of the field variables ξ_i. [The latter are defined by Eqs. (8).] A Hamiltonian expression of the model, in terms of a Hamiltonian functional and generalized Poisson bracket, is given by Eqs. (9) and (15)–(18).

Only the dissipationless form of the model is presented. In many applications such dissipative processes as resistivity and viscosity are appropriately included in the conventional way—for example, by appending ηJ to the rhs of Eq. (3). The Hamiltonian property is then lost, but it remains significant in that dissipation has been introduced in an explicit and physical way. As discussed in the Introduction, there is no fake dissipation.

In large part because of its Hamiltonian property the present four-field model conserves not only total energy, but also four generalized helicities, or Casimir invariants. These constants of the motion, which are given by Eqs. (7), have considerable value in applications.

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