MEMORANDUM

To: file
From: MVU
Subject: Shifted circle geometry for BOUT
Updated: January 13, 2009

The Grad-Shafranov equation for a low β equilibrium ordering, $\beta \sim \epsilon^2$, can be solved order by order in the small parameter ϵ. To lowest order flux surfaces are concentric circles, to second order they are shifted circles, to third order they have elliptical and triangular distortion. Assuming that flux surfaces consist of shifted circles one can verify that the solution to Grad-Shafranov equation can be obtained dropping terms $O(\epsilon^2)$ and higher.

0.1 Geometry of flux surfaces

Start from

$$\begin{aligned} R &= R_0 + \Delta(r) + r\cos(\theta) \\ Z &= r\sin(\theta) \end{aligned} \tag{1}$$

Here the parameters are:
R_0 - major radial coordinate of the center of outermost flux surface
a - minor radius of the outermost flux surface
Δ_0 - shift in outward direction from center of outermost flux surface to the center of innermost surface (i.e. magnetic axis)
Non-dimensional minor radial coordinate, ρ, is

$$\rho = r/a \tag{2}$$

The aspect ratio, ϵ, is defined as

$$\epsilon = a/R_0 \tag{3}$$

The profiles of pressure, $p(\rho)$, and safety factor, $q(\rho)$, are given. The normalizing value for pressure is p_0 and $p_1(0) = 1$, so that

$$p = p_0 p_1(\rho) \tag{4}$$

The reference value for the toroidal field B_t is B_0, so that

$$B_t(r, \theta) = B_0 \frac{R_0}{R}, \tag{5}$$

The safety factor is

$$q(r) = \frac{d\Psi_t/dr}{d\Psi_p/dr}, \tag{6}$$

and within our approximation, i.e. dropping terms $O(\epsilon^2)$, $\Psi_t = B_0 r^2/2$, so the poloidal flux is

$$\psi_p(r) = \int_0^r \frac{1}{q} \frac{d\Psi_t}{dr} dr = B_0 \int_0^r \frac{r}{q(r)} dr \tag{7}$$

Next, as

$$q(r) \approx \frac{r B_t}{R_0 B_p}, \tag{8}$$

the poloidal field, B_p, is approximately defined by the q profile

$$B_p(r, \theta) \approx \frac{r B_t}{R_0 q(r)} = B_0 \frac{r}{q(r) R}, \tag{9}$$

This formula accounts for toroidal bending but one also needs a correction for the shift. Using

$$|B_p| = |\nabla\phi \times \nabla\psi_p| = \frac{|\partial\psi_p/\partial r| \, |\nabla r|}{R} \tag{10}$$

and

$$|\nabla r| = \frac{1}{1 + \Delta' \cos\theta}, \tag{11}$$

one finds

$$B_p(r, \theta) = \frac{B_0 r/qR}{1 + \Delta' \cos(\theta)}, \tag{12}$$

Within our ordering the equation for the Shafranov shift $\Delta(r)$ is

$$\frac{d\Delta}{dr} = \epsilon\frac{q^2}{\rho^3}\int_0^\rho dx \;\; x(\beta_p x\frac{\partial p_1}{\partial x} - \frac{x^2}{q^2}) \tag{13}$$

where the poloidal beta, β_p, is

$$\beta_p = \frac{8\pi p_0}{\epsilon^2 B_0^2} \tag{14}$$

and boundary conditions are

$$\begin{aligned} \Delta'(0) &= 0 \\ \Delta(0) &= \Delta_0 \\ \Delta(a) &= 0. \end{aligned} \tag{15}$$

Another integration is needed to find $\Delta(r)$

$$\Delta(r) = \Delta_0 + \int_0^r \frac{d\Delta}{dr} dr \tag{16}$$

0.2 Construction of orthogonal coordinates

The ρ, θ coordinates are not orthogonal, so a new poloidal coordinate, $\tilde{\theta}$ is constructed instead.
First note

$$\begin{aligned} \frac{\partial R}{\partial r} &= \cos(\theta) + \Delta' \\ \frac{\partial R}{\partial \theta} &= -r\sin(\theta) \\ \frac{\partial Z}{\partial r} &= \sin(\theta) \\ \frac{\partial Z}{\partial \theta} &= r\cos(\theta) \end{aligned} \tag{17}$$

Now use identity

$$\begin{pmatrix} \partial r/\partial R & \partial r/\partial Z \\ \partial\theta/\partial R & \partial\theta/\partial Z \end{pmatrix} = \begin{pmatrix} \partial R/\partial r & \partial R/\partial\theta \\ \partial Z/\partial\theta & \partial Z/\partial\theta \end{pmatrix}^{-1} = \frac{1}{J}\begin{pmatrix} \partial Z/\partial\theta & -\partial R/\partial\theta \\ -\partial Z/\partial r & \partial R/\partial r \end{pmatrix}, \tag{18}$$

where the Jacobian, J, is

$$J = \frac{\partial R}{\partial r}\frac{\partial Z}{\partial \theta} - \frac{\partial R}{\partial \theta}\frac{\partial Z}{\partial r} = r(1 + \Delta' \cos(\theta)) \tag{19}$$

The first row can be identified as ∇r, the second as $\nabla \theta$, so that

$$\nabla r = \frac{1}{r(1 + \Delta' \cos(\theta))}(r\cos(\theta), r\sin(\theta)) \tag{20}$$

and

$$\nabla \theta = \frac{1}{r(1 + \Delta' \cos(\theta))}(-\sin(\theta), \cos(\theta) + \Delta') \tag{21}$$

Integration along the ∇r produces a $\tilde{\theta}$=const line. At $\rho = a$ the $\tilde{\theta}$ is defined to be equal to θ. Now, considering $\theta(\tilde{\theta}, r)$, and using $\nabla\tilde{\theta} \cdot \nabla r = 0$

$$\nabla\theta \cdot \nabla r = \left(\frac{\partial \theta}{\partial r}\nabla r + \frac{\partial \Theta}{\partial \tilde{\theta}}\nabla\tilde{\theta}\right) \cdot \nabla r = \frac{\partial \theta}{\partial r}(\nabla r)^2 \tag{22}$$

Then, for constant $\tilde{\theta}$,

$$\frac{\partial \theta}{\partial r} = \frac{\nabla r \cdot \nabla \theta}{\nabla r \cdot \nabla r} = \frac{\Delta'}{r}\sin(\theta) \tag{23}$$

Integrating inwards from the outer boundary produces $\theta(r, \tilde{\theta})$

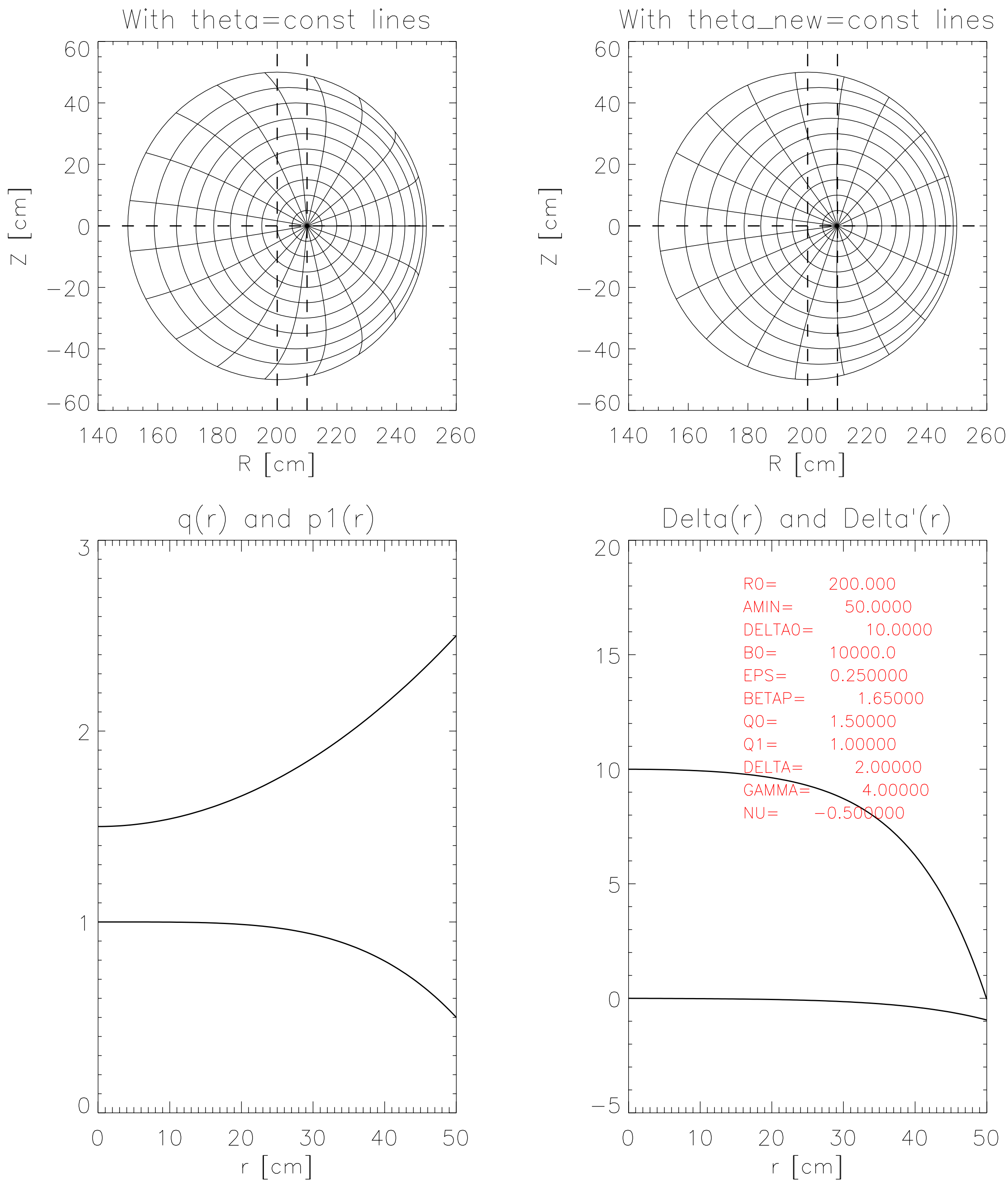
$$\theta(r, \tilde{\theta}) = \tilde{\theta} - \int_a^r \theta_r dr \tag{24}$$

Note: As suggested by I. Joseph [17-Oct-08] one can carry out the θ inegration analytically

$$\int \frac{d\theta}{\sin(\theta)} = \int \frac{\Delta'}{r} dr = f(r) \tag{25}$$

so

$$\tan(\theta/2) = \tan(\tilde{\theta}/2)\exp(f(r)) \tag{26}$$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With theta_new=const lines
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Delta(r) and Delta'(r)
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AMIN= 50.0000
DELTA0= 10.0000
B0= 10000.0
EPS= 0.250000
BETAP= 1.65000
Q0= 1.50000
Q1= 1.00000
DELTA= 2.00000
GAMMA= 4.00000
NU= −0.500000