6-Field Models (V3.4.4)

Tianyang Xia[1,2,*], Jianguo Chen[2,#], Xueqiao Xu[3]

February 9, 2018

[1]Institute of Plasma Physics, Chinese Academy of Sciences, Hefei, China

[2]School of Physics, Peking University, Beijing, 100871, China

[3]Lawrence Livermore National Laboratory, Livermore, CA 94550, USA

Email: *xiaty@ipp.ac.cn, #cjgls@pku.edu.cn

Contents

1 MHD equations **1**

2 Full set of equations **3**
2.1 Curvature vector . . . 3
2.2 Radial Force Balance . . . 4
2.3 Vorticity . . . 5
2.4 Density equation . . . 7
2.5 Parallel velocity equations . . . 9
2.6 Ohm's Law . . . 11
2.7 Ion temperature equation . . . 12
2.8 Electron temperature equation . . . 15
2.9 Magnetic Flutter in Parallel Thermal Conduction . . . 17

3 Normalization **19**

4 Normalization by $\bar{B}$ (Recommended) **24**

5 Results **29**

6 Conclusion **29**

7 Reduced to 3Field model **34**
7.1 Pressure Equation 34
7.2 Ohm's Law 34
7.3 Vorticity Equation 34

1 MHD equations

The original equations we want to use in the simulations are[1]:

$$\frac{\partial}{\partial t} n_i + \nabla \cdot (n_i \boldsymbol{V}_i) = 0,$$

$$\frac{\partial}{\partial t}(m_j n_j \boldsymbol{V}_j) + \boldsymbol{\nabla} P_j + \boldsymbol{\nabla} \cdot \left(\overleftrightarrow{\boldsymbol{\pi}}_j + m_j n_j \boldsymbol{V}_j \boldsymbol{V}_j\right) = Z_j e n_j (\boldsymbol{E} + \boldsymbol{V}_j \times \boldsymbol{B}) + \boldsymbol{R}_j,$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} P_j + \frac{1}{2} m_j n_j V_j^2\right) + \nabla \cdot \boldsymbol{Q}_j = Z_j e n_j \boldsymbol{V}_j \cdot \boldsymbol{E} + \boldsymbol{V}_i \cdot \boldsymbol{R}_i + W_j.$$

Here, the energy flux is given by Branginskii

$$\boldsymbol{Q}_j = \left(\frac{5}{2} k_B n_j T_j + \frac{1}{2} m_j n_j V_j^2\right) \boldsymbol{V}_j + \overleftrightarrow{\pi} \cdot \boldsymbol{V}_j + \boldsymbol{q}_j$$

with

$$\boldsymbol{q}_i = -\kappa_{\parallel i} \nabla_{\parallel} T_i + \frac{5 P_i}{2 m_i \Omega_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i - \kappa_{\perp i} \nabla_{\perp} T_i \tag{1.1}$$

$$\boldsymbol{q}_e = -\kappa_{\parallel e} \nabla_{\parallel} T_e + \frac{5 P_e}{2 m_e \Omega_e} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e - \kappa_{\perp e} \nabla_{\perp} T_e - 0.71 \frac{T_e}{e} J_{\parallel} + \frac{3 \nu_e}{2 \Omega_e} \frac{T_e \hat{\boldsymbol{b}} \times \boldsymbol{J}}{e} \tag{1.2}$$

and the friction force

$$\boldsymbol{R}_e = -\boldsymbol{R}_i = e n_e \left(\frac{\boldsymbol{J}_{\parallel}}{\sigma_{\parallel}} + \frac{\boldsymbol{J}_{\perp}}{\sigma_{\perp}}\right) - 0.71 k_B n_e \boldsymbol{\nabla}_{\parallel} T_e + \frac{3 \nu_e}{2 \Omega_e} k_B n_e \boldsymbol{b} \times \boldsymbol{\nabla} T_e.$$

The energy exchange term

$$
\begin{aligned}
W_i &= \frac{3m_e}{m_i}\frac{k_B n_e}{\tau_e}\left(T_e - T_i\right), \\
W_e &= -W_i - \boldsymbol{R}_e \cdot \left(\boldsymbol{V}_e - \boldsymbol{V}_i\right). \\
\tau_e &= \frac{3\sqrt{m_e}\left(k_B T_e\right)^{\frac{3}{2}}}{4\sqrt{2\pi} n_e e^4 \ln\Lambda}
\end{aligned}
$$

The viscous stress tensor $\overleftrightarrow{\boldsymbol{\pi}}_j$ is composed by three parts, sp-caleed "parallel" part $\overleftrightarrow{\boldsymbol{\pi}}_{cj}$, "perpendicular" part $\overleftrightarrow{\boldsymbol{\pi}}_{\perp j}$ and "gyroviscous" part $\overleftrightarrow{\boldsymbol{\pi}}_{gj}$. The "parallel" viscous stress tensor yields magnetic pumping term which can damps the plasma flow shear [2]. The expression of $\overleftrightarrow{\boldsymbol{\pi}}_{cj}$ is given by Ref. [1] [3] as

$$
\overleftrightarrow{\boldsymbol{\pi}}_{cj} = \left(\boldsymbol{bb} - \frac{1}{3}\overleftrightarrow{\boldsymbol{I}}\right)\pi_{cj}, \tag{1.3}
$$

and

$$
\begin{aligned}
\pi_{cj} &= \eta_j^0\left[\left(\boldsymbol{V}_E + \boldsymbol{V}_{Pj}\right)\cdot\boldsymbol{\kappa} - \frac{2}{\sqrt{B}}\nabla_\parallel\left(\sqrt{B}V_{\parallel j}\right)\right] \\
&+\eta_j^0\left[\left(1.61 k_B \frac{\boldsymbol{b}\times\nabla T_j}{Z_j e B}\right)\cdot\boldsymbol{\kappa} + \left(\frac{7.09}{5P_{ij}\sqrt{B}}\right)\nabla_\parallel\left(\sqrt{B}\kappa_{\parallel i}\nabla_\parallel T_i\right)\right] \\
&-\eta_j^0\left[\frac{2.44\kappa_{\parallel i}\nabla_\parallel T_i}{5P_j}\left(2.27\nabla_\parallel \ln T_j - \nabla_\parallel \ln P_j\right)\right] \\
&-\eta_j^0\left[\left(\frac{\mu_0}{B^3}\right)\boldsymbol{b}\times\nabla\left(P_i + P_e\right)\cdot\left(\nabla\Phi + \frac{\nabla P_j}{Z_j e n} + 1.61\frac{\nabla T_j}{Z_j e}\right)\right].
\end{aligned} \tag{1.4}
$$

The "perpendicular" part is a factor of ν_j/Ω_j smaller than the "gyroviscous" part [1].

For those equations, we can get the momentum and energy of Braginskii equations as

$$
m_j n_j \frac{d}{dt}\boldsymbol{V}_j + \boldsymbol{\nabla}P_j + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_j = Z_j e n_j\left(\boldsymbol{E} + \boldsymbol{V}_j\times\boldsymbol{B}\right) + \boldsymbol{R}_j, \tag{1.5}
$$

$$
\frac{d}{dt}\left(\frac{3}{2}P_j\right) + \frac{5}{2}P_j\boldsymbol{\nabla}\cdot\boldsymbol{V}_i = -\overleftrightarrow{\boldsymbol{\pi}}_j : \boldsymbol{\nabla}\boldsymbol{V}_j - \boldsymbol{\nabla}\cdot\boldsymbol{q}_j + W_j, \tag{1.6}
$$

Here $d/dt = \partial/\partial t + \boldsymbol{V}_j\cdot\boldsymbol{\nabla}$. The electron velocity V_e is written as

$$
V_{\parallel e} = V_{\parallel i} - \frac{J_\parallel}{e n_e}. \tag{1.7}
$$

With the quasineutral condition,

$$Z_i n_i = n_e. \tag{1.8}$$

if the ion and electron have same temperature $T_i = T_e$:

$$P_0 = P_i + P_e = n_i * T_i + n_e * T_e = P_i * (1 + Z_i) \tag{1.9}$$

The linearizing expressions use the definations of variables as

$$\begin{aligned} n_j &= n_{j0} + n_{j1}, \\ P_j &= P_{j0} + p_{j1}, \\ P = P_i + P_e &= P_0 + p_1 = (P_{i0} + P_{e0}) + (p_{i1} + p_{e1}), \\ \Phi &= \Phi_0 + \phi, \\ J_\parallel &= J_{\parallel 0} + J_{\parallel 1}, \\ \boldsymbol{b} &= \boldsymbol{b}_0 - \boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi. \end{aligned}$$

Notice that the flute reduction $k_\parallel \sim \epsilon k_\perp$ is adopted in this work.

2 Full set of equations

2.1 Curvature vector

The field-line curvature vector κ is defined as

$$\kappa \equiv \mathbf{b} \cdot \nabla \mathbf{b} = (\nabla \times \mathbf{b}) \times \mathbf{b} \tag{2.1.1}$$

due to

$$\nabla(\mathbf{b} \cdot \mathbf{b}) = 2 \left[\mathbf{b} \times (\nabla \times \mathbf{b}) + (\mathbf{b} \cdot \nabla)\mathbf{b}\right] = 0$$

This can be re-written as

$$\kappa = \left[\frac{1}{B}\nabla \times \mathbf{B} - \mathbf{B} \times \nabla\left(\frac{1}{B}\right)\right] \times \mathbf{b}$$

Using $\nabla\left(\frac{1}{B}\right) = -\frac{\nabla B}{B^2}$, this becomes

$$\begin{aligned} \kappa &= \frac{\mu_0}{B^2}\mathbf{J} \times \mathbf{B} + \frac{1}{B}(\mathbf{b} \times \nabla B) \times \mathbf{b} \\ &= \frac{\mu_0}{B^2}\mathbf{J} \times \mathbf{B} + \frac{1}{B}\nabla_\perp B \end{aligned} \tag{2.1.2}$$

$$
\begin{aligned}
\mathbf{b}\cdot\nabla\times\mathbf{b} &= \mathbf{b}\cdot\nabla\times\frac{\mathbf{B}}{B} = \mathbf{b}\cdot\left[\frac{1}{B}\nabla\times\mathbf{B} + \frac{1}{B^2}\mathbf{B}\times\nabla B\right] \\
&= \mathbf{b}\cdot\frac{\nabla\times\mathbf{B}}{B} && (2.1.3) \\
\frac{\nabla\times\mathbf{b}}{B} &= \frac{1}{B^2}\nabla\times\mathbf{B} + \frac{1}{B^2}\mathbf{b}\times\nabla B && (2.1.4) \\
\frac{2}{B}\mathbf{b}\times\kappa &= \frac{2}{B}\mathbf{b}\times[(\nabla\times\mathbf{b})\times\mathbf{b}] \\
&= \frac{2}{B}\nabla\times\mathbf{b} - \frac{2}{B^2}\mathbf{b}\cdot(\nabla\times\mathbf{B})\,\mathbf{b} && (2.1.5) \\
\nabla\times\frac{\mathbf{b}}{B} &= \frac{1}{B}\nabla\times\mathbf{b} + \frac{1}{B^2}\mathbf{b}\times\nabla B \\
&= \frac{2}{B}\nabla\times\mathbf{b} - \frac{1}{B^2}\nabla\times\mathbf{B} \\
&= \frac{2}{B}\mathbf{b}\times\kappa - \frac{1}{B^2}\nabla\times\mathbf{B} + \frac{2}{B^2}\mathbf{b}\cdot(\nabla\times\mathbf{B})\,\mathbf{b} \\
&= \frac{2}{B}\mathbf{b}\times\kappa + \frac{\mu_0(\mathbf{J}_\parallel - \mathbf{J}_\perp)}{B^2} && (2.1.6) \\
&\simeq \frac{2}{B}\mathbf{b}\times\kappa + O(\beta) && (2.1.7)
\end{aligned}
$$

For the last step (2.1.7), looking into [1] APPENDIX A for details.

2.2 Radial Force Balance

According to the radial force balance, the radial electric field can be expressed as:

$$E_r = \left(\frac{\nabla P_i}{Z_i e n_i}\right)_r - V_{\theta i}B_\phi + V_{\phi i}B_\theta \qquad (2.2.1)$$

We define part of equilibrium E_{r0}:

$$E_{r,dia0} = -\left(\nabla\phi_{dia0}\right)_r = \left(\frac{\nabla P_i}{Z_i e n_i}\right)_r \qquad (2.2.2)$$

which makes the $\boldsymbol{E}_{r,dia0}\times\boldsymbol{B}$ flow balance ion diamagnetic flow:

$$\frac{\boldsymbol{E}_{r,dia0}\times\boldsymbol{B}}{B^2} = -\frac{\boldsymbol{b}\times\nabla P_i}{Z_i e n_i B} \qquad (2.2.3)$$

The remained part of E_{r0} is called net $E_{r0,net}$ introducing the net flow:

$$\boldsymbol{V}_{e0,net} = \frac{\boldsymbol{E}_{r0,net}\times\boldsymbol{B}}{B^2} = \frac{\boldsymbol{B}\times\nabla\Phi_{0,net}}{B^2} \qquad (2.2.4)$$

2.3 Vorticity

First, we can add the ion and electron momentum equations together and obtain

$$m_i n_i \frac{d}{dt}\boldsymbol{V}_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} + \boldsymbol{\nabla}P \;=\; \boldsymbol{J}\times\boldsymbol{B}. \tag{2.3.1}$$

Here the electron momentum is negalected for $m_e \ll m_i$ and the viscosity is kept. Multiply the term $\boldsymbol{b}\cdot\boldsymbol{\nabla}\times$ on Eq. (2.3.1), we derive

$$\begin{aligned}
\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\left[m_i n_i \frac{d}{dt}\boldsymbol{V}_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}\right] &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\times(\boldsymbol{J}\times\boldsymbol{B}) \\
&= B^2\boldsymbol{b}\cdot\nabla\left(\frac{J_\parallel}{B}\right) - \frac{1}{B^2}\boldsymbol{b}\times(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_\perp B^2 \\
&= B^2\boldsymbol{b}\cdot\nabla\left(\frac{J_\parallel}{B}\right) - \frac{1}{\mu_0}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}_\perp B^2.
\end{aligned}$$

Because $\boldsymbol{\kappa}\equiv\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} = \frac{\mu_0}{B^2}\boldsymbol{J}\times\boldsymbol{B} + \frac{1}{B}\boldsymbol{\nabla}_\perp B$, so the last equation becomes

$$\boldsymbol{b}\cdot(\boldsymbol{\nabla}-2\boldsymbol{\kappa})\times\left[m_i n_i \frac{d}{dt}\boldsymbol{V}_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}\right] \;=\; B^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\left(\frac{J_\parallel}{B}\right) + 2\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}P.$$

Negalect the "parallel" voscous term and the curvature on momentum, then take the time derivative operator out of the spatial differencial, the vorticity equation is obtained

$$\begin{aligned}
\frac{d}{dt}\varpi &= \frac{\partial}{\partial t}\varpi + (\boldsymbol{V}_E + \boldsymbol{V}_{\parallel i} + \boldsymbol{V}_{Pi})\cdot\nabla\varpi \\
&= B^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\left(\frac{J_\parallel}{B}\right) + 2\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}P_1 \\
&\quad -\frac{1}{2\Omega_i}\left[n_i Z_i e\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\Phi\right) - m_i\Omega_i\boldsymbol{b}\times\boldsymbol{\nabla}n_i\cdot\boldsymbol{\nabla}V_E^2\right] \\
&\quad +\frac{1}{2\Omega_i}\left[\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2 P_i\right) - \nabla_\perp^2\left(\boldsymbol{V}_E\cdot\boldsymbol{\nabla}P_i\right)\right].
\end{aligned}$$

Here the vorticity is defined as

$$\begin{aligned}
\varpi &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\times m_i n_i V_i \\
&\simeq n_{i0}\frac{m_i}{B_0}\left(\nabla_\perp^2\phi + \frac{1}{n_{i0}}\nabla_\perp\phi\cdot\nabla_\perp n_{i0} + \frac{1}{n_{i0}Z_i e}\nabla_\perp^2 p_{i1}\right).
\end{aligned} \tag{2.3.2}$$

The ion velocity is composed by $E\times B$ drift $\boldsymbol{V}_E$, diamagnetic drift $\boldsymbol{V}_D$, polarization drift $\boldsymbol{V}_{Pi}$ (Eq.(2.4.1)) and perturbed parallel volocity $V_{\parallel i}\boldsymbol{b}$

$$
\begin{aligned}
\boldsymbol{V}_i &= \boldsymbol{V}_{\perp i} + V_{\|i}\boldsymbol{b} = \boldsymbol{V}_E + \boldsymbol{V}_D + V_{\|i}\boldsymbol{b} + \boldsymbol{V}_{Pi} = \\
&= \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp \Phi + \frac{1}{Z_i e n_i B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp P_i + V_{\|i}\boldsymbol{b} + \boldsymbol{V}_{Pi} \\
&= \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp \phi + \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp \Phi_{0,net} + \frac{1}{Z_i e n_i B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp p_{i1} + V_{\|i}\boldsymbol{b}_0 + \boldsymbol{V}_{Pi} \qquad (2.3.3)\\
\boldsymbol{V}^2_{E\times B} &= \left(\frac{\boldsymbol{b}\times\nabla\Phi}{B}\right)^2 = \left(\frac{\nabla_\perp\Phi}{B}\right)^2 = \frac{(\nabla\Phi)^2 - (\nabla_\|\Phi)^2}{B^2} \\
&= \frac{1}{B^2}\left[(\nabla\Phi_0)^2 + 2\nabla\Phi_0\cdot\nabla\phi + (\nabla\phi)^2 - (\nabla_\|\Phi_0)^2 - 2\nabla_\|\Phi_0\cdot\nabla_\|\phi - (\nabla_\|\phi)^2\right] \\
&= \frac{1}{B^2}\left[(\nabla_\perp\Phi_0)^2 + 2\nabla_\perp\Phi_0\cdot\nabla_\perp\phi + (\nabla_\perp\phi)^2\right] \qquad (2.3.4)\\
\boldsymbol{V}_{0,net} &= \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}_\perp \Phi_{0,net} \qquad (2.3.5)\\
\varpi_0 &\simeq n_{i0}\frac{m_i}{B_0}\left(\nabla^2_\perp\Phi_0 + \frac{1}{n_{i0}}\nabla_\perp\Phi_0\cdot\nabla_\perp n_{i0} + \frac{1}{n_{i0}Z_i e}\nabla^2_\perp P_{i0}\right) \\
&= n_{i0}\frac{m_i}{B_0}\left(\nabla^2_\perp\Phi_{0,net} + \frac{1}{n_{i0}}\nabla_\perp\Phi_0\cdot\nabla_\perp n_{i0}\right) \qquad (2.3.6)\\
\nabla_\perp\Phi &= \frac{\boldsymbol{b}\times\nabla_\perp\Phi}{B}\times\boldsymbol{B} = \boldsymbol{V}_{E\times B}\times\boldsymbol{B} \qquad (2.3.7)
\end{aligned}
$$

Notice that the cancellation of zero-order of $E\times B$ and diamagnetic drift is applied here for equilibrium. For the last step ,we use the linearized expression

$$
\begin{aligned}
\frac{\partial}{\partial t}\varpi \quad = \quad & B_0^2\boldsymbol{\nabla}_{\|0}\left(\frac{J_{\|1}}{B}\right) - B_0^2\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{J_{\|0}}{B_0}\right) + 2\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}p_1 \\
& -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\Phi_0\cdot\boldsymbol{\nabla}\varpi - \frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\nabla\varpi_0 - V_{\|i}\nabla_{\|0}\varpi_0 \\
& -\frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}p_{i1}\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\Phi_0\right) - \frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}P_{i0}\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\phi\right) \\
& +\frac{1}{2\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i1}\cdot\boldsymbol{\nabla}\left(\frac{\nabla\Phi_0}{B_0}\right)^2 + \frac{1}{\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i0}\cdot\boldsymbol{\nabla}\left(\frac{\nabla\Phi_0\cdot\nabla\phi}{B_0^2}\right) \\
& -\frac{1}{2\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i1}\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\|\Phi_0}{B_0}\right)^2 - \frac{1}{\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i0}\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\|\Phi_0\nabla_\|\phi}{B_0^2}\right) \\
& +\frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2P_{i0}\right) + \frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\Phi_0\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2p_{i1}\right) \\
& -\frac{1}{2\Omega_i}\nabla_\perp^2\left(\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\Phi_0\cdot\boldsymbol{\nabla}p_{i1}\right) - \frac{1}{2\Omega_i}\nabla_\perp^2\left(\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P_{i0}\right) \\
& -B_0^2\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{J_{\|1}}{B}\right) \\
& -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\boldsymbol{\nabla}\varpi - V_{\|i}\nabla_{\|0}\varpi + V_{\|i}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\nabla\varpi_0 \\
& -\frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}p_{i1}\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\phi\right) \\
& +\frac{1}{2\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i0}\cdot\boldsymbol{\nabla}\left(\frac{\nabla\phi}{B_0}\right)^2 + \frac{1}{\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i1}\cdot\boldsymbol{\nabla}\left(\frac{\nabla\Phi_0\cdot\nabla\phi}{B_0^2}\right) \\
& -\frac{1}{2\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i0}\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\|\phi}{B_0}\right)^2 - \frac{1}{\Omega_i}Z_ieB_0\boldsymbol{b}_0\times\boldsymbol{\nabla}n_{i1}\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\|\Phi_0\nabla_\|\phi}{B_0^2}\right) \\
& +\frac{1}{2\Omega_i}\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2p_{i1}\right) \\
& -\frac{1}{2\Omega_i}\nabla_\perp^2\left(\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}p_{i1}\right).
\end{aligned} \tag{2.3.8}
$$

2.4 Density equation

The density equation can be written

$$
\frac{\partial}{\partial t}n_i + \boldsymbol{V}_i\cdot\nabla n_i \quad = \quad -n_i\boldsymbol{\nabla}\cdot\boldsymbol{V}_i.
$$

It is convient to use the exact identities

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{V}_E &= \left(\boldsymbol{\nabla} \times \frac{\boldsymbol{b}}{B}\right) \cdot \boldsymbol{\nabla}\Phi \\
&\simeq \frac{2}{B}\boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla}\Phi, \\
\boldsymbol{\nabla} \cdot (n_i \boldsymbol{V}_D) &= \frac{1}{Z_i e}\left(\boldsymbol{\nabla} \times \frac{\boldsymbol{b}}{B}\right) \cdot \boldsymbol{\nabla} P_i \\
&\simeq \frac{2}{Z_i e B}\boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} P_i, \\
\boldsymbol{\nabla} \cdot \left(V_{\|i}\boldsymbol{b}\right) &= B\nabla_{\|}\left(\frac{V_{\|i}}{B}\right).
\end{aligned}
$$

Here, $\boldsymbol{\nabla} \times \frac{\boldsymbol{b}}{B} \simeq \frac{2}{B}\boldsymbol{b} \times \boldsymbol{\kappa}$ (2.1.7). The polarization drift is defined as

$$
\boldsymbol{V}_P = \frac{m_j}{Z_j e B^2}\frac{d\boldsymbol{E}}{dt} \tag{2.4.1}
$$

and it is the small correction to the $E \times B$ and diamagnetic velocities. For ion it is of the order δ^2 and for electron $\sqrt{m_e/m_i}$ samller than the other two. ?????

We can rewrite the density equation as

$$
\begin{aligned}
\frac{\partial}{\partial t} n_i = &-\frac{1}{B_0}\boldsymbol{b} \times \boldsymbol{\nabla}_{\perp}\Phi \cdot \nabla n_i - \frac{2n_i}{B}\boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla}_{\perp}\Phi \\
&-\frac{2}{Z_i e B}\boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla}_{\perp} P_i \\
&-V_{\|i}\boldsymbol{b} \cdot \nabla n_i - n_i B \nabla_{\|}\left(\frac{V_{\|i}}{B}\right) \\
&-\boldsymbol{\nabla} \cdot (n_i \boldsymbol{V}_{Pi})
\end{aligned} \tag{2.4.2}
$$

After Linearizing

$$
\begin{aligned}
\frac{\partial}{\partial t}n_i \quad = \quad & -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\nabla n_{i0}-\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\Phi_0\cdot\nabla n_i \\
& -\frac{2n_{i0}}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}_\perp\phi-\frac{2n_i}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}_\perp\Phi_{0,net} \\
& -\frac{2}{Z_ieB_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}_\perp p_{i1} \qquad (2.4.3) \\
& -V_{\|i}\nabla_{\|0}n_{i0}-n_{i0}B_0\nabla_{\|0}\left(\frac{V_{\|i}}{B_0}\right) \\
& -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\nabla n_i-\frac{2n_i}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}_\perp\phi \\
& -V_{\|i}\nabla_{\|0}n_i+V_{\|i}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}n_{i0} \\
& -n_iB_0\nabla_{\|0}\left(\frac{V_{\|i}}{B_0}\right)+n_{i0}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{V_{\|i}}{B_0}\right) \\
& -\boldsymbol{\nabla}\cdot\left(n_i\boldsymbol{V}_{Pi}\right). \qquad (2.4.4)
\end{aligned}
$$

2.5 Parallel velocity equations

Start from the momentum equation, $\boldsymbol{b}\cdot$ Eq. (2.3.1) can get

$$
m_in_i\frac{d}{dt}V_{\|i}+\boldsymbol{b}\cdot\boldsymbol{\nabla}P+\boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\right) \quad = \quad 0.
$$

Here we negalect $m_in_i\boldsymbol{V}_i\cdot\partial\boldsymbol{b}/\partial t$ and $m_in_i\boldsymbol{\nabla}\boldsymbol{b}\cdot\boldsymbol{V}_i$ for they are smaller than the other terms. Because the "perpendicular" part of viscous stress tensor is much smaller than the other two parts, it is negalected in our discussion. According to Ref. [1], from the expression of parallel viscocity (1.3),

$$
\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{ci} \quad = \quad \left[\boldsymbol{b}\left(\boldsymbol{\nabla}\cdot\boldsymbol{b}\right)+\boldsymbol{\kappa}\right]\pi_{ci}+\boldsymbol{b}\boldsymbol{\nabla}_\|\pi_{ci}-\frac{1}{3}\boldsymbol{\nabla}\pi_{ci},
$$

so that

$$
\boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{ci}\right) \quad = \quad \frac{2}{3}B^{\frac{3}{2}}\nabla_\|\left(\frac{\pi_{ci}}{B^{\frac{3}{2}}}\right).
$$

For the "gyroviscous" contributions, we only consider the case of a straight, homogeneous, time-independent magnetic field and neglect parallel derivatives ($\boldsymbol{b}\cdot\boldsymbol{\nabla}\to 0$) to obtain

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{gi} \quad \simeq \quad & -m_in_i\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}\boldsymbol{V}_i+\boldsymbol{b}\times\boldsymbol{\nabla}\left(\frac{P_i}{2\Omega_i}\boldsymbol{\nabla}\cdot\boldsymbol{V}_\perp+\frac{1}{5\Omega_i}\boldsymbol{\nabla}\cdot\boldsymbol{q}_{\perp i}\right) \\
& +\boldsymbol{\nabla}_\perp\left[\frac{P_i}{2\Omega_i}\boldsymbol{\nabla}\cdot\left(\boldsymbol{b}\times\boldsymbol{V}_i\right)+\frac{1}{5\Omega_i}\boldsymbol{\nabla}\cdot\left(\boldsymbol{b}\times\boldsymbol{q}_i\right)\right]. \qquad (2.5.1)
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\frac{P_i}{2\Omega_i}\boldsymbol{\nabla}\cdot\boldsymbol{V}_\perp + \frac{1}{5\Omega_i}\boldsymbol{\nabla}\cdot\boldsymbol{q}_{\perp i} &\simeq -\frac{P_i}{2\Omega_i}\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}\ln n_i - \frac{n_i k_B}{2\Omega_i}\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}T_i \\
&\simeq 0,
\end{aligned}
$$

and

$$
\frac{P_i}{2\Omega_i}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{V}_i) + \frac{1}{5\Omega_i}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{q}_i) \simeq -\frac{1}{2m_i\Omega_i}\left[P_i\boldsymbol{\nabla}_\perp\cdot\left(e\boldsymbol{\nabla}_\perp\Phi + \frac{\nabla_\perp P_i}{n_i}\right) + \boldsymbol{\nabla}_\perp\cdot(P_i\boldsymbol{\nabla}_\perp T_i)\right]
$$

So the parallel part of the divergence of the ion "gyroviscous" is

$$
\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{gi} \simeq -m_i n_i\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}\boldsymbol{V}_i - \boldsymbol{\nabla}_\perp\left\{\frac{1}{2m_i\Omega_i^2}\left[P_i\boldsymbol{\nabla}_\perp\cdot\left(e\boldsymbol{\nabla}_\perp\Phi + \frac{\boldsymbol{\nabla}_\perp P_i}{n_i}\right) + \boldsymbol{\nabla}_\perp\cdot(P_i\boldsymbol{\nabla}_\perp T_i)\right]\right\}
$$

$$
\boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{gi}\right) \simeq -m_i n_i\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}V_{\parallel i}.
$$

The parallel divergence of viscous term can be written as

$$
\begin{aligned}
\boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\right) &\simeq \boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{ci}\right) + \boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{gi}\right) \\
&\simeq \frac{2}{3}B^{\frac{3}{2}}\nabla_\parallel\left(\frac{\pi_{ci}}{B^{\frac{3}{2}}}\right) - m_i n_i\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}V_{\parallel i}.
\end{aligned}
$$

Therefore, we can get the parallel velocity as

$$
\begin{aligned}
\frac{\partial}{\partial t}V_{\parallel i} + \boldsymbol{V}_i\cdot\boldsymbol{\nabla}V_{\parallel i} &= -\frac{1}{m_i n_i}\boldsymbol{b}\cdot\boldsymbol{\nabla}P - \boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\right). \\
\frac{\partial}{\partial t}V_{\parallel i} &= -\left(\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\parallel i}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}V_{\parallel i} - \frac{1}{m_i n_i}\boldsymbol{b}\cdot\boldsymbol{\nabla}P \\
&\quad -\frac{2}{3m_i n_i}B^{\frac{3}{2}}\nabla_\parallel\left(\frac{\pi_{ci}}{B^{\frac{3}{2}}}\right) - \boldsymbol{V}_{Pi}\cdot\boldsymbol{\nabla}V_{\parallel i}. \qquad (2.5.2)
\end{aligned}
$$

Linearizing derives

$$\begin{aligned}
\frac{\partial}{\partial t}V_{\|i} &= -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\Phi_0\cdot\nabla V_{\|i} \\
&-\frac{1}{m_in_{i0}}\boldsymbol{\nabla}_{\|0}p_1+\frac{1}{m_in_i}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}P_0 \\
&-\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\nabla V_{\|i}-V_{\|i}\nabla_\| V_{\|i} \\
&+\frac{1}{m_in_{i0}}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}p_1 \\
&-\boldsymbol{V}_{Pi}\cdot\boldsymbol{\nabla}V_{\|i}-\frac{2}{3m_in_i}B^{\frac{3}{2}}\nabla_\|\left(\frac{\pi_{ci}}{B^{\frac{3}{2}}}\right).
\end{aligned} \tag{2.5.3}$$

2.6 Ohm's Law

For the parallel part of Eq. (1.5) for electron,

$$m_en_e\frac{d}{dt}V_{\|e}+\boldsymbol{b}\cdot\boldsymbol{\nabla}P_e = -en_eE_\|+R_{\|e}-\boldsymbol{b}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_e\right).$$

Here the electron-ion friction is written as

$$\boldsymbol{R}_e = en_e\left(\frac{\boldsymbol{J}_\|}{\sigma_\|}+\frac{\boldsymbol{J}_\perp}{\sigma_\perp}\right)-0.71k_Bn_e\boldsymbol{\nabla}_\|T_e+\frac{3\nu_e}{2\Omega_e}k_Bn_e\boldsymbol{b}\times\boldsymbol{\nabla}T_e.$$

So negalect the inertial terms and viscous stress tensor, we have

$$\nabla_\|P_e = -en_e\left(-\nabla_\|\Phi-\frac{\partial}{\partial t}A_\|\right)+en_e\frac{J_{\|1}}{\sigma_\|}-0.71k_Bn_e\nabla_\|T_e.$$

Then we can obtain the Ohm's law with the substitution $A_\| = B\psi$ and $J_{\|1} = -\frac{1}{\mu_0}B_0\nabla_\perp^2\psi$,

$$\frac{\partial}{\partial t}\psi = -\frac{1}{B}\nabla_\|\Phi+\frac{\eta_\|}{\mu_0}\nabla_\perp^2\psi+\frac{1}{en_eB}\nabla_\|P_e+\frac{0.71k_B}{eB}\nabla_\|T_e. \tag{2.6.1}$$

After linearized,

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi \;\; = \;\; & -\frac{1}{B_0}\nabla_{\|0}\phi + \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\Phi_0 \\
& + \frac{\eta_\|}{\mu_0}\nabla_\perp^2\psi \\
& + \frac{1}{en_{e0}B_0}\nabla_{\|0}p_{e1} - \frac{1}{en_eB_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}P_{e0} \\
& + \frac{0.71k_B}{eB_0}\nabla_{\|0}T_{e1} - \frac{0.71k_B}{eB_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}T_{e0} \\
& + \frac{1}{B_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi - \frac{1}{en_eB_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}p_{e1} \\
& - \frac{0.71k_B}{eB_0}\boldsymbol{b}_0 \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}T_{e1}. \qquad (2.6.2)
\end{aligned}
$$

2.7 Ion temperature equation

For simplicity , we treat the time partial differential on kinetic term simply on velocity since it is 2nd order quantity at least. Then we have

$$
\frac{d}{dt}\left(\frac{3}{2}P_j\right) + \frac{5}{2}P_j\boldsymbol{\nabla}\cdot\boldsymbol{V}_i \;\; = - \;\; \overleftrightarrow{\boldsymbol{\pi}}_i : \boldsymbol{\nabla}\boldsymbol{V}_i - \boldsymbol{\nabla}\cdot\boldsymbol{q}_j + W_j,
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{3}{2}p_i\right) \;\; = \;\; & -\frac{5}{2}P_i\boldsymbol{\nabla}\cdot\boldsymbol{V}_i - \overleftrightarrow{\boldsymbol{\pi}}_i : \boldsymbol{\nabla}\boldsymbol{V}_i - \boldsymbol{\nabla}\cdot\left(\frac{5P_i}{2m_i\Omega_i}\boldsymbol{b}\times\boldsymbol{\nabla}T_i\right) \\
& + \nabla_\|\left(\kappa_{\|i}\nabla_\|T_i\right) + \nabla_\perp\left(\kappa_{\perp i}\nabla_\perp T_i\right) \\
& + \frac{3m_e}{m_i}\frac{k_Bn_e}{\tau_e}\left(T_e - T_i\right).
\end{aligned}
$$

The ion temperature can be written as

$$
\begin{aligned}
\frac{\partial}{\partial t}T_i + \boldsymbol{V}_i\cdot\boldsymbol{\nabla}T_i \;\; = \;\; & -\frac{2}{3}T_i\boldsymbol{\nabla}\cdot\boldsymbol{V} + \frac{2}{3n_ik_B}\nabla_\|\left(\kappa_{\|i}\nabla_\|T_i\right) + \frac{2}{3n_ik_B}\nabla_\perp\left(\kappa_{\perp i}\nabla_\perp T_i\right) \\
& + \frac{2m_e}{m_i}\frac{Z_i}{\tau_e}\left(T_e - T_i\right) - \frac{2}{3k_Bn_i}\overleftrightarrow{\boldsymbol{\pi}}_i : \boldsymbol{\nabla}\boldsymbol{V}_i - \frac{5}{3k_Bn_i}\boldsymbol{\nabla}\cdot\left(\frac{P_i}{m_i\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_i\right).
\end{aligned}
$$

For the gyrofrequency term in the energy flux,

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\frac{5P_i}{2m_i\Omega_i}\boldsymbol{b}\times\boldsymbol{\nabla}T_i\right) \;\; = \;\; & \frac{5}{2Z_ie}\left[P_i\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{b}\times\boldsymbol{\nabla}T_i}{B}\right) + \frac{\boldsymbol{b}\times\boldsymbol{\nabla}T_i}{B}\cdot\boldsymbol{\nabla}P_i\right] \\
= \;\; & \frac{5}{2Z_ie}P_i\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}T_i - \frac{5}{2}\frac{\boldsymbol{b}\times\boldsymbol{\nabla}P_i}{Z_ieB}\cdot\boldsymbol{\nabla}T_i
\end{aligned}
$$

The last term of RHS is canceled by both the convectional term and the $\boldsymbol{\nabla}\cdot\boldsymbol{V}_{Di}$.

For the viscous term, the "perpendicular" part is neglected for ions, then we can write the "parallel" viscous term as

$$\begin{aligned}
\overleftrightarrow{\boldsymbol{\pi}}_{ci}:\boldsymbol{\nabla}\boldsymbol{V}_i &= \pi_{ci}\left(\boldsymbol{bb}-\frac{1}{3}\overleftrightarrow{\boldsymbol{I}}\right):\boldsymbol{\nabla}\boldsymbol{V}_i \\
&\simeq \frac{\pi_{ci}}{3}\left[2\nabla_{\parallel}V_{\parallel i}-3\boldsymbol{\kappa}\cdot\boldsymbol{V}_{\perp i}-\boldsymbol{\nabla}\cdot\boldsymbol{V}_{\perp i}-V_{\parallel i}\left(\boldsymbol{\nabla}\cdot\boldsymbol{b}\right)\right] \\
&\simeq \frac{\pi_{ci}}{3}\left[\frac{2}{\sqrt{B}}\nabla_{\parallel}\left(\sqrt{B}V_{\parallel i}\right)-\boldsymbol{\kappa}\cdot\left(\boldsymbol{V}_E+\boldsymbol{V}_{Di}\right)-\frac{k_B}{Z_ien_iB}\boldsymbol{b}\cdot\boldsymbol{\nabla}n_i\times\boldsymbol{\nabla}T_i\right] \\
&\quad+\frac{\pi_{ci}}{3}\left[\frac{\mu_0}{B^3}\boldsymbol{b}\times\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla}\phi+\frac{\mu_0}{Z_ien_iB^3}\boldsymbol{b}\times\boldsymbol{\nabla}P_e\cdot\boldsymbol{\nabla}P_i\right]. \qquad (2.7.1)
\end{aligned}$$

with the leading order expression. For simplicity, the last two terms of RHS can be dropped since they come from the second term of RHS of

$$\left(\boldsymbol{\nabla}\times\frac{\boldsymbol{b}}{B}\right) = \frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}-\frac{\mu_0\boldsymbol{b}\times\boldsymbol{\nabla}P}{B^3}+O(\beta).$$

For the calculation of viscous term, the approximation of streight and homogeneous magnetic field is adopted, thus Eq. (2.7.1) can be written as

$$\overleftrightarrow{\boldsymbol{\pi}}_{ci}:\boldsymbol{\nabla}\boldsymbol{V}_i = \frac{\pi_{ci}}{3}\left[\frac{2}{\sqrt{B}}\nabla_{\parallel}\left(\sqrt{B}V_{\parallel i}\right)-\frac{k_B}{Z_ien_iB}\boldsymbol{b}\cdot\boldsymbol{\nabla}n_i\times\boldsymbol{\nabla}T_i\right]$$

The "gyroviscous" term have the relations as

$$\overleftrightarrow{\boldsymbol{\pi}}_{gi}\cdot\boldsymbol{V}_i \simeq \frac{P_i}{2\Omega_i}\boldsymbol{b}\times\boldsymbol{\nabla}V_{\parallel i}^2.$$

Within Eq. (2.5.1), we can get

$$\begin{aligned}
\overleftrightarrow{\boldsymbol{\pi}}_{gi}:\boldsymbol{\nabla}\boldsymbol{V}_i &\simeq \boldsymbol{\nabla}\cdot\left(\overleftrightarrow{\boldsymbol{\pi}}_{gi}\cdot\boldsymbol{V}_i\right)-\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_{gi}\right)\cdot\boldsymbol{V}_i \\
&\simeq \frac{m_i}{Z_ie}P_iV_{\parallel i}\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}V_{\parallel i}-m_in_iV_{\parallel i}\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}V_{\parallel i} \\
&\quad+m_in_iV_{\parallel i}\boldsymbol{V}_{Di}\cdot\boldsymbol{\nabla}V_{\parallel i}+\boldsymbol{V}_{\perp i}\cdot\boldsymbol{\nabla}_{\perp}\left(\frac{1}{2m_i\Omega_i^2}A_{\pi gi}\right),
\end{aligned}$$

where

$$A_{\pi gi} = eP_i\nabla_\perp^2\phi + k_BT_i\nabla_\perp^2 P_i - k_BT_i\boldsymbol{\nabla}_\perp P_i\cdot\frac{\boldsymbol{\nabla}_\perp n_i}{n_i} + k_B\boldsymbol{\nabla}_\perp P_i\cdot\boldsymbol{\nabla}_\perp T_i + k_BP_i\nabla_\perp^2 T_i.$$

According to Ref. [1], we drop the $\overleftrightarrow{\boldsymbol{\pi}}_{gi} : \boldsymbol{\nabla}\boldsymbol{V}_i$ and obtain

$$\overleftrightarrow{\boldsymbol{\pi}}_{gi} : \boldsymbol{\nabla}\boldsymbol{V}_i \simeq \frac{m_i}{Z_ie}P_iV_{\parallel i}\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}V_{\parallel i}$$

After all, the temperature equation can be written as

$$\begin{aligned}
\frac{\partial}{\partial t}T_i = & -\left(\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\parallel i}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}T_i \\
& -\frac{2}{3}T_i\left[\left(\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi + \frac{1}{Z_ien_i}\boldsymbol{\nabla}P_i + \frac{5}{2}\frac{k_B}{Z_ie}\boldsymbol{\nabla}T_i\right) + B\nabla_\parallel\left(\frac{V_{\parallel i}}{B}\right)\right] \\
& +\frac{2}{3n_ik_B}\nabla_\parallel\left(\kappa_{\parallel i}\nabla_\parallel T_i\right) + \frac{2}{3n_ik_B}\nabla_\perp\left(\kappa_{\perp i}\nabla_\perp T_i\right) \\
& +\frac{2m_e}{m_i}\frac{Z_i}{\tau_e}\left(T_e - T_i\right) \\
& -\frac{2\pi_{ci}}{9k_Bn_i}\left[\frac{2}{\sqrt{B}}\nabla_\parallel\left(\sqrt{B}V_{\parallel i}\right) - \frac{k_B}{Z_ien_iB}\boldsymbol{b}\cdot\boldsymbol{\nabla}n_i\times\boldsymbol{\nabla}T_i\right] \\
& -\frac{4}{3\Omega_i}T_iV_{\parallel i}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}V_{\parallel i} \\
& -\frac{2}{3}T_i\boldsymbol{\nabla}\cdot\boldsymbol{V}_{Pi} - \boldsymbol{V}_{Pi}\cdot\boldsymbol{\nabla}T_i.
\end{aligned}$$

Linearizing

$$
\begin{aligned}
\frac{\partial}{\partial t}T_i \quad = \quad & -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\boldsymbol{\nabla}T_{i0}-\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\Phi_0\cdot\boldsymbol{\nabla}T_{i1}-V_{\|i}\nabla_{\|0}T_{i0} \\
& -\frac{2}{3}T_{i0}\left[\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\phi+\frac{1}{Z_ien_{i0}}\boldsymbol{\nabla}p_{i1}+\frac{5}{2}\frac{k_B}{Z_ie}\boldsymbol{\nabla}T_{i1}\right)+B_0\nabla_{\|0}\left(\frac{V_{\|i}}{B_0}\right)\right] \\
& -\frac{2}{3}T_{i1}\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi_0+\frac{1}{Z_ien_{i0}}\boldsymbol{\nabla}P_{i0}+\frac{5}{2}\frac{k_B}{Z_ie}\boldsymbol{\nabla}T_{i0}\right) \\
& +\frac{2}{3n_{i0}k_B}\nabla_\|\left(\kappa_{\|i}\nabla_\|T_{i1}\right)+\frac{2}{3n_{i0}k_B}\nabla_\perp\left(\kappa_{\perp i}\nabla_\perp T_{i1}\right) \\
& +\frac{2m_e}{m_i}\frac{Z_i}{\tau_e}\left(T_{e1}-T_{i1}\right) \\
& -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\boldsymbol{\nabla}T_{i1}-V_{\|i}\nabla_{\|0}T_{i1}+V_{\|i}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}T_{i0} \\
& -\frac{2}{3}T_{i1}\left[\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\phi+\frac{1}{Z_ien_{i0}}\boldsymbol{\nabla}p_{i1}+\frac{5}{2}\frac{k_B}{Z_ie}\boldsymbol{\nabla}T_{i1}\right)+B_0\nabla_{\|0}\left(\frac{V_{\|i}}{B_0}\right)\right] \\
& +\frac{2}{3}T_{i0}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{V_{\|i}}{B_0}\right) \\
& -\frac{2}{3k_Bn_i}\overleftrightarrow{\boldsymbol{\pi}}_i:\boldsymbol{\nabla}\boldsymbol{V}_i \\
& +\frac{2}{3}T_i\boldsymbol{\nabla}\cdot\boldsymbol{V}_{Pi}-\boldsymbol{V}_{Pi}\cdot\boldsymbol{\nabla}T_i
\end{aligned}
\tag{2.7.2}
$$

2.8 Electron temperature equation

For electron temperature, the Braginskii equation is

$$
\begin{aligned}
\frac{3}{2}n_e\frac{d}{dt}\left(k_BT_e\right) \quad = - \quad & P_e\boldsymbol{\nabla}\cdot\boldsymbol{V}_e-\boldsymbol{\nabla}\cdot\left[-0.71\frac{k_BT_e\boldsymbol{J}_\|}{e}+\frac{5P_e}{2eB}\boldsymbol{b}\times\boldsymbol{\nabla}T_e+\frac{3k_B\nu_e}{2\Omega_e}\frac{T_e\boldsymbol{b}\times\boldsymbol{J}}{e}\right] \\
& +\nabla_\|\left(\kappa_{\|e}\nabla_\|T_e\right)+\nabla_\perp\left(\kappa_{\perp e}\nabla_\perp T_e\right) \\
& -\frac{3m_e}{m_i}\frac{n_ek_B}{\tau_e}\left(T_e-T_i\right)-\boldsymbol{R}_e\cdot\left(\boldsymbol{V}_e-\boldsymbol{V}_i\right) \\
& -\frac{2}{3k_Bn_e}\overleftrightarrow{\boldsymbol{\pi}}_e:\boldsymbol{\nabla}\boldsymbol{V}_e.
\end{aligned}
$$

For the terms in energy flux,

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\left[-0.71\frac{k_BT_e\boldsymbol{J}_\|}{e}+\frac{3k_B\nu_e}{2\Omega_e}\frac{T_e\boldsymbol{b}\times\boldsymbol{J}}{e}\right] \quad = \quad & -0.71\frac{k_BT_e}{e}B\nabla_\|\left(\frac{J_\|}{B}\right)-0.71\frac{k_BJ_\|}{e}\nabla_\|T_e \\
& +\frac{3k_Bm_e\nu_eT_e}{e^2B^2}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}P+\frac{3k_Bm_e}{2e^2B}\boldsymbol{b}\times\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla}\left(\frac{\nu_eT_e}{B}\right)
\end{aligned}
$$

Notice that the last two terms in the RHD of last equation has the order of $\nu_e/\Omega_e \sim \epsilon\sqrt{m_e/m_i} \ll 1$ as the first two, so they can be neglected in the following calculations in flute reduction. The friction force term in the energy exchange term can be written as

$$
\begin{aligned}
\boldsymbol{R}_e \cdot (\boldsymbol{V}_e - \boldsymbol{V}_i) &= \left[Z_i e n_i \left(\frac{\boldsymbol{J}_\parallel}{\sigma_\parallel} + \frac{\boldsymbol{J}_\perp}{\sigma_\perp} \right) - 0.71 k_B n_e \boldsymbol{\nabla}_\parallel T_e + \frac{3\nu_e}{2\Omega_e} k_B n_e \boldsymbol{b} \times \boldsymbol{\nabla} T_e \right] \\
&\quad \cdot \left[-\frac{1}{Z_i e n_i B} \boldsymbol{b} \times \boldsymbol{\nabla} P - \frac{1}{Z_i e n_i} J_\parallel \boldsymbol{b}_\parallel \right] \\
&= -\eta_\parallel J_\parallel^2 + 0.71 \frac{k_B}{e} J_\parallel \nabla_\parallel T_e - \frac{\eta_\perp}{B} \boldsymbol{b} \times \boldsymbol{\nabla} P \cdot \boldsymbol{J}_\perp - \frac{3\nu_e k_B}{2\Omega_e e B} (\boldsymbol{b} \times \boldsymbol{\nabla} T_e) \cdot (\boldsymbol{b} \times \boldsymbol{\nabla} P) \\
&\simeq -\eta_\parallel J_\parallel^2 + 0.71 \frac{k_B}{e} J_\parallel \nabla_\parallel T_e.
\end{aligned}
$$

Notice that the polarization drift is neglected here for it is much smaller than the other velocities. The term of perpendicular current is zero because $\boldsymbol{J}_\perp \simeq \boldsymbol{\nabla} P/B$. The last term with gyrofrequency is also dropped for the order $\nu_e/\Omega_e \sim \epsilon\sqrt{m_e/m_i}$. Then Neglect the electron viscocity, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} T_e + \boldsymbol{V}_e \cdot \boldsymbol{\nabla} T_e &= -\frac{2}{3} T_e \boldsymbol{\nabla} \cdot \boldsymbol{V}_e + 0.71 \frac{2T_e}{3en_e} B \nabla_\parallel \left(\frac{J_\parallel}{B} \right) \\
&\quad + \frac{2}{3n_e k_B} \nabla_\parallel \left(\kappa_{\parallel e} \nabla_\parallel T_e \right) + \frac{2}{3n_e k_B} \nabla_\perp \left(\kappa_{\perp e} \nabla_\perp T_e \right) \\
&\quad - \frac{2m_e}{m_i} \frac{1}{\tau_e} (T_e - T_i) + \frac{2}{3n_e k_B} \eta_\parallel J_\parallel^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t} T_e &= -\left(\frac{1}{B_0} \boldsymbol{b} \times \boldsymbol{\nabla}_\perp \Phi + V_{\parallel e} \boldsymbol{b} \right) \cdot \boldsymbol{\nabla} T_e \\
&\quad -\frac{2}{3} T_e \left[\left(\frac{2}{B} \boldsymbol{b} \times \boldsymbol{\kappa} \right) \cdot \left(\boldsymbol{\nabla} \Phi - \frac{1}{en_e} \boldsymbol{\nabla} P_e - \frac{5}{2} \frac{k_B}{e} \boldsymbol{\nabla} T_e \right) + B \nabla_\parallel \left(\frac{V_{\parallel e}}{B} \right) \right] \\
&\quad +0.71 \frac{2T_e}{3en_e} B \nabla_\parallel \left(\frac{J_\parallel}{B} \right) \\
&\quad + \frac{2}{3n_e k_B} \nabla_\parallel \left(\kappa_{\parallel e} \nabla_\parallel T_e \right) + \frac{2}{3n_e k_B} \nabla_\parallel \left(\kappa_{\perp e} \nabla_\perp T_e \right) \\
&\quad - \frac{2m_e}{m_i} \frac{1}{\tau_e} (T_e - T_i) + \frac{2}{3n_e k_B} \eta_\parallel J_\parallel^2 \\
&\quad - \frac{2}{3} T_e \boldsymbol{\nabla} \cdot \boldsymbol{V}_{Pe} - \boldsymbol{V}_{Pe} \cdot \boldsymbol{\nabla} T_e.
\end{aligned}
$$

After linearizing

$$
\begin{aligned}
\frac{\partial}{\partial t}T_e \quad = \quad & -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\boldsymbol{\nabla}T_{e0}-\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\Phi_0\cdot\boldsymbol{\nabla}T_{e1}-V_{\parallel e}\boldsymbol{\nabla}_{\parallel 0}T_{e0}\\
& -\frac{2}{3}T_{e0}\left[\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\phi-\frac{1}{en_{e0}}\boldsymbol{\nabla}p_{e1}-\frac{5}{2}\frac{k_B}{e}\boldsymbol{\nabla}T_{e1}\right)+B_0\nabla_{\parallel 0}\left(\frac{V_{\parallel e}}{B_0}\right)\right]\\
& -\frac{2}{3}T_{e1}\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi_0-\frac{1}{en_{e0}}\boldsymbol{\nabla}P_{e0}-\frac{5}{2}\frac{k_B}{e}\boldsymbol{\nabla}T_{e0}\right)\\
& +0.71\frac{2}{3e}\frac{T_{e0}}{n_{e0}}B_0\nabla_{\parallel 0}\left(\frac{J_{\parallel 1}}{B_0}\right)+0.71\frac{2}{3e}\frac{T_{e1}}{n_{e0}}B_0\nabla_{\parallel 0}\left(\frac{J_{\parallel 0}}{B_0}\right)\\
& -0.71\frac{2}{3e}\frac{T_{e0}}{n_{e0}}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{J_{\parallel 0}}{B_0}\right)\\
& +\frac{2}{3n_{e0}k_B}\nabla_\parallel\left(\kappa_{\parallel e}\nabla_\parallel T_{e1}\right)+\frac{2}{3n_{e0}k_B}\nabla_\perp\left(\kappa_{\perp e}\nabla_\perp T_{e1}\right)\\
& -\frac{2m_e}{m_i}\frac{1}{\tau_e}\left(T_{e1}-T_{i1}\right)+\frac{4}{3n_{e0}k_B}\eta_\parallel J_{\parallel 0}J_{\parallel 1}\\
& -\frac{1}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}_\perp\phi\cdot\boldsymbol{\nabla}T_{e1}-V_{\parallel e}\nabla_{\parallel 0}T_{e1}+V_{\parallel e}\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}T_{e0}\\
& -\frac{2}{3}T_{e1}\left[\left(\frac{2}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\phi-\frac{1}{en_{e0}}\boldsymbol{\nabla}p_{e1}-\frac{5}{2}\frac{k_B}{e}\boldsymbol{\nabla}T_{e1}\right)+B_0\nabla_{\parallel 0}\left(\frac{V_{\parallel e}}{B_0}\right)\right]\\
& +\frac{2}{3}T_{e0}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{V_{\parallel e}}{B_0}\right)\\
& +0.71\frac{2}{3e}\frac{T_{e1}}{n_{e0}}B_0\nabla_{\parallel 0}\left(\frac{J_{\parallel 1}}{B_0}\right)-0.71\frac{2}{3e}\frac{T_{e0}}{n_{e0}}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{J_{\parallel 1}}{B_0}\right)\\
& -0.71\frac{2}{3e}\frac{T_{e1}}{n_{e0}}B_0\boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{J_{\parallel 0}}{B_0}\right)+\frac{2}{3n_{e0}k_B}\eta_\parallel J_{\parallel 1}^2\\
& -\frac{2}{3}T_e\boldsymbol{\nabla}\cdot\boldsymbol{V}_{Pe}-\boldsymbol{V}_{Pe}\cdot\boldsymbol{\nabla}T_e
\end{aligned}
\tag{2.8.1}
$$

2.9 Magnetic Flutter in Parallel Thermal Conduction

The magnetic flutter has strong impact on the distribution of heat fluxes towards divertor targets[4]. Considering the magnetic field unit vector:

$$
\mathbf{b} \quad = \quad \mathbf{b}_0+\mathbf{b}_1=\mathbf{b}_0+\nabla A_\parallel\times\mathbf{b}_0/B\simeq\mathbf{b}_0-\mathbf{b}_0\times\nabla\psi \tag{2.9.1}
$$

$$
\nabla_\parallel \quad = \quad \mathbf{b}\cdot\nabla=\mathbf{b}_0\cdot\nabla-\mathbf{b}_0\times\nabla\psi\cdot\nabla=\nabla_{\parallel 0}-B_0[\psi,\ \cdot] \tag{2.9.2}
$$

the higher order (>1) of the thermal conduction in temperature equations (2.7.2) and (2.8.1) can be expressed as:

$$
\begin{aligned}
\nabla_{\|}(\kappa_{\|j}\nabla_{\|}T_j) =& \nabla_{\|0}(\kappa_{\|j}\nabla_{\|0}T_{j1}) \\
& -\mathbf{b}_0 \times \nabla\psi \cdot \nabla[\kappa_{\|j}\nabla_{\|0}T_{j1}] - \nabla_{\|0}[\kappa_{\|j}\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})] \\
& + \mathbf{b}_0 \times \nabla\psi \cdot \nabla[\kappa_{\|j}\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})] \\
=& \nabla_{\|0}\kappa_{\|j}\nabla_{\|0}T_{j1} + \kappa_{\|j}\nabla^2_{\|0}T_{j1} \\
& -\mathbf{b}_0 \times \nabla\psi \cdot \nabla\kappa_{\|j}(\nabla_{\|0}T_{j1}) - \kappa_{\|j}\mathbf{b}_0 \times \nabla\psi \cdot \nabla(\nabla_{\|0}T_{j1}) \\
& -\nabla_{\|0}\kappa_{\|j}[\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})] - \kappa_{\|j}\nabla_{\|0}[\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})] \\
& + \mathbf{b}_0 \times \nabla\psi \cdot \nabla\kappa_{\|j}[\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})] \\
& + \kappa_{\|j}\mathbf{b}_0 \times \nabla\psi \cdot \nabla[\mathbf{b}_0 \times \nabla\psi \cdot \nabla(T_{j0}+T_{j1})]
\end{aligned}
\tag{2.9.3}
$$

NOTE: The equilibrium temperature profiles are flux functions $T_{j0}(\psi)$, so $\nabla_{\|0}T_{j0} = 0$.

3 Normalization

For numerical simulation, the normalization are necessary

$$
\begin{aligned}
\hat{T}_j = \frac{T_j}{\bar{T}_j}, \quad & \hat{n} = \frac{n_i}{\bar{n}}, \quad \hat{L} = \frac{L}{\bar{L}}, \\
\hat{t} = \frac{t}{\bar{t}}, \quad & \hat{B} = \frac{B}{\bar{B}}, \quad \hat{J} = \frac{\mu_0 \bar{L}}{B_0} J, \\
\hat{\psi} = \frac{\psi}{\bar{L}}, \quad & \hat{\phi} = \frac{\bar{t}}{\bar{L}^2 B_0}\phi, \quad \hat{\varpi} = \frac{\bar{t}}{m_i \bar{n}}\varpi, \\
\tau = \frac{\bar{T}_i}{\bar{T}_e}, \quad & \hat{V} = \frac{V}{V_A}, \quad V_A = \frac{\bar{L}}{\bar{t}} = \frac{\bar{B}}{\sqrt{\mu_0 m_i n_i}}, \qquad (3.1) \\
\hat{P}_j = \frac{P_j}{k_B \bar{n} \bar{T}_j} \quad & \hat{\kappa} = \bar{L}\kappa, \quad \hat{\bigtriangledown} = \bar{L}\bigtriangledown \qquad (3.2)
\end{aligned}
$$

so we have

$$
\begin{aligned}
\hat{P} = (\tau \hat{P}_i + \hat{P}_e) &= \frac{P}{(k_B \bar{n} \bar{T}_e)}, \\
&= \tau \hat{P}_i * (1 + Z_i), \text{ if } T_i = T_e \qquad (3.3) \\
\hat{E} &= \frac{E}{V_A \bar{B}} = \boxed{-\hat{\nabla}(\hat{B}_0 \hat{\phi})} \qquad (3.4) \\
\hat{\boldsymbol{V}}_{E\times B} &= \frac{\boldsymbol{V}_{E\times B}}{V_A} = \frac{\hat{\boldsymbol{E}}_r \times \hat{\boldsymbol{B}}_0}{\hat{B}_0^2} \simeq \boxed{\mathbf{b}_0 \times \hat{\nabla}\hat{\phi}} \qquad (3.5)
\end{aligned}
$$

The last step in Eq.(3.8), the B0 is assumed constant.

$$
\begin{aligned}
\hat{E} &= \frac{E}{V_A \bar{B} \hat{B}_0} = \boxed{-\frac{\hat{\nabla}(\hat{B}_0 \hat{\phi})}{\hat{B}_0} \simeq -\hat{\nabla}\hat{\phi}} \qquad (3.6) \\
\hat{\boldsymbol{V}}_{E\times B} &= \frac{\boldsymbol{V}_{E\times B}}{V_A} = \frac{\hat{\boldsymbol{E}}_r \times \hat{\boldsymbol{B}}_0}{\hat{B}_0} \simeq \boxed{\mathbf{b}_0 \times \hat{\nabla}\hat{\phi}} \qquad (3.7) \\
\hat{\nabla}_\perp (\hat{B}_0 \hat{\Phi}_0) &= \left[\mathbf{b}_0 \times \hat{\nabla}(\hat{B}_0 \hat{\Phi}_0)\right] \times \mathbf{b}_0 = \hat{\boldsymbol{V}}_{E\times B} \times \mathbf{B}_0 \qquad (3.8)
\end{aligned}
$$

NOTE: In the code of version that using normalization factor B_0, in some place the B_0 is considered as constant value. In the gyroviscous terms in vorticity equation, for example:

$$
\left[\frac{\hat{\nabla}_\perp \left(\hat{B}_0 \hat{\Phi}_0\right)}{\hat{B}_0}\right]^2 \simeq \left[\hat{\nabla}_\perp \hat{\Phi}_0\right]^2 \qquad (3.9)
$$

The linearized equations are

$$\hat{\varpi} = \boldsymbol{b}\cdot\hat{\boldsymbol{\nabla}}\times\hat{n}_i\hat{V}_i \simeq \frac{\hat{n}_{i0}}{\hat{B}_0}\left[\hat{\nabla}_\perp^2(\hat{B}_0\hat{\phi}) + \frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp(\hat{B}_0\hat{\phi})\cdot\hat{\nabla}_\perp\hat{n}_{i0} + \frac{k_B T_i}{Z_i e\bar{L}\bar{B}V_A}\frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp^2\hat{p}_{i1}\right] \tag{3.10}$$

$$U_{\text{para0}} = \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},\ U_{\text{para1}} = \frac{k_B\bar{T}_e}{m_i V_A^2},\ U_{\text{para2}} = \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},\ U_{\text{para3}} = 1.0 \tag{3.11}$$

$$\begin{aligned}
\frac{\partial}{\partial\hat{t}}\hat{\varpi} \quad = \quad & \hat{B}_0^2\hat{\nabla}_{\parallel 0}\hat{J}_{\parallel 1} - \hat{B}_0^2\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{J}_{\parallel 0} \\
& +2\frac{k_B\bar{T}_e}{m_i V_A^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{\nabla}}\hat{p}_1 - \frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi} \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\left(\hat{B}_0\hat{\Phi}_0\right)\right] \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{P}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\left(\hat{B}_0\hat{\phi}\right)\right] \\
& +\frac{1}{2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)}{\hat{B}_0}\right]^2 + \boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\nabla}_\perp\left(\hat{B}_0\hat{\phi}\right)}{\hat{B}_0^2}\right] \\
& +\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{P}_{i0}\right) + \frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{p}_{i1}\right) \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left[\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\right] \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left[\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{P}_{i0}\right] \\
& -\hat{B}_0^2\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{J}_{\parallel 1} \\
& -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi} - \hat{V}_{\parallel i}\hat{\nabla}_{\parallel 0}\hat{\varpi} \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\left(\hat{B}_0\hat{\phi}\right)\right] + \frac{1}{2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\left(\hat{B}_0\hat{\phi}\right)}{\hat{B}_0}\right]^2 \\
& +\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\nabla}_\perp\left(\hat{B}_0\hat{\phi}\right)}{\hat{B}_0^2}\right] \\
& +\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{p}_{i1}\right) \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left(\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\right),
\end{aligned}$$

$$Ni_{\text{para1}} = \frac{k_B \bar{T}_i}{Z_i e \bar{B} \bar{L} V_A},\ Vi_{\text{para}} = \frac{\mu_0 k_B \bar{T}_e \bar{n}}{\bar{B}^2} = \frac{k_B \bar{T}_e}{m_i V_A^2}, \tag{3.12}$$

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}} \hat{n}_i \quad = \quad & -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\phi} \right) \cdot \hat{\boldsymbol{\nabla}} \hat{n}_{i0} \\
& -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\Phi}_0 \right) \cdot \hat{\boldsymbol{\nabla}} n_i - \hat{V}_{\parallel i} \hat{\nabla}_{\parallel 0} \hat{n}_{i0} \\
& -\frac{2\hat{n}_{i0}}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\phi} \right) \boxed{-\frac{2\hat{n}_i}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\Phi}_{0,net} \right)} \\
& -\frac{2 k_B \bar{T}_e}{Z_i e \bar{B} \bar{L} V_A} \frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \hat{p}_{i1} - \hat{n}_{i0} \hat{B}_0 \hat{\nabla}_{\parallel 0} \left(\frac{V_{\parallel i}}{B_0} \right) \\
& -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\phi} \right) \cdot \hat{\boldsymbol{\nabla}} \hat{n}_i - \hat{V}_{\parallel i} \hat{\nabla}_{\parallel 0} \hat{n}_i + \hat{V}_{\parallel i} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{n}_{i0} \\
& -\frac{2\hat{n}_i}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\phi} \right) \\
& -\hat{n}_i \hat{B}_0 \hat{\nabla}_{\parallel 0} \left(\frac{\hat{V}_{\parallel i}}{\hat{B}_0} \right) + \hat{n}_{i0} \hat{B}_0 \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \left(\frac{\hat{V}_{\parallel i}}{\hat{B}_0} \right), \\
\frac{\partial}{\partial t} \hat{V}_{\parallel i} \quad = \quad & -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\Phi}_0 \right) \cdot \hat{\boldsymbol{\nabla}} \hat{V}_{\parallel i} \\
& -\frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \hat{\nabla}_{\parallel 0} \hat{p}_1 + \frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{P}_0 \\
& -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \left(\hat{B}_0 \hat{\phi} \right) \cdot \hat{\boldsymbol{\nabla}} \hat{V}_{\parallel i} \\
& +\frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{p}_1,
\end{aligned}$$

$$
\begin{aligned}
\psi_{\text{para1}} &= \frac{k_B \bar{T}_e}{e\bar{B}\bar{L}V_A}, \\
Ti_{\text{para1}} &= \frac{2}{3}\frac{1}{\bar{L}V_A},\ Ti_{\text{para2}} = \frac{k_B \bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \hat{t}}\hat{\psi} &= -\frac{1}{\hat{B}_0}\hat{\nabla}_{\|0}\left(\hat{B}_0\hat{\phi}\right) + \frac{\eta_\|}{\mu_0 V_A \bar{L}}\hat{\nabla}_\perp^2\hat{\psi} \\
&+\frac{1}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\Phi}_0\right) \\
&+\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\hat{\nabla}_{\|0}\hat{p}_{e1} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{P}_{e0} \\
&+\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\hat{\nabla}_{\|0}\hat{T}_{e1} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e0} \\
&+\frac{1}{B_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right) - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{e1} \\
&-\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e1}, \\
\frac{\partial}{\partial \hat{t}}\hat{T}_i &= -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i0} - \frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i1} - \hat{V}_{\|i}\hat{\nabla}_{\|0}\hat{T}_{i0} \\
&-\frac{2}{3}\hat{T}_{i0}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right) + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{p}_{i1} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i1}\right] \\
&-\frac{2}{3}\hat{T}_{i0}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|i}}{\hat{B}_0}\right) \\
&-\frac{2}{3}\hat{T}_{i1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\boxed{\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\Phi}_0\right) + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{P}_{i0}} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i0}\right] \\
&+\frac{2}{3\bar{n}_i\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\nabla}_\|\left(\kappa_{\|i}\hat{\nabla}_\|\hat{T}_{i1}\right) + \frac{2}{3\bar{n}_i\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp\left(\kappa_{\perp i}\hat{\nabla}_\perp\hat{T}_{i1}\right) \\
&+\frac{2Z_i m_e\bar{t}}{m_i\tau_e}\left(\hat{T}_{e1}-\hat{T}_{i1}\right) \\
&-\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i1} - \hat{V}_{\|i}\hat{\nabla}_{\|0}\hat{T}_{i1} + \hat{V}_{\|i}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i0} \\
&-\frac{2}{3}\hat{T}_{i1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right) + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{p}_{i1} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i1}\right] \\
&-\frac{2}{3}\hat{T}_{i1}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{V_{\|i}}{B_0}\right) + \frac{2}{3}\hat{T}_{i0}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\hat{V}_{\|i}}{\hat{B}_0}\right),
\end{aligned}
$$

$$Te_{\mathrm{para1}} = \frac{2}{3}\frac{1}{Z_i\bar{L}V_A},\ Te_{\mathrm{para2}} = \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A},\ Te_{\mathrm{para3}} = \frac{\bar{B}}{e\mu_0\bar{n}_e\bar{L}V_A},\ Te_{\mathrm{para4}} = \frac{\bar{B}^2}{\mu_0 k_B\bar{n}_e\bar{T}_e}, \tag{3.14}$$

$$\begin{aligned}
\frac{\partial}{\partial\hat{t}}\hat{T}_e \quad = \quad & -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e0} - \frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\Phi}_0\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e1} - \hat{V}_{\|e}\hat{\nabla}_{\|0}\hat{T}_{e0} \\
& -\frac{2}{3}\hat{T}_{e0}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right) - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{p}_{e1} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e1}\right] \\
& -\frac{2}{3}\hat{T}_{e0}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) \\
& -\frac{2}{3}\hat{T}_{e1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\boxed{\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\Phi}_0\right) - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{P}_{e0}} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e0}\right] \\
& +0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\hat{J}_{\|1} + 0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\hat{J}_{\|0} \\
& -0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}_{\|0}\hat{J}_{\|0} \\
& +\frac{2}{3\bar{n}_e\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\nabla}_\|\left(\kappa_{\|e}\hat{\nabla}_\|\hat{T}_{e1}\right) + \frac{2}{3\bar{n}_e\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\nabla}_\perp\left(\kappa_{\perp e}\hat{\nabla}_\perp\hat{T}_{e1}\right) \\
& -\frac{2m_e\bar{t}}{m_i\tau_e}\left(\hat{T}_{e1} - \hat{T}_{i1}\right) + \frac{4\bar{B}^2}{3\mu_0\bar{n}k_B\bar{T}_e}\frac{\hat{B}_0^2}{\hat{n}_{e0}}\hat{\eta}_\|\hat{J}_{\|0}\hat{J}_{\|1} \\
& -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\left(\hat{B}_0\hat{\phi}\right)\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e1} - \hat{V}_{\|e}\hat{\nabla}_{\|0}\hat{T}_{e1} + \hat{V}_{\|e}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e0} \\
& -\frac{2}{3}\hat{T}_{e1}\left(\frac{2}{B_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\left(\hat{B}_0\hat{\phi}\right) - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{p}_{e1} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e1}\right] \\
& -\frac{2}{3}\hat{T}_{e1}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) \\
& +\frac{2}{3}\hat{T}_{e0}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) \\
& +0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\hat{J}_{\|1} - 0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\nabla}\hat{J}_{\|1} \\
& -0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\nabla}\hat{J}_{\|0} \\
& +\frac{2\bar{B}^2}{3\mu_0\bar{n}k_B\bar{T}_e}\frac{\hat{B}_0^2}{\hat{n}_{e0}}\hat{\eta}_\|\hat{J}_{\|1}^2.
\end{aligned}$$

4 Normalization by $\bar{B}$ (Recommended)

Using $\bar{B}$ in the normalization factors in ϕ and J.

$$\hat{T}_j = \frac{T_j}{\bar{T}_j}, \qquad \hat{n} = \frac{n_i}{\bar{n}}, \qquad \hat{L} = \frac{L}{\bar{L}},$$
$$\hat{t} = \frac{t}{\bar{t}}, \qquad \hat{B} = \frac{B}{\bar{B}}, \qquad \hat{J} = \frac{\mu_0 \bar{L}}{\bar{B}} J,$$
$$\hat{\psi} = \frac{\psi}{\bar{L}}, \qquad \hat{\phi} = \frac{\bar{t}}{\bar{L}^2 \bar{B}} \phi, \qquad \hat{\varpi} = \frac{\bar{t}}{m_i \bar{n}} \varpi,$$
$$\tau = \frac{\bar{T}_i}{\bar{T}_e}, \qquad \hat{V} = \frac{V}{V_A}, \qquad V_A = \frac{\bar{L}}{\bar{t}} = \frac{\bar{B}}{\sqrt{\mu_0 m_i n_i}},$$
$$\hat{P}_j = \frac{P_j}{k_B \bar{n} \bar{T}_j}, \qquad \hat{\kappa} = \bar{L}\kappa, \qquad \hat{\nabla} = \bar{L}\nabla$$
$$\hat{\eta}_\parallel = \frac{\eta_\parallel}{\mu_0 V_A \bar{L}}, \qquad \hat{\kappa}_{\parallel,\perp} = \frac{2\kappa_{\parallel,\perp}}{3 V_A \bar{L} \bar{n}},$$

so we have

$$\begin{aligned} \hat{P} = (\tau \hat{P}_i + \hat{P}_e) &= \frac{P}{(k_B \bar{n} T_e)}, \\ &= \tau \hat{P}_i * (1 + Z_i), \text{ if } T_i = T_e && (4.1) \\ \hat{E} &= \frac{E}{V_A \bar{B}} = -\hat{\nabla}\hat{\phi} && (4.2) \\ \hat{\boldsymbol{V}}_{E\times B} &= \frac{\boldsymbol{V}_{E\times B}}{V_A} = \frac{\hat{\boldsymbol{E}}_r \times \hat{\boldsymbol{B}}_0}{\hat{B}_0^2} = \boxed{\frac{\mathbf{b}_0 \times \hat{\nabla}\hat{\phi}}{\hat{B}_0}} && (4.3) \\ \hat{\nabla}_\perp \hat{\Phi}_0 &= \left[\mathbf{b}_0 \times \hat{\nabla}\hat{\Phi}_0\right] \times \mathbf{b}_0 = \hat{\boldsymbol{V}}_{E\times B} \times \mathbf{B}_0 && (4.4) \end{aligned}$$

After ignoring polarization drift (2.4.1) $\boldsymbol{V}_P$ and taking $\boldsymbol{b}\cdot\nabla = \boldsymbol{b}_0\cdot\nabla - \boldsymbol{b}_0 \times \nabla\psi\cdot\nabla$, The linearized equations are

$$\hat{\varpi} = \boldsymbol{b}\cdot\hat{\boldsymbol{\nabla}}\times\hat{n}_i\hat{V}_i \simeq \frac{\hat{n}_{i0}}{\hat{B}_0}\left(\hat{\nabla}_\perp^2\hat{\phi} + \frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp\hat{\phi}\cdot\hat{\nabla}_\perp\hat{n}_{i0} + \frac{k_B\bar{T}_i}{Z_i e\bar{L}\bar{B}V_A}\frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp^2\hat{p}_{i1}\right) \tag{4.5}$$

$$U_{\mathrm{para0}} = \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},\ U_{\mathrm{para1}} = \frac{k_B\bar{T}_e}{m_i V_A^2},\ U_{\mathrm{para2}} = \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},\ U_{\mathrm{para3}} = 1.0 \tag{4.6}$$

$$\begin{aligned}
\frac{\partial}{\partial\hat{t}}\hat{\varpi} \quad = \quad & \hat{B}_0^2\hat{\nabla}_{\parallel 0}\left(\frac{\hat{J}_{\parallel 1}}{\hat{B}_0}\right) - \hat{B}_0^2\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\hat{J}_{\parallel 0}}{\hat{B}_0}\right) + 2\frac{k_B\bar{T}_e}{m_i V_A^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{\nabla}}\hat{p}_1 \\
& -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\Phi}_0\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi} - \frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi}_0 - \hat{V}_{\parallel i}\hat{\nabla}_{\parallel 0}\hat{\varpi}_0 \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\hat{\Phi}_0\right] - \frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{P}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\hat{\phi}\right] \\
& +\frac{1}{2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\hat{\Phi}_0}{\hat{B}_0}\right]^2 + \boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\hat{\Phi}_0\cdot\hat{\nabla}_\perp\hat{\phi}}{\hat{B}_0^2}\right] \\
& +\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{P}_{i0}\right) + \frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\Phi}_0\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{p}_{i1}\right) \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left[\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\Phi}_0\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\right] - \frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left[\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{P}_{i0}\right] \\
& -\hat{B}_0^2\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\hat{J}_{\parallel 1}}{\hat{B}_0}\right) \\
& -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi} - \hat{V}_{\parallel i}\hat{\nabla}_{\parallel 0}\hat{\varpi} + \hat{V}_{\parallel i}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{\varpi}_0 \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\hat{\nabla}_\perp^2\hat{\phi}\right] \\
& +\frac{1}{2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i0}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\hat{\phi}}{\hat{B}_0}\right]^2 + \boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{n}_{i1}\cdot\hat{\boldsymbol{\nabla}}\left[\frac{\hat{\nabla}_\perp\hat{\Phi}_0\cdot\hat{\nabla}_\perp\hat{\phi}}{\hat{B}_0^2}\right] \\
& +\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0^2}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\left(\hat{\nabla}_\perp^2\hat{p}_{i1}\right) \\
& -\frac{k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{B}_0}\hat{\nabla}_\perp^2\left(\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{i1}\right),
\end{aligned}$$

$$Ni_{\mathrm{para1}} = \frac{k_B \bar{T}_i}{Z_i e \bar{B} \bar{L} V_A},\; Vi_{\mathrm{para}} = \frac{\mu_0 k_B \bar{T}_e \bar{n}}{\bar{B}^2} = \frac{k_B \bar{T}_e}{m_i V_A^2}, \tag{4.7}$$

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}} \hat{n}_i \quad = \quad & -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \cdot \hat{\boldsymbol{\nabla}} \hat{n}_{i0} - \frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\Phi}_0 \cdot \hat{\boldsymbol{\nabla}} n_i \\
& -\frac{2\hat{n}_{i0}}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \boxed{-\frac{2\hat{n}_i}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \hat{\Phi}_{0,net}} \\
& -\frac{2 k_B \bar{T}_e}{Z_i e \bar{B} \bar{L} V_A} \frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \hat{p}_{i1} \\
& -\hat{V}_{\parallel i} \hat{\nabla}_{\parallel 0} \hat{n}_{i0} - \hat{n}_{i0} \hat{B}_0 \hat{\nabla}_{\parallel 0} \left(\frac{V_{\parallel i}}{B_0} \right) \\
& -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \cdot \hat{\boldsymbol{\nabla}} \hat{n}_i - \frac{2\hat{n}_i}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}} \cdot \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \\
& -\hat{V}_{\parallel i} \hat{\nabla}_{\parallel 0} \hat{n}_i + \hat{V}_{\parallel i} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{n}_{i0} \\
& -\hat{n}_i \hat{B}_0 \hat{\nabla}_{\parallel 0} \left(\frac{\hat{V}_{\parallel i}}{\hat{B}_0} \right) + \hat{n}_{i0} \hat{B}_0 \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \left(\frac{\hat{V}_{\parallel i}}{\hat{B}_0} \right), \\
\frac{\partial}{\partial t} \hat{V}_{\parallel i} \quad = \quad & -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\Phi}_0 \cdot \hat{\boldsymbol{\nabla}} \hat{V}_{\parallel i} \\
& -\frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \hat{\nabla}_{\parallel 0} \hat{p}_1 + \frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{P}_0 \\
& -\frac{1}{\hat{B}_0} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \cdot \hat{\boldsymbol{\nabla}} \hat{V}_{\parallel i} \\
& +\frac{k_B \bar{T}_e}{m_i V_A^2} \frac{1}{\hat{n}_{i0}} \boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}} \hat{\psi} \cdot \hat{\boldsymbol{\nabla}} \hat{p}_1,
\end{aligned}$$

$$
\begin{aligned}
\psi_{\mathrm{para1}} &= \frac{k_B \bar{T}_e}{e\bar{B}\bar{L}V_A}, \\
Ti_{\mathrm{para1}} &= \frac{2}{3}\frac{1}{\bar{L}V_A},\ Ti_{\mathrm{para2}} = \frac{k_B \bar{T}_i}{Z_i e\bar{B}\bar{L}V_A},
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \hat{t}}\hat{\psi} &= -\frac{1}{\hat{B}_0}\hat{\nabla}_{\|0}\hat{\phi} + \frac{1}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}_\perp\hat{\Phi}_0 + \frac{\eta_\|}{\mu_0 V_A \bar{L}}\left(\frac{\hat{J}_{\|1}}{\hat{B}_0}\right) \\
&+\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\hat{\nabla}_{\|0}\hat{p}_{e1} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{P}_{e0} \\
&+\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\hat{\nabla}_{\|0}\hat{T}_{e1} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e0} \\
&+\frac{1}{B_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}_\perp\hat{\phi} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{p}_{e1} \\
&-\frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{0.71}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{e1}, \\
\frac{\partial}{\partial \hat{t}}\hat{T}_i &= -\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i0} - \frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\Phi}_0\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i1} - \hat{V}_{\|i}\hat{\nabla}_{\|0}\hat{T}_{i0} \\
&-\frac{2}{3}\hat{T}_{i0}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\hat{\phi} + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{p}_{i1} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i1}\right] \\
&-\frac{2}{3}\hat{T}_{i0}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|i}}{\hat{B}_0}\right) \\
&-\frac{2}{3}\hat{T}_{i1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\boxed{\hat{\boldsymbol{\nabla}}\hat{\Phi}_0 + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{P}_{i0}} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i0}\right] \\
&+\frac{1}{\hat{n}_{i0}}\hat{\nabla}_\|\left(\hat{\kappa}_{\|i}\hat{\nabla}_\|\hat{T}_{i1}\right) + \frac{1}{\hat{n}_{i0}}\hat{\nabla}_\perp\left(\hat{\kappa}_{\perp i}\hat{\nabla}_\perp\hat{T}_{i1}\right) \\
&+\frac{2Z_i m_e\bar{t}}{m_i\tau_e}\left(\hat{T}_{e1} - \hat{T}_{i1}\right) \\
&-\frac{1}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}_\perp\hat{\phi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i1} - \hat{V}_{\|i}\hat{\nabla}_{\|0}\hat{T}_{i1} + \hat{V}_{\|i}\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\hat{T}_{i0} \\
&-\frac{2}{3}\hat{T}_{i1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0\times\hat{\boldsymbol{\kappa}}\right)\cdot\left[\hat{\boldsymbol{\nabla}}\hat{\phi} + \frac{k_B\bar{T}_i}{Z_i e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{i0}}\hat{\boldsymbol{\nabla}}\hat{p}_{i1} + \frac{5k_B\bar{T}_i}{2Z_i e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{i1}\right] \\
&-\frac{2}{3}\hat{T}_{i1}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{V_{\|i}}{B_0}\right) + \frac{2}{3}\hat{T}_{i0}\hat{B}_0\boldsymbol{b}_0\times\hat{\boldsymbol{\nabla}}\hat{\psi}\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\hat{V}_{\|i}}{\hat{B}_0}\right),
\end{aligned}
$$

$$Te_{\text{para1}} = \frac{2}{3}\frac{1}{Z_i \bar{L} V_A},\ Te_{\text{para2}} = \frac{k_B \bar{T}_e}{e\bar{B}\bar{L}V_A},\ Te_{\text{para3}} = \frac{\bar{B}}{e\mu_0 \bar{n}_e \bar{L} V_A},\ Te_{\text{para4}} = \frac{\bar{B}^2}{\mu_0 k_B \bar{n}_e \bar{T}_e}, \tag{4.9}$$

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}}\hat{T}_e \quad = \quad & -\frac{1}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\phi} \cdot \hat{\boldsymbol{\nabla}} \hat{T}_{e0} - \frac{1}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp \hat{\Phi}_0 \cdot \hat{\boldsymbol{\nabla}} \hat{T}_{e1} - \hat{V}_{\|e}\hat{\nabla}_{\|0}\hat{T}_{e0} \\
& -\frac{2}{3}\hat{T}_{e0}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}}\right) \cdot \left[\hat{\boldsymbol{\nabla}}\hat{\phi} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{p}_{e1} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e1}\right] \\
& -\frac{2}{3}\hat{T}_{e0}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) \\
& -\frac{2}{3}\hat{T}_{e1}\left(\frac{2}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}}\right) \cdot \left[\boxed{\hat{\boldsymbol{\nabla}}\hat{\Phi}_0 - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{P}_{e0}} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e0}\right] \\
& +0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{J}_{\|1}}{\hat{B}_0}\right) + 0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{J}_{\|0}}{\hat{B}_0}\right) \\
& -0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi} \cdot \hat{\boldsymbol{\nabla}}_{\|0}\left(\frac{\hat{J}_{\|0}}{\hat{B}_0}\right) \\
& +\frac{1}{\hat{n}_{e0}}\hat{\nabla}_\|\left(\hat{\kappa}_{\|e}\hat{\nabla}_\|\hat{T}_{e1}\right) + \frac{1}{\hat{n}_{e0}}\hat{\nabla}_\perp\left(\hat{\kappa}_{\perp e}\hat{\nabla}_\perp\hat{T}_{e1}\right) \\
& -\frac{2m_e\bar{t}}{m_i\tau_e}\left(\hat{T}_{e1} - \hat{T}_{i1}\right) + \frac{4\bar{B}^2}{3\mu_0\bar{n}k_B\bar{T}_e}\frac{\hat{\eta}_\|}{\hat{n}_{e0}}\hat{J}_{\|0}\hat{J}_{\|1} \\
& -\frac{1}{\hat{B}_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}_\perp\hat{\phi} \cdot \hat{\boldsymbol{\nabla}}\hat{T}_{e1} - \hat{V}_{\|e}\hat{\nabla}_{\|0}\hat{T}_{e1} + \hat{V}_{\|e}\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi} \cdot \hat{\boldsymbol{\nabla}}\hat{T}_{e0} \\
& -\frac{2}{3}\hat{T}_{e1}\left(\frac{2}{B_0}\boldsymbol{b}_0 \times \hat{\boldsymbol{\kappa}}\right) \cdot \left[\hat{\boldsymbol{\nabla}}\hat{\phi} - \frac{k_B\bar{T}_e}{e\bar{B}\bar{L}V_A}\frac{1}{\hat{n}_{e0}}\hat{\boldsymbol{\nabla}}\hat{p}_{e1} - \frac{5k_B\bar{T}_e}{2e\bar{B}\bar{L}V_A}\hat{\boldsymbol{\nabla}}\hat{T}_{e1}\right] \\
& -\frac{2}{3}\hat{T}_{e1}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) + \frac{2}{3}\hat{T}_{e0}\hat{B}_0\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi} \cdot \hat{\boldsymbol{\nabla}}\left(\frac{\hat{V}_{\|e}}{\hat{B}_0}\right) \\
& +0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\hat{\nabla}_{\|0}\left(\frac{\hat{J}_{\|1}}{\hat{B}_0}\right) \\
& -0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e0}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi} \cdot \hat{\nabla}\left(\frac{\hat{J}_{\|1}}{\hat{B}_0}\right) - 0.71\frac{2\bar{B}}{3e\mu_0\bar{n}_e\bar{L}V_A}\frac{\hat{T}_{e1}}{\hat{n}_{e0}}\hat{B}_0\boldsymbol{b}_0 \times \hat{\boldsymbol{\nabla}}\hat{\psi} \cdot \hat{\nabla}\left(\frac{\hat{J}_{\|0}}{\hat{B}_0}\right) \\
& +\frac{2\bar{B}^2}{3\mu_0\bar{n}k_B\bar{T}_e}\frac{\hat{\eta}_\|}{\hat{n}_{e0}}\hat{J}_{\|1}^2.
\end{aligned}$$

5 Results

6 Conclusion

The equations of 6-field are listed below when we drop the polarization velocity and viscous terms

$$
\begin{aligned}
\frac{\partial}{\partial t}\varpi &= -\left(\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\|i}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}\varpi \\
&\quad + B^2\nabla_\|\left(\frac{J_\|}{B}\right) + 2\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}p_1 \\
&\quad - \frac{1}{2\Omega_i}\left[\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}P_i\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\Phi\right) - Z_ieB\boldsymbol{b}\times\boldsymbol{\nabla}n_i\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\perp\Phi}{B}\right)^2\right] \\
&\quad + \frac{1}{2\Omega_i}\left[\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2 P_i\right) - \nabla_\perp^2\left(\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}P_i\right)\right], && (6.1) \\
\frac{\partial}{\partial t}n_i &= -\left(\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\|i}\boldsymbol{b}\right)\cdot\nabla n_i \\
&\quad - \frac{2n_i}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\Phi - \frac{2}{Z_ieB}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}P_i - n_iB\nabla_\|\left(\frac{V_{\|i}}{B}\right), && (6.2) \\
\frac{\partial}{\partial t}V_{\|i} &= -\left(\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\|i}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}V_{\|i} - \frac{1}{m_in_i}\boldsymbol{b}\cdot\boldsymbol{\nabla}P, && (6.3) \\
\frac{\partial}{\partial t}\psi &= -\frac{1}{B}\nabla_\|\phi + \frac{\eta}{\mu_0}\nabla_\perp^2\psi + \frac{1}{en_eB}\nabla_\|P_e + \frac{0.71}{eB}\nabla_\|T_e, && (6.4) \\
\frac{\partial}{\partial t}T_i &= -\left(\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\|i}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}T_i \\
&\quad - \frac{2}{3}T_i\left[\left(\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi + \frac{1}{Z_ien_i}\boldsymbol{\nabla}P_i + \frac{5}{2}\frac{1}{Z_ie}\boldsymbol{\nabla}T_i\right) + B\nabla_\|\left(\frac{V_{\|i}}{B}\right)\right] \\
&\quad + \frac{2}{3n_i}\nabla_\|\left(\kappa_{\|i}\nabla_\|T_i\right) + \frac{2}{3n_i}\nabla_\perp\left(\kappa_{\perp i}\nabla_\perp T_i\right) \\
&\quad + \frac{2m_e}{m_i}\frac{Z_i}{\tau_e}\left(T_e - T_i\right), && (6.5) \\
\frac{\partial}{\partial t}T_e &= -\left(\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi + V_{\|e}\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}T_e \\
&\quad - \frac{2}{3}T_e\left[\left(\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi - \frac{1}{en_e}\boldsymbol{\nabla}P_e - \frac{5}{2}\frac{1}{e}\boldsymbol{\nabla}T_e\right) + B\nabla_\|\left(\frac{V_{\|e}}{B}\right)\right] \\
&\quad + 0.71\frac{2T_e}{3en_e}B\nabla_\|\left(\frac{J_\|}{B}\right) \\
&\quad + \frac{2}{3n_e}\nabla_\|\left(\kappa_{\|e}\nabla_\|T_e\right) + \frac{2}{3n_e}\nabla_\perp\left(\kappa_{\perp e}\nabla_\perp T_e\right) \\
&\quad - \frac{2m_e}{m_i}\frac{1}{\tau_e}\left(T_e - T_i\right) + \frac{2}{3n_e}\eta_\|J_\|^2 && (6.6)
\end{aligned}
$$

The definations make the closure of the equations

$$
\begin{aligned}
Z_i n_{i0} + Z_{im} n_{im} &= n_{e0} \\
n_j &= n_{j0} + n_{j1}, \\
P_j &= P_{j0} + p_{j1}, \\
P = P_i + P_e &= P_0 + p_1 = (P_{i0} + P_{e0}) + (p_{i1} + p_{e1}), \\
P_t &= P_i + P_e + P_{im}, \\
\Phi &= \Phi_0 + \phi, \\
J_\parallel &= J_{\parallel 0} + J_{\parallel 1}, \\
V_{\parallel e} &= \frac{Z_i n_i + Z_{im} n_{im}}{n_e} V_{\parallel i} - \frac{J_{\parallel 1}}{e n_e} = V_{\parallel i} - \frac{J_{\parallel 1}}{e n_e}, \\
\boldsymbol{b} &= \boldsymbol{b}_0 - \boldsymbol{b}_0 \times \boldsymbol{\nabla} \psi, \\
\varpi &= n_{i0} \frac{m_i}{B_0} \left(\nabla_\perp^2 \phi + \frac{1}{n_{i0}} \nabla_\perp \phi \cdot \nabla_\perp n_{i0} + \frac{1}{n_{i0} Z_i e} \nabla_\perp^2 p_{i1} \right) \\
J_{\parallel 1} &= -\frac{1}{\mu_0} B_0 \nabla_\perp^2 \psi.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\varpi &= -\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi\cdot\boldsymbol{\nabla}\varpi + B^2\nabla_\parallel\left(\frac{J_\parallel}{B}\right) + 2\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}p_1 \\
&\quad -\frac{1}{2\Omega_i}\left[\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}P_i\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2\Phi\right) - Z_ieB\boldsymbol{b}\times\boldsymbol{\nabla}n_i\cdot\boldsymbol{\nabla}\left(\frac{\nabla_\perp\Phi}{B}\right)^2\right] \\
&\quad +\frac{1}{2\Omega_i}\left[\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\left(\nabla_\perp^2 P_i\right) - \nabla_\perp^2\left(\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}P_i\right)\right] + \mu_{\parallel i}\nabla_{\parallel 0}^2\varpi, \qquad (6.7) \\
\frac{\partial}{\partial t}n_i &= -\frac{1}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi\cdot\nabla n_i - \frac{2n_i}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\Phi - \frac{2}{Z_ieB}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}P_i - n_iB\nabla_\parallel\left(\frac{V_{\parallel i}}{B}\right), \qquad (6.8) \\
\frac{\partial}{\partial t}V_{\parallel i} &= -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi\cdot\boldsymbol{\nabla}V_{\parallel i} - \frac{1}{m_in_i}\boldsymbol{b}\cdot\boldsymbol{\nabla}P, \qquad (6.9) \\
\frac{\partial}{\partial t}\psi &= -\frac{1}{B}\nabla_\parallel\phi + \frac{\eta}{\mu_0}\nabla_\perp^2\psi + \frac{1}{en_eB}\nabla_\parallel P_e + \frac{0.71}{eB}\nabla_\parallel T_e, \qquad (6.10) \\
\frac{\partial}{\partial t}T_i &= -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi\cdot\boldsymbol{\nabla}T_i \\
&\quad -\frac{2}{3}T_i\left[\left(\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi + \frac{1}{Z_ien_i}\boldsymbol{\nabla}P_i + \frac{5}{2}\frac{1}{Z_ie}\nabla T_i\right) + B\nabla_\parallel\left(\frac{V_{\parallel i}}{B}\right)\right] \\
&\quad -\frac{2}{3n_i}\nabla_{\parallel 0}q_{\parallel i} + \frac{2m_e}{m_i}\frac{Z_i}{\tau_e}\left(T_e - T_i\right), \qquad (6.11) \\
\frac{\partial}{\partial t}T_e &= -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_\perp\Phi\cdot\boldsymbol{\nabla}T_e \\
&\quad -\frac{2}{3}T_e\left[\left(\frac{2}{B}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\boldsymbol{\nabla}\Phi - \frac{1}{en_e}\boldsymbol{\nabla}P_e - \frac{5}{2}\frac{1}{e}\nabla T_e\right) + B\nabla_\parallel\left(\frac{V_{\parallel e}}{B}\right)\right] \\
&\quad +0.71\frac{2T_e}{3en_e}B\nabla_\parallel\left(\frac{J_\parallel}{B}\right) - \frac{2}{3n_e}\nabla_{\parallel 0}q_{\parallel e} - \frac{2m_e}{m_i}\frac{1}{\tau_e}\left(T_e - T_i\right) + \frac{2}{3n_e}\eta_\parallel J_\parallel^2 \qquad (6.12)
\end{aligned}
$$

BOUT++ two fluid 6 field model($\varpi, n_i, V_{\parallel i}, Ti, Te, \psi$): based on Braginskii Equations, the density, momentum and energy of ion and electron are described in drift ordering.

Continuity Terms
Compressional Terms
Electron Hall
Gyroviscous Terms
Thermal Force
Energy Exchange
Energy Flux
Thermal Conduction

$$\nabla_\parallel q_{\parallel i,e} = \nabla_{\parallel 0}q_{\parallel i,e} - \boldsymbol{b}_0\times\boldsymbol{\nabla}\psi\cdot\nabla q_{\parallel i,e} \qquad (6.13)$$

References

[1] A. N. Simakov and P. J. Catto, Phys. Plasmas**10**, 4744 (2003). http://dx.doi.org/10.1063/1.1703527

[2] X. Q. Xu, W. M. Nevins, R. H. Cohen, J. R. Myra, P. B. Snyder, New J. Phys. **4**, 53.1 (2002).

[3] X. Q. Xu et al., Commun. Comput. Phys. **4**, 949 (2008).

[4] T. Y. Xia, X. Q. Xu, Nucl. Fusion **55**, 113030 (2015). http://iopscience.iop.org/0029-5515/55/11/113030