# 3+1 gyro-Landau Fluid Module User Manual

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## Contents



# 1 Introduction

This manual describes the 3+1 gyro-Landau fluid simulation using the `glfkbm3-1` module in BOUT++ framework. The `glfkbm3-1` module can perform gyro-Landau fluid simulation in both electrostatic and electromagnetic limits. It is well benchmarked with other fluid and gyrokinetic codes.

This manual is organized as follows. Section 2 describes the way to get the `glfkbm3-1` code and compile it. Section 3 describes the options for typical run. Section 4 describes the 3+1 gyro-Landau fluid model we used and the details of implementation. Section 5 talks about the options in the input file.

# 2 Installation

We are using git version control system. You can use the following command to get the code:

```
git clone ssh://user@portal-auth.nersc.gov/project/projectdirs/
    bout_glf/www/git/bout_glf.git
```

and switch to `modomegad` branch:

```
git checkout bout_modomegad
```

To compile the code, check the `configure` file of your machine and run the following command:

```
./configure
make
cd examples/glfkbm3-1
make
```

# 3 Running

This section talks about running the 3+1 code and focus on the unique

## 3.1 Electromagnetic KBM simulations for cbm grids

To run code in electromagnetic code, you should make sure

```
electrostatic = false
```

to open the electromagnetic mode. These options also should be set correctly to be consistent with six-field:

```
curv_model = 1                # Use the b0xcv curvature model
phi_constant_density = false  # include the cross-term in Poisson
                              # equation
```

Then open all the physics

```
eHall = true        #Pe term in Ohm's law
gyroaverage = true     # include gyro average effect
FLR_effect = true       # include FLR effect (phi_f terms)
continuity = true      # use continuity equation
compression = true
energy_flux = true
isotropic = false
```

where `isotropic` should be set to `false`. Also the Landau closures and toroidal closures

```
Landau_damping_i = true
Landau_damping_e = true
Lpar = 24.2

toroidal_closure1 = true  #toroidal closure in pressure equation
toroidal_closure2 = true
toroidal_closure3 = true

nu1r = 1.232
nu1i = 0.437
nu2r = -0.912
nu2i = 0.362
nu3r = -1.164
nu3i = 0.294
nu4r = 0.478
nu4i = -1.926
nu5r = 0.515
nu5i = -0.958
```

The density and temperature profiles are controlled by the profile control section:

```
##################################################
## profile control
Equilibrium_case = 4

############### 4 cbm tanh #######################

fit_pressure = false

#hyperbolic tanh profile, N0 = N0tanh(n0_height*Nbar, n0_ave*Nbar, n0_width, n0_
n0_fake_prof = true
n0_height = 0.400        #the total height of profile of N0, in percentage of Ni
n0_ave = 0.200          #the constant tail of N0 profile, in percentage of Ni_x
n0_width = 0.1         #the width of the gradient of N0, in percentage of x
n0_center = 0.633      #the the center of N0, in percentage of x
n0_bottom_x = 0.81     #the start of flat region of N0 on SOL side, in percentag
T0_const = 1000
```

The `Equilibrium_case` is set to 4 which is closest to the six-field setting. The units for `n0_height` and `n0_ave` are $10^{20}\mathrm{m}^{-3}$.

## 3.2 Electrostatic ITG benchmark for cyclone case

For the electrostatic ITG simulation, first set

```
electrostatic = true
```

to open the electrostatic mode. Also set

```
curv_model = 3                # Use the b0xcv curvature model
phi_constant_density = true   # include the cross-term in Poisson
                              # equation
```

for the core simulations. The physics part is the same as the electromagnetic runs. For the cyclone grid, the equilibrium density and temperature profiles are set as

```
####################################################
## profile control
Equilibrium_case = 4

#*************** 2 cyclone ********************#

n0_cyclone = 1.0e20
```

where the unit of `n0_cyclone` is $\mathrm{m}^{-3}$.

# 4 3+1 gyro-Landau fluid model

## 4.1 Physics model

### Ion Equations

$$\frac{\partial \tilde{n}_i}{\partial t}+\boldsymbol{v}_\Phi\cdot\nabla\tilde{n}_i+\boldsymbol{v}_{\Phi_0}\cdot\nabla\tilde{n}_i+B\tilde{\nabla}_\parallel\frac{n_0\tilde{u}_{\parallel i}}{B}+\frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\boldsymbol{v}_\Phi]\cdot\nabla\tilde{T}_{\perp i}-\frac{1}{2T_0}[\hat{\nabla}_\perp^2\boldsymbol{v}_{\bar{A}_\parallel}]\cdot\nabla\tilde{q}_{\perp i}$$
$$-n_0\left(1+\frac{1}{2}\eta_i\hat{\nabla}_\perp^2\right)i\omega_{G*}\frac{e\Phi}{T_0}+n_0\left(2+\frac{1}{2}\hat{\nabla}_\perp^2\right)i\omega_d\frac{e\Phi}{T_0}+\frac{1}{T_0}i\omega_d\left(\tilde{p}_{\parallel i}+\tilde{p}_{\perp i}\right)=0 \tag{1}$$

$$n_0\frac{\partial\tilde{u}_{\parallel i}}{\partial t}+n_0\boldsymbol{v}_\Phi\cdot\nabla\tilde{u}_{\parallel i}+n_0\boldsymbol{v}_{\Phi_0}\cdot\nabla\tilde{u}_{\parallel i}+\frac{n_0e}{m_i}\tilde{\nabla}_\parallel\Phi+\frac{n_0e}{m_i}\delta\bar{\boldsymbol{b}}\cdot\nabla\Phi_0+\frac{B}{m_i}\tilde{\nabla}_\parallel\frac{\tilde{p}_{\parallel i}}{B}+\frac{1}{2T_0}\hat{\nabla}_\perp^2\boldsymbol{v}_\Phi\cdot\nabla\tilde{q}_{\perp i}$$
$$-\frac{n_0}{2m_i}\hat{\nabla}_\perp^2\boldsymbol{v}_{\bar{A}}\cdot\nabla\tilde{T}_{\perp i}+\frac{n_0e}{m_i}\frac{\partial\bar{A}_\parallel}{\partial t}+\frac{n_0T_0}{m_i}\left(1+\eta_i+\frac{\eta_i}{2}\hat{\nabla}_\perp^2\right)i\omega_{G*}\frac{e\bar{A}_\parallel}{T_0}+\left(\frac{\tilde{p}_{\perp i}}{m_i}+\frac{n_0e}{2m_i}\hat{\nabla}_\perp^2\Phi\right)\nabla_\parallel\ln B$$
$$+\frac{1}{T_0}i\omega_d\left(\tilde{q}_{\parallel i}+\tilde{q}_{\perp i}+4p_{i0}\tilde{u}_{\parallel i}\right)=0 \quad (2)$$

$$\frac{\mathrm{d}\tilde{p}_{\parallel i}}{\mathrm{d}t}+B\tilde{\nabla}_\parallel\frac{\tilde{q}_{\parallel i}+3p_{0i}\tilde{u}_{\parallel i}}{B}+\frac{n_0}{2}\hat{\nabla}_\perp^2\boldsymbol{v}_\phi\cdot\nabla\tilde{T}_{\perp i}+2\left(\tilde{q}_{\perp i}+p_{i0}\tilde{u}_{\parallel i}\right)\nabla_\parallel\ln B$$
$$-n_0T_0\left(1+\eta_i+\frac{\eta_i}{2}\hat{\nabla}_\perp^2\right)i\omega_{G*}\frac{e\Phi}{T_0}+n_0T_0\left(4+\frac{1}{2}\hat{\nabla}_\perp^2\right)i\omega_d\frac{e\Phi}{T_0}+i\omega_d\left(\tilde{r}_{\parallel,\parallel}+\tilde{r}_{\parallel,\perp}\right)=0 \quad (3)$$

$$\frac{\mathrm{d}\tilde{p}_{\perp i}}{\mathrm{d}t}+B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_{\perp i}+p_{0i}\tilde{u}_{\parallel i}}{B^2}+\frac{1}{2}\hat{\nabla}_\perp^2\boldsymbol{v}_\Phi\cdot\nabla\tilde{p}_{\perp i}+\hat{\hat{\nabla}}_\perp^2\boldsymbol{v}_{\bar{A}}\cdot\nabla\tilde{T}_{\perp i}-\frac{p_{0i}}{2}\hat{\nabla}_\perp^2\boldsymbol{v}_{\bar{A}}\cdot\nabla\tilde{u}_{\parallel i}$$
$$-\frac{1}{2}\hat{\nabla}_\perp^2\boldsymbol{v}_{\bar{A}}\cdot\nabla\tilde{q}_{\perp i}-n_0T_0\left[1+\frac{1}{2}\hat{\nabla}_\perp^2+\eta_i\left(1+\frac{1}{2}\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2\right)\right]i\omega_{G*}\frac{e\Phi}{T_0}$$
$$+n_0T_0\left(3+\frac{3}{2}\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2\right)i\omega_d\frac{e\Phi}{T_0}+i\omega_d\left(\tilde{r}_{\parallel,\perp}+\tilde{r}_{\perp,\perp}\right)=0 \quad (4)$$

Where

$$\boldsymbol{v}_\Phi=\frac{1}{B}\boldsymbol{b}\times\nabla\Phi \quad (5)$$
$$\boldsymbol{v}_{\bar{A}_\parallel}=\frac{1}{B}\boldsymbol{b}\times\nabla\bar{A}_\parallel \quad (6)$$
$$\left(\Phi,\bar{A}_\parallel\right)=\Gamma_0^{\frac{1}{2}}\left(\phi,A_\parallel\right) \quad (7)$$
$$i\omega_*=\frac{T_0}{eBn_0}\boldsymbol{b}\times\nabla n_0\cdot\nabla \quad (8)$$
$$i\omega_d=\frac{T_0}{eB^3}\boldsymbol{B}\times\nabla B\cdot\nabla=\frac{1}{2}\frac{T_0}{eB}\left(\frac{\boldsymbol{b}\times\nabla B}{B}\cdot\nabla+\boldsymbol{b}\times\kappa\cdot\nabla\right) \quad (9)$$
$$\tilde{\nabla}_\parallel=\nabla_\parallel-\boldsymbol{v}_{\bar{A}_\parallel}\cdot\nabla \quad (10)$$
$$\nabla_\parallel=\boldsymbol{b}_0\cdot\nabla \quad (11)$$
$$\hat{\nabla}_\perp^2\Phi=2b\frac{\partial\Gamma_0^{\frac{1}{2}}}{\partial b}\phi \quad (12)$$
$$\hat{\hat{\nabla}}_\perp^2\Phi=b\frac{\partial^2}{\partial b^2}\left(b\Gamma_0^{\frac{1}{2}}\right)\phi \quad (13)$$

$$\Omega_i = \frac{eB}{m_i} \tag{14}$$

$$\eta_i = L_n / L_T \tag{15}$$

$$T_0 = T_{\perp 0} = 2T_{\parallel 0} = m_i v_{th}^2 \tag{16}$$

Closures $i\omega_d \left(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}\right)$, $i\omega_d \left(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}\right)$ and only keep $\tilde{q}_{\parallel i}$, $\tilde{q}_{\perp i}$ in parallel derivative.

$$\tilde{q}_{\parallel i} = -n_0 \sqrt{\frac{8}{\pi}} v_{T_{th}} \frac{ik_\parallel \tilde{T}_{\parallel i}}{|k_\parallel|} \tag{17}$$

$$\tilde{q}_{\perp i} = -n_0 \sqrt{\frac{2}{\pi}} v_{T_{th}} \frac{ik_\parallel}{|k_\parallel|} \left(\tilde{T}_{\perp i} + \frac{e}{2} \hat{\nabla}_\perp^2 \Phi\right) \tag{18}$$

mirror closure terms

$$i\omega_d \left(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}\right) = i\omega_d \left(7\tilde{p}_\parallel + \tilde{p}_\perp - 4T_0 \tilde{n} - 2i \frac{|\omega_d|}{\omega_d} \left(\nu_1 \tilde{T}_\parallel + \nu_2 \tilde{T}_\perp\right)\right) \tag{19}$$

$$i\omega_d \left(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}\right) = i\omega_d \left(\tilde{p}_\parallel + 5\tilde{p}_\perp - 3T_0 \tilde{n} - 2i \frac{|\omega_d|}{\omega_d} \left(\nu_3 \tilde{T}_\parallel + \nu_4 \tilde{T}_\perp\right)\right) \tag{20}$$

$$i\omega_d \left(\tilde{q}_{\parallel i} + \tilde{q}_{\perp i}\right) = 2n_0 T_0 \nu_5 |\omega_d| \tilde{u}_{\parallel i} \tag{21}$$

where

$$\nu = \nu_r + i\nu_i \frac{|\omega_d|}{\omega_d} \Rightarrow \begin{cases} \nu_1 & = (1.232, 0.437) \\ \nu_2 & = (-0.912, 0.362) \\ \nu_3 & = (-1.164, 0.294) \\ \nu_4 & = (0.478, -1.926) \\ \nu_5 & = (0.515, -0.958) \end{cases} \tag{22}$$

## Electron equations

$T_{e0} = T_{i0} = T_0,\ J_{0\parallel} = -en_0 u_{\parallel e0}$

$$\frac{\partial \tilde{n}_e}{\partial t} + \boldsymbol{v}_{ET} \cdot \nabla \tilde{n}_e + B \tilde{\nabla}_\parallel \frac{n_0 \tilde{u}_{\parallel e}}{B} - \frac{B}{e} \delta \boldsymbol{b} \cdot \nabla \frac{J_{0\parallel}}{B} - n_0 i\omega_* \frac{e\phi}{T_0} + 2n_0 i\omega_d \frac{e\phi}{T_0} - \frac{1}{T_0} i\omega_d \left(\tilde{p}_{\parallel e} + \tilde{p}_{\perp e}\right) = 0 \tag{23}$$

$$\frac{\partial A_\parallel}{\partial t} + \tilde{\nabla}_\parallel \phi + \delta \boldsymbol{b} \cdot \nabla \phi_0 - \frac{B}{n_0 e} \tilde{\nabla}_\parallel \frac{\tilde{p}_{\parallel e}}{B} - (1 + \eta_i)\, i\omega_* A_\parallel = \eta \tilde{J}_\parallel \tag{24}$$

$$\frac{\partial \tilde{p}_{\parallel e}}{\partial t} + \boldsymbol{v}_{ET} \cdot \nabla \tilde{p}_{\parallel e} + B\tilde{\nabla}_{\parallel} \frac{\tilde{q}_{\parallel e} + 3p_{0e}\tilde{u}_{\parallel e}}{B} - \frac{B}{e}\delta \boldsymbol{b} \cdot \nabla \frac{3T_0 J_{0\parallel}}{B} - en_0(1+\eta_i)i\omega_* \phi + 4en_0 i\omega_d \phi = 0 \tag{25}$$

$$\frac{\partial \tilde{p}_{\perp e}}{\partial t} + \boldsymbol{v}_{ET} \cdot \nabla \tilde{p}_{\perp e} + B^2\tilde{\nabla}_{\parallel} \frac{\tilde{q}_{\parallel e} + p_{0e}\tilde{u}_{\parallel e}}{B^2} - \frac{B^2}{e}\delta \boldsymbol{b} \cdot \nabla \frac{T_0 J_{0\parallel}}{B} - en_0(1+\eta_i)i\omega_* \phi + 3en_0 i\omega_d \phi = 0 \tag{26}$$

where

$$\boldsymbol{v}_{ET} = \frac{1}{B}\boldsymbol{b} \times \nabla(\phi + \phi_0) \tilde{\nabla}_{\parallel} = \nabla_{\parallel} - \boldsymbol{v}_{A_{\parallel}} \cdot \nabla \delta \boldsymbol{b} = -\boldsymbol{b} \times \nabla \Psi = -\boldsymbol{v}_{A_{\parallel}} \tag{27}$$

Landau damping for electron

$$\tilde{q}_{\parallel e} = -n_0 \sqrt{\frac{8}{\pi}} v_{T_{th}} \frac{ik_{\parallel} \tilde{T}_{\parallel e}}{|k_{\parallel}|} \tag{28}$$

$$\tilde{q}_{\perp e} = -n_0 \sqrt{\frac{2}{\pi}} v_{T_{th}} \frac{ik_{\parallel} \tilde{T}_{\perp e}}{|k_{\parallel}|} \tag{29}$$

# Poisson's Equation and the current

Padé approximation

$$\begin{cases} \Gamma_0^{\frac{1}{2}}(b) \approx \frac{1}{1+b/2} \\ \Gamma_0(b) \approx \frac{1}{1+b} \\ \Gamma_0 - \Gamma_1 \approx 1 \end{cases} \tag{30}$$

$$b = k_{\perp}^2 \rho_i^2 = -\rho_i^2 \nabla_{\perp}^2 \tag{31}$$

$$b = -\rho_i^2 \nabla_{\perp}^2 \tag{32}$$

quasi-neutrality

$$\varpi_G = eB \left[ \bar{n}_i - \tilde{n}_i - n_0 \left(1 - \Gamma_0\right) \frac{e\phi}{T_0} + \frac{e\rho_i^2}{T_0} \nabla n_0 \cdot \nabla (\Gamma_0 - \Gamma_1)\phi \right] \tag{33}$$

$$\bar{n}_i - n_0 \left(1 - \Gamma_0\right) \frac{e\phi}{T_0} + \frac{e\rho_i^2}{T_0} \nabla n_0 \cdot \nabla (\Gamma_0 - \Gamma_1)\phi = \tilde{n}_e \tag{34}$$

$$\Gamma_0^{\frac{1}{2}} n_{G0} - n_0(1 - \Gamma_0)\frac{e\phi_0}{T_0} + \frac{e\rho_i^2}{T_0} \nabla n_0 \cdot \nabla (\Gamma_0 - \Gamma_1)\phi_0 = n_0 \tag{35}$$

Simple Padé approximation:

$$\bar{n}_i = \frac{1}{1+\frac{b}{2}}\tilde{n}_i - \frac{n_0 2b}{T_0(2+b)^2}\tilde{T}_{\perp i} \tag{36}$$

### Current

$$\nabla_\perp^2 A_\parallel = -\mu_0 e n_0(\bar{u}_{\parallel i} - \tilde{u}_{\parallel e}) = -\mu_0 \tilde{J}_\parallel \tag{37}$$

Padé approximation:

$$\bar{u}_{\parallel i} = \Gamma_0^{\frac{1}{2}}\tilde{u}_{\parallel i} - \frac{2b}{(2+b)^2}\tilde{q}_{\perp i} \tag{38}$$

### Voriticity equation

Define

$$\varpi_G = eB\left(\tilde{n}_e - \tilde{n}_i\right) \tag{39}$$
$$\varpi_{G0} = eB\left(n_0 - n_{G0}\right) \tag{40}$$
$$\phi_f = \Phi - \phi \tag{41}$$

$$\begin{aligned}\frac{\partial \varpi_G}{\partial t} &+ \boldsymbol{v}_E\cdot\nabla\varpi_{G0} + \boldsymbol{v}_E\cdot\nabla\varpi_G + \boldsymbol{v}_{E0}\cdot\nabla\varpi_G - e\boldsymbol{b}\times\nabla\phi_f\cdot\nabla\tilde{n}_i \\ &= B^2\tilde{\nabla}_\parallel\frac{\tilde{J}_\parallel}{B} + B^2\delta\boldsymbol{b}\cdot\nabla\frac{J_{0\parallel}}{B} + \frac{eB}{T_0}i\omega_d(\tilde{p}_{\parallel e} + \tilde{p}_{\parallel i} + \tilde{p}_{\perp e} + \tilde{p}_{\perp i}) \\ -eB^2\tilde{\nabla}_\parallel\frac{n_0(\bar{u}_{\parallel i} - \tilde{u}_{\parallel i})}{B} &+ eB^2(\delta\bar{\boldsymbol{b}} - \delta\boldsymbol{b})\cdot\nabla\frac{n_0\tilde{u}_{\parallel i}}{B} - eBn_0 i\omega_{G*}\frac{e\phi_f}{T_0} + 2eBn_0 i\omega_d\frac{e\phi_f}{T_0} \\ &+ \frac{en_0}{2T_0}\boldsymbol{b}\times\nabla\boldsymbol{\Phi}^a\cdot\nabla\tilde{T}_{\perp i} - eB\frac{\eta_i}{2}n_0 i\omega_{G*}\frac{e\boldsymbol{\Phi}^a}{T_0} + \frac{eB}{2}n_0 i\omega_d\frac{e\boldsymbol{\Phi}^a}{T_0}\end{aligned} \tag{42}$$

## 4.2 Normalization

### Basic normalized quantities

- Normalization parameters $(\bar{L}, \bar{T}, \bar{N}, \bar{B}), \bar{V} = \bar{L}/\bar{T}, \bar{V}^2 = V_A^2 = \bar{B}^2/\mu_0 m_i \bar{N}, \bar{\Omega} = e\bar{B}/m_i, C_{nor} = \bar{\Omega}\bar{T}$ and

$$\hat{t} = \frac{t}{\bar{T}}, \quad \hat{B} = \frac{B}{\bar{B}}, \quad \hat{\nabla} = \bar{L}\nabla, \quad \hat{\kappa} = \bar{L}\kappa. \tag{43}$$

- Notations $\psi = A_\parallel/B, \Psi = \bar{A}_\parallel/B, U = \tilde{\varpi}_G/m_i$

The evolving variables are then normalized as:

$$\hat{n} = \frac{\tilde{n}}{\bar{N}}, \tag{44}$$

$$\hat{u}_\parallel = \frac{\tilde{u}_\parallel}{\bar{V}}, \tag{45}$$

$$\hat{p} = \frac{\tilde{p}}{m_i \bar{V}^2 \bar{N}}, \tag{46}$$

$$\hat{U} = \frac{\bar{T}}{\bar{N}} U, \tag{47}$$

$$\hat{\psi} = \frac{\psi}{\bar{L}}. \tag{48}$$

Other important variables are normalized as:

$$\hat{T} = \frac{\tilde{T}}{m_i \bar{V}^2}, \tag{49}$$

$$\hat{\eta} = \frac{\bar{T}}{\bar{L}^2 \mu_0} \eta, \tag{50}$$

$$\hat{J}_{\parallel c} = \bar{L} J_{\parallel c}, \tag{51}$$

$$\hat{\varpi}_G = \frac{\varpi_G}{e \bar{B} \bar{N}}, \tag{52}$$

$$\hat{\phi} = \frac{\phi}{\bar{L} \bar{V} \bar{B}}. \tag{53}$$

## Ion density equation

Rewrite $n_i$ equation to:

$$\begin{aligned}\frac{\partial \tilde{n}_i}{\partial t} &= -\frac{1}{B_0} b_0 \times \nabla\Phi \cdot \nabla n_i - \frac{1}{eB_0} b_0 \times \kappa \cdot \nabla (p_{\parallel i} + p_{\perp i}) \\ &- \frac{n_i}{B_0} b_0 \times \kappa \cdot \nabla \left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right)\Phi - n_0 B_0 \tilde{\nabla}_\parallel \frac{\tilde{u}_{\parallel i}}{B_0} - \frac{n_0}{2T_0 B_0} b_0 \times \nabla \left(\hat{\nabla}_\perp^2 \Phi\right) \cdot \nabla T_{\perp i}\end{aligned} \tag{54}$$

Linearize the $n_i$ equation:

$$\begin{aligned}\frac{\partial \tilde{n}_i}{\partial t} &= -[\Phi, n_{G0}] - [\Phi_0, n_{G0}] - [\Phi, \tilde{n}_i] - \frac{1}{eB_0} b_0 \times \kappa \cdot \nabla (p_{\parallel i} + p_{\perp i}) \\ &- \frac{n_0}{B} b_0 \times \kappa \cdot \nabla \left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right)\Phi - \frac{\tilde{n}_i}{B} b_0 \times \kappa \cdot \nabla \left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right)\Phi\end{aligned}$$

$$-n_0 B_0 \tilde{\nabla}_\| \frac{\tilde{u}_{\|i}}{B_0} - n_i B_0 \tilde{\nabla}_\| \frac{\tilde{u}_{\|i}}{B_0} - \frac{n_0}{2T_0}[\tilde{\nabla}_\perp^2 \Phi, \tilde{T}_{i0}] - \frac{1}{2}\left[\hat{\nabla}_\perp^2 \Phi, \frac{n_0 \tilde{T}_{\perp i}}{T_{i0}}\right] \quad (55)$$

Normalized $n_i$ equation:

$$\frac{\partial \hat{n}_i}{\partial \hat{t}} = -[\hat{\Phi}, \hat{n}_{G0}] - [\hat{\Phi}_0, \hat{n}_{G0}] - [\hat{\Phi}, \hat{n}_i] - \frac{1}{C_{nor}\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}(\hat{p}_{\|i} + \hat{p}_{\perp i})$$
$$- \frac{\hat{n}_0}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right)\hat{\Phi} - \frac{\hat{n}_i}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right)\hat{\Phi}$$
$$- \hat{n}_0 \hat{B}_0 \hat{\nabla}_\| \frac{\hat{u}_{\|i}}{\hat{B}_0} - \hat{n}_i \hat{B}_0 \hat{\nabla}_\| \frac{\hat{u}_{\|i}}{\hat{B}_0} - \frac{\hat{n}_0}{2\hat{T}_0}[\hat{\nabla}_\perp^2 \hat{\Phi}, \hat{T}_{i0}] - \frac{1}{2}\left[\hat{\nabla}_\perp^2 \hat{\Phi}, \frac{\hat{n}_0 \hat{T}_{\perp i}}{\hat{T}_{i0}}\right] \quad (56)$$

## Ion parallel velocity equation

Combine with Ampere's law, we get a simplified form of the equation:

$$n_0 \frac{\partial \tilde{u}_{\|i}}{\partial t} + n_0 v_\Phi \cdot \nabla \tilde{u}_{\|i} + n_0 v_{\Phi_0} \cdot \nabla \tilde{u}_{\|i} + \frac{B}{m_i}\tilde{\nabla}_\| \frac{\tilde{p}_\|}{B}$$
$$+ \left(\frac{\tilde{p}_{\perp i}}{m_i} + \frac{n_0 e}{2m_i}\hat{\nabla}_\perp^2 \Phi\right)\nabla_\| \ln B + \frac{1}{T_0} i\omega_d (\tilde{q}_{\|i} + \tilde{q}_{\|e}) + \frac{4}{T_0} i\omega_d p_{i0} \tilde{u}_{\|i} = 0 \quad (57)$$

Rewrite the equation to:

$$\frac{\partial \tilde{u}_{\|i}}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla\Phi \cdot \nabla \tilde{u}_{\|i} - \frac{B_0}{n_0 m_i}\tilde{\nabla}_\| \frac{\tilde{p}_\|}{B} - \frac{4}{eB_0} b_0 \times \kappa \cdot \nabla T_{i0} \tilde{u}_{\|i}$$
$$- \left(\frac{\tilde{p}_{\perp i}}{n_0 m_i} + \frac{e}{2m_i}\hat{\nabla}_\perp^2 \Phi\right)\nabla_\| \log B_0 - \frac{1}{p_{i0}} i\omega_d (\tilde{q}_{\|i} + \tilde{q}_{\perp i}) \quad (58)$$

Linearized equation:

$$\frac{\partial \tilde{u}_{\|i}}{\partial t} = -[\Phi, \tilde{u}_{\|i}] - \frac{B}{n_0 m_i}\tilde{\nabla}_\| \frac{\tilde{p}_\|}{B} + \frac{B_0}{n_0 m_i}[\psi, P_0]$$
$$- \frac{4}{eB_0} b_0 \times \kappa \cdot \nabla T_{i0} \tilde{u}_{\|i} - \frac{\tilde{p}_{\perp i}}{n_0 m_i}\nabla_\| \ln B - \frac{e}{2m_i}\hat{\nabla}_\perp^2 \Phi \nabla_\| \ln B - \frac{1}{p_{i0}} i\omega_d (\tilde{q}_{\|i} + \tilde{q}_{\perp i}) \quad (59)$$

Normalized equation:

$$\frac{\partial \hat{u}_{\parallel i}}{\partial \hat{t}} = -[\hat{\Phi}, \hat{u}_{\parallel i}] - \frac{1}{\hat{n}_0}\hat{\nabla}_{\parallel}\hat{p}_{\parallel} + \frac{\hat{B}_0}{\hat{n}_0}[\hat{\psi}, \hat{P}_0]$$
$$- \frac{4}{C_{nor}\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\hat{T}_{i0}\hat{u}_{\parallel i} - \frac{\hat{p}_{\perp i}}{\hat{n}_0\hat{B}_0}\nabla_{\parallel 0}B_0 - \frac{C_{nor}}{B_0}\hat{\nabla}_{\perp}^2\hat{\Phi}\hat{\nabla}_{\parallel 0}B - \frac{1}{\hat{p}_{i0}} i\omega_d(\hat{q}_{\parallel i} + \hat{q}_{\perp i}) \tag{60}$$

## Ion parallel pressure equation

Rewrite the equation to:

$$\frac{\partial \tilde{p}_{\parallel i}}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla\Phi \cdot \nabla p_{\parallel i} - \frac{n_0}{2B_0} b_0 \times \nabla\left(\hat{\nabla}_{\perp}^2\Phi\right) \cdot \nabla T_{\perp i}$$
$$- \frac{p_{\parallel i}}{B_0} b_0 \times \kappa \cdot \nabla\left(4 + \frac{1}{2}\hat{\nabla}_{\perp}^2\right)\Phi - 3B_0\tilde{\nabla}_{\parallel}\frac{p_{i0}\tilde{u}_{\parallel i}}{B_0} - i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}) - B_0\tilde{\nabla}_{\parallel}\frac{\tilde{q}_{\parallel i}}{B_0} \tag{61}$$

Linearized equation:

$$\begin{aligned}\frac{\partial \tilde{p}_{\parallel i}}{\partial t} &= -[\Phi, p_{i0}] - [\Phi, \tilde{p}_{\parallel i}] - \frac{n_0}{2}[\hat{\nabla}_{\perp}^2\Phi, T_{i0}] - \frac{n_0}{2}[\hat{\nabla}_{\perp}^2\Phi, T_{\perp i}] \\ &\quad - \frac{p_{i0}}{B_0} b_0 \times \kappa \cdot \nabla\left(4 + \frac{1}{2}\hat{\nabla}_{\perp}^2\right)\Phi - \frac{\tilde{p}_{\parallel i}}{B_0} b_0 \times \kappa \cdot \nabla\left(4 + \frac{1}{2}\hat{\nabla}_{\perp}^2\right)\Phi \\ &\quad - 3B_0\tilde{\nabla}_{\parallel}\frac{p_{i0}\tilde{u}_{\parallel i}}{B_0} - 3B_0\nabla_{\parallel 0}\frac{p_{\parallel i}\tilde{u}_{\parallel i}}{B_0} - i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}) - B_0\tilde{\nabla}_{\parallel}\frac{\tilde{q}_{\parallel i}}{B_0}\end{aligned} \tag{62}$$

Normalized equation:

$$\begin{aligned}\frac{\partial \hat{p}_{\parallel i}}{\partial \hat{t}} &= -[\hat{\Phi}, \hat{p}_{i0}] - [\hat{\Phi}, \hat{p}_{\parallel i}] - \frac{\hat{n}_0}{2}[\hat{\nabla}_{\perp}^2\Phi, \hat{T}_{i0}] - \frac{\hat{n}_0}{2}[\hat{\nabla}_{\perp}^2\hat{\Phi}, \hat{T}_{\perp i}] \\ &\quad - \frac{\hat{p}_{i0}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\left(4 + \frac{1}{2}\hat{\nabla}_{\perp}^2\right)\hat{\Phi} - \frac{\hat{p}_{\parallel i}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\left(4 + \frac{1}{2}\hat{\nabla}_{\perp}^2\right)\hat{\Phi} \\ &\quad - 3\hat{B}_0\hat{\nabla}_{\parallel}\frac{\hat{p}_{i0}\hat{u}_{\parallel i}}{\hat{B}_0} - 3\hat{B}_0\hat{\nabla}_{\parallel 0}\frac{\hat{p}_{\parallel i}\hat{u}_{\parallel i}}{\hat{B}_0} - i\omega_d(\hat{r}_{\parallel,\parallel} + \hat{r}_{\parallel,\perp}) - \hat{B}_0\hat{\nabla}_{\parallel}\frac{\hat{q}_{\parallel i}}{\hat{B}_0}\end{aligned} \tag{63}$$

## Ion perpendicular pressure equation

Rewrite the equation to:

$$\frac{\partial \tilde{p}_{\perp i}}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla\Phi_2 \cdot \nabla p_{\perp i} - \frac{n_0}{B_0} b_0 \times \left(\hat{\hat{\nabla}}_{\perp}^2\Phi\right) \cdot \nabla T_{\perp i}$$

$$-\frac{p_{\perp i}}{B_0}b_0\times\kappa\cdot\nabla\left(3+\frac{3}{2}\tilde{\nabla}_\perp^2+\hat{\tilde{\nabla}}_\perp^2\right)\Phi-B_0^2\tilde{\nabla}_\parallel\frac{p_{\perp i}\tilde{u}_{\parallel i}}{B_0^2}-i\omega_d(\tilde{r}_{\parallel,\perp}+\tilde{r}_{\perp,\perp})-B_0^2\nabla_\parallel\frac{\tilde{q}_{\perp i}}{B_0^2} \tag{64}$$

Linearized equation:

$$\begin{aligned}\frac{\partial\tilde{p}_{\perp i}}{\partial t} &= -[\Phi_2,p_{i0}]-[\Phi_2,\tilde{p}_{\parallel i}]-n_0[\hat{\tilde{\nabla}}_\perp^2\Phi,T_{i0}]-n_0[\hat{\tilde{\nabla}}_\perp^2\Phi,\tilde{T}_{\perp i}]\\ -\frac{p_{i0}}{B_0}b_0\times\kappa\cdot\nabla&\left(3+\frac{3}{2}\tilde{\nabla}_\perp^2+\hat{\tilde{\nabla}}_\perp^2\right)\Phi-\frac{\tilde{p}_{\perp i}}{B_0}b_0\times\kappa\cdot\nabla\left(3+\frac{3}{2}\hat{\nabla}_\perp^2+\hat{\tilde{\nabla}}_\perp^2\right)\Phi\\ &-B_0^2\tilde{\nabla}_\parallel\frac{p_{i0}\tilde{u}_{\parallel i}}{B_0^2}-B_0^2\tilde{\nabla}_\parallel\frac{\tilde{p}_{\perp i}\tilde{u}_{\parallel i}}{B_0^2}-i\omega_d(\tilde{r}_{\parallel,\perp}+\tilde{r}_{\perp,\perp})-B_0^2\nabla_{\parallel 0}\frac{\tilde{q}_{\perp i}}{B_0^2}\end{aligned} \tag{65}$$

Normalized equation:

$$\begin{aligned}\frac{\partial\hat{p}_{\perp i}}{\partial\hat{t}} &= -[\hat{\Phi}_2,\hat{p}_{i0}]-[\hat{\Phi}_2,\hat{p}_{\parallel i}]-\hat{n}_0[\hat{\tilde{\nabla}}_\perp^2\hat{\Phi},\hat{T}_{i0}]-\hat{n}_0[\hat{\tilde{\nabla}}_\perp^2\hat{\Phi},\hat{T}_{\perp i}]\\ -\frac{\hat{p}_{i0}}{\hat{B}_0}b_0\times\hat{\kappa}\cdot\hat{\nabla}&\left(3+\frac{3}{2}\hat{\nabla}_\perp^2+\hat{\tilde{\nabla}}_\perp^2\right)\hat{\Phi}-\frac{\hat{p}_{\perp i}}{\hat{B}_0}b_0\times\hat{\kappa}\cdot\hat{\nabla}\left(3+\frac{3}{2}\hat{\nabla}_\perp^2+\hat{\tilde{\nabla}}_\perp^2\right)\hat{\Phi}\\ &-\hat{B}_0^2\hat{\nabla}_\parallel\frac{\hat{p}_{i0}\hat{u}_{\parallel i}}{\hat{B}_0^2}-\hat{B}_0^2\hat{\nabla}_{\parallel 0}\frac{\hat{p}_{\perp i}\hat{u}_{\parallel i}}{B_0^2}-i\omega_d(\hat{r}_{\parallel,\perp}+\hat{r}_{\perp,\perp})-\hat{B}_0^2\hat{\nabla}_{\parallel 0}\frac{\hat{q}_{\perp i}}{\hat{B}_0^2}\end{aligned} \tag{66}$$

## Ampere's Law

$$\frac{\partial A_\parallel}{\partial t}=-\tilde{\nabla}_\parallel\phi+\frac{B_0}{n_0e}\tilde{\nabla}_\parallel\frac{\tilde{p}_{\parallel e}}{B_0}+\frac{\eta}{\mu_0}\nabla_\perp^2A_\parallel-\frac{\eta_H}{\mu_0}\nabla_\perp^4A_\parallel \tag{67}$$

Linearized equation ($\psi=A_\parallel/B_0$):

$$\frac{\partial\psi}{\partial t}=-\frac{1}{B_0}\tilde{\nabla}_\parallel\phi+\frac{1}{n_0eB_0}\tilde{\nabla}_\parallel\tilde{p}_{\parallel e}+\frac{1}{n_0e}[\psi,P_{e0}]+\frac{\eta}{\mu_0}\nabla_\perp^2\psi-\frac{\eta_H}{\mu_0}\nabla_\perp^4\psi \tag{68}$$

Normalized equation:

$$\frac{\partial\hat{\psi}}{\partial\hat{t}}=-\frac{1}{\hat{B}_0}\hat{\nabla}_\parallel\hat{\phi}+\frac{1}{C_{nor}\hat{n}_0\hat{B}_0}\hat{\nabla}_\parallel\hat{p}_{\parallel e}+\frac{1}{C_{nor}\hat{n}_0}[\hat{\psi},\hat{P}_{e0}]+\hat{\eta}\hat{J}_{\parallel c}-\hat{\eta}_H\hat{\nabla}_\perp^2\hat{J}_{\parallel c} \tag{69}$$

## Electron parallel pressure equation

Rewrite equation to:

$$\frac{\partial \tilde{p}_{\parallel e}}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla\phi \cdot \nabla p_{\parallel e} - \frac{4 p_{\parallel e}}{B_0} b_0 \times \kappa \cdot \nabla\phi - 3 B_0 \tilde{\nabla}_\parallel \frac{p_{\parallel e}\tilde{u}_{\parallel e}}{B_0} + \frac{T_{\parallel e}}{e B_0} b_0 \times \kappa \cdot \nabla(7\tilde{p}_{\parallel e} + \tilde{p}_{\perp e} - 4 T_{e0}\tilde{n}_e) - B_0 \nabla_\parallel \frac{\tilde{q}_{\parallel e}}{B_0} \quad (70)$$

Linearized equation:

$$\frac{\partial \tilde{p}_{\parallel e}}{\partial t} = -[\phi, p_{e0}] - [\phi, \tilde{p}_{\parallel e}] - \frac{4 p_{e0}}{B_0} b_0 \times \kappa \cdot \nabla\phi - \frac{4 \tilde{p}_{\parallel e}}{B_0} b_0 \times \kappa \cdot \nabla\phi - 3 B_0 \tilde{\nabla}_\parallel \frac{p_{e0}\tilde{u}_{\parallel e}}{B_0} - 3 B_0 \tilde{\nabla}_{\parallel 0} \frac{\tilde{p}_{\parallel e}\tilde{u}_{\parallel e}}{B_0} + \frac{T_{e0}}{e B_0} b_0 \times \kappa \cdot \nabla(7\tilde{p}_{\parallel e} + \tilde{p}_{\perp e} - 4 T_{e0}\tilde{n}_e) + \frac{\tilde{T}_{\parallel e}}{e B_0} b_0 \times \kappa \cdot \nabla(7\tilde{p}_{\parallel e} + \tilde{p}_{\perp e} - 4 T_{e0}\tilde{n}_e) - B_0 \nabla_\parallel \frac{\tilde{q}_{\parallel e}}{B_0} \quad (71)$$

Normalized equation:

$$\frac{\partial \hat{p}_{\parallel e}}{\partial \hat{t}} = -[\hat{\phi}, \hat{p}_{e0}] - [\hat{\phi}, \hat{p}_{\parallel e}] - \frac{4 \hat{p}_{e0}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\hat{\phi} - \frac{4 \hat{p}_{\parallel e}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}\hat{\phi} - 3 \hat{B}_0 \hat{\nabla}_\parallel \frac{\hat{p}_{e0}\hat{u}_{\parallel e}}{B_0} - 3 \hat{B}_0 \hat{\nabla}_{\parallel 0} \frac{\hat{p}_{\parallel e}\hat{u}_{\parallel e}}{B_0} + \frac{\hat{T}_{e0}}{C_{nor}\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}(7\hat{p}_{\parallel e} + \hat{p}_{\perp e} - 4 \hat{T}_{e0}\hat{n}_e) + \frac{\hat{T}_{\parallel e}}{C_{nor}\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla}(7\hat{p}_{\parallel e} + \hat{p}_{\perp e} - 4 \hat{T}_{e0}\hat{n}_e) - \hat{B}_0 \hat{\nabla}_\parallel \frac{\hat{q}_{\parallel e}}{\hat{B}_0} \quad (72)$$

## Electron perpendicular pressure equation

Rewrite equation to:

$$\frac{\partial \tilde{p}_{\perp e}}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla\phi \cdot \nabla p_{\perp e} - \frac{3 p_{\perp e}}{B_0} b_0 \times \kappa \cdot \nabla\phi - B_0^2 \tilde{\nabla}_\parallel \frac{p_{\perp e}\tilde{u}_{\parallel e}}{B_0^2} + \frac{T_{\perp e}}{e B_0} b_0 \times \kappa \cdot \nabla(\tilde{p}_{\parallel e} + 5\tilde{p}_{\perp e} - T_{e0}\tilde{n}_e) - B_0^2 \nabla_\parallel \frac{\tilde{q}_{\perp e}}{B_0^2} \quad (73)$$

Linearized equation:

$$\frac{\partial \tilde{p}_{\perp e}}{\partial t} = -[\phi, p_{e0}] - [\phi, \tilde{p}_{\perp e}] - \frac{3 p_{e0}}{B_0} b_0 \times \kappa \cdot \nabla\phi - \frac{3 \tilde{p}_{\perp e}}{B_0} b_0 \times \kappa \cdot \nabla\phi$$

$$
- B_0^2 \tilde{\nabla}_\parallel \frac{p_{e0} \tilde{u}_{\parallel e}}{B_0^2} - B_0^2 \tilde{\nabla}_{\parallel 0} \frac{\tilde{p}_{\perp e} \tilde{u}_{\parallel e}}{B_0^2}
$$

$$
+ \frac{T_{e0}}{eB_0} b_0 \times \kappa \cdot \nabla (\tilde{p}_{\parallel e} + 5\tilde{p}_{\perp e} - T_{e0} \tilde{n}_e) + \frac{\tilde{T}_{\perp e}}{eB_0} b_0 \times \kappa \cdot \nabla (\tilde{p}_{\parallel e} + 5\tilde{p}_{\perp e} - T_{e0} \tilde{n}_e) - B_0^2 \nabla_\parallel \frac{\tilde{q}_{\perp e}}{B_0^2} \tag{74}
$$

Normalized equation:

$$
\frac{\partial \hat{p}_{\perp e}}{\partial \hat{t}} = -[\hat{\phi}, \hat{p}_{e0}] - [\hat{\phi}, \hat{p}_{\perp e}] - \frac{3\hat{p}_{e0}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla} \hat{\phi} - \frac{3\hat{p}_{\perp e}}{\hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla} \hat{\phi}
$$

$$
- \hat{B}_0^2 \hat{\nabla}_\parallel \frac{\hat{p}_{e0} \hat{u}_{\parallel e}}{\hat{B}_0^2} - \hat{B}_0^2 \hat{\nabla}_{\parallel 0} \frac{\hat{p}_{\perp e} \hat{u}_{\parallel e}}{\hat{B}_0^2}
$$

$$
+ \frac{\hat{T}_{e0}}{C_{nor} \hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla} (\hat{p}_{\parallel e} + 5\hat{p}_{\perp e} - \hat{T}_{e0} \hat{n}_e) + \frac{\hat{T}_{\perp e}}{C_{nor} \hat{B}_0} b_0 \times \hat{\kappa} \cdot \hat{\nabla} (\hat{p}_{\parallel e} + 5\hat{p}_{\perp e} - \hat{T}_{e0} \hat{n}_e) - \hat{B}_0^2 \hat{\nabla}_\parallel \frac{\hat{q}_{\perp e}}{\hat{B}_0^2} \tag{75}
$$

## Vorticity equation

Rewrite equation to:

$$
\frac{\partial \tilde{\varpi}_G}{\partial t} = -\frac{1}{B_0} b_0 \times \nabla \phi \cdot \nabla \varpi_G + B_0^2 \tilde{\nabla}_\parallel \frac{\tilde{J}_\parallel}{B_0} + b_0 \times \kappa \cdot \nabla (\tilde{p}_{\parallel i} + \tilde{p}_{\perp i} + \tilde{p}_{\parallel e} + \tilde{p}_{\perp e})
$$

$$
+ eB_0 b_0 \times \nabla \phi_f \cdot \nabla n_i + \frac{eB_0 n_0}{T_{i0}} b_0 \times \nabla \left( \hat{\nabla}_\perp^2 \Phi \right) \cdot \nabla T_{\perp i} + eB^2 (\delta \bar{\boldsymbol{b}} - \delta \boldsymbol{b}) \cdot \nabla \frac{n_0 u_{\parallel i}}{B_0} + en_0 b_0 \times \kappa \cdot \nabla \left( 2\phi_f + \hat{\nabla}_\perp^2 \Phi \right) \tag{76}
$$

Linearized equation:

$$
\frac{\partial \tilde{\varpi}_G}{\partial t} = -[\phi, \varpi_{G0}] - [\phi, \varpi_G] - \frac{B_0^2}{\mu_0} \tilde{\nabla}_\parallel \tilde{J}_{\parallel c} - B_0^3 [\psi, J_0] + b_0 \times \kappa \cdot \nabla (\tilde{p}_{\parallel i} + \tilde{p}_{\perp i} + \tilde{p}_{\parallel e} + \tilde{p}_{\perp e})
$$

$$
+ eB_0 [\phi_f, n_0] + eB_0 [\phi_f, \tilde{n}_i] + \frac{eB_0 n_0}{T_{i0}} [\hat{\nabla}_\perp^2 \Phi, T_{i0}] + \frac{eB_0 n_0}{T_{i0}} [\hat{\nabla}_\perp^2 \Phi, \tilde{T}_{\perp i}]
$$

$$
- eB_0^3 \left[ \Psi - \psi, \frac{n_0 \tilde{u}_{\parallel i}}{B_0} \right] + en_0 b_0 \times \kappa \cdot \nabla \left( 2\phi_f + \hat{\nabla}_\perp^2 \Phi \right) \tag{77}
$$

where $J_{\parallel c} = \nabla_\perp^2 \psi = -\mu_0 J_\parallel / B_0$.

We define $U$ as

$$
U = \frac{\varpi_G}{m_i}, \tag{78}
$$

and the relationship between normalized variables are

$$\hat{\varpi}_G = \hat{B}(\hat{n}_e - \hat{n}_i), \tag{79}$$
$$\hat{U} = C_{nor}\hat{\varpi}_G. \tag{80}$$

We get the normalized vorticity equation as:

$$\frac{\partial \hat{U}}{\partial \hat{t}} = -[\hat{\phi}, \hat{U}_0] - [\hat{\phi}, \hat{U}] - \frac{V_A^2}{\bar{V}^2}\hat{B}_0^2\hat{\nabla}_\parallel \hat{J}_{\parallel c} - \hat{B}_0^3[\hat{\psi}, \hat{J}_0] + b_0 \times \hat{\kappa} \cdot \hat{\nabla}(\hat{p}_{\parallel i} + \hat{p}_{\perp i} + \hat{p}_{\parallel e} + \hat{p}_{\perp e})$$
$$+ C_{nor}\hat{B}_0[\hat{\phi}_f, \hat{n}_0] + C_{nor}\hat{B}_0[\hat{\phi}_f, \hat{n}_i] + \frac{C_{nor}\hat{B}_0\hat{n}_0}{\hat{T}_{i0}}[\hat{\nabla}_\perp^2\hat{\Phi}, \hat{T}_{i0}] + \frac{C_{nor}\hat{B}_0\hat{n}_0}{\hat{T}_{i0}}[\hat{\nabla}_\perp^2\hat{\Phi}, \hat{T}_{\perp i}]$$
$$- C_{nor}\hat{B}_0^3\left[\hat{\Psi} - \hat{\psi}, \frac{\hat{n}_0\hat{u}_{\parallel i}}{\hat{B}_0}\right] + C_{nor}\hat{n}_0 b_0 \times \hat{\kappa} \cdot \hat{\nabla}\left(2\hat{\phi}_f + \hat{\nabla}_\perp^2\hat{\Phi}\right) \tag{81}$$

## 4.3 Implementation

### Solving gyro-kinetic Poisson equation

The gyro-kinetic Poisson equation is

$$\varpi_G = eB\left[\bar{n}_i - \tilde{n}_i - n_0\left(1 - \Gamma_0\right)\frac{e\phi}{T_0} + \frac{e\rho_i^2}{T_0}\nabla n_0 \cdot \nabla(\Gamma_0 - \Gamma_1)\phi\right]. \tag{82}$$

Substituted to $U = \varpi_G/m_i$:

$$U = \frac{eB}{m_i}\left[\bar{n}_i - \tilde{n}_i - n_0\left(1 - \Gamma_0\right)\frac{e\phi}{T_0} + \frac{e\rho_i^2}{T_0}\nabla n_0 \cdot \nabla(\Gamma_0 - \Gamma_1)\phi\right]. \tag{83}$$

The normalized equation is

$$\hat{U} = C_{nor}\hat{B}_0\left[\hat{\bar{n}}_i - \hat{n}_i - C_{nor}\frac{\hat{n}_0}{\hat{T}_0}\left(1 - \Gamma_0\right)\hat{\phi} + C_{nor}\frac{\hat{\rho}_i^2}{\hat{T}_0}\hat{\nabla}\hat{n}_0 \cdot \hat{\nabla}(\Gamma_0 - \Gamma_1)\hat{\phi}\right]. \tag{84}$$

We define $\hat{n}_{mid}$ as

$$\hat{n}_{mid} = \frac{\hat{T}_0}{C_{nor}\hat{n}_0}\left(\hat{\bar{n}}_i - \hat{n}_i - \frac{\hat{U}}{C_{nor}\hat{B}_0}\right), \tag{85}$$

and Eq. (84) becomes

$$\left(1 - \Gamma_0\right)\hat{\phi} - \frac{\hat{\rho}_i^2}{\hat{n}_0}\hat{\nabla}\hat{n}_0 \cdot \hat{\nabla}(\Gamma_0 - \Gamma_1)\hat{\phi} = \hat{n}_{mid}. \tag{86}$$

We utilize Padé approximation. Eq. (86) becomes

$$b\hat{\phi} - (1+b)\frac{\hat{\rho}_i^2}{\hat{n}_0}\hat{\nabla}\hat{n}_0 \cdot \hat{\nabla}\hat{\phi} = (1+b)\hat{n}_{mid}, \tag{87}$$

where $b = -\hat{\rho}_i^2\hat{\nabla}_\perp^2$. We ignore the $b$ in the second term of Eq. (87). Now it is simplified as

$$\hat{\nabla}_\perp^2\hat{\phi} + \frac{1}{\hat{n}_0}\hat{\nabla}\hat{n}_0 \cdot \hat{\nabla}\hat{\phi} = -\frac{1}{\hat{\rho}_i^2}\hat{n}_{mid} + \hat{\nabla}_\perp^2\hat{n}_{mid}. \tag{88}$$

Eq. (88) can be solved by the `invert_laplace` function in BOUT++.

## Gyro-average operators

We have three major gyro-average operators in 3+1 code to be solved:

$$\Phi = \Gamma_0^{\frac{1}{2}}\phi \tag{89}$$

$$\hat{\nabla}_\perp^2\Phi = 2b\frac{\partial\Gamma_0^{\frac{1}{2}}}{\partial b}\phi \tag{90}$$

$$\hat{\hat{\nabla}}_\perp^2\Phi = b\frac{\partial^2}{\partial b^2}\left(b\Gamma_0^{\frac{1}{2}}\right)\phi \tag{91}$$

The gyroaverage operator $\Gamma_0^{\frac{1}{2}}$ can be solved by the `invert_laplace` function in BOUT++. From Padé approximation,

$$\Phi = \Gamma_0^{\frac{1}{2}}\phi = \frac{1}{1+\frac{b}{2}}\phi, \tag{92}$$

which can be simplified as

$$-\frac{1}{2}\rho_i^2\nabla_\perp^2\Phi + \Phi = \phi. \tag{93}$$

For the modified Laplacian operator, we have

$$\begin{aligned}
\hat{\nabla}_\perp^2\Phi =& 2b\frac{\partial\Gamma_0^{\frac{1}{2}}}{\partial b}\phi \\
=& -\frac{b}{\left(1+\frac{b}{2}\right)^2}\phi \\
=& 2\left[\frac{1}{\left(1+\frac{b}{2}\right)^2} - \frac{1}{1+\frac{b}{2}}\right]\phi \\
=& 2\left[\Phi_2 - \Phi\right],
\end{aligned} \tag{94}$$

and

$$
\begin{aligned}
\hat{\hat{\nabla}}_{\perp}^{2}\Phi =& b\frac{\partial^2}{\partial b^2}\left(b\Gamma_0^{\frac{1}{2}}\right)\phi \\
=& -\frac{b}{\left(1+\frac{b}{2}\right)^3}\phi \\
=& 2\left[\frac{1}{\left(1+\frac{b}{2}\right)^3}-\frac{1}{\left(1+\frac{b}{2}\right)^2}\right]\phi \\
=& 2\left[\Phi_3-\Phi_2\right],
\end{aligned} \tag{95}
$$

where $\Phi_2 = \Gamma_0^{\frac{1}{2}}\Phi$ and $\Phi_3 = \Gamma_0^{\frac{1}{2}}\Phi_2$. $\Phi_2$ and $\Phi_3$ can be solved the same as $\Phi$ in Eq. (93).

**Landau closures**

$$
\frac{ik_{\parallel}}{\left|k_{\parallel}\right|}\hat{T} = \mathtt{isign\_kpar}(\mathtt{T},\mathtt{k0}) \tag{96}
$$

**Toroidal closures**

# 5 Input File

# References