Gyrofluid for ETG

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Gyrokinetic equation for electron

Ordering:

$$\frac{\omega}{\Omega_e} \sim \frac{k_\parallel v_{te}}{\Omega_e} \sim \frac{e\phi}{T} \sim \frac{\delta B}{B_0} \sim \frac{F_1}{F_0} \sim \frac{\rho_e}{L} \sim \epsilon \ll 1,\ \rho_e k_\perp \sim 1$$

Gyrokinetic Equation:

$$\frac{\partial F(\mathbf{X}, v_\parallel, \mu, t)}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F + \dot{v}_\parallel \frac{\partial F}{\partial v_\parallel} = C(F)$$

$$\begin{cases} \dot{\mathbf{X}} = v_\parallel(\mathbf{b}_0 + \frac{\langle\delta\mathbf{B}\rangle}{\mathbf{B}_0}) + \mathbf{v}_E + \mathbf{v}_d & \begin{cases} \mathbf{v}_d = \frac{v_\parallel^2 + \mu B}{\Omega B^2}\mathbf{B} \times \nabla B, & low\ \beta \\ \mathbf{v}_E = \frac{c}{B}\mathbf{b}_0 \times \nabla\left\langle \phi + \phi_0 \right\rangle, & \langle \delta\mathbf{B} \rangle = -\mathbf{b}_0 \times \nabla \left\langle A_\parallel \right\rangle \end{cases} \\ \dot{v}_\parallel = -\frac{e}{m}\widetilde{E}_\parallel - \mu\tilde{\mathbf{b}} \cdot \nabla B + v_\parallel \kappa \cdot \mathbf{v}_E & \begin{cases} \widetilde{E}_\parallel = -\frac{1}{c}\frac{\partial\left\langle A_\parallel \right\rangle}{\partial t} - \mathbf{b} \cdot \nabla \left\langle \phi + \phi_0 \right\rangle, \mathbf{b} = \mathbf{b}_0 + \delta\mathbf{b} \\ \langle \phi + \phi_0 \rangle = J_0(\alpha)(\phi + \phi_0),\ \alpha = -i\frac{\sqrt{2\mu B}}{\Omega}\nabla_\perp = \quad \frac{\sqrt{2\mu B}}{v_{ti}}\rho_i k_\perp \end{cases} \end{cases}$$

Gyroaverage operator (only operates on ϕ and $A_\parallel$):

$$J_0(\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}(\frac{\sqrt{2\mu B}}{2\Omega})^{2n}\nabla_\perp^{2n}$$

Notations:

$$\begin{cases} i\omega_d = \frac{v_t^2}{\Omega B^2}\mathbf{B} \times \nabla B \cdot \nabla & \Rightarrow B\mathbf{v}_d \cdot \nabla F = i\omega_d[FB(v_\parallel^2 + \mu B)] \\ \nabla_\parallel = \mathbf{b} \cdot \nabla & \mathbf{b} = \mathbf{b}_0 + \frac{\langle\delta\mathbf{B}\rangle}{B} \\ \mathbf{v}_\phi = \frac{c}{B}\mathbf{b} \times \nabla(\phi + \phi_0) & \mathbf{v}_{A_\parallel} = \frac{c}{B}\mathbf{b} \times \nabla A_\parallel \end{cases}$$

Get ride of the lowest and first order terms, then the gyrokinetic equation is $O(\epsilon^2)$ equation evolving $\tilde{f}$, ϕ and $A_\parallel$.
Local Maxwellian distribution(with parallel flow for electron):

$$F_0 = F_M = \frac{n_0}{(2\pi v_t^2)^{3/2}}e^{-(v_\parallel - u_{0\parallel})^2/2v_t^2 - \mu B/v_t^2}$$

$T = mv_t^2$, $T_{\perp 0} = 2T_{\parallel 0} = mv_t^2 = T_0$

$$
\begin{aligned}
\frac{\partial}{\partial t}B\tilde{f}+B\nabla_{\parallel}\tilde{f}v_{\parallel}+\mathbf{v}_{\phi}\cdot\nabla[(F_0+\tilde{f})BJ_0]-\mathbf{v}_{A_{\parallel}}\cdot\nabla[(F_0+\tilde{f})B\frac{v_{\parallel}}{c}J_0] \\
-2F_0BJ_0i\omega_d\frac{e\phi}{T}+2F_0B\frac{v_{\parallel}}{c}J_0i\omega_d\frac{eA_{\parallel}}{T}-F_0BJ_1\frac{\alpha}{2}i\omega_d\frac{e\phi}{T}+F_0B\frac{v_{\parallel}}{c}J_1\frac{\alpha}{2}i\omega_d\frac{eA_{\parallel}}{T} \\
+\frac{i\omega_d}{v_t^2}[\tilde{fB}(v_{\parallel}^2+\mu B)]+\frac{e}{mc}\frac{\partial F_0}{\partial v_{\parallel}}BJ_0\frac{\partial A_{\parallel}}{\partial t} \\
+\frac{e}{m}\nabla_{\parallel}(\frac{\partial F_0}{\partial v_{\parallel}}BJ_0\phi)-\frac{e}{m}J_0\phi\frac{\partial F_0}{\partial v_{\parallel}}B(1-\frac{\mu B}{v_t^2})\nabla_{\parallel}\ln B \\
-\frac{e}{m}\frac{\partial F_0}{\partial v_{\parallel}}J_{0A_{\parallel}}J_{0\phi}(\hat{\mathbf{b}}\times\nabla A_{\parallel})\cdot\nabla\phi-\mu B^2\frac{\partial\tilde{f}}{\partial v_{\parallel}}\nabla_{\parallel}\ln B \\
+\frac{\partial F_0}{\partial v_{\parallel}}BJ_0\frac{\mu B}{c}i\omega_d\frac{eA_{\parallel}}{T}+\frac{\partial}{\partial v_{\parallel}}(F_0BJ_0v_{\parallel})i\omega_d\frac{e\phi}{T}-\frac{\partial}{\partial v_{\parallel}}(F_0BJ_0v_{\parallel}^2)i\omega_d\frac{eA_{\parallel}}{cT}=0
\end{aligned}
$$

- Meaning of terms: parallel motion, $\mathbf{E}\times\mathbf{B}$ drift, curvature and ∇B drift, electric acceleration, mirror force acceleration, phase conservation term
  - part of the phase conservation term is $F_0BJ_0i\omega_d\frac{e\phi}{T}$ which is absorbed in $\mathbf{E}\times\mathbf{B}$ drift term
  - to keep phase space conservation, the corresponding term is added

Electron gyrofluid equation

- Definitions

$$
F=F_{0M}+\tilde{f}
$$

$$
\begin{cases}
\tilde{n}=\int\tilde{f}d^3v & n_0\tilde{u}_{\parallel}=\int\tilde{f}v_{\parallel}d^3v \\
\tilde{P}_{\parallel}=m\int\tilde{f}v_{\parallel}^2d^3v & \tilde{P}_{\perp}=m\int\tilde{f}\mu d^3v
\end{cases}
$$

Where $d^3v=2\pi Bdv_{\parallel}d\mu$.

- Moment
  - Take moments of gyrokinetic equation to get gyrofluid equation. Define $\langle A\rangle=2\pi\int Adv_{\parallel}Bd\mu$
- Electron continuity equation (parallel acceleration terms disappear except part of phase conservation term)

$$
\frac{\partial\tilde{n}}{\partial t}+B\nabla_{\parallel}\frac{n_0\tilde{u}_{\parallel}}{B}+\mathbf{v}_{\phi}\cdot\nabla\langle FJ_0\rangle-\frac{1}{c}\mathbf{v}_{A_{\parallel}}\cdot\langle Fv_{\parallel}J_0\rangle-\left\langle F_0(2J_0+J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{e\phi}{T_0}+\frac{1}{T_0}i\omega_d(\tilde{p}_{\parallel}+\tilde{p}_{\perp})=0
$$

  - Finite Lamor Radius terms: Terms contain J_0 need special treatment

$$
\begin{aligned}
\langle F_0 J_0\rangle &= n_0\Gamma_0^{1/2}\\
\left\langle F_0 v_\parallel J_0\right\rangle &= 0\\
\langle F_0 J_1\alpha\rangle &= -\hat{\nabla}_\perp^2\\
\mathbf{v}_\phi\cdot\nabla\langle FJ_0\rangle &= \mathbf{v}_\phi\cdot\nabla(n\Gamma_0^{1/2}(b))\\
&= n_0 i\omega_*\Gamma_0^{1/2}\frac{e\phi}{T_0}+\frac{n_0}{2}\eta_i\hat{\nabla}_\perp^2 i\omega_*\frac{e\Phi}{T_0}-n_0\hat{\nabla}_\perp^2 i\omega_d\frac{e\phi}{T_0}+NL\\
NL &= \mathbf{v}_\Phi\cdot\nabla\tilde{n}+\frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp\\
\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle F v_\parallel J_0\right\rangle &= \mathbf{v}_{A_\parallel}\cdot\nabla(n\Gamma_0^{1/2}(b))\\
&= \frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}_\parallel}]\cdot\nabla\tilde{q}_\perp
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{v}_\phi\cdot\nabla\langle FBJ_0\rangle=\mathbf{v}_\phi\cdot\nabla(nB\Gamma_0^{1/2}(b))\\
&=n_0 i\omega_*\Gamma_0^{1/2}\frac{e\phi}{T_0}+\frac{n_0}{2}\eta_i\hat{\nabla}_\perp^2 i\omega_*\frac{e\Phi}{T_0}-n_0\hat{\nabla}_\perp^2 i\omega_d\frac{e\phi}{T_0}+NL
\end{aligned}
$$

  * $\hat{\nabla}_\perp^2\Phi=2b\frac{\partial\Gamma_0^{1/2}}{\partial b}\phi$, $b=\frac{1}{\Omega}\sqrt{\frac{T_\perp}{m}}$, $\eta_i=L_n/L_T$
  * Diamagnetic frequency $i\omega_*=(cT_0/eBn_0)\nabla n_0\cdot\mathbf{b}\times\nabla$

- Substitute FLR terms

$$
\begin{aligned}
&\frac{\partial\tilde{n}}{\partial t}+\mathbf{v}_\Phi\cdot\nabla\tilde{n}+B\tilde{\nabla}_\parallel\frac{n_0\tilde{u}_\parallel}{B}+\frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp-\frac{1}{c}\frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}_\parallel}]\cdot\nabla\tilde{q}_\perp+n_0(1+\frac{1}{2}\eta_i\hat{\nabla}_\perp^2)i\omega_*\frac{e\Phi}{T_0}\\
&-n_0(2+\frac{1}{2}\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0}+\frac{1}{T_0}i\omega_d(\tilde{p}_\parallel+\tilde{p}_\perp)=0
\end{aligned}
$$

  * last term contains curvature drift via $i\omega_d\tilde{p}_\parallel$ and ∇B drift via $i\omega_d\tilde{p}_\perp$
  Where $\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}_\Phi\cdot\nabla$, $\mathbf{V}_\Phi=\frac{c}{B}\mathbf{b}\times\nabla\Phi=\frac{c}{B}\mathbf{b}\times\nabla(\Gamma_0^{1/2}\phi)$, $i\omega_*=(cT_0/eBn_0)\nabla n_0\cdot\mathbf{b}\times\nabla$, $i\omega_d=\frac{v_t^2}{\Omega_e B^2}\mathbf{B}\times\nabla B\cdot\nabla$, $\tilde{\nabla}_\parallel=\nabla_\parallel-\mathbf{v}_{\bar{A}_\parallel}\cdot\nabla=\nabla_\parallel-\hat{\mathbf{b}}\times\nabla\bar{A}_\parallel\cdot\nabla$. Already consider electron charge

- Electron parallel momentum equation

$$
\begin{aligned}
&\frac{\partial n_0\tilde{u}_\parallel}{\partial t}+\frac{B}{m}\nabla_\parallel\frac{\tilde{p}_\parallel}{B}+\mathbf{v}_\phi\cdot\nabla\left\langle Fv_\parallel J_0\right\rangle-\frac{1}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle Fv_\parallel^2 J_0\right\rangle-\left\langle F_0 v_\parallel^2(2J_0+J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{eA_\parallel}{cT_0}\\
&+\frac{1}{T_0}i\omega_d(\tilde{q}_\parallel+\tilde{q}_\perp+4p_0\tilde{u}_\parallel)+\langle F_0J_0\rangle\frac{e}{mc}\frac{\partial A_\parallel}{\partial t}+\frac{e}{m}\nabla_\parallel\langle F_0J_0\rangle\phi-\frac{e}{m}\phi\left\langle F_0J_0(1-\frac{\mu B}{v_t^2})\right\rangle\nabla_\parallel\ln B\\
&-\frac{e}{mB}\left\langle F_0J_{0A_\parallel}J_{0\phi}\right\rangle\mathbf{b}\times\nabla A_\parallel\cdot\nabla\phi+\frac{\tilde{p}_\perp}{m}\nabla_\parallel\ln B+\langle F_0\mu BJ_0\rangle i\omega_d\frac{eA_\parallel}{cT_0}-\left\langle F_0BJ_0v_\parallel^2\right\rangle i\omega_d\frac{eA_\parallel}{cT}=0
\end{aligned}
$$

$$
\begin{aligned}
&1)\mathbf{v}_\phi\cdot\nabla\left\langle Fv_\parallel J_0\right\rangle = NL(v_\parallel)=\mathbf{v}_\Phi\cdot\nabla n_0\tilde{u}_\parallel+\frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp\\
&2)\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle Fv_\parallel^2 J_0\right\rangle=\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle n_0v_t^2\Gamma_0^{1/2}\right\rangle+NL(v_\parallel^2)\\
&\quad=-n_0v_t^2\Gamma_0^{1/2}i\omega_*\frac{eA_\parallel}{T_0}-\frac{1}{2}n_0v_t^2\eta_i\hat{\nabla}_\perp^2 i\omega_*\frac{eA_\parallel}{T_0}+n_0\hat{\nabla}_\perp^2 i\omega_d\frac{eA_\parallel}{T_0}+\mathbf{v}_{\bar{A}}\cdot\nabla\tilde{p}_\parallel+\frac{n_0}{2}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}]\cdot\nabla\tilde{T}_\perp\\
&3)\left\langle F_0v_\parallel^2(2J_0+J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{eA_\parallel}{cT_0}=2n_0v_t^2\Gamma_0^{1/2}i\omega_d\frac{eA_\parallel}{cT_0}-\frac{v_t^2}{2}\hat{\nabla}_\perp^2 i\omega_d\frac{eA_\parallel}{cT_0}\\
&4)\left\langle F_0J_0\right\rangle\frac{e}{mc}\frac{\partial A_\parallel}{\partial t}=\frac{n_0e}{mc}\frac{\partial\bar{A}_\parallel}{\partial t}\\
&5)\frac{e}{m}\nabla_\parallel\left\langle F_0J_0\right\rangle\phi=\frac{e}{m}\nabla_\parallel n_0\Phi\\
&6)\frac{e}{m}\phi\left\langle F_0J_0(1-\frac{\mu B}{v_t^2})\right\rangle\nabla_\parallel\ln B=\frac{e}{m}n_0\Phi\nabla_\parallel\ln B-\frac{n_0e}{2m}(2\Phi+\hat{\nabla}_\perp^2\Phi)\nabla_\parallel\ln B=-\frac{n_0e}{2m}\hat{\nabla}_\perp^2\Phi\nabla_\parallel\ln B\\
&7)\frac{e}{mB}\left\langle F_0J_{0A_\parallel}J_{0\phi}\right\rangle\mathbf{b}\times\nabla A_\parallel\cdot\nabla\phi=\frac{n_0e}{mB}\Gamma_{0A_\parallel}^{1/2}\Gamma_{0\phi}^{1/2}\mathbf{b}\times\nabla A_\parallel\cdot\nabla\phi\\
&8)\left\langle F_0\mu BJ_0\right\rangle i\omega_d\frac{eA_\parallel}{cT_0}=\frac{v_t^2}{2}(2\Gamma_0^{1/2}+\hat{\nabla}_\perp^2)i\omega_d\frac{eA_\parallel}{cT_0}
\end{aligned}
$$

$$
\begin{aligned}
&n_0\frac{\partial\tilde{u}_\parallel}{\partial t}+n_0\mathbf{v}_\Phi\cdot\nabla\tilde{u}_\parallel-\frac{n_0e}{m}\tilde{\nabla}_\parallel\Phi+\frac{B}{m}\tilde{\nabla}_\parallel\frac{\tilde{p}_\parallel}{B}+\frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp-\frac{n_0}{2m}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}]\cdot\nabla\tilde{T}_\perp\\
&-\frac{n_0e}{mc}\frac{\partial\bar{A}_\parallel}{\partial t}-n_0v_t^2(1+\eta_e+\frac{\eta_e}{2}\hat{\nabla}_\perp^2)i\omega_*\frac{e\bar{A}_\parallel}{cT_0}+(\frac{\tilde{p}_\perp}{m}-\frac{n_0e}{2m}\hat{\nabla}_\perp^2\Phi)\nabla_\parallel\ln B+\frac{1}{T_0}i\omega_d(\tilde{q}_\parallel+\tilde{q}_\perp+4p_0\tilde{u}_\parallel)=0
\end{aligned}
$$

- electron parallel pressure equation

$$
\begin{aligned}
&\frac{\partial\tilde{p}_\parallel}{\partial t}+B\nabla_\parallel\frac{\tilde{q}_\parallel+3p_0\tilde{u}_\parallel}{B}+m\mathbf{v}_\phi\cdot\nabla\left\langle Fv_\parallel^2J_0\right\rangle-\frac{m}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle Fv_\parallel^3J_0\right\rangle-m\left\langle F_0v_\parallel^2(2J_0+J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{e\phi}{T_0}\\
&+\frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\parallel}+\tilde{r}_{\parallel,\perp})+2(\tilde{q}_\perp+p_0\tilde{u}_\parallel)\nabla_\parallel\ln B-2m\left\langle F_0v_\parallel^2J_0\right\rangle i\omega_d\frac{e\phi}{T_0}=0
\end{aligned}
$$

Gyroaveraged terms

$$
\begin{aligned}
&1.m_e\mathbf{v}_\phi\cdot\nabla\left\langle Fv_\parallel^2J_0\right\rangle=m_e\mathbf{v}_\phi\cdot\nabla(n_0v_t^2\Gamma_0^{1/2})+NL\\
&\quad=m_en_0v_t^2\Gamma_0^{1/2}i\omega_*\frac{e\phi}{T_0}+m_en_0\eta_eb\frac{\partial(v_t^2\Gamma_0^{1/2})}{\partial b}i\omega_*\frac{e\phi}{T_0}-2m_en_0b\frac{\partial(v_t^2\Gamma_0^{1/2})}{\partial b}i\omega_d\frac{e\phi}{T_0}+\mathbf{v}_\Phi\cdot\nabla\tilde{p}_\parallel+\frac{n_0}{2}\left[\nabla_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp\\
&2.\frac{m_e}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle Fv_\parallel^3J_0\right\rangle=NL\\
&\quad=\frac{1}{c}\mathbf{v}_{\bar{A}}\cdot\nabla\tilde{q}_\parallel+\frac{1}{c}3p_0\mathbf{v}_{\bar{A}}\cdot\nabla\tilde{u}_\parallel+\frac{3}{2c}\left[\nabla_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp\\
&3.m\left\langle F_0v_\parallel^2(2J_0+J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{e\phi}{T_0}=2mn_0v_t^2i\omega_d\frac{e\Phi}{T_0}-\frac{m}{2}n_0v_t^2\hat{\nabla}_\perp^2 i\omega_d\frac{e\phi}{T_0}\\
&4.2m\left\langle F_0v_\parallel^2J_0\right\rangle i\omega_d\frac{e\phi}{T_0}=2mn_0v_t^2i\omega_d\frac{e\Phi}{T_0}
\end{aligned}
$$

Add gyroaveraged terms

$$
\begin{aligned}
&\frac{\partial \tilde{p}_\parallel}{\partial t} + B\nabla_\parallel \frac{\tilde{q}_\parallel + 3p_0\tilde{u}_\parallel}{B} + m_e n_0 v_t^2 \Gamma_0^{1/2} i\omega_* \frac{e\phi}{T_0} + m_e n_0 \eta_e b \frac{\partial (v_t^2 \Gamma_0^{1/2})}{\partial b} i\omega_* \frac{e\phi}{T_0} - 2m_e n_0 b \frac{\partial (v_t^2 \Gamma_0^{1/2})}{\partial b} i\omega_d \frac{e\phi}{T_0} + \mathbf{v}_\Phi \cdot \nabla \tilde{p}_\parallel \\
&+ \frac{n_0}{2}\left[\nabla_\perp^2 \mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp - \frac{1}{c}\mathbf{v}_{\tilde{A}}\cdot\nabla\tilde{q}_\parallel - \frac{1}{c}3p_0\mathbf{v}_{\tilde{A}}\cdot\nabla\tilde{u}_\parallel - \frac{3}{2c}\left[\nabla_\perp^2\mathbf{v}_{\tilde{A}}\right]\cdot\nabla\tilde{q}_\perp - 2mn_0v_t^2 i\omega_d\frac{e\Phi}{T_0} + \frac{m}{2}v_t^2\hat{\nabla}_\perp^2 i\omega_d\frac{e\phi}{T_0} \\
&+ \frac{1}{v_t^2} i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}) + 2(\tilde{q}_\perp + p_0\tilde{u}_\parallel)\nabla_\parallel \ln B - 2mn_0v_t^2 i\omega_d \frac{e\Phi}{T_0} = 0 \\
\Rightarrow &\frac{d\tilde{p}_\parallel}{dt} + B\tilde{\nabla}_\parallel \frac{\tilde{q}_\parallel + 3p_0\tilde{u}_\parallel}{B} + \frac{m_e n_0}{2}\left[\nabla_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp + 2(\tilde{q}_\perp + p_0\tilde{u}_\parallel)\nabla_\parallel \ln B \\
&+ m_e n_0 v_t^2(1 + 2\eta_e + \frac{\eta_e}{2}\hat{\nabla}_\perp^2) i\omega_* \frac{e\Phi}{T_0} - mn_0v_t^2(4 + \frac{1}{2}\hat{\nabla}_\perp^2) i\omega_d \frac{e\Phi}{T_0} + \frac{1}{v_t^2} i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}) = 0
\end{aligned}
$$

$-\frac{3m_e}{2c}\left[\nabla_\perp^2\mathbf{v}_{\tilde{A}}\right]\cdot\nabla\tilde{q}_\perp$ is neglected.

- electron perpendicular pressure equation

$$
\begin{aligned}
&\frac{\partial \tilde{p}_\perp}{\partial t} + B^2\nabla_\parallel\left[\frac{1}{B^2}(\tilde{q}_\perp + p_0\tilde{u}_\parallel)\right] + mB\mathbf{v}_\phi\cdot\nabla\left\langle F\mu J_0\right\rangle \\
&- \frac{mB}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle F\mu v_\parallel J_0\right\rangle - mB\left\langle F_0\mu(2J_0 + J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{e\phi}{T_0} + \frac{1}{v_t^2} i\omega_d(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}) = 0
\end{aligned}
$$

Gyroaveraged terms

$$
\begin{aligned}
&1.mB\mathbf{v}_\phi\cdot\nabla\left\langle F\mu J_0\right\rangle = mB\mathbf{v}_\phi\cdot\nabla\left[\frac{n_0v_t^2}{B}\frac{\partial}{\partial b}(b\Gamma_0^{1/2})\right] + NL \\
&\quad = mn_0v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2}) i\omega_*\frac{e\phi}{T_0} + mn_0\eta_e b\frac{\partial}{\partial b}(v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2})) i\omega_*\frac{e\phi}{T_0} - 2mn_0 b\frac{\partial}{\partial b}(v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2})) i\omega_d\frac{e\phi}{T_0} \\
&\quad + \mathbf{v}_\Phi\cdot\nabla\tilde{p}_\perp + \frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{p}_\perp \\
&2.\frac{mB}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\left\langle F\mu v_\parallel J_0\right\rangle = NL \\
&\quad = \frac{1}{c}\mathbf{v}_{\tilde{A}}\cdot\nabla\tilde{q}_\perp + \frac{1}{c}p_0\mathbf{v}_{\tilde{A}}\cdot\nabla\tilde{u}_\parallel + \frac{1}{c}\frac{p_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\tilde{A}}\right]\cdot\nabla\tilde{u}_\parallel + \frac{1}{c}\frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\tilde{A}}\right]\cdot\nabla\tilde{q}_\perp + \frac{1}{c}\left[\hat{\hat{\nabla}}_\perp^2\mathbf{v}_{\tilde{A}}\right]\cdot\nabla\tilde{q}_\perp \\
&3.mB\left\langle F_0\mu(2J_0 + J_1\frac{\alpha}{2})\right\rangle i\omega_d\frac{e\phi}{T_0} = 2mB\frac{n_0v_t^2}{B}\frac{\partial}{\partial b}(b\Gamma_0^{1/2}) i\omega_d\frac{e\phi}{T_0} - mB\frac{v_t^2}{B}\hat{\hat{\nabla}}_\perp^2 i\omega_d\frac{e\phi}{T_0} \\
&\quad = mn_0v_t^2(2 + \hat{\nabla}_\perp^2) i\omega_d\frac{e\Phi}{T_0} - mn_0v_t^2\hat{\hat{\nabla}}_\perp^2 i\omega_d\frac{e\phi}{T_0}
\end{aligned}
$$

Add gyroaveraged term we get

$$
\begin{aligned}
&\frac{\partial \tilde{p}_\perp}{\partial t} + B^2\nabla_\parallel\left[\frac{1}{B^2}(\tilde{q}_\perp + p_0\tilde{u}_\parallel)\right] + mn_0v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2})i\omega_*\frac{e\phi}{T_0} + mn_0\eta_e b\frac{\partial}{\partial b}(v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2}))i\omega_*\frac{e\phi}{T_0}\\
&-2mn_0b\frac{\partial}{\partial b}(v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2}))i\omega_d\frac{e\phi}{T_0} + \mathbf{v}_\Phi\cdot\nabla\tilde{p}_\perp + \frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{p}_\perp\\
&-\frac{1}{c}\mathbf{v}_{\bar{A}}\cdot\nabla\tilde{q}_\perp - \frac{1}{c}p_0\mathbf{v}_{\bar{A}}\cdot\nabla\tilde{u}_\parallel - \frac{1}{c}\frac{p_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{u}_\parallel - \frac{1}{c}\frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp - \frac{1}{c}\left[\hat{\hat{\nabla}}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp\\
&-mn_0v_t^2(2+\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0} + mn_0v_t^2\hat{\hat{\nabla}}_\perp^2 i\omega_d\frac{e\phi}{T_0} + \frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}) = 0\\
\Rightarrow&\frac{d\tilde{p}_\perp}{dt} + B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_\perp + p_0\tilde{u}_\parallel}{B^2} + \frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{p}_\perp - \frac{1}{c}\left[\hat{\hat{\nabla}}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp\\
&-\frac{1}{c}\frac{p_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{u}_\parallel - \frac{1}{c}\frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp\\
&+mn_0v_t^2\left[1 + \frac{1}{2}\hat{\nabla}_\perp^2 + \eta_e(2 + \hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)\right]i\omega_*\frac{e\phi}{T_0} - 2mn_0v_t^2(4 + 2\hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)i\omega_d\frac{e\phi}{T_0} + \frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}) = 0
\end{aligned}
$$

$$
b\frac{\partial}{\partial b}(v_t^2\frac{\partial}{\partial b}(b\Gamma_0^{1/2})) = v_t^2(2 + \hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)
$$

Final equations

- Original

$$
\begin{aligned}
&\frac{d\tilde{n}}{dt} + B\tilde{\nabla}_\parallel\frac{n_0\tilde{u}_\parallel}{B} + \frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp - \frac{1}{c}\frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}_\parallel}]\cdot\nabla\tilde{q}_\perp\\
&+n_0(1 + \frac{1}{2}\eta_e\hat{\nabla}_\perp^2)i\omega_*\frac{e\Phi}{T_0} - n_0(2 + \frac{1}{2}\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0} + \frac{1}{T_0}i\omega_d(\tilde{p}_\parallel + \tilde{p}_\perp) = 0
\end{aligned}
$$

$$
\begin{aligned}
&n_0\frac{d\tilde{u}_\parallel}{dt} - \frac{n_0e}{m}\tilde{\nabla}_\parallel\Phi + \frac{B}{m}\tilde{\nabla}_\parallel\frac{\tilde{p}_\parallel}{B} + \frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp - \frac{n_0}{2m}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}]\cdot\nabla\tilde{T}_\perp\\
&-\frac{n_0e}{mc}\frac{\partial\bar{A}_\parallel}{\partial t} - n_0v_t^2(1 + \eta_e + \frac{\eta_e}{2}\hat{\nabla}_\perp^2)i\omega_*\frac{e\bar{A}_\parallel}{cT_0} + (\frac{\tilde{p}_\perp}{m} - \frac{n_0e}{2m}\hat{\nabla}_\perp^2\Phi)\nabla_\parallel\ln B + \frac{1}{T_0}i\omega_d(\tilde{q}_\parallel + \tilde{q}_\perp + 4p_0\tilde{u}_\parallel) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d\tilde{p}_\parallel}{dt} + B\tilde{\nabla}_\parallel\frac{\tilde{q}_\parallel + 3p_0\tilde{u}_\parallel}{B} + \frac{n_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp + 2(\tilde{q}_\perp + p_0\tilde{u}_\parallel)\nabla_\parallel\ln B\\
&+m_en_0v_t^2(1 + 2\eta_e + \frac{\eta_e}{2}\hat{\nabla}_\perp^2)i\omega_*\frac{e\Phi}{T_0} - mn_0v_t^2(4 + \frac{1}{2}\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0} + \frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d\tilde{p}_\perp}{dt} + B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_\perp + p_0\tilde{u}_\parallel}{B^2} + \frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{p}_\perp - \frac{n_0}{c}\left[\hat{\hat{\nabla}}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{T}_\perp\\
&-\frac{1}{c}\frac{p_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{u}_\parallel - \frac{1}{c}\frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{q}_\perp + mn_0v_t^2\left[1 + \frac{1}{2}\hat{\nabla}_\perp^2 + \eta_e(2 + \hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)\right]i\omega_*\frac{e\phi}{T_0}\\
&-2mn_0v_t^2(4 + 2\hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)i\omega_d\frac{e\phi}{T_0} + \frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp}) = 0
\end{aligned}
$$

Where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_\Phi \cdot \nabla$, $\mathbf{V}_\Phi = \frac{c}{B}\mathbf{b}\times\nabla\Phi$, $\mathbf{V}_{\bar{A}_\parallel} = \frac{c}{B}\mathbf{b}\times\nabla\bar{A}_\parallel$,$(\Phi, \bar{A}_\parallel) = \Gamma_0^{1/2}(\phi, A_\parallel)$, $i\omega_* = \frac{cT_0}{eBn_0}\nabla n_0 \cdot \mathbf{b}\times\nabla$, $i\omega_d = \frac{v_t^2}{\Omega_e B^2}\mathbf{B}\times\nabla B\cdot\nabla$, $\tilde{\nabla}_\parallel = \nabla_\parallel - \frac{1}{c}\mathbf{V}_{\bar{A}_\parallel}\cdot\nabla$,$\hat{\nabla}_\perp^2\Phi = 2b\frac{\partial\Gamma_0^{1/2}}{\partial b}\phi$, $\hat{\hat{\nabla}}_\perp^2\Phi = b\frac{\partial^2}{\partial b^2}(b\Gamma_0^{1/2})\phi$, $\Omega_e = -\frac{eB}{m_e}$, $\eta_e = L_n/L_T$, $T_0 = T_{\perp 0} = 2T_{\parallel 0}$

- Closures
  Drop mirror closure terms $i\omega_d(\tilde{r}_{\parallel,\perp} + \tilde{r}_{\perp,\perp})$, $i\omega_d(\tilde{r}_{\parallel,\parallel} + \tilde{r}_{\parallel,\perp})$ and only keep $\tilde{q}_\parallel, \tilde{q}_\perp$in parallel derivative.

$$\tilde{q}_\parallel = -n_0\sqrt{\frac{8}{\pi}}v_{t\parallel}\frac{ik_\parallel\tilde{T}_\parallel}{|k_\parallel|}$$
$$\tilde{q}_\perp = -n_0\sqrt{\frac{2}{\pi}}v_{t\parallel}\frac{ik_\parallel\tilde{T}_\perp}{|k_\parallel|}$$

- Simplified and use SI units (include resistivity)

$$\frac{d\tilde{n}}{dt} + B\tilde{\nabla}_\parallel\frac{n_0\tilde{u}_\parallel}{B} + \frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp + n_0(1+\frac{1}{2}\eta_e\hat{\nabla}_\perp^2)i\omega_*\frac{e\Phi}{T_0} - n_0(2+\frac{1}{2}\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0} + \frac{1}{T_0}i\omega_d(\tilde{p}_\parallel + \tilde{p}_\perp) = 0$$

$$\begin{aligned}&n_0\frac{d\tilde{u}_\parallel}{dt} - \frac{n_0e}{m}\tilde{\nabla}_\parallel\Phi + \frac{B}{m}\tilde{\nabla}_\parallel\frac{\tilde{p}_\parallel}{B} - \frac{n_0}{2m}[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}]\cdot\nabla\tilde{T}_\perp - \frac{n_0e}{m}\frac{\partial\bar{A}_\parallel}{\partial t}\\&- n_0v_t^2(1+\eta_e+\frac{\eta_e}{2}\hat{\nabla}_\perp^2)i\omega_*\frac{e\bar{A}_\parallel}{cT_0} + (\frac{\tilde{p}_\perp}{m} - \frac{n_0e}{2m}\hat{\nabla}_\perp^2\Phi)\nabla_\parallel\ln B + \frac{4}{T_0}i\omega_d(p_0\tilde{u}_\parallel) = \frac{en_e\eta}{m_e}J_\parallel\end{aligned}$$

$$\begin{aligned}&\frac{d\tilde{p}_\parallel}{dt} + \tilde{\nabla}_\parallel\tilde{q}_\parallel + B\tilde{\nabla}_\parallel\frac{3p_0\tilde{u}_\parallel}{B} + \frac{n_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp + 2p_0\tilde{u}_\parallel\nabla_\parallel\ln B\\&+ m_en_0v_t^2(1+2\eta_e+\frac{\eta_e}{2}\hat{\nabla}_\perp^2)i\omega_*\frac{e\Phi}{T_0} - mn_0v_t^2(4+\frac{1}{2}\hat{\nabla}_\perp^2)i\omega_d\frac{e\Phi}{T_0} = 0\end{aligned}$$

$$\begin{aligned}&\frac{d\tilde{p}_\perp}{dt} + B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_\perp + p_0\tilde{u}_\parallel}{B^2} + \frac{1}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{p}_\perp - n_0\left[\hat{\hat{\nabla}}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla\tilde{T}_\perp - \frac{p_0}{2}\left[\hat{\nabla}_\perp^2\mathbf{v}_{\bar{A}}\right]\cdot\nabla\tilde{u}_\parallel\\&+ mn_0v_t^2\left[1+\frac{1}{2}\hat{\nabla}_\perp^2+\eta_e(2+\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2)\right]i\omega_*\frac{e\Phi}{T_0} - 2mn_0v_t^2(4+2\hat{\nabla}_\perp^2+\hat{\hat{\nabla}}_\perp^2)i\omega_d\frac{e\Phi}{T_0} = 0\end{aligned}$$

Ion equation, Poisson's Equation and Ampere's Law

- Adiabatic ion

$$\tilde{n}_i = -n_{i0}\frac{Ze\phi}{T_{0i}}$$

- For ETG ($k_\perp\lambda_D \ll 1$), quasi-neutrality

$$\bar{n}_e + n_{e0}(1-\Gamma_0)\frac{e\phi}{T_{e0}} = \tilde{n}_i$$

– Simple Pade approximation

$$\bar{n}_e = \frac{1}{1+b/2}\tilde{n}_e - \frac{n_{e0}2b}{T_0(2+b)^2}\tilde{T}_{\perp e}$$

where $b = k_\perp^2\rho_e^2 = -\rho_e^2\nabla_\perp^2$ and $k_\perp = -i\nabla_\perp$.

- Ampere's law

$$\nabla_\perp^2 A_\parallel = -\mu_0 e n_0 \bar{u}_{\parallel e}$$

Pade approximation:

$$\bar{u}_{\parallel e} = \Gamma_0^{1/2}(b)\tilde{u}_{\parallel e} - \frac{2b}{(2+b)^2}\tilde{q}_{\perp e}$$

Normalization and implementation

- Normalization parameters $(\bar{L},\bar{T},\bar{N},\bar{B})$, $\Phi_c=\Phi/B$, $\psi=A_\parallel/B$, $\Psi=\bar{A}_\parallel/B$, $\Phi_c^a=\hat{\nabla}_\perp^2\Phi_c$,$\hat{\nabla}_\perp^2\Psi=\Psi^a$, $\hat{\hat{\nabla}}_\perp^2\Psi=\Psi^b$, $V_{Ae}^2=\bar{B}^2/\mu_0 m_e\bar{N}$, $\bar{\Omega}_e=e\bar{B}/m_e$, $J_{\parallel c}=-\frac{\mu_0 J_\parallel}{B}=\nabla_\perp^2\psi$
  $\tilde{n}_e=\hat{n}_e\bar{N}$, $\Phi_c=\bar{L}^2\hat{\Phi}_c/\bar{T}$, $\Psi=\hat{\Psi}\bar{L}$, $T_0=\hat{T}_0 m_e\bar{v}_{th}^2$, $\Omega_e=-\frac{e\bar{B}}{m_e}\hat{\Omega}_e$,$\tilde{P}=\bar{N}m_e\bar{v}_{th}^2\hat{P}_1$, $\Lambda=\bar{v}_{th}(\hat{u}_{\parallel e}-\frac{\bar{L}\bar{\Omega}_e}{\bar{v}_{th}}\hat{B}\hat{\Psi})=\bar{v}_{th}\hat{\Lambda}$, $J_{\parallel c}=\hat{\nabla}_\perp^2\hat{\psi}/\bar{L}$, $\eta=\hat{\eta}\bar{L}^2\mu_0/\bar{T}$

$$\tilde{T}_\perp=\frac{\tilde{p}_\perp-T_0\tilde{n}}{n_0}=m_e\bar{v}_{th}^2\frac{\hat{p}_\perp-\hat{T}_0\hat{n}}{\hat{n}_0}$$

$$\begin{aligned}
&\frac{\partial\tilde{n}_e}{\partial t}+\mathbf{b}\times\nabla\Phi_c\cdot\nabla\tilde{n}_e+B\partial_\parallel\frac{n_0\tilde{u}_{\parallel e}}{B}-\mathbf{b}\times\nabla\Psi\cdot\nabla n_0\tilde{u}_\parallel+\frac{n_0}{2T_0}\mathbf{b}\times\nabla\hat{\nabla}_\perp^2\Phi_c\cdot\nabla\tilde{T}_\perp-\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c-\frac{\eta_e}{2}\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c^a\\
&-\frac{2n_0v_t^2}{\Omega_e}\mathbf{b}\times\nabla B\cdot\nabla\frac{e\Phi_c}{T_0}-\frac{n_0v_t^2}{2\Omega_e}\mathbf{b}\times\nabla B\cdot\nabla\frac{e\hat{\nabla}_\perp^2\Phi_c}{T_0}+\frac{v_t^2}{T_0\Omega_eB}\mathbf{b}\times\nabla B\cdot\nabla(\tilde{p}_\parallel+\tilde{p}_\perp)=0\\
\Rightarrow&\frac{\partial\hat{n}_e}{\partial\hat{t}}+\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{n}_e+\hat{B}\partial_\parallel\frac{\hat{n}_0\hat{u}_{\parallel e}}{\hat{B}}-\mathbf{b}\times\hat{\nabla}\hat{\Psi}\cdot\hat{\nabla}\hat{n}_0\hat{u}_{\parallel e}+\frac{\hat{n}_0}{2\hat{T}_0}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c^a\cdot\hat{\nabla}\hat{T}_\perp-\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_c\\
&-\frac{\eta_e}{2}\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_c^a+\frac{2\hat{n}_0}{\hat{\Omega}_e}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c+\frac{\hat{n}_0}{2\hat{\Omega}_e}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c^a-\frac{1}{\bar{T}\bar{\Omega}_e}\frac{1}{\hat{\Omega}_e\hat{B}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}(\hat{p}_\parallel+\hat{p}_\perp)=0
\end{aligned}$$

Rewrite parallel motion equation $\Lambda=\tilde{u}_{\parallel e}-\frac{Be}{m_e}\Psi\rightarrow\Lambda+\frac{Be}{m_e}\Psi=\tilde{u}_{\parallel e}$

$$\begin{aligned}
&n_0\frac{\partial\tilde{u}_\parallel}{\partial t}+n_0\mathbf{b}\times\nabla\Phi_c\cdot\nabla\tilde{u}_{\parallel e}-\frac{n_0e}{m_e}\partial_\parallel\Phi+\frac{n_0e}{m_e}\mathbf{b}\times\nabla\Psi\cdot\nabla(\Phi_cB)+\frac{B}{m_e}\partial_\parallel\frac{\tilde{p}_\parallel}{B}\\
&-\frac{1}{m_e}\mathbf{b}\times\nabla\Psi\cdot\nabla\tilde{p}_\parallel-\frac{n_{0e}}{2m_e}[\mathbf{b}\times\nabla\Psi^a]\cdot\nabla\tilde{T}_\perp-\frac{n_0e}{m_e}\frac{\partial\bar{A}_\parallel}{\partial t}+v_t^2(1+\eta_e)\mathbf{b}\times\nabla n_0\cdot\nabla\Psi+v_t^2\frac{\eta_e}{2}\mathbf{b}\times\nabla n_0\cdot\nabla\Psi^a\\
&+\frac{\tilde{p}_\perp}{Bm_e}\partial_\parallel B-\frac{n_0e}{2m_e}\Phi_c^a\partial_\parallel B+\frac{4}{T_0}\frac{v_t^2}{\Omega_eB}\mathbf{b}\times\nabla B\cdot\nabla(p_0\tilde{u}_\parallel)=\frac{en_e\eta}{m_e}J_\parallel\\
\Rightarrow&\frac{\partial\Lambda}{\partial t}+\mathbf{b}\times\nabla\Phi_c\cdot\nabla\Lambda+\mathbf{b}\times\nabla\Phi_c\cdot\nabla(\frac{Be}{m_e}\Psi)-\frac{e}{m_e}\partial_\parallel\Phi+\frac{1}{n_{0e}}\mathbf{b}\times\nabla\Psi\cdot\nabla(\Phi_cB)+\frac{B}{m_en_{0e}}\partial_\parallel\frac{\tilde{p}_\parallel}{B}\\
&-\frac{1}{m_en_{0e}}\mathbf{b}\times\nabla\Psi\cdot\nabla\tilde{p}_\parallel-\frac{1}{2m_e}[\mathbf{b}\times\nabla\Psi^a]\cdot\nabla\tilde{T}_\perp+\frac{1}{n_{e0}}v_t^2(1+\eta_e)\mathbf{b}\times\nabla n_0\cdot\nabla\Psi+\frac{1}{n_{e0}}v_t^2\frac{\eta_e}{2}\mathbf{b}\times\nabla n_0\cdot\nabla\Psi^a\\
&+\frac{\tilde{p}_\perp}{n_{e0}Bm_e}\partial_\parallel B-\frac{e}{2m_e}\Phi_c^a\partial_\parallel B+\frac{4}{n_{e0}T_0}\frac{v_t^2}{\Omega_eB}\mathbf{b}\times\nabla B\cdot\nabla(p_0\tilde{u}_\parallel)=-\frac{Be\eta}{\mu_0m_e}J_{\parallel c}\\
\Rightarrow&\frac{\partial\hat{\Lambda}}{\partial\hat{t}}+\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{\Lambda}+\bar{\Omega}_e\bar{T}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}(\hat{B}\hat{\Psi})-\bar{\Omega}_e\bar{T}\hat{\partial}_\parallel(\hat{\Phi}_c\hat{B})+\bar{\Omega}_e\bar{T}\mathbf{b}\times\hat{\nabla}\hat{\Psi}\cdot\hat{\nabla}(\hat{\Phi}_c\hat{B})+\frac{\hat{B}}{\hat{n}_{e0}}\hat{\partial}_\parallel\frac{\hat{p}_\parallel}{\hat{B}}\\
&-\frac{1}{\hat{n}_{0e}}\mathbf{b}\times\hat{\nabla}\hat{\Psi}\cdot\hat{\nabla}\hat{p}_\parallel-\frac{1}{2}[\mathbf{b}\times\hat{\nabla}\hat{\Psi}^a]\cdot\hat{\nabla}\hat{T}_\perp+\frac{\hat{T}_0}{\hat{n}_{e0}}(1+\eta_e)\mathbf{b}\times\hat{\nabla}\hat{n}_{0e}\cdot\hat{\nabla}\hat{\Psi}+\frac{\hat{T}_0}{\hat{n}_{e0}}\frac{\eta_e}{2}\mathbf{b}\times\hat{\nabla}\hat{n}_{0e}\cdot\hat{\nabla}\hat{\Psi}^a\\
&+\frac{\hat{p}_\perp}{\hat{n}_{e0}\hat{B}}\hat{\partial}_\parallel\hat{B}-\bar{\Omega}_e\bar{T}\frac{1}{2}\hat{\Phi}_c^a\hat{\partial}_\parallel\hat{B}-\frac{1}{\bar{\Omega}_e\bar{T}}\frac{4}{\hat{n}_{e0}}\frac{1}{\hat{\Omega}_e\hat{B}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}(\hat{p}_0\hat{u}_{\parallel e})=-\bar{\Omega}_e\bar{T}\hat{B}\hat{\eta}\hat{J}_{\parallel c}
\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial \tilde{p}_\parallel}{\partial t} + \mathbf{b}\times\nabla\Phi_c\cdot\nabla\tilde{p}_\parallel + \tilde{\nabla}_\parallel\tilde{q}_\parallel + B\partial_\parallel\frac{3p_0\tilde{u}_\parallel}{B} - 3\mathbf{b}\times\nabla\Psi\cdot\nabla p_0\tilde{u}_\parallel + \frac{n_0}{2}\mathbf{b}\times\nabla\Phi_c^a\cdot\nabla\tilde{T}_\perp + \frac{2p_0\tilde{u}_{\parallel e}}{B}\partial_\parallel B \\
&- m_e v_t^2(1+2\eta_e)\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c - m_e v_t^2\frac{\eta_e}{2}\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c^a - \frac{4n_0 e v_t^2}{\Omega_e}\mathbf{b}\times\nabla B\cdot\nabla\Phi_c - \frac{n_0 e v_t^2}{2\Omega_e}\mathbf{b}\times\nabla B\cdot\nabla\Phi_c^a = 0 \\
\Rightarrow &\frac{\partial \hat{p}_\parallel}{\partial \hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{p}_\parallel + \tilde{\nabla}_\parallel\tilde{q}_\parallel + \hat{B}\partial_\parallel\frac{3\hat{p}_0\hat{u}_{\parallel e}}{\hat{B}} - 3\mathbf{b}\times\hat{\nabla}\hat{\Psi}\cdot\hat{\nabla}\hat{p}_0\hat{u}_{\parallel e} + \frac{\hat{n}_0}{2}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c^a\cdot\hat{\nabla}\hat{T}_\perp + \frac{2\hat{p}_0\hat{u}_{\parallel e}}{\hat{B}}\hat{\partial}_\parallel\hat{B} \\
&- \hat{T}_0(1+2\eta_e)\mathbf{b}\times\hat{\nabla}\hat{n}_{0e}\cdot\hat{\nabla}\hat{\Phi}_c - \hat{T}_0\frac{\eta_e}{2}\mathbf{b}\times\hat{\nabla}\hat{n}_{0e}\cdot\hat{\nabla}\hat{\Phi}_c^a + \frac{4\hat{n}_{0e}\hat{T}_0}{\hat{B}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c + \frac{\hat{n}_{0e}\hat{T}_0}{2\hat{B}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c^a = 0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \tilde{p}_\perp}{\partial t} + \mathbf{b}\times\nabla\Phi_c\cdot\nabla\tilde{p}_\perp + B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_\perp}{B^2} + B^2\partial_\parallel\frac{p_0\tilde{u}_\parallel}{B^2} - B^2\mathbf{b}\times\nabla\Psi\cdot\nabla\frac{p_0\tilde{u}_\parallel}{B^2} + \frac{1}{2}\mathbf{b}\times\nabla\Phi_c^a\cdot\nabla\tilde{p}_\perp \\
&- n_0\mathbf{b}\times\nabla\Phi_c^b\cdot\nabla\tilde{T}_\perp - \frac{p_0}{2}\mathbf{b}\times\nabla\Psi_c^a\cdot\nabla\tilde{u}_\parallel - T_0(1+2\eta_e)\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c - T_0(\frac{1}{2}+\eta_e)\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c^a \\
&- T_0\eta_e\mathbf{b}\times\nabla n_0\cdot\nabla\Phi_c^b + \frac{2}{B}n_0T_0(4\mathbf{b}\times\nabla B\cdot\nabla\Phi_c + 2\mathbf{b}\times\nabla B\cdot\nabla\Phi_c^a + \mathbf{b}\times\nabla B\cdot\nabla\Phi_c^b) = 0 \\
\Rightarrow &\frac{\partial \hat{p}_\perp}{\partial \hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{p}_\perp + B^2\tilde{\nabla}_\parallel\frac{\tilde{q}_\perp}{B^2} + \hat{B}^2\hat{\partial}_\parallel\frac{\hat{p}_0\hat{u}_{\parallel e}}{\hat{B}^2} - \hat{B}^2\mathbf{b}\times\hat{\nabla}\hat{\Psi}\cdot\hat{\nabla}\frac{\hat{p}_0\hat{u}_{\parallel e}}{\hat{B}^2} + \frac{1}{2}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c^a\cdot\hat{\nabla}\hat{p}_\perp \\
&- \hat{n}_0\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c^b\cdot\hat{\nabla}\hat{T}_\perp - \frac{\hat{p}_0}{2}\mathbf{b}\times\hat{\nabla}\hat{\Psi}_c^a\cdot\hat{\nabla}\hat{u}_{\parallel e} - \hat{T}_0(1+2\eta_e)\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_c - \hat{T}_0(\frac{1}{2}+\eta_e)\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_c^a \\
&- \hat{T}_0\eta_e\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_c^b + \frac{2}{\hat{B}}\hat{n}_0\hat{T}_0(4\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c + 2\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c^a + \mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\Phi}_c^b) = 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{q}_\parallel &= -n_0\sqrt{\frac{8}{\pi}}v_{t\parallel}\frac{ik_\parallel\tilde{T}_\parallel}{|k_\parallel|} \\
\tilde{q}_\perp &= -n_0\sqrt{\frac{2}{\pi}}v_{t\parallel}\frac{ik_\parallel\tilde{T}_\perp}{|k_\parallel|}
\end{aligned}
$$

- Get electric field from quasi-neutrality
  Padé approximation, $\Gamma_0^{1/2}(b) = 1/(1+b/2)$, $\Gamma_0(b) = 1/(1+b)$, $b = k_\perp^2\rho_i^2 = -\rho_i^2\nabla_\perp^2 = -(\rho_i^2/\bar{L}^2)\hat{\nabla}_\perp^2$

$$
\begin{aligned}
Step3 :& \bar{\hat{n}}_e = \frac{1}{1+b/2}\hat{n}_e - \frac{\hat{n}_{e0}2b}{\hat{T}_0(2+b)^2}\hat{T}_{\perp e} = \hat{n}_s - \frac{\hat{n}_{e0}}{\hat{T}_0}\hat{T}_s \\
step1a :& (1+\frac{b}{2})\hat{n}_s = \hat{n}_e \\
step1b :& (2+b)\hat{T}_{s1} = -4\hat{T}_{\perp e} \\
step2 :& (2+b)\hat{T}_s = \hat{T}_{s1} + 2\hat{T}_{\perp e}
\end{aligned}
$$

$$
\hat{n}_i = -\frac{\bar{L}}{\bar{\rho}_e}\hat{n}_{i0}\frac{Z\hat{\phi}}{\hat{T}_{0i}}
$$

$$
\begin{aligned}
&\hat{\bar{n}}_e + \frac{\bar{L}}{\bar{\rho}_e}\hat{n}_{e0}(1-\Gamma_0)\frac{\hat{\phi}}{\hat{T}_{e0}} = \hat{n}_i \\
\Rightarrow &\hat{\bar{n}}_e + \frac{\bar{L}}{\bar{\rho}_e}\frac{\hat{n}_{e0}}{\hat{T}_{e0}}(\frac{b}{1+b} + \frac{1}{\tau})\hat{\phi} = 0 \\
\Rightarrow &[1+(\tau+1)b]\,(\frac{\tau}{\tau+1}\frac{\bar{\rho}_e\hat{T}_{e0}}{\bar{L}\hat{n}_{e0}}\hat{\bar{n}}_e + \hat{\phi}) = -\frac{\tau^2}{\tau+1}\frac{\bar{\rho}_e\hat{T}_{e0}}{\bar{L}\hat{n}_{e0}}\hat{\bar{n}}_e \\
&\phi_{sour} = -\frac{\tau^2}{\tau+1}\frac{\bar{\rho}_e\hat{T}_{e0}}{\bar{L}\hat{n}_{e0}}\hat{\bar{n}}_e \\
&\hat{\phi}_i = \frac{\tau}{\tau+1}\frac{\bar{\rho}_e\hat{T}_{e0}}{\bar{L}\hat{n}_{e0}}\hat{\bar{n}}_e + \hat{\phi} \\
&\hat{\phi}_c = \frac{1}{\hat{B}}(\hat{\phi}_i - \frac{\tau}{\tau+1}\frac{\bar{\rho}_e\hat{T}_{e0}}{\bar{L}\hat{n}_{e0}}\hat{\bar{n}}_e)
\end{aligned}
$$

where $\omega_{pe}^2 = Ne^2/m_e\epsilon_0$, $\bar{\rho}_e = m_e\bar{v}_{th}/e\bar{B}$, $\tau = T_{i0}/T_{e0}$. For simplicity, assume $\tau = 1$ now.

- Get magnetic field

$$
\begin{aligned}
step2 :&\hat{\nabla}_\perp^2(\hat{\psi}\hat{B}) = -\frac{\bar{v}_{th}\bar{\Omega}_e\bar{L}}{V_{Ae}^2}\hat{n}_{0e}\hat{\bar{u}}_{\parallel e} \\
\bar{u}_{\parallel e} &= \Gamma_0^{1/2}(b)\tilde{u}_{\parallel e} - \frac{2b}{(2+b)^2}\tilde{q}_{\perp e} \\
step1 :&(1+\frac{b}{2})\hat{\bar{u}}_{\parallel e} = \hat{u}_{\parallel e}
\end{aligned}
$$

- Gyroaveraged quantities

$$
\begin{aligned}
\Phi_c = \Gamma_0^{1/2}\phi &\Rightarrow (1+\frac{b}{2})\hat{\Phi}_c = \hat{\phi}_c \\
\Psi = \Gamma_0^{1/2}\psi &\Rightarrow (1+\frac{b}{2})\hat{\Psi} = \hat{\psi} \\
\hat{\nabla}_\perp^2\phi = \Phi_c^a = 2b\frac{\partial\Gamma_0^{1/2}}{\partial b}\phi_c &\Rightarrow (1+\frac{b}{2})(2\phi_c - (1+\frac{b}{2})\Phi_c^a) = 2\phi_c \\
&\Rightarrow (1+\frac{b}{2})\hat{\Phi}_s^a = 2\hat{\phi}_c \\
&\Rightarrow (1+\frac{b}{2})\hat{\Phi}_c^a = 2\hat{\phi}_c - \hat{\Phi}_s^a \\
\hat{\hat{\nabla}}_\perp^2\phi = \Phi_c^b = b\frac{\partial^2}{\partial b^2}(b\Gamma_0^{1/2})\phi &\Rightarrow \Phi_c^b = \Phi_c^a + \Phi_s^b \\
&\Rightarrow \Phi_s^b = \frac{1}{2(1+\frac{b}{2})^3}\phi_c \\
&\Rightarrow (1+\frac{b}{2})\hat{\Phi}_{s2}^b = \frac{1}{2}\hat{\phi}_c \\
&\Rightarrow (1+\frac{b}{2})\hat{\Phi}_{s1}^b = \hat{\Phi}_{s2}^b \\
&\Rightarrow (1+\frac{b}{2})\hat{\Phi}_s^b = \hat{\Phi}_{s1}^b
\end{aligned}
$$

Electrostatic virsion

In density, parallel and perpendicular pressure equation, just drop terms associated with Ψ.

$$
\begin{aligned}
&n_0\frac{\partial \tilde{u}_\parallel}{\partial t}+n_0\mathbf{b}\times\nabla\Phi_c\cdot\nabla\tilde{u}_{\parallel e}-\frac{n_0e}{m_e}\partial_\parallel\Phi+\frac{B}{m_e}\partial_\parallel\frac{\tilde{p}_\parallel}{B}\\
&+\frac{\tilde{p}_\perp}{Bm_e}\partial_\parallel B-\frac{n_0e}{2m_e}\Phi_c^a\partial_\parallel B+\frac{4}{T_0}\frac{v_t^2}{\Omega_eB}\mathbf{b}\times\nabla B\cdot\nabla(p_0\tilde{u}_\parallel)=\frac{en_e\eta}{m_e}J_\parallel\\
\Rightarrow&\frac{\partial\hat{u}_{\parallel e}}{\partial\hat{t}}+\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{u}_{\parallel e}-\bar{\Omega}_e\bar{T}\hat{\partial}_\parallel(\hat{\Phi}_c\hat{B})+\frac{\hat{B}}{\hat{n}_{e0}}\hat{\partial}_\parallel\frac{\hat{p}_\parallel}{\hat{B}}\\
&+\frac{\hat{p}_\perp}{\hat{n}_{e0}\hat{B}}\hat{\partial}_\parallel\hat{B}-\bar{\Omega}_e\bar{T}\frac{1}{2}\hat{\Phi}_c^a\hat{\partial}_\parallel\hat{B}-\frac{1}{\bar{\Omega}_e\bar{T}}\frac{4}{\hat{n}_{e0}}\frac{1}{\hat{\Omega}_e\hat{B}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}(\hat{p}_0\hat{u}_{\parallel e})=-\bar{\Omega}_e\bar{T}\hat{B}\hat{\eta}\hat{J}_{\parallel c}
\end{aligned}
$$

Don't need Ampere's law anymore, so current is calculated from parallel velocity

$$
\begin{aligned}
J_\parallel &= -en_{e0}u_{\parallel e}\\
J_{\parallel c} &= \frac{\mu_0 en_{e0}}{B}u_{\parallel e}\\
\hat{J}_{\parallel c} &= \frac{\bar{L}\bar{V}\bar{\Omega}_e}{\bar{V}_{Ae}^2}\frac{\hat{n}_{e0}}{\hat{B}}\hat{u}_{\parallel e}
\end{aligned}
$$