3-field with electron inertial in Ohm's law and parallel current in Pressure equation

$$\frac{d\varpi}{dt} - eB(\mathbf{V}_\Phi - \mathbf{v}_E)\cdot\nabla n_i = B\nabla_\parallel J_\parallel + 2\mathbf{b}_0\times\kappa\cdot\nabla\tilde{P}$$
$$\frac{dP}{dt} + T_0(\mathbf{v}_\Phi - \mathbf{v}_E)\cdot\nabla n_i - \nabla_\parallel\left(\frac{P_0}{2}\frac{J_\parallel}{en_0}\right) = 0$$
$$-\frac{m_e}{e}\frac{d\tilde{u}_{\parallel e}}{dt} - \frac{m_e}{e}\mathbf{V}_E\cdot\nabla u_{\parallel e0} + \frac{dA_\parallel}{dt} + \partial_\parallel^0\phi - \frac{1}{n_0 e}\nabla_\parallel\tilde{p}_e - \frac{1}{2en_{0e}}\delta\mathbf{b}\cdot\nabla P_0 = \eta J_\parallel$$
$$\varpi = eB(\tilde{n}_e - \tilde{n}_i) = eB(\Gamma_0^{1/2}\tilde{n}_i - n_{i0}(1-\Gamma_0)\frac{e\phi}{T_{i0}} - \tilde{n}_i)$$

where$d/dt = \partial/\partial t + \mathbf{V}_E\cdot\nabla$, $\mathbf{V}_\Phi = \frac{1}{B}\mathbf{b}\times\nabla\Phi = \frac{1}{B}\mathbf{b}\times\nabla\Gamma_0^{1/2}\phi$, $\mathbf{v}_E = \frac{1}{B}\mathbf{b}_0\times\nabla\phi$, $J_{0\parallel} = -en_0 u_{0\parallel e}$, $\tilde{J}_\parallel = -en_0\tilde{u}_{\parallel e} = -\nabla_\perp^2 A_\parallel/\mu_0$, $\delta\mathbf{b} = \frac{1}{B}\nabla A_\parallel\times\mathbf{b}$,$\partial_\parallel = \partial_\parallel^0 + \delta\mathbf{b}\cdot\nabla$, $\nabla_\parallel f = B\partial_\parallel\frac{f}{B}$.

- Expression with quantities used in code
  $U = \frac{\varpi}{m_i}$, $\tilde{J} = -\frac{\mu_0}{B_0}\tilde{J}_\parallel = \nabla_\perp^2\psi$, $J_0 = -\frac{\mu_0}{B_0}J_{0\parallel}$, $\phi_c = \frac{\phi}{B_0}$, $\Phi_{fc} = \Phi_c - \phi_c$, $\Lambda = -\frac{m_e}{e}\tilde{u}_{\parallel e} + A_\parallel$, $\Lambda_c = \frac{\Lambda}{B_0} = -\frac{m_e}{eB_0}\tilde{u}_{\parallel e} + \frac{A_\parallel}{B_0} = -\frac{m_e}{e^2\mu_0 n_0}\tilde{J} + \psi$, $u_{\parallel e0} = \frac{B_0}{e\mu_0 n_0}J_0$

$$m_i\frac{dU}{dt} - eB\mathbf{b}\times\nabla\Phi_{fc}\cdot\nabla n_0 - eB\mathbf{b}\times\nabla\Phi_{fc}\cdot\nabla\tilde{n}_i$$
$$= -B^2\nabla_\parallel\tilde{J} - B^2\mathbf{b}_0\times\nabla\psi\cdot\nabla J_0 + 2\mathbf{b}_0\times\kappa\cdot\nabla\tilde{P}$$
$$\frac{dP}{dt} + T_0\mathbf{b}\times\nabla\Phi_{fc}\cdot\nabla n_0 + T_0\mathbf{b}\times\nabla\Phi_{fc}\cdot\nabla\tilde{n}_i + B\partial_\parallel\left(T_0\frac{\tilde{J}}{\mu_0 e}\right) + B\mathbf{b}\times\nabla\left(T_0\frac{J_0}{\mu_0 e}\right)\cdot\nabla\psi = 0$$
$$\frac{d\Lambda_c}{dt} - \frac{m_e}{eB}\mathbf{V}_E\cdot\nabla\left(\frac{B_0}{e\mu_0 n_0}J_0\right) + \frac{1}{B}\partial_\parallel^0\phi - \frac{1}{Bn_0 e}\nabla_\parallel\tilde{p}_e - \frac{1}{2eBn_{0e}}\delta\mathbf{b}\cdot\nabla P_0 = \eta J_\parallel$$
$$\varpi = eB(\tilde{n}_e - \tilde{n}_i) = eB(\Gamma_0^{1/2}\tilde{n}_i - n_{i0}(1-\Gamma_0)\frac{e\phi}{T_{i0}} - \tilde{n}_i)$$

- Normalization
  $U = \bar{N}\hat{U}/\bar{T}$, $\nabla = \hat{\nabla}/\bar{L}$, $J_0 = \hat{J}_0/\bar{L}$, $\tilde{J} = \hat{J}_1/\bar{L} = \hat{\nabla}_\perp^2\hat{\psi}/\hat{L}$, $\psi = \hat{\psi}\bar{L}$, $\tilde{P} = \bar{B}^2\hat{P}_1/2\mu_0$, $\tilde{u}_{\parallel e} = \frac{B}{\mu_0 en_{e0}}\tilde{J} = \frac{v_a^2}{\Omega_i\bar{L}}\frac{\hat{B}}{\hat{n}_e}\hat{J}_1$, $\phi_c = \bar{L}^2\hat{\phi}_c/\bar{T}$, $\eta = \hat{\eta}\bar{L}^2\mu_0/\bar{T}$, $\tilde{P}_e = \frac{\bar{B}^2}{2\mu_0}[\frac{1}{2}(\hat{P}_1 + \frac{1}{C_{nor}}\frac{\hat{T}_0\hat{U}}{\hat{B}})] = \frac{\bar{B}^2}{2\mu_0}\hat{P}_{e1}$, $\tilde{n}_i = \frac{\bar{N}}{2}(\frac{\hat{P}_1}{\hat{T}_0} - \frac{1}{C_{nor}}\frac{\hat{U}}{\hat{B}}) = \bar{N}\hat{n}_{i1}$,$C_{nor} = \Omega_i\bar{T}$, $T_0 = \frac{1}{2}m_i V_a^2\hat{T}_0$, $n_{i0} = \bar{N}\frac{\hat{P}_0}{2\hat{T}_0}$, $\Lambda_c = \bar{L}(\hat{\psi} - \frac{m_e}{m_i}\frac{1}{C_{nor}^2}\frac{\hat{J}_1}{\hat{n}_0}) = \bar{L}\hat{\Lambda}_c$

$$\frac{\partial\hat{U}}{\partial\hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{U} + C_{nor}\hat{B}\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_{fc} - C_{nor}\hat{B}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_{fc}\cdot\hat{\nabla}\hat{n}_{i1}$$
$$= -\hat{\nabla}_\parallel\hat{J}_1 + \hat{B}^2\mathbf{b}_0\times\hat{\nabla}\hat{\psi}\cdot\hat{\nabla}\hat{J}_{0\parallel} + \mathbf{b}_0\times\hat{\kappa}\cdot\nabla\hat{P}_1$$
$$\frac{\partial\hat{P}_1}{\partial\hat{t}} - \mathbf{b}\times\hat{\nabla}\hat{P}_0\cdot\hat{\nabla}\hat{\phi}_c + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{P}_1$$
$$-\hat{T}_0\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_{fc} + \hat{T}_0\mathbf{b}\times\hat{\nabla}\hat{\Phi}_{fc}\cdot\hat{\nabla}\hat{n}_{i1} + \frac{1}{C_{nor}}\hat{B}\hat{\partial}_\parallel(\hat{T}_0\hat{J}_1) + \frac{1}{C_{nor}}\hat{B}\mathbf{b}\times\hat{\nabla}(\hat{T}_0\hat{J}_0)\cdot\hat{\nabla}\hat{\psi} = 0$$
$$\frac{\partial\hat{\Lambda}_c}{\partial\hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{\Lambda}_c - \frac{m_e}{m_i}\frac{1}{C_{nor}^2}\mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\left(\frac{\hat{J}_0}{\hat{n}_0}\right) + \frac{1}{\hat{B}}\hat{\partial}_\parallel^0(\hat{\phi}_c * \hat{B})$$
$$-\frac{1}{4C_{nor}\hat{n}_0\hat{B}}\mathbf{b}_0\times\hat{\nabla}\hat{P}_0\cdot\hat{\nabla}\hat{\psi} - \frac{1}{2C_{nor}\hat{n}_0}\hat{\nabla}_\parallel\hat{P}_{e1} = \hat{\eta}\hat{J}_1$$
$$\hat{U} = C_{nor}\frac{\hat{B}\hat{n}_0}{\hat{T}_0}\left(\frac{\hat{T}_0}{\hat{n}_0}\Gamma_0^{1/2}\hat{n}_{i1} - 2C_{nor}(1-\Gamma_0)\hat{\phi} - \frac{\hat{T}_0}{\hat{n}_0}\hat{n}_{i1}\right)$$

4-Field Gyrofluid

March 2, 2012

$$\frac{d\varpi}{dt} - eB(\mathbf{V}_\Phi - \mathbf{v}_E)\cdot\nabla n_i = B\nabla_\parallel J_\parallel + 2\mathbf{b}_0\times\kappa\cdot\nabla\tilde{P} - \frac{2J_{0\parallel}}{B^2}\mathbf{B}\times\nabla B\cdot\nabla A_\parallel$$

$$-B\nabla_\parallel e n_0(\bar{u}_{\parallel i} - \tilde{u}_{\parallel i}) + eB(\bar{\nabla}_\parallel - \nabla_\parallel) n_0\tilde{u}_{\parallel i} + \frac{2en_0}{B^2}\mathbf{B}\times\nabla B\cdot\nabla(\frac{\Phi}{Z} - \phi)$$

$$\frac{dP}{dt} + T_i(\mathbf{v}_\Phi - \mathbf{v}_E)\cdot\nabla n_i + \nabla_\parallel\frac{P_0}{2}(\tilde{u}_{\parallel e} + \tilde{u}_{\parallel i}) + (\bar{\nabla}_\parallel - \nabla_\parallel)\frac{P_0\tilde{u}_{\parallel i}}{2} - \frac{T_e}{e}\nabla_\parallel J_{0\parallel}$$

$$+ \frac{2T_eJ_{0\parallel}}{eB^3}\mathbf{B}\times\nabla B\cdot\nabla A_\parallel + \frac{P_0}{B^3}\mathbf{B}\times\nabla B\cdot\nabla(\Phi + \phi) = 0$$

$$m_in_0\frac{\partial\tilde{u}_{\parallel i}}{\partial t} + m_in_0\mathbf{V}_\Phi\cdot\nabla\tilde{u}_{\parallel i} + \nabla_\parallel P + en_0\frac{\partial(\bar{A}_\parallel - A_\parallel)}{\partial t} + en_0(\bar{\partial}_\parallel\Phi - \partial_\parallel\phi)$$

$$+ (\bar{\nabla}_\parallel - \nabla_\parallel)\tilde{p}_i + B(\delta\bar{\mathbf{b}} - \delta\mathbf{b})\cdot\nabla\frac{P_0}{2B} + \frac{2m_i}{eB^2}\mathbf{b}\times\nabla B\cdot\nabla P_0\tilde{u}_{\parallel i} = 0$$

$$\frac{\partial A_\parallel}{\partial t} + \partial_\parallel\phi - \frac{1}{n_0e}\nabla_\parallel\tilde{p}_e - \frac{B}{2en_0}\delta\mathbf{b}\cdot\nabla\frac{P_0}{B} = \eta J_\parallel$$

$$\varpi = eB(\tilde{n}_e - \tilde{n}_i) = eB(\Gamma_0^{1/2}\tilde{n}_i - n_{0i}(1 - \Gamma_0)\frac{e\phi}{T_0} - \tilde{n}_i)$$

$$\tilde{J}_\parallel = n_0e(\bar{u}_i - \tilde{u}_{e\parallel}) = -\nabla_\perp^2 A_\parallel/\mu_0$$

$$\tilde{n}_i = \frac{1}{2}(\frac{\tilde{P}}{T_e} - \frac{\varpi}{eB})$$

$$\tilde{p}_e = \frac{1}{2}(P + \frac{T_e\varpi}{eB})$$

where $d/dt = \partial/\partial t + \mathbf{V}_E\cdot\nabla$, $\mathbf{v}_E = \frac{1}{B}\mathbf{b}_0\times\nabla\phi$, $\mathbf{V}_\Phi = \frac{1}{B}\mathbf{b}\times\nabla\Phi = \frac{1}{B}\mathbf{b}\times\nabla\Gamma_0^{1/2}\phi$, $J_\parallel = J_0 + \tilde{J}_\parallel$, $J_{0\parallel} = -en_{0e}u_{0\parallel e}$, $\bar{u}_i = \Gamma_0^{1/2}\tilde{u}_{\parallel i}$ $\delta\mathbf{b} = \frac{1}{B}\nabla A_\parallel\times\mathbf{b}$, $\delta\bar{\mathbf{b}} = \frac{1}{B}\nabla\bar{A}_\parallel\times\mathbf{b}$, $\bar{A}_\parallel = \Gamma_0^{1/2}A_\parallel$. And the approximation $\mathbf{B}\times\nabla B/B^2 \approx \mathbf{b}_0\times\kappa$ is used for low β plasma. $n_0 = n_{e0} = n_{i0}$, $T_i = T_e = T_0$, $\nabla_\parallel f = B\partial_\parallel\frac{f}{B}$, $\partial_\parallel = \partial_\parallel^0 + \delta\mathbf{b}\cdot\nabla$

- Normalization
  $U = \frac{\varpi}{m_i}$, $\psi = \frac{A_\parallel}{B_0}$, $\tilde{J} = -\frac{\mu_0}{B_0}\tilde{J}_\parallel = \nabla_\perp^2\psi$, $J_0 = -\frac{\mu_0}{B}J_{\parallel 0}$ $\phi_c = \frac{\phi}{B_0}$, $\Phi_{fc} = \Phi_c - \phi_c$, $\psi_f = \bar{\psi} - \psi$

$$
\begin{aligned}
&\frac{\partial U}{\partial t}+\mathbf{V}_E\cdot\nabla U-\frac{eB}{m_i}(\mathbf{V}_\Phi-\mathbf{v}_E)\cdot\nabla n_i=-\frac{B^2}{\mu_0 m_i}\partial_\parallel\tilde{J}-\frac{B^2}{\mu_0 m_i}\delta\mathbf{b}\cdot\nabla J_0+\frac{2}{m_i}\mathbf{b}_0\times\kappa\cdot\nabla\tilde{P}\\
&+\frac{2J_0B}{\mu_0 m_i}\mathbf{b}\times\nabla B\cdot\nabla\psi-\frac{B^2}{m_i}\partial_\parallel\frac{en_0(\bar{u}_{\parallel i}-\tilde{u}_{\parallel i})}{B}+\frac{eB^2}{m_i}\partial_{\parallel f}\frac{n_0\tilde{u}_{\parallel i}}{B}+\frac{2en_0}{m_i}\mathbf{b}\times\nabla B\cdot\nabla(\Phi-\phi)\\
&\frac{dP}{dt}+T_i(\mathbf{v}_\Phi-\mathbf{v}_E)\cdot\nabla n_i+2(\tilde{u}_{\parallel e}+\tilde{u}_{\parallel i})+\nabla_{\parallel f}\frac{P_0\tilde{u}_{\parallel i}}{2}+\frac{BT_e}{e\mu_0}\nabla_\parallel J_0\\
&-\frac{2T_eJ_0}{\mu_0 e}\mathbf{b}\times\nabla B\cdot\nabla\psi+\frac{P_0}{B}\mathbf{b}\times\nabla B\cdot\nabla(\Phi_c+\phi_c)=0\\
&m_in_0\frac{\partial\tilde{u}_{\parallel i}}{\partial t}+m_in_0\mathbf{V}_\Phi\cdot\nabla\tilde{u}_{\parallel i}+\nabla_\parallel P+Ben_0\frac{\partial\psi_f}{\partial t}+en_0B(\bar{\partial}_\parallel\Phi_c-\partial_\parallel\phi_c)\\
&+\nabla_{\parallel f}\tilde{p}_i+B\delta\mathbf{b}_f\cdot\nabla\frac{P_0}{2B}+\frac{2m_i}{eB^2}\mathbf{b}\times\nabla B\cdot\nabla P_0\tilde{u}_{\parallel i}=0\\
&\frac{\partial\psi}{\partial t}+\partial_\parallel\phi_c-\frac{1}{n_0e}\partial_\parallel\frac{\tilde{p}_e}{B}-\frac{1}{2en_0}\delta\mathbf{b}\cdot\nabla\frac{P_0}{B}=-\frac{\eta}{\mu_0}\tilde{J}\\
&U=\frac{eB}{m_i}(\tilde{n}_e-\tilde{n}_i)=\frac{eB}{m_i}(\Gamma_0^{1/2}\tilde{n}_i-n_{i0}(1-\Gamma_0)\frac{e\phi}{T_i}-\tilde{n}_i)\\
&\tilde{J}=-\frac{\mu_0n_0e}{B}(\bar{u}_i-\tilde{u}_{e\parallel})=\nabla_\perp^2\psi\\
&\tilde{n}_i=\frac{1}{2}(\frac{\tilde{P}}{T_e}-\frac{\varpi}{eB})\\
&\tilde{p}_e=\frac{1}{2}(P+\frac{T_e\varpi}{eB})
\end{aligned}
$$

$U=\bar{N}\hat{U}/\bar{T}$, $\nabla=\hat{\nabla}/\bar{L}$, $J_0=\hat{J}_0/\bar{L}$, $\tilde{J}=\hat{J}_1/\bar{L}=\hat{\nabla}_\perp^2\hat{\psi}/\hat{L}$, $\psi=\hat{\psi}\bar{L}$, $\tilde{P}=\bar{B}^2\hat{P}_1/2\mu_0$, $\tilde{u}_{\parallel e}=\frac{B}{\mu_0 en_{e0}}\tilde{J}=\frac{v_a^2}{\Omega_i\bar{L}}\frac{\hat{B}}{\hat{n}_e}\hat{J}_1$, $\phi_c=\bar{L}^2\hat{\phi}_c/\bar{T}$, $\eta=\hat{\eta}\bar{L}^2\mu_0/\bar{T}$, $\tilde{P}_e=\frac{\bar{B}^2}{2\mu_0}[\frac{1}{2}(\hat{P}_1+\frac{1}{C_{nor}}\frac{\hat{T}_0\hat{U}}{\hat{B}})]=\frac{\bar{B}^2}{2\mu_0}\hat{P}_{e1}$, $\tilde{n}_i=\frac{\bar{N}}{2}(\frac{\hat{P}_1}{\hat{T}_0}-\frac{1}{C_{nor}}\frac{\hat{U}}{\hat{B}})=\bar{N}\hat{n}_{i1}$,$C_{nor}=\Omega_i\bar{T}$, $T_0=\frac{1}{2}m_iV_a^2\hat{T}_0$, $n_{i0}=\bar{N}\frac{\hat{P}_0}{2\hat{T}_0}$

$$
\begin{aligned}
&\frac{\partial \hat{U}}{\partial \hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{U} + C_{nor}\hat{B}\mathbf{b}\times\hat{\nabla}\hat{n}_0\cdot\hat{\nabla}\hat{\Phi}_{fc} - C_{nor}\hat{B}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_{fc}\cdot\hat{\nabla}\hat{n}_{i1} = \\
&-\hat{\nabla}_{\|}\hat{J}_1 + \hat{B}^2\mathbf{b}_0\times\hat{\nabla}\hat{\psi}\cdot\hat{\nabla}\hat{J}_{0\|} + \mathbf{b}_0\times\hat{\kappa}\cdot\nabla\hat{P}_1 \\
&+2\hat{J}_0\hat{B}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\psi} - C_{nor}\hat{B}^2\hat{\partial}_{\|}\frac{\hat{n}_0\hat{u}_{\|if}}{\hat{B}} - C_{nor}\hat{B}^2\mathbf{b}\times\hat{\nabla}\hat{\psi}_f\cdot\hat{\nabla}\frac{\hat{n}_0\hat{u}_{\|i}}{\hat{B}} + C_{nor}\hat{n}_0\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}(\hat{\Phi}_c - \hat{\phi}_c) \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \hat{P}_1}{\partial \hat{t}} - \mathbf{b}\times\hat{\nabla}\hat{P}_0\cdot\hat{\nabla}\hat{\phi}_c + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{P}_1 - \frac{1}{2}\mathbf{b}\times\hat{\nabla}\hat{P}_0\cdot\hat{\nabla}\hat{\Phi}_{fc} + \hat{T}_i\mathbf{b}\times\hat{\nabla}\hat{\Phi}_{fc}\cdot\hat{\nabla}\hat{n}_{i1} \\
&+\hat{B}\hat{\partial}_{\|}\frac{\hat{P}_0}{2\hat{B}}(2\hat{u}_{\|i} + \hat{u}_{\|if} + \frac{\hat{B}}{C_{nor}\hat{n}_0}\hat{J}_1) - \hat{B}\mathbf{b}\times\hat{\nabla}\hat{\psi}_f\cdot\hat{\nabla}\frac{\hat{P}_0\hat{u}_{\|i}}{2\hat{B}} \\
&+\frac{\hat{T}_e\hat{B}}{C_{nor}}\mathbf{b}\times\hat{\nabla}\hat{J}_0\cdot\hat{\nabla}\hat{\psi} - \frac{2\hat{T}_e\hat{J}_0}{C_{nor}}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{\psi} + \frac{\hat{P}_0}{\hat{B}}\mathbf{b}\times\hat{\nabla}B\cdot\hat{\nabla}(\hat{\Phi}_c + \hat{\phi}_c) = 0 \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \hat{u}_{\|i}}{\partial \hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{u}_{\|i} + \frac{\hat{B}}{2\hat{n}_0}\hat{\partial}_{\|}\frac{\hat{P}_1}{\hat{B}} + \frac{\hat{B}}{2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{P}_0}{\hat{B}}\cdot\hat{\nabla}\hat{\psi} + C_{nor}\hat{B}\frac{\partial\hat{\psi}_f}{\partial\hat{t}} + C_{nor}\hat{B}(\hat{\bar{\partial}}_{\|}\hat{\Phi}_c - \hat{\partial}_{\|}\hat{\phi}_c) \\
&+\frac{\hat{T}_i\hat{B}}{2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{n}_{i1}}{\hat{B}}\cdot\hat{\nabla}\hat{\psi}_f + \frac{\hat{B}}{4\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{P}_0}{\hat{B}}\cdot\hat{\nabla}\hat{\psi}_f + \frac{1}{C_{nor}\hat{B}^2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{P}_0\hat{u}_{\|i} = 0 \qquad (3)
\end{aligned}
$$

$$
\frac{\partial\hat{\psi}}{\partial\hat{t}} + \hat{\partial}_{\|}\hat{\phi}_c - \frac{1}{4C_{nor}\hat{n}_0}\mathbf{b}_0\times\hat{\nabla}\frac{\hat{P}_0}{\hat{B}}\cdot\hat{\nabla}\hat{\psi} - \frac{1}{2C_{nor}\hat{n}_0}\hat{\partial}_{\|}\frac{\hat{P}_{e1}}{\hat{B}} = \hat{\eta}\hat{J}_1 \qquad (4)
$$

$$
\hat{U} = C_{nor}\frac{\hat{B}\hat{n}_0}{\hat{T}_i}(\frac{\hat{T}_i}{\hat{n}_0}\Gamma_0^{1/2}\hat{n}_{i1} - 2C_{nor}(1-\Gamma_0)\hat{\phi} - \frac{\hat{T}_i}{\hat{n}_0}\hat{n}_{i1}) \qquad (5)
$$

$$
\hat{J}_1 = \frac{C_{nor}\hat{n}_0}{\hat{B}}(\hat{u}_{\|e} - \hat{u}_{\|i}) = \hat{\nabla}_{\perp}^2\hat{\psi} \qquad (6)
$$

Rewrite Equ. 3 in new variable: $\hat{\Lambda}_i = \hat{u}_{\|i} + C_{nor}\hat{B}\hat{\psi}_f$

$$
\begin{aligned}
&\frac{\partial\hat{\Lambda}_i}{\partial\hat{t}} + \mathbf{b}\times\hat{\nabla}\hat{\phi}_c\cdot\hat{\nabla}\hat{\Lambda}_i + \mathbf{b}\times\hat{\nabla}\hat{\Phi}_{fc}\cdot\hat{\nabla}\hat{\Lambda}_i - C_{nor}\hat{B}\mathbf{b}\times\hat{\nabla}\hat{\Phi}_c\cdot\hat{\nabla}\hat{\psi}_f + \frac{\hat{B}}{2\hat{n}_0}\hat{\partial}_{\|}\frac{\hat{P}_1}{\hat{B}} + \frac{\hat{B}}{2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{P}_0}{\hat{B}}\cdot\hat{\nabla}\hat{\psi} \\
&+C_{nor}\hat{B}(\hat{\bar{\partial}}_{\|}\hat{\Phi}_{fc} + \mathbf{b}\times\hat{\nabla}\hat{\phi}\cdot\hat{\nabla}\hat{\psi}_f) + \frac{\hat{T}_i\hat{B}}{2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{n}_{i1}}{\hat{B}}\cdot\hat{\nabla}\hat{\psi}_f + \frac{\hat{B}}{4\hat{n}_0}\mathbf{b}\times\hat{\nabla}\frac{\hat{P}_0}{\hat{B}}\cdot\hat{\nabla}\hat{\psi}_f \qquad (7) \\
&+\frac{1}{C_{nor}\hat{B}^2\hat{n}_0}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{P}_0\hat{\Lambda}_i - \frac{1}{\hat{B}\hat{n}_0}\mathbf{b}\times\hat{\nabla}\hat{B}\cdot\hat{\nabla}\hat{P}_0\hat{\psi}_f = 0
\end{aligned}
$$

– Evolving quantities $(\hat{U}, \hat{P}_1, \hat{\Lambda}_i, \hat{\psi})$

– substitute quantities

$$
\begin{cases}
\hat{n}_{i1} = \frac{1}{2}(\frac{\hat{P}_1}{\hat{T}_e} - \frac{1}{C_{nor}}\frac{\hat{U}}{\hat{B}}) \\
\hat{P}_{e1} = \frac{1}{2}(\hat{P}_1 + \frac{1}{C_{nor}}\frac{\hat{T}_e\hat{U}}{\hat{B}}) \\
\hat{J}_1 = \hat{\nabla}_{\perp}^2\hat{\psi} \\
\hat{u}_{\|i} = \hat{\Lambda}_i - C_{nor}\hat{B}\hat{\psi}_f
\end{cases}
$$

– invert quantities $n_s = \frac{m_i V_a^2}{2e}\frac{\hat{T}_i}{\hat{n}_0}(Z\hat{\bar{n}}_i - Z\hat{n}_{i1} - \frac{\hat{U}}{C_{nor}\hat{B}}) = \frac{m_i V_a^2}{2e}\hat{n}_s$

$$\begin{cases} \hat{n}_{i1} = (1 - \frac{\rho_i^2}{2L^2}\hat{\nabla}_\perp^2)\hat{n}_i \\ \hat{n}_s = -\frac{\rho_i^2}{L^2}\hat{\nabla}_\perp^2(2C_{nor}\hat{\phi}_c * \hat{B} - \hat{n}_s) \\ \hat{\phi}_c = (1 - \frac{\rho_i^2}{2L^2}\hat{\nabla}_\perp^2)\hat{\Phi}_c \\ \hat{\hat{\psi}} = (1 - \frac{\rho_i^2}{2L^2}\hat{\nabla}_\perp^2)\hat{\psi} \\ \hat{\hat{u}}_{\|i} = (1 - \frac{\rho_i^2}{2L^2}\hat{\nabla}_\perp^2)\hat{u}_{\|i} \end{cases}$$

- Code changes

Symbol	meaning	Symbol	meaning
bool Parallel_ion	evolving ion parallel motion	Field3D gyroPsi	$\Gamma_0^{1/2}\hat{\psi}$
bool GradB_terms	keep Grad B terms	Field3D gyroPsi_d	$\Gamma_0^{1/2}\hat{\psi} - \hat{\psi}$
Field3D Lambdai	Evoloving: $\hat{\Lambda}_i = \hat{u}_{\|i} + ZC_{nor}\hat{B}\hat{\psi}_f$	Int gyroPsi_flags	inversion flag
Field3d Vipar	$\hat{u}_{\|i}$		
Field3D gyroVipar	$\Gamma_0^{1/2}\hat{u}_{\|i}$		
Field3D gyroVipar_d	$\Gamma_0^{1/2}\hat{u}_{\|i} - \hat{u}_{\|i}$		

  - new parallel derivative function Grad_parP_gyro(), which the nonlinear term comes from $\hat{\psi}_f$
  - boundary conditions:

$\hat{\Lambda}_i$	$\Gamma_0^{1/2}\hat{u}_{\|i}$	$\Gamma_0^{1/2}\hat{\psi}$
zerolaplace	dirichlet	neumann

- Resonant

$$\omega_{*i}(\omega_{*i} - \omega_{*e}) = (1 + 2q^2)\omega_s^2$$

where $\omega_{*i} = \frac{1}{n_i e B}\mathbf{b} \times \nabla P_i \cdot \nabla = \frac{nP_i'}{n_i e}$ and $\omega_s = \frac{\sqrt{(T_i+T_e)/m_i}}{Rq}$. In our isothermal model, assuming that T_i and T_e are constant and $T_i = T_e = T_0$. Define a function for wave resonant

$$\begin{aligned} f_{res} &= (|\omega_{*i}(\omega_{*i} - \omega_{*e}) - (1 + 2q^2)\omega_s^2|)^{1/2} \\ &= (|2\omega_{*i}^2 - (1 + 2q^2)\omega_s^2|)^{1/2} \\ &= (|\left(\frac{T_i n P_0'}{P_0 e}\right)^2 2 - (1 + 2q^2)\frac{T_i 2}{R^2 q^2 m_i}|)^{1/2} \\ &= \sqrt{2T_i}(|\left(\frac{nP_0'}{P_0 e}\right)^2 T_i - (\frac{1}{q^2} + 2)\frac{1}{R^2 m_i}|)^{1/2} \end{aligned}$$

The changeable parameters are T_i, P_0, q.

Drift ballooning

$$-\omega(\omega - \omega_i^*)[(1 + 2q^2)\omega_s^2 - \omega(\omega - \omega_e^*)] = (1 + 2q^2)\gamma_I^2[\omega_s^2 - \omega(\omega - \omega_e^*)]$$
$$\omega^4 - (\omega_i^* + \omega_e^*)\omega^3 + (\omega_i^*\omega_e^* + (1 + 2q^2)(\gamma_I^2 - \omega_s^2))\omega^2 + (1 + 2q^2)(\omega_s^2\omega_i - \gamma_I^2\omega_e^*)\omega - (1 + 2q^2)\gamma_I^2\omega_s^2 = 0$$