Simulation Equations for L-H transition

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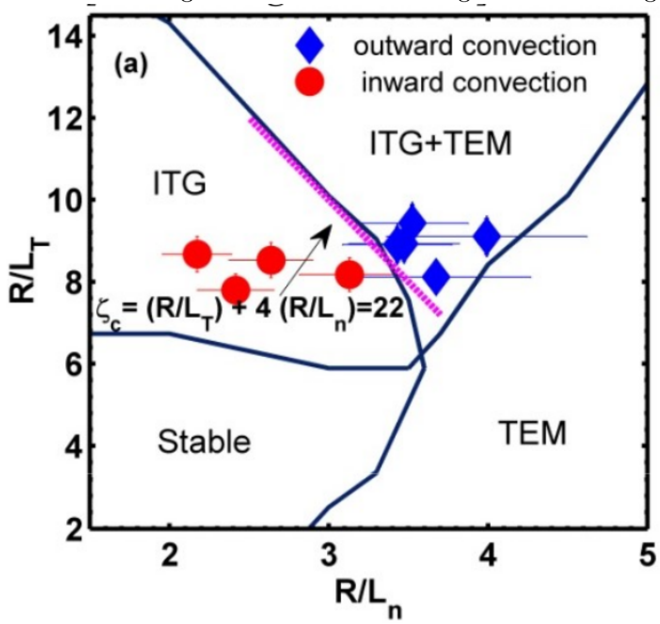
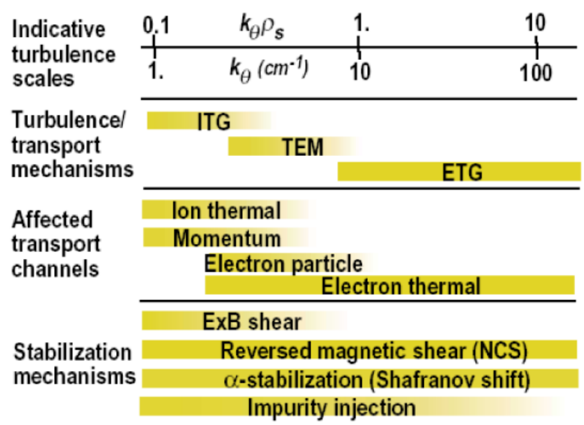
Abstract

In this article I give the equations we are using in the L-H transition simulation.

1 Turbulence Specification

Our whole simulation area is on a very dangerous $q = 3$ rational surface, where instabilities are easy to grow. For tokamaks, we have good curvature on average, so that we don't need to worry about global interchange instabilities. MHD kink modes arise from the term $\frac{1}{2}\int \mathrm{d}\boldsymbol{x}\left[J_{0\|}/c\hat{b}\cdot(\xi_\perp^*\times\boldsymbol{B}_{1\perp})\right]$ in the energy integral, which is rigorously zero in our electrostatic model. Tearing modes are also electromagnetic. Up to now we can see that the above instabilities won't occur even though we have no magnetic shear, which enables us to grow micro-instabilities. But unfortunately, micro-instabilities grow much slower than global interchange instabilities, and are much weaker.





The two pictures above show that our model is adequate to ITG simulation. They further gives that our turbulence spacial scale is a few gyro-radii, and the possible ways for us to realize L-H transition are enhancing $\boldsymbol{E}\times\boldsymbol{B}$ shear flow and inducing Shafranov shift as Drake did[1].

2 Simulation Equations

- According to [2], to simulate ballooning modes, we have to separate n from p.
- We advance $\ln p_i$, $\ln p_e$ and $\ln n$ instead of p_i, p_e and n, so that they are more accurately solved when they are small.
- In this section I give the simulation equations without the constant-n assumption.
- I don't write source and dissipation terms for convenience.

2.1 Continuity Equation

The electron continuity equation is

$$\partial_t n + \nabla \cdot \left[n \left(\boldsymbol{v}_E + \boldsymbol{v}_{de} + \boldsymbol{v}_{\|e} \right) \right] = 0 \tag{1}$$

$$\left(\partial_t + \boldsymbol{v}_E \cdot \nabla \right) n = -n \nabla \cdot \boldsymbol{v}_E - \nabla \cdot \left(n \boldsymbol{v}_{de} \right) - \nabla_{\|} \left(n v_{\|e} \right) \tag{2}$$

Under our normalization,

$$\frac{\mathrm{d}}{\mathrm{d}t} n = \hat{C} p_e - n \hat{C} \phi - \nabla_{\|} \left(n v_{\|e} \right) \tag{3}$$

where

$$\mathrm{d}_t n = \partial_t n + \frac{a}{\rho_s} \boldsymbol{v}_E \cdot \nabla n = \partial_t n + \frac{a}{\rho_s} \left[\phi, n \right] \tag{4}$$

$$\hat{C} = -2 \frac{a}{R_0} \left[\left(\cos\theta + \theta \hat{s} \sin\theta \right) \frac{\partial}{\partial y} + \sin\theta \frac{\partial}{\partial x} \right]$$

i.e.

$$\begin{aligned}
\mathrm{e}^{\ln n} \frac{\mathrm{d}}{\mathrm{d}t} \ln n &= \mathrm{e}^{\ln p_e} \hat{C} \ln p_e - \mathrm{e}^{\ln n} \hat{C} \phi - v_{\|e} \mathrm{e}^{\ln n} \nabla_{\|} \ln n - \mathrm{e}^{\ln n} \nabla_{\|} v_{\|e} \\
\frac{\mathrm{d}}{\mathrm{d}t} \ln n &= T_e \hat{C} \ln p_e - \hat{C} \phi - v_{\|e} \nabla_{\|} \ln n - \nabla_{\|} v_{\|e}
\end{aligned}$$

and $v_{\|e}$ is obtained by

$$j_{\|} = n \left(v_{\|i} - v_{\|e} \right) \tag{5}$$

2.2 Ion Pressure Equation

Under our approximation, the original ion pressure equation given by Zeiler was

$$\frac{3}{2} \left(\partial_t + \boldsymbol{v}_E \cdot \nabla + \boldsymbol{v}_{\|i} \cdot \nabla \right) p_i + \frac{5}{2} p_i \nabla \cdot \left(\boldsymbol{v}_E + \boldsymbol{v}_{\|i} \right) = 0 \tag{6}$$

which is, under our normalization,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i = -\frac{5}{3} p_i \hat{C} \phi - \frac{5}{3} p_i \nabla_{\|} v_{\|i} - v_{\|i} \nabla_{\|} p_i \tag{7}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln p_i = -\frac{5}{3} \left(\hat{C} \phi + \nabla_{\|} v_{\|i} \right) - v_{\|i} \nabla_{\|} \ln p_i \tag{8}$$

2.3 Electron Pressure Equation

Under our approximation, the original electron pressure equation given by Zeiler was

$$\frac{3}{2} \left(\partial_t + \boldsymbol{v}_E \cdot \nabla + \boldsymbol{v}_{\|e} \cdot \nabla \right) p_e + \frac{5}{2} p_e \nabla \cdot \left(\boldsymbol{v}_E + \boldsymbol{v}_{\|e} \right) = 0 \tag{9}$$

which is, under our normalization,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_e = -\frac{5}{3} p_e \hat{C} \phi - \frac{5}{3} p_e \nabla_{\|} v_{\|e} - v_{\|e} \nabla_{\|} p_e \tag{10}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln p_e = -\frac{5}{3} \left(\hat{C} \phi + \nabla_{\|} v_{\|e} \right) - v_{\|e} \nabla_{\|} \ln p_e \tag{11}$$

2.4 Vorticity Equation

The vorticity equation is also the current continuity equation

$$\nabla \cdot \boldsymbol{j}_{pol} + \nabla \cdot \boldsymbol{j}_d + \nabla_\| j_\| = 0 \tag{12}$$

with

$$\boldsymbol{j}_{pol} = \frac{enc}{B\Omega_i}\left(\frac{\mathrm{d}}{\mathrm{d}t} + v_{\|i}\nabla_\|\right)\left(-\nabla_\perp \phi - \frac{\nabla_\perp p_i}{en}\right) \tag{13}$$

from which we can get the normalized vorticity equation

$$\nabla_\perp \cdot \left[n\left(\frac{\mathrm{d}}{\mathrm{d}t} + v_{\|i}\nabla_\|\right)\left(\nabla_\perp \phi + \frac{\nabla_\perp p_i}{n}\right)\right] = \hat{C}\left(p_i + p_e\right) + \nabla_\| j_\| \tag{14}$$

just for simplicity, it is reduced to

$$\mathrm{d}_t \nabla_\perp^2 \left(n\phi + p_i\right) = \hat{C}\left(p_i + p_e\right) + \nabla_\| j_\|$$

We define the vorticity as

$$w = \nabla_\perp \cdot \left[\frac{c}{eB\Omega_i}\left(e\nabla_\perp n\phi + \nabla_\perp p_i\right)\right] \tag{15}$$

So that the normalized vorticity equation can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} w = \hat{C}\left(p_i + p_e\right) + \nabla_\| j_\| \tag{16}$$

2.5 Ion Parallel Momentum Equation

The ion momentum equation given by Zeiler is

$$m_i n \left(\partial_t + \boldsymbol{v}_E \cdot \nabla + \boldsymbol{v}_{\|i} \cdot \nabla\right) v_{\|i} = -\nabla_\| \left(p_i + p_e\right) - p_i \nabla \times \frac{\boldsymbol{b}}{\omega_{ci}} \cdot \nabla v_{\|i} \tag{17}$$

Thus, our normalized ion momentum equation should be

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{\|i} = -\frac{1}{n}\nabla_\| \left(p_i + p_e\right) - v_{\|i}\nabla_\| v_{\|i} - \frac{p_i}{n}\hat{C} v_{\|i} \tag{18}$$

2.6 Ohm's Law

The electron momentum equation given by Zeiler is

$$0 = -\nabla p_e - en\left[\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}_e \times \boldsymbol{B}\right] + \boldsymbol{R}_{ei} \tag{19}$$

where

$$R_{ei\|} = \left(ne\eta_\| j_\| - 0.71 n \nabla_\| T_e\right) \tag{20}$$

The Spitzer resistivity

$$\eta_\| = \frac{m_e \nu_e}{ne^2} \tag{21}$$

So that the parallel component of the electron momentum equation gives our Ohm's law

$$\frac{m_e}{e}\nu_e j_\| = \nabla_\| p_e - ne\nabla_\| \phi \tag{22}$$

where

$$\nu_e = 2.906 \times 10^{-6} \ln\Lambda \frac{n}{\mathrm{cm}^{-3}}\left(\frac{T_e}{\mathrm{eV}}\right)^{-3/2} \mathrm{Hz} \tag{23}$$

$$\ln\Lambda = 22.36 + \frac{3}{2}\ln\frac{T_e}{\mathrm{eV}} - \frac{1}{2}\ln\frac{n}{\mathrm{cm}^{-3}} \tag{24}$$

in our simulation, we normalize ν_e to $1/t_0$, $j_\|$ to $en_0 c_{s0}$. The normalized Ohm's law is

$$j_\| = \frac{1}{\nu_e}\frac{m_i}{m_e}\left(\nabla_\| p_e - n\nabla_\| \phi\right) \tag{25}$$

2.7 Conclusion

The gathering all simulation equations, we have the time-advancing equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln n = T_e\hat{C}\ln p_e - \hat{C}\phi - v_{\|e}\nabla_{\|}\ln n - \nabla_{\|}v_{\|e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln p_i = -\frac{5}{3}\left(\hat{C}\phi + \nabla_{\|}v_{\|i}\right) - v_{\|i}\nabla_{\|}\ln p_i$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln p_e = -\frac{5}{3}\left(\hat{C}\phi + \nabla_{\|}v_{\|e}\right) - v_{\|e}\nabla_{\|}\ln p_e$$

$$\frac{\mathrm{d}}{\mathrm{d}t}w = \hat{C}\left(p_i + p_e\right) + \nabla_{\|}j_{\|}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v_{\|i} = -\frac{1}{n}\nabla_{\|}\left(p_i + p_e\right) - v_{\|i}\nabla_{\|}v_{\|i} - \frac{p_i}{n}\hat{C}v_{\|i}$$

and the relations

$$w = \nabla_{\perp}^2\left(n\phi + p_i\right)$$

$$j_{\|} = \frac{1}{\nu_e}\frac{m_i}{m_e}\left(\nabla_{\|}p_e - n\nabla_{\|}\phi\right)$$

$$v_{\|e} = v_{\|i} - \frac{j_{\|}}{n}$$

2.8 Sheath Boundary Condition

According to Bohm's sheath theory,

$$\begin{aligned} j_{\|} &= j_{si} - j_e = en_ec_s\mathrm{e}^{-1/2}\left[1 - \exp\left(\frac{1}{2}\left(1 + \ln\frac{m_i}{2\pi m_e}\right) + \frac{e\left(V - \phi_p\right)}{T_e}\right)\right] \\ &\approx enc_s\mathrm{e}^{-1/2}\left[1 - \exp\left(3 + \frac{e\left(V - \phi_p\right)}{T_e}\right)\right] \end{aligned}$$

where j_{si} is saturated ion current, V is the electric potential of the ideal conductor, ϕ_p is the potential just inside the sheath.

3 Energy Transfer Rules

References

[1] Drake J F, Lau Y T, Guzdar P N, et al. Local negative shear and the formation of transport barriers[J]. Physical review letters, 1996, 77(3): 494.

[2] M.Francisquez, B.Zhu and B.N.Rogers, Nucl. Fusion 57, 116049 (2017)