University of Limerick
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Exercise Sheet 1

MA 6011 (Cryptographic Mathematics)

week 2: Problems 1–4 at lab; Problems 5–10 at tutorial.

Problem 1: The Fibonacci sequence is defined as follows:

$$F_0 = 0, F_1 = 1, F_2 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}\,.$$

Calculate the first 10 terms of the sequence.

Problem 2: Calculate the first 10 terms of the sequence of integers given by

$$P_0 = 0, P_1 = 1, P_2 = 2 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1}\,.$$

For $n = 1, 2, \ldots, 9$ calculate the Pythagorean triple (a, b, c) where we define

$$a = P_{n+1}^2 - P_n^2, \quad b = 2P_{n+1}P_n, \quad c = P_{n+1}^2 + P_n^2\,.$$

Prove that for each $n \geq 1$ the numbers a and b in such a triple differ by one.

Problem 3: Find three Pythagorean triples with $c = 85$.

Problem 4: In 1770 Lagrange proved that every positive integer can be written as a sum of four integer squares. Find a way of writing the year of your birth and 2012 as a sum of four integer squares.

Problem 5: The first few triangular numbers are $1, 3, 6, 10, \ldots$ and the first few square numbers are $1, 4, 9, 16, \ldots$.
Find a formula for the nth triangular number and the nth square number.

Problem 6: Find families of Pythagorean triples in which b and c differ by 1.

Problem 7: Find all (x, y) satisfying $x^2 + y^2 = 10$ where x and y are rational.

Problem 8: Find the gcd of the following pairs of numbers.

(i) $a = 2009$ and $b = 9002$

(ii) $a = 13579$ and $b = 2468$.

(iii) $a = 123456789$ and $b = 987654321$

(iv) $a = 233$ and $b = 144$

(v) $a = 234$ and $b = 143$

In each case, write the $\gcd(a, b)$ in the form $ax + by$.

Problem 9: Prove that if $\gcd(a, b) = 1$ and $1 = ax + by$ then $\gcd(x, y) = 1$. Verify in the appropriate cases above.

Problem 10: Referring to Problem 2, verify that $\gcd(P_{10}, P_9) = 1$. Prove in general that $\gcd(P_{n+1}, P_n) = 1$ and find integers r and s such that $1 = rP_{n+1} + sP_n$.