

**Problem 1: Basic Vector Operations** (12 points, 4 points/each)

Let two vectors  $\mathbf{a} = (1 \ 2 \ 3)^T$  and  $\mathbf{b} = (-8 \ 1 \ 2)^T$ , answer the following equations:

- (1) Calculate the  $\ell_2$  norm of  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\|\mathbf{a}\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \quad \|\mathbf{b}\|_2 = \sqrt{(-8)^2 + 1^2 + 2^2} = \sqrt{69}$$

- (2) Calculate the Euclidean distance between  $\mathbf{a}$  and  $\mathbf{b}$  (i.e.  $\ell_2$  norm of  $\mathbf{a} - \mathbf{b}$ ).

$$\mathbf{a} - \mathbf{b} = (9, 1, 1)^T$$

$$\|\mathbf{a} - \mathbf{b}\|_2 = \sqrt{83}$$

- (3) Are  $\mathbf{a}$  and  $\mathbf{b}$  orthogonal? State your reason.

They are orthogonal, because  $\mathbf{a} \cdot \mathbf{b} = -8 + 2 + 6 = 0$  and  $\|\mathbf{a}\|_2 \neq 0$  &  $\|\mathbf{b}\|_2 \neq 0$ .

**Problem 2: Basic Matrix Operations** (40 points, 4 points/each)

Suppose  $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ , answer the following questions:

- (1) Calculate  $A^{-1}$  and  $\det(A)$ .

$$\det(A) = -20 - 54 - 54 + 90 + 18 + 36 = 16$$

$$\text{From } [AI] = [IA^{-1}], \text{ we have that } A^{-1} = \begin{bmatrix} -1/8 & -3/8 & 3/8 \\ 3/8 & -7/8 & 3/8 \\ 3/4 & -3/4 & 1/4 \end{bmatrix}$$

- (2) The Rank of  $A$  is?

We can convert the matrix  $A$  into a stepped matrix to solve the rank, but notice that  $\det(A) \neq 0$ , so  $\text{rank}(A) = 3$ .

- (3) The trace of  $A$  is?

$$\text{tr}(A) = 1 - 5 + 4 = 0$$

- (4) Calculate  $A + A^T$ .

$$A + A^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & 3 \\ 9 & -3 & 8 \end{bmatrix}$$

- (5) Is  $A$  an orthogonal matrix? State your reason.

$$A \text{ is not an orthogonal matrix, because } AA^T = \begin{bmatrix} 19 & 27 & 36 \\ 27 & 43 & 60 \\ 36 & 60 & 88 \end{bmatrix} \neq I.$$

- (6) Calculate all the eigenvalue  $\lambda$  and corresponding eigenvectors of  $A$ .

Calculating the eigenvalues is equivalent to finding all  $\lambda$  such that the eigen equation  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  have nontrivial solutions. By the inverse matrix theorem, we only need to find all  $\lambda$  such that  $A - \lambda I$  is irreversible, so

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_{2,3} = -2 \end{cases}$$

Next, let  $\mathbf{x} = (a \ b \ c)^T$ , simplify the augmented matrix of  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  by row transformation.

When  $\lambda = 4$ ,  $\mathbf{x} = (1 \ 1 \ 2)^T$ . When  $\lambda = -2$ ,  $\mathbf{x} = (1 \ 1 \ 0)^T$  or  $\mathbf{x} = (-1 \ 0 \ 1)^T$

- (7) Diagonalize the matrix  $A$ .

Matrix A can be diagonalized, if there is an invertible matrix P and diagonal matrix D, such that  $A = PDP^{-1}$ .

The diagonal matrix D is consists of eigenvalues.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The invertible matrix P is consists of eigen vectors.

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

Therefor,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

(8) Calculate the  $\ell_{2,1}$  norm  $\|A\|_{2,1}$  and the Frobenius norm (i.e.  $\ell_2$  norm)  $\|A\|_F$ .

$$\|A\|_{2,1} = \sum_{i=1}^3 \sqrt{\sum_{j=1}^3 a_{i,j}^2} = \sqrt{19} + \sqrt{43} + \sqrt{88}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^3 \sum_{j=1}^3 a_{i,j}^2} = 5\sqrt{6}$$

(9) Calculate the nuclear norm  $\|A\|_*$  and the spectral norm  $\|A\|_2$ .

$$\det(A^T A - \lambda I) = \det \begin{bmatrix} 19 - \lambda & 27 & 36 \\ 27 & 43 - \lambda & 60 \\ 36 & 60 & 88 - \lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 \approx 145.56 \\ \lambda_2 = 4 \\ \lambda_3 \approx 0.44 \end{cases}$$

For the nuclear norm  $\|A\|_*$ , according to its definition, that is, the sum of singular value, we have

$$\|A\|_* = \sum_{i=1}^3 \sqrt{\text{eig}(A^T A)} = \sqrt{145.56} + \sqrt{4} + \sqrt{0.44} \approx 14.72$$

For the spectral norm  $\|A\|_2$ , according to its definition, that is, the max singular value ,we have

$$\|A\|_2 = \max_{\lambda_i} \left( \sqrt{\text{eig}(A^T A)} \right) = \sqrt{145.56} \approx 12.06$$

### Problem 3: Linear Equations (48 points, 4 points/each)

Please give some proper steps to show how you get the answer.

Let  $x = (x_1, x_2, x_3)^T$  and

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1 \\ x_1 - x_2 = -1 \\ -x_1 + 2x_2 + x_3 = 2 \end{cases}$$

Answer the following questions:

- (1) Solve the linear equations 1 (6 points) (6 points)

By basic elimination, we can get simply

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

- (2) Write it into matrix form(i.e.  $Ax = b$ ) and we will use the same A and b in the following questions.

$$Ax = b$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

- (3) The Rank of A is?

There are many ways to obtain  $\text{rank}(A) = 3$ , for example, we can show that  $\det(A) \neq 0$ (see below) or calculate the number of linearly independent rows by elementary transformation.

- (4) Calculate  $A^{-1}$  and  $\det(A)$ .

$$\text{From } [AI] = [IA^{-1}], \text{ we have that } A^{-1} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}$$

$$\det(A) = -2 + 6 - 3 - 2 = -1$$

- (5) Use (4) to solve the linear equations

First of all, write the augmented matrix

$$A^* = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 2 \end{bmatrix}$$

Then, transform A into a reduced row echelon matrix.

$$A^* = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, we have

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

- (6) Calculate the inner product and outer product of x and b.(i.e.  $\langle \mathbf{x}, \mathbf{b} \rangle$  and  $\mathbf{x} \times \mathbf{b}$ )

$$\langle \mathbf{x}, \mathbf{b} \rangle = -1 + 2 = 1, \mathbf{x} \times \mathbf{b} = \mathbf{x}\mathbf{b}^T = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

- (7) Calculate the  $\ell_1$ ,  $\ell_2$  and  $\ell_\infty$  norm of b

$$\ell_1 = 1 + |-1| + 2 = 4$$

$$\ell_2 = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$\ell_\infty = \max b_i = 2$$

- (8) Suppose  $\mathbf{y} = (y_1, y_2, y_3)^T$ , calculate  $\mathbf{y}^T A \mathbf{y}$ ,  $\nabla_{\mathbf{y}} \mathbf{y}^T A \mathbf{y}$ .

$$y^T A y = (y_1, y_2, y_3) \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2y_1^2 - y_2^2 - y_3^2 + 3y_1y_2 + 2y_1y_3 + 2y_2y_3$$

$$\nabla_y y^T A y = A y + y^T A = (A + A^T) y$$

(9) We add one linear equation  $-x_1 + 2x_2 + x_3 = 2$  into linear equations above.

Write it into matrix form (i.e.  $A_1 x = b_1$ )

$$A_1 x = b_1$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}$$

(10) The rank of  $A_1$  is?

Transform  $A_1$  into a row echelon matrix to find the number of linearly independent rows.

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & -3/2 \\ 0 & 2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,  $\text{rank}(A_1) = 3$ .

(11) Could these linear equations  $A_1 x = b_1$  be solved? State reasons.

It is solvable and the result is still

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$