Problem 1: Basic Vector Operations (12 points, 4 points/each)

Let two vectors $a = (1\ 2\ 3)^T$ and $b = (-8\ 1\ 2)^T$, answer the following equations:

(1) Calculate the ℓ_2 norm of a and b.

$$\|\boldsymbol{a}\|_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14},\quad \|\boldsymbol{b}\|_2 = \sqrt{(-8)^2+1^2+2^2} = \sqrt{69}$$

(2) Calculate the Euclidean distance between a and b (i.e. ℓ_2 norm of $a - b$).

$$\boldsymbol{a} - \boldsymbol{b} = (9,1,1)^T$$

$$\|\boldsymbol{a} - \boldsymbol{b}\|_2 = \sqrt{83}$$

(3) Are a and b orthogonal? State you reason.

They are orthogonal, because $\boldsymbol{a} \cdot \boldsymbol{b} = -8 + 2 + 6 = 0$ and $\|\boldsymbol{a}\|_2 \neq 0\ \&\ \|\boldsymbol{b}\|_2 \neq 0$.

Problem 2: Basic Matrix Operations (40 points, 4 points/each)

Suppose $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$, answer the following questions:

(1) Calculate A^{-1} and $det(A)$.

$$\det(A) = -20 - 54 - 54 + 90 + 18 + 36 = 16$$

From $[AI] = [IA^{-1}]$, we have that $A^{-1} = \begin{bmatrix} -1/8 & -3/8 & 3/8 \\ 3/8 & -7/8 & 3/8 \\ 3/4 & -3/4 & 1/4 \end{bmatrix}$

(2) The Rank of A is?

We can convert the matrix A into a stepped matrix to solve the rank, but notice that $det(A) \neq 0$, so $rank(A) = 3$.

(3) The trace of A is?

$$tr(A) = 1 - 5 + 4 = 0$$

(4) Calculate $A + A^T$.

$$A + A^T = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 9 \\ 0 & -10 & 3 \\ 9 & -3 & 8 \end{bmatrix}$$

(5) Is A an orthogonal matrix? State your reason.

A is not an orthogonal matrix, because $AA^T = \begin{bmatrix} 19 & 27 & 36 \\ 27 & 43 & 60 \\ 36 & 60 & 88 \end{bmatrix} \neq I$.

(6) Calculate all the eigenvalue λ and corresponding eigenvectors of A.

Calculating the eigenvalues is equivalent to finding all λ such that the eigen equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ have nontrivial solutions. By the inverse matrix theorem, we only need to find all λ such that $A - \lambda I$ is irreversible, so

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_{2,3} = -2 \end{cases}$$

Next, let $\boldsymbol{x} = (a\ b\ c)^T$, simplify the augmented matrix of $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ by row transformation.

When $\lambda = 4$, $\boldsymbol{x} = (1\ 1\ 2)^T$. When $\lambda = 4$, $\boldsymbol{x} = (1\ 1\ 0)^T$ or $\boldsymbol{x} = (-1\ 0\ 1)^T$

(7) Diagonalize the matrix A.

Matrix A can be diagonalized, if there is an invertible matrix P and diagonal matrix D, such that $A = PDP^{-1}$.

The diagonal matrix D is consists of eigenvalues.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The invertible matrix P is consists of eigen vectors.

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

Therefor,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

(8) Calculate the $\ell_{2,1}$ norm $\|A\|_{2,1}$ and the Frobenius norm (i.e. ℓ_2 norm) $\|A\|_F$.

$$\|A\|_{2,1} = \sum_{i=1}^{3} \sqrt{\sum_{j=1}^{3} a_{i,j}^2} = \sqrt{19} + \sqrt{43} + \sqrt{88}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} a_{i,j}^2} = 5\sqrt{6}$$

(9) Calculate the nuclear norm $\|A\|_*$ and the spectral norm $\|A\|_2$.

$$\det(A^T A - \lambda I) = \det \begin{bmatrix} 19-\lambda & 27 & 36 \\ 27 & 43-\lambda & 60 \\ 36 & 60 & 88-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 \approx 145.56 \\ \lambda_2 = 4 \\ \lambda_3 \approx 0.44 \end{cases}$$

For the nuclear norm $\|A\|_*$, according to its definition, that is, the sum of singular value, we have

$$\|A\|_* = \sum_{i=1}^{3} \sqrt{\mathrm{eig}(A^T A)} = \sqrt{145.56} + \sqrt{4} + \sqrt{0.44} \approx 14.72$$

For the spectral norm $\|A\|_2$, according to its definition, that is, the max singular value ,we have

$$\|A\|_2 = \max_{\lambda_i} \left(\sqrt{\mathrm{eig}(A^T A)} \right) = \sqrt{145.56} \approx 12.06$$

Problem 3: Linear Equations (48 points, 4 points/each)

Please give some proper steps to show how you get the answer.

Let $x = (x_1, x_2, x_3)^T$ and

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1 \\ x_1 - x_2 = -1 \\ -x_1 + 2x_2 + x_3 = 2 \end{cases}$$

Answer the following questions:

(1) Solve the linear equations 1 (6 points) (6 points)

By basic elimination, we can get simply

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

(2) Write it into matrix form(i.e. $Ax = b$) and we will use the same A and b in the following questions.

$$Ax = b$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

(3) The Rank of A is?

There are many ways to obtain $rank(A) = 3$, for example, we can show that $\det(A) \neq 0$(see below) or calculate the number of linearly independent rows by elementary transformation.

(4) Calculate A^{-1} and $det(A)$.

From $[AI] = [IA^{-1}]$, we have that $A^{-1} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}$

$$\det(A) = -2 + 6 - 3 - 2 = -1$$

(5) Use (4) to solve the linear equations

First of all, write the augmented matrix

$$A^* = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 2 \end{bmatrix}$$

Then, transform A into a reduced row echelon matrix.

$$A^* = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, we have

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

(6) Calculate the inner product and outer product of x and b.(i.e. $\langle \boldsymbol{x}, \boldsymbol{b} \rangle$ and $\boldsymbol{x} \times \boldsymbol{b}$)

$$\langle \boldsymbol{x}, \boldsymbol{b} \rangle = -1 + 2 = 1, \boldsymbol{x} \times \boldsymbol{b} = \boldsymbol{x}\boldsymbol{b}^T = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

(7) Calculate the ℓ_1, ℓ_2 and ℓ_∞ norm of b

$$\ell_1 = 1 + |-1| + 2 = 4$$

$$\ell_2 = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$\ell_\infty = \max b_i = 2$$

(8) Suppose $y = (y_1, y_2, y_3)^T$, calculate $y^T A y$, $\nabla_y \, y^T A y$.

$$y^T A y = (y_1, y_2, y_3)\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2y_1^2 - y_2^2 - y_3^2 + 3y_1y_2 + 2y_1y_3 + 2y_2y_3$$

$$\nabla_y\, y^T A y = Ay + y^T A = (A + A^T)y$$

(9) We add one linear equation $-x_1 + 2x_2 + x_3 = 2$ into linear equations above. Write it into matrix form(i.e. $A_1 x = b_1$)

$$A_1 x = b_1$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}$$

(10) The rank of A_1 is?

Transform A_1 into a row echelon matrix to find the number of linearly independent rows.

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & -3/2 \\ 0 & 2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $rank(A_1) = 3$.

(11) Could these linear equations $A_1 x = b_1$ be solved? State reasons.

It is solvable and the result is still

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$