1. 第二型曲面积分

直接计算　投影到坐标面上

化成第一型曲面积分

利用高斯公式

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练习三十六/一(1)

设$\partial\Omega$是立体$\Omega: x^2+y^2\le 1, 0\le z\le 4$表面的外侧,函数$f(y,z)$连续,则曲面积分$\oiint\limits_{\partial\Omega} f(y,z)dxdz$化为直角坐标系下的二重积分为______.

分析$: z=0$与$z=4$都是母线平行于y轴的柱面.

$y=\pm\sqrt{1-x^2}$ 其中$D_{zx}:\ |x|\le 1, 0\le z\le 4.$

$$\oiint\limits_{\partial\Omega} f(y,z)dzdx$$

$$=\iint\limits_{D_{zx}} f(\sqrt{1-x^2},z)dzdx-\iint\limits_{D_{zx}} f(-\sqrt{1-x^2},z)dzdx$$

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练习三十六/一(3)

设S为抛物面$z=\frac{1}{R}(R^2-x^2-y^2)$上$z\geq 0$部分的上侧,则$\iint\limits_S x^2y^2z^2dxdz=$______.

分析：$y=\pm\sqrt{R^2-x^2-Rz}$

$$\iint\limits_S x^2y^2z^2dxdz=\iint\limits_{D_{zx}} x^2(\sqrt{R^2-x^2-Rz})^2z^2dxdz$$

$$-\iint\limits_{D_{zx}} x^2(-\sqrt{R^2-x^2-Rz})^2z^2dxdz\ =0$$

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例： 计算 $\iint\limits_{\Sigma}(2x+z)dydz+zdxdy$,

其中 Σ 为有向曲面 $z=x^2+y^2\ (0\le z\le 1)$,

其法向量与 z 轴正向夹锐角.

解法一：投影到坐标面上 Σ： $z=x^2+y^2$

Σ_1： $x=\sqrt{z-y^2}$ Σ_2： $x=-\sqrt{z-y^2}$

$$原式=\iint\limits_{\Sigma_1}(2x+z)dydz+\iint\limits_{\Sigma_2}(2x+z)dydz+\iint\limits_{\Sigma}zdxdy$$

$$= -\iint_{D_{yz}} (2\sqrt{z-y^2} + z)dydz + \iint_{D_{yz}} (-2\sqrt{z-y^2} + z)dydz$$

$$+ \iint_{D_{xy}} (x^2 + y^2)dxdy$$

$$= -4\iint_{D_{yz}} \sqrt{z-y^2}\,dydz + \iint_{D_{xy}} (x^2+y^2)dxdy$$

$$= -4\int_{-1}^{1} dy \int_{y^2}^{1} \sqrt{z-y^2}\,dz + \int_0^{2\pi} d\theta \int_0^1 \rho^2 \cdot \rho d\rho$$

$$= -\frac{\pi}{2}$$

解法二：化成第一型曲面积分

$\vec{n} = \pm\{2x, 2y, -1\}$　　　取负号

$$\vec{n^0} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\{-2x, -2y, 1\}$$

$$原式 = \iint_{\Sigma}\{2x + z, 0, z\}\cdot\vec{n^0}\, dS = \iint_{\Sigma}\frac{-4x^2 - 2xz + z}{\sqrt{4x^2 + 4y^2 + 1}}\, dS$$

$$= \iint_{D_{xy}}\frac{-4x^2 - 2x(x^2 + y^2) + (x^2 + y^2)}{\sqrt{4x^2 + 4y^2 + 1}}\cdot$$

$$\cdot\sqrt{(2x)^2 + (2y)^2 + 1}\, dxdy$$

$$= \iint_{D_{xy}} (-3x^2 \underline{- 2x^3 - 2xy^2} + y^2) dxdy$$

(利用对称性)

$$= \iint_{D_{xy}} (-3x^2 + y^2) dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (-3\rho^2 \cos^2 \theta + \rho^2 \sin^2 \theta) \cdot \rho d\rho$$

$$= -\frac{\pi}{2}$$

解法三：利用高斯公式

设 Σ_1：$z=1 \quad (x^2+y^2\le 1)$ 下侧

$$原式=\oiint\limits_{\Sigma+\Sigma_1}(2x+z)dydz+zdxdy-\iint\limits_{\Sigma_1}dxdy$$

$$=-\iiint\limits_{\Omega}(2+1)dv+\iint\limits_{D_{xy}}dxdy$$

$$=-3\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_{\rho^2}^1 dz+\pi$$

$$=-\frac{3\pi}{2}+\pi=-\frac{\pi}{2}$$

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练习三十六/四：

计算 $\oiint\limits_{\Sigma}\frac{xdydz+z^2dxdy}{x^2+y^2+z^2}$,其中$\Sigma$是由

曲面 $x^2+y^2=R^2$ 及两平面 $z=\pm R\ (R>0)$

所围成立体表面的外侧.

解： $\Sigma_1: z=R$， $\Sigma_2: z=-R$， $\Sigma_3: x^2+y^2=R^2$

$$\iint\limits_{\Sigma_1}\frac{xdydz}{x^2+y^2+z^2}=\iint\limits_{\Sigma_2}\frac{xdydz}{x^2+y^2+z^2}=0$$

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$$\iint_{\Sigma_3} \frac{z^2 dxdy}{x^2 + y^2 + z^2} = 0$$

$$\iint_{\Sigma_1 + \Sigma_2} \frac{z^2 dxdy}{x^2 + y^2 + z^2}$$

$$= \iint_{D_{xy}} \frac{R^2 dxdy}{x^2 + y^2 + R^2} - \iint_{D_{xy}} \frac{(-R)^2 dxdy}{x^2 + y^2 + (-R)^2}$$

$$= 0$$

$$\text{原式} = \iint_{\Sigma_3} \frac{x\,dydz}{x^2+y^2+z^2} \qquad (x = \pm\sqrt{R^2-y^2})$$

$$= \iint_{D_{yz}} \frac{\sqrt{R^2-y^2}}{R^2+z^2} dydz - \iint_{D_{yz}} \frac{-\sqrt{R^2-y^2}}{R^2+z^2} dydz$$

$$= 2\int_{-R}^{R} \sqrt{R^2-y^2}\, dy \int_{-R}^{R} \frac{1}{R^2+z^2} dz$$

$$= \frac{1}{2}\pi^2 R$$

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练习三十六/五

计算曲面积分

$$\iint_S \{x+f(x,y,z), y+2f(x,y,z), z+f(x,y,z)\}\cdot d\vec{S},$$

其中$f\in C^1$, S为平面$x-y+z=1$在第四卦限部分的上侧.

解:记$\vec{F}(x,y,z)=(x+f)\vec{i}+(y+2f)\vec{j}+(z+f)\vec{k}$,

$$\vec{n}=\{1,-1,1\},\qquad \vec{n^0}=\{\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\}$$

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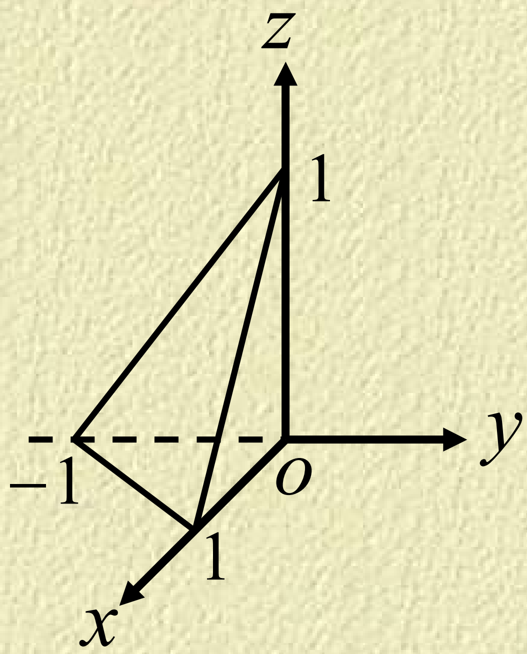
$$\text{原式} = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot \overrightarrow{n^0} dS$$



$$= \frac{1}{\sqrt{3}} \iint_S (x - y + z) dS$$

$$= \frac{1}{\sqrt{3}} \iint_S dS$$

$$= \frac{1}{\sqrt{3}} \iint_{D_{xy}} \sqrt{1 + (-1)^2 + 1^2}\, dxdy = \frac{1}{2}$$

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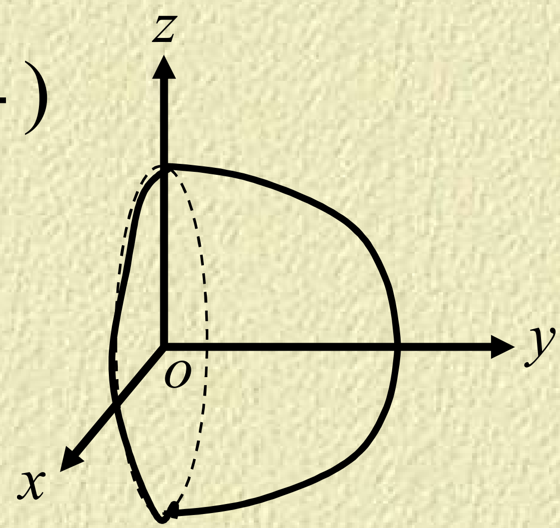
练习三十六/六

计算曲面积分$\iint\limits_{S} yzdydz + (x^2 + z^2)ydzdx + xydxdy$,

其中S为曲面$y = 4 - x^2 - z^2$上$y \geq 0$部分的右侧.

解：设 $S_1 : y = 0 \quad (x^2 + z^2 \leq 4)$

左侧



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$$原式 = \oiint_{S+S_1} - \iint_{S_1}$$

$$= \iiint_{\Omega} (0 + x^2 + z^2 + 0)\,dv - \iint_{S_1} 0\,dzdx$$

$$= \int_0^{2\pi} d\theta \int_0^2 \rho\,d\rho \int_0^{4-\rho^2} \rho^2\,dy$$

$$= 2\pi \int_0^2 \rho^3 (4 - \rho^2)\,d\rho$$

$$= \frac{32\pi}{3}$$

练习三十六/十二

设$\partial\Omega$为单连通区域Ω的边界曲面,$\{\cos\alpha,\cos\beta,\cos\gamma\}$为$\partial\Omega$单位外法向,函数$P,Q,R$在$\Omega$上有二阶连续偏导数,试证明

$$\oiint_{\partial\Omega}[(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\cos\alpha+$$

$$+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\cos\beta+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\cos\gamma]\,dS=0.$$

化成第二型曲面积分,再利用高斯公式.

证：左式 $=\oiint\limits_{\partial\Omega}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dydz+$

$$+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dzdx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$

$$=\iiint\limits_{\Omega}[\frac{\partial}{\partial x}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})+$$

$$+\frac{\partial}{\partial z}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})]dv$$

$$=\iiint\limits_{\Omega}0dv=0$$

练习三十六/十五

试证明$\displaystyle\oiint\limits_{\partial\Omega} u\frac{\partial v}{\partial \vec{n}}dS=\iiint\limits_{\Omega}u\Delta v dv+\iiint\limits_{\Omega}\nabla u\cdot\nabla v dv$,其中

$u\in C^1,v\in C^2$,光滑曲面$\partial\Omega$是立体Ω表面的外侧,

$\dfrac{\partial v}{\partial \vec{n}}$是函数$v$在$\partial\Omega$上沿$\partial\Omega$外法向$\vec{n}$的方向导数.

证: $\displaystyle\oiint\limits_{\partial\Omega} u\frac{\partial v}{\partial \vec{n}}dS=\oiint\limits_{\partial\Omega} u\nabla v\cdot\overrightarrow{n^0}\,dS$

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$$= \oiint_{\partial\Omega} u \nabla v \cdot d\vec{S}$$

$$= \oiint_{\partial\Omega} (u \frac{\partial v}{\partial x} \vec{i} + u \frac{\partial v}{\partial y} \vec{j} + u \frac{\partial v}{\partial z} \vec{k}) \cdot d\vec{S}$$

(利用高斯公式)

$$= \iiint_{\Omega} [\frac{\partial}{\partial x}(u \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(u \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(u \frac{\partial v}{\partial z})] dv$$

$$= \iiint_{\Omega} (\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} +$$

$$+ u \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial z} + u \frac{\partial^2 v}{\partial z^2}) \, dv$$

$$= \iiint_{\Omega} [\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\} \cdot \{\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\} +$$

$$+ u(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2})] \, dv$$

$$= \iiint_{\Omega} \nabla u \cdot \nabla v \, dv + \iiint_{\Omega} u \Delta v \, dv$$

练习三十七/六

求向量场$\vec{A}=\{z-2y, x-2z, y-2x\}$沿曲线$C$的环流量，其中$C$为$\begin{cases} 4x^2+4y^2+z^2=32+8xy \\ 2x+2y+z=0 \end{cases}$，从$z$轴正向看来是反时针方向.

解：$\Phi=\int_C \vec{A}\cdot d\vec{s}$

$$=\int_C (z-2y)\,dx+(x-2z)dy+(y-2x)dz$$

利用斯托克斯公式

$$\Phi = \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z-2y & x-2z & y-2x \end{vmatrix} \mathrm{d}S$$

其中 Σ: $2x+2y+z=0$ ，上侧

$$\vec{n}=\{2,2,1\} \qquad \vec{n^0}=\left\{\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\}$$

$$= 5\iint_{\Sigma} dS = 5\iint_{D_{xy}} 3dxdy$$

$$\Phi = 5\iint_{\Sigma} dS = 5\iint_{D_{xy}} 3dxdy$$

将 $z = -2x - 2y$ 代入 $4x^2 + 4y^2 + z^2 = 32 + 8xy$

得 $x^2 + y^2 = 4$ $\qquad \Phi = 15 \cdot 4\pi = 60\pi$

也可利用参数方程直接计算第二型曲线积分

$$x = 2\cos t, y = 2\sin t, z = -4\cos t - 4\sin t$$

逆时针 $t : 0 \to 2\pi$