§2.1 参数曲线

1. 将一个半径为 r 的圆盘在 XY 平面内沿 X 轴作无滑动的滚动, 写出圆盘上一点的轨迹方程（此曲线称为旋轮线, or 摆线）.

解：设初始位置时, 圆盘中心 $\mathrm{C}(0,\mathrm{r})$, 考虑点 $\mathrm{M}(0,0)$ 的运动轨迹. 设 CM 转过的弧度为 t , C 与 M 在 X 轴上的投影为 C' 、 M' , M 在 CC' 上的投影为 N , 则若设 $\mathrm{M}=(x(t), y(t))$, 有

$x(t)=|\mathrm{OC}'|-|\mathrm{M}'\mathrm{C}'|=\overset{\frown}{\mathrm{MC}'}-|\mathrm{M}'\mathrm{C}'|=rt-r\sin t$

$y(t)=|\mathrm{CC}'|-|\mathrm{CN}|=r-r\cos t$

所以, $\mathrm{M}=(rt-r\sin t, r-r\cos t)$.

2. 证明：曲线的切线与某个确定的方向成定角.

证明： $\vec{r}(t)=(3,6t,6t^2)$, 单位切向量 $\vec{r}'_*(t)=\frac{1}{1+2t^2}(1,2t,2t^2)$, 若 $\vec{r}'_*(t)$ 与单位常向量 $\vec{C}=(c_1,c_2,c_3)$ 成定角, 则

$$\cos\angle(\vec{r}'_*(t),\vec{C})=\vec{r}'_*(t)\cdot\vec{C}=\frac{1}{1+2t^2}(c_1+2c_2t+2c_3t^2)\equiv a \text{ , } a \text{ 为常数}$$

$$c_1{}^2+c_2{}^2+c_3{}^2=1$$

则 $c_1=c_3=a=\frac{\sqrt{2}}{2}, c_2=0$.

所以, $\vec{r}(t)$ 的切线与 $(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})$ 的方向始终成定角 $\frac{\pi}{4}$.

3. 设平面曲线 c 与同一平面的一条曲线 l 相交于正则点 P , 且落在直线 l 的一侧. 证明: l 是曲线 c 在点 P 的切线.

证明：设曲线 c ： $\vec{r}=\vec{r}(t)$, 点 P 对应 $t=t_0$.

在 c 与 l 所在平面内, 作 $l_1//l$, 记 $l_1\cap c=\left\{\vec{r}=\vec{r}(t)\,|\,t=t_0+\vartriangle t_{11}, t_0+\vartriangle t_{12}\right\}$. 再作 $l_i//l$, s. t. $dist(l_{i-1},l_i)=dist(l,l_i)$, 记 $l_i\cap c=\left\{\vec{r}=\vec{r}(t)\,|\,t=t_0+\vartriangle t_{i1}, t_0+\vartriangle t_{i2}\right\}$, $i=2,3,4,\cdots$

这样有, $l//l_1//l_2//\cdots//l_n//\cdots$, $l_n\to l$.

$\frac{\vec{r}(t_0+\vartriangle t_{n2})-\vec{r}(t_0-\vartriangle t_{n1})}{\vartriangle t_{n1}+\vartriangle t_{n2}}//l$.

由 P 为正则点, 可知 $\vec{r}'(t_0)$ 存在, $\dfrac{\vec{r}(t_0+\triangle t_{n2})-\vec{r}(t_0-\triangle t_{n1})}{\triangle t_{n1}+\triangle t_{n2}}=\vec{r}'(t_0)$

$\therefore l//\vec{r}'(t_0)$, 即 l 是 c 在点 P 的切线.

4. 证明：若曲线 $\vec{r}(t)$ 在点 t_0 有 $x'(t_0)\neq 0$, 则该曲线在 t_0 的一个邻域内可表示成 $y=f(x)$, $z=g(x)$.

证明：因 $x'(t_0)\neq 0$, 不妨设 $x'(t_0)>0$, 则存在 t_0 的一个邻域 $\cup(t_0)$, 使得 $x=x(t)$ 在 $\cup(t_0)$ 内连续且严格递增. 从而在 $\cup(t_0)$ 内存在 $x=x(t)$ 的反函数, 设为 $t=h(x)$. 所以, 在 $\cup(t_0)$ 内,

$$y=y(t)=y(h(x))\triangleq f(x),\ z=z(t)=z(h(x))\triangleq g(x).$$

即曲线在 t_0 的一个邻域内可表示成 $y=f(x)$, $z=g(x)$.

5. 求曲线 $\begin{cases}x^2+y^2+z^2=1\\ x^2+y^2=x\end{cases}$, $z\geq 0$ 的参数方程.

解：$\begin{cases}x^2+y^2+z^2=1\\ x^2+y^2=x\end{cases}\Rightarrow\begin{cases}x^2+y^2+z^2=1\\ (x-\dfrac{1}{2})^2+y^2=\dfrac{1}{4}\end{cases}$

令 $y=\dfrac{1}{2}\sin t$, 则 $x=\dfrac{1}{2}+\dfrac{1}{2}\cos t$, $z=\sqrt{\dfrac{1}{2}-\dfrac{1}{2}\cos t}$, $0\leq t<2\pi$.

所以, 该曲线的参数方程为 $\vec{r}(t)=(\dfrac{1}{2}+\dfrac{1}{2}\cos t,\dfrac{1}{2}\sin t,\sqrt{\dfrac{1}{2}-\dfrac{1}{2}\cos t})$.

§2.2 曲线的弧长

1.设下面的常数 $a>0$,求曲线在指定范围内的弧长：

（1）$\vec{r}(t)=(acht, asht, at)$, $0\le t\le b$.

（2）悬链线 $y=ach\dfrac{x}{a}$, $[0,x]$.

（3）曳物线 $\vec{r}(t)=(a\cos t, a\ln(\sec t+\tan t)-a\sin t)$, $[0,t]$.

解：（1）$\vec{r}'(t)=(asht, acht, a), |\vec{r}'(t)|=\sqrt{a^2(sh^2t+ch^2t+1)}=\sqrt{2}acht.$

$$\therefore s=\int_0^b|\vec{r}'(t)|dt=\sqrt{2}a\int_0^b chtdt=\sqrt{2}ashb.$$

（2）令 $x=t$, 则 $y=ach\dfrac{t}{a}, \vec{r}(t)=(t, ach\dfrac{t}{a}), \vec{r}'(t)=(1, sh\dfrac{t}{a}).$

$$\therefore s=\int_0^x|\vec{r}'(t)|dt=\int_0^x\sqrt{1+sh^2\frac{t}{a}}dt=ash\frac{x}{a}.$$

（3）$\vec{r}'(t)=(-a\sin t, a(\dfrac{1}{\cos t}-\cos t))$

$$\therefore s=\int_0^t|\vec{r}'(t)|dt=\int_0^t\sqrt{a^2\sin^2 t+a^2(\frac{1}{\cos^2 t}+\cos^2 t-2)}dt=-a\ln\cos t.$$

2. 求下列曲线的单位切向量场 $\dfrac{d\vec{r}}{ds}$：

（1）圆螺旋线 $\vec{r}(t)=(a\cos t, a\sin t, bt), a>0.$

（2）$\vec{r}(t)=(\cos^3 t, \sin^3 t, \cos 2t).$

解：（1）$\vec{r}'(t)=(-a\sin t, a\cos t, b)$

$$\therefore \frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}=\frac{d\vec{r}}{dt}\cdot\frac{1}{|\vec{r}'(t)|}=\frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b).$$

（2）$\vec{r}'(t)=(-3\cos^2 t\sin t, 3\sin^2 t\cos t, -2\sin 2t)$

$$\therefore \frac{d\vec{r}}{ds}=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}=\frac{1}{|5\sin t\cos t|}(-3\cos^2 t\sin t, 3\sin^2 t\cos t, -2\sin 2t).$$

3.设曲线 c 是下面两个曲面的交线：$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1, x=ach\dfrac{z}{a}, a,b>0$. 求 c 从点 $(a,0,0)$ 到点 (x,y,z) 的弧长.

解：：令 $z=t$,则 $x=ach\dfrac{t}{a}, y=bsh\dfrac{t}{a}$

$\therefore c$ 的参数方程为 $\vec{r}(t)=(ach\dfrac{t}{a}, bsh\dfrac{t}{a}, t)$

$\vec{r}'(t)=(sh\dfrac{t}{a}, \dfrac{b}{a}ch\dfrac{t}{a}, 1)$

$\therefore s=\int_0^z|\vec{r}'(t)|dt=\int_0^z\dfrac{\sqrt{a^2+b^2}}{a}ch\dfrac{t}{a}dt=\sqrt{a^2+b^2}\cdot sh\dfrac{z}{a}$

4.求曲线 $\vec{r}=\vec{r}(t)$,使得 $\vec{r}(0)=(1,0,-5), \vec{r}'(t)=(t^2,t,e^t)$.

解：由 $\vec{r}'(t)=(t^2,t,e^t)$ 可得 $\vec{r}(t)=(\dfrac{1}{3}t^3,\dfrac{1}{2}t^2,e^t)+\vec{c}$,$\vec{c}$ 为常向量.

当 $t=0$, $\vec{r}(0)=(0,0,1)+\vec{c}=(1,0,-5)$

$\therefore \vec{c}=(1,0,-6)$.

$\therefore \vec{r}(t)=(\dfrac{1}{3}t^3+1,\dfrac{1}{2}t^2,e^t-6)$.

§2.3 曲线的曲率和 Frenet 标架

1.求曲线的曲率：

（1）$\vec{r}=\left(at, a\sqrt{2}\ln t, \dfrac{a}{t}\right).(a>0)$

（2）$\vec{r}=\left(3t-t^3, 3t^2, 3t+t^3\right).$

（3）$\vec{r}=\left(a\left(t-\sin t\right), a\left(1-\cos t\right), bt\right).(a>0)$

（4）$\vec{r}=\left(\cos^3 t, \sin^3 t, \cos 2t\right).$

解：（1）$\vec{r}'(t)=(a, \dfrac{\sqrt{2}a}{t}, -\dfrac{a}{t^2}), \vec{r}''(t)=\left(0, -\dfrac{\sqrt{2}a}{t^2}, \dfrac{2a}{t^3}\right),$

$$\vec{r}'(t)\times\vec{r}''(t)=\left(\frac{\sqrt{2}a^4}{t^4}, -\frac{2a^2}{t^3}, -\frac{\sqrt{2}a^2}{t^2}\right).$$

$$\therefore \kappa=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}=\frac{\sqrt{2}t^2}{a\left(t^2+1\right)^2}.$$

（2）$\vec{r}'(t)=(3-3t^2, 6t, 3+3t^2), \vec{r}''(t)=\left(-6t, 6, 6t\right),$

$$\vec{r}'(t)\times\vec{r}''(t)=8\left(t^2-1, -2t, t^2+1\right).$$

$$\therefore \kappa=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}=\frac{1}{3\left(t^2+1\right)^2}.$$

（3）$\vec{r}'(t)=(a\left(1-\cos t\right), a\sin t, b), \vec{r}''(t)=\left(a\sin t, a\cos t, 0\right),$

$$\vec{r}'(t)\times\vec{r}''(t)=\left(-ab\cos t, ab\sin t, a^2\left(\cos t-1\right)\right).$$

$$\therefore \kappa=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}=\frac{a\sqrt{b^2+4a^2\sin^4\dfrac{t}{2}}}{\left(b^2+4a^2\sin^2\dfrac{t}{2}\right)^{\frac{3}{2}}}.$$

（4）$\vec{r}'(t)=(-3\cos^2 t\sin t, 3\sin^2 t\cos t, -2\sin 2t),$

$$\vec{r}''(t)=\left(6\cos t\sin^2 t-3\cos^3 t, 6\sin t\cos^2 t-3\sin^3 t, -4\cos 2t\right),$$

$$\vec{r}'(t)\times\vec{r}''(t)=\left(12\sin^2 t\cos^3 t,-12\sin^3 t\cos^2 t,-9\sin^2 t\cos^2 t\right).$$

$$\therefore \kappa=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}=\frac{3}{25|\sin t\cos t|}.$$

2.求曲线的密切平面方程：

（1）$\vec{r}(t)=\left(a\cos t,a\sin t,bt\right),a^2+b^2\neq 0.$

（2）$\vec{r}(t)=\left(a\cos t,b\sin t,e^t\right)$,在 $t=0$ 处,其中 $ab\neq 0$.

解：（1）$\vec{r}'(t)=\left(-a\sin t,a\cos t,b\right),\vec{r}''(t)=\left(-a\cos t,-a\sin t,0\right),$

$$\vec{r}'(t)\times\vec{r}''(t)=\left(ab\sin t,-ab\cos t,a^2\right).$$

密切平面 $\left(\overrightarrow{X}-\vec{r}\right)\cdot\vec{\gamma}=0$,即 $\left(\overrightarrow{X}-\vec{r}\right)\cdot\left(\vec{r}'(t)\times\vec{r}''(t)\right)=0$,

亦即 $b\sin t\cdot x-b\cos t\cdot y+az-abt=0$.

（2）$\vec{r}'(t)=\left(-a\sin t,b\cos t,e^t\right),\vec{r}''(t)=\left(-a\cos t,-b\sin t,e^t\right),$

$$\vec{r}'(t)\times\vec{r}''(t)=\left(be^t\left(\cos t+\sin t\right),ae^t\left(\sin t-\cos t\right),ab\right).$$

密切平面 $\left(\overrightarrow{X}-\vec{r}\right)\cdot\vec{\gamma}=0$,即 $\left(\overrightarrow{X}-\vec{r}\right)\cdot\left(\vec{r}'(t)\times\vec{r}''(t)\right)=0$,

当 $t=0$ 时,$\vec{r}=\left(a,0,1\right),\vec{r}'\times\vec{r}''=\left(b,-a,ab\right).$

此时,密切平面为 $\dfrac{x}{a}-\dfrac{y}{b}+z=2$.

3.求曲线 $\begin{cases} x+shx=y+\sin y \\ z+e^z=\left(x+1\right)+\ln\left(x+1\right)\end{cases}$,在 $\left(0,0,0\right)$ 处的曲率和 Frenet 标架.

解：设曲线的参数方程为：$x=x\left(s\right),y=y\left(s\right),z=z\left(s\right)$,其中 s 是弧长参数,且 $s=0$ 对应于点 $\left(0,0,0\right)$.因此函数 $x\left(s\right),y\left(s\right),z\left(s\right)$ 满足下列方程组：

$$\begin{cases} x+shx=y+\sin y & \cdots\cdots[1] \\ z+e^z=(x+1)+\ln\left(x+1\right) & \cdots\cdots[2] \\ \dot{x}^2+\dot{y}^2+\dot{z}^2=1 & \cdots\cdots[3]\end{cases}$$

$[1],[2]$式关于 s 求导得到,

$$\begin{cases} \dot{x}+chx\cdot\dot{x}=\dot{y}+\cos y\cdot y & \cdots\cdots[4] \\ \dot{z}+e^{z}\cdot\dot{z}=\dot{x}+\dfrac{\dot{x}}{x+1} & \cdots\cdots[5] \end{cases}$$

令 $s=0$,可得到 $\dot{x}(0)=\dot{y}(0)=\dot{z}(0)=\dfrac{1}{\sqrt{3}}$.

$\therefore \vec{\alpha}(0)=\dot{\vec{r}}(0)=(\dfrac{\sqrt{3}}{3},\dfrac{\sqrt{3}}{3},\dfrac{\sqrt{3}}{3})$.

$[3],[4],[5]$式再关于 s 求导,得

$$\begin{cases} \dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}=0 \\ \ddot{x}+shx\cdot\dot{x}^{2}+chx\cdot\ddot{x}=\ddot{y}-\sin y\cdot\dot{y}^{2}+\cos y\cdot\ddot{y} & \cdots\cdots(*) \\ \ddot{z}+e^{z}\dot{z}^{2}+e^{z}\ddot{z}=\ddot{x}+\dfrac{\ddot{x}}{x+1}-\dfrac{\dot{x}^{2}}{(x+1)^{2}} \end{cases}$$

令 $s=0$,得到 $\ddot{x}(0)=\ddot{y}(0)=\dfrac{1}{9},\ddot{z}(0)=-\dfrac{2}{9},\ddot{\vec{r}}(0)=\left(\dfrac{1}{9},\dfrac{1}{9},-\dfrac{2}{9}\right)$.

$\therefore \kappa=|\ddot{\vec{r}}(0)|=\dfrac{\sqrt{6}}{9}$,

$$\vec{\beta}(0)=\frac{\ddot{\vec{r}}(0)}{|\ddot{\vec{r}}(0)|}=\left(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},-\frac{\sqrt{6}}{3}\right),\vec{\gamma}(0)=\vec{\alpha}(0)\times\vec{\beta}(0)=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right).$$

4.求曲线 $\begin{cases} x^{2}+y^{2}+z^{2}=9 \\ x^{2}-z^{2}=3 \end{cases}$ 在 $(2,2,1)$ 处的曲率和密切平面方程.

解:　设曲线的参数方程是 $x=x(s),y=y(s),z=z(s)$,其中 s 是弧长参数,且 $s=0$ 对应于点 $(2,2,1)$.因此函数 $x(s),y(s),z(s)$ 满足:

$$\begin{cases} x^{2}+y^{2}+z^{2}=9 & \cdots\cdots[1] \\ x^{2}-z^{2}=3 & \cdots\cdots[2] \\ \dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}=1 & \cdots\cdots[3] \end{cases}$$

$[1],[2]$式关于 s 求导,得

$$\begin{cases} x\dot{x}+y\dot{y}+z\dot{z}=0 & \cdots\cdots[4] \\ x\dot{x}-z\dot{z}=0 & \cdots\cdots[5] \end{cases}$$

令 $s=0$,得到 $\dot{x}(0)=\frac{1}{3},\dot{y}(0)=-\frac{2}{3},\dot{z}(0)=\frac{2}{3}$.

$\therefore \vec{\alpha}(0)=\dot{\vec{r}}(0)=\left(\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$.

$[3],[4],[5]$式再求导,得

$$\begin{cases} \dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}=0 \\ \dot{x}^2+x\ddot{x}+\dot{y}^2+y\ddot{y}+\dot{z}^2+z\ddot{z}=0 \\ \dot{x}^2+x\ddot{x}-\dot{z}^2-z\ddot{z}=0 \end{cases}$$

令 $s=0$,得到 $\ddot{x}(0)=0,\ddot{y}(0)=-\frac{1}{3},\ddot{z}(0)=-\frac{1}{3},\therefore \ddot{\vec{r}}(0)=\left(0,-\frac{1}{3},-\frac{1}{3}\right)$

$\therefore \kappa=|\ddot{\vec{r}}(0)|=\frac{\sqrt{2}}{3},\vec{\beta}(0)=\frac{\ddot{\vec{r}}(0)}{|\ddot{\vec{r}}(0)|}=\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$,

$\vec{\gamma}(0)=\vec{\alpha}(0)\times\vec{\beta}(0)=\left(\frac{2\sqrt{2}}{3},\frac{\sqrt{2}}{6},-\frac{\sqrt{2}}{6}\right)$.

密切平面：$\frac{2\sqrt{2}}{3}(x-2)+\frac{\sqrt{2}}{6}(y-2)-\frac{\sqrt{2}}{6}(z-1)=0$,即 $4x+y-z-9=0$.

5.设曲线的方程 $\vec{r}(t)=\begin{cases} \left(e^{-1/t^2},t,0\right),t<0 \\ (0,0,0),t=0 \\ \left(0,t,e^{-1/t^2}\right),t>0 \end{cases}$,

证明：这是一条正则曲线,且在 $t=0$ 处的曲率为 0 $t\neq 0$ $t\to\pm 0$

证明：$t<0,\vec{r}'(t)=\left(\frac{2}{t^3}e^{-1/t^2},1,0\right)$

$t>0,\vec{r}'(t)=\left(0,1,\frac{2}{t^3}e^{-1/t^2}\right)$

$t=0, \vec{r}'_{+}(0)=(0,1,0), \vec{r}'_{-}(0)=(0,1,0), \therefore \vec{r}'(0)=(0,1,0).$

$\therefore \forall t, \vec{r}'(t)\neq 0. \therefore$ 这是一条正则曲线.

$$t<0, \vec{r}''(t)=\left(\left(\frac{4}{t^6}-\frac{6}{t^4}\right)e^{-1/t^2},0,0\right)$$

$$t>0, \vec{r}''(t)=\left(0,0,\left(\frac{4}{t^6}-\frac{6}{t^4}\right)e^{-1/t^2}\right)$$

$t=0, \vec{r}''_{+}(0)=(0,0,0), \vec{r}''_{-}(0)=(0,0,0), \therefore \vec{r}''(0)=(0,0,0).$

$\therefore$ 曲线在 $t=0$ 处的曲率为 0 .

$$\therefore t<0 \text{ 时}, \vec{\alpha}(t)=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}=\frac{1}{\sqrt{1+\frac{4}{t^6}e^{-1/t^2}}}\left(\frac{2}{t^3}e^{-1/t^2},1,0\right)$$

$$\vec{\gamma}(t)=\frac{\vec{r}'(t)\times\vec{r}''(t)}{|\vec{r}'(t)\times\vec{r}''(t)|}=\operatorname{sgn}\left(4-6t^2\right)(0,0,-1)$$

$$\vec{\beta}(t)=\vec{\gamma}(t)\times\vec{\alpha}(t)=\frac{\operatorname{sgn}\left(4-6t^2\right)}{\sqrt{1+\frac{4}{t^6}e^{-1/t^2}}}\left(1,-\frac{2}{t^3}e^{-1/t^2},0\right)$$

$$t>0 \text{ 时}, \vec{\alpha}(t)=\frac{1}{\sqrt{1+\frac{4}{t^6}e^{-1/t^2}}}\left(0,1,\frac{2}{t^3}e^{-1/t^2}\right)$$

$$\vec{\gamma}(t)=\operatorname{sgn}\left(4-6t^2\right)(1,0,0)$$

$$\vec{\beta}(t)=\frac{\operatorname{sgn}\left(4-6t^2\right)}{\sqrt{1+\frac{4}{t^6}e^{-1/t^2}}}\left(0,-\frac{1}{t^3}e^{-1/t^2},1\right)$$

$t\to+0, \vec{\alpha}=(0,1,0), \vec{\beta}=(0,0,1), \vec{\gamma}=(1,0,0)$

$t\to-0, \vec{\alpha}=(0,1,0), \vec{\beta}=(1,0,0), \vec{\gamma}=(0,0,-1)$

§2.4 挠率和 Frenet 公式

1. 计算§3 习题 1 中各曲线的挠率.

（1）$\vec{r}=\left(at, a\sqrt{2}\ln t, \frac{a}{t}\right).(a>0)$

（2）$\vec{r}=\left(3t-t^3, 3t^2, 3t+t^3\right).$

（3）$\vec{r}=\left(a(t-\sin t), a(1-\cos t), bt\right).(a>0)$

（4）$\vec{r}=\left(\cos^3 t, \sin^3 t, \cos 2t\right).$

解：（1）$\vec{r}'(t)=(a, \frac{\sqrt{2}a}{t}, -\frac{a}{t^2}), \vec{r}''(t)=\left(0, -\frac{\sqrt{2}a}{t^2}, \frac{2a}{t^3}\right),$

$\vec{r}'''(t)=\left(0, \frac{2\sqrt{2}a}{t^3}, -\frac{6a}{t^4}\right)$

$$\therefore \tau=\frac{\left(\vec{r}'(t), \vec{r}''(t), \vec{r}'''(t)\right)}{|\vec{r}'(t)\times\vec{r}''(t)|^2}=\frac{\sqrt{2}t^2}{a\left(t^2+1\right)^2}$$

（2）$\vec{r}'(t)=(3-3t^2, 6t, 3+3t^2), \vec{r}''(t)=\left(-6t, 6, 6t\right),$

$\vec{r}'''(t)=\left(-6, 0, 6\right)$

$$\therefore \tau=\frac{\left(\vec{r}'(t), \vec{r}''(t), \vec{r}'''(t)\right)}{|\vec{r}'(t)\times\vec{r}''(t)|^2}=\frac{1}{3\left(t^2+1\right)^2}$$

（3）$\vec{r}'(t)=(a(1-\cos t), a\sin t, b), \vec{r}''(t)=\left(a\sin t, a\cos t, 0\right),$

$\vec{r}'''(t)=\left(a\cos t, -a\sin t, 0\right)$

$$\therefore \tau=\frac{\left(\vec{r}'(t), \vec{r}''(t), \vec{r}'''(t)\right)}{|\vec{r}'(t)\times\vec{r}''(t)|^2}=\frac{-b}{b^2+a^2\left(\cos t-1\right)^2}$$

（4）$\vec{r}'(t)=(-3\cos^2 t\sin t, 3\sin^2 t\cos t, -2\sin 2t),$

$$\vec{r}'''(t)=\left(-6\sin^3 t+21\cos^2 t\sin t,6\cos^3 t-21\sin^2 t\cos t,8\sin 2t\right)$$

$$\therefore \tau=\frac{\left(\vec{r}'(t),\vec{r}''(t),\vec{r}'''(t)\right)}{|\vec{r}'(t)\times\vec{r}''(t)|^2}=\frac{8}{25\sin 2t}$$

$$\vec{r}''(t)=\left(6\cos t\sin^2 t-3\cos^3 t,6\sin t\cos^2 t-3\sin^3 t,-4\cos 2t\right),$$

2.求§3 习题 3 中的曲线在$\left(0,0,0\right)$处的挠率.

解：曲线$\begin{cases} x+shx=y+\sin y \\ z+e^z=\left(x+1\right)+\ln\left(x+1\right)\end{cases}$

$$\dot{\vec{r}}(0)=(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}),\ddot{\vec{r}}\left(0\right)=\left(\frac{1}{9},\frac{1}{9},-\frac{2}{9}\right).$$

原题中的方程组$\left(*\right)$再求导,得

$$\begin{cases} \dot{x}^2+\dot{x}\ddot{x}+\dot{y}^2+\dot{y}\ddot{y}+\dot{z}^2+\dot{z}\ddot{z}=0 \\ \dddot{x}+chx\cdot\dot{x}^3+3shx\cdot\dot{x}\ddot{x}+chx\cdot\dddot{x}=\dddot{y}-\cos y\cdot\dot{y}^3-3\sin y\cdot\dot{y}\ddot{y}+\cos y\cdot\dddot{y} \\ \dddot{z}+e^z\dot{z}^3+3e^z\dot{z}\ddot{z}+e^z\dddot{z}=\dddot{x}+\dfrac{\ddot{x}\left(x+1\right)-\dot{x}\ddot{x}}{\left(x+1\right)^2}-\dfrac{2\dot{x}\ddot{x}\left(x+1\right)^2-2\dot{x}^3\left(x+1\right)}{\left(x+1\right)^4}\end{cases}$$

令$s=0$,得到$\dddot{\vec{r}}\left(0\right)=\left(-\frac{8}{81}\sqrt{3},\frac{\sqrt{3}}{81},\frac{\sqrt{3}}{81}\right)$

$$\therefore \tau\left(0\right)=\frac{\left(\dot{\vec{r}}\left(0\right),\ddot{\vec{r}}\left(0\right),\dddot{\vec{r}}\left(0\right)\right)}{|\ddot{\vec{r}}\left(0\right)|^2}=\frac{1}{2}$$

3.设曲线$\vec{r}=\vec{r}(s)$的挠率是非零常数,求曲线$\tilde{\vec{r}}=\frac{1}{\tau}\vec{\beta}(s)-\int\vec{\gamma}(s)ds$的曲率和挠率.

解：$\tilde{\vec{r}}'=\frac{1}{\tau}\dot{\vec{\beta}}-\vec{\gamma}=-\frac{\kappa}{\tau}\vec{\alpha},\tilde{\vec{r}}''=-\frac{\dot{\kappa}}{\tau}\vec{\alpha}-\frac{\kappa}{\tau}\dot{\vec{\alpha}}=-\frac{1}{\tau}\left(\dot{\kappa}\vec{\alpha}+\kappa^2\vec{\beta}\right),$

$$\tilde{\vec{r}}'''=-\frac{1}{\tau}\left(\ddot{\kappa}\vec{\alpha}+\dot{\kappa}\dot{\vec{\alpha}}+2\kappa\dot{\kappa}\vec{\beta}+\kappa^2\dot{\vec{\beta}}\right)=-\frac{1}{\tau}\left\{\left(\ddot{\kappa}-\kappa^3\right)\vec{\alpha}+\left(\dot{\kappa}\kappa+2\kappa\right)\vec{\beta}+\kappa^2\tau\vec{\gamma}\right\}$$

$$\therefore \vec{\tilde{r}}' \times \vec{\tilde{r}}'' = \frac{\kappa^3}{\tau^2}\vec{\gamma}, \left(\vec{\tilde{r}}', \vec{\tilde{r}}'', \vec{\tilde{r}}'''\right) = -\frac{\kappa^5}{\tau^2}$$

$$\therefore \tilde{\kappa} = \frac{|\vec{\tilde{r}}' \times \vec{\tilde{r}}''|}{|\vec{\tilde{r}}'|^3} = |\tau|, \tilde{\tau} = \frac{\left(\vec{\tilde{r}}', \vec{\tilde{r}}'', \vec{\tilde{r}}'''\right)}{|\vec{\tilde{r}}' \times \vec{\tilde{r}}''|^2} = -\frac{\tau^2}{\kappa}.$$

4.证明：满足条件$\left(\frac{1}{\kappa}\right)^2 + \left[\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right]^2 =$常数的空间挠曲线或是常曲率的曲线或是球面上的一条曲线.

证明：$\frac{d}{ds}\left[\vec{r} + \frac{1}{\kappa}\vec{\beta} + \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\vec{\gamma}\right] = \left[\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right)\right]\vec{\gamma}$

因$\left(\frac{1}{\kappa}\right)^2 + \left[\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right]^2 =$常数,故两边对$s$求导,

$$\frac{2}{\kappa}\frac{d}{ds}\left(\frac{1}{\kappa}\right) + \frac{2}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right] = 0$$

两边同数乘$\frac{\tau}{2}$,

$$\frac{\tau}{\kappa}\frac{d}{ds}\left(\frac{1}{\kappa}\right) + \frac{d}{ds}\left(\frac{1}{\kappa}\right)\cdot\frac{d}{ds}\left[\frac{1}{\tau}\cdot\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right] = 0$$

①$\frac{d}{ds}\left(\frac{1}{\kappa}\right) \neq 0$时,$\frac{\tau}{\kappa} + \frac{d}{ds}\left[\frac{1}{\tau}\cdot\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right] = 0$,从而

$$\frac{d}{ds}\left[\vec{r} + \frac{1}{\kappa}\vec{\beta} + \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\vec{\gamma}\right] = 0$$

$\therefore \vec{r} + \frac{1}{\kappa}\vec{\beta} + \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\vec{\gamma} = \vec{r_0}, \vec{r_0}$为常向量.

$\therefore |\vec{r} - \vec{r_0}| = c$, c为常数.即曲线是球面上的一条曲线.

②$\frac{d}{ds}\left(\frac{1}{\kappa}\right) = 0$时,$\kappa$为常数,即曲线为常曲率的曲线

5.试求沿曲线定义的向量场$\vec{\rho}(s)$,使得以下各式同时成立：

$\dot{\vec{\alpha}}(s)=\vec{\rho}(s)\times\vec{\alpha}(s),\dot{\vec{\beta}}(s)=\vec{\rho}(s)\times\vec{\beta}(s),\dot{\vec{\gamma}}(s)=\vec{\rho}(s)\times\vec{\gamma}(s)$

解：因$\vec{\rho}(s)$沿曲线定义,可设$\vec{\rho}(s)=a(s)\vec{\alpha}(s)+b(s)\vec{\beta}(s)+c(s)\vec{\gamma}(s)$,则有

$\kappa\vec{\beta}=\dot{\vec{\alpha}}(s)=\left(a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}\right)\times\vec{\alpha}(s)=-b\vec{\gamma}+c\vec{\beta}$

$-\kappa\vec{\alpha}+\tau\vec{\gamma}=\dot{\vec{\beta}}(s)=\left(a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}\right)\times\vec{\beta}(s)=a\vec{\gamma}-c\vec{\alpha}$

$-\tau\vec{\beta}=\dot{\vec{\gamma}}(s)=\left(a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}\right)\times\vec{\gamma}(s)=-a\vec{\beta}+b\vec{\alpha}$

$\therefore a=\tau,b=0,c=\kappa$

$\therefore\vec{\rho}(s)=\tau(s)\vec{\alpha}(s)+\kappa(s)\vec{\gamma}(s)$

6.证明：（1）若曲线在每一点处的切线都经过一个定点,则该曲线必是一条直线；

（2）若曲线在每一点处的密切平面都经过一个定点,则该曲线必是一条平面曲线；

（3）若曲线在每一点处的法平面都经过一个定点,则该曲线必是一条球面曲线.

证明：（1）设定点为$\vec{c}$,则有$\left(\vec{r}(s)-\vec{c}\right)\times\dot{\vec{r}}(s)=0$,即$\left(\vec{r}(s)-\vec{c}\right)\times\vec{\alpha}(s)=0$.

对上式求导,有$\left(\vec{r}(s)-\vec{c}\right)\times\ddot{\vec{r}}(s)=0$,即$\left(\vec{r}(s)-\vec{c}\right)\times\kappa\vec{\beta}=0$.

$\because\vec{\alpha}\perp\vec{\beta}$,故$\vec{r}(s)-\vec{c}=0$或$\kappa=0$,总有该曲线是一条直线.

（2）设定点为$\vec{c}$,则有$\left(\vec{r}-\vec{c}\right)\cdot\vec{\gamma}=0$.

对上式求导,得到$-\tau\left(\vec{r}-\vec{c}\right)\cdot\vec{\beta}=0$.

$\therefore\tau=0$或$\left(\vec{r}-\vec{c}\right)\times\vec{\alpha}=0$（后一种情况为题 1）,总有该曲线是平面曲线.

（3）设定点为$\vec{c}$,则有$\left(\vec{r}-\vec{c}\right)\cdot\vec{\alpha}=0$,即$\left(\vec{r}-\vec{c}\right)\cdot\dot{\vec{r}}=0$,

也等价于$\left(\vec{r}-\vec{c}\right)\cdot\frac{d}{ds}\left(\vec{r}-\vec{c}\right)=0$,即$|\vec{r}-\vec{c}|=c$,该曲线是球面曲线.

7.设$\left\{\vec{r}(s);\overrightarrow{\alpha_1}(s),\overrightarrow{\alpha_2}(s),\overrightarrow{\alpha_3}(s)\right\}$是定义在曲线$\vec{r}(s)$上的单位正交标架场,命

$\frac{d\overrightarrow{\alpha_i}}{ds}=\sum_{j=1}^{3}\lambda_{ij}\overrightarrow{\alpha_j}$,$1\le i\le 3$,证明：$\lambda_{ij}+\lambda_{ji}=0$.

证明：$0=\left(\overrightarrow{\alpha_i}(s)\cdot\overrightarrow{\alpha_j}(s)\right)'=\frac{d\overrightarrow{\alpha_i}}{ds}\cdot\overrightarrow{\alpha_j}+\frac{d\overrightarrow{\alpha_j}}{ds}\cdot\overrightarrow{\alpha_i}=\lambda_{ij}+\lambda_{ji}$.

8.证明：曲线$\vec{r}(s)=\left(\frac{(1+s)^{3/2}}{3},\frac{(1-s)^{3/2}}{3},\frac{s}{\sqrt{2}}\right)$,$-1<s<1$,以$s$

证明： $\vec{r}'(s)=\left(\frac{\sqrt{1+s}}{2},-\frac{\sqrt{1-s}}{2},\frac{1}{\sqrt{2}}\right),\therefore|\vec{r}'(s)|=1$

$\therefore s$ 为曲线弧参.

$$\ddot{\vec{r}}(s)=\left(\frac{1}{4\sqrt{1+s}},\frac{1}{4\sqrt{1-s}},0\right),\therefore\kappa=|\ddot{\vec{r}}(s)|=\frac{\sqrt{2}}{4\sqrt{1-s^2}}.$$

$$\dddot{\vec{r}}(s)=\left(-\frac{1}{8}(1+s)^{-3/2},\frac{1}{8}(1-s)^{-3/2},0\right),\therefore\tau=\frac{\left(\dot{\vec{r}},\ddot{\vec{r}},\dddot{\vec{r}}\right)}{|\ddot{\vec{r}}|^2}=\frac{\sqrt{2}}{4\sqrt{1-s^2}}.$$

$$\vec{\alpha}(s)=\dot{\vec{r}}(s)=\left(\frac{\sqrt{1+s}}{2},-\frac{\sqrt{1-s}}{2},\frac{1}{\sqrt{2}}\right)$$

$$\vec{\beta}(s)=\frac{\ddot{\vec{r}}(s)}{|\ddot{\vec{r}}(s)|}=\left(\sqrt{\frac{1-s}{2}},\sqrt{\frac{1+s}{2}},0\right)$$

$$\vec{\gamma}(s)=\vec{\alpha}(s)\times\vec{\beta}(s)=\left(-\sqrt{\frac{1+s}{2}},\sqrt{\frac{1-s}{2}},\frac{1}{\sqrt{2}}\right).$$

9. 如果 $\vec{\sigma}=\vec{\alpha}(s)$ 是曲线 $\vec{r}=\vec{r}(s)$ 的切线象. 证明：该曲线的曲率和挠率分别是

$\kappa_\sigma=\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}$, $\tau_\sigma=\frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}}$,并求它的 Frenet 标架场.

证明：$\dot{\vec{\alpha}}=\kappa\vec{\beta},\ddot{\vec{\alpha}}=-\kappa^2\vec{\alpha}+\dot{\kappa}\vec{\beta}+\kappa\tau\vec{\gamma},$

$$\dddot{\vec{\alpha}}=-3\kappa\dot{\kappa}\vec{\alpha}+\left(\ddot{\kappa}-\kappa^3-\kappa\tau^2\right)\vec{\beta}+\left(2\dot{\kappa}\tau+\kappa\dot{\tau}\right)\vec{\gamma}$$

$$\kappa_\sigma=\frac{|\dot{\vec{\alpha}}\times\ddot{\vec{\alpha}}|}{|\dot{\vec{\alpha}}|^3}=\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}.$$

$$\tau_\sigma=\frac{\left(\dot{\vec{\alpha}},\ddot{\vec{\alpha}},\dddot{\vec{\alpha}}\right)}{|\dot{\vec{\alpha}}\times\ddot{\vec{\alpha}}|^2}=\frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}}.$$

$$\overrightarrow{\alpha_\sigma}=\frac{\dot{\vec{\alpha}}}{|\dot{\vec{\alpha}}|}=\vec{\beta}.$$

$$\overrightarrow{\gamma_\sigma}=\frac{\dot{\vec{\alpha}}\times\ddot{\vec{\alpha}}}{|\dot{\vec{\alpha}}\times\ddot{\vec{\alpha}}|}=\frac{1}{\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}}\left(\vec{\gamma}+\frac{\tau}{\kappa}\vec{\alpha}\right).$$

$$\overrightarrow{\beta_\sigma}=\overrightarrow{\gamma_\sigma}\times\overrightarrow{\alpha_\sigma}=\frac{1}{\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}}\left(-\vec{\alpha}+\frac{\tau}{\kappa}\vec{\gamma}\right).$$

10. $\vec{r}=\vec{r}(t)$ $\left\{\vec{r}(t);\vec{\alpha}(t),\vec{\beta}(t),\vec{\gamma}(t)\right\}$ $\left(\vec{\alpha},\vec{\alpha}',\vec{\alpha}''\right)\cdot\left(\vec{\gamma},\vec{\gamma}',\vec{\gamma}''\right)=\varepsilon|\vec{\alpha}'|^3|\vec{\gamma}'|^3$ $\varepsilon=\operatorname{sgn}\tau$

证明：$\left(\vec{\alpha},\vec{\alpha}',\vec{\alpha}''\right)=\left(\vec{\alpha},\kappa|\vec{r}'|\vec{\beta},-|\vec{r}'|^2\kappa^2\vec{\alpha}+\left(|\vec{r}'|^2\dot{\kappa}+\kappa|\vec{r}'||\vec{r}'|'\right)\vec{\beta}+|\vec{r}'|^2\kappa\tau\vec{\gamma}\right)$

$=\kappa^2\tau|\vec{r}'|^3$,

$\left(\vec{\gamma},\vec{\gamma}',\vec{\gamma}''\right)=\left(\vec{\gamma},-\tau|\vec{r}'|\vec{\beta},|\vec{r}'|^2\kappa\tau\vec{\alpha}-\left(|\vec{r}'|^2\dot{\tau}+|\vec{r}'|'\tau\right)\vec{\beta}-|\vec{r}'|^2\tau^2\vec{\gamma}\right)$

$=\kappa\tau^2|\vec{r}'|^3$,

$|\vec{\alpha}'|=|\dot{\vec{\alpha}}||\vec{r}'|=\kappa|\vec{r}'|$, $|\vec{\gamma}'|=|\dot{\vec{\gamma}}||\vec{r}'|=(\operatorname{sgn}\tau)\tau|\vec{r}'|$.

$\therefore\left(\vec{\alpha},\vec{\alpha}',\vec{\alpha}''\right)\cdot\left(\vec{\gamma},\vec{\gamma}',\vec{\gamma}''\right)=\varepsilon|\vec{\alpha}'|^3|\vec{\gamma}'|^3,\varepsilon=\operatorname{sgn}\tau$.

§2.5 曲线论基本定理

1.如果一条曲线的切向量与一个固定的方向成定角,则称该曲线为定倾曲线,或一般螺线（这样的曲线可以看成是柱面上与直母线成定角的曲线）,证明：曲线$\left(\kappa>0\right)$是定倾曲线的充要条件是它的挠率与曲率之比是常数.

证明：$\Rightarrow$

设曲线$\vec{r}=\vec{r}(s)$, s为弧参,是一定倾曲线.则$\exists\vec{a}, s.t.\ \vec{\alpha}(s)\cdot\vec{a}=const$.

对上式求导,得$\kappa\vec{\beta}(s)\cdot\vec{a}=0$,即$\vec{\beta}(s)\cdot\vec{a}=0$ $\left(\kappa>0\right)$,即$\vec{\beta}(s)$与一固定方向垂直.

$$\therefore 0=\left(\vec{\beta},\dot{\vec{\beta}},\ddot{\vec{\beta}}\right)=\begin{vmatrix}0 & 1 & 0\\ -\kappa & 0 & \tau\\ -\dot{\kappa} & -\kappa^2-\tau^2 & \dot{\tau}\end{vmatrix}=\kappa\dot{\tau}-\tau\dot{\kappa}$$

$$\therefore \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)=\frac{\dot{\tau}\kappa-\tau\dot{\kappa}}{\kappa^2}=0,\text{即}\frac{\tau}{\kappa}=const.$$

$\Leftarrow$

若$\dfrac{\tau}{\kappa}=c$, c为常数,则$\dot{\vec{\gamma}}=-\tau\vec{\beta}=-c\kappa\vec{\beta}=-c\dot{\vec{\alpha}}$.

两边对s求积分,得$\vec{\gamma}=-c\vec{\alpha}+\vec{a}$（$\vec{a}$为常向量）.

数乘$\vec{\alpha}$, $0=\vec{\alpha}\cdot\vec{\gamma}=-c+\vec{\alpha}\cdot\vec{a}$,即$\vec{\alpha}\cdot\vec{a}=c$, $\vec{r}(s)$为定倾曲线.

2.设$\tau=c\kappa$， c为常数。写出这条曲线的参数方程。

证明：令$t(s)=\int_0^s\kappa(s)ds$,则$\dfrac{dt}{ds}=\kappa(s)$.

$$\begin{cases}\dot{\vec{\alpha}}=\kappa\vec{\beta}\\ \dot{\vec{\beta}}=-\kappa\vec{\alpha}+\tau\vec{\gamma}\\ \dot{\vec{\gamma}}=-c\kappa\vec{\beta}\end{cases}\Rightarrow\begin{cases}\dfrac{d\vec{\alpha}}{dt}=\dot{\vec{\alpha}}\cdot\dfrac{ds}{dt}=\vec{\beta}\\ \dfrac{d\vec{\beta}}{dt}=-\vec{\alpha}+c\vec{\gamma}\\ \dfrac{d\vec{\gamma}}{dt}=-c\vec{\beta}\end{cases}$$

$$\therefore\frac{d^2\vec{\beta}}{dt^2}=-\frac{d\vec{\alpha}}{dt}+c\frac{d\vec{\gamma}}{dt}=-\left(1+c^2\right)\vec{\beta},\text{解得}\vec{\beta}=\vec{A}\cos\sqrt{1+c^2}t+\vec{B}\sin\sqrt{1+c^2}t.$$

$$\because|\vec{\beta}|^2=1\qquad\therefore\cos^2\sqrt{1+c^2}t\cdot\vec{A}^2+\sin^2\sqrt{1+c^2}t\cdot\vec{B}^2+2\vec{A}\vec{B}\cos\sqrt{1+c^2}t\sin\sqrt{1+c^2}t=1$$

$$\therefore \begin{cases} \vec{A}^2 = 1 \\ \vec{B}^2 = 1 \\ \vec{A}\vec{B} = 0 \end{cases} \text{可取} \begin{cases} \vec{A} = -\vec{e_1} \\ \vec{B} = -\vec{e_2} \end{cases}, \text{则} \vec{\beta} = \left(-\cos\sqrt{1+c^2}t, -\sin\sqrt{1+c^2}t, 0\right)$$

对 $\frac{d\vec{\alpha}}{dt} = \vec{\beta}$ 两边关于 t 积分,得 $\vec{\alpha} = \frac{1}{\sqrt{1+c^2}}\left(-\sin\sqrt{1+c^2}t, \cos\sqrt{1+c^2}t, 0\right) + \vec{a}$

$\because |\vec{\alpha}|^2 = 1 \therefore \frac{1}{1+c^2} + |\vec{a}|^2 + \frac{2}{\sqrt{1+c^2}}\left(-\sin\sqrt{1+c^2}t \cdot \vec{e_1} + \cos\sqrt{1+c^2}t \cdot \vec{e_2}\right) \cdot \vec{a} = 1$

令上式中 $t = 0$ 及 $t = \frac{\pi}{\sqrt{1+c^2}}$,可得 $\vec{e_2} \cdot \vec{a} = 0, |\vec{a}| = \frac{|c|}{\sqrt{1+c^2}}$

令 $t = \frac{\pi/2}{\sqrt{1+c^2}}$ 及 $t = \frac{3\pi/2}{\sqrt{1+c^2}}$,可得 $\vec{e_1} \cdot \vec{a} = 0$

于是有 $\vec{a}$ 与 $\vec{e_1}$、$\vec{e_2}$ 均垂直,$\therefore \vec{a} = \frac{c}{\sqrt{1+c^2}}\vec{e_3}$ $\therefore \vec{\alpha} = \frac{1}{\sqrt{1+c^2}}\left(-\sin\sqrt{1+c^2}t, \cos\sqrt{1+c^2}t, c\right)$

关于 s 求积分,最终得到

$$\vec{r} = \frac{1}{\sqrt{1+c^2}}\left(\int -\sin\left(\sqrt{1+c^2}\int_0^s \kappa(s)ds\right)ds, \int \cos\left(\sqrt{1+c^2}\int_0^s \kappa(s)ds\right)ds, cs\right)$$

3.证明:曲线 $\vec{r}(t) = \left(t+\sqrt{3}\sin t, 2\cos t, \sqrt{3}t - \sin t\right)$ 和 $\vec{r_1}(u) = \left(2\cos\frac{u}{2}, 2\sin\frac{u}{2}, -u\right)$ 是合同的.

证明:$\vec{r}'(t) = \left(1+\sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t\right)$,

$\vec{r}''(t) = \left(-\sqrt{3}\sin t, -2\cos t, \sin t\right)$

$\vec{r}'''(t) = \left(-\sqrt{3}\cos t, 2\sin t, \cos t\right)$,

$\vec{r}'(t) \times \vec{r}''(t) = \left(2\sqrt{3}\cos t - 2, -4\sin t, -2\cos t - 2\sqrt{3}\right)$,

$\left(\vec{r}', \vec{r}'', \vec{r}'''\right) = -8$.

$$\therefore \kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{1}{4}, \tau = \frac{\left(\vec{r}', \vec{r}'', \vec{r}'''\right)}{|\vec{r}' \times \vec{r}''|^2} = -\frac{1}{4}.$$

$$\vec{r}_1'(u)=\left(-\sin\frac{u}{2},\cos\frac{u}{2},-1\right),$$

$$\vec{r}_1''(u)=\left(-\frac{1}{2}\cos\frac{u}{2},-\frac{1}{2}\sin\frac{u}{2},0\right),$$

$$\vec{r}_1'''(u)=\left(\frac{1}{4}\sin\frac{u}{2},-\frac{1}{4}\cos\frac{u}{2},0\right),$$

$$\vec{r}_1'(u)\times\vec{r}_1''(u)=\left(-\frac{1}{2}\sin\frac{u}{2},\frac{1}{2}\cos\frac{u}{2},\frac{1}{2}\right),$$

$$\left(\vec{r}',\vec{r}'',\vec{r}'''\right)=-8.$$

$$\therefore \kappa_1=\frac{|\vec{r}_1'\times\vec{r}_1''|}{|\vec{r}_1'|^3}=\frac{1}{4},\tau_1=\frac{\left(\vec{r}_1',\vec{r}_1'',\vec{r}_1'''\right)}{|\vec{r}_1'\times\vec{r}_1''|^2}=-\frac{1}{4}.$$

$\therefore \kappa_1=\kappa,\tau_1=\tau$.

$\therefore \vec{r}(t)$ 与 $\vec{r}_1(u)$ 合同.

4.证明：曲线 $c_1:\vec{r}=(cht,sht,t)$ 与曲线 $c_2:\vec{r}=\left(\frac{e^{-u}}{\sqrt{2}},\frac{e^{u}}{\sqrt{2}},u+1\right)$ 在空间 E^3 的一个刚体运动下是合同的,试求使 c_1 与 c_2 合同的刚体运动.

解：$(cht,sht,t)=\left(\frac{e^t+e^{-t}}{2},\frac{e^t-e^{-t}}{2},t\right)$

$$=(0,0,-1)+\left(\frac{e^{-t}}{\sqrt{2}},\frac{e^{t}}{\sqrt{2}},t+1\right)\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}$$

且$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$是正交阵,故$c_1$和$c_2$合同.

§2.6 曲线在一点的标准展开

1.若在两条曲线之间可以建立一个点对应,使得在对应点这两条曲线有公共的主法线,则称这两条曲线互为共轭曲线.若一条曲线有非平凡的共轭曲线,则称它为 Bertrand 曲线.证明：在互为共轭的曲线 c_1, c_2 的对应点之间的距离为常数,并且在对应点处的切线成定角.

证明：①设其中一条曲线 c_1 的 Frenet 标架为 $\left\{\overrightarrow{r_1}(s); \overrightarrow{\alpha_1}(s), \overrightarrow{\beta_1}(s), \overrightarrow{\gamma_1}(s)\right\}$, 另一条曲线 c_2 以 c_1 的弧参 s 为参数, 可记做 $\overrightarrow{r_2}(s) = \overrightarrow{r_1}(s) + \lambda(s)\overrightarrow{\beta_1}(s)$

两边关于 s 求导, 得

$$|\overrightarrow{r_2}'|\overrightarrow{\alpha_2} = \overrightarrow{\alpha_1} + \lambda'\overrightarrow{\beta_1} + \lambda\dot{\overrightarrow{\beta_1}} = (1-\lambda\kappa_1)\overrightarrow{\alpha_1} + \lambda'\overrightarrow{\beta_1} + \lambda\tau_1\overrightarrow{\gamma_1}$$

两边数乘 $\overrightarrow{\beta_1}$, 得 $0 = \lambda'(s)$. $\therefore \lambda(s) \equiv c$ （常数）

$\therefore |\overrightarrow{r_2}(s) - \overrightarrow{r_1}(s)| = |\lambda(s)| = |c|$, 即 c_1 与 c_2 在对应点之间的距离为常数.

②$$\frac{d\left(\overrightarrow{\alpha_1}\cdot\overrightarrow{\alpha_2}\right)}{ds} = \dot{\overrightarrow{\alpha_1}}\cdot\overrightarrow{\alpha_2} + \overrightarrow{\alpha_1}\cdot\overrightarrow{\alpha_2}' = \kappa_1\overrightarrow{\beta_1}\cdot\overrightarrow{\alpha_2} + \kappa_2|\overrightarrow{r_2}'|\overrightarrow{\alpha_1}\cdot\overrightarrow{\beta_2} = 0\left(\overrightarrow{\beta_1}//\overrightarrow{\beta_2}, \therefore \overrightarrow{\beta_2}\perp\overrightarrow{\alpha_1}\right)$$

$\therefore \overrightarrow{\alpha_1}(s)\cdot\overrightarrow{\alpha_2}(s) \equiv c$ （常数）, 即 c_1 与 c_2 在对应点处的切线成定角.

2. 证明：曲率 κ 和挠率 τ 均不为 0 的曲线是 Bertrand 曲线的充要条件是：∃常数 λ, μ $(\lambda \neq 0)$, $s.t. \lambda\kappa + \mu\tau = 1$.

证明：设曲线 $c: \vec{r} = \vec{r}(s)$, s 为其弧参, 且曲率 k 和挠率 τ 均不为 0.

$\Rightarrow$

若 $\vec{r}(s)$ 为 Bertand 曲线, 则由上题知, 其非平凡共轭曲线为 $\vec{r_1}(s) = \vec{r}(s) + \lambda\vec{\beta}(s)$, 其中 λ 为非 0 常数.

两边关于 s 求导, 得 $\vec{r_1}' = (1-\lambda\kappa)\vec{\alpha} + \lambda\tau\vec{\gamma}$

$$\therefore \overrightarrow{\alpha_1} = \frac{\vec{r_1}'}{|\vec{r_1}'|} = \frac{1}{\sqrt{(1-\lambda\kappa)^2 + \lambda^2\tau^2}}\left[(1-\lambda\kappa)\vec{\alpha} + \lambda\tau\vec{\gamma}\right]$$

由题 1 的结论可知, $\vec{\alpha}\cdot\overrightarrow{\alpha_1} = \dfrac{1-\lambda\kappa}{\sqrt{(1-\lambda\kappa)^2 + \lambda^2\tau^2}} = \cos\angle\left(\vec{\alpha}, \overrightarrow{\alpha_1}\right) \equiv \cos\theta_0$ （θ_0 为一定值）

于是有 $\dfrac{1-\lambda\kappa}{\lambda\tau} = ctg\theta_0$, 即 $\lambda\kappa + \tau\lambda ctg\theta_0 = 1$.

取 $\mu = \lambda ctg\theta_0$, 即得结论成立.

$\Leftarrow$

令 $\vec{r_1}(s)=\vec{r}(s)+\lambda\vec{\beta}(s)$, 只需证明 $\vec{r_1}$ 为 $\vec{r}$ 的共轭曲线, 即 $\vec{\beta_1}//\vec{\beta}$.

已知 $\vec{\alpha_1}=\dfrac{1}{\sqrt{(1-\lambda\kappa)^2+\lambda^2\tau^2}}\left[(1-\lambda\kappa)\vec{\alpha}+\lambda\tau\vec{\gamma}\right]$

$\because \lambda\kappa+\mu\tau=1, \therefore \vec{\alpha_1}=\dfrac{\operatorname{sgn}\tau}{\sqrt{\lambda^2+\mu^2}}\left(\mu\vec{\alpha}+\lambda\vec{\gamma}\right)$.

两边关于 s 求导, 可得, $\kappa_1|\vec{r_1}'|\vec{\beta_1}=\dfrac{\operatorname{sgn}\tau}{\sqrt{\lambda^2+\mu^2}}(\mu\kappa-\lambda\tau)\vec{\beta}$

$\therefore \vec{\beta_1}//\vec{\beta}$, 即 $\vec{r_1}(s)$ 为 $\vec{r}(s)$ 的一非平凡共轭曲线, 从而 $\vec{r}(s)$ 为 Bertrand 曲线.

3. 若在曲线 c_1 c_2 c_1 c_2 c_2 c_1 c_1 c_2 c_1 $\vec{r}=\vec{r_1}(s)$ s c_1 c_2 $\vec{r_2}(s)=\vec{r_1}(s)+(c-s)\vec{\alpha_1}(s)$ c $\vec{\alpha_1}(s)$ c_1

证明：设 $c_2:\vec{r_2}(s)=\vec{r_1}(s)+\lambda(s)\vec{\alpha_1}(s)$, 其中 s 是 $\vec{r_1}$ 的弧参.

已知 $\vec{r_2}$ 法线与 $\vec{\alpha_1}$ 平行, 则 $\vec{\alpha_2}$ 与 $\vec{\alpha_1}$ 垂直, 也即 $\vec{r_2}'$ 与 $\vec{\alpha_1}$ 垂直.

$\because \vec{r_2}'=(1+\dot{\lambda})\vec{\alpha_1}+\lambda\kappa_1\vec{\beta_1}$ $\therefore \vec{r_2}'\cdot\vec{\alpha_1}=1+\dot{\lambda}=0$, 即 $\lambda=-s+c$. 得证.

4. 设 c_1 的方程是 $\vec{r}=\vec{r_1}(s)$, 试求 c_1 的渐缩线 c_2 的方程（提示：设 c_2 的方程为 $\vec{r_2}(s)=\vec{r_1}(s)+\lambda(s)\vec{\beta_1}(s)+\mu(s)\vec{\gamma_1}(s)$, 且要求 $\vec{r_2}'(s)//\left(\lambda\vec{\beta_1}+\mu\vec{\gamma_1}\right)$, 以此确定 λ 和 μ）.

证明：由题意, 设 $c_2:\vec{r_2}(s)=\vec{r_1}(s)+\lambda(s)\vec{\beta_1}(s)+\mu(s)\vec{\gamma_1}(s)$

则 $\vec{r_2}'=(1\text{-}\lambda\kappa_1)\vec{\alpha_1}+(\lambda'\text{-}\tau_1\mu)\vec{\beta_1}+(\lambda\tau_1+\mu')\vec{\gamma_1}$

因 c_2 为 c_1 的渐缩线, 故有 $\lambda(s)\vec{\beta_1}(s)+\mu(s)\vec{\gamma_1}(s)//\vec{r_2}'(s)$

$\therefore \left(\lambda\vec{\beta_1}+\mu\vec{\gamma_1}\right)\times\vec{r_2}'=0$, 即

$(0,\lambda,\mu)\times(1\text{-}\lambda\kappa_1,\lambda'\text{-}\tau_1\mu,\lambda\tau_1+\mu')=\left(\lambda^2\tau_1+\lambda\mu'-\lambda'\mu+\tau_1\mu^2,\mu-\mu\lambda\kappa_1,\lambda^2\kappa_1-\lambda\right)=0$

$\therefore 1\text{-}\lambda\kappa_1=0, \lambda^2\tau_1+\lambda\mu'-\lambda'\mu+\tau_1\mu^2=0$

$\therefore \lambda=\dfrac{1}{\kappa_1}, \tau_1=\dfrac{\lambda'\mu-\lambda\mu'}{\lambda^2+\mu^2}=-\left(\arctan\dfrac{\mu}{\lambda}\right)'$

$\therefore \mu = -\dfrac{1}{\kappa_1}\tan\left(\int \tau_1(s)ds\right)$

$\therefore \overrightarrow{r_2}(s) = \overrightarrow{r_1}(s) + \dfrac{1}{\kappa_1(s)}\overrightarrow{\beta_1}(s) - \dfrac{1}{\kappa_1(s)}\tan\left(\int \tau_1(s)ds\right)\overrightarrow{\gamma_1}(s)$

5. 证明：若平面曲线的曲率中心轨迹是正则曲线, 则它是原曲线的一条渐缩线.

证明：设平面曲线为 $\vec{r}=\vec{r}(s)$, s 为弧参, 则 $\overrightarrow{r_1}(s) = \vec{r}(s) + \dfrac{1}{\kappa(s)}\overrightarrow{\beta}(s)$.

两边关于 s 求导, 得 $\overrightarrow{r_1}'(s) = \left(\dfrac{1}{\kappa(s)}\right)'\overrightarrow{\beta}(s) + \dfrac{\tau(s)}{\kappa(s)}\vec{\gamma}(s)$

因 $\vec{r}(s)$ 为平面曲线, $\tau(s) \equiv 0$

$\therefore \overrightarrow{r_1}'(s) = \left(\dfrac{1}{\kappa(s)}\right)'\overrightarrow{\beta}(s)$

$\because \overrightarrow{r_1}(s)$ 为正则曲线

$\therefore 0 \neq \overrightarrow{r_1}'(s) // \overrightarrow{\beta}(s)$, 从而曲率中心轨迹是原曲线的一条渐缩线.

6. 经过曲率中心, 并与密切平面垂直的直线称为曲率轴. 证明：球心在点 $s=0$ 的曲率轴上、经过点 $\vec{r}(0)$ 的球面与曲线 $\vec{r}=\vec{r}(s)$ 在 $s=0$ 处有二阶以上的切触（提示：只要证明

$\lim\limits_{s\to 0}\dfrac{1}{s^2}\left\{|\vec{r}(s)-\left(\vec{r}(0)+\dfrac{1}{\kappa_0}\overrightarrow{\beta}(0)+c\vec{\gamma}(0)\right)|-\sqrt{\left(\dfrac{1}{\kappa_0}\right)^2+c^2}\right\}=0$ ）.

证明： $\vec{r}(s)=\vec{r}(0)+s\overrightarrow{\alpha}(0)+\dfrac{s^2}{2}\kappa_0\overrightarrow{\beta}(0)+o\left(s^2\right)$ ，曲率轴上的点可表示为 $\vec{r}(0)+\dfrac{1}{\kappa_0}\overrightarrow{\beta}(0)+c\vec{\gamma}(0)$, 故只需证明题中提示.

$$\lim_{s\to 0}\frac{1}{s^2}\left\{|\vec{r}(s)-\left(\vec{r}(0)+\frac{1}{\kappa_0}\overrightarrow{\beta}(0)+c\vec{\gamma}(0)\right)|-\sqrt{\left(\frac{1}{\kappa_0}\right)^2+c^2}\right\}$$

$$=\lim_{s\to 0}\frac{1}{s^2}\left\{|s\overrightarrow{\alpha}(0)+\frac{s^2}{2}\kappa_0\overrightarrow{\beta}(0)-c\vec{\gamma}(0)|-\sqrt{\left(\frac{1}{\kappa_0}\right)^2+c^2}\right\}$$

$$=\lim_{s\to 0}\frac{1}{s^2}\left\{\sqrt{s^2+\left(\frac{\kappa_0}{2}s^2-\frac{1}{\kappa_0}\right)+c^2}-\sqrt{\left(\frac{1}{\kappa_0}\right)^2+c^2}\right\}$$

$$=\lim_{s\to 0}\frac{\kappa_0{}^2s^2/4}{\sqrt{s^2+\left(\frac{\kappa_0}{2}s^2-\frac{1}{\kappa_0}\right)+c^2}+\sqrt{\left(\frac{1}{\kappa_0}\right)^2+c^2}}=0$$

7. 与曲线在一点有三阶以上切触的球面称为密切球面. 试求曲线$\vec{r}=\vec{r}(s)$在点s处的密切球面的中心.

解：设$\vec{r}=\vec{r}(s)$在点s处的密切球面的中心:

$\vec{r_1}(s)=\vec{r}(s)+\lambda_1(s)\vec{\alpha}(s)+\lambda_2(s)\vec{\beta}(s)+\lambda_3(s)\vec{\gamma}(s)$, 则

球面半径$R(s)=\sqrt{\lambda_1{}^2(s)+\lambda_2{}^2(s)+\lambda_3{}^2(s)}$, 且$\lim\limits_{\triangle s\to 0}\dfrac{\left(\vec{r}\left(s+\triangle s\right)-\vec{r_1}(s)\right)^2-R^2(s)}{\triangle s^3}=0$.

Taylor 展开$\vec{r}\left(s+\triangle s\right)$, 有

$$\vec{r}\left(s+\triangle s\right)-\vec{r_1}(s)=\left(\triangle s-\frac{\kappa^2(s)}{6}\triangle s^3-\lambda_1(s)\right)\vec{\alpha}(s)+\left(\frac{\kappa(s)}{2}\triangle s^2+\frac{\dot{\kappa}(s)}{6}\triangle s^3-\lambda_2(s)\right)\vec{\beta}(s)+$$

$$\left(\frac{\kappa(s)\tau(s)}{6}\triangle s^3-\lambda_3(s)\right)\vec{\gamma}(s)+o\left(\triangle s^3\right)$$

$$\therefore\left(\vec{r}\left(s+\triangle s\right)-\vec{r_1}(s)\right)^2=-2\lambda_1(s)\triangle s+\left(1-\lambda_2(s)\kappa(s)\right)\triangle s^2+$$

$$\left(\frac{\lambda_1(s)}{3}\kappa^2(s)-\frac{\lambda_2(s)}{3}\dot{\kappa}(s)-\frac{\lambda_3(s)}{3}\kappa(s)\tau(s)\right)\triangle s^3+o\left(\triangle s^3\right)+\lambda_1{}^2(s)+\lambda_2{}^2(s)+\lambda_3{}^2(s)$$

$$\therefore -2\lambda_1(s)=0, 1-\lambda_2(s)\kappa(s)=0, \frac{\lambda_1(s)}{3}\kappa^2(s)-\frac{\lambda_2(s)}{3}\dot{\kappa}(s)-\frac{\lambda_3(s)}{3}\kappa(s)\tau(s)$$

$$\therefore \lambda_1(s)=0, \lambda_2(s)=\frac{1}{\kappa(s)}, \lambda_3(s)=-\frac{\dot{\kappa}(s)}{\kappa^2(s)\tau(s)}$$

$$\therefore \vec{r_1}(s)=\vec{r}(s)+\frac{1}{\kappa(s)}\vec{\alpha}(s)+\lambda_2(s)\vec{\beta}(s)-\frac{\dot{\kappa}(s)}{\kappa^2(s)\tau(s)}\vec{\gamma}(s)$$

§2.7 平面曲线

1. 求下列平面的相对曲率κ_r：

（1）椭圆$\vec{r}=(a\cos t,b\sin t),0\le t<2\pi$

解：$\vec{r}'=(-a\sin t,b\cos t),\vec{r}''=(-a\cos t,-b\sin t)$

$$\therefore \kappa_r=\frac{xy''-x'y'}{\left(x'^2+y'^2\right)^{3/2}}=\frac{ab}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}$$

（2）双曲线$\vec{r}=(acht,bsht)$

解：$\vec{r}'=(asht,bcht),\vec{r}''=(acht,bsht)$

$$\therefore \kappa_r=\frac{-ab}{\left(a^2sh^2t+b^2ch^2t\right)^{3/2}}$$

（3）抛物线$\vec{r}=(t,t^2)$

解：$\vec{r}'=(1,2t),\vec{r}''=(0,2),\therefore \kappa_r=\dfrac{2}{\left(1+4t^2\right)^{3/2}}$

（4）摆线$\vec{r}=(a(t-\sin t),a(1-\cos t))$

解：$\vec{r}'=(a(1-\cos t),a\sin t),\vec{r}''=(a\sin t,a\cos t)$

$$\therefore \kappa_r=-\frac{1}{2a\sqrt{2-2\cos t}}$$

（5）悬链线$\vec{r}=\left(t,ach\dfrac{t}{a}\right)$

解：$\vec{r}'=\left(1,sh\dfrac{t}{a}\right),\vec{r}''=\left(0,\dfrac{1}{a}ch\dfrac{t}{a}\right),\therefore \kappa_r=\dfrac{1}{a}\left(ch\dfrac{t}{a}\right)^{-2}$

（6）曳物线$\vec{r}=\left(a\cos\varphi,a\ln(\sec\varphi+\tan\varphi)-a\sin\varphi\right),0\le\varphi<\dfrac{\pi}{2}$

解：$\vec{r}'=\left(-a\sin\varphi,\dfrac{a}{\cos\varphi}-a\cos\varphi\right),\vec{r}''=\left(-a\cos\varphi,a\dfrac{\sin\varphi}{\cos^2\varphi}+a\sin\varphi\right)$

$$\therefore \kappa_r=-\frac{\tan\varphi}{a}$$

2.设在平面极坐标系下,曲线方程为 $\rho=\rho(\theta)$, θ 为极角, ρ 为极距.求曲线的相对曲率的表达式.

解: $\vec{r}(\theta)=\left(\rho(\theta)\cos\theta,\rho(\theta)\sin\theta\right)$

$\vec{r}'(\theta)=\left(-\rho\sin\theta+\rho'\cos\theta,\rho\cos\theta+\rho'\sin\theta\right)$

$\vec{r}''(\theta)=\left((\rho''-\rho)\cos\theta-2\rho'\sin\theta,(\rho''-\rho)\sin\theta+2\rho'\cos\theta\right)$

$$\therefore \kappa_r=\frac{2\rho'^2-\rho\rho''+\rho^2}{\left(\rho'^2+\rho^2\right)^{3/2}}$$

3.已知曲线的相对曲率为 $\kappa_r(s)=\dfrac{1}{1+s^2}$,其中 s 为弧参,求此平面曲线的参数方程.

解：不妨设$x(0)=0,y(0)=0,\theta(0)=0$，则

$\theta(s)=\theta(0)+\int_0^s\frac{1}{1+s^2}ds=\arctan s$

$x(s)=x(0)+\int_0^s\cos(\arctan s)ds=\ln|s+\sqrt{1+s^2}|$

$y(s)=y(0)+\int_0^s\sin(\arctan s)ds=\sqrt{1+s^2}-1$

$\therefore \vec{r}(s)=\left(\ln|s+\sqrt{1+s^2}|,\sqrt{1+s^2}-1\right)$

4.求第 1 题中各类曲线的曲率中心轨迹.

（1）椭圆 $\vec{r}=(a\cos t,b\sin t),0\le t<2\pi.$

解: $\vec{\beta}(t)=\dfrac{1}{|\vec{r}'|}(-y',x')=\dfrac{1}{\sqrt{a^2\sin^2t+b^2\cos^2t}}(-b\cos t,-a\sin t)$

$\therefore$曲率中心轨迹为

$$\vec{r}(t)+\frac{1}{\kappa_r}\vec{\beta}(t)=\left(a^2-b^2\right)\left(\frac{\cos^3t}{a},-\frac{\sin^3t}{b}\right)$$

（2）双曲线 $\vec{r}=(acht,bsht)$.

解: $\vec{\beta}(t)=\dfrac{1}{|\vec{r}'|}(-y',x')=\dfrac{1}{\sqrt{b^2ch^2t+a^2sh^2t}}(-bcht,asht)$

$\therefore$曲率中心轨迹为

$$\vec{r}(t)+\frac{1}{\kappa_r}\vec{\beta}(t)=\left(\frac{a^2+b^2}{a}ch^3t,\frac{b^2-a^2}{b}sh^3t\right)$$

（3）抛物线 $\vec{r}=(t,t^2)$.

解：$\vec{\beta}(t)=\frac{1}{|\vec{r}'|}(-y',x')=\frac{1}{\sqrt{1+4t^2}}(-2t,1)$

$\therefore$曲率中心轨迹为

$$\vec{r}(t)+\frac{1}{\kappa_r}\vec{\beta}(t)=\left(-4t^3,\frac{1}{2}+3t^2\right)$$

（4）摆线$\vec{r}=\left(a\left(t-\sin t\right),a\left(1-\cos t\right)\right)$.

解：$\vec{\beta}(t)=\frac{1}{|\vec{r}'|}(-y',x')=\frac{1}{\sqrt{a^2\left(2-2\cos t\right)}}\left(-a\sin t,a\left(1-\cos t\right)\right)$

$\therefore$曲率中心轨迹为

$$\vec{r}(t)+\frac{1}{\kappa_r}\vec{\beta}(t)=\left(a\left(t+\sin t\right),-a\left(1-\cos t\right)\right)$$

（5）悬链线$\vec{r}=\left(t,ach\frac{t}{a}\right)$.

解：$\vec{\beta}(t)=\frac{1}{|\vec{r}'|}(-y',x')=\frac{1}{\sqrt{1+sh^2\frac{t}{a}}}\left(-sh\frac{t}{a},1\right)$

$\therefore$曲率中心轨迹为

$$\vec{r}(t)+\frac{1}{\kappa_r}\vec{\beta}(t)=\left(t-\frac{a}{2}sh\frac{2t}{a},2ach\frac{t}{a}\right)$$

（6）曳物线$\vec{r}=\left(a\cos\varphi,a\ln\left(\sec\varphi+\tan\varphi\right)-a\sin\varphi\right),0\le\varphi<\frac{\pi}{2}$.

解：$\vec{\beta}(\varphi)=\frac{1}{|\vec{r}'|}\left(a\cos\varphi-\frac{a}{\cos\varphi},-a\sin\varphi\right)$

$\therefore$曲率中心轨迹为

$$\vec{r}(\varphi)+\frac{1}{\kappa_r}\vec{\beta}(\varphi)=\left(a\sec\varphi,a\ln\left(\sec\varphi+\tan\varphi\right)\right)$$

5.求下列曲线的渐伸线.

（1）圆周：$x^2+y^2=a^2$.

解:$\vec{r}(t)=(a\cos t, a\sin t)$

$s=\int_0^t|\vec{r}'(t)|dt=at, \therefore t=\dfrac{s}{a}$

$\therefore \vec{r}(s)=\left(a\cos\dfrac{s}{a}, a\sin\dfrac{s}{a}\right), \vec{\alpha}(s)=\dot{\vec{r}}(s)=\left(-\sin\dfrac{s}{a}, \cos\dfrac{s}{a}\right)$

$\therefore$ 所求渐伸线方程为：

$$\vec{r_2}(s)=\vec{r}(s)+(c-s)\vec{\alpha}(s)=\left(a\cos\frac{s}{a}-(c-s)\sin\frac{s}{a}, a\sin\frac{s}{a}+(c-s)\cos\frac{s}{a}\right)$$

（2）悬链线： $y=ach\dfrac{x}{a}$.

解:$\vec{r}(t)=\left(t, ach\dfrac{t}{a}\right)$

$s=\int_0^t|\vec{r}'(t)|dt=ash\dfrac{t}{a}$

$\therefore$ 所求渐伸线方程为：

$$\vec{r_2}(t)=\vec{r}(t)+\left(c-ash\frac{t}{a}\right)\frac{\vec{r}'(t)}{|\vec{r}'(t)|}=\left(t+\frac{c-ash\frac{t}{a}}{ch\frac{t}{a}}, \frac{c+ash\frac{t}{a}}{ch\frac{t}{a}}\right)$$

（3）摆线： $\vec{r}(t)=(t-\sin t, 1-\cos t)$.

解:$\vec{r}'(t)=(1-\cos t, \sin t)$

$s=\int_0^t|\vec{r}'(t)|dt=-4\cos\dfrac{t}{2}$

$\therefore$ 所求渐伸线方程为：

$$\vec{r_2}(t)=\vec{r}(t)+\left(c+4\cos\frac{t}{2}\right)\frac{\vec{r}'(t)}{|\vec{r}'(t)|}=\left(t+\sin t+c\sin\frac{t}{2}, 3+\cos t+c\cos\frac{t}{2}\right)$$

§ 3.1 曲面的定义

1.写出椭球面、单叶双曲面、双叶双曲面、椭圆抛物面、双曲抛物面的参数方程.

解:椭球面:$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\vec{r}(\varphi,\theta)=(a\cos\varphi\cos\theta,b\cos\varphi\sin\theta,c\sin\varphi)$,其中$-\frac{\pi}{2}<\varphi<\frac{\pi}{2},-\pi<\theta<\pi$

单叶双曲面:$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

$\vec{r}(\varphi,\theta)=(a\sec\varphi\cos\theta,b\sec\varphi\sin\theta,c\tan\varphi)$,其中$-\frac{\pi}{2}<\varphi<\frac{\pi}{2},-\pi<\theta<\pi$

双叶双曲面:$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

$\vec{r}(\varphi,\theta)=(a\tan\varphi\cos\theta,b\tan\varphi\sin\theta,c\sec\varphi)$,其中$0<\varphi<\frac{\pi}{2},-\pi<\theta<\pi$

椭圆抛物面:$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$

$\vec{r}(u,v)=\left(au\cos v,bu\sin v,\frac{1}{2}u^2\right)$

双曲抛物面:$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$

$\vec{r}(u,v)=(a(u+v),b(u-v),2uv)$

2.在球面$x^2+y^2+z^2=1$上, 命$N=(0,0,1)$, $S=(0,0,-1)$. 对于赤道平面上的任一点$P=(u,v,0)$, 可作唯一的一条直线经过N、P两点, 它与球面有唯一的一个交点P′.

(1)证点P′的坐标是$x=\frac{2u}{u^2+v^2+1},y=\frac{2v}{u^2+v^2+1},z=\frac{u^2+v^2-1}{u^2+v^2+1}$.
它给出了球面上去掉北极N的剩余部分的正则参数表示.
(2)求球面上去掉南极S的剩余部分的类似的参数表示.
(3)求上面两种参数表示在公共部分所给出的参数变换.
(3)对于P′点, 记对应的南极投影(u,v), 北极投影(u',v')

解:(1)令$t=\frac{2}{u^2+v^2+1}$,则$x=tu,y=tv,z=1-t$.
$\because x^2+y^2+z^2=t^2(u^2+v^2+1)-2t+1=1$
$\therefore$ P′在球面上
$\because (u,v,-1)\times(tu,tv,-t)=0$
$\therefore \overrightarrow{NP}//\overrightarrow{NP'}$

(2)由对称性, 过点S、P的直线与球面有唯一交点$P''=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{1-u^2-v^2}{u^2+v^2+1}\right)$

(3)对于P'点, 记对应的南极投影(u,v), 北极投影(u',v')

由(1)知$u=\dfrac{-x}{z-1}, v=\dfrac{-y}{z-1}\cdots\cdots(*)$

由(2)知$x=\dfrac{2u'}{u'^2+v'^2+1}, y=\dfrac{2v'}{u'^2+v'^2+1}, z=\dfrac{1-u'^2-v'^2}{u'^2+v'^2+1}$

代入$(*)$,有$u=\dfrac{u'}{u'^2+v'^2}, v=\dfrac{v'}{u'^2+v'^2}$

3.把单叶双曲面、双曲抛物面写成直纹面形式的参数方程.

解:单叶双曲面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$，即$\left(\dfrac{x}{a}+\dfrac{z}{c}\right)\left(\dfrac{x}{a}-\dfrac{z}{c}\right)=\left(1+\dfrac{y}{b}\right)\left(1-\dfrac{y}{b}\right)$

一直母线为$\begin{cases}\dfrac{x}{a}+\dfrac{z}{c}=u\left(1+\dfrac{y}{b}\right)\\ u\left(\dfrac{x}{a}-\dfrac{z}{c}\right)=1-\dfrac{y}{b}\end{cases}$

令$v=\dfrac{y}{b}$,则$x=\dfrac{a}{2}\left(u+\dfrac{1}{u}\right)+v\left[\dfrac{a}{2}\left(u-\dfrac{1}{u}\right)\right], z=\dfrac{c}{2}\left(u-\dfrac{1}{u}\right)+v\left[\dfrac{c}{2}\left(u+\dfrac{1}{u}\right)\right]$

$\therefore \vec{r}=\left(\dfrac{a}{2}\left(u+\dfrac{1}{u}\right),0,\dfrac{c}{2}\left(u-\dfrac{1}{u}\right)\right)+v\left(\dfrac{a}{2}\left(u-\dfrac{1}{u}\right),b,\dfrac{c}{2}\left(u+\dfrac{1}{u}\right)\right)$

双曲抛物面:$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$,即$\left(\dfrac{x}{a}+\dfrac{y}{b}\right)\left(\dfrac{x}{a}-\dfrac{y}{b}\right)=2z$

一直母线为$\begin{cases}\dfrac{x}{a}-\dfrac{y}{b}=c\\ c\left(\dfrac{x}{a}+\dfrac{y}{b}\right)=2z\end{cases}$

令$v=z, u=c$,则$x=\dfrac{ua}{2}+\dfrac{va}{u}, y=-\dfrac{ub}{2}+\dfrac{vb}{u}$

$\therefore \vec{r}=\left(\dfrac{ua}{2},-\dfrac{ub}{2},0\right)+v\left(\dfrac{a}{u},\dfrac{b}{u},1\right)$

4.已知空间E^3中四个点$P_i\ (1\le i\le 4)$的坐标(x_i,y_i,z_i),过线段P_1P_2与P_3P_4上有相同分比的点所作的直线构成一直纹面, 写出此直纹面的参数方程. 考察它是正则曲面片的条件.

解:$\overrightarrow{OP}(\lambda,t)=(1-\lambda)\overrightarrow{OP_1}+\lambda\overrightarrow{OP_2}+t\left[(1-\lambda)\overrightarrow{P_1P_3}+\lambda\overrightarrow{P_2P_4}\right]\triangleq\vec{r}(\lambda,t)$

$\overrightarrow{r_\lambda}=-\overrightarrow{OP_1}+\overrightarrow{OP_2}+t\left(-\overrightarrow{P_1P_3}+\overrightarrow{P_2P_4}\right)$

$\overrightarrow{r_t}=(1-\lambda)\overrightarrow{P_1P_3}+\lambda\overrightarrow{P_2P_4}$

故正则只需$\overrightarrow{r_\lambda}\times\overrightarrow{r_t}=(1-\lambda)\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}+\lambda\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_4}-t\overrightarrow{P_1P_3}\times\overrightarrow{P_2P_4}\neq 0$

5.求正螺旋面$\vec{r}=\vec{r}(u,v)=(u\cos v, u\sin v, bv)$与圆柱面$(x-a)^2+y^2=a^2$的交线, 及其曲率、挠率.

解:将正螺旋面的参数表示代入圆柱面方程

$(u\cos v-a)^2+(u\sin v)^2=a^2$

得到$u^2=2au\cos v \Rightarrow u=0$,或$u=2a\cos v$

代回正螺旋面的参数表示, 交线c_1, c_2分别是

$\vec{r_1}(v)=(0,0,bv), \kappa_1=\tau_1=0$

$\vec{r_2}(v)=(2a\cos^2 v, 2a\cos v\sin v, bv), \kappa_2=\dfrac{4a}{4a^2+b^2}, \tau_2=\dfrac{2b}{4a^2+b^2}$

§ 3.2 切平面和法线

1.证明:一个曲面是球面$\Leftrightarrow$它的所有法线通过一个定点.

证明: $\Rightarrow$ $\vec{r}^2 = c \Rightarrow \vec{r}\cdot d\vec{r} = 0$

因$d\vec{r}$是切向量，可知$\vec{r}$是法向量, 则法向量过点$(0,0)$

"$\Leftarrow$" 移动坐标轴使所有法线过点$(0,0)$,则$\vec{r}$也是法向量

$\Rightarrow \vec{r}\cdot d\vec{r} = 0 \Rightarrow \vec{r}^2 = c$

2.证明:一个曲面是旋转面的充分必要条件是它的所有法线与一条固定的直线都相交.

证明:"$\Rightarrow$"设$\vec{r}(u,v) = \left(f(v)\cos u, f(v)\sin u, g(v)\right)$

$\vec{r_u} = \left(-f(v)\sin u, f(v)\cos u, 0\right), \vec{r_v} = \left(f'(v)\cos u, f'(v)\sin u, g'(v)\right)$

$\vec{r_u}\times\vec{r_v} = \left(f(v)g'(v)\cos u, f(v)g'(v)\sin u, -f(v)f'(v)\right)$

法线的参数方程为：

$$\frac{x - f(v)\cos u}{f(v)g'(v)\cos u} = \frac{y - f(v)\sin u}{f(v)g'(v)\sin u} = \frac{z - g(v)}{-f(v)f'(v)}$$

显然$\left(0, 0, g(v) + \dfrac{f(v)f'(v)}{g'(v)}\right)$在法线上, 也在$Z$轴上, 即法线总与Z轴相交.

"$\Leftarrow$" 不妨设曲面S的所有法线与Z轴重合, 法线与Z轴的交点为$\left(0, 0, h(u,v)\right)$,曲面$S$的方程为:$\vec{r}(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$,则$\left[\vec{r}(u,v) - \left(0, 0, h(u,v)\right)\right] // \left(\vec{r_u}\times\vec{r_v}\right)$,即

$$\begin{cases} x\dfrac{\partial x}{\partial u} + y\dfrac{\partial y}{\partial u} + (z-h)\dfrac{\partial z}{\partial u} = 0 \\ x\dfrac{\partial x}{\partial v} + y\dfrac{\partial y}{\partial v} + (z-h)\dfrac{\partial z}{\partial v} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial\left(x^2+y^2\right)}{\partial u} = -2(z-h)\dfrac{\partial z}{\partial u} \\ \dfrac{\partial\left(x^2+y^2\right)}{\partial v} = -2(z-h)\dfrac{\partial z}{\partial v} \end{cases}$$

$$\therefore \begin{vmatrix} \dfrac{\partial\left(x^2+y^2\right)}{\partial u} & \dfrac{\partial\left(x^2+y^2\right)}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0$$

$\therefore x^2 + y^2$与z是函数相关的，故$\exists$函数$f, s.t. x^2 + y^2 = f(z)$

$\therefore S$的方程可表示为$\vec{r} = \left(\sqrt{f(z)}\cos\theta, \sqrt{f(z)}\sin\theta, z\right)$,即$S$为旋转面

3.证明:一个曲面是锥面的充要条件是它的所有切平面都经过一个定点.

证明: $\Rightarrow$ $\vec{r}(u,v) = \vec{a} + v\vec{l}(u)$, $\vec{r_u} = v\vec{l}'(u), \vec{r_v} = \vec{l}(u)$

点(u,v)处的切平面:$\vec{X}(\lambda, \mu) = \vec{a} + (v+\mu)\vec{l}(u) + \lambda v\vec{l}'(u)$

取$\lambda=0, \mu=-v$,总有$\vec{a}$在$\vec{X}$上.

"$\Leftarrow$" 不妨设定点为原点$(0,0,0)$, 曲面$\vec{r} = \left(x, y, f(x,y)\right)$, 则

$\overrightarrow{r_x}=(1,0,f_x),\overrightarrow{r_y}=(0,1,f_y)$

依题意,$\vec{r}-\vec{0}$与$\overrightarrow{r_x},\overrightarrow{r_y}$共面,即$\left(\vec{r},\overrightarrow{r_x},\overrightarrow{r_y}\right)=0$

$$\begin{vmatrix}1&0&f_x\\0&1&f_y\\x&y&f(x,y)\end{vmatrix}=0\Rightarrow f(x,y)=xf_x+yf_y$$

$\therefore f(x,y)=F\left(\dfrac{x}{y}\right)\therefore$ 该曲面为锥面

4.假定在方程$\dfrac{x^2}{a-\lambda}+\dfrac{y^2}{b-\lambda}+\dfrac{z^2}{c-\lambda}=1$中,$a,b,c$为常数且$a>b>c,\lambda$为参数.当$\lambda\in(-\infty,c)$时,方程给出一族椭球圆;当$\lambda\in(c,b)$时,方程给出一族单叶双曲面;当$\lambda\in(b,a)$时,方程给出一族双叶双曲面. 证明:过空间中不在各坐标轴上的任一点有且恰有分别属于这三族曲面的三个二次曲面，且它们沿交线是彼此正交的.

证明:对于空间中任一点(x_0,y_0,z_0),令$f(\lambda)=\dfrac{{x_0}^2}{a-\lambda}+\dfrac{{y_0}^2}{b-\lambda}+\dfrac{{z_0}^2}{c-\lambda}-1$,其中$\lambda\in(-\infty,c)$
$\cup(c,b)\cup(b,a)$.

$f(\lambda)$分别在$(-\infty,c),(c,b),(b,a)$上单增,且$f(-\infty)f_-(c)<0,f_+(c)f_-(b)<0,f_+(b)f_-(a)$
<0,故存在唯一确定的$\lambda_1\in(-\infty,c),\lambda_2\in(c,b),\lambda_3\in(b,a)$,使得$f(\lambda_i)=0,i=1,2,3$

这样就确定了三个二次曲面:

椭球面:$\dfrac{x^2}{a-\lambda_1}+\dfrac{y^2}{b-\lambda_1}+\dfrac{z^2}{c-\lambda_1}=1$

单叶双曲面:$\dfrac{x^2}{a-\lambda_2}+\dfrac{y^2}{b-\lambda_2}+\dfrac{z^2}{c-\lambda_2}=1$

双叶双曲面:$\dfrac{x^2}{a-\lambda_3}+\dfrac{y^2}{b-\lambda_3}+\dfrac{z^2}{c-\lambda_3}=1$

三个二次曲面在点(x_0,y_0,z_0)处的法向量记为$\overrightarrow{n_i}=\left(\dfrac{x_0}{a-\lambda_i},\dfrac{y_0}{b-\lambda_i},\dfrac{z_0}{c-\lambda_i}\right),i=1,2,3$,则

$$\overrightarrow{n_i}\cdot\overrightarrow{n_j}=\frac{{x_0}^2}{(a-\lambda_i)(a-\lambda_j)}+\frac{{y_0}^2}{(b-\lambda_i)(b-\lambda_j)}+\frac{{z_0}^2}{(c-\lambda_i)(c-\lambda_j)}$$

$$=\frac{1}{\lambda_i-\lambda_j}\left(\frac{{x_0}^2}{a-\lambda_i}-\frac{{x_0}^2}{a-\lambda_j}+\frac{{y_0}^2}{b-\lambda_i}-\frac{y^2}{b-\lambda_j}+\frac{z^2}{c-\lambda_i}-\frac{z^2}{c-\lambda_j}\right)=0,(i\neq j)$$

故它们沿交线是彼此正交的.

5.设S是圆锥面:$\vec{r}=(v\cos u,v\sin u,v)$,$c$为S上一条曲线,方程为$u=\sqrt{2}t,v=e^t$.

(1)将c的切向量用$\overrightarrow{r_u},\overrightarrow{r_v}$的线性组合表示出来.

(2)证明:c的切向量平分了$\overrightarrow{r_u}$与$\overrightarrow{r_v}$的夹角.

解：(1) $\vec{r}'(t)=\frac{d\vec{r}}{dt}=\vec{r_u}\frac{du}{dt}+\vec{r_v}\frac{dv}{dt}=\sqrt{2}\vec{r_u}+e^t\vec{r_v}$

(2) $\vec{r_u}=(-v\sin u, v\cos u, v), \vec{r_v}=(\cos u, \sin u, 1)$

$\therefore \cos\angle\left(\vec{r}'(t),\vec{r_u}(t)\right)=\frac{\vec{r}'(t)\cdot\vec{r_u}(t)}{|\vec{r}'(t)||\vec{r_u}(t)|}=\frac{\sqrt{2}}{2}, \cos\angle\left(\vec{r}'(t),\vec{r_v}(t)\right)=\frac{\vec{r}'(t)\cdot\vec{r_v}(t)}{|\vec{r}'(t)||\vec{r_v}(t)|}=\frac{\sqrt{2}}{2}$

得证 $\angle\left(\vec{r}'(t),\vec{r_u}(t)\right)=\angle\left(\vec{r}'(t),\vec{r_v}(t)\right)=\frac{\pi}{4}$

§ 3.3 曲面的第一基本形式

1. 求下列曲面的第一基本形式:

$(1).\vec{r}=(u\cos v,u\sin v,\varphi(v)).$

$(2).\vec{r}=(u\cos v,u\sin v,\varphi(u)+av)$,其中$a$是常数.

解: (1). $\vec{r}_u=(\cos v,\sin v,0)$

$$\vec{r}_v=(-u\sin v,u\cos v,\varphi^{'}(v))$$

$$\therefore E=\vec{r}_u^{\,2}=\cos^2 v+\sin^2 v=1,\quad F=\vec{r}_u\vec{r}_v=0,\quad G=\vec{r}_v^{\,2}=u^2+\varphi^{'2}(V)$$

$$\therefore I=Edu^2+2Fdudv+Gdv^2=du^2+(u^2+\varphi^{'2}(v))dv^2$$

(2). $\vec{r}_u=(\cos v,\sin v,\varphi^{'}(u))$

$$\vec{r}_v=(-u\sin v,u\cos v,a)$$

$$\therefore E=\vec{r}_u^{\,2}=1+\varphi^{'2}(u),\quad F=\vec{r}_u\vec{r}_v=a\varphi^{'}(u),\quad G=\vec{r}_v^{\,2}=u^2+a^2$$

$$\therefore I=Edu^2+2Fdudv+Gdv^2=(1+\varphi^{'2}(u))du^2+2a\varphi^{'}(u)dudv+(u^2+a^2)dv^2$$

2. 设曲面的参数方程是

$$\bar{r}=\left(\frac{2au}{u^2+v^2+a^2},\frac{2av}{u^2+v^2+a},\frac{u^2+v^2-a}{u^2+v^2+a}\right),$$

求它的第一基本形式.

解: $\bar{r}_u=\left(\frac{2a(-u^2+v^2+a^2)}{(u^2+v^2+a^2)^2},\frac{-4auv}{(u^2+v^2+a^2)^2},\frac{4a^2u}{(u^2+v^2+a^2)^2}\right)$

$$\bar{r}_v=\left(\frac{-4auv}{(u^2+v^2+a^2)^2},\frac{2a(u^2-v^2+a^2)}{(u^2+v^2+a^2)^2},\frac{4a^2v}{(u^2+v^2+a^2)^2}\right)$$

$$\therefore E=\bar{r}_u^2=\frac{4a^2}{(u^2+v^2+a^2)^2},\quad F=0,\quad G=\bar{r}_v^2=\frac{4a^2}{(u^2+v^2+a^2)^2}$$

$$\therefore I=\frac{4a^2}{(u^2+v^2+a^2)^2}(du^2+dv^2)$$

3. 设在曲面上一点,由二次方程

$$Pdu^2+2Qdudv+Rdv^2=0$$

确定了两个切方向. 证明: 这两个正方向彼此正交的充分必要条件是

$$ER-2FQ+GP=0.$$

证明： $Pdu^2+2Qdudv+Rdv^2=0 \quad \Rightarrow \quad P\left(\frac{du}{dv}\right)^2+2Q\left(\frac{du}{dv}\right)+R=0$

$$\Rightarrow \quad \frac{d_1u}{d_1v}+\frac{d_2u}{d_2v}=-\frac{2Q}{P}, \quad \frac{d_1u}{d_1v}\cdot\frac{d_2u}{d_2v}=\frac{R}{P}$$

两个切方向$d_1\bar{r}, d_2\bar{r}$正交 $\quad \Leftrightarrow \quad d_1\bar{r}\cdot d_2\bar{r}=Ed_1ud_2u+F(d_1ud_2v+d_1vd_2u)+Gd_1vd_2v$

$$\Leftrightarrow E\frac{d_1u}{d_1v}\frac{d_2u}{d_2v}+F(\frac{d_1u}{d_1v}+\frac{d_2u}{d_2v})+G=0$$

$$\Leftrightarrow E\frac{R}{P}+F(-\frac{2Q}{P})+G=0$$

$$\Leftrightarrow ER-2FQ+GP=0$$

4. 求球面上与经线交成定角的轨线方程.

解： 设球面 $\bar{r}=(a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, a\sin\varphi)$

$E=a^2\cos^2\varphi, \quad F=0, \quad G=a^2, \quad I=a^2\cos^2\varphi d\theta^2+a^2d\varphi^2$

对于球面上的经线 $\theta=\theta_0$（常数）, $\mathrm{d}\theta=0$

设所求曲线（方向向量($\delta\theta, \delta\varphi$)）与经线（方向向量($\mathrm{d}\theta, \mathrm{d}\varphi$)）的夹角为定角$\alpha_0$，则

$$\cos\alpha_0=\frac{d\bar{r}\cdot\delta\bar{r}}{|d\bar{r}|\cdot|\delta\bar{r}|}=\frac{a^2d\varphi\delta\varphi}{\sqrt{a^2\cos^2\varphi\delta\theta^2+a^2\delta\varphi^2}\cdot\sqrt{a^2d\varphi^2}}=\frac{\delta\varphi}{\sqrt{\cos^2\varphi\delta\theta^2+\delta\varphi^2}}$$

$$\Rightarrow \tan^2\alpha_0=\frac{1-\cos^2\alpha_0}{\cos^2\alpha_0}=\frac{\cos^2\varphi\delta\theta^2}{\delta\varphi^2}$$

$\because 0\le\alpha_0\le\frac{\pi}{2}, \quad \delta\theta=\tan\alpha_0\frac{1}{\cos\varphi}\delta\varphi,$ 两边积分，得 $\theta=\tan\alpha_0\ln|\sec\varphi+\tan\varphi|+c$

$$\therefore \bar{r}=(\varphi, \tan\alpha_0\ln|\sec\varphi+\tan\varphi|+c)$$

5. 已知曲面的第一基本形式为$I=du^2+(u^2+a^2)dv^2$，求：

(1) 曲线 $C_1: u+v=0$与$C_2: u-v=0$的交角.

(2) 曲线 $C_1: u=\frac{a}{2}v^2, C_2: u=-\frac{a}{2}v^2, C_3: v=1$所构成的曲边三角形的边长和各个内角.

(3) 曲线$u=av, u=-av$ 和 $v=1$ 所围成的曲边三角形的面积.

解：由$I=du^2+(u^2+a^2)dv^2$ 得 $E=1,\quad F=0,\quad G=u^2+a^2$

(1). 对$u+v=0$两边微分,得$du+dv=0 \quad \Rightarrow \quad du=-dv$

对$u-v=0$两边微分,得$\delta u=\delta v$

设C_1,C_2的交角为θ,则

$$\cos\theta=\frac{d\bar{r}\cdot\delta\bar{r}}{|d\bar{r}|\cdot|\delta\bar{r}|}=\frac{du\delta u+(u^2+a^2)dv\delta v}{\sqrt{du^2+(u^2+a^2)dv^2}\sqrt{\delta u^2+(u^2+a^2)\delta v^2}}$$

$$=\frac{-dv\delta v+(u^2+a^2)dv\delta v}{\sqrt{dv^2+(u^2+a^2)dv^2}\sqrt{\delta v^2+(u^2+a^2)\delta v^2}}$$

在交点(0,0)处, $\cos\theta=\dfrac{(a^2-1)dv\delta v}{\sqrt{(a^2+1)dv^2}\sqrt{(a^2+1)\delta v^2}}=\pm\dfrac{a^2-1}{a^2+1}$

$\therefore$交角为: $\theta=\arccos\dfrac{a^2-1}{a^2+1},\pi-\theta.$

(2). C_1,C_2,C_3的交点为 $O=(0,0),\quad A=(-\frac{a}{2},1),\quad B=(\frac{a}{2},1)$

OA的弧长$=\displaystyle\int_0^1\sqrt{\left(\frac{du}{dv}\right)^2+(u^2+a^2)}dv=\int_0^1\frac{|a|}{2}\cdot\sqrt{v^4+4v^2+4}dv=\frac{7}{6}|a|$

OB的弧长$=\dfrac{7}{6}|a|,\quad |AB|=|a|$

在C_1上，$d_1u=ad_1v,\quad$ 在C_2上，$d_2u=-ad_2v,\quad$ 在C_3上，$d_3v=0$

$$\cos\angle A=\frac{d_1ud_3u}{\sqrt{(d_1u)^2+(u^2+a^2)d_1v^2}\cdot\sqrt{d_3u^2}}=\frac{-v}{\sqrt{v^2+1+\frac{v^4}{4}}}=-\frac{2}{3}$$

$\therefore\angle A=\arccos\dfrac{2}{3},\quad \angle B=\angle A=\arccos\dfrac{2}{3}$

$$\cos\angle O=\frac{d_1ud_2u+a^2d_1vd_2v}{\sqrt{d_1u^2+a^2d_1v^2}\cdot\sqrt{d_2u^2+a^2d_2v^2}}=1$$

$\therefore\angle O=0$

(3). C_1,C_2,C_3的交点为 $O=(0,0),\quad A=(-a,1),\quad B=(a,1)$

$$A=\iint_D\sqrt{EG-F^2}dudv=\iint_D\sqrt{u^2+a^2}dudv=2\int_0^a\sqrt{u^2+a^2}du\int_{\frac{u}{a}}^1dv=[\frac{2}{3}-\frac{\sqrt{2}}{3}+\ln(\sqrt{2}+1)]a^2$$

§ 3.5 保长对应和保角对应

1. 证明:在悬链面$\bar{r}=(ach\frac{t}{a}\cos\theta, ach\frac{t}{a}\sin\theta, t), -\infty<t<+\infty, 0<\theta<2\pi$ 与正螺旋面$\bar{r}=(v\cos u, v\sin u, au), 0<u<2\pi, -\infty<v<+\infty$之间, 存在保长对应.

证明: 悬链面的第一基本形式为 $I=ch^2\frac{t}{a}dt^2+a^2ch^2\frac{t}{a}d\theta^2$

正螺面的第一基本形式为 $I^*=(v^2+a^2)du^2+dv^2$

令$a^2ch^2\frac{t}{a}=v^2+a^2, u=\theta$,则有$v=ash\frac{t}{a}, u=\theta$,从而可算得$I=I^*$

因此,悬链面与正螺旋面之间有保长对应:$u=\theta, v=ash\frac{t}{a}$.

2. 证明:曲面$\bar{r}=(a(conu+\cos v), a(\sin u+\sin v), b(u+v))$和一个旋转面能够建立保长对应.

证明: 曲面可化为 $\bar{r}=(2a\cos\frac{u-v}{2}\cos\frac{u+v}{2}, 2a\cos\frac{u-v}{2}\sin\frac{u+v}{2}, 2b(\frac{u+v}{2}))$

令$u^{'}=\frac{u+v}{2}, v^{'}=2a\cos\frac{u-v}{2}$, 则曲面又化为$\bar{r}=(v^{'}\cos u^{'}, v^{'}\sin u^{'}, 2bu^{'})$

$I=(v^{'}+4b^2)du^{'}+dv^{'}$

设旋转面为$\bar{r}=(f(\tilde{v})\cos\tilde{u}, f(\tilde{v})\sin\tilde{u}, g(\tilde{v}))$, 则

$\tilde{I}=f^2(\tilde{v})d\tilde{u}^2+(f^{'2}(\tilde{v})+g^{'2}(\tilde{v}))d\tilde{v}^2$

令$\tilde{u}=u^{'}=\frac{u+v}{2}, \tilde{v}=v^{'}=2a\cos\frac{u-v}{2}$, 则

只需满足 $f^2(\tilde{v})=\tilde{v}^2+4b^2, f^{'2}(\tilde{v})+g^{'2}(\tilde{v})=1$

可取$f(\tilde{v})=\sqrt{\tilde{v}^2+4b^2}, g(\tilde{v})=2b\ln\left|\tilde{v}+\sqrt{\tilde{v}^2+4b^2}\right|$, 便有$I=\tilde{I}$.

因此曲面和旋转面$\bar{r}=(\sqrt{\tilde{v}^2+4b^2}\cos\tilde{u}, \sqrt{\tilde{v}^2+4b^2}\sin\tilde{u}, 2b\ln\left|\tilde{v}+\sqrt{\tilde{v}^2+4b^2}\right|)$

可建立保长对应:$\tilde{u}=\frac{u+v}{2}, \tilde{v}=2a\cos\frac{u-v}{2}$.

3. 证明:平面到它自身的任意一个保长对应必定是平面上的一个刚体运动(或与关于一条直线的反射的合成).

证明： 设$\vec{r}(u,v)=(u,v,0)$, $I=du^2+dv^2$

$\vec{r}^*(\tilde{u},\tilde{v})=(\tilde{u},\tilde{v},0)$, $I^*=d\tilde{u}^2+d\tilde{v}^2$

设σ: $\tilde{u}=f(u,v),\tilde{v}=g(u,v)$为两平面之间的一个保长对应,且记

$$J=\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial g}{\partial u}\\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v}\end{pmatrix},\quad I^*=\begin{pmatrix}d\tilde{u}\,d\tilde{v}\end{pmatrix}\begin{pmatrix}d\tilde{u}\\ d\tilde{v}\end{pmatrix}=\begin{pmatrix}du\,dv\end{pmatrix}JJ^T\begin{pmatrix}du\\ dv\end{pmatrix},\quad I=\begin{pmatrix}du\,dv\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}$$

由于$I=I^*$,故有$JJ^T=1$,即J为正交矩阵,从而保长对应为平面上一刚体运动(当$\det J=1$时)或刚体运动与关于一条直线的反射的合成(当$\det J=-1$时).

4. 试建立旋转面$\vec{r}=(f(u)\cos v, f(u)\sin v, g(u))$与平面的保角对应.

解： 旋转面的第一基本形式为$I=(f'^2(u)+g'^2(u))du^2+f^2(u)dv^2$

$$=f^2(u)\left(\frac{f'^2(u)+g'^2(u)}{f^2(u)}du^2+dv^2\right)$$

平面$\vec{r}=(\tilde{u},\tilde{v},0)$的第一基本形式为$\tilde{I}=d\tilde{u}^2+d\tilde{v}^2$

令$d\tilde{u}^2=\dfrac{f'^2(u)+g'^2(u)}{f^2(u)}du^2, d\tilde{v}^2=dv^2$，可取$\tilde{u}=\displaystyle\int_0^u\sqrt{\frac{f'^2(t)+g'^2(t)}{f^2(t)}}dt,\tilde{v}=v$,

则有$I=f^2(u)\tilde{I}$

故所求的保角对应为$\tilde{u}=\displaystyle\int_0^u\sqrt{\frac{f'^2(t)+g'^2(t)}{f^2(t)}}dt,\tilde{v}=v.$

5. 试建立第2题中的曲面与平面的保角对应.

解： 设平面$\vec{\tilde{r}}=(\tilde{u},\tilde{v},0)$,其第一基本形式为$\tilde{I}=d\tilde{u}^2+d\tilde{v}^2$

曲面$\vec{r}=(2a\cos\frac{u-v}{2}\cos\frac{u+v}{2},2a\cos\frac{u-v}{2}\sin\frac{u+v}{2},2b(\frac{u+v}{2}))$

令$u_1=\dfrac{u+v}{2},v_1=\dfrac{u-v}{2}$， 则曲面的第一基本形式为

$$I=(4a^2\cos^2 v_1+4b^2)du_1^2+4a^2\sin^2 v_1 dv_1^2$$

$$=(4a^2\cos^2 v_1+4b^2)\left(du_1^2+\frac{a^2\sin^2 v_1}{a^2\cos^2 v_1+b^2}dv_1^2\right)$$

令$d\tilde{u}^2=du_1^2$，$d\tilde{v}^2=\dfrac{a^2\sin^2 v_1}{a^2\cos^2 v_1+b^2}dv_1^2$

可取$\tilde{u}=u_{1},\tilde{v}=-\ln\left|\cos v_1+\sqrt{\cos^2 v_1+\frac{b^2}{a^2}}\right|$

则有$I=(4a^2\cos^2 v_1+4b^2)\tilde{I}$

故所求的保长对应为$\tilde{u}=\frac{u+v}{2},\tilde{v}=-\ln\left|\cos\frac{u-v}{2}+\sqrt{\cos^2\frac{u-v}{2}+\frac{b^2}{a^2}}\right|.$

§ 3.6 可展曲面

1.(1) 证明：曲面$\vec{r}=\left(u^2+\dfrac{v}{3},2u^3+uv,u^4+\dfrac{2u^2v}{3}\right)$是可展曲面.

证明：$\vec{r}=\left(u^2,2u^3,u^4\right)+v\left(\dfrac{1}{3},u,\dfrac{2u^2}{3}\right)\triangleq\vec{\alpha}(u)+v\vec{l}(u)$

$\vec{\alpha}'(u)=\left(2u,6u^2,4u^3\right)\Rightarrow\vec{\alpha}'(u)=6u\vec{l}(u)\Rightarrow\left(\vec{\alpha}'(u),\vec{l}(u),\vec{l}'(u)\right)=0$

(2) 证明：$\vec{r}=\left(\cos u-(u+v)\sin v,\sin v+(u+v)\cos v,u+2v\right)$是可展曲面，它是哪一类可展曲面？

证明：$\vec{r}=(\cos v,\sin v,v)+(u+v)(-\sin v,\cos v,1)\triangleq\vec{\alpha}(v)+t\vec{l}(v)$

$\vec{\alpha}'(v)=(-\sin v,\cos v,1)=\vec{l}(v)$

$\therefore\left(\vec{\alpha}'(u),\vec{l}(u),\vec{l}'(u)\right)=0$且$\vec{r}=\vec{\alpha}(v)+t\vec{\alpha}'(v)$，即$\vec{r}$为切线面.

(3) 证明：$\vec{r}=\left(a(u+v),b(u-v),2uv\right)$不是可展曲面.

证明：$\vec{r}=(au,bu,0)+v(a,-b,2u)\triangleq\vec{\alpha}(u)+v\vec{l}(u)$

$\vec{\alpha}'(u)=(a,b,0),\vec{l}'(u)=(0,0,2)\Rightarrow\left(\vec{\alpha}'(u),\vec{l}(u),\vec{l}'(u)\right)=-4ab\neq0$

2.证明：挠率不为0的曲线的主法线和次法线分别生成的直纹面都不是可展曲面.

证明：设曲线$\vec{r}=\vec{r}(s)$，s为弧参.

主法线和次法线分别生成的直纹面为：$\vec{r}_1(s)=\vec{r}(s)+t\vec{\beta}(s),\vec{r}_2(s)=\vec{r}(s)+t\vec{\gamma}(s)$

$\because\dot{\vec{r}}(s)=\vec{\alpha}(s),\dot{\vec{\beta}}(s)=-\kappa(s)\vec{\alpha}(s)+\tau(s)\vec{\gamma}(s),\dot{\vec{\gamma}}(s)=-\tau(s)\vec{\beta}(s)$

$\therefore\left(\dot{\vec{r}}(s),\vec{\beta}(s),\dot{\vec{\beta}}(s)\right)=\tau(s)\neq0,\left(\dot{\vec{r}}(s),\vec{\gamma}(s),\dot{\vec{\gamma}}(s)\right)=\tau(s)\neq0$，得证.

3.对于挠率不为0的曲线，是否有单参数法线族构成可展曲面？若有，求出所有可能的这种可展曲面.

解：单参数法线生成的直纹面：$\vec{r}_1(s,t)=\vec{r}(s)+t\left(\lambda(s)\vec{\beta}(s)+\mu(s)\vec{\gamma}(s)\right)\triangleq\vec{a}(s)+t\vec{l}(s)$

则$\vec{a}'=\dot{\vec{r}}=\vec{\alpha},\vec{l}'=-\kappa\lambda\vec{\alpha}+\left(\dot{\lambda}-\tau\mu\right)\vec{\beta}+\left(\dot{\mu}+\lambda\tau\right)\vec{\gamma}$

$\therefore\left(\vec{a}',\vec{l},\vec{l}'\right)=\lambda\dot{\mu}-\dot{\lambda}\mu+\left(\lambda^2+\mu^2\right)\tau$

若为可展曲面，则$\lambda\dot{\mu}-\dot{\lambda}\mu+\left(\lambda^2+\mu^2\right)\tau=0$

若$\mu\neq 0$,则$\left(\dfrac{\lambda}{\mu}\right)'=\left[\left(\dfrac{\lambda}{\mu}\right)^2+1\right]\tau$,即$\dfrac{\lambda}{\mu}=\tan\int\tau(s)ds$,也即$\lambda=\mu\tan\int\tau(s)ds$.($\mu=0$时也满足)

$\therefore$所有可能的可展曲面为:$\vec{r}_1(s,t)=\vec{r}(s)+t\mu(s)\left[\left(\tan\int\tau(s)ds\right)\vec{\beta}(s)+\vec{\gamma}(s)\right]$

4.已知空间挠曲线$\vec{r}=\vec{r}(s)$,s为弧参,求定义在曲线上的向量场$\vec{l}(s)=\lambda(s)\vec{\alpha}(s)+\mu(s)\vec{\gamma}(s)$,使得由$\vec{l}(s)$生成的,以已知曲线为准线的直纹面是可展曲面.

解:$\vec{r}_1(s,t)=\vec{r}(s)+t\vec{l}(s)=\vec{r}(s)+t\left(\lambda(s)\vec{\alpha}(s)+\mu(s)\vec{\gamma}(s)\right)$

$\dot{\vec{r}}=\vec{\alpha},\vec{l}'=\dot{\lambda}\vec{\alpha}+(\lambda\kappa-\mu\tau)\vec{\beta}+\dot{\mu}\vec{\gamma}\Rightarrow\left(\dot{\vec{r}},\vec{l},\vec{l}'\right)=\mu(-\lambda\kappa+\mu\tau)$

若为可展曲面,$\mu(-\lambda\kappa+\mu\tau)=0$,则$\mu=0$,或$\mu\tau=\lambda\kappa$

即$\vec{l}(s)=\lambda(s)\vec{\alpha}(s)$,或$\vec{l}(s)=\lambda(s)\left(\vec{\alpha}(s)+\dfrac{\kappa(s)}{\tau(s)}\vec{\gamma}(s)\right)$

5.设c为直纹面S上与直母线处处正交的一条曲线,曲面S沿曲线c的法线生成另一直纹面$\tilde{S}$. 证明:$\tilde{S}$是可展曲面$\Leftrightarrow S$是可展曲面.

证明:设$S:\vec{r}_1(u,v)=\vec{r}(u)+v\vec{l}(u)$,$c:\vec{r}=\vec{r}(u)$,其中$\vec{r}(u)$与$\vec{l}(u)$处处正交,即$\vec{r}'(u)\cdot\vec{l}(u)=0$.

$\tilde{S}:\vec{r}_2(u,t)=\vec{r}(u)+t\vec{n}(u)$,其中$\vec{n}(u)$为曲面$S$沿曲线$c$的法向量,不妨设$\vec{n}(u)=\vec{r}'(u)\times\vec{l}(u)$.

$\because\left(\vec{r}',\vec{n},\vec{n}'\right)=\left(\vec{r}',\vec{r}'\times\vec{l},\vec{r}''\times\vec{l}+\vec{r}\times\vec{l}'\right)=\left[\vec{r}'\times\left(\vec{r}'\times\vec{l}\right)\right]\cdot\left(\vec{r}''\times\vec{l}+\vec{r}\times\vec{l}'\right)$

$=\left[\left(\vec{r}'\cdot\vec{l}\right)\vec{r}'-\left(\vec{r}'\cdot\vec{r}'\right)\vec{l}\right]\cdot\left(\vec{r}''\times\vec{l}+\vec{r}\times\vec{l}'\right)=|\vec{r}'|^2\left(\vec{r},\vec{l},\vec{l}'\right)$

$\therefore\tilde{S}$是可展曲面$\Leftrightarrow S$是可展曲面

§4.1 第二基本形式

1.求下列曲面的第二基本形式.

$(1)\vec{r}=(a\cos\varphi\cos\theta,a\cos\varphi\sin\theta,b\sin\varphi)$

$(2)\vec{r}=\left(u,v,\frac{1}{2}(u^2+v^2)\right)$

$(3)\vec{r}=(a(u+v),a(u-v),2uv)$

解:$(1)\overrightarrow{r_\varphi}=(-a\sin\varphi\cos\theta,-a\sin\varphi\sin\theta,b\cos\varphi),\overrightarrow{r_\theta}=(-a\cos\varphi\sin\theta,a\cos\varphi\cos\theta,0)$

$\therefore\vec{n}=\pm(b\cos\varphi\cos\theta,b\cos\varphi\sin\theta,a\sin\varphi)\Big/\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}$

$\overrightarrow{r_{\varphi\varphi}}=(-a\cos\varphi\cos\theta,-a\cos\varphi\sin\theta,-b\sin\varphi),\overrightarrow{r_{\varphi\theta}}=(a\sin\varphi\sin\theta,-a\sin\varphi\cos\theta,0)$

$\overrightarrow{r_{\theta\theta}}=(-a\cos\varphi\cos\theta,-a\cos\varphi\sin\theta,0)$

$\therefore L=\overrightarrow{r_{\varphi\varphi}}\times\vec{n}=\mp ab\Big/\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi},M=\overrightarrow{r_{\varphi\theta}}\times\vec{n}=0$

$N=\overrightarrow{r_{\theta\theta}}\times\vec{n}=-ab\cos^2\varphi\Big/\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}$

$\therefore \mathrm{II}=-ab(\cos^2\varphi d\theta^2+d\varphi^2)\Big/\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}$

$(2)\overrightarrow{r_u}=(1,0,u),\overrightarrow{r_v}=(0,1,v)$

$\therefore\vec{n}=(-u,-v,1)\Big/\sqrt{1+u^2+v^2},\overrightarrow{r_{uu}}=(0,0,1),\overrightarrow{r_{uv}}=(0,0,0),\overrightarrow{r_{vv}}=(0,0,1)$

$\therefore L=\overrightarrow{r_{uu}}\times\vec{n}=\pm1\Big/\sqrt{1+u^2+v^2},M=\overrightarrow{r_{uv}}\times\vec{n}=0,N=\overrightarrow{r_{vv}}\times\vec{n}=\pm1\Big/\sqrt{1+u^2+v^2}$

$\therefore \mathrm{II}=\pm(du^2+dv^2)\Big/\sqrt{1+u^2+v^2}$

$(3)\overrightarrow{r_u}=(a,a,2v),\overrightarrow{r_v}=(a,-a,2u)$

$\therefore\vec{n}=(u+v,v-u,-a)\Big/\sqrt{2u^2+2v^2+a^2}$

$\overrightarrow{r_{uu}}=(0,0,0),\overrightarrow{r_{uv}}=(0,0,2),\overrightarrow{r_{vv}}=(0,0,0)$

$\therefore L=\overrightarrow{r_{uu}}\times\vec{n}=0,M=\overrightarrow{r_{uv}}\times\vec{n}=-a\Big/\sqrt{2u^2+2v^2+a^2},N=\overrightarrow{r_{vv}}\times\vec{n}=0$

$\therefore \mathrm{II}=-4adudv\Big/\sqrt{2u^2+2v^2+a^2}$

2.求曲线$\vec{r}=\vec{r}(s)$的切线面的第二基本形式,其中s是曲线的弧长参数.

解:$\vec{r}=\vec{r}(s)$的切线面:$\vec{r_1}(s,t)=\vec{r}(s)+t\vec{\alpha}(s)$

$\overrightarrow{r_{1s}}=\vec{\alpha}+t\kappa\vec{\beta},\overrightarrow{r_{1t}}=\vec{\alpha}$

$\therefore\vec{n}=\pm\vec{\gamma},\overrightarrow{r_{1ss}}=-t\dot{\kappa}\vec{\alpha}+(\kappa+t\dot{\kappa})\vec{\beta}+t\kappa\tau\vec{\gamma},\overrightarrow{r_{1st}}=\kappa\vec{\beta},\overrightarrow{r_{1tt}}=0$

$\therefore L=\overrightarrow{r_{1ss}}\cdot\vec{n}=\pm t\kappa\tau,M=\overrightarrow{r_{1st}}\cdot\vec{n}=0,N=\overrightarrow{r_{1tt}}\cdot\vec{n}=0$

$\therefore \mathrm{II}=t\kappa\tau ds^2$

3.求曲面$z=f(x,y)$的第一、第二基本形式.

解: $\vec{r}(x,y)=\left(x,y,f(x,y)\right),\overrightarrow{r_x}=\left(1,0,f_x\right),\overrightarrow{r_y}=\left(0,1,f_y\right)$

$\therefore \vec{n}=\left(-f_x,-f_y,1\right)\Big/\sqrt{1+f_x{}^2+f_y{}^2},\overrightarrow{r_{xx}}=\left(0,0,f_{xx}\right),\overrightarrow{r_{xy}}=\left(0,0,f_{xy}\right),\overrightarrow{r_{yy}}=\left(0,0,f_{yy}\right)$

$\therefore E=\overrightarrow{r_x}\cdot\overrightarrow{r_x}=1+f_x{}^2,F=\overrightarrow{r_x}\cdot\overrightarrow{r_y}=f_xf_y,G=\overrightarrow{r_y}\cdot\overrightarrow{r_y}=1+f_y{}^2$

$L=\overrightarrow{r_{xx}}\cdot\vec{n}=\pm f_{xx}\Big/\sqrt{1+f_x{}^2+f_y{}^2},M=\overrightarrow{r_{xy}}\cdot\vec{n}=\pm f_{xy}\Big/\sqrt{1+f_x{}^2+f_y{}^2},$

$N=\overrightarrow{r_{yy}}\cdot\vec{n}=\pm f_{yy}\Big/\sqrt{1+f_x{}^2+f_y{}^2}$

$\therefore \mathrm{I}=\left(1+f_x{}^2\right)dx^2+f_xf_ydxdy+\left(1+f_y{}^2\right)dy^2$

$\mathrm{II}=\pm\left(f_{xx}dx^2+2f_{xy}dxdy+f_{yy}dy^2\right)\Big/\sqrt{1+f_x{}^2+f_y{}^2}$

4.证明:当曲面在空间E^3中作刚体运动时, 它的Ⅰ、Ⅱ是不变的.

证明:刚体运动$f:S\to S^*,\vec{r}^*=f\left(\vec{r}\right)=\vec{r}T+\overrightarrow{r_0}$,其中$TT^T=E$,且$\det T=1$

$\therefore \overrightarrow{r_u}^*=\overrightarrow{r_u}\cdot T,\overrightarrow{r_v}^*=\overrightarrow{r_v}\cdot T,\vec{n}^*=\vec{n}\cdot T$

$\therefore d\vec{r}^*=d\vec{r}\cdot T,d\vec{n}^*=d\vec{n}\cdot T$

$\therefore \mathrm{I}^*=\left(d\vec{r}^*\right)^2=\left(d\vec{r}\cdot T\right)^2=\left(d\vec{r}\right)^2=\mathrm{I}$

$\mathrm{II}^*=-d\vec{r}^*\cdot d\vec{n}^*=-\left(d\vec{r}\cdot T\right)\cdot\left(d\vec{n}\cdot T\right)=\mathrm{II}$

5.直接证明:若在可展曲面S上存在两个不同的单参数直线族, 则S必定是平面.

证明:$S:\vec{r}(u,v)=\vec{\alpha}(u)+v\vec{l}(u)=\vec{\beta}(v)+u\vec{m}(v)$,则$\overrightarrow{r_u}=\vec{\alpha}'(u)+v\vec{l}'(u)=\vec{m}(v),\overrightarrow{r_v}=\vec{l}(u)$

从而$\overrightarrow{r_{uu}}=\overrightarrow{r_{vv}}=0,\overrightarrow{r_{uv}}=\vec{l}'(u)$,

又$\because \left(\overrightarrow{r_u}\times\overrightarrow{r_v}\right)\cdot\overrightarrow{r_{uv}}=\left(\vec{\alpha}'+v\vec{l}',\vec{l},\vec{l}'\right)=\left(\vec{\alpha}',\vec{l},\vec{l}'\right)=0$

$\therefore M=\vec{n}\cdot\overrightarrow{r_{uv}}=0,L=\overrightarrow{r_{uu}}\cdot\vec{n}=0,N=\overrightarrow{r_{vv}}\cdot\vec{n}=0$

$\therefore \mathrm{II}=0$,即S必定是平面

§ 4.2 法曲率

1.设悬链面方程为$\vec{r}=\left(\sqrt{u^2+a^2}\cos v,\sqrt{u^2+a^2}\sin v,a\ln\left(u+\sqrt{u^2+a^2}\right)\right)$,求它的Ⅰ和Ⅱ,并求它在点$(0,0)$沿切向量$d\vec{r}=2\vec{r_u}+\vec{r_v}$的法向量.

解:$\vec{r_u}=(u\cos v,u\sin v,a)\Big/\sqrt{u^2+a^2},\vec{r_v}=\sqrt{u^2+a^2}(-\sin v,\cos v,0)$

$\therefore \vec{n}=\pm(-a\cos v,-a\sin v,u)\Big/\sqrt{u^2+a^2},\overrightarrow{r_{uu}}=\left(a^2\cos v,a^2\sin v,-au\right)\Big/\sqrt{\left(u^2+a^2\right)^3}$

$\overrightarrow{r_{uv}}=(-u\sin v,u\cos v,0)\Big/\sqrt{u^2+a^2},\overrightarrow{r_{vv}}=\sqrt{u^2+a^2}(-\cos v,-\sin v,0)$

$\therefore E=\vec{r_u}^{\,2}=1,F=\vec{r_u}\cdot\vec{r_v}=0,G=\vec{r_v}^{\,2}=u^2+a^2$

$L=\overrightarrow{r_{uu}}\cdot\vec{n}=\mp\dfrac{a}{u^2+a^2},M=\overrightarrow{r_{uv}}\cdot\vec{n}=0,N=\overrightarrow{r_{vv}}\cdot\vec{n}=0$

$\therefore \mathrm{I}=du^2+\left(u^2+a^2\right)dv^2,\ \mathrm{II}=\pm\left(-\dfrac{a}{u^2+a^2}du^2+adv^2\right)$

切向量$d\vec{r}=2\vec{r_u}+\vec{r_v}$的方向为$(du,dv)=(2,1),\kappa_n=\left.\dfrac{\mathrm{II}}{\mathrm{I}}\right|_{(2,1)}=\dfrac{-4+a^2}{a\left(4+a^2\right)}$

2.证明:曲面上一条曲线在任意一点的法曲率等于该曲线在该点由其切向量决定的法截面上的投影曲线在该点的相对曲率.

证明:法截面由$\left\{\vec{\alpha},\vec{n}\right\}$张成,则曲线$\vec{r}(s)$在法截面上的投影为

$\vec{r_1}(s)=\left(\vec{r}(s)\vec{\alpha}(s_0),\vec{r}(s)\vec{n}(s_0)\right)\triangleq(x,y)$

$\therefore \vec{r_1}'(s)=\left(\vec{\alpha}(s)\cdot\vec{\alpha}(s_0),\vec{\alpha}(s)\cdot\vec{n}(s_0)\right),\vec{r_1}''(s)=\left(\kappa(s)\vec{\beta}(s)\cdot\vec{\alpha}(s_0),\kappa(s)\vec{\beta}(s)\cdot\vec{n}(s_0)\right)$

$\therefore \vec{r_1}'(s_0)=(1,0),\vec{r_1}''(s_0)=(0,\kappa_n(s_0))$

$\therefore \kappa_r=\begin{vmatrix}x' & y'\\ x'' & y''\end{vmatrix}\Big/|\vec{r}'|^3=\kappa_n$

3.求下列曲面上的渐近曲线:

(1)正螺旋面:$\vec{r}=(u\cos v,u\sin v,bv)$

(2)双曲抛物面:$\vec{r}=\left(\dfrac{u+v}{2},\dfrac{u-v}{2},\dfrac{uv}{2}\right)$

解:$(1)(2)L=N=0$,参数曲线网为渐近曲线网.

4.设c为曲面上一非直线的渐近曲线,其参数方程为$u=u(s),v=v(s)$,其中s为弧参.

证明:c的挠率等于$\tau=\dfrac{1}{\sqrt{EG-F^2}}\begin{vmatrix}(\dot v)^2 & -\dot u\dot v & (\dot u)^2\\ E & F & G\\ L & M & N\end{vmatrix}$

证明:因c为非直线的渐近曲线,故由定理3知,$\vec\beta\cdot\vec n=0$,又因$\vec\alpha\cdot\vec n=0$,故$\vec n=\pm\vec\gamma$.

$$\tau=-\dot{\vec\gamma}\cdot\vec\beta=-\left(\pm\dot{\vec n}\right)\cdot\left(\pm\vec n\times\vec\alpha\right)=-\dot{\vec n}\cdot\left(\vec n\times\vec\alpha\right)=\left(\dot{\vec r},\vec n,\dot{\vec n}\right)=\frac{1}{|\vec r_u\times\vec r_v|}\left(\dot u\vec r_u+\dot v\vec r_v,\vec r_u\times\vec r_v,\dot u\vec n_u+\dot v\vec n_v\right)$$

$$=\frac{1}{|\vec r_u\times\vec r_v|}\left\{\left[\left(\dot u\vec r_u+\dot v\vec r_v\right)\cdot\vec r_v\right]\left[\vec r_u\left(\dot u\vec n_u+\dot v\vec n_v\right)\right]-\left[\left(\dot u\vec r_u+\dot v\vec r_v\right)\cdot\vec r_u\right]\left[\vec r_v\left(\dot u\vec n_u+\dot v\vec n_v\right)\right]\right\}$$

$$=\frac{1}{\sqrt{EG-F^2}}\left[\left(F\dot u+G\dot v\right)\left(L\dot u+M\dot v\right)+\left(E\dot u+F\dot v\right)\left(M\dot u+N\dot v\right)\right]=\text{所求}.$$

5.设n为正整数,则$\overrightarrow{\alpha_n}=\left(r\cos t,r\sin t,\operatorname{sgn}t\cdot|t|^n\right)$落在圆柱面$x^2+y^2=r^2$上,试求曲线$\overrightarrow{\alpha_n}$在$t=0$处的法曲率.验证:当$n\geq 2$时,$\overrightarrow{\alpha_n}$在$t=0$处的曲率中心在一个圆周上,写出这个圆周的方程.

解:圆柱面S的方程是:$\vec r=\left(r\cos\dfrac{u}{r},r\sin\dfrac{u}{r},v\right)$,$\mathrm{I}=du^2+dv^2$,$\mathrm{II}=\pm\dfrac{1}{r}du^2$

$$\overrightarrow{\alpha_n}\text{中},\begin{cases}u=rt\\ v=\operatorname{sgn}t\cdot|t|^n\end{cases}\Rightarrow\begin{cases}u_t=r\\ v_t=\begin{cases}n|t|^{n-1}\left(n\geq 2\right)\\ 1\left(n=1\right)\end{cases}\end{cases}$$

$$\therefore\kappa_n\big|_{t=0}=\frac{\mathrm{II}}{\mathrm{I}}\bigg|_{t=0}=\begin{cases}\left.\dfrac{\pm\frac{1}{r}r^2}{r^2+n^2t^{2n-2}}\right|_{t=0}=\pm\dfrac{1}{r}\left(n\geq 2\right)\\ \left.\dfrac{\pm\frac{1}{r}r^2}{r^2+1}\right|_{t=0}=\pm\dfrac{r}{r^2+1}\left(n=1\right)\end{cases}$$

已知曲率中心C在以$\overrightarrow{\alpha_n}(t)-\dfrac{1}{2\kappa_n}\overrightarrow{n_n}(t)$为中心,$\dfrac{1}{2\kappa_n}$为半径的圆$c_n$上

$n\geq 2,t=0$时,$\because\overrightarrow{\alpha_n}(t)=\left(r,0,0\right),\overrightarrow{\alpha_n}'(t)=\left(0,r,0\right)\therefore\overrightarrow{n_n}(t)=\left(1,0,0\right)$

$\therefore c_n$都可表示为$\left(x-\dfrac{r}{2}\right)^2+z^2=\left(\dfrac{r}{2}\right)^2$,得证.

§4.3 Gauss 映射和 Weingarten 映射

1.证明:在曲面上任意一点，任意两个彼此正交的切方向上的法曲率之和是一常数.

证明:曲面上任一点P, 设$\{e_1,e_2\}$是P的两个彼此正交的主方向单位向量, 对应的主曲率是κ_1,κ_2,则在点P沿两个彼此正交的切向量$e^{(1)}=e_1\cos\theta+e_2\sin\theta, e^{(2)}=e_1\cos\left(\theta\pm\frac{\pi}{2}\right)+e_2\sin\left(\theta\pm\frac{\pi}{2}\right)$的法曲率分别是:

$\kappa^{(1)}=\kappa_1\cos^2\theta+\kappa_2\sin^2\theta, \kappa^{(2)}=\kappa_1\sin^2\theta+\kappa_2\cos^2\theta$

$\therefore \kappa^{(1)}+\kappa^{(2)}=\kappa_1+\kappa_2=const$

2.设曲面S_1,S_2的交线c_1的曲率是κ,曲线c_1在曲面S_i上的法曲率是${\kappa_n}^{(i)}\left(i=1,2\right)$.假定$S_1$和$S_2$在交点的法线之间的夹角是$\theta$.证明:

$\kappa^2\sin^2\theta=\left({\kappa_n}^{(1)}\right)^2+\left({\kappa_n}^{(2)}\right)^2-2{\kappa_n}^{(1)}{\kappa_n}^{(2)}\cos\theta$

证明:设c_1的主法向量为$\vec{\beta}$, S_1、S_2在交线上的单位法向量分别是$\overrightarrow{n_1}$、$\overrightarrow{n_2}$，则$\angle\left(\overrightarrow{n_1},\overrightarrow{n_2}\right)=\theta$

且记$\angle\left(\vec{\beta},\overrightarrow{n_1}\right)=\theta_1$，$\angle\left(\vec{\beta},\overrightarrow{n_2}\right)=\theta_2$，由于$\vec{\beta},\overrightarrow{n_1},\overrightarrow{n_2}$都在法平面上，有$\theta_1=\theta_2\pm\theta$

$\because {\kappa_n}^{(1)}=\kappa\cos\theta_1=\kappa\cos\left(\theta_2\pm\theta\right), {\kappa_n}^{(2)}=\kappa\cos\theta_2$

$\therefore \left({\kappa_n}^{(1)}\right)^2+\left({\kappa_n}^{(2)}\right)^2-2{\kappa_n}^{(1)}{\kappa_n}^{(2)}\cos\theta=\kappa^2\left(\cos^2\left(\theta_2\pm\theta\right)+\cos^2\theta_2-2\cos\left(\theta_2\pm\theta\right)\cos\theta_2\cos\theta\right)$

$=\kappa^2\sin^2\theta$

3.证明:在可展曲面上, 直母线既是渐近线, 又是曲率线;直母线的正交轨线是另一族曲率线.

证明；直母线是直线$\Rightarrow$直母线是渐近线

可展曲面沿直母线的切平面不变, 故法向量不变, 从而曲面沿直母线的法线展成平面(一种特殊的可展曲面)$\Rightarrow$直母线是曲率线

两确定的主方向正交, 若不确定, 任一主方向的正交方向也是主方向$\Rightarrow$直母线的正交轨线是另一族曲率线

4.设曲面上的一条曲率线不是渐近曲线,并且它的密切平面与曲面的切平面交成定角,证明该曲线必是平面曲线.

证明:该曲线是曲率线,$\therefore \vec{n}'//\vec{r}'//\vec{\alpha}$

又因为密切平面与切平面交成定角,$\therefore 0=\left(\vec{\gamma}\cdot\vec{n}\right)'=\vec{\gamma}'\cdot\vec{n}+\vec{\gamma}\cdot\vec{n}'=\vec{\gamma}'\cdot\vec{n}=-\tau|\vec{r}'|\vec{\beta}\cdot\vec{n}$

已知$|\vec{r}'|\neq 0$,若$\vec{\beta}\cdot\vec{n}=0$,由4.2节定理3知, 曲线为渐近线, 与题意矛盾,$\therefore \tau=0$,得证

5.假定两个可展曲面相交成一条曲线,并且这条曲线与两个可展曲面的直母线分别正交. 证明:这两条曲面在各交点交成定角.

证明:由题3,这条交线是两曲面的曲率线,$\therefore \overrightarrow{n_i}' // \vec{r}'\ (i=1,2)$

又$\because \overrightarrow{n_i} \perp \vec{r}', \therefore \overrightarrow{n_i} \cdot \overrightarrow{n_j}' = 0\ (i,j=1,2)$

$\therefore \left(\overrightarrow{n_1} \cdot \overrightarrow{n_2}\right)' = \overrightarrow{n_1}' \cdot \overrightarrow{n_2} + \overrightarrow{n_1} \cdot \overrightarrow{n_2}' = 0, \therefore \overrightarrow{n_1} \cdot \overrightarrow{n_2} = const$,即两条曲面在各交点交成定角

6.证明:在曲面上任意一点P的某个邻域内都能取正交参数系(u,v), 使得参数曲线在该点的切方向是彼此正交的主方向.

证明:设曲面$S: \vec{r}=\vec{r}(u,v)$,对曲面上任一点P,在T_PS上总是可取单位正交基$\left\{\overrightarrow{c_1}, \overrightarrow{c_2}\right\}$,使得$\overrightarrow{c_1}, \overrightarrow{c_2}$是曲面$S$在$P$点的主方向

另一方面, 将$\left\{\overrightarrow{r_u}, \overrightarrow{r_v}\right\}$ $Schmidt$正交化,得到$\left\{\overrightarrow{e_1}, \overrightarrow{e_2}\right\}$, 作为$T_PS$的活动基底.设$P=P(u_0,v_0)$,则$\exists a_1, a_2, b_1, b_2, s.t.$

$$\begin{cases} \overrightarrow{c_1} = a_1\overrightarrow{e_1}(u_0,v_0) + b_1\overrightarrow{e_2}(u_0,v_0) \\ \overrightarrow{c_2} = a_2\overrightarrow{e_1}(u_0,v_0) + b_2\overrightarrow{e_2}(u_0,v_0) \end{cases}$$

根据这样的a_1, a_2, b_1, b_2,设

$$\begin{cases} \overrightarrow{d_1}(u,v) = a_1\overrightarrow{e_3}(u,v) + b_1\overrightarrow{e_4}(u,v) \\ \overrightarrow{d_2}(u,v) = a_2\overrightarrow{e_3}(u,v) + b_2\overrightarrow{e_4}(u,v) \end{cases}$$

$\because \overrightarrow{c_1} \cdot \overrightarrow{c_2} = 0 \therefore a_1a_2 + b_1b_2 = 0 \therefore \overrightarrow{d_1}(u,v) \cdot \overrightarrow{d_2}(u,v) = 0$

从而$\overrightarrow{d_1}, \overrightarrow{d_2}$是曲面上两个处处线性无关的连续可微的切向量场,且$\overrightarrow{d_1}(P)=\overrightarrow{c_1}, \overrightarrow{d_2}(P) = \overrightarrow{c_2}$

从而在点P的某个邻域上存在新参数系$(\tilde{u},\tilde{v})$,使得$\vec{r}_{\tilde{u}} // \overrightarrow{d_1}, \vec{r}_{\tilde{v}} // \overrightarrow{d_2}$,满足题意

7.设在曲面上一个固定点与一个主方向的夹角为的切方向所对应的法曲率, 记为$\kappa_n(\theta)$.证明:$\dfrac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta = H$.其中$H=\dfrac{1}{2}(\kappa_1+\kappa_2)$.

证明:$\dfrac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta = \dfrac{1}{2\pi}\int_0^{2\pi} \left(\kappa_1\cos^2\theta + \kappa_2\sin^2\theta\right) d\theta = \dfrac{1}{2\pi}(\kappa_1\pi + \kappa_2\pi) = \dfrac{1}{2}(\kappa_1+\kappa_2)$
$= H$

8.在非脐点处, 如果夹角为θ_0的任意两个切方向的法曲率之和为常数, 则该夹角θ_0必为$\dfrac{\pi}{2}$.

证明:不妨设两切向量为$e^{(1)} = e_1\cos\theta + e_2\sin\theta, e^{(2)} = e_1\cos(\theta+\theta_0) + e_2\sin(\theta+\theta_0)$,其中$\{e_1, e_2\}$是曲面在该非脐点处的主方向单位向量,$0 \le \theta_0 \le \pi$

则$\kappa_n^{(1)} + \kappa_n^{(2)} = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta + \kappa_1\cos^2(\theta+\theta_0) + \kappa_2\sin^2(\theta+\theta_0) = const$

$$\therefore 0=\frac{d\left(\kappa_n^{(1)}+\kappa_n^{(2)}\right)}{d\theta}=\left(\kappa_2-\kappa_1\right)\left(\sin 2\theta+\sin 2\left(\theta+\theta_0\right)\right)$$

由非脐点知$\kappa_1 \neq \kappa_2$,

$\therefore 0=\sin 2\theta+\sin 2\left(\theta+\theta_0\right)=2\sin\left(\theta+\theta_0\right)\cos\theta_0$,对一切$\theta$都成立

$$\therefore \theta_0=\frac{\pi}{2}$$

§ 4.4 主方向和主曲率的计算

1. 求螺面 $\vec{r}=(u\cos v,u\sin v,u+v)$ 的 Gauss 曲率和平均曲率.

解： $\vec{r}_u=(\cos v,\sin v,1),\quad \vec{r}_v=(-u\sin v,u\cos v,1)$

$$\vec{n}=\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}=\frac{1}{\sqrt{2u^2+1}}(\sin v-u\cos v,-\cos v-u\sin v,u)$$

$$\vec{r}_{uu}=0,\quad \vec{r}_{uv}=(-\sin v,\cos v,0),\quad \vec{r}_{vv}=(-u\cos v,-u\sin v,0)$$

$$\therefore E=\vec{r}_u^{\,2}=2,\quad F=\vec{r}_u\vec{r}_v=1,\quad G=\vec{r}_v^{\,2}=u^2+1$$

$$L=\vec{r}_{uu}\vec{n}=0,\quad M=\vec{r}_{uv}\vec{n}=-\frac{1}{\sqrt{2u^2+1}},\quad N=\vec{r}_{vv}\vec{n}=\frac{u^2}{\sqrt{2u^2+1}}$$

$$\therefore K=\frac{LN-M^2}{EG-F^2}=-\frac{1}{(2u^2+1)^2}\qquad H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{u^2+1}{\sqrt{(2u^2+1)^3}}$$

2. 设曲面 S 在一点的两两夹角为 $\frac{2\pi}{m}$ 的 m 个切向量所对应的法曲率为 $k_n^{(1)},\cdots,k_n^{(m)}$.

证明: 当 $m>2$ 时有 $H=\frac{1}{m}(k_n^{(1)}+\cdots+k_n^{(m)})$.

证明： 设$\{\vec{e}_1,\vec{e}_2\}$是在一点P的两个彼此正交的主方向单位向量，对应的的主曲率为k_1,k_2. m个切向量为$\vec{d}_1,\cdots,\vec{d}_m$,其中$\vec{d}_1$与$\vec{e}_1$的夹角为$\theta$，则

$$k_n^{(1)}=k_1\cos^2\theta+k_2\sin^2\theta$$

$$k_n^{(i)}=k_1\cos^2(\theta+\frac{(i-1)\cdot 2\pi}{m})+k_2\sin^2(\theta+\frac{(i-1)\cdot 2\pi}{m}),2\le i\le m.$$

$$\therefore \sum_{i=1}^m k_n^{(i)}=k_1\sum_{i=1}^m\cos^2(\theta+\frac{(i-1)\cdot 2\pi}{m})+k_2\sum_{i=1}^m\sin^2(\theta+\frac{(i-1)\cdot 2\pi}{m})$$

$$=\frac{k_1+k_2}{2}m+\frac{k_1}{2}\sum_{i=1}^m\cos(2\theta+\frac{(i-1)\cdot 4\pi}{m})-\frac{k_2}{2}\sum_{i=1}^m\cos(2\theta+\frac{(i-1)\cdot 4\pi}{m})$$

$$=m\frac{k_1+k_2}{2}+(\frac{k_1}{2}-\frac{k_2}{2})\sum_{i=1}^m\cos(2\theta+\frac{4(i-1)\pi}{m})$$

$$\text{又}\because \sum_{i=1}^{m}\cos(2\theta+\frac{4(i-1)\pi}{m})=\frac{1}{2\sin\frac{2\pi}{m}}\sum_{i=1}^{m}2\sin\frac{2\pi}{m}\cos(2\theta+\frac{4(i-1)\pi}{m})$$

$$=\frac{1}{2\sin\frac{2\pi}{m}}\sum_{i=1}^{m}\left[\sin(2\theta+\frac{2\pi+4(i-1)\pi}{m})-\sin(2\theta+\frac{4(i-1)\pi-2\pi}{m})\right]$$

$$=\frac{1}{2\sin\frac{2\pi}{m}}\left[-\sin(2\theta-\frac{2\pi}{m})+\sin(2\theta+\frac{2\pi+4(m-1)\pi}{m})\right]$$

$$=0$$

$$\therefore \sum_{i=1}^{m}k_n^{(i)}=m\frac{k_1+k_2}{2}=mH \quad \Rightarrow H=\frac{1}{m}\sum_{i=1}^{m}k_n^{(i)}$$

3. 求双曲抛物面$\vec{r}=(a(u+v),b(u-v),2uv)$的Gauss曲率，平均曲率$H$，主曲率$k_1,k_2$及对应的主方向.

证明: $\vec{r}_u=(a,b,2v)$, $\vec{r}_v=(a,-b,2u)$

$$\vec{r}_{uu}=\vec{r}_{vv}=0,\quad \vec{r}_{uv}=(0,0,2),\quad \vec{n}=\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}=\frac{(b(u+v),a(v-u),-ab)}{\sqrt{b^2(u+v)^2+a^2(v-u)^2+a^2b^2}}$$

$$\Rightarrow E=a^2+b^2+4v^2,\quad F=a^2-b^2+4uv,\quad G=a^2+b^2+4u^2$$

$$L=N=0,\quad M=\frac{-2ab}{\sqrt{b^2(u+v)^2+a^2(v-u)^2+a^2b^2}}$$

$$\Rightarrow K=\frac{LN-M^2}{EG-F^2}=\frac{-a^2b^2}{\left[a^2b^2+(u^2+v^2)(a^2+b^2)-2uv(a^2-b^2)\right]^2}$$

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{ab(a^2-b^2+4uv)}{2\left[a^2b^2+(u^2+v^2)(a^2+b^2)-2uv(a^2-b^2)\right]^{3/2}}$$

$$\text{主方向满足:}\begin{vmatrix}\delta v^2 & -\delta u\delta v & \delta u^2\\ a^2+b^2+4v^2 & a^2-b^2+4uv & a^2+b^2+4u^2\\ 0 & M & 0\end{vmatrix}=0$$

$$\Rightarrow\frac{\delta u}{\delta v}=\frac{\sqrt{a^2+b^2+4u^2}}{\sqrt{a^2+b^2+4v^2}},\quad \text{对应的主曲率为}k_1=\frac{II}{I}=\frac{-2ab}{\sqrt{EG-F^2}(\sqrt{EG}+F)}$$

$$\frac{\delta u}{\delta v}=-\frac{\sqrt{a^2+b^2+4u^2}}{\sqrt{a^2+b^2+4v^2}},\quad \text{对应的主曲率为}k_2=\frac{II}{I}=\frac{-2ab}{\sqrt{EG-F^2}(\sqrt{EG}-F)}$$

4. 设在曲线$\vec{r}=\vec{r}(s)$的所有法线上截取长度为λ的一段,它的端点的轨迹构成一个管状面,其方程可以表为

$\bar{r}(s,\theta)=\bar{r}(s)+\lambda(\cos\theta\beta(s)+\sin\theta\gamma(s)),$

其中β,γ分别是曲线$\bar{r}(s)$的主法向量和次法向量.求该曲面上各点的主曲率k_1,k_2及Gauss曲率和平均曲率.

解：$\bar{r}_s=\bar{\alpha}+\lambda\left[\cos\theta(-k\bar{\alpha}+\tau\bar{\gamma})+\sin(-\tau\bar{\beta})\right]$

$$=(1-\lambda k\cos\theta)\bar{\alpha}-\lambda\tau\sin\theta\bar{\beta}+\lambda\tau\cos\theta\bar{\gamma}$$

$$\bar{r}_\theta=-\lambda\sin\theta\bar{\beta}+\lambda\cos\theta\bar{\gamma}$$

$$\bar{n}=\frac{1}{\lambda(\lambda k\cos\theta-1)}(\lambda\cos\theta(\lambda k\cos\theta-1)\bar{\beta}+\lambda\sin\theta(\lambda k\cos\theta-1)\bar{\gamma})$$

$$=\cos\theta\bar{\beta}+\sin\theta\bar{\gamma}$$

$$\bar{r}_{ss}=(-\lambda\dot{k}\cos\theta+\lambda k\tau\sin\theta)\bar{\alpha}+(k-\lambda k^2\cos\theta-\lambda\dot{\tau}\sin\theta-\lambda\tau^2\cos\theta)\bar{\beta}+$$
$$(\lambda\dot{\tau}\cos\theta-\lambda\tau^2\sin\theta)\bar{\gamma}$$

$$\bar{r}_{s\theta}=\lambda k\sin\theta\bar{\alpha}-\lambda\tau\cos\theta\bar{\beta}-\lambda\tau\sin\theta\bar{\gamma}$$

$$\bar{r}_{\theta\theta}=-\lambda\cos\theta\bar{\beta}-\lambda\sin\theta\bar{\gamma}$$

$$\therefore E=(1-\lambda k\cos\theta)^2+\lambda^2\tau^2\sin^2\theta+\lambda^2\tau^2\cos^2\theta=(1-\lambda k\cos\theta)^2+\lambda^2\tau^2$$

$$F=\lambda^2\tau,\quad G=\lambda^2$$

$$L=\bar{r}_{ss}\bar{n}=k\cos\theta-\lambda k^2\cos^2\theta-\lambda\tau^2,\quad M=\bar{r}_{s\theta}\bar{n}=-\lambda\tau,\quad N=\bar{r}_{\theta\theta}\bar{n}=-\lambda$$

$$\therefore K=\frac{LN-M^2}{EG-F^2}=\frac{k\cos\theta}{\lambda(\lambda k\cos\theta-1)},\quad H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{1-2\lambda k\cos\theta}{2\lambda(\lambda k\cos\theta-1)}$$

$$k_1=H-\sqrt{H^2-K}=-\frac{1}{\lambda},\quad k_2=H+\sqrt{H^2-K}=-\frac{k\cos\theta}{\lambda k\cos\theta-1}$$

5. 在曲面$\bar{r}=\bar{r}(u,v)$上每一点沿法线方向截取长度为λ的一段（假定λ充分小），其端点的轨迹构成曲面$\bar{r}^*(u,v)=\bar{r}(u,v)+\lambda\bar{n}(u,v)$.从点$\bar{r}(u,v)$到点$\bar{r}^*(u,v)$的对应记作$\sigma$.

(1) 证明:两个曲面在对应点的切平面互相平行.

(2) 证明:σ把曲面$\bar{r}(u,v)$上的曲率线映为曲面$\bar{r}^*(u,v)$上的曲率线.

(3) 在对应点的Gauss曲率和平均曲率有下列关系:

$$K^*=\frac{K}{1-2\lambda H+\lambda^2K},\quad H^*=\frac{H-\lambda K}{1-2\lambda H+\lambda^2K}.$$

证明:(1). $\bar{r}_u^*=\bar{r}_u+\lambda\bar{n}_u,\quad \bar{r}_v^*=\bar{r}_v+\lambda\bar{n}_v$

$$\bar{r}_u^*\times\bar{r}_v^*=\bar{r}_u\times\bar{r}_v+\lambda\bar{r}_u\times\bar{n}_v+\lambda\bar{n}_u\times\bar{r}_v+\lambda^2\bar{n}_u\times\bar{n}_v$$

$\because\left|\bar{n}\right|=1,\quad\therefore\bar{n}_u\cdot\bar{n}=\bar{n}_v\cdot\bar{n}=0,$ 即$\bar{n}_u\perp\bar{n},\bar{n}_v\perp\bar{n}$

又$\because \vec{r}_u \perp \vec{n}, \vec{r}_v \perp \vec{n}, \quad \therefore \vec{r}_u \times \vec{r}_v, \vec{r}_u \times \vec{n}_v, \vec{n}_u \times \vec{r}, \vec{n}_u \times \vec{n}_v$均与$\vec{n}$平行

$\therefore \vec{r}_u^* \times \vec{r}_v^*$与$\vec{n}$平行,从而$\vec{n}^*$与$\vec{n}$平行,因此两曲面在对应点的切平面互相平行.

(2). 由(1)知, $\vec{n}^*$与$\vec{n}$平行,又因$\left|\vec{n}^*\right| = |\vec{n}| = 1, \quad$ 故$\vec{n}^* = \pm \vec{n} \quad \Rightarrow d\vec{n}^*$与$d\vec{n}$平行

若$\vec{r}(u(t), v(t))$为$\vec{r}(u,v)$上的曲率线,则

$d\vec{r}^* = d\vec{r} + \lambda d\vec{n} = (-\frac{1}{k_n} + \lambda) d\vec{n} \quad \Rightarrow d\vec{r}^*$与$d\vec{n}$平行,故$d\vec{r}^*$与$d\vec{n}^*$平行

由$Rodriques$定理知, $\vec{r}^*(u(t), v(t))$也是曲率线.

(3). 由(2)得, $d\vec{n}^* = \dfrac{1}{\lambda - \dfrac{1}{k_n}} d\vec{r}^* = \dfrac{k_n}{\lambda k_n - 1} d\vec{r}^* \quad \Rightarrow k_n^* = \dfrac{k_n}{1 - \lambda k_n}$

$$\Rightarrow k_1^* = \frac{k_1}{1-\lambda k_1}, k_2^* = \frac{k_2}{1-\lambda k_2}$$

$$\Rightarrow k^* = k_1^* \cdot k_2^* = \frac{k_1 k_2}{1 - \lambda(k_1+k_2) + \lambda^2 k_1 k_2} = \frac{K}{1 - 2\lambda H + \lambda^2 K}$$

$$H^* = \frac{k_1^* + k_2^*}{2} = \frac{(k_1+k_2) - 2\lambda k_1 k_2}{2[1 - \lambda(k_1+k_2) + \lambda^2 k_1 k_2]} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}$$

6. 证明:(1) $\begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \dfrac{1}{\sqrt{EG-F^2}} \begin{pmatrix} -\vec{n}_u \cdot (\vec{r}_v \times \vec{n}) & \vec{n}_u \cdot (\vec{r}_u \times \vec{n}) \\ -\vec{n}_v \cdot (\vec{r}_v \times \vec{n}) & \vec{n}_v \cdot (\vec{r}_u \times \vec{n}) \end{pmatrix}$.

(2) $\begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \cdot \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

证明:(1) $\begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \dfrac{1}{EG-F^2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

$$= \frac{1}{EG-F^2} \begin{pmatrix} LG - MF & -LF + ME \\ MG - NF & -MF + NE \end{pmatrix}$$

而$\begin{pmatrix} -\vec{n}_u \cdot (\vec{r}_v \times \vec{n}) & \vec{n}_u \cdot (\vec{r}_u \times \vec{n}) \\ -\vec{n}_v \cdot (\vec{r}_v \times \vec{n}) & \vec{n}_v \cdot (\vec{r}_u \times \vec{n}) \end{pmatrix} = \dfrac{1}{\sqrt{EG-F^2}} \begin{pmatrix} -\vec{n}_u \cdot [\vec{r}_v \times (\vec{r}_u \times \vec{r}_v)] & \vec{n}_u \cdot [\vec{r}_u \times (\vec{r}_u \times \vec{r}_v)] \\ -\vec{n}_v \cdot [\vec{r}_v \times (\vec{r}_u \times \vec{r}_v)] & \vec{n}_v \cdot [\vec{r}_u \times (\vec{r}_u \times \vec{r}_v)] \end{pmatrix}$

$$= \frac{1}{\sqrt{EG-F^2}} \begin{pmatrix} \vec{n}_u \cdot [(\vec{r}_u \vec{r}_v)\vec{r}_v - (\vec{r}_v \vec{r}_v)\vec{r}_u] & -\vec{n}_u \cdot [(\vec{r}_u \vec{r}_u)\vec{r}_v - (\vec{r}_v \vec{r}_u)\vec{r}_u] \\ \vec{n}_v \cdot [(\vec{r}_u \vec{r}_v)\vec{r}_v - (\vec{r}_v \vec{r}_v)\vec{r}_u] & -\vec{n}_v \cdot [(\vec{r}_u \vec{r}_u)\vec{r}_v - (\vec{r}_v \vec{r}_u)\vec{r}_u] \end{pmatrix}$$

$= \dfrac{1}{\sqrt{EG-F^2}} \begin{pmatrix} LG - MF & -LF + ME \\ MG - NF & -MF + NE \end{pmatrix}$ 得证.

(2) $\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \vec{n}_u \\ \vec{n}_v \end{pmatrix} \begin{pmatrix} \vec{n}_u & \vec{n}_v \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \vec{r}_u \\ \vec{r}_v \end{pmatrix} \begin{pmatrix} \vec{r}_u & \vec{r}_v \end{pmatrix} \left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \right)^T$

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}^T = \begin{pmatrix} L & M \\ M & N \end{pmatrix}\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^T\right)^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

得证.

§ 4.5 Dupin 标形和曲面在一点的标准展开

1. 设旋转曲面的经线有水平切线,证明:这些切点都是曲面的抛物点.

证明: 设旋转曲面的参数方程为$\bar{r}(u,v)=(f(v)\cos u, f(v)\sin u, g(v)),(f(v)>0)$

因S的经线为$v-$曲线,即$u=u_0$, $\bar{r}(u_0,v)=(f(v)\cos u_0, f(v)\sin u_0, g(v))$

经线的切线为$\bar{r}_v=(f^{'}(v)\cos u_0, f^{'}(v)\sin u_0, g^{'}(v))$.又因经线有水平切线,故

$g^{'}(v)=0$,从而$K=\dfrac{g^{'}(g^{"}f^{'}-f^{"}g^{'})}{f(f^{'}+g^{'})^2}=0$

因此这些切点都是曲面的抛物点.

2. 求曲面$\bar{r}=(u^3,v^3,u+v)$上的抛物点轨迹.

解: $\bar{r}_u=(3u^2,0,1)$, $\bar{r}_v=(0,3v^2,1)$, $\bar{n}=\dfrac{1}{\sqrt{u^4+v^4+9u^4v^4}}(v^2,u^2,-3u^2v^2)$

$\bar{r}_{uu}=(6u,0,0)$, $\bar{r}_{uv}=(0,0,0)$, $\bar{r}_{vv}=(0,6v,0)$

$\therefore L=\bar{r}_{uu}\bar{n}=\dfrac{6uv^2}{\sqrt{EG-F^2}}$, $M=0$, $N=\bar{r}_{vv}\bar{n}=\dfrac{6u^2v}{\sqrt{EG-F^2}}$

抛物点$\Leftrightarrow K=\dfrac{LN-M^2}{EG-F^2}=0\Leftrightarrow LN-M^2=0\Leftrightarrow 36u^3v^3=0\Leftrightarrow u=0$或$v=0$

故所求抛物点的轨迹为$\bar{r}_1=(u^3,0,u)$,$\bar{r}_2=(0,v^3,v)$.

3. 研究4.4的习题4中管状曲面上,各种类型点的分布.

解: $K=\dfrac{-k\cos\theta}{\lambda(1-\lambda k\cos\theta)}$

(i). 当$k=0$时,$K=0$,为抛物点

(ii). 当$k\neq 0$时,K的符号由$\cos\theta$决定

当$\theta=\dfrac{\pi}{2}$或$\dfrac{3\pi}{2}$时,$K=0$,为抛物点

当$\dfrac{\pi}{2}<\theta<\dfrac{3\pi}{2}$时,$K>0$,为椭圆点

当$0\le\theta<\dfrac{\pi}{2}$或$\dfrac{3\pi}{2}<\theta\le 2\pi$时,$K<0$,为双曲点.

4. 设θ是曲面上的一个双曲点的两个渐进方向的夹角. 证明:

(1) $tg\theta=\dfrac{\sqrt{-K}}{H}$

(2) $\cos\theta=\pm\dfrac{EN-2FM+GL}{\sqrt{(EN-GL)^2+4(EM-FL)(GM-FN)^2}}$,

其中$E,F,G;L,M,N$分别是曲面在该点的第一类、第二类基本量.

证明:(1) 在双曲点上,$LN-M^2<0$,故方程$Ldu^2+2Mdudv+Ndv^2=0$

在该双曲点的一个邻域U内有两个不同的解,

即U上每一点都存在两个渐进方向.

$\therefore$在曲面上可取渐进曲线网为参数曲线网,从而有$L=N=0$

$$\therefore\cos\theta=\cos\angle(\vec{r}_u,\vec{r}_v)=\frac{\vec{r}_u\vec{r}_v}{|\vec{r}_u||\vec{r}_v|}=\frac{F}{\sqrt{EG}},\quad tg\theta=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}=\frac{\sqrt{EG-F^2}}{F}$$

$$\text{又}\because H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{-MF}{EG-F^2},\quad K=\frac{LN-M^2}{EG-F^2}=\frac{-M^2}{EG-F^2}$$

$$\therefore\frac{\sqrt{-K}}{H}=\frac{\mp M}{\sqrt{EG-F^2}}\cdot\frac{EG-F^2}{-MF}=\pm\frac{\sqrt{EG-F^2}}{F},\quad\therefore tg\theta=\frac{\sqrt{-K}}{H}$$

(2) $\cos\theta=\dfrac{1}{\sec\theta}=\pm\dfrac{1}{\sqrt{1+tg^2\theta}}=\pm\dfrac{H}{\sqrt{H^2-K}}$

将$H=\dfrac{LG-2MF+NE}{2(EG-F^2)},K=\dfrac{LN-M^2}{EG-F^2}$代入上式,即可得证.

5. 求下列曲面在原点处的近似曲面:

(1) $z=\exp(x^2+y^2)-1$;

(2) $z=\ln\cos x-\ln\cos y$;

(3) $z=(x+3y)^3$.

解:(1) $\vec{r}(u,v)=(x,y,\exp(x^2+y^2))$

则$\vec{r}_x=(1,0,2xe^{x^2+y^2}),\quad \vec{r}_y=(0,1,2ye^{x^2+y^2})$

$$\vec{n}=\frac{1}{\sqrt{1+4(x^2+y^2)e^{x^2+y^2}}}(-2xe^{x^2+y^2},-2ye^{x^2+y^2},1)$$

$$\vec{r}_{xx}=(0,0,(2+4x^2)e^{x^2+y^2}),\quad \vec{r}_{xy}=(0,0,4xye^{x^2+y^2}),\quad \vec{r}_{yy}=(0,0,(2+4y^2)e^{x^2+y^2})$$

在原点$(0,0)$处, $\vec{n}=(0,0,1)$, $\vec{r}_x=(1,0,0)$, $\vec{r}_y=(0,1,0)$

$\vec{r}_{xx}=(0,0,2)$, $\vec{r}_{xy}=(0,0,0)$, $\vec{r}_{yy}=(0,0,2)$

$\therefore E=G=1, F=0;\quad L=N=2, M=0$

$\therefore H=2, K=4, k_1=H+\sqrt{H^2-K}=2, k_2=H-\sqrt{H^2-K}=2$

$\therefore$ 近似曲线为: $z=\dfrac{1}{2}(k_1x^2+k_2y^2)=x^2+y^2$

(2) 同(1),可解得在原点处的近似曲线为: $z=\dfrac{1}{2}(x^2+y^2)$.

(3) 同(1),可解得在原点处的近似曲线为: $z=0$.

6. 求曲面$z=e^{-(x^2+y^2)}$的Gauss曲率,画出它的草图,并指出椭圆点和双曲点的区域.

解: $\vec{r}(u,v)=(x,y,e^{-\frac{x+y}{2}})$, 则$\vec{r}_x=(1,0,-xe^{-\frac{x+y}{2}}), \vec{r}_x=(0,1,-ye^{-\frac{x+y}{2}})$

$$\vec{n}=\frac{1}{\sqrt{1+(x^2+y^2)e^{-\frac{x+y}{2}}}}(xe^{-\frac{x+y}{2}}, ye^{-\frac{x+y}{2}}, 1)$$

$\vec{r}_{xx}=(0,0,(x^2-1)e^{-\frac{x+y}{2}})$, $\vec{r}_{xy}=(0,0,xye^{-\frac{x+y}{2}})$, $\vec{r}_{yy}=(0,0,(y^2-1)e^{-\frac{x+y}{2}})$

$\therefore E=1+x^2e^{-(x^2+y^2)}, F=xye^{-(x^2+y^2)}, G=1+y^2e^{-(x^2+y^2)}$

$$L=\frac{(x^2-1)e^{-\frac{x+y}{2}}}{\sqrt{1+(x^2+y^2)e^{-(x^2+y^2)}}}, M=\frac{xye^{-\frac{x+y}{2}}}{\sqrt{1+(x^2+y^2)e^{-(x^2+y^2)}}}, N=\frac{(y^2-1)e^{-\frac{x+y}{2}}}{\sqrt{1+(x^2+y^2)e^{-(x^2+y^2)}}}$$

$$\therefore K=\frac{LN-M^2}{EG-F^2}=\frac{(1-x^2-y^2)e^{-(x^2+y^2)}}{[1+(x^2+y^2)e^{-(x^2+y^2)}]^2}$$

椭圆点 $\Leftrightarrow K>0 \Leftrightarrow 1-x^2-y^2>0$

$\therefore x^2+y^2<1$, 故椭圆点区域为: $\{x^2+y^2<1\}$

双曲点 $\Leftrightarrow K<0 \Leftrightarrow x^2+y^2>1$

$\therefore$ 双曲点的区域为: $\{x^2+y^2>1\}$

7. 证明: 如果曲面在一点有三个渐进方向,它们两两不共线, 则该点必定是平点.

证明: 设该点不是平点, 则若是椭圆点, 有$K>0$, 即$LN-M^2>0$, 无渐近方向.

若是双曲点,则$K<0$, 即$LN-M^2<0$, 有2个渐近方向.

若是非平点的抛物点,则$K=0$,即$LN-M^2=0$, 有1个渐近方向.

$\therefore$ 该点只能是平点.

§ 4.6 某些特殊曲面

1. 证明：$z = c \cdot arctg\dfrac{y}{x}$是极小曲面.并求它的主曲率.

证明：$\vec{r} = (x, y, c \cdot arctg\dfrac{y}{x}),\quad \vec{r}_x = (1, 0, -\dfrac{cy}{x^2 + y^2}),\quad \vec{r}_y = (0, 1, \dfrac{cx}{x^2 + y^2})$

$$\vec{n} = \frac{1}{\sqrt{c^2x^2 + c^2y^2 + (x^2 + y^2)^2}}(cy, -cx, x^2 + y^2)$$

$$\vec{r}_{xx} = (0, 0, \frac{2cxy}{(x^2 + y^2)^2}),\quad \vec{r}_{xy} = (0, 0, \frac{c(y^2 - x^2)}{(x^2 + y^2)^2}),\quad \vec{r}_{yy} = (0, 0, \frac{-2cxy}{(x^2 + y^2)^2})$$

$$\therefore E = 1 + \frac{c^2y^2}{(x^2 + y^2)^2},\quad F = -\frac{c^2xy}{(x^2 + y^2)^2},\quad G = 1 + \frac{c^2x^2}{(x^2 + y^2)^2}$$

证明：$\vec{r} = (x, y, c \cdot arctg\dfrac{y}{x}),\quad \vec{r}_x = (1, 0, -\dfrac{cy}{x^2 + y^2}),\quad \vec{r}_y = (0, 1, \dfrac{cx}{x^2 + y^2})$

$$\vec{n} = \frac{1}{\sqrt{c^2x^2 + c^2y^2 + (x^2 + y^2)^2}}(cy, -cx, x^2 + y^2)$$

$$\vec{r}_{xx} = (0, 0, \frac{2cxy}{(x^2 + y^2)^2}),\quad \vec{r}_{xy} = (0, 0, \frac{c(y^2 - x^2)}{(x^2 + y^2)^2}),\quad \vec{r}_{yy} = (0, 0, \frac{-2cxy}{(x^2 + y^2)^2})$$

$$\therefore E = 1 + \frac{c^2y^2}{(x^2 + y^2)^2},\quad F = -\frac{c^2xy}{(x^2 + y^2)^2},\quad G = 1 + \frac{c^2x^2}{(x^2 + y^2)^2}$$

$$L = \frac{2cxy}{\sqrt{c^2x^2 + c^2y^2 + (x^2 + y^2)^2}(x^2 + y^2)}$$

$$M = \frac{c(y^2 - x^2)}{\sqrt{c^2x^2 + c^2y^2 + (x^2 + y^2)^2}(x^2 + y^2)}$$

$$N = \frac{-2cxy}{\sqrt{c^2x^2 + c^2y^2 + (x^2 + y^2)^2}(x^2 + y^2)}$$

故$LG - 2MF + NE = 0$,从而$H = 0$，即曲面为极小曲面.

又$K = \dfrac{LN - M^2}{EG - F^2} = \dfrac{-c^2}{(c^2 + x^2 + y^2)^2} = k_1k_2 = -k_1^2$

故$k_1 = \dfrac{c}{c^2 + x^2 + y^2}$，$\quad k_2 = -\dfrac{c}{c^2 + x^2 + y^2}$.

2. 假定一个极小曲面的方程可以写成$z = f(x) + g(y)$的形状.证明:除了一个附加的任意常数外,它必定是$z = \dfrac{1}{a}\ln\dfrac{\cos ay}{\cos ax}$,其中$a$是常数.此曲面成为Scherk曲面.

证明： $\bar{r}=(x,y,f(x)+g(y))$, $\begin{cases}E=1+f'\\F=f'g',\\G=1+g'\end{cases}$ $\begin{cases}L=\pm\dfrac{f''}{\sqrt{1+f'^2+g'^2}}\\M=0\\N=\pm\dfrac{g''}{\sqrt{1+f'^2+g'^2}}\end{cases}$

极小曲面 $\Rightarrow H=0\Rightarrow LG-2MF+NE=0\Rightarrow f''(1+g'^2)+g''(1+f'^2)=0$

$\Rightarrow \dfrac{f''}{1+f'^2}=\dfrac{g''}{1+g'^2}=a$ （a为常数),积分可得 $arctgf'(x)=ax+c_1$,

$arctgg'(y)=-ay+c_2$

$\Rightarrow f'(x)=tg(ax+c_1),g'(y)=tg(-ay+c_2)$, 再积分得

$f(x)=-\dfrac{1}{a}\ln\cos(ax+c_1)+c_1'$, $g(y)=\dfrac{1}{a}\ln\cos(-ay+c_2)+c_2'$

当取$c=c_1=c_2=c_1'=c_2'=0$时, $f(x)=-\dfrac{1}{a}\ln\cos ax$, $g(y)=\dfrac{1}{a}\ln\cos ay$

$\therefore z=\dfrac{1}{a}\ln\dfrac{\cos ay}{\cos ax}$

3. 证明:$\bar{r}=(3u(1+v^2)-u^3,3v(1+u^2)-v^3,3(u^2-v^2))$是极小曲面.它称为Ennerper曲面. 证明它的曲率是平面曲线, 并求曲率线所在平面.

证明: $\begin{cases}E=9(1+u^2+v^2)^2\\F=0\\G=9(1+u^2+v^2)^2\end{cases}$, $\begin{cases}L=\dfrac{1}{\triangle}(1+u^2+v^2)^2\\M=0\\N=-\dfrac{1}{\triangle}(1+u^2+v^2)^2\end{cases}$

其中:取$\bar{n}=\dfrac{1}{\sqrt{4u^2+4v^2+(1-u^2-v^2)^2}}(-2u,-2v,1-u^2-v^2)$, $\triangle=\sqrt{EG-F^2}$

$\Rightarrow LG-2MF+NE=0\Rightarrow H=0\Rightarrow$曲面为极小曲面.

曲率线方程为$\begin{vmatrix}\delta v^2 & -\delta u\delta v & \delta u^2\\E & F & G\\L & M & N\end{vmatrix}=0$

因$F=M=0$,故方程为$(EN-LG)\delta u\delta v=0\Leftrightarrow(1+u^2+v^2)^3\delta u\delta v=0\Leftrightarrow\delta u\delta v=0$

$\Rightarrow u\equiv const$或$v\equiv const$,即曲率线为$v-$曲线和$u-$曲线.

又因$(\vec{r}_u,\vec{r}_{uu},\vec{r}_{uuu})=0$,故$\vec{r}_u$始终与$\vec{r}_{uu}\times\vec{r}_{uuu}=36(0,-1,v)$垂直

即$u-$曲线(即$v=v_0$)为平面曲线.

该平面为$X:(\vec{X}-\vec{r})\cdot(0,-1,v_0)=0$,即$-y+v_0z+3v_0+2v_0^3=0$(与$u$无关)

同理,$(\vec{r}_v,\vec{r}_{vv}\vec{r}_{vvv})=0$,故$\vec{r}_v$始终与$\vec{r}_{vv}\times\vec{r}_{vvv}=-36(1,0,u)$垂直

即$v-$曲线(即$u=u_0$)为平面曲线.

该平面为$X:(\vec{X}-\vec{r})\cdot(1,0,u_0)=0$,即$x+u_0z-3u_0-2u_0^3=0$(与$v$无关)

4. 证明:正螺面$\vec{r}=(u\cos v,u\sin v,bv)$是极小曲面.并证明:除了平面之外,直纹极小曲面都是正螺面.

证明: (1). $\begin{cases} E=1 \\ F=0 \\ G=u^2+b^2 \end{cases}$, $\begin{cases} L=0 \\ M=\pm\dfrac{b}{\sqrt{u^2+b^2}} \\ N=0 \end{cases}$

$\Rightarrow LG-2MF+NE=0\Rightarrow H=0\Rightarrow$ 正螺面为极小曲面.

(2). 设直纹面$\Sigma:\vec{r}(u,v)=\vec{a}(u)+v\vec{l}(u)$,

其中u为$\vec{r}_1(u)=\vec{a}(u)$的弧长参数,$\left|\vec{l}(u)\right|=1,\dot{\vec{a}}(u)\perp\vec{l}(u)$.

记曲线$\vec{r}_1(u)$的$Frenet$标架为$\left\{\vec{r}_1;\vec{\alpha},\vec{\beta},\vec{\gamma}\right\}$,曲率及饶率分别为$k,\tau$.

$\vec{r}_u=\vec{\alpha}(u)+v\vec{l}^{'}(u)$, $\vec{r}_v=\vec{l}(u)$, $\vec{r}_{uu}=k\vec{\beta}+v\vec{l}^{''}$, $\vec{r}_{uv}=\vec{l}^{'}(u),\vec{r}_{vv}=0$

$\therefore E=\vec{r}_u\vec{r}_u$, $F=\vec{r}_u\vec{r}_v=(\vec{\alpha}+v\vec{l}^{'})\vec{l}=\vec{\alpha}\vec{l}+v\vec{l}^{'}\vec{l}=0$, $G=\vec{r}_v\vec{r}_v=\vec{l}^2=1$

$N=\vec{r}_{vv}\vec{n}=0$. 若Σ为极小曲面,则$H\equiv0$ $\Rightarrow LG-2MF+NE=0$

$\Rightarrow LG=0$ $\Rightarrow L=0$ $\Rightarrow\vec{r}_{uu}\cdot(\vec{r}_u\times\vec{r}_v)=0$对任意的$v$成立.

特别地,$\vec{r}_{uu}\cdot(\vec{r}_u\times\vec{r}_v)\big|_{v=0}=0$, 即$k(\vec{\beta},\vec{\alpha},\vec{l})=0$ $\Rightarrow k=0$或$(\vec{\beta},\vec{\alpha},\vec{l})=0$

$\Rightarrow k=0$或$\vec{l}=\pm\vec{\beta}$

1°.当$k=0$时,$\vec{r}_1(u)=\vec{a}(u)$为直线.

若$\vec{l}(u)=\vec{l}_0$(常向量), 则$\vec{r}(u,v)$为平面.

若$\vec{l}(u)\neq\vec{l}_0$, 可设$\vec{a}(u)=(0,0,bu)$为z轴, $\vec{l}(u)=(\cos u,\sin u,0)$

则$\vec{r}(u,v)=(v\cos u,v\sin u,bu)$为正螺面.

2°.当$\vec{l}=\pm\vec{\beta}$时,不失一般性地, 可设$\vec{l}=\vec{\beta}$(否则,只需在Σ的方程中,用$-\vec{l}(u)$替换$\vec{l}(u)$). 则$\vec{l}^{'}=\vec{\beta}^{'}=-k\vec{\alpha}+\tau\vec{\gamma}$, $\vec{l}^{''}=-\dot{k}\vec{\alpha}-(k^2+\tau^2)\vec{\beta}+\dot{\tau}\vec{\gamma}$

$\vec{r}_{uu}=k\vec{\beta}+v[-\dot{k}\vec{\alpha}-(k^2+\tau^2)\vec{\beta}+\dot{\tau}\vec{\gamma}]=-v\dot{k}\vec{\alpha}+(k-vk^2-v\tau^2)\vec{\beta}+v\dot{\tau}\vec{\gamma}$

$\vec{r}_u \times \vec{r}_v = -v\vec{\alpha} + (1-vk)\vec{\gamma}$

因$\vec{r}_{uu} \cdot (\vec{r}_u \times \vec{r}_v) = v[\dot{\tau} + v(\dot{k}\tau - k\dot{\tau})] = 0$对$\forall v$成立

故对$\forall v$,有$\dot{\tau} + v(\dot{k}\tau - k\dot{\tau}) = 0 \quad \Rightarrow \dot{\tau} = 0$且$\dot{k}\tau - k\dot{\tau} = 0 \quad \Rightarrow \dot{\tau} = 0$且$\dot{k}\tau = 0$

$\Rightarrow \tau = 0$或$\dot{\tau} = \dot{k} = 0$

i).若$\tau = 0$,则$\vec{r}_1(u) = \vec{a}(u)$为平面曲线,从而$\Sigma$为一平面.

ii).若$\dot{\tau} = \dot{k} = 0$,则$k = c_1, \tau = c_2$(c_1, c_2均为常数)

当$k = c_1 = 0$时,为情形1°.

当$k = c_1 \neq 0$时,$\vec{r}_1(u) = \vec{a}(u)$为圆螺旋线,此时可设

$\vec{r}_1(u) = \vec{a}(u) = (a\cos u, a\sin u, bu)$

则$\vec{l}(u) = \vec{\beta}(u) = \dfrac{\ddot{\vec{r}}_1(u)}{\left|\ddot{\vec{r}}_1(u)\right|} = \dfrac{1}{a}(-a\cos u, -a\sin u, 0) = (-\cos u, -\sin u, 0)$

$\therefore \vec{r}(u,v) = ((a-v)\cos u, (a-v)\sin u, bu)$为正螺面.

5. 证明:如果Gauss映射是曲面到单位球面的保角对应,则该曲面或者是球面,或者是极小曲面.

证明: 曲面S的第一基本形式为$I = d\vec{r} \cdot d\vec{r}$, 单位球面$\Sigma$的第一基本形式为

$I' = d\vec{n} \cdot d\vec{n} = \rho(u,v)I, (\rho(u,v) > 0)$

故曲面的第三基本形式为$III = d\vec{n} \cdot d\vec{n} = I' = \rho(u,v)I$

由$III - 2HII + KI = 0$,即$\rho I - 2HII + KI = 0 \quad \Rightarrow (\rho + K)I = 2HII$

(i).若$H \equiv 0$,则S为极小曲面.

(ii).若$H \neq 0$,则$II = \dfrac{\rho + K}{2H} I, \forall P \in S$, P沿任一单位切向量的法曲率均为

$k_n = \dfrac{II}{I} = \dfrac{\rho + K}{2H}$,即$k_n = k_1 = k_2$,故$S$上每一点均为脐点.

若$k_1 = k_2 = 0$,则S上的点为平点,从而S为平面,当然是极小曲面.

若$k_1 = k_2 \neq 0$,则S上的点为圆点,从而S为球面.

6. 求伪球面(见方程(15))的全面积.(由结果可知:伪球面尽管向无穷远处延伸, 但是它的全面积是有限的)

解: $\vec{r}(\varphi, \theta) = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, \pm a[\ln(\sec\varphi + tg\varphi) - \sin\varphi])$

其中$0 \le \varphi < \dfrac{\pi}{2}, 0 \le \theta < 2\pi$.

$\vec{r}_\varphi = (-a\sin\varphi\cos\theta, -a\sin\varphi\sin\theta, \pm a(\sec\varphi - \cos\varphi))$

$$\vec{r}_\theta = (-a\cos\varphi\sin\theta, a\cos\varphi\cos\theta, 0)$$

$$\therefore E = a^2\sin^2\varphi + a^2(\sec^2\varphi + \cos^2\varphi - 2) = a^2 tg^2\varphi, \quad F = 0, \quad G = a^2\cos^2\varphi$$

$$\therefore S = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} a^2\left|\sin\varphi\right| d\varphi d\theta = 2a^2\pi.$$

§ 5.1 自然标架的运动公式

1. 设有参数变换$u^{\alpha}=u^{\alpha}(u^{1'},u^{2'})$,命$a^{\alpha}_{\alpha'}=\dfrac{\partial u^{\alpha}}{\partial u^{\alpha'}}$,假定$\det(a^{\alpha}_{\alpha'})>0$.证明：

$$g_{\alpha'\beta'}=g_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'},\quad b_{\alpha'\beta'}=b_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}.$$

证明：$g_{\alpha'\beta'}=\bar{r}_{\alpha'}\cdot\bar{r}_{\beta'}=(\bar{r}_{\alpha}\cdot a^{\alpha}_{\alpha'})\cdot(\bar{r}_{\beta}\cdot a^{\beta}_{\beta'})=g_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}$.

$$b_{\alpha'\beta'}=-\bar{r}_{\alpha'}\cdot\bar{n}_{\beta'}=-(\bar{r}_{\alpha}\cdot a^{\alpha}_{\alpha'})\cdot(\bar{n}_{\beta}\cdot a^{\beta}_{\beta'})=-\bar{r}_{\alpha}\bar{n}_{\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}=b_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}.$$

2. 证明：在上题的参数变换下,$(g_{\alpha\beta})$的逆矩阵$(g^{\alpha\beta})$的变换规律是

$$g^{\alpha\beta}=g^{\alpha'\beta'}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}.$$

证明：由上题知，$g_{\alpha'\beta'}=g_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}\quad\Rightarrow g_{\alpha'\beta'}\cdot g^{\alpha\beta}=a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}\quad\Rightarrow g^{\alpha\beta}=g^{\alpha'\beta'}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}$.

3. 如果用$\Gamma^{\gamma'}_{\alpha'\beta'}$记关于$(g_{\alpha'\beta'})$的*Christoffel*记号，证明：在习题1的参数变换下有变换规律$\Gamma^{\gamma'}_{\alpha'\beta'}=\Gamma^{\gamma}_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}a^{\gamma'}_{\gamma}+\dfrac{\partial a^{\gamma}_{\alpha'}}{\partial u^{\beta'}}a^{\gamma'}_{\gamma}$,其中$(a^{\alpha'}_{\alpha})$是$(a^{\alpha}_{\alpha'})$的逆矩阵，即$a^{\alpha'}_{\alpha}=\dfrac{\partial u^{\alpha'}}{\partial u^{\alpha}}$.

证明：$\Gamma^{\gamma'}_{\alpha'\beta'}=g^{\gamma'\xi'}\Gamma_{\xi'\alpha'\beta'}=g^{\gamma'\xi'}\bar{r}_{\alpha'\beta'}\bar{r}_{\xi'}$

$$\because g^{\gamma'\xi'}=a^{\xi'}_{\xi}a^{\gamma'}_{\gamma}g^{\gamma\xi},\quad \bar{r}_{\alpha'\beta'}=\frac{\partial}{\partial u^{\beta'}}(\bar{r}_{\alpha}\cdot a^{\alpha}_{\alpha'})=\bar{r}_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}+\bar{r}_{\alpha}\frac{\partial a^{\alpha}_{\alpha'}}{\partial u^{\beta'}},\quad \bar{r}_{\xi'}=\bar{r}_{\xi}a^{\xi}_{\xi'}$$

$$\therefore \Gamma^{\gamma'}_{\alpha'\beta'}=a^{\xi'}_{\xi}a^{\gamma'}_{\gamma}g^{\gamma\xi}(\bar{r}_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}+\bar{r}_{\alpha}\frac{\partial a^{\alpha}_{\alpha'}}{\partial u^{\beta'}})\bar{r}_{\xi}a^{\xi}_{\xi'}=g^{\gamma\xi}\bar{r}_{\alpha\beta}\bar{r}_{\xi}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}a^{\gamma'}_{\gamma}+\frac{\partial a^{\alpha}_{\alpha'}}{\partial u^{\beta'}}g^{\gamma\xi}g_{\alpha\xi}a^{\gamma'}_{\gamma}$$

$$=\Gamma^{\gamma}_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}a^{\gamma'}_{\gamma}+\frac{\partial a^{\gamma}_{\alpha'}}{\partial u^{\beta'}}a^{\gamma'}_{\gamma}.(\text{因}g^{\gamma\xi}g_{\alpha\xi}=\delta^{\gamma}_{\alpha})$$

4. 验证：曲面的平均曲率H可以表示成$H=\dfrac{1}{2}b_{\alpha\beta}g^{\alpha\beta}$，并且$H$在习题1的参数变换下是不变的.

证明：$H=\dfrac{LG-2MF+NE}{2(EG-F^2)}=\dfrac{b_{11}g_{22}-2b_{12}g_{21}+b_{22}g_{11}}{2g}=\dfrac{1}{2}(b_{11}g^{11}+2b_{12}g^{12}+b_{22}g^{22})$

$$=\frac{1}{2}b_{\alpha\beta}g^{\alpha\beta}$$

$$H'=\frac{1}{2}b_{\alpha'\beta'}g^{\alpha'\beta'}=\frac{1}{2}b_{\alpha\beta}a^{\alpha}_{\alpha'}a^{\beta}_{\beta'}a^{\beta'}_{\beta}a^{\alpha'}_{\alpha}g^{\alpha\beta}=\frac{1}{2}b_{\alpha\beta}g^{\alpha\beta}=H.$$

5. 证明下列恒等式:

(1) $g^{\gamma\xi}\Gamma_{\xi\alpha}^{\beta}+g^{\beta\xi}\Gamma_{\xi\alpha}^{\gamma}=-\dfrac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}$.

(2) $\dfrac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}-\dfrac{\partial g_{\alpha\gamma}}{\partial u^{\beta}}=g_{\beta\xi}\Gamma_{\alpha\gamma}^{\xi}-g_{\gamma\xi}\Gamma_{\alpha\beta}^{\xi}$.

(3) $\Gamma_{\alpha\beta}^{\beta}=\dfrac{1}{2}\dfrac{\partial\ln g}{\partial u^{\alpha}}$,其中$g=g_{11}g_{22}-(g_{12})^{2}$.

证明:(1) $g^{\gamma\beta}g_{\beta l}=\delta_{l}^{\gamma},u^{\alpha},g_{\beta l}\dfrac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}+g^{\gamma\beta}\dfrac{\partial g_{\beta l}}{\partial u^{\alpha}}=0\quad\Rightarrow g_{\beta l}\dfrac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}=-g^{\gamma\beta}\dfrac{\partial g_{\beta l}}{\partial u^{\alpha}}$

$$\Rightarrow g_{\beta l}\frac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}=-g^{\gamma\beta}(\Gamma_{\beta\alpha}^{\delta}g_{\delta l}+\Gamma_{l\alpha}^{\delta}g_{\delta\beta})\quad\Rightarrow\frac{\partial g^{\gamma\xi}}{\partial u^{\alpha}}=-g^{\gamma\beta}(\Gamma_{\beta\alpha}^{\delta}g_{\delta l}+\Gamma_{l\alpha}^{\delta}g_{\delta\beta})g^{\xi l}$$

$$(\xi\leftrightarrow\beta)\quad\frac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}=-g^{\gamma\xi}\Gamma_{\xi\alpha}^{\beta}-\Gamma_{l\alpha}^{\gamma}g^{l\beta}$$

$$(l\leftrightarrow\xi)\quad\frac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}=-g^{\gamma\xi}\Gamma_{\xi\alpha}^{\beta}-g^{\beta\xi}\Gamma_{\xi\alpha}^{\gamma}.$$

(2) $g_{\beta\xi}\Gamma_{\alpha\gamma}^{\xi}-g_{\gamma\xi}\Gamma_{\alpha\beta}^{\xi}=g_{\beta\xi}g^{\xi\eta}\Gamma_{\eta\alpha\gamma}-g_{\gamma\xi}g^{\xi\eta}\Gamma_{\eta\alpha\beta}=\delta_{\beta}^{\eta}\Gamma_{\eta\alpha\gamma}-\delta_{\gamma}^{\eta}\Gamma_{\eta\alpha\beta}=\Gamma_{\beta\alpha\gamma}-\Gamma_{\gamma\alpha\beta}$

$$=\Gamma_{\beta\alpha\gamma}+\Gamma_{\alpha\beta\gamma}-\Gamma_{\alpha\beta\gamma}-\Gamma_{\gamma\alpha\beta}=\Gamma_{\beta\alpha\gamma}+\Gamma_{\alpha\beta\gamma}-\Gamma_{\alpha\gamma\beta}-\Gamma_{\gamma\alpha\beta}=\frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}-\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}}.$$

(3) $\dfrac{1}{2}\dfrac{\partial\ln g}{\partial u^{\alpha}}=\dfrac{1}{2g}\dfrac{\partial g}{\partial u^{\alpha}}=\dfrac{1}{2g}(\dfrac{\partial g_{11}}{\partial u^{\alpha}}g_{22}+g_{11}\dfrac{\partial g_{22}}{\partial u^{\alpha}}-2g_{12}\dfrac{\partial g_{12}}{\partial u^{\alpha}})$

$$=\frac{1}{2}(\frac{\partial g_{11}}{\partial u^{\alpha}}g^{11}+g^{22}\frac{\partial g_{22}}{\partial u^{\alpha}}+g^{12}\frac{\partial g_{12}}{\partial u^{\alpha}}+g^{21}\frac{\partial g_{21}}{\partial u^{\alpha}})$$

$$=\frac{1}{2}g^{\beta\xi}\frac{\partial g_{\beta\xi}}{\partial u^{\alpha}}=\frac{1}{2}g^{\beta\xi}\frac{\partial(\vec{r}_{\beta}\cdot\vec{r}_{\xi})}{\partial u^{\alpha}}=\frac{1}{2}g^{\beta\xi}(\vec{r}_{\beta\alpha}\cdot\vec{r}_{\xi}+\vec{r}_{\xi\alpha}\cdot\vec{r}_{\beta})$$

$$=\frac{1}{2}g^{\beta\xi}(\Gamma_{\xi\beta\alpha}+\Gamma_{\beta\xi\alpha})=\frac{1}{2}(g^{\beta\xi}\Gamma_{\xi\beta\alpha}+g^{\beta\xi}\Gamma_{\beta\xi\alpha})=\frac{1}{2}(\Gamma_{\beta\alpha}^{\beta}+\Gamma_{\xi\alpha}^{\xi})=\Gamma_{\alpha\beta}^{\beta}.$$

§ 5.2 曲面的唯一性定理

1. 推导函数$f_{\alpha\beta}(u), f_\alpha(u), f(u)$所满足的方程组(4).

证明: $f_{\alpha\beta}(u)=(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})\cdot(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)}),\quad f_\alpha(u)=(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})\cdot(\vec{n}^{(1)}-\vec{n}^{(2)})$

$$f(u)=(\vec{n}^{(1)}-\vec{n}^{(2)})^2.$$

$$\frac{\partial f_{\alpha\beta}}{\partial u^\gamma}=\frac{\partial(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})}{\partial u^\gamma}\cdot(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)})+(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})\cdot\frac{\partial(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)})}{\partial u^\gamma}$$

$$=(\Gamma_{\alpha\gamma}^\delta\vec{r}_\delta^{(1)}+b_{\alpha\gamma}\vec{n}^{(1)}-\Gamma_{\alpha\gamma}^\delta\vec{r}_\delta^{(2)}-b_{\alpha\gamma}\vec{n}^{(2)})\cdot(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)})+(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})\cdot(\Gamma_{\beta\gamma}^\delta\vec{r}_\delta^{(1)}+$$
$$b_{\beta\gamma}\vec{n}^{(1)}-\Gamma_{\beta\gamma}^\delta\vec{r}_\delta^{(2)}-b_{\beta\gamma}\vec{n}^{(2)})$$

$$=\Gamma_{\alpha\gamma}^\delta(\vec{r}_\delta^{(1)}-\vec{r}_\delta^{(2)})\cdot(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)})+\Gamma_{\beta\gamma}^\delta(\vec{r}_\delta^{(1)}-\vec{r}_\delta^{(2)})\cdot(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})+b_{\alpha\gamma}(\vec{n}^{(1)}-\vec{n}^{(2)})\cdot$$
$$(\vec{r}_\beta^{(1)}-\vec{r}_\beta^{(2)})+b_{\beta\gamma}(\vec{n}^{(1)}-\vec{n}^{(2)})\cdot(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})$$

$$=\Gamma_{\alpha\gamma}^\delta f_{\delta\beta}+\Gamma_{\beta\gamma}^\delta f_{\alpha\delta}+b_{\gamma\alpha}f_\beta+b_{\gamma\beta}f_\alpha$$

$$\frac{\partial f_\alpha}{\partial u^\gamma}=\frac{\partial(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})}{\partial u^\gamma}\cdot(\vec{n}^{(1)}-\vec{n}^{(2)})+(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})\cdot\frac{\partial(\vec{n}^{(1)}-\vec{n}^{(2)})}{\partial u^\gamma}$$

$$=\Gamma_{\alpha\gamma}^\delta(\vec{r}_\delta^{(1)}-\vec{r}_\delta^{(2)})\cdot(\vec{n}^{(1)}-\vec{n}^{(2)})+b_{\alpha\gamma}(\vec{n}^{(1)}-\vec{n}^{(2)})^2+(-b_\gamma^\delta\vec{r}_\delta^{(1)}+b_\gamma^\delta\vec{r}_\delta^{(2)})\cdot(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})$$

$$=\Gamma_{\gamma\alpha}^\delta f_\delta+b_{\gamma\delta}f-b_\gamma^\delta f_{\delta\alpha}$$

$$\frac{\partial f}{\partial u^\gamma}=2(\vec{n}^{(1)}-\vec{n}^{(2)})\cdot\frac{\partial(\vec{n}^{(1)}-\vec{n}^{(2)})}{\partial u^\gamma}=2(\vec{n}^{(1)}-\vec{n}^{(2)})\cdot(-b_\gamma^\alpha)(\vec{r}_\alpha^{(1)}-\vec{r}_\alpha^{(2)})=-2b_\gamma^\alpha f_\alpha$$

2. 已知函数$f_{\alpha\beta}(u), f_\alpha(u), f(u)$满足方程组(4).命

$$F(u)\equiv g^{\alpha\gamma}g^{\beta\delta}f_{\alpha\beta}f_{\gamma\delta}+2g^{\alpha\gamma}f_\alpha f_\gamma+f^2,\text{证明:}\frac{\partial F(u)}{\partial u^\xi}=0.$$

证明: $\dfrac{\partial F(u)}{\partial u^\xi}=\dfrac{\partial g^{\alpha\gamma}}{\partial u^\xi}g^{\beta\delta}f_{\alpha\beta}f_{\gamma\delta}+\dfrac{\partial g^{\beta\delta}}{\partial u^\xi}g^{\alpha\gamma}f_{\alpha\beta}f_{\gamma\delta}+\dfrac{\partial f_{\alpha\beta}}{\partial u^\xi}g^{\alpha\gamma}g^{\beta\delta}f_{\gamma\delta}+\dfrac{\partial f_{\gamma\delta}}{\partial u^\xi}g^{\alpha\gamma}g^{\beta\delta}f_{\alpha\beta}$

$$+2\frac{g^{\alpha\gamma}}{\partial u^\xi}f_\alpha f_\gamma+2\frac{f_\alpha}{\partial u^\xi}g^{\alpha\gamma}f_\gamma+2\frac{\partial f_\gamma}{\partial u^\xi}g^{\alpha\gamma}f_\alpha+2f\frac{\partial f}{\partial u^\xi}$$

$$=(-g^{\alpha\eta}\Gamma_{\eta\xi}^\gamma-g^{\gamma\eta}\Gamma_{\eta\xi}^\alpha)g^{\beta\delta}f_{\alpha\beta}f_{\gamma\delta}+(g^{\beta\eta}\Gamma_{\eta\xi}^\beta-g^{\delta\eta}\Gamma_{\eta\xi}^\beta)g^{\alpha\gamma}f_{\alpha\beta}f_{\gamma\delta}+(\Gamma_{\xi\alpha}^\eta f_{\eta\beta}+\Gamma_{\xi\beta}^\eta f_{\alpha\eta}$$
$$+b_{\xi\alpha}f_\beta+b_{\xi\beta}f_\alpha)g^{\alpha\gamma}g^{\beta\delta}f_{\gamma\delta}+(\Gamma_{\xi\gamma}^\eta f_{\eta\delta}+\Gamma_{\xi\delta}^\eta f_{\eta\gamma}+b_{\xi\gamma}f_\delta+b_{\xi\delta}f_\gamma)g^{\alpha\gamma}g^{\beta\delta}f_{\alpha\beta}+$$
$$2(-g^{\alpha\eta}\Gamma_{\eta\xi}^\gamma-g^{\gamma\eta}\Gamma_{\eta\xi}^\alpha)f_\alpha f_\gamma+(-b_\xi^\eta f_{\eta\alpha}+\Gamma_{\xi\alpha}^\eta f_\eta+b_{\xi\alpha}f)g^{\alpha\gamma}f_\gamma+2(-b_\xi^\eta f_{\eta\lambda}+\Gamma_{\xi\gamma}^\eta f_\eta+$$
$$b_{\xi\gamma}f)g^{\alpha\gamma}f_\alpha+2f(-2b_\xi^\alpha f_\alpha)=0$$

(最后一个等式利用公式$\Gamma_{\alpha\beta\xi}g^{\xi\gamma}=\Gamma_{\alpha\beta}^\gamma,\quad b_{\alpha\xi}g^{\xi\gamma}=b_\alpha^\gamma$)

§ 5.3 曲面论基本方程

1. 验证方程(13)和(8)的等价性.

证明: 方程(13): $\dfrac{\partial b_{\alpha\beta}}{\partial u^{\gamma}}-\dfrac{\partial b_{\alpha\gamma}}{\partial u^{\beta}}=b_{\beta\delta}\Gamma^{\delta}_{\alpha\gamma}-b_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta}$

方程(8): $\dfrac{\partial b^{\delta}_{\beta}}{\partial u^{\gamma}}-\dfrac{\partial b^{\delta}_{\gamma}}{\partial u^{\beta}}=-b^{\eta}_{\beta}\Gamma^{\delta}_{\eta\gamma}+b^{\eta}_{\gamma}\Gamma^{\delta}_{\eta\beta}$

$(13)\Rightarrow(8)$: 若 $\dfrac{\partial b_{\alpha\beta}}{\partial u^{\gamma}}-\dfrac{\partial b_{\alpha\gamma}}{\partial u^{\beta}}=b_{\beta\delta}\Gamma^{\delta}_{\alpha\gamma}-b_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta}$,两边同乘$g^{\delta\alpha}$,并对$\alpha$求和,得

$$0=g^{\delta\alpha}\left(\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}}-\frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}}+b_{\gamma\eta}\Gamma^{\eta}_{\alpha\beta}-b_{\beta\eta}\Gamma^{\eta}_{\alpha\gamma}\right)=\frac{\partial(g^{\delta\alpha}b_{\alpha\beta})}{\partial u^{\gamma}}-\frac{\partial(g^{\delta\alpha}b_{\alpha\gamma})}{\partial u^{\beta}}-\frac{\partial g^{\delta\alpha}}{\partial u^{\gamma}}b_{\alpha\beta}+\frac{\partial g^{\delta\alpha}}{\partial u^{\beta}}b_{\alpha\gamma}$$
$$+g^{\delta\alpha}\Gamma^{\eta}_{\alpha\beta}b_{\eta\gamma}-g^{\delta\alpha}\Gamma^{\eta}_{\alpha\gamma}b_{\eta\beta}$$

由5.5习题5(1)的结论: $\dfrac{\partial g^{\gamma\beta}}{\partial u^{\alpha}}=-g^{\gamma\xi}\Gamma^{\beta}_{\xi\alpha}-g^{\beta\xi}\Gamma^{\gamma}_{\xi\alpha}$,可得

$$0=\frac{\partial b^{\delta}_{\beta}}{\partial u^{\gamma}}-\frac{\partial b^{\delta}_{\gamma}}{\partial u^{\beta}}+(g^{\delta\xi}\Gamma^{\alpha}_{\xi\gamma}+g^{\alpha\xi}\Gamma^{\delta}_{\xi\gamma})b_{\alpha\beta}-(g^{\delta\xi}\Gamma^{\alpha}_{\xi\beta}+g^{\alpha\xi}\Gamma^{\delta}_{\xi\beta})b_{\alpha\gamma}$$
$$+g^{\delta\alpha}\Gamma^{\eta}_{\alpha\beta}b_{\eta\gamma}-g^{\delta\alpha}\Gamma^{\eta}_{\alpha\gamma}b_{\eta\beta}$$
$$=\frac{\partial b^{\delta}_{\beta}}{\partial u^{\gamma}}-\frac{\partial b^{\delta}_{\gamma}}{\partial u^{\beta}}+b^{\xi}_{\beta}\Gamma^{\delta}_{\xi\gamma}-b^{\xi}_{\gamma}\Gamma^{\delta}_{\xi\beta}$$
$$\therefore\frac{\partial b^{\delta}_{\beta}}{\partial u^{\gamma}}-\frac{\partial b^{\delta}_{\gamma}}{\partial u^{\beta}}=-b^{\eta}_{\beta}\Gamma^{\delta}_{\eta\gamma}+b^{\eta}_{\gamma}\Gamma^{\delta}_{\eta\beta}\quad(\xi\leftrightarrow\eta)$$

反之,$(8)\Rightarrow(13)$: (8)式两边同乘以$g_{\delta\alpha}$,并对δ求和,同理可得(13)式.

2. 证明：若(u,v)是曲面上的参数系,使得参数曲线网是正交的曲率线网，则主曲率k_1,k_2满足下列方程:

$$\begin{cases}\dfrac{\partial k_1}{\partial v}=\dfrac{1}{2}\dfrac{E_v}{E}(k_2-k_1),\\ \dfrac{\partial k_2}{\partial v}=\dfrac{1}{2}\dfrac{G_u}{G}(k_1-k_2).\end{cases}$$

证明： $k_1=\dfrac{L}{E}$， $k_2=\dfrac{N}{G}$

$$\frac{\partial k_1}{\partial v}=\frac{1}{E^2}(E\frac{\partial L}{\partial v}-L\frac{\partial E}{\partial v})=\frac{1}{E}(\frac{N}{2G}\frac{\partial E}{\partial v}+\frac{L}{2E}\frac{\partial E}{\partial v})-\frac{L}{E^2}\frac{\partial E}{\partial v}$$
$$=\frac{1}{2}\frac{1}{E}\frac{\partial E}{\partial v}(\frac{N}{G}+\frac{L}{E})-\frac{L}{E^2}\frac{\partial E}{\partial v}=\frac{1}{2}\frac{E_v}{E}(\frac{N}{G}+\frac{L}{E}-\frac{2L}{E})$$
$$=\frac{1}{2}\frac{E_v}{E}(\frac{N}{G}-\frac{L}{E})=\frac{1}{2}\frac{E_v}{E}(k_2-k_1)$$
$$\frac{\partial k_2}{\partial v}=\frac{1}{G^2}(G\frac{\partial N}{\partial u}-N\frac{\partial G}{\partial u})=\frac{1}{G}(\frac{N}{2G}\frac{\partial G}{\partial u}+\frac{L}{2E}\frac{\partial G}{\partial u})-\frac{N}{G^2}\frac{\partial G}{\partial u}$$
$$=\frac{G_u}{2G}(\frac{N}{G}+\frac{L}{E}-\frac{2N}{G})=\frac{G_u}{2G}(\frac{L}{E}-\frac{N}{G})=\frac{1}{2}\frac{G_u}{G}(k_1-k_2)$$

3. 证明：平均曲率为常数的曲面或是平面，或是球面，或是它的第一基本形式和第二基本形式可以表示成

$$I=\lambda[(du)^2+(dv)^2],\quad II=(1+\lambda H)(du)^2-(1-\lambda H)(dv)^2.$$

证明： $H=\dfrac{1}{2}(k_1+k_2)$

$i)$.若$k_1=k_2=0$,则曲面为平面.

$ii)$.若$k_1=k_2=c\neq 0$,则曲面为球面.

$iii)$.若$k_1\neq k_2$,且$H=\dfrac{1}{2}(k_1+k_2)\equiv c'$,不妨设$k_1>H>k_2$.

取曲面S的正交曲率线网作为参数曲线网,则$\dfrac{\partial L}{\partial v}=H\dfrac{\partial E}{\partial v},\dfrac{\partial N}{\partial u}=H\dfrac{\partial G}{\partial u}$

故可设$L=HE+\varphi(u),N=HG+\psi(v)$

则$k_1=\dfrac{L}{E}=H+\dfrac{\varphi(u)}{E},k_2=\dfrac{N}{G}=H+\dfrac{\psi(v)}{G}$. 显然有$\dfrac{\varphi(u)}{E}>0,\dfrac{\psi(v)}{G}<0$且

$\dfrac{\varphi(u)}{E}+\dfrac{\psi(v)}{G}=0$. 设$E=\lambda(u,v)\varphi(u)$,则$G=-\lambda(u,v)\psi(v)$

$$I=\lambda(u,v)(\varphi(u)du^2-\psi(v)dv^2)=\lambda(u(u^*,v^*),v(u^*,v^*))(du^{*2}+dv^{*2})$$

其中$du^*=\sqrt{\varphi(u)}du$, $dv^*=\sqrt{-\psi(v)}dv$.

$$II=(1+\lambda H)\varphi(u)du^2-(1-\lambda H)\psi(v)dv^2=(1+\lambda H)du^{*2}+(1-\lambda H)dv^{*2}.$$

4. 设S是E^3中的一块曲面,它的主曲率是两个不相等的常值函数.证明:S是圆柱面的一部分.

证明： 圆柱面$\bar{r}=(a\cos\frac{u}{a},a\sin\frac{u}{a},v)$的第一、第二基本形式分别为

$I_1=du^2+dv^2,II_1=-\frac{1}{a}du^2$

故只需证明S与该圆柱面有相同的第一、第二基本形式.从而在E^3的一个刚体运动下S与圆柱面重合.

取正交的曲率线网作为S的参数曲线网,$k_1\neq k_2$

$$\begin{cases}\frac{\partial L}{\partial v}=k_1\frac{\partial E}{\partial v}=H\frac{\partial E}{\partial v}\\ \frac{\partial N}{\partial u}=k_2\frac{\partial G}{\partial u}=H\frac{\partial G}{\partial u}\end{cases},\quad \text{又因}H\neq k_1,H\neq k_2,\text{故}\begin{cases}\frac{\partial E}{\partial v}=0\\ \frac{\partial G}{\partial u}=0\end{cases}\Rightarrow\begin{cases}E=f(u)>0\\ G=g(v)>0\end{cases}$$

故可设$I=f(u)du^2+g(v)dv^2$, 此时$R_{1212}=0=-LN$.

不妨设$N=0$, 则$II=k_1f(u)du^2$

令$du_1=\sqrt{f(u)}du,dv_1=\sqrt{g(v)}dv$, 则$I=du_1{}^2+dv_1{}^2,II=k_1du_1{}^2$

记$a=-\frac{1}{k_1}$, 有$I=I_1,II=II_1$.

5. 已知曲面的第一基本形式和第二基本形式分别为

$$I=u^2((du)^2+(dv)^2),\quad II=A(u,v)(du)^2+B(u,v)(dv)^2.$$

证明:(1) $A\cdot B\equiv1$; (2) A和B只是u的函数.

证明:(1) $\because F=M=0$ $\therefore$该参数系是由正交曲率线网构成的.

$$R_{1212}=\sqrt{EG}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}=-LN$$

由$I=u^2(du^2+dv^2)$,得$E=G=u^2$,则上式化为$LN=1$,即$A\cdot B\equiv1$

(2) $\frac{\partial L}{\partial v}=H\frac{\partial E}{\partial v}=0$,故$L=f(u)$,即$A=f(u),A\cdot B\equiv1\quad\Rightarrow B=g(u)$

$\therefore A$和B只是u的函数.

§ 5.4 曲面的存在性定理

1. 验证$f_{\alpha\beta}(u^1,u^2)$, $f_\alpha(u^1,u^2)$, $f(u^1,u^2)$满足方程组(12).

证明： $f_{\alpha\beta}(u^1,u^2)=\vec{r}_\alpha(u^1,u^2)\cdot\vec{r}_\beta(u^1,u^2)-g_{\alpha\beta}(u^1,u^2)$

$$f_\alpha(u^1,u^2)=\vec{r}_\alpha(u^1,u^2)\cdot\vec{n}(u^1,u^2),\quad f(u^1,u^2)=\vec{n}(u^1,u^2)\cdot\vec{n}(u^1,u^2)-1$$

$$\frac{\partial f_{\alpha\beta}}{\partial u^\gamma}=\frac{\partial\vec{r}_\alpha}{\partial u^\gamma}\cdot\vec{r}_\beta+\frac{\partial\vec{r}_\beta}{\partial u^\gamma}\cdot\vec{r}_\alpha-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}$$

$$=(\Gamma^\delta_{\alpha\gamma}\vec{r}_\delta+b_{\alpha\gamma}\vec{n})\cdot\vec{r}_\beta+(\Gamma^\delta_{\beta\gamma}\vec{r}_\delta+b_{\beta\gamma}\vec{n})\cdot\vec{r}_\alpha-(\Gamma_{\alpha\beta\gamma}+\Gamma_{\beta\alpha\gamma})$$

$$=\Gamma^\delta_{\alpha\gamma}f_{\delta\beta}+\Gamma^\delta_{\beta\gamma}f_{\delta\alpha}+b_{\alpha\gamma}f_\beta+b_{\beta\gamma}f_\alpha$$

$$\frac{\partial f_\alpha}{\partial u^\gamma}=\frac{\partial\vec{r}_\alpha}{\partial u^\gamma}\cdot\vec{n}+\vec{r}_\alpha\cdot\frac{\partial\vec{n}}{\partial u^\gamma}=(\Gamma^\delta_{\alpha\gamma}\vec{r}_\delta+b_{\alpha\gamma}\vec{n})\cdot\vec{n}+\vec{r}_\alpha\cdot(-b^\delta_\gamma\vec{r}_\delta)=-b^\delta_\gamma f_{\delta\alpha}+\Gamma^\delta_{\gamma\alpha}f_\delta+b_{\gamma\alpha}f$$

$$\frac{\partial f}{\partial u^\gamma}=2\vec{n}\cdot\frac{\partial\vec{n}}{\partial u^\gamma}=2\vec{n}\cdot(-b^\alpha_\gamma\vec{r}_\alpha)=-2b^\alpha_\gamma f_\alpha$$

2. 判断下列给出的二次微分形式φ,ψ能否作为E^3中一块曲面的第一基本形式和第二基本形式？说明理由.

(1) $\varphi=du^2+dv^2,\quad \psi=du^2-dv^2;$

(2) $\varphi=du^2+\cos^2 u dv^2,\quad \psi=\cos^2 u du^2+dv^2.$

解:(1) 不能.因为$Gauss$方程不成立.

$$E=G=1,F=0;L=1,M=0,N=-1,$$

则$R_{1212}=\sqrt{EG}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}=0$

而$b_{11}b_{22}-b_{12}^2=LN-M^2=-1\neq0.$ 即$R_{1212}\neq-(b_{11}b_{22}-b_{12}^2)$

(2) $\because F=M=0,$

$$\therefore b_{11}b_{22}-b_{12}^2=\cos^2 u,\quad R_{1212}=\sqrt{EG}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}=-\cos^2 u$$

$\therefore R_{1212}=-(b_{11}b_{22}-b_{12}^2)$,即$Gauss$方程成立.

但$\frac{\partial N}{\partial u}=0,\quad \frac{\partial G}{\partial u}=-\sin 2u,\quad H=\frac{1}{2}(\frac{N}{G}+\frac{L}{E})=\frac{1+\cos^4 u}{2\cos^2 u}$

$\therefore\frac{\partial N}{\partial u}\neq H\frac{\partial G}{\partial u}$, 即$Codazzi$方程不成立.

3. 求曲面，使它的第一基本形式和第二基本形式分别为

$$I=(1+u^2)du^2+u^2dv^2,\quad II=\frac{1}{\sqrt{1+u^2}}(du^2+u^2dv^2).$$

解：设所求曲面$S:\vec{r}=\vec{r}(u,v)$. 记$\vec{\alpha}_1=\frac{\vec{r}_u}{E},\vec{\alpha}_2=\frac{\vec{r}_v}{G},\vec{\alpha}_3=\vec{\alpha}_1\times\vec{\alpha}_2$, 则

$$\begin{cases}(\vec{\alpha}_1)_u=-\frac{E_v}{2\sqrt{EG}}\vec{\alpha}_2+\frac{L}{\sqrt{E}}\vec{\alpha}_3=\frac{1}{1+u^2}\vec{\alpha}_3\cdots(1)\\ (\vec{\alpha}_2)_u=\frac{E_v}{2\sqrt{EG}}\vec{\alpha}_1+\frac{M}{\sqrt{G}}\vec{\alpha}_3=0\cdots(2)\\ (\vec{\alpha}_3)_u=-\frac{L}{\sqrt{E}}\vec{\alpha}_1-\frac{M}{\sqrt{G}}\vec{\alpha}_2=-\frac{1}{1+u^2}\vec{\alpha}_1\cdots(3)\end{cases}$$

$$\begin{cases}(\vec{\alpha}_1)_v=\frac{G_u}{2\sqrt{EG}}\vec{\alpha}_2+\frac{M}{\sqrt{E}}\vec{\alpha}_3=\frac{1}{\sqrt{1+u^2}}\vec{\alpha}_2\cdots(4)\\ (\vec{\alpha}_2)_v=\frac{G_u}{2\sqrt{EG}}\vec{\alpha}_1+\frac{N}{\sqrt{G}}\vec{\alpha}_3=\frac{1}{\sqrt{1+u^2}}\vec{\alpha}_1+\frac{u}{\sqrt{1+u^2}}\vec{\alpha}_3\cdots(5)\\ (\vec{\alpha}_3)_v=-\frac{M}{\sqrt{E}}\vec{\alpha}_1-\frac{N}{\sqrt{G}}\vec{\alpha}_2=-\frac{u}{\sqrt{1+u^2}}\vec{\alpha}_2\cdots(6)\end{cases}$$

由(2),得$\vec{\alpha}_2=\vec{\alpha}_2(v)$, 不妨设曲线$C:\vec{r}_1=\vec{r}_1(v)$,以$v$为其弧长参数,

有$\dot{\vec{r}}_1=\vec{\alpha}=\vec{\alpha}_2(v)$,其$Frenet$标架$\{\vec{r};\vec{\alpha},\vec{\beta},\vec{\gamma}\}$,从而$\begin{cases}\vec{\alpha}_1=\vec{\beta}\cos\theta+\vec{\gamma}\sin\theta\cdots(7)\\ \vec{\alpha}_2=\vec{\alpha}\cdots(8)\\ \vec{\alpha}_3=\vec{\beta}\sin\theta-\vec{\gamma}\cos\theta\cdots(9)\end{cases}$

(7)式对v求导,得$(\vec{\alpha}_1)_v=-k\vec{\alpha}_2\cos\theta+(\tau+\theta_v)\vec{\alpha}_3$,代入(4)式,得$\theta_v=-\tau\cdots(10)$

$$\frac{1}{\sqrt{1+u^2}}=-k\cos\theta\cdots(11)$$

(8)式对v求导,代入(5)式,得

$$\begin{cases}u\cos\theta-\sin\theta=0\\ \cos\theta+u\sin\theta=-k\sqrt{1+u^2}\end{cases}\Rightarrow\begin{cases}\cos\theta=-\dfrac{k}{\sqrt{1+u^2}}\\ \sin\theta=-\dfrac{uk}{\sqrt{1+u^2}}\end{cases}$$

由$\cos^2\theta+\sin^2\theta=1$,可得$k=1$.

$$\therefore\begin{cases}\cos\theta=-\dfrac{1}{\sqrt{1+u^2}}\\ \sin\theta=-\dfrac{u}{\sqrt{1+u^2}}\end{cases}\cdots(12)$$

对(12)中两式关于v求导,可得$\theta_v=0$,从而由(10)式知$\tau=0$

因此曲线C为圆$(k=1\tau=0)$.

于是,可选取坐标系,使得$C:\vec{r}_1=(\cos v,\sin v,0)$

$\vec{\alpha}=\vec{\alpha}_2=(-\sin v,\cos v,0),\vec{\beta}=(-\cos v,-\sin v,0),\vec{\gamma}=(0,0,-1)$.

$$\therefore\begin{cases}\vec{\alpha}_1=(-\cos\theta\cos v,-\cos\theta\cos v,-\sin\theta)\\ \vec{\alpha}_2=(-\sin v,\cos v,0)\end{cases}$$

又由(12)式得$\begin{cases}\vec{\alpha}_1=\dfrac{1}{\sqrt{1+u^2}}(\cos v,\sin v,u)\\ \vec{\alpha}_2=(-\sin v,\cos v,0)\end{cases}$

$$\therefore\begin{cases}\vec{r}_u=\sqrt{E}\vec{\alpha}_1=(\cos v,\sin v,u)\\ \vec{r}_v=\sqrt{G}\vec{\alpha}_2=(-u\sin v,u\cos v,0)\end{cases}$$

解方程组,可得$\vec{r}=(u\cos v,u\sin v,\dfrac{u^2}{2})$,这是抛物线$z=\dfrac{1}{2}x^2,y=0$绕$z$轴旋转所得的旋转抛物面.

4. 已知$\varphi=E(u,v)du^2+G(u,v)dv^2,\quad\psi=\lambda(u,v)\cdot\varphi$,其中$E>0,G>0$.若$\varphi,\psi$能够作为曲面的第一基本形式和第二基本形式,则函数$E,G,\lambda$应该满足什么条件?

假定$E=G$,写出满足上述条件的E,G,λ的具体表达式.

解: 若φ,ψ能够作为曲面的第一、第二基本形式,则φ,ψ的系数需满足$Gauss-Codazzi$方程. $\because F=M=0\quad\therefore$该曲面的参数曲线网为正交的曲率线网.

故E,G,λ需满足方程:

$$\begin{cases} -\lambda^2 EG = \sqrt{EG}\left\{\left(\dfrac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\dfrac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\} \cdots (1) \\ \dfrac{\partial(\lambda E)}{\partial v} = \lambda \dfrac{\partial E}{\partial v} \cdots (2) \\ \dfrac{\partial(\lambda G)}{\partial u} = \lambda \dfrac{\partial G}{\partial u} \cdots (3) \end{cases}$$

由(2),(3)可得$\begin{cases} E\dfrac{\partial\lambda}{\partial v} = 0 \\ G\dfrac{\partial\lambda}{\partial u} = 0 \end{cases}$.　又因$E > 0, G > 0$,　$\therefore \begin{cases} \dfrac{\partial\lambda}{\partial v} = 0 \\ \dfrac{\partial\lambda}{\partial u} = 0 \end{cases}$　$\Rightarrow \lambda = c(c)$

故E, G, λ需满足(1)式及$\lambda = const$.

若$E = G$,由则(1)式,$\lambda^2 E^2 = -E\left\{\left(\dfrac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\dfrac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\} = -E\Delta \log\sqrt{E}$

即$\Delta \log\sqrt{E} = -\lambda^2 E$,其中$\lambda = const$.

§ 5.5　Gauss 定理

1. 已知曲面的第一基本形式如下所示, 求它们的$Gauss$曲率.

(1)　$I=\dfrac{du^2+dv^2}{[1+\frac{c}{4}(u^2+v^2)]^2}$,　c是常数.

(2)　$I=\dfrac{a^2(du^2+dv^2)}{v^2}$,　$v>0, a$是常数.

(3)　$I=\dfrac{du^2+dv^2}{(u^2+v^2+c)^2}$,　$c>0$是常数.

(4)　$I=du^2+e^{\frac{2u}{a}}dv^2$,　a是常数.

(5)　$I=du^2+ch^2\dfrac{u}{a}dv^2$,　a是常数.

解:(1)　$\because F=0$,　$\therefore$曲面的参数曲线网为正交的.

$$\therefore K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}$$

$$\because E=G=\frac{1}{[1+\frac{c}{4}(u^2+v^2)]^2}\qquad \therefore K=-\frac{1}{E}(\frac{\partial}{\partial u^2}+\frac{\partial}{\partial v^2})\log\sqrt{E}$$

$$\frac{\partial}{\partial u^2}\log\sqrt{E}=\frac{\frac{c^2}{4}u^2-\frac{c}{2}[1+\frac{c}{4}(u^2+v^2)]}{[1+\frac{c}{4}(u^2+v^2)]^2}$$

$$\therefore K=-\frac{c^2}{4}(u^2+v^2)+c+\frac{c^2}{4}(u^2+v^2)=c$$

(2) $E=G=\frac{a^2}{v^2}, F=0$

$$\frac{\partial}{\partial u^2}\log\sqrt{E}=0,\quad \frac{\partial}{\partial v^2}\log\sqrt{E}=\frac{1}{v^2}$$

$$(1), K=-\frac{1}{E}(\frac{\partial}{\partial u^2}+\frac{\partial}{\partial v^2})\log\sqrt{E}=-\frac{v^2}{a^2}\cdot\frac{1}{v^2}=-\frac{1}{a^2}$$

(3) $E=G=\frac{1}{(u^2+v^2+c)^2}, F=0$

$$\frac{\partial}{\partial u^2}\log\sqrt{E}=\frac{2(u^2-v^2-c)}{(u^2+v^2+c)^2},\quad \frac{\partial}{\partial v^2}\log\sqrt{E}=\frac{2(v^2-u^2-c)}{(u^2+v^2+c)^2}$$

$$\therefore K=-(u^2+v^2+c)^2\frac{-4c}{(u^2+v^2+c)^2}=4c$$

(4) $E=1, F=0, G=e^{\frac{2u}{a}}.\quad (\sqrt{G})_u=\frac{1}{a}e^{\frac{u}{a}}, \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u=(\sqrt{G})_{uu}=\frac{1}{a^2}e^{\frac{u}{a}}$

$$\therefore K=-e^{-\frac{u}{a}}\left\{0+\frac{1}{a^2}e^{\frac{u}{a}}\right\}=-\frac{1}{a^2}$$

(5) $E=1, F=0, G=ch^2\frac{u}{a}.\quad \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u=(\sqrt{G})_{uu}=\frac{1}{a^2}ch\frac{u}{a}$

$$\therefore K=-\frac{1}{ch\frac{u}{a}}\frac{1}{a^2}ch\frac{u}{a}=-\frac{1}{a^2}$$

2. 证明在下列曲面之间不存在等距对应:

(1)球面; (2)柱面; (3)双曲抛物面$z=x^2-y^2$.

证明: 设球面:$\vec{r}_1(\varphi,\theta)=(a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, a\sin\varphi)\quad (a>0)$

柱面:$\vec{r}_2(u,v)=(b\cos u, b\sin u, v)\quad (b>0)$

双曲抛物面:$\vec{r}_3(\bar{u},\bar{v})=(\bar{u},\bar{v},\bar{u}^2-\bar{v}^2)$

则球面的$Gauss$曲率为$K_1=\frac{1}{a^2}$

柱面的$Gauss$曲率为$K_2=0$

双曲抛物面的$Gauss$曲率为$K_3=\dfrac{-4}{(1+4\overline{u}^2+4\overline{v}^2)^2}$

故三曲面之间不存在等距对应.

3. 设曲面S和$\overline{S}$的第一基本形式分别为

$$I=du^2+(1+u^2)dv^2,\quad \overline{I}=\frac{\overline{u}^2}{\overline{u}^2-1}d\overline{u}^2+\overline{u}^2d\overline{v}^2,$$

试问:在S与$\overline{S}$之间是否存在保长对应?

解: S和$\overline{S}$的$Gauss$曲率分别为$K=-\dfrac{1}{(1+u^2)^2},\overline{K}=-\dfrac{1}{\overline{u}^4}$

令$\begin{cases}\overline{u}=\sqrt{1+u^2}\\ \overline{v}=v\end{cases}$, 则$\overline{I}=\dfrac{\overline{u}^2}{\overline{u}^2-1}d\overline{u}^2+\overline{u}^2d\overline{v}^2=\dfrac{1+u^2}{u^2}(\dfrac{u}{\sqrt{1+u^2}})du^2+(\sqrt{1+u^2})^2dv^2$

$=du^2+(1+u^2)dv^2=I$

$\therefore S$与$\overline{S}$之间存在保长对应$:\overline{u}=\sqrt{1+u^2},\overline{v}=v.$

4. 设曲面S和$\overline{S}$的方程分别为$\vec{r}=(u\cos v,u\sin v,\ln u)$和$\vec{r}=(\overline{u}\cos\overline{v},\overline{u}\sin\overline{v},\overline{v})$.证明:在$\overline{u}=u,\overline{v}=v$的对应下曲面$S$和$\overline{S}$有相同的$Gauss$曲率,但是在$S$和$\overline{S}$之间不存在保长对应.

证明: $S:\vec{r}_u=(\cos v,\sin v,\dfrac{1}{u}),\quad \vec{r}_v=(-u\sin v,u\cos v,0)$

$E=1+\dfrac{1}{u^2},F=0,G=u^2\quad \therefore I=(1+\dfrac{1}{u^2})du^2+u^2dv^2$

$R_{1212}=\sqrt{u^2+1}\cdot\dfrac{1}{(u^2+1)^{\frac{3}{2}}}=\dfrac{1}{u^2+1},\quad K=-\dfrac{R_{1212}}{EG-F^2}=-\dfrac{1}{(u^2+1)^2}$

$\overline{S}:\overline{I}=d\overline{u}^2+(1+\overline{u}^2)d\overline{v}^2,\quad \overline{R}_{1212}=\sqrt{1+\overline{u}^2}\cdot\dfrac{1}{(1+\overline{u}^2)^{\frac{3}{2}}}=\dfrac{1}{1+\overline{u}^2}$

$\overline{K}=-\dfrac{\overline{R}_{1212}}{\overline{E}\overline{G}-\overline{F}^2}=-\dfrac{1}{(\overline{u}^2+1)^2}$

$\therefore$在对应$\overline{u}=u,\overline{v}=v$下$,K=\overline{K}.$

假设S与$\overline{S}$之间存在保长读$\overline{u}=\overline{u}(u,v),\overline{v}=\overline{v}(u,v)$,则由$Gauss$定理$,K=\overline{K}$

从而$\overline{u}^2=u^2.$

$$J=\begin{pmatrix}\dfrac{\partial\overline{u}}{\partial u}&\dfrac{\partial\overline{v}}{\partial u}\\ \dfrac{\partial\overline{u}}{\partial v}&\dfrac{\partial\overline{v}}{\partial v}\end{pmatrix}=\begin{pmatrix}\pm1&\dfrac{\partial\overline{v}}{\partial u}\\ 0&\dfrac{\partial\overline{v}}{\partial v}\end{pmatrix}$$

当$\dfrac{\partial\overline{u}}{\partial u}=1$时,由$J\begin{pmatrix}1 & 0\\ 0 & \overline{u}^2+1\end{pmatrix}J^T=\begin{pmatrix}\dfrac{1}{u^2}+1 & 0\\ 0 & u^2\end{pmatrix}$,

即$$\begin{pmatrix}(\overline{u}^2+1)\left(\dfrac{\partial\overline{v}}{\partial u}\right)^2 & (\overline{u}^2+1)\dfrac{\partial\overline{v}}{\partial u}\dfrac{\partial\overline{v}}{\partial v}\\ (\overline{u}^2+1)\dfrac{\partial\overline{v}}{\partial v}\dfrac{\partial\overline{v}}{\partial u} & (\overline{u}^2+1)\left(\dfrac{\partial\overline{v}}{\partial v}\right)^2\end{pmatrix}=\begin{pmatrix}\dfrac{1}{u^2}+1 & 0\\ 0 & u^2\end{pmatrix}$$

$$\therefore\begin{cases}1+(\overline{u}^2+1)\left(\dfrac{\partial\overline{v}}{\partial u}\right)^2=1+\dfrac{1}{u^2}\cdots(1)\\ \dfrac{\partial\overline{v}}{\partial v}\dfrac{\partial\overline{v}}{\partial u}=0\cdots(2)\\ (\overline{u}^2+1)\left(\dfrac{\partial\overline{v}}{\partial v}\right)^2=u^2\cdots(3)\end{cases}$$

由(1),$\left(\dfrac{\partial\overline{v}}{\partial u}\right)^2=\dfrac{1}{u^2(u^2+1)}$, 由(3),$\left(\dfrac{\partial\overline{v}}{\partial v}\right)^2=\dfrac{u^2}{u^2+1}$ $\Rightarrow\dfrac{\partial\overline{v}}{\partial v}\dfrac{\partial\overline{v}}{\partial u}=\dfrac{1}{(\overline{u}^2+1)^2}\neq 0$

与(2)式矛盾.

当$\dfrac{\partial\overline{u}}{\partial u}=-1$时,同理可得出矛盾.

$\therefore S$与$\overline{S}$之间不存在保长对应.

5. 设曲面S和$\overline{S}$的第一基本形式分别为

$$I=e^{2v}[du^2+a^2(1+u^2)dv^2],\overline{I}=e^{2\overline{v}}[d\overline{u}^2+b^2(1+\overline{u}^2)d\overline{v}^2],$$

其中$a^2\neq b^2$.证明:在对应$\overline{u}=u,\overline{v}=v$下这两个曲面有相同的$Gauss$曲率,但是该对应不是保长对应.

证明: $\because F=\overline{F}=0$ $\therefore R_{1212}=\sqrt{EG}\left\{\left(\dfrac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\dfrac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}=\dfrac{e^{2v}a^2}{1+u^2}$

$$K=-\frac{R_{1212}}{EG-F^2}=-\frac{1}{(1+u^2)^2}e^{-2v}$$

$$\overline{R}_{1212}=\frac{e^{2\overline{v}}b^2}{1+\overline{u}^2},\quad \overline{K}=-\frac{\overline{R}_{1212}}{\overline{E}\overline{G}-\overline{F}^2}=-\frac{1}{(1+\overline{u}^2)^2}e^{-2\overline{v}}$$

$\therefore$在对应$\overline{u}=u,\overline{v}=v$下,$K=\overline{K}$.

但因$a^2\neq b^2$,故在对应$\overline{u}=u,\overline{v}=v$下,$I\neq\overline{I}$.

$\therefore$该对应不是保长对应.

6. 证明: 曲面在一般的参数系(u,v)下,$Gauss$曲率有下面的表达式:

$$K=\frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} -\frac{G_{uu}}{2}+F_{uv}-\frac{E_{vv}}{2} & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix}-\begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right\}.$$

证明: $L=\vec{r}_{uu}\vec{n}=\vec{r}_{uu}\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}=\frac{(\vec{r}_{uu},\vec{r}_u,\vec{r}_v)}{|\vec{r}_u\times\vec{r}_v|}$, $\quad M=\vec{r}_{uv}\vec{n}=\frac{(\vec{r}_{uv},\vec{r}_u,\vec{r}_v)}{|\vec{r}_u\times\vec{r}_v|}$

$N=\vec{r}_{vv}\vec{n}=\frac{(\vec{r}_{vv},\vec{r}_u,\vec{r}_v)}{|\vec{r}_u\times\vec{r}_v|}$, $\quad |\vec{r}_u\times\vec{r}_v|^2=(\vec{r}_u\times\vec{r}_v)(\vec{r}_u\times\vec{r}_v)=(\vec{r}_u\vec{r}_u)(\vec{r}_v\vec{r}_v)-(\vec{r}_u\vec{r}_v)^2=EG-F^2$

$$LN-M^2=\frac{(\vec{r}_{uu},\vec{r}_u,\vec{r}_v)(\vec{r}_{vv},\vec{r}_u,\vec{r}_v)-(\vec{r}_{uv},\vec{r}_u,\vec{r}_v)^2}{|\vec{r}_u\times\vec{r}_v|^2}$$

$$\therefore K=\frac{LN-M^2}{EG-F^2}=\frac{(\vec{r}_{uu},\vec{r}_u,\vec{r}_v)(\vec{r}_{vv},\vec{r}_u,\vec{r}_v)-(\vec{r}_{uv},\vec{r}_u,\vec{r}_v)^2}{(EG-F^2)^2}$$

$$\Rightarrow K(EG-F^2)^2=(\vec{r}_{uu},\vec{r}_u,\vec{r}_v)(\vec{r}_{vv},\vec{r}_u,\vec{r}_v)-(\vec{r}_{uv},\vec{r}_u,\vec{r}_v)^2$$

$$=\begin{vmatrix} \vec{r}_{uu}\vec{r}_{vv} & \vec{r}_u\vec{r}_{vv} & \vec{r}_v\vec{r}_{vv} \\ \vec{r}_{uu}\vec{r}_u & \vec{r}_u\vec{r}_u & \vec{r}_v\vec{r}_u \\ \vec{r}_{uu}\vec{r}_v & \vec{r}_u\vec{r}_v & \vec{r}_v\vec{r}_v \end{vmatrix}-\begin{vmatrix} \vec{r}_{uv}\vec{r}_{uv} & \vec{r}_u\vec{r}_{uv} & \vec{r}_v\vec{r}_{uv} \\ \vec{r}_{uv}\vec{r}_u & \vec{r}_u\vec{r}_u & \vec{r}_v\vec{r}_u \\ \vec{r}_{uv}\vec{r}_v & \vec{r}_u\vec{r}_v & \vec{r}_v\vec{r}_v \end{vmatrix}$$

$$=\begin{vmatrix} \vec{r}_{uu}\vec{r}_{vv} & F_v-\frac{G_u}{2} & \frac{G_v}{2} \\ \frac{E_u}{2} & E & F \\ F_u-\frac{E_v}{2} & F & G \end{vmatrix}-\begin{vmatrix} \vec{r}_{uv}\vec{r}_{uv} & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}$$

$$=(\vec{r}_{uu}\vec{r}_{vv}-\vec{r}_{uv}\vec{r}_{uv})(EG-F^2)+\begin{vmatrix} 0 & F_v-\frac{G_u}{2} & \frac{G_v}{2} \\ \frac{E_u}{2} & E & F \\ F_u-\frac{E_v}{2} & F & G \end{vmatrix}-\begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}$$

$$=(F_{uv}-\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu})(EG-F^2)+\begin{vmatrix} 0 & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix}-\begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}$$

$$=\begin{vmatrix} -\frac{G_{uu}}{2}+F_{uv}-\frac{E_{vv}}{2} & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}$$

$$\therefore K=\frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} -\frac{G_{uu}}{2}+F_{uv}-\frac{E_{vv}}{2} & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right\}$$

7. 若在定理3中S_1的$Gauss$曲率$K\equiv 0$,则定理的结论是否成立?举例说明.

解: 若定理3中S_1的$Gauss$曲率$K\equiv 0$,定理的结论不成立.

因此时的S_1为可展曲面,而使$I=C_1du^2+C_2dv^2, II=D_1du^2+D_2dv^2$的曲面是不存在的,这与$\sigma$保持在每一点沿每一个切方向的法曲率不变, 从而$\sigma_*$处处非退化矛盾.

§ 6.1 测地曲率和测地饶率

1. 证明：旋转面上纬线的测地曲率是常数，它的倒数等于在经线的切线上从切点到它与旋转轴的交点之间的线段之长.

证明： 设旋转面方程为$\vec{r}=(f(v)\cos u, f(v)\sin u, g(v))$, $\quad I=f^2du^2+(f^{'2}+g^{'2})dv^2$

纬线即$u-$曲线$:v=v_0$(常数)

其测地曲率$k_{g_u}=-\dfrac{1}{2\sqrt{G}}\cdot\dfrac{\partial\ln E}{\partial v}=-\dfrac{f^{'}(v_0)}{f(v_0)\sqrt{f^{'2}(v_0)+g^{'2}(v_0)}}$为常数

切点$P(u_0,v_0)$, 过点P的切线$:\vec{r}_v(u_0,v_0)=(f^{'}(v_0)\cos u_0, f^{'}(v_0)\sin u_0, g^{'}(v_0))$

设切线与旋转轴(即z轴)交与点$P^{'}=(0,0,z)$,则

$\overrightarrow{PP^{'}}$与$\vec{r}_v(u_0,v_0)$平行,从而$\overrightarrow{PP^{'}}\times\vec{r}_v(u_0,v_0)=0$,即

$$(-f(v_0)\cos u_0,-f(v_0)\sin u_0,z-g(v_0))\times(f^{'}(v_0)\cos u_0, f^{'}(v_0)\sin u_0, g^{'}(v_0))=0$$

$$\Rightarrow f(v_0)g^{'}(v_0)\sin u_0-f^{'}(v_0)\sin u_0\cdot z+f^{'}(v_0)g(v_0)\sin u_0=0$$

$$\Rightarrow z=\frac{f^{'}(v_0)g(v_0)-f(v_0)g^{'}(v_0)}{f^{'}(v_0)}$$

$$\therefore\left|\overrightarrow{PP^{'}}\right|=\sqrt{f^2(v_0)+\frac{f^2(v_0)g^{'2}(v_0)}{f^{'2}(v_0)}}=\left|\frac{f(v_0)\sqrt{f^{'2}(v_0)+g^{'2}(v_0)}}{f^{'}(v_0)}\right|=\frac{1}{\left|k_{g_u}\right|}$$

2. 证明$:$在球面$\vec{r}=(a\cos u\cos v, a\cos u\sin v, a\sin u)\quad(-\dfrac{\pi}{2}\le u\le\dfrac{\pi}{2}, 0\le v<2\pi)$上,曲线的测地曲率可以表成$k_g=\dfrac{d\theta}{ds}-\sin u\dfrac{dv}{ds}$,其中$\theta$是曲线与经线(即$u-$曲线)之间的夹角.

证明： $E=a^2, F=0, G=a^2\cos^2 u$

$$\therefore k_g=\frac{d\theta}{ds}-\frac{1}{2\sqrt{G}}\frac{\partial\ln E}{\partial v}\cos\theta+\frac{1}{2\sqrt{E}}\frac{\partial\ln G}{\partial u}\sin\theta=\frac{d\theta}{ds}-\frac{1}{|a|}\frac{\sin u}{\cos u}\sin\theta$$

$$\because\sin\theta=\sqrt{G}\frac{dv}{ds}=|a|\cos u\frac{dv}{ds}$$

$$\therefore k_g=\frac{d\theta}{ds}-\frac{1}{|a|}\frac{\sin u}{\cos u}|a|\cos u\frac{dv}{ds}=\frac{d\theta}{ds}-\sin u\frac{dv}{ds}.$$

3. 证明:在曲面的一般参数(u,v)下,曲线$u=u(s),v=v(s)$的测地曲率是

$$k_g=\sqrt{g}(B\dot{u}-A\dot{v}+\dot{u}\ddot{v}-\dot{v}\ddot{u}),$$

其中$g=EG-F^2, A=\Gamma_{11}^1(\dot{u})^2+2\Gamma_{12}^1\dot{u}\dot{v}+\Gamma_{22}^1(\dot{v})^2, B=\Gamma_{11}^2(\dot{u})^2+2\Gamma_{12}^2\dot{u}\dot{v}+\Gamma_{22}^2(\dot{v})^2.$

特别是,参数曲线的测地曲率分别为$k_{g_1}=\sqrt{g}\Gamma_{11}^2(\dot{u})^3, k_{g_2}=-\sqrt{g}\Gamma_{22}^1(\dot{v})^3.$

证明: S上的曲线C的参数方程为$\vec{r}=\vec{r}(u(s),v(s)), s$为$C$的弧长参数.

$\vec{n}$为S沿C的法向量.

$$\dot{\vec{r}}=\vec{r}_u\frac{du}{ds}+\vec{r}_v\frac{dv}{ds},\quad \ddot{\vec{r}}=\vec{r}_{uu}\left(\frac{du}{ds}\right)^2+2\vec{r}_{uv}\frac{du}{ds}\frac{dv}{ds}+\vec{r}_{vv}\left(\frac{dv}{ds}\right)^2+\vec{r}_u\frac{d^2u}{ds^2}+\vec{r}_v\frac{d^2v}{ds^2}$$

$$k_g=(\vec{n},\dot{\vec{r}},\ddot{\vec{r}})=(\vec{r}_u,\vec{r}_{uu},\vec{n})\left(\frac{du}{ds}\right)^3+[2(\vec{r}_u,\vec{r}_{uv},\vec{n})+(\vec{r}_v,\vec{r}_{uv},\vec{n})]\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right)+$$

$$[(\vec{r}_u,\vec{r}_{vv},\vec{n})+2(\vec{r}_v,\vec{r}_{uv},\vec{n})]\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2+(\vec{r}_v,\vec{r}_{vv},\vec{n})\left(\frac{dv}{ds}\right)^3+(\vec{r}_u,\vec{r}_v,\vec{n})(\frac{du}{ds}\frac{d^2v}{ds^2}-\frac{d^2u}{ds^2}\frac{dv}{ds})$$

由$Gauss$方程:$\vec{r}_{uu}=\Gamma_{11}^1\vec{r}_u+\Gamma_{11}^2\vec{r}_v+L\vec{n}$ 可得

$$(\vec{r}_u,\vec{r}_{uu},\vec{n})=\Gamma_{11}^1(\vec{r}_u,\vec{r}_u,\vec{n})+\Gamma_{11}^2(\vec{r}_u,\vec{r}_v,\vec{n})+L(\vec{r}_u,\vec{n},\vec{n})=\Gamma_{11}^2(\vec{r}_u,\vec{r}_v,\vec{n})$$

又因$(\vec{r}_u,\vec{r}_v,\vec{n})=(\vec{r}_u\times\vec{r}_v)\cdot\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}=|\vec{r}_u\times\vec{r}_v|=\sqrt{EG-F^2}$

故$(\vec{r}_u,\vec{r}_{uu},\vec{n})=\Gamma_{11}^2\sqrt{EG-F^2}$

类似可得$(\vec{r}_v,\vec{r}_{uu},\vec{n})=-\Gamma_{11}^1\sqrt{EG-F^2},(\vec{r}_u,\vec{r}_{uv},\vec{n})=\Gamma_{12}^2\sqrt{EG-F^2},$

$(\vec{r}_v,\vec{r}_{uv},\vec{n})=-\Gamma_{12}^1\sqrt{EG-F^2},(\vec{r}_u,\vec{r}_{vv},\vec{n})=-\Gamma_{12}^1\sqrt{EG-F^2},(\vec{r}_v,\vec{r}_{vv},\vec{n})=-\Gamma_{22}^1\sqrt{EG-F^2}.$

将其代如k_g的方程,得

$$k_g=[\Gamma_{11}^2\left(\frac{du}{ds}\right)^3+(2\Gamma_{11}^2-\Gamma_{11}^1)\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right)+(\Gamma_{22}^2-2\Gamma_{22}^1)\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2-\Gamma_{22}^1\left(\frac{dv}{ds}\right)^3$$

$$+\frac{du}{ds}\frac{d^2v}{ds^2}-\frac{d^2u}{ds^2}\frac{dv}{ds}]\sqrt{EG-F^2}=\sqrt{g}(B\dot{u}-A\dot{v}+\dot{u}\ddot{v}-\dot{v}\ddot{u})$$

其中$g=EG-F^2, A=\Gamma_{11}^1(\dot{u})^2+2\Gamma_{12}^1\dot{u}\dot{v}+\Gamma_{22}^1(\dot{v})^2, B=\Gamma_{11}^2(\dot{u})^2+2\Gamma_{12}^2\dot{u}\dot{v}+\Gamma_{22}^2(\dot{v})^2.$

特别地,$u-$曲线(即$v=const$)的测地曲率为$k_{g_1}=\sqrt{g}\Gamma_{11}^2(\dot{u})^3$

$v-$曲线(即$u=const$)的测地曲率为$k_{g_2}=-\sqrt{g}\Gamma_{22}^1(\dot{v})^3.$

4. 假定Φ是曲面S上的保长变换构成的变换群,并且保持曲面S上的一条C不变.
证明:如果Φ限制在C上的作用是传递的,则曲线C的测地曲率必为常数.

证明: $\because \Phi$保持S上C不变 $\therefore$对$\forall\varphi\in\Phi,\forall P\in C$,有$\varphi(P)\in C$

$\therefore \forall P,Q\in C,\exists\varphi_1\in\Phi$,设$\varphi_1(P)=P_1,\cdots,\varphi_1(P_i)=P_{i+1},\cdots,\varphi_1(P_n)=Q$

由于Φ在C上的作用是传递的,故$\exists\psi\in\Phi$,使得$\psi(P)=Q$

又因ψ为保长变换,故$k_{g_P}=k_{g_Q}$

由P,Q的任意性知,C上的测地曲率$k_g\equiv const.$

5. 设$\vec{e}_1,\vec{e}_2$是曲面在一点的两个彼此正交的主方向,对应的主曲率分别为k_1,k_2.证明:曲面在该点与$\vec{e}_1$成θ角的切方向的测地饶率是$\tau_g=\dfrac{1}{2}(k_2-k_1)\sin 2\theta=\dfrac{1}{2}\dfrac{dk_n(\theta)}{d\theta}$.

证明: 在该点附近取正交曲率线网作为参数曲线网,并且有$\vec{r}_u\parallel\vec{e}_1,\vec{r}_v\parallel\vec{e}_2$.

则$I=Edu^2+Gdv^2, II=k_1Edu^2+k_2Gdv^2$

$$\frac{d\vec{r}}{ds}=\vec{r}_u\frac{du}{ds}+\vec{r}_v\frac{dv}{ds}=\sqrt{E}\frac{du}{ds}\vec{e}_1+\sqrt{G}\frac{dv}{ds}\vec{e}_2=\vec{e}_1\cos\theta+\vec{e}_2\sin\theta$$

$$\Rightarrow\frac{du}{ds}=\frac{\cos\theta}{|\vec{r}_u|}=\frac{\cos\theta}{\sqrt{E}},\quad \frac{dv}{ds}=\frac{\sin\theta}{|\vec{r}_v|}=\frac{\sin\theta}{\sqrt{G}}$$

$$\therefore\tau_g=\frac{1}{\sqrt{g}}\begin{vmatrix}\left(\frac{dv}{ds}\right)^2 & -\frac{du}{ds}\cdot\frac{dv}{ds} & \left(\frac{du}{ds}\right)^2\\ E & F & G\\ L & M & N\end{vmatrix}=\frac{1}{\sqrt{EG}}\frac{du}{ds}\cdot\frac{dv}{ds}(k_2-k_1)EG$$

$$=\sqrt{EG}\frac{\cos\theta}{\sqrt{E}}\frac{k_2\sin\theta}{\sqrt{G}}(k_2-k_1)=\frac{1}{2}(k_2-k_1)\sin 2\theta$$

$$\frac{dk_n(\theta)}{d\theta}=\frac{d(k_1\cos^2\theta+k_2\sin^2\theta)}{d\theta}=(k_2-k_1)\sin 2\theta$$

$$\therefore\tau_g=\frac{1}{2}\frac{dk_n(\theta)}{d\theta}$$

6. 假定曲面上经过一个双曲点的两条渐进曲线在该点的曲率不为零.证明:这两条曲线在该点的饶率的绝对值相等,符号相反,并且这两个饶率之积等于曲面在该点的$Gauss$曲率K.

证明： 设曲面在该双曲点的两个彼此正交的主方向为$\bar{e}_1,\bar{e}_2$,对应的主曲率分别为k_1,k_2,且其中一条渐进曲线与$\bar{e}_1$成θ角,则另一渐进曲线与$\bar{e}_1$成$-\theta$角.

由上题结论知,曲面在该点沿两渐进方向的测地饶率分别为

$$\tau_{g_1}=\frac{1}{2}(k_2-k_1)\sin 2\theta,\quad \tau_{g_2}=\frac{1}{2}(k_2-k_1)\sin(-2\theta)=-\frac{1}{2}(k_2-k_1)\sin 2\theta=-\tau_{g_1}$$

又因两渐进曲线在该双曲点处曲率不为零,故两渐进曲线在该点的饶率分别为τ_{g_1},τ_{g_2},从而两条曲线在该点的饶率的绝对值相等,符号相反.

由4.5节的习题4(1)的结论知,$tg2\theta=\dfrac{\sqrt{-K}}{H}$,即

$$K=-H^2tg^2 2\theta=\frac{-\frac{1}{4}(k_1+k_2)^2\sin^2 2\theta}{\cos^2 2\theta}$$

$$\Rightarrow K=\frac{-\frac{1}{4}(k_1{}^2+k_2{}^2)tg^2 2\theta}{\frac{1}{2}tg^2 2\theta+1}=(k_1{}^2+k_2{}^2)\frac{\sin^2 2\theta}{-2\sin^2 2\theta-4\cos^2 2\theta}$$

$$\therefore k_1{}^2+k_2{}^2=\frac{-2\sin^2 2\theta-4\cos^2 2\theta}{\sin^2 2\theta}K,$$

$$\therefore \tau_{g_1}\tau_{g_2}=-\frac{1}{4}(k_2-k_1)^2\sin^2 2\theta=-\frac{1}{4}(k_1{}^2+k_2{}^2-2K)\sin^2 2\theta$$

$$=-\frac{1}{4}(-2K\sin^2 2\theta-4K\cos^2 2\theta-2K\sin^2 2\theta)=K$$

7. 证明:$k_n^2+\tau_g^2-2Hk_n+k=0.$

证明：取正交的曲率线网作为参数曲线网,$\bar{e}_1,\bar{e}_2$为主方向,k_1,k_2为对应的主曲率,切方向与$\bar{e}_1$成θ角.

由$Euler$公式,$k_n=k_1\cos^2\theta+k_2\sin^2\theta$

由习题5结论,$\tau_g=\dfrac{1}{2}(k_2-k_1)\sin 2\theta$

又$\because H=\dfrac{1}{2}(k_1+k_2),K=k_1k_2,\quad \therefore k_n^2+\tau_g^2-2Hk_n+k=0$

8. 证明:任何两个正交方向的测地饶率之和为零.

证明： $\vec{e}_1,\vec{e}_2,k_1,k_2$同习题5,设两个正交方向与$\vec{e}_1$的夹角分别为$\theta$及$\theta+\frac{\pi}{2}$.

则$\tau_{g_1}=\frac{1}{2}(k_2-k_1)\sin 2\theta,\tau_{g_2}=\frac{1}{2}(k_2-k_1)\sin(2\theta+\pi)=-\frac{1}{2}(k_2-k_1)\sin 2\theta$

$\therefore \tau_{g_1}+\tau_{g_2}=0$

§ 6.2　测地线

1. 证明:柱面上的测地线必定是定倾曲线.

证明:　不妨设柱面的直母线与oz轴平行,故曲面方程可取为$\bar{r}=\bar{r}(u,v)=(f(u),$
$g(u),v)$,其中v为准线的弧厂参数.现在求形如$v=v(u)$的测地线方程.此时,

$\bar{n}=\bar{r}_u\times\bar{r}_v=(g',-f',0),\bar{r}_u=(f',g',v'),\bar{r}_{uu}=(f'',g'',v'')$

对于测地线,有$\begin{vmatrix} g' & -f' & 0 \\ f' & g' & v' \\ f'' & g'' & v'' \end{vmatrix}=0$,即$(g'^2+f'^2)v''-(g'g''+f'f'')v'=0$

因$|\bar{n}|^2=f'^2+g'^2=1$,　故$g'g''+f'f''=\dfrac{1}{2}(g'^2+f'^2)=0$,从而$v''=0,v=c_1u+c_2$

$\therefore$测地线族的方程为$\bar{r}=(f(u),g(u),c_1u+c_2)$

$$\therefore\cos\theta=\cos(\bar{r}_u,\bar{\gamma})=\frac{\bar{r}_u\cdot\bar{\gamma}}{|\bar{r}_u|}=\frac{c_1}{\sqrt{g'^2+f'^2+c_1^2}}=\frac{c_1}{\sqrt{1+c_1^2}}$$

$\therefore$即测地线与oz轴(即直母线)成定角,从而形如$v=v(u)$的测地线为定倾曲线.
又因直母线也是测地线,且与oz轴平行,故直母线也是定倾曲线.

$\therefore$柱面上的测地线必定是定倾曲线.

2. 设曲线C是旋转面$\bar{r}(u,v)=(f(u)\cos v,f(u)\sin v,g(u))$上的一条测地线,用$\theta$表示曲线$C$与经线的交角.证明:沿测地线$C$成立恒等式$f(u)\cdot\sin\theta=$常数.

证明:$I=(f'^2(u)+g'^2(u))du^2+f^2(u)dv^2,F=0$,　由测地线方程,有

$$\begin{cases}\dfrac{d\theta}{ds}=\dfrac{1}{2\sqrt{G}}\dfrac{\partial\ln E}{\partial v}\cos\theta-\dfrac{1}{2\sqrt{E}}\dfrac{\partial\ln G}{\partial u}\sin\theta=-\dfrac{f'^2(u)}{f(u)\sqrt{f'^2(u)+g'^2(u)}}\sin\theta \\ \dfrac{du}{ds}=\dfrac{1}{\sqrt{E}}\cos\theta=\dfrac{1}{\sqrt{f'^2(u)+g'^2(u)}}\cos\theta\end{cases}$$

$$\Rightarrow\frac{d(f(u)\cdot\sin\theta)}{ds}=f'(u)\frac{du}{ds}\sin\theta+f(u)\cos\theta\frac{d\theta}{ds}=0$$

$$\Rightarrow f(u)\cdot\sin\theta=\text{常数}$$

3. 设在旋转曲上存在一条测地线C与经线交成定角θ,并且$\theta\neq0^\circ,90^\circ$.证明:此旋转面比为圆柱面.

证明: 设旋转面方程为$\vec{r}=(f(v)\cos u, f(v)\sin u, v)$,则$I=f^2(v)du^2+(1+f'^2(v))dv^2$

测地线方程为$\begin{cases}\dfrac{d\theta}{ds}=\dfrac{1}{2\sqrt{1+f'^2(v)}}\dfrac{2f'(v)}{f(v)}\cos(\dfrac{\pi}{2}+\theta)=-\dfrac{f'(v)}{f(v)\sqrt{1+f'^2(v)}}\sin\theta \\ \dfrac{du}{ds}=\dfrac{1}{f(v)}\cos(\dfrac{\pi}{2}+\theta)=-\dfrac{1}{f(v)}\sin\theta \\ \dfrac{dv}{ds}=\dfrac{1}{\sqrt{1+f'^2(v)}}\sin(\dfrac{\pi}{2}+\theta)=\dfrac{1}{\sqrt{1+f'^2(v)}}\cos\theta\end{cases}$

$\therefore \dfrac{d\theta}{dv}=-\dfrac{f'(v)}{f(v)}tg\theta$,因$\theta\equiv c$(常数),且$\theta\neq 0^\circ, 90^\circ$,故$\dfrac{d\theta}{dv}=0$,

从而$f'(v)=0$, $f(v)=const$,因此曲面为圆柱面.

4. 证明:(1) 若曲面上一条曲线既是测地线,又是渐进曲线,则它必定是直线.

(2) 若曲面上一条曲线既是测地线,又是曲率线,则它必定是平面曲线.

(3) 若曲面上一条测地线是非直线的平面曲线, 则它必定是曲率线.

证明:(1) 因曲线为测地线,故$k_g=0$,又由曲线为渐进曲线,可知$k_n=0$

$k^2=k_g^2+k_n^2=0, K=0$ $\quad\therefore$曲线为直线.

(2) 设曲线C既是测地线又是曲率线,则若C为直线,当然是平面曲线;

若C不是直线,由C为测地线,知$\vec{\beta}=\pm\vec{n}$,从而$\dot{\vec{\beta}}=-k\vec{\alpha}+\tau\vec{\gamma}=\pm\dot{\vec{n}}$,

又因C为曲率线,故依$Rodriques$定理,有$\dot{\vec{n}}\parallel\vec{\alpha}$,即$\dot{\vec{n}}=\lambda\vec{\alpha}$($\lambda$为某一确定常数)

$\therefore -k\vec{\alpha}+\tau\vec{\gamma}=\pm\lambda\vec{\alpha}$,即$(\pm\lambda+k)\vec{\alpha}-\tau\vec{\gamma}=0$,故$\tau=0$ $\quad\therefore C$是平面曲线.

(3) 因曲线C为非直线的测地线,故$\vec{\beta}=\pm\vec{n}$

从而$\pm d\vec{n}=d\vec{\beta}=\dot{\vec{\beta}}ds=(-k\vec{\alpha}+\tau\vec{\gamma})ds=-k\vec{\alpha}ds=-kd\vec{r}$(因$C$为平面曲线,故$\tau=0$)

即$d\vec{n}\parallel d\vec{r}$, $\quad\therefore C$是曲率线.

5. 证明：若曲面上所有的测地线都是平面曲线，则该曲面必是全脐点曲面.

证明：因对$\forall P\in S$及点P的任一单位切向量$\bar{v}$,均存在唯一的一条测地线过点P,

且以$\bar{v}$为其在P处的切向量.

故S上任一点处均存在至少三条测地线是非直线的平面曲线.

$\forall P\in S$,设C_1,C_2,C_3为过点P的三条非直线的测地线,对应的在点P处的单位切向量分别为$\bar{v}_1,\bar{v}_2,\bar{v}_3$.

由习题4(3)的结论,知C_1,C_2,C_3均为曲率线,从而$\bar{v}_1,\bar{v}_2,\bar{v}_3$均为点$P$处的主方向.故由$P$的任意性知,曲面$S$在每一点处均有三个不同的主方向,

而这只有在脐点处才会产生.

因此,S为全脐点曲面.

6. 已知曲面的第一基本形式如下,求曲面上的测地线：

(1) $I=v(du^2+dv^2)$;

(2) $I=\dfrac{a^2}{v^2}(du^2+dv^2)$.

证明:(1) 测地线方程:$\begin{cases}\dfrac{d\theta}{ds}=\dfrac{\cos\theta}{2v^{3/2}}\\ \dfrac{du}{ds}=\dfrac{1}{\sqrt{v}}\cos\theta\\ \dfrac{dv}{ds}=\dfrac{1}{\sqrt{v}}\sin\theta\end{cases}\Rightarrow\begin{cases}\dfrac{dv}{d\theta}=2v\cdot tg\theta\cdots(1)\\ \dfrac{dv}{du}=tg\theta\cdots(2)\end{cases}$

由(1)$\Rightarrow\sqrt{v}\cos\theta=c\quad\Rightarrow tg\theta=\dfrac{\sqrt{v-c^2}}{c}$

由(2)$\Rightarrow\dfrac{dv}{du}=\dfrac{\sqrt{v-c^2}}{c}\quad\Rightarrow u=2c\sqrt{v-c^2}+c_1$

(2) 测地线方程:$\begin{cases}\dfrac{d\theta}{ds}=-\dfrac{1}{a}\cos\theta\\ \dfrac{du}{ds}=\dfrac{v}{a}\cos\theta\\ \dfrac{dv}{ds}=\dfrac{v}{a}\sin\theta\end{cases}\Rightarrow\begin{cases}\dfrac{dv}{d\theta}=-v\cdot tg\theta\\ \dfrac{dv}{du}=tg\theta\end{cases}$

$\Rightarrow\dfrac{v}{\cos\theta}=c\quad\Rightarrow tg\theta=\dfrac{\sqrt{c^2-v^2}}{v}$

$\Rightarrow u=\pm\sqrt{c^2-v^2}+c_1$

7. 若在曲面上存在两族测地线, 它们彼此正交成定角, 则该曲面必是可展曲面.

证明: 取其中一族测地线C_1为$u-$曲线,建立正交参数系(u,v),设另一族测地线C_2与$u-$曲线的夹角为θ,则

$$0=k_{g_1}=-\frac{1}{2\sqrt{G}}\cdot\frac{\partial \ln E}{\partial v} \quad \Rightarrow E_v=0$$

$$0=k_{g_2}=\frac{d\theta}{ds}+\frac{1}{2\sqrt{E}}\frac{\partial \ln G}{\partial u}\sin\theta. \quad \text{又}\theta=const,\text{且}\theta\in(0,\pi) \quad \Rightarrow G_u=0$$

代入公式: $K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}$, 得$K=0$ $\therefore$曲面可展.

8. 证明:曲面上的测地线的饶率恰是曲面沿曲线的切方向的测地饶率.

证明: 测地线$\Rightarrow k_g=0$,其标架场$\{\bar{r}(s);\vec{e}_1,\vec{e}_2,\vec{e}_3\}$的运动公式为

$$\begin{cases}\dfrac{d\bar{r}}{ds}=\vec{e}_1\\ \dfrac{d\vec{e}_1}{ds}=k_n\vec{e}_3\\ \dfrac{d\vec{e}_2}{ds}=\tau_g\vec{e}_3\\ \dfrac{d\vec{e}_3}{ds}=-k_n\vec{e}_1-\tau_g\vec{e}_2\end{cases}$$

令$\vec{e}_1^*=\vec{e}_1,\vec{e}_2^*=\vec{e}_3,\vec{e}_3^*=-\vec{e}_2$,则有
$$\begin{cases}\dfrac{d\bar{r}}{ds}=\vec{e}_1^*\\ \dfrac{d\vec{e}_1^*}{ds}=k_n\vec{e}_2^*\\ \dfrac{d\vec{e}_2^*}{ds}=-k_n\vec{e}_1^*+\tau_g\vec{e}_3^*\\ \dfrac{d\vec{e}_3^*}{ds}=-\tau_g\vec{e}_2^*\end{cases}$$

当$k_n=0$时,由定理4知结论成立.

当$k_n\neq 0$时,$\{\bar{r};\vec{e}_1^*,\varepsilon\vec{e}_2^*,\varepsilon\vec{e}_3^*\}$恰好是曲线的$Frenet$标架,其中$\varepsilon=sign k_n$.

由曲线论基本定理知,$\tau=\tau_g$.

9. 假定曲面S_1和S_2沿曲线C相切,证明:若C是S_1上的测地线,则C也必定是S_2上的测地线.

如果C是S_1上的曲率线或渐进曲线,又如何?

证明:(1) 因S_1, S_2沿C相切,故S_1, S_2沿C的单位法向量$\vec{n}_1, \vec{n}_2$平行,即$\vec{n}_1 = \pm\vec{n}_2$

若C是直线,则C既是S_1上的测地线,也是S_2上的测地线.

若C不是直线,则因C是S_1上的测地线,故C的主法向量$\vec{\beta} = \pm\vec{n}_1$,从而$\vec{\beta} = \pm\vec{n}_2$

故C也是S_2上的测地线.

(2) 若C是S_1上的曲率线,则有$d\vec{n}_1 \parallel d\vec{r}$,从而$d\vec{n}_2 \parallel d\vec{r}$,即$C$也是$S_2$上的曲率线.

若C是S_1上的渐进曲线,此时,若C为直线,则显然C也是S_2上的渐进曲线.

若C不是直线,则$d\vec{n}_1 \perp d\vec{r}$,从而$d\vec{n}_2 \perp d\vec{r}$,故$C$也是$S_2$上的渐进曲线.

§ 6.3　测地坐标系

1. 设曲面的第一基本形式为$I=du^2+G(u,v)dv^2$,求$\Gamma^{\alpha}_{\beta\gamma}$及$Gauss$曲率$K$.

解：$\because F(u,v)=0$,　$\therefore$有正交的参数曲线网

$$\therefore \Gamma^1_{11}=\Gamma^1_{12}=\Gamma^1_{21}=\Gamma^2_{11}=0,\quad \Gamma^1_{22}=-\frac{1}{2E}\frac{\partial G}{\partial u}=-\frac{1}{2}G_u,\quad \Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2}\frac{\partial \ln G}{\partial u}$$

$$\Gamma^2_{22}=\frac{1}{2}\frac{\partial \ln G}{\partial v}$$

$$K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}=-\frac{1}{\sqrt{G}}(\sqrt{G})_{uu}$$

2. 设曲面的第一基本形式为$I=du^2+G(u,v)dv^2$,并且$G(u,v)$满足条件$G(0,v)=1$, $G_u(0,v)=0$.证明:$G(u,v)=1-u^2K(0,v)+o(u^2)$.

证明：　由上题知,$K(0,v)=\dfrac{G_u^2(0,v)-G(0,v)G_{uu}(0,v)}{2G^2(0,v)}=-\dfrac{1}{2}G_{uu}(0,v)$

对$G(u,v)$关于u在$u=0$处$Taylor$展开,有

$$G(u,v)=G(0,v)+G_u(0,v)u+\frac{1}{2}G_{uu}(0,v)u^2+o(u^2)=1-u^2K(0,v)+o(u^2)$$

3. 设曲面上以点P为中心、以r为半径的测地圆的周长为L_r,所围面积是A_r,

证明:点P处的$Gauss$曲率是$K_0=\lim\limits_{r\to 0}\dfrac{3}{\pi}\cdot\dfrac{2\pi r-L_r}{r^3}=\lim\limits_{r\to 0}\dfrac{12}{\pi}\cdot\dfrac{\pi r^2-A_r}{r^4}$.

证明：　在点P附近取测地极坐标系(s,θ),则有

$$I=ds^2+G(s,\theta)d\theta^2,\text{其中}\lim_{s\to 0}\sqrt{G(s,\theta)}=0,\quad \lim_{s\to 0}\frac{\partial}{\partial s}\sqrt{G(s,\theta)}=1$$

$$\therefore K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_\theta}{\sqrt{G}}\right)_\theta+\left(\frac{(\sqrt{G})_s}{\sqrt{E}}\right)_s\right\}=-\frac{1}{\sqrt{G}}(\sqrt{G})_{ss}$$

$$\Rightarrow (\sqrt{G})_{ss}=-K\sqrt{G},\text{两边关于}s\text{求导,得}\frac{\partial^3\sqrt{G}}{\partial s^3}=-K\frac{\partial\sqrt{G}}{\partial s}-\frac{\partial K}{\partial s}\sqrt{G}$$

$$\therefore \frac{\partial^3\sqrt{G(0,\theta)}}{\partial s^3}=\lim_{s\to 0}\frac{\partial^3\sqrt{G(s,\theta)}}{\partial s^3}=-K_0\lim_{s\to 0}\frac{\partial\sqrt{G(s,\theta)}}{\partial s}-\lim_{s\to 0}\frac{\partial K}{\partial s}\cdot\lim_{s\to 0}\sqrt{G(s,\theta)}$$
$$=-K_0$$

对$\sqrt{G(s,\theta)}$关于s在$s=0$处$Taylor$展开,得

$$\sqrt{G(s,\theta)} = \sqrt{G(0,\theta)} + \frac{\partial\sqrt{G(s,\theta)}}{\partial s}\Big|_{s=0} s + \frac{1}{2}\frac{\partial^2\sqrt{G(s,\theta)}}{\partial s^2}\Big|_{s=0} s^2 + \frac{1}{6}\frac{\partial^3\sqrt{G(s,\theta)}}{\partial s^3}\Big|_{s=0} \cdot$$

$$s^3 + o(s^3)R(\theta) = s + \frac{1}{2}\frac{\partial^2\sqrt{G(s,\theta)}}{\partial s^2}\Big|_{s=0} s^2 - \frac{1}{6}K_0 s^3 + o(s^3)R(\theta)$$

又 $\because \lim\limits_{s\to 0}\frac{\partial}{\partial s}\sqrt{G(s,\theta)} = 1 \quad \therefore \lim\limits_{s\to 0}\frac{\partial^2\sqrt{G(s,\theta)}}{\partial s^2} = 0$

$$\therefore \sqrt{G(s,\theta)} = s - \frac{1}{6}K_0 s^3 + o(s^3)R(\theta)$$

$$\therefore L_r = \int_0^{2\pi}\sqrt{G(r,\theta)}d\theta = 2\pi r - \frac{K_0}{6}r^3 2\pi + o(r^3)\int_0^{2\pi}R(\theta)d\theta$$

$$\Rightarrow K_0 = \lim_{r\to 0}\frac{2\pi r - L_r + o(r^3)\int_0^{2\pi}R(\theta)d\theta}{\frac{\pi}{3}r^3} = \lim_{r\to 0}\frac{3}{\pi}\cdot\frac{2\pi r - L_r}{r^3}$$

又 $\because A_r = \int_0^r\int_0^{2\pi}\sqrt{G(s,\theta)}dsd\theta = \pi r^2 - \frac{\pi K_0}{12}r^4 + \int_0^r o(s^3)ds\int_0^{2\pi}R(\theta)d\theta$

$$\therefore K_0 = \lim_{r\to 0}\frac{12}{\pi}\cdot\frac{\pi r^2 - A_r}{r^4}$$

§ 6.4 常曲率曲面

1. 试在测地极坐标系下写出常曲率曲面的第一基本形式.

解： 常曲率曲面S的$Gauss$曲率$K\equiv const$.在S上取测地极坐标系(s,θ),则

$$I=ds^2+G(s,\theta)d\theta^2,\text{且}\lim_{s\to 0}\sqrt{G(s,\theta)}=0,\lim_{s\to 0}\frac{\partial\sqrt{G(s,\theta)}}{\partial s}=1$$

$$K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_\theta}{\sqrt{G}}\right)_\theta+\left(\frac{(\sqrt{G})_s}{\sqrt{E}}\right)_s\right\}=-\frac{1}{\sqrt{G}}(\sqrt{G})_{ss}$$

$$\Rightarrow(\sqrt{G})_{ss}+K\sqrt{G}=0$$

$i)$.当$K>0$时,$\sqrt{G}=f_1(\theta)\cos(\sqrt{K}s)+f_2(\theta)\sin(\sqrt{K}s)$

因$\lim\limits_{s\to 0}\sqrt{G}=0$,故$f_1(\theta)=0$

又因$\dfrac{\partial\sqrt{G}}{\partial s}=\sqrt{K}f_2(\theta)\cos(\sqrt{K}s),\lim\limits_{s\to 0}\dfrac{\partial\sqrt{G}}{\partial s}=1$,故$f_2(\theta)=\dfrac{1}{\sqrt{K}}$

于是$\sqrt{G}=\dfrac{1}{\sqrt{K}}$, $\quad I=ds^2+\dfrac{1}{K}\sin^2(\sqrt{K}s)d\theta^2$

$ii)$.当$K=0$时, $\quad(\sqrt{G})_{ss}=0$, $\quad$从而$\sqrt{G}=f_1(\theta)+f_2(\theta)s$

因$\lim\limits_{s\to 0}\sqrt{G}=0$,故$f_1(\theta)=0$

又因$\lim\limits_{s\to 0}\dfrac{\partial\sqrt{G}}{\partial s}=1$,故$f_2(\theta)=1$, $\quad\therefore\sqrt{G}=s$

从而$I=ds^2+s^2d\theta^2$

$iii)$.当$K<0$时, $\quad\sqrt{G}=f_1(\theta)ch(\sqrt{-K}s)+f_2(\theta)sh(\sqrt{-K}s)$

由$\lim\limits_{s\to 0}\sqrt{G}=0$,得$f_1(\theta)=0$, $\quad$又由$\lim\limits_{s\to 0}\dfrac{\partial\sqrt{G}}{\partial s}=1$,得$f_2(\theta)=\dfrac{1}{\sqrt{-K}}$

$\therefore\sqrt{G}=\dfrac{1}{\sqrt{-K}}sh(\sqrt{-K}s)$, $\quad$从而$I=ds^2-\dfrac{1}{K}sh^2(\sqrt{-K}s)d\theta^2$

2. 证明:在常曲率曲面上,以点P为中心的测地圆具有常测地曲率.

证明： 在S上取测地极坐标系(s,θ),则$I=ds^2+G(s,\theta)d\theta^2$

测地圆为$\theta-$曲线，即$s=s_0$（常数），其测地曲率为$k_g=\dfrac{1}{2\sqrt{E}}\dfrac{\partial\ln G}{\partial u}=\dfrac{1}{2}\dfrac{G_s}{G}$

因S为常曲率曲面，故S的第一基本形式为下列三种情况之一：

$$I=ds^2+\frac{1}{K}\sin^2(\sqrt{K}s)d\theta^2\quad(K>0)$$

$$I = ds^2 + s^2 d\theta^2$$

$$I = ds^2 - \frac{1}{K}\sin^2(\sqrt{-K}s)d\theta^2 (K<0)$$

而在上述三种情况下，$k_g = \frac{G_s}{2G}$均与θ无关,即$k_g \equiv const.$

因此，在常曲率曲面上，测地圆有常测地曲率.

3. 已知常曲率曲面的第一基本形式是

$$I = \begin{cases} du^2 + \frac{1}{K}\sin^2(\sqrt{K}u)dv^2, K>0, \\ du^2 - \frac{1}{K}sh^2(\sqrt{-K}u)dv^2, K<0. \end{cases}$$

证明:该曲面大会的测地线可以分别表示为:$A\sin(\sqrt{K}u)\cos v + B\sin(\sqrt{K}u)\sin v + C\cos(\sqrt{K}u)=0$,及$Ash(\sqrt{-K}u)\cos v + Bsh(\sqrt{-K}u)\sin v + Cch(\sqrt{-K}u)=0$,其中$A,B,C$是不全为零的常数.

证明: 当$K>0$时,测地线方程为

$$\begin{cases} \frac{d\theta}{ds} = -\sqrt{K}\frac{\cos(\sqrt{K}u)}{\sin(\sqrt{K}u)}\sin\theta \\ \frac{du}{ds} = \cos\theta \\ \frac{dv}{ds} = \frac{\sqrt{K}}{\sin(\sqrt{K}u)}\sin\theta \end{cases} \Rightarrow \begin{cases} \frac{d\theta}{du} = -\sqrt{K}\frac{\cos(\sqrt{K}u)}{\sin(\sqrt{K}u)}tg\theta \\ \frac{dv}{du} = \frac{\sqrt{K}}{\sin(\sqrt{K}u)}tg\theta \end{cases}$$

$$\Rightarrow \sin\theta = \frac{c}{\sin(\sqrt{K}u)} \quad \Rightarrow tg\theta = \frac{c_1}{\sqrt{\sin^2(\sqrt{K}u)-c^2}}$$

$$\Rightarrow \frac{dv}{du} = \frac{c_1\sqrt{K}}{\sin(\sqrt{K}u)\sqrt{\sin^2(\sqrt{K}u)-c^2}} \quad \Rightarrow dv = \frac{c_1\sqrt{K}}{\sin(\sqrt{K}u)\sqrt{\sin^2(\sqrt{K}u)-c^2}}du,$$

积分上式即可证得.

当$K<0$时,同理可得到测地线方程.

4. 试求$Klein$圆:$u^2+v^2<1, I = \frac{du^2+dv^2}{[1-(u^2+v^2)]^2}$内的测地线.

解: 令$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ $(0<r<1, 0\le\theta<2\pi)$,则$I = \frac{1}{(1-r^2)^2}dr^2 + \frac{r^2}{(1-r^2)^2}d\theta^2$

$$测地线方程:\begin{cases} \dfrac{d\alpha}{ds} = -\dfrac{1+r^2}{r}\sin\alpha \\ \dfrac{dr}{ds} = (1-r^2)\cos\alpha \\ \dfrac{d\theta}{ds} = \dfrac{1-r^2}{r}\sin\alpha \end{cases}$$

其中α为该测地线与$r-$曲线的夹角,s为测地线的弧长参数.

$$\therefore \begin{cases} \dfrac{d\alpha}{dr} = -\dfrac{1+r^2}{r(1-r^2)} tg\alpha \cdots (1) \\ \dfrac{dr}{d\theta} = r\cdot ctg\alpha \cdots (2) \end{cases}$$

$$由(1)式 \Rightarrow \ln|\sin\alpha| = \ln\frac{1-r^2}{r} + c^{'} \Rightarrow \frac{r}{1-r^2}\sin\alpha = c \Rightarrow \sin\alpha = \frac{c(1-r^2)}{r}$$

$$\Rightarrow ctg\alpha = \pm\frac{\sqrt{1-\dfrac{c^2(1-r^2)^2}{r^2}}}{\dfrac{c(1-r^2)}{r}} \quad (c, c^{'}为积分常数)$$

$$\therefore \frac{dr}{d\theta} = \pm\frac{r\sqrt{r^2-c^2(1-r^2)^2}}{c(1-r^2)} \Rightarrow \theta = \pm\int\frac{c(1-r^2)}{r\sqrt{r^2-c^2(1-r^2)^2}}dr$$

$$令 x = \frac{a(1+r^2)}{r}, 其中 a = \frac{c}{\sqrt{1+4c^2}}, 则有$$

$$1-x^2 = \frac{r^2-a^2(1+r^2)}{r^2} = \frac{r^2-c^2(r^2-1)^2}{r^2(1+4c^2)}, \quad dx = \frac{-a(1-r^2)}{r^2}dr$$

$$\therefore \theta = \mp\int\frac{dx}{\sqrt{1-x^2}} = \pm\arccos x + \theta_0, (\theta_0为积分常数)$$

$$\therefore x = \cos(\theta-\theta_0) = \frac{a(1+r^2)}{r} \Rightarrow r(\cos\theta\cos\theta_0 - \sin\theta\sin\theta_0) = \frac{c}{\sqrt{1+4c^2}}(1+r^2)$$

$$\Rightarrow u\cos\theta_0 - v\sin\theta_0 = \frac{c}{\sqrt{1+4c^2}}(1+u^2+v^2)$$

$$\Rightarrow u^2 - \frac{\sqrt{1+4c^2}}{c}u\cos\theta_0 + v^2 + \frac{\sqrt{1+4c^2}}{c}v\sin\theta_0 + 1 = 0$$

$$\Rightarrow (u - \frac{\sqrt{1+4c^2}}{2c}\cos\theta_0)^2 + (v + \frac{\sqrt{1+4c^2}}{2c}\sin\theta_0)^2 = \frac{1}{4c^2}$$

5. 试求$Klein$圆$: u^2+v^2<1, I = \dfrac{du^2+dv^2}{[1-(u^2+v^2)]^2}$和

$Poincare$上半平面$: y>0, I=\frac{1}{4y^2}(x^2+y^2)$之间的保长对应.

解： 记$\omega=u+iv, z=x+iy$， 考虑分式线形变换$:\omega=a\frac{z+b}{z+c}$,则

$$d\omega=\frac{a(c-b)}{(z+c)^2}dz,\quad I_1=\frac{|d\omega|^2}{(1-|\omega|^2)^2},\quad I_2=\frac{|dz|^2}{-(z-\overline{z})^2}$$

为使$I_1=I_2$， 即$\frac{|d\omega|^2}{(1-|\omega|^2)^2}=\frac{|a(c-b)|^2|dz|^2}{|(z+c)^2|^2(1-\frac{|a(z+b)|^2}{|z+c|^2})^2}=\frac{|a(c-b)|^2|dz|^2}{[|z+c|^2-|a(z+b)|^2]}$

$$=\frac{|a|^2|c-b|^2|dz|^2}{[(z+c)(\overline{z}+\overline{c})-|a|^2(z+b)(\overline{z}+\overline{b})]^2}=-\frac{|dz|^2}{(z-\overline{z})^2}$$

令$|a|=1, b=\overline{c}$， 则有$\frac{|c-b|^2}{[z\overline{c}+c\overline{z}-zc-\overline{c}\overline{z}]^2}=-\frac{1}{(z-\overline{z})^2}$

即$\frac{|\overline{c}-c|^2}{(z-\overline{z})^2(\overline{c}-c)^2}=-\frac{1}{(z-\overline{z})^2}\quad\Rightarrow\frac{|\overline{c}-c|^2}{(\overline{c}-c)^2}=-1$

$\therefore c$必是虚数,不妨设$c=i, b=-i$,且取$a=1$,从而有$I_1=I_2$.

此时$\omega=\frac{z-i}{z+i}$， 即$u+iv=\frac{x+iy-i}{x+iy+i}=\frac{x^2+y^2-1}{x^2+(y+1)^2}+i\frac{-2x}{x^2+(y+1)^2}$

$\therefore\begin{cases}u=\frac{x^2+y^2-1}{x^2+(y+1)^2}\\ v=\frac{-2x}{x^2+(y+1)^2}\end{cases}$为$Klein$圆和$Poincare$上半平面之间的一个保长对应.

6.第一基本形式如下的曲面都具有常数$Gauss$曲率$-\frac{1}{a}$试求它们之间的保长对应:

(1) $I=\frac{a^2}{v^2}(du^2+dv^2)\quad(v>0)$

(2) $I=du^2+e^{\frac{2u}{a}}dv^2$

(3) $I=du^2+ch^2\frac{u}{a}dv^2$.

解： (1)与(2)$:$令$\begin{cases}u_1=v_2\\ v_1=ae^{-\frac{u_2}{a}}\end{cases}$， 则$I_1=I_2$

(1)与(3)：令$\begin{cases} u_1 = r\cos\theta \\ v_1 = r\sin\theta \end{cases}$，则

$$I_1 = \frac{a^2}{r^2}\csc^2\theta dr^2 + a^2\csc^2\theta d\theta^2 = \csc^2\theta(da\ln r)^2 + a^2\csc^2\theta d\theta^2$$

令$\begin{cases} v_3 = a\ln r \\ ch\dfrac{u_3}{a} = \csc\theta \end{cases}$，即$\begin{cases} r = e^{\frac{v_3}{a}} \\ \theta = \arcsin\dfrac{1}{ch\dfrac{u_3}{a}} \end{cases}$，则有

$$I = du_3{}^2 + ch^2\frac{u_3}{a}dv_3{}^2 = I_3$$

$\therefore \begin{cases} u_1 = e^{\frac{v_3}{a}} th\dfrac{u_3}{a} \\ v_1 = e^{\frac{v_3}{a}}\dfrac{1}{ch\dfrac{u_3}{a}} = e^{\frac{v_3}{a}}\sec h\dfrac{u_3}{a} \end{cases}$ 为I_1与I_2之间的一个保长对应.

(2)与(3)：$\begin{cases} u_2 = a\ln(ae^{-\frac{v_3}{a}}ch\dfrac{u_3}{a}) \\ v_2 = e^{\frac{v_3}{a}}th\dfrac{u_3}{a} \end{cases}$，则有$I_2 = I_3$

§ 6.5　曲面上向量场的平行移动

1.　证明:若$x^\alpha = x^\alpha(u^1,u^2)$是偏微分方程组$\dfrac{\partial x^\alpha}{\partial u^\beta} = -\Gamma^\alpha_{\beta\gamma} x^\gamma$的非零解,则$(i)\, f = g_{\alpha\beta}x^\alpha x^\beta$是非零常数;$(ii)\,\vec{X} = x^\alpha(u^1,u^2)\cdot \vec{r}_\alpha(u^1,u^2)$是曲面上的切向量场,它沿曲面上任意一条曲线是平行的.

证明:(i)

$$\begin{aligned}\frac{\partial f}{\partial u^1} &= \frac{\partial g_{\alpha\beta}}{\partial u^1}x^\alpha x^\beta + \frac{\partial x^\alpha}{\partial u^1} g_{\alpha\beta}x^\beta + \frac{\partial x^\beta}{\partial u^1} g_{\alpha\beta}x^\alpha \\ &= \Gamma_{\beta\alpha 1}x^\alpha x^\beta + \Gamma_{\alpha\beta 1}x^\alpha x^\beta - \Gamma^\alpha_{1\gamma}x^\gamma g_{\alpha\beta}x^\beta - \Gamma^\beta_{1\gamma}x^\gamma g_{\alpha\beta}x^\alpha \\ &= \Gamma_{\beta\alpha 1}x^\alpha x^\beta + \Gamma_{\alpha\beta 1}x^\alpha x^\beta - \Gamma_{\beta 1\gamma}x^\gamma x^\beta - \Gamma_{\alpha 1\gamma}x^\gamma x^\alpha \\ &= \Gamma_{\beta\alpha 1}x^\alpha x^\beta + \Gamma_{\alpha\beta 1}x^\alpha x^\beta - \Gamma_{\beta\gamma 1}x^\gamma x^\beta - \Gamma_{\alpha\gamma 1}x^\gamma x^\alpha \\ &= 0\end{aligned}$$

同理可得$\dfrac{\partial f}{\partial u^2} = 0.$　$\therefore\ f = const$　又$\because\ x^\alpha \neq 0, x^\beta \neq 0$　$\therefore\ f \neq 0.$

(ii)　设$C: u^\alpha = u^\alpha(t)$为曲面上任一条曲线,因$\dfrac{dx^\alpha}{dt} = \dfrac{\partial x^\alpha}{\partial u^\beta}\cdot\dfrac{du^\beta}{dt},\quad \alpha = 1,2$

$$\begin{aligned}\therefore \frac{dx^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\frac{du^\gamma}{dt} &= \frac{\partial x^\alpha}{\partial u^\beta}\cdot\frac{du^\beta}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\frac{du^\gamma}{dt} = -\Gamma^\alpha_{\beta\gamma}x^\gamma\frac{du^\beta}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\frac{du^\gamma}{dt} \\ &= -\Gamma^\alpha_{\gamma\beta}x^\gamma\frac{du^\beta}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\frac{du^\gamma}{dt} = 0,\quad \alpha = 1,2\end{aligned}$$

$\therefore\ \vec{X}$沿曲面上任一条曲线平行.

2.　证明:在曲面上存在一个非零的、与路径无关的平行切向量场, 当且仅当该曲面的$Gauss$曲率为零.

证明:$\Leftarrow$)　当曲面S的$Gauss$曲率$K \equiv 0$时,可取参数系(u,v),使得$I = du^2 + dv^2$

从而$\Gamma^\alpha_{\beta\gamma} \neq 0.$　取切向量场$\vec{X}(t) = x^\alpha\cdot\vec{r}_\alpha$,其中:$x^\alpha = 1, \alpha = 1,2.$

则$\vec{X}(t) \neq 0$沿S上任一曲线$u^\gamma = u^\gamma(t)$,有$\dfrac{D\vec{X}(t)}{dt} = (\dfrac{dx^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\dfrac{du^\gamma}{dt})\vec{r}_\alpha = 0$

即$\vec{X}(t)$为非零的平行切向量场.

$\Rightarrow$)　在S上取正交参数曲线网,$\vec{X}(u^1(t),u^2(t)) = x^\alpha(u^1(t),u^2(t))\cdot\vec{r}_\alpha(u^1(t),u^2(t))$为非零的、与路径无关的平行切向量场.

不妨设$x^2 \neq 0$,则对S上任一曲线$u^\gamma = u^\gamma(t)$,有

$$\frac{D\vec{X}(t)}{dt} = (\frac{dx^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}x^\beta\frac{du^\gamma}{dt})\vec{r}_\alpha = (\frac{\partial x^\alpha}{\partial u^\gamma} + \Gamma^\alpha_{\beta\gamma}x^\beta)\frac{du^\gamma}{dt}\vec{r}_\alpha = 0$$

当取$u^1=t,u^2=1$时, 有$\frac{du^1}{dt}=1,\frac{du^2}{dt}=0,$ 从而$\frac{\partial x^\alpha}{\partial u^1}+\Gamma^\alpha_{\beta 1}x^\beta=0 \quad (\alpha=1,2)$

当取$u^1=1,u^2=t$时, 有$\frac{du^1}{dt}=0,\frac{du^2}{dt}=1,$ 从而$\frac{\partial x^\alpha}{\partial u^2}+\Gamma^\alpha_{\beta 2}x^\beta=0 \quad (\alpha=1,2)$

记$u=u^1,v=u^2$,则有

$$\frac{\partial x^1}{\partial u}=-\Gamma^1_{11}x^1-\Gamma^1_{21}x^2=-\frac{E_u}{2E}x^1-\frac{E_v}{2E}x^2,\quad \frac{\partial x^1}{\partial v}=-\frac{E_v}{2E}x^1+\frac{E_u}{2E}x^2,$$

$$\frac{\partial x^2}{\partial u}=\frac{E_u}{2G}x^1-\frac{G_u}{2G}x^2,\quad \frac{\partial x^2}{\partial v}=-\frac{G_u}{2G}x^1-\frac{G_v}{2G}x^2.$$

又因$\frac{\partial^2 x^1}{\partial u\partial v}=\frac{\partial^2 x^1}{\partial v\partial u}$, 即$\frac{\partial}{\partial v}(-\frac{E_u}{2E}x^1-\frac{E_v}{2E}x^2)=\frac{\partial}{\partial u}(-\frac{E_v}{2E}x^1+\frac{E_u}{2E}x^2)$

$$\therefore (E_{uv}x^1+E_u\frac{\partial x^1}{\partial v}+E_{vv}x^2+E_v\frac{\partial x^2}{\partial u})2E-(E_ux^1+E_vx^2)2E_v$$

$$=(E_{uv}x^1+E_v\frac{\partial x^1}{\partial u}-G_{uu}x^2-G_u\frac{\partial x^2}{\partial u})\cdot 2E-(E_vx^1-G_ux^2)2E_u$$

$$\Rightarrow [2EG(E_{vv}+G_{uu})-EE_vG_v-GG_uE_u-GE_v^2-EG_u^2]x^2=0$$

$$\because x^2\neq 0 \quad \therefore 2EG(E_{vv}+G_{uu})-EE_vG_v-GG_uE_u-GE_v^2-EG_u^2=0$$

$$K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}$$

$$=-\frac{1}{4E^2G^2}[2EG(E_{vv}+G_{uu})-(EE_vG_v+GG_uE_u+GE_v^2+EG_u^2)]=0$$

3. 证明:曲面S上的$u^\alpha-$曲线的单位切向量沿曲线$C:u^\gamma=u^\gamma(t)$是平行的充分必要条件是沿曲线C成立$\Gamma^\alpha_{\beta\gamma}\frac{du^\gamma}{dt}=0 \quad (\beta\neq\alpha)$.

证明:$\Rightarrow$) S上的$u^\alpha-$曲线的单位切向量为$\vec{X}=\frac{\vec{r}_\alpha}{|\vec{r}_\alpha|}=x^\beta\cdot\vec{r}_\beta$

则$x^\beta=\begin{cases}\frac{1}{|\vec{r}_\alpha|}, & \beta=\alpha\\ 0, & \beta\neq\alpha\end{cases}$ $\quad\therefore \frac{dx^\beta}{dt}=0,\beta\neq\alpha$

因$\vec{X}$沿C平行,故$\frac{dx^\beta}{dt}+\Gamma^\beta_{\delta\gamma}x^\delta\frac{du^\gamma}{dt}=0 \quad (\beta\neq\alpha)$

$$\therefore \Gamma^\beta_{\alpha\gamma}\frac{1}{|\vec{r}_\alpha|}\frac{du^\gamma}{dt}+\Gamma^\beta_{\beta\gamma}x^\beta\frac{du^\gamma}{dt}=0 \quad (\beta\neq\alpha)$$

$$\therefore \frac{1}{|\bar{r}_\alpha|}\Gamma_{\alpha\gamma}^{\beta}\frac{du^\gamma}{dt}=0 \quad \Rightarrow \Gamma_{\alpha\gamma}^{\beta}\frac{du^\gamma}{dt}=0 \quad (\beta\neq\alpha)$$

$\Leftarrow$) $\bar{X}=\dfrac{\bar{r}_\alpha}{|\bar{r}_\alpha|}$, 当$\alpha=1$,即当$\bar{X}$为$u^1-$曲线的单位切向量时,

$$x^1=\frac{1}{|\bar{r}_1|}=\frac{1}{\sqrt{g_{11}}}, x^2=0. \quad \therefore \frac{dx^2}{dt}+\Gamma_{\beta\gamma}^{2}x^\beta\frac{du^\gamma}{dt}=0$$

(现要证:$\dfrac{dx^1}{dt}+\Gamma_{\beta\gamma}^{1}x^\beta\dfrac{du^\gamma}{dt}=0$.因$x^2=0$,故只需证:$\dfrac{dx^1}{dt}+\Gamma_{1\gamma}^{1}\dfrac{1}{\sqrt{g_{11}}}\dfrac{du^\gamma}{dt}=0$)

$$\frac{dx^1}{dt}+\frac{1}{\sqrt{g_{11}}}\Gamma_{1\gamma}^{1}\frac{du^\gamma}{dt}=\frac{-\frac{\frac{dg_{11}}{dt}}{2\sqrt{g_{11}}}}{g_{11}}+\frac{1}{\sqrt{g_{11}}}\Gamma_{1\gamma}^{1}\frac{du^\gamma}{dt}=\frac{1}{\sqrt{g_{11}}}(-\frac{1}{2g_{11}}\frac{dg_{11}}{dt}+\Gamma_{1\gamma}^{1}\frac{du^\gamma}{dt})$$

$$=\frac{1}{\sqrt{g_{11}}}(-\frac{\Gamma_{11\gamma}}{g_{11}}+\Gamma_{1\gamma}^{1})\frac{du^\gamma}{dt}=\frac{1}{\sqrt{g_{11}}}(-\frac{\Gamma_{1\gamma}^{1}g_{11}-\Gamma_{1\gamma}^{2}g_{21}}{g_{11}}+\Gamma_{1\gamma}^{1})\frac{du^\gamma}{dt}$$

$$=\frac{1}{\sqrt{g_{11}}}\frac{1}{g_{11}}g_{21}\Gamma_{1\gamma}^{2}\frac{du^\gamma}{dt}=0$$

同理可证$\alpha=2$的情形.