第十讲 三重积分、第一型曲线积分、第一型曲面积分的计算、多元函数积分学的应用

1. 三重积分的计算

直角坐标　　先单后重，先重后单

柱面坐标

球面坐标

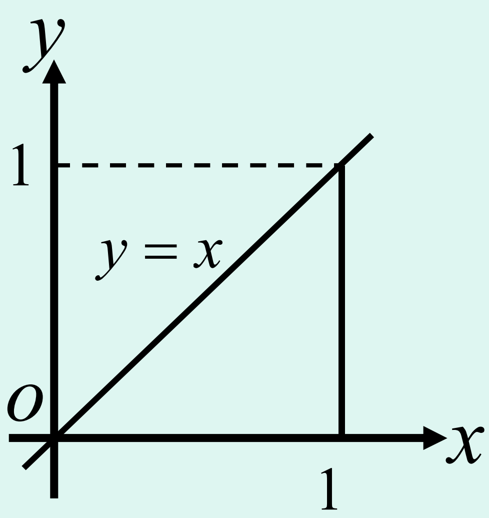
练习三十一/一(2)

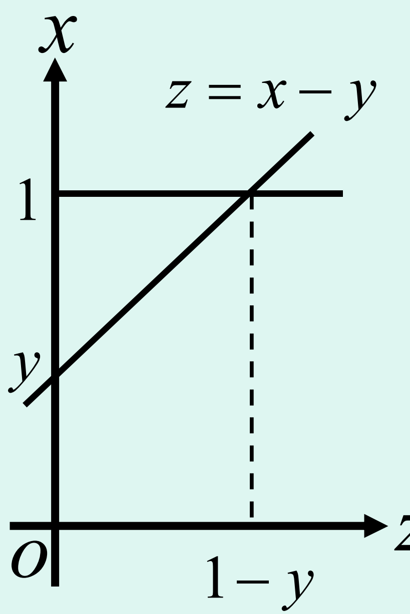
设函数$f(x,y,z)$连续,

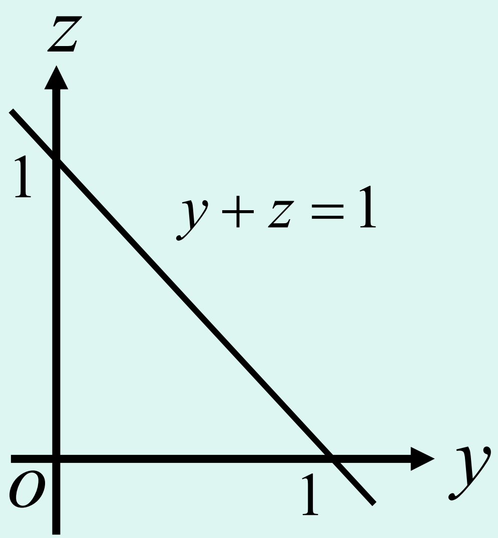
则$\int_0^1 dx\int_0^x dy\int_0^{x-y} f(x,y,z)\,dz = (C)$.

$$\int_0^1 dx\int_0^x dy\int_0^{x-y} f\,dz \;=\int_0^1 dy\int_y^1 dx\int_0^{x-y} f\,dz$$

$$=\int_0^1 dy\int_0^{1-y} dz\int_{y+z}^1 f\,dx \;=\int_0^1 dz\int_0^{1-z} dy\int_{y+z}^1 f\,dx$$







练习三十一/二(1)

设$\Omega: x\ge 0, y\ge 0, z\ge 0, x+y+z\le 1,$

则$\iiint\limits_{\Omega}\frac{1+x^3}{3+x^3+y^3+z^3}dv=$ ________.

分析:利用轮换不变性,

$$\because 3\iiint\limits_{\Omega}\cdots dv=\iiint\limits_{\Omega}\frac{(1+x^3)+(1+y^3)+(1+z^3)}{3+x^3+y^3+z^3}dv$$

$$\therefore \iiint\limits_{\Omega}\cdots dv=\frac{1}{3}\iiint\limits_{\Omega}dv=\frac{1}{18}$$

练习十一/二(2)

已知$\Omega=\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le 1, 0\le z\le 1\}$,

而函数$F(x,y,z)=(1+x+z)^{1+y+z}$在Ω上有三阶

连续偏导数,则$\iiint\limits_{\Omega} F'''_{xyz}(x,y,z)dv=$ ________.

分析:$\iiint\limits_{\Omega} F'''_{xyz}(x,y,z)dv=\int_0^1 dx\int_0^1 dy\int_0^1 F'''_{xyz}(x,y,z)dz$

$$=\int_0^1 dx\int_0^1 [F''_{xy}(x,y,1)-F''_{xy}(x,y,0)]dy$$

$$= \int_0^1 [F_x'(x,1,1) - F_x'(x,0,1) - F_x'(x,1,0) + F_x'(x,0,0)]dx$$

$$= F(1,1,1) - F(0,1,1) - F(1,0,1) + F(0,0,1)$$

$$- F(1,1,0) + F(0,1,0) + F(1,0,0) - F(0,0,0)$$

$$= 12$$

练习三十一/三

设$\Omega: |x| \le z, |y| \le z, 0 \le z \le 1$, 求$\iiint\limits_{\Omega}(z-x)(z-y)dv$.

解：原式$= \int_0^1 dz \iint\limits_{|x|\le z, |y|\le z} (z-x)(z-y)\,dxdy$

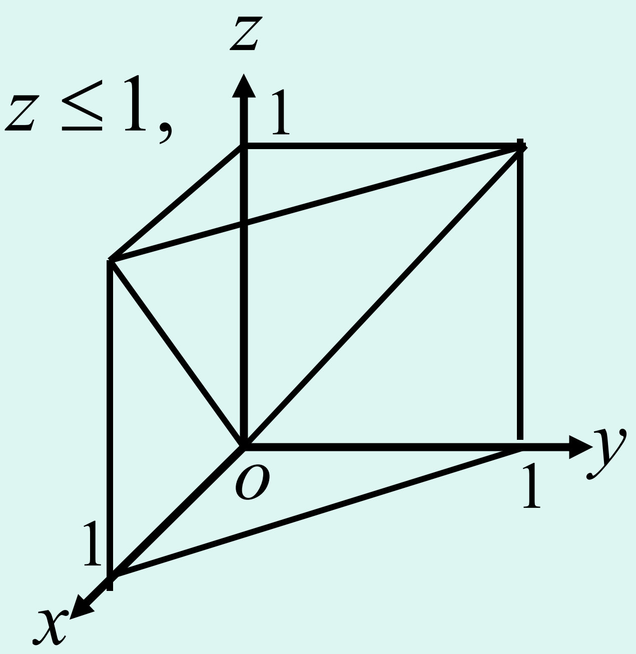
$$= \int_0^1 dz \int_{-z}^{z} (z-x)dx \int_{-z}^{z} (z-y)dy$$

$$= \int_0^1 dz \int_{-z}^{z} z dx \int_{-z}^{z} z dy = \int_0^1 2z^2 \cdot 2z^2 dz = \frac{4}{5}$$

练习十一/四

设$\Omega: x \geq 0, 0 \leq y \leq 1-x, 0 \leq z \leq 1$,

求$\iiint\limits_{\Omega} |x+y-z| dv$.



解：原式

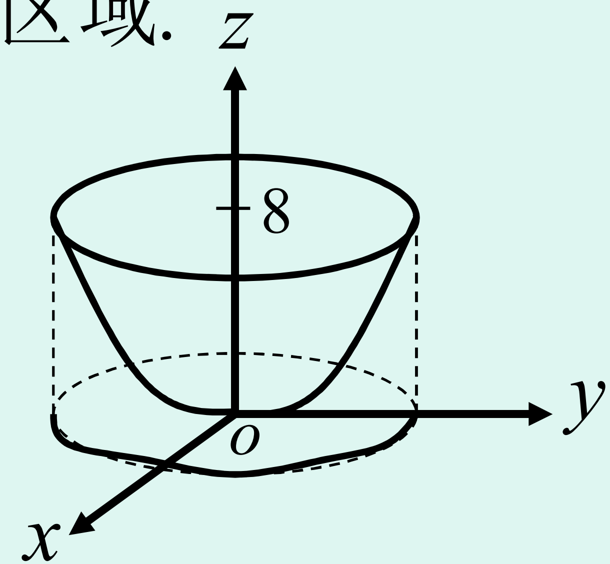
$$= \int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} (x+y-z) dz$$

$$+ \int_0^1 dx \int_0^{1-x} dy \int_{x+y}^1 (z-x-y) dz = \frac{1}{6}$$

例：计算$I=\iiint\limits_{\Omega}(x^2+y^2)dv$，其中$\Omega$为平面

曲线$\begin{cases} y^2=2z \\ x=0 \end{cases}$绕$z$轴旋转一周形成的

曲面与平面$z=8$所围成的区域.



解：曲面方程 $x^2+y^2=2z$

$$I=\int_0^{2\pi}d\theta\int_0^4\rho d\rho\int_{\frac{\rho^2}{2}}^8\rho^2 dz$$

$$=2\pi\int_0^4\rho^3(8-\frac{\rho^2}{2})d\rho=\frac{1024}{3}\pi$$

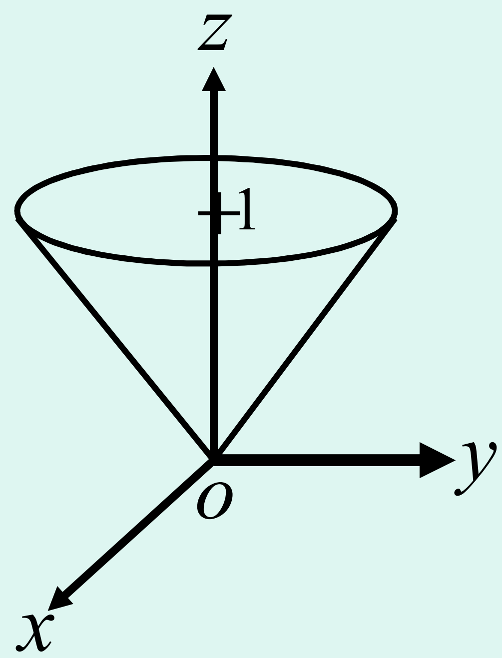
$$\text{或}I=\int_0^8 dz\iint\limits_{x^2+y^2\le 2z}(x^2+y^2)dxdy$$

$$=\int_0^8 dz\int_0^{2\pi}d\theta\int_0^{\sqrt{2z}}\rho^2\cdot\rho d\rho$$

$$=2\pi\int_0^8\frac{1}{4}(2z)^2 dz=\frac{1024}{3}\pi$$

练习三十一/八

求$\iiint\limits_{\Omega} e^{-z^3} dv$,其中$\Omega = \{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\}$.



柱面坐标,先积**z**(先单后重)

$$\iiint\limits_{\Omega} e^{-z^3} dv = \int_0^{2\pi} d\theta \int_0^1 \rho d\rho \int_{\rho}^1 e^{-z^3} dz$$

无法积出

柱面坐标,先重后单

$$\iiint\limits_{\Omega} e^{-z^3} dv = \int_0^1 dz \int_0^{2\pi} d\theta \int_0^z e^{-z^3} \rho d\rho$$

$$= \pi \int_0^1 e^{-z^3} z^2 dz = \frac{\pi}{3}(1-e^{-1})$$

球面坐标

$$\iiint\limits_{\Omega} e^{-z^3} dv = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^{\sec\varphi} e^{-r^3\cos^3\varphi} r^2 dr$$

$$= \frac{2\pi}{3}(1-e^{-1}) \int_0^{\frac{\pi}{4}} \tan\varphi \sec^2\varphi d\varphi = \frac{\pi}{3}(1-e^{-1})$$

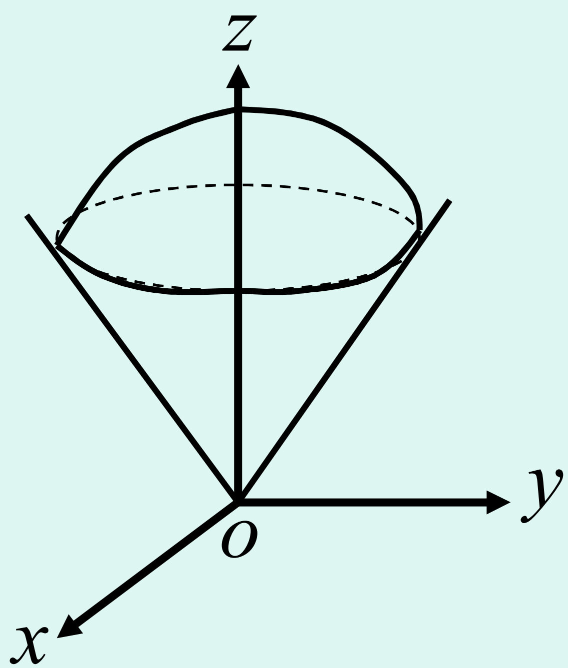
练习三十一/九

设$\Omega=\{(x,y,z)\mid\sqrt{x^2+y^2}\le z\le 1+\sqrt{1-x^2-y^2}\}$,

求$\iiint\limits_{\Omega}\frac{1}{\sqrt{x^2+y^2+z^2}}[\frac{1}{\sqrt{x^2+y^2}}\arctan\frac{\sqrt{x^2+y^2}}{z}+\frac{1}{z}]dv.$

解：$z=1+\sqrt{1-x^2-y^2}$

$\Rightarrow r=2\cos\varphi$



$$原式=\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi$$

$$\int_0^{2\cos\varphi} \frac{1}{r}[\frac{1}{r\sin\varphi}\arctan(\tan\varphi)+\frac{1}{r\cos\varphi}]r^2 dr$$

$$=\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{2\cos\varphi} (\varphi+\tan\varphi)\, dr$$

$$=2\pi\int_0^{\frac{\pi}{4}} (2\varphi\cos\varphi+2\sin\varphi)\, d\varphi$$

$$=\frac{\pi^2}{\sqrt{2}}$$

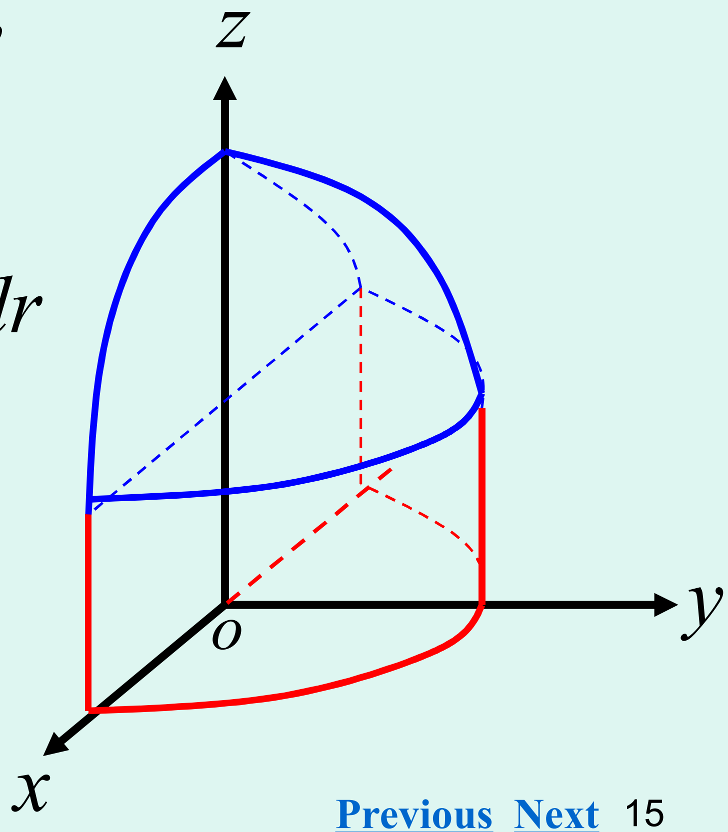
例：计算三次积分

$$I=\int_{-1}^{1}dx\int_{0}^{\sqrt{1-x^2}}dy\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}dz.$$

解：$I=\iiint\limits_{\Omega}\frac{1}{\sqrt{x^2+y^2+z^2}}dv$

$$=\int_0^{\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi d\varphi\int_{\sec\varphi}^{2\cos\varphi}\frac{1}{r}\cdot r^2dr$$

$$=\frac{\pi}{2}\left(\frac{7}{3}-\frac{4\sqrt{2}}{3}\right)$$



练习三十一/十

求$[0,+\infty)$上的连续函数$f(t)$，使满足

$$f(t)=1+\iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})dv.$$

解：$f(t)=1+\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^t f(r)r^2dr$

$$=1+4\pi\int_0^t r^2 f(r)dr$$

有 $f(0)=1$

$$f'(t) = 4\pi t^2 f(t)$$

$$\frac{df(t)}{f(t)} = 4\pi t^2 dt$$

$$\ln f(t) = \frac{4}{3}\pi t^3 + \ln C$$

$$f(t) = Ce^{\frac{4}{3}\pi t^3}$$

$$f(0) = C = 1$$

$$\therefore f(t) = e^{\frac{4}{3}\pi t^3}$$

练习三十一/十一

已知 $f(x)$ 是连续函数, 且当 $x \to 0$ 时, 有 $f(x) \sim x$,

求极限 $$\lim_{t\to 0^+} \frac{1}{t^5} \iiint\limits_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)\,\mathrm{d}V$$

解： **原极限**$$= \lim_{t\to 0^+} \frac{1}{t^5} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\varphi\,\mathrm{d}\varphi \int_0^t f(r^2)\cdot r^2\,\mathrm{d}r$$

$$= \lim_{t\to 0^+} \frac{4\pi \int_0^t f(r^2)\cdot r^2\,\mathrm{d}r}{t^5}$$

$$= \lim_{t\to 0^+} \frac{4\pi f(t^2)\cdot t^2}{5t^4} = \frac{4\pi}{5}$$

练习三十一/十二

已知$f(t)$在$[0,1]$上连续，证明

$$\int_0^1 dx\int_0^x dy\int_0^y f(x)f(y)f(z)dz=\frac{1}{3!}[\int_0^1 f(t)dt]^3.$$

证：令$\varphi(u)=\int_0^u dx\int_0^x dy\int_0^y f(x)f(y)f(z)\,dz$

$$-\frac{1}{3!}[\int_0^u f(t)\,dt]^3$$

则$\varphi'(u)=\int_0^u dy\int_0^y f(u)f(y)f(z)\,dz$

$$-\frac{1}{2!}[\int_0^u f(t)\,dt]^2\cdot f(u)$$

$$\varphi'(u) = f(u)\{\int_0^u dy \int_0^y f(y)f(z)dz - \frac{1}{2}[\int_0^u f(t)dt]^2\}$$

令 $\psi(u) = \int_0^u dy \int_0^y f(y)f(z)dz - \frac{1}{2}[\int_0^u f(t)dt]^2$

则 $\psi'(u) = \int_0^u f(u)f(z)\,dz - \int_0^u f(t)\,dt \cdot f(u)$

$$= f(u)[\int_0^u f(z)\,dz - \int_0^u f(t)\,dt] = 0$$

$\psi(u)$为常数，$\psi(u) = \psi(0) = 0.$

$\varphi'(u) = f(u)\psi(u) = 0$，$\varphi(u)$为常数，

$\varphi(u) = \varphi(0) = 0$，有 $\varphi(1) = 0$，证毕.

2. 第一型曲线积分

计算　　化为定积分

$$\int_L f(x,y)\,ds \qquad (L: y = y(x))$$

$$= \int_a^b f[x, y(x)]\sqrt{1+[y'(x)]^2}\,dx$$

例：计算$\int_L (x+y+1)ds$，其中L是由点$A=(0,2)$

到点$B=(0,-2)$的曲线段$x=\sqrt{4-y^2}$.

解：$ds=\sqrt{1+(\frac{dx}{dy})^2}\,dy=\frac{2}{\sqrt{4-y^2}}dy$

$$\int_L (x+y+1)ds=\int_{-2}^{2}(\sqrt{4-y^2}+y+1)\cdot\frac{2}{\sqrt{4-y^2}}dy$$

$$=8+2\pi$$

练习十二/二(2)

设L为椭圆$\dfrac{x^2}{2}+\dfrac{y^2}{3}=1$,已知其周长为$a$,

则$\oint\limits_L (3x^2+5xy+2y^2)ds=$ ________.

分析：利用对称性

$$\oint\limits_L (3x^2+5xy+2y^2)ds=\oint\limits_L (3x^2+2y^2)ds$$

$$=\oint\limits_L 6ds \;=6a$$

练习十二/三　计算曲线积分$\oint_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds$,

其中L为星形线$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ (a > 0)$.

解：令 $x = a\cos^3 t,\ y = a\sin^3 t \ \ (0 \le t \le 2\pi)$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = 3a\,|\sin t \cos t|\, dt$$

$$\oint_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds = 4\int_{L/4} (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds$$

$$= 4\int_0^{\frac{\pi}{2}} (a^{\frac{4}{3}} \cos^4 t + a^{\frac{4}{3}} \sin^4 t) \cdot 3a \sin t \cos t dt = 4a^{\frac{7}{3}}$$

3. 第一型曲面积分

计算　化为二重积分

$$\iint_{\Sigma} f(x,y,z)\,dS \qquad (\Sigma : z = z(x,y))$$

$$= \iint_{D_{xy}} f[x,y,z(x,y)]\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy$$

例：计算 $\iint\limits_{\Sigma}(ax+by+cz)^2\,dS$，

其中 Σ： $x^2+y^2+z^2=R^2$.

解：原式

$$=\iint\limits_{\Sigma}(a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz)\,dS$$

（利用对称性）

$$=\iint\limits_{\Sigma}(a^2x^2+b^2y^2+c^2z^2)\,dS$$

$$= a^2 \iint_{\Sigma} x^2 dS + b^2 \iint_{\Sigma} y^2 dS + c^2 \iint_{\Sigma} z^2 dS$$

(利用轮换不变性)

$$= (a^2 + b^2 + c^2) \iint_{\Sigma} x^2 dS$$

$$= \frac{a^2 + b^2 + c^2}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{a^2 + b^2 + c^2}{3} \iint_{\Sigma} R^2 dS = \frac{4}{3} \pi R^4 (a^2 + b^2 + c^2)$$

练习十二/十

计算$I=\iint\limits_S \dfrac{dS}{\sqrt{1-x^2-y^2}}$，其中$S$为锥面

$z=\sqrt{x^2+y^2}$上被柱面$z^2=x$所截下的部分.

解：$\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=x \end{cases}$

消去z，得$x^2+y^2=x$

S在xoy坐标面上的投影$D_{xy}: x^2+y^2 \le x.$

$$S: z=\sqrt{x^2+y^2}$$

$$dS=\sqrt{1+z_x^2+z_y^2}\,dxdy=\sqrt{2}\,dxdy$$

$$I=\iint\limits_{D_{xy}}\frac{1}{\sqrt{1-x^2-y^2}}\cdot\sqrt{2}\,dxdy$$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\cos\theta}\frac{1}{\sqrt{1-\rho^2}}\cdot\rho d\rho=\sqrt{2}(\pi-2)$$

练习十二/十二

计算曲面积分$\iint\limits_{\Sigma}\frac{z}{x^2+y^2+z^2}dS$,其中积分区域为曲面

$\Sigma=\{(x,y,z)\,|\,x^2+y^2=R^2,0\le z\le H\},(R>0,H>0)$

解：$\Sigma: x=\pm\sqrt{R^2-y^2}$

$$dS=\sqrt{1+x_y^2+x_z^2}\,dydz=\frac{R}{\sqrt{R^2-y^2}}dydz$$

$$原式=2\iint\limits_{D_{yz}}\frac{z}{R^2+z^2}\cdot\frac{R}{\sqrt{R^2-y^2}}dydz$$

$$=2R\int_0^H\frac{z}{R^2+z^2}dz\int_{-R}^{R}\frac{1}{\sqrt{R^2-y^2}}dy=\pi R\ln\frac{R^2+H^2}{R^2}$$

4. 多元函数积分的应用

几何应用

平面图形的面积　　立体的体积

曲线弧长　　　曲面的面积

物理应用

平面薄片或立体的质量,质心,转动惯量

曲线的质量,质心,转动惯量

曲面的质量,质心,转动惯量

练习十二/八

曲面 $z=13-x^2-y^2$ 将球面 $x^2+y^2+z^2=25$ 分成三部分，求这三部分曲面面积之比.

解： $z=\pm\sqrt{25-x^2-y^2}$

$$dS=\sqrt{1+z_x^2+z_y^2}\,dxdy=\frac{5}{\sqrt{25-x^2-y^2}}\,dxdy$$

$\begin{cases} z=13-x^2-y^2 \\ x^2+y^2+z^2=25 \end{cases}$ 消去 z，得 $\begin{array}{l} x^2+y^2=9 \\ x^2+y^2=16 \end{array}$

$$S_1 = \iint_{x^2+y^2\le 9} \frac{5}{\sqrt{25-x^2-y^2}}\, dxdy$$

$$= 5\int_0^{2\pi} d\theta \int_0^3 \frac{5}{\sqrt{25-\rho^2}}\rho\, d\rho = 10\pi$$

$$S_3 = \iint_{x^2+y^2\le 16} \frac{5}{\sqrt{25-x^2-y^2}}\, dxdy$$

$$= 5\int_0^{2\pi} d\theta \int_0^4 \frac{5}{\sqrt{25-\rho^2}}\rho\, d\rho = 20\pi$$

$$S = S_1 + S_2 + S_3 = 100\pi$$

$$S_2 = 70\pi$$

$$S_1 : S_2 : S_3 = 10\pi : 70\pi : 20\pi = 1:7:2$$

练习十二/五　　利用曲线积分计算

柱面$x^2+y^2=Rx$含在$0\le z\le \frac{1}{R}(x^2+y^2)$内的面积.

解：$A=\oint_L \frac{1}{R}(x^2+y^2)ds=\oint_L xds$

$L: x^2+y^2=Rx$ 或 $(x-\frac{R}{2})^2+y^2=(\frac{R}{2})^2$

令$x=\frac{R}{2}+\frac{R}{2}\cos t, y=\frac{R}{2}\sin t\ (0\le t\le 2\pi)$

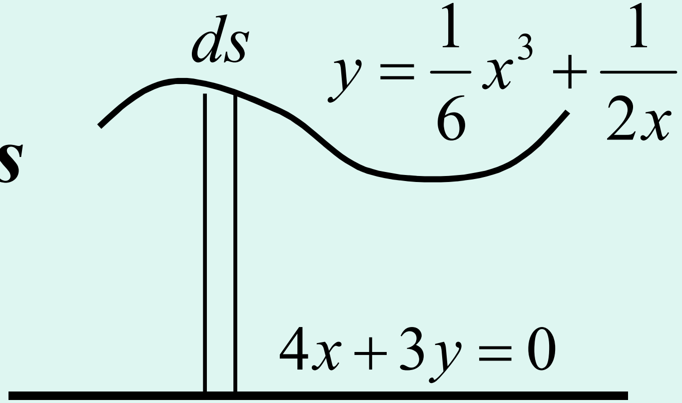
$$A=\int_0^{2\pi}\frac{R}{2}(1+\cos t)\cdot\frac{R}{2}dt=\frac{1}{2}\pi R^2$$

练习十二/六　　利用曲线积分,

求曲线$C: y=\frac{1}{6}x^3+\frac{1}{2x}\ (\frac{1}{2}\le x\le 2)$绕直线

$L: 4x+3y=0$旋转所得的旋转曲面的面积.

解：$\boldsymbol{dS=2\pi\cdot\frac{|4x+3y|}{5}ds}$



$$S=\int_C \frac{2\pi}{5}|4x+3y|ds$$

在曲线C上, $4x+3y>0.$

$$S=\frac{2\pi}{5}\int_C (4x+3y)ds$$

$$=\frac{2\pi}{5}\int_{\frac{1}{2}}^{2}(4x+\frac{x^3}{2}+\frac{3}{2x})\sqrt{(\frac{x^2}{2}+\frac{1}{2x^2})^2}\,dx$$

$$=\frac{8\pi}{5}\ln 2+\frac{1425\pi}{256}$$

Previous