3.11、(1) $\begin{cases} u_t - a^2 u_{xx} = f(x,t), & x>0, t>0 \\ u(x,0) = \phi(x), & x>0 \\ u_x(0,t) = 0 \end{cases}$

解：我们做偶延拓,把$\phi(x), f(x,t)$偶延拓成$\Phi(x), F(x,t)$,

$$F(x,t) = \begin{cases} f(x,t), & x \ge 0, t \ge 0 \\ f(-x,t), & x<0, t \ge 0 \end{cases} \quad \Phi(x) = \begin{cases} \phi(x), & x \ge 0 \\ \phi(-x), & x<0 \end{cases}$$

考虑初值问题

$$\begin{cases} U_t - a^2 U_{xx} = F(x,t), & x \in R, t>0 \\ U(x,0) = \Phi(x), & x \in R \end{cases}$$

对于$U(x,t)$

$$U(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_R \Phi(y) e^{-\frac{(x-y)^2}{4a^2 t}} dy + \int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}} \int_R F(y,\tau) e^{-\frac{(x-y)^2}{4a^2(t-\tau)}} dy d\tau$$

$$= \frac{1}{2a\sqrt{\pi t}} \int_0^\infty \phi(y) e^{-\frac{(x-y)^2}{4a^2 t}} dy + \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^0 \Phi(y) e^{-\frac{(x-y)^2}{4a^2 t}} dy$$

$$+ \int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}} \int_0^\infty f(y,\tau) e^{-\frac{(x-y)^2}{4a^2(t-\tau)}} dy d\tau + \int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}} \int_{-\infty}^0 F(y,\tau) e^{-\frac{(x-y)^2}{4a^2(t-\tau)}} dyd$$

其中

$$\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{0}\Phi(y)e^{-\frac{(x-y)^2}{4a^2t}}dy$$

$$+\int_0^t\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{0}F(y,\tau)e^{-\frac{(x-y)^2}{4a^2(t-\tau)}}dyd\tau$$

$$\overset{y=-\eta}{=}\frac{1}{2a\sqrt{\pi t}}\int_0^{\infty}\phi(\eta)\exp(-\frac{(x+\eta)^2}{4a^2t})d\eta$$

$$\int_0^t\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_0^{+\infty}f(\eta,\tau)e^{-\frac{(x+\eta)^2}{4a^2(t-\tau)}}d\eta d\tau$$

所以半无界区域上的解为

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_0^{\infty}\phi(y)(e^{-\frac{(x-y)^2}{4a^2t}}+e^{-\frac{(x+y)^2}{4a^2t}})dy$$

$$+\int_0^t\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_0^{\infty}f(y,\tau)(e^{-\frac{(x-y)^2}{4a^2(t-\tau)}}+e^{-\frac{(x+y)^2}{4a^2(t-\tau)}})dyd\tau$$