《微分几何》课程电子课件

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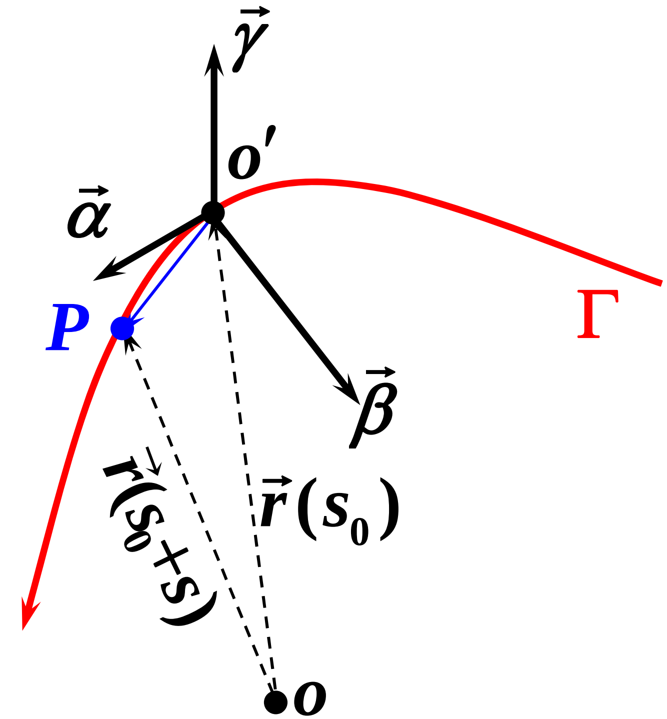
四、空间曲线在一点邻近的结构

在切点s_0附近，C^3类曲线$\vec{r}=\vec{r}(s_0+s)$在局部坐标系

$[\vec{r}(s_0);\vec{\alpha}(s_0),\vec{\beta}(s_0),\vec{\gamma}(s_0)]$中的方程为

$$\begin{cases} \xi = s-\frac{1}{6}k^2(s_0)s^3+o(s^3) \\ \eta=\frac{1}{2}k(s_0)s^2+\frac{1}{6}k'(s_0)s^3+o(s^3), \\ \zeta=\frac{1}{6}k(s_0)\tau(s_0)s^3+o(s^3) \end{cases}$$



其中$o(s^3)$为$s\to 0$时s^3的高阶无穷小.

该公式称为Bouquet公式或曲线在邻域内的局部规范式.

证 (考虑向量$\vec{r}(s_0+s)-\vec{r}(s_0)$的新坐标)

将$\vec{r}(s_0+s)-\vec{r}(s_0)$写成$\vec{\alpha}(s_0),\vec{\beta}(s_0),\vec{\gamma}(s_0)$的线性组合.

由Taylor公式,

$$\vec{r}(s_0+s)-\vec{r}(s_0)=\dot{\vec{r}}(s_0)s+\tfrac{1}{2!}\ddot{\vec{r}}(s_0)s^2+\tfrac{1}{3!}[\dddot{\vec{r}}(s_0)+o(\vec{1})]s^3$$

其中$\dot{\vec{r}}(s_0)=\vec{\alpha}(s_0)$, $\ddot{\vec{r}}(s_0)=k(s_0)\vec{\beta}(s_0)$,

$$\dddot{\vec{r}}(s_0)=\dot{k}(s_0)\vec{\beta}(s_0)+k(s_0)\dot{\vec{\beta}}(s_0)$$

$$=\dot{k}(s_0)\vec{\beta}(s_0)+k(s_0)[-k(s_0)\vec{\alpha}(s_0)+\tau(s_0)\vec{\gamma}(s_0)]$$

$$=-k^2(s_0)\vec{\alpha}(s_0)+\dot{k}(s_0)\vec{\beta}(s_0)+k(s_0)\tau(s_0)\vec{\gamma}(s_0)$$

以下记上述$o(\vec{1})=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$, 并省掉$(s_0)$, 则

$$\vec{r}(s_0+s)-\vec{r}=$$

$$[s+\tfrac{1}{6}(\varepsilon_1-k^2)s^3]\vec{\alpha}+[\tfrac{1}{2}ks^2+\tfrac{1}{6}(\dot{k}+\varepsilon_2)s^3]\vec{\beta}+\tfrac{1}{6}(k\tau+\varepsilon_3)s^3\vec{\gamma}$$

以$[\vec{r}(s_0);\vec{\alpha}(s_0),\vec{\beta}(s_0),\vec{\gamma}(s_0)]$为新坐标系,则有

$$\begin{cases}\xi=s-\frac{1}{6}k^2(s_0)s^3+o(s^3)\\ \eta=\frac{1}{2}k(s_0)s^2+\frac{1}{6}k'(s_0)s^3+o(s^3)\\ \zeta=\frac{1}{6}k(s_0)\tau(s_0)s^3+o(s^3)\end{cases}\xrightarrow{\text{近似}}\begin{cases}x=s\\ y=\frac{1}{2}k(s_0)s^2\\ z=\frac{1}{6}k(s_0)\tau(s_0)s^3\end{cases}$$

只保留各分量中的最低阶无穷小,则得到一近似曲线.

该近似曲线能刻画曲线在这一点邻近的近似形状,

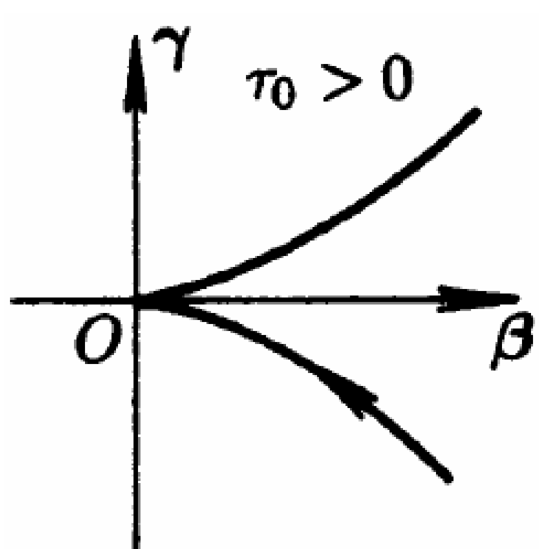
它为多项式曲线,且完全由该点的曲率和挠率决定.

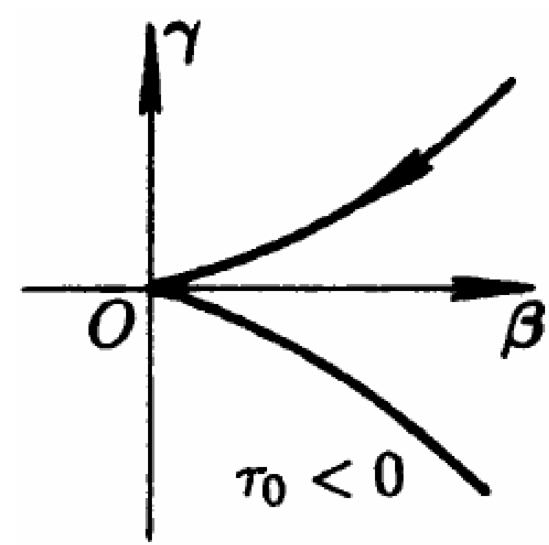
该近似曲线在法平面 $x=0$ 上的投影为

$$(x=0),\ \ y=\tfrac{1}{2}k(s_0)s^2,\ \ z=\tfrac{1}{6}k(s_0)\tau(s_0)s^3$$

从中消去参数 s 得到 $(x=0), z^2=\dfrac{2\tau^2(s_0)}{9k(s_0)}y^3$

它是半立方抛物线.



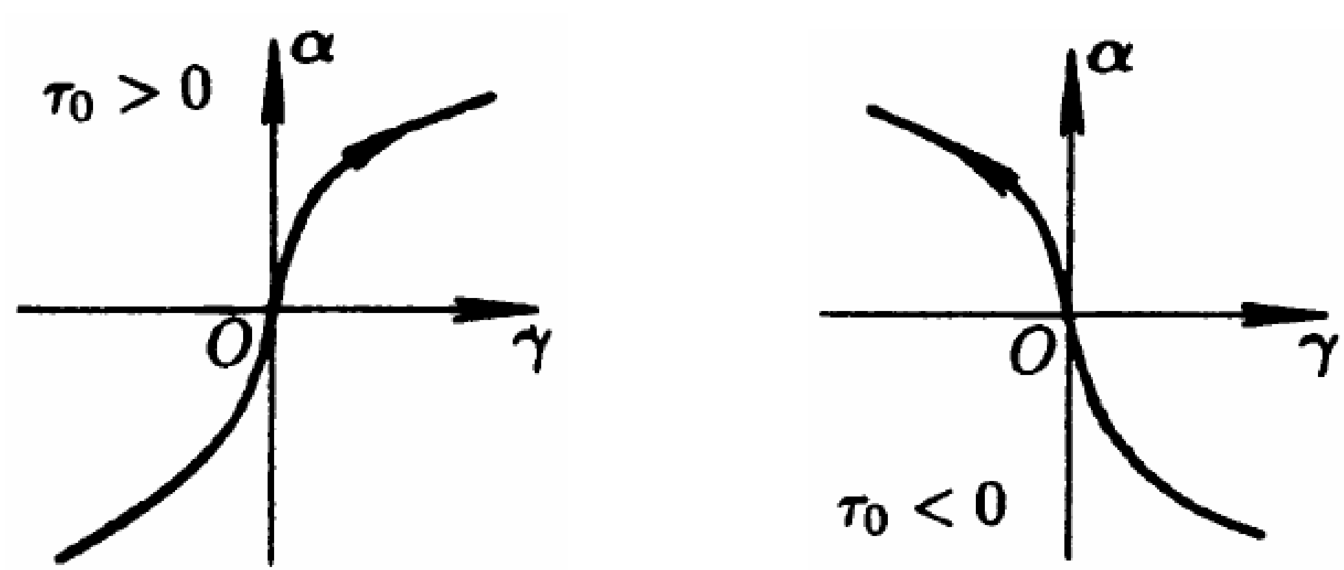


该近似曲线在从切平面 $y=0$ 上的投影为

$$(y=0),\ \ x=s,\ \ z=\tfrac{1}{6}k(s_0)\tau(s_0)s^3$$

从中消去参数 s 得到 $(y=0), z=\tfrac{1}{6}k(s_0)\tau(s_0)x^3$

它是立方抛物线.

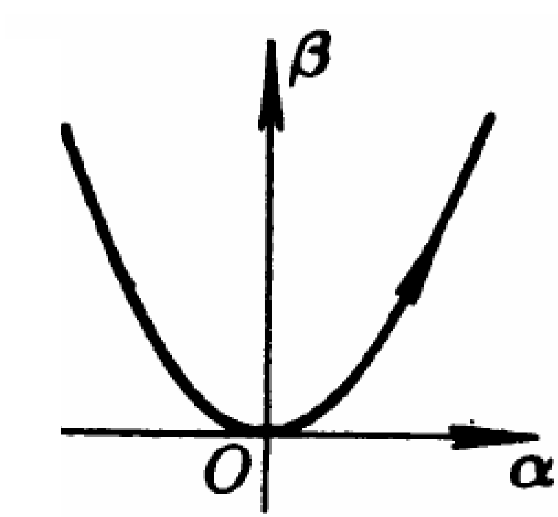


该近似曲线在密切平面$z=0$上的投影为

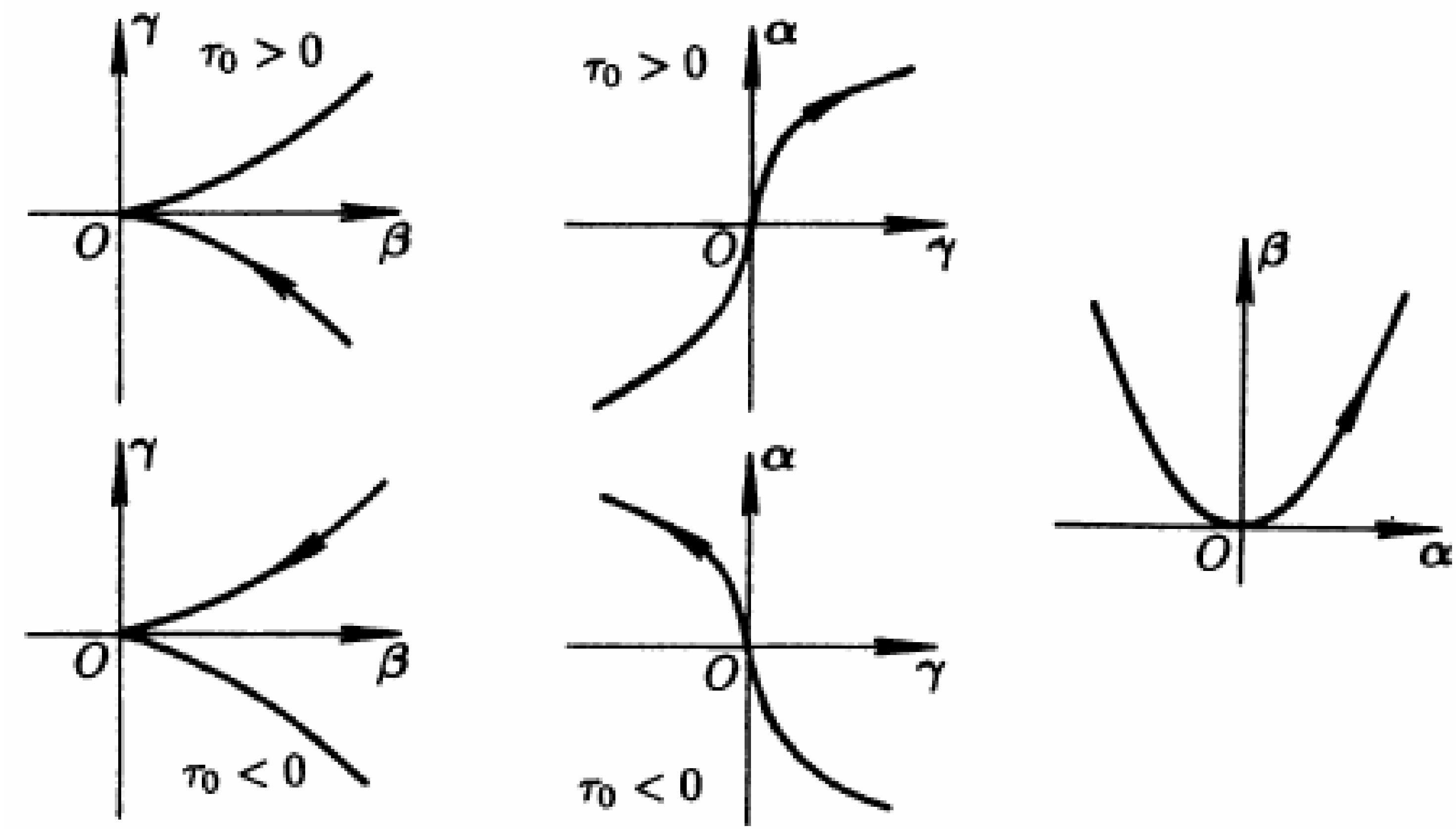
$$(z=0),\ \ x=s,\ \ y=\tfrac{1}{2}k(s_0)s^2$$

从中消去参数s得到$(z=0),\ y=\tfrac{1}{2}k(s_0)x^2$

它是抛物线.



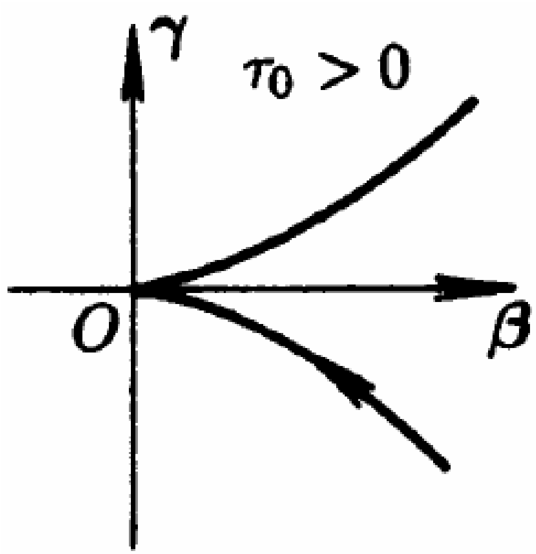
局部曲线图形特点

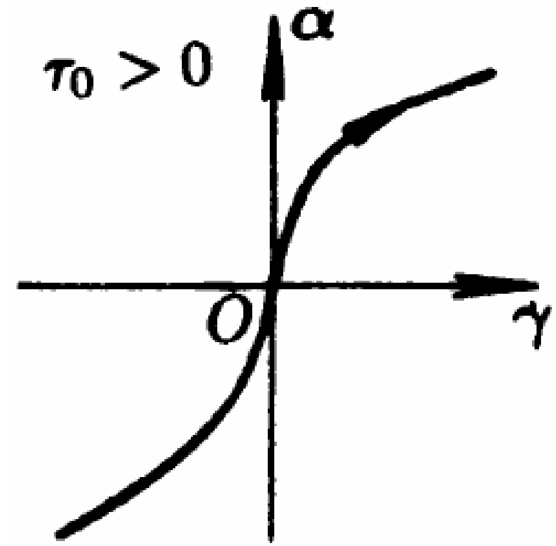


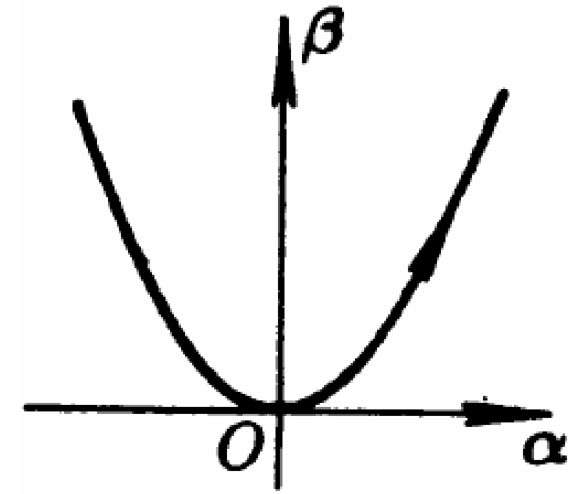
1. 曲线穿过法平面和密切平面,但不穿过从切平面;

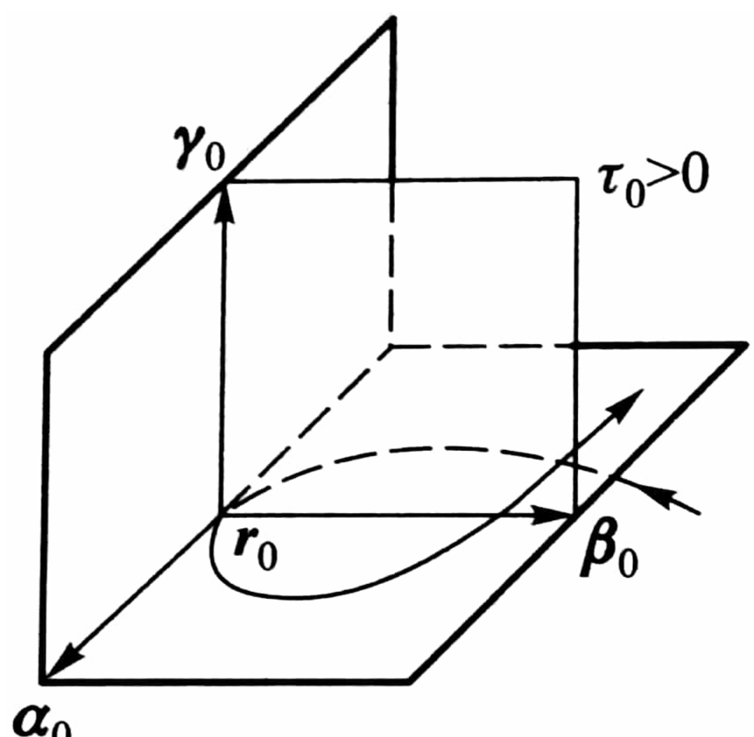
2. 主法向量总是指向曲线开口的方向;

3. 挠率的符号对曲线的影响:





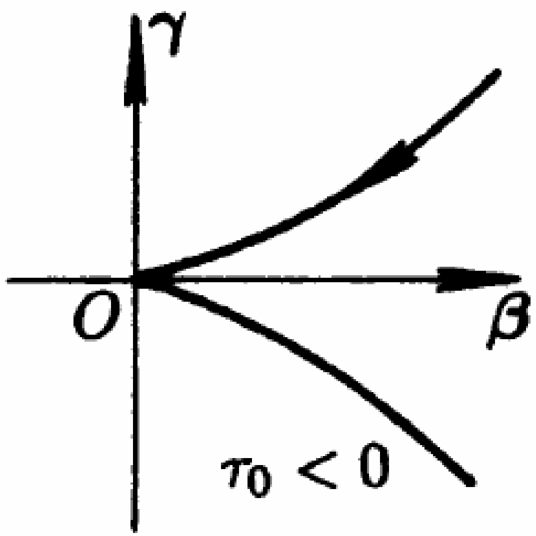


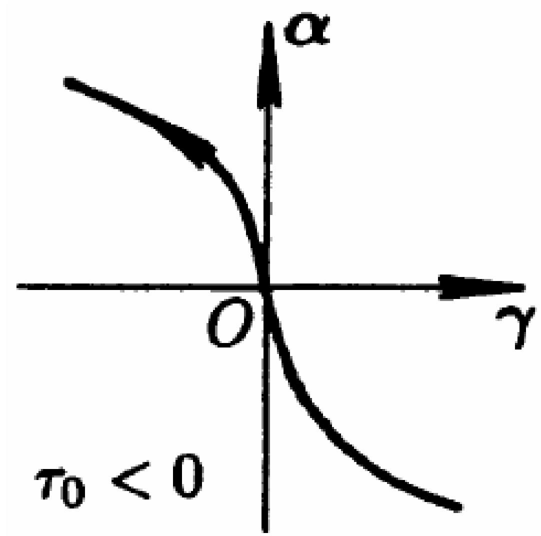


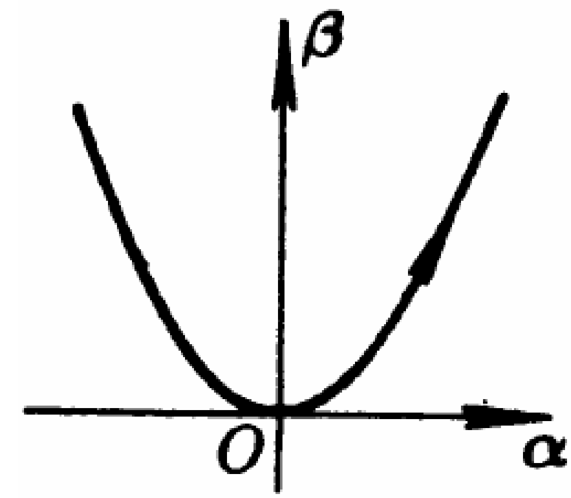
$\tau(s_0) > 0$时,

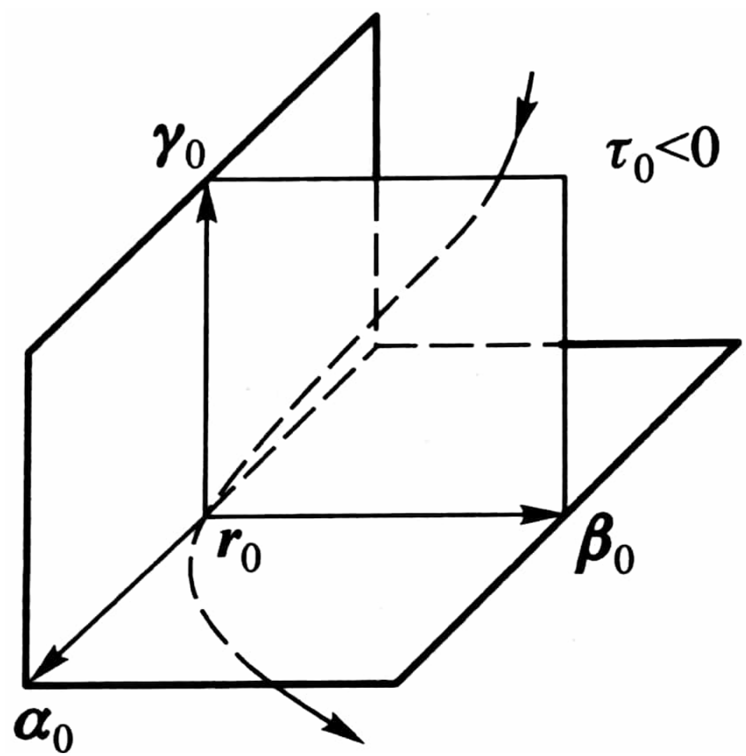
曲线从第六卦限经过

$\vec{r}(s_0)$进入第一卦限;









$\tau(s_0) < 0$时,

曲线从第二卦限经过

$\vec{r}(s_0)$进入第五卦限.