《微分几何》课程电子课件

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七、曲面在一点邻近的结构

通过法截线在一点P邻近的形状来分析

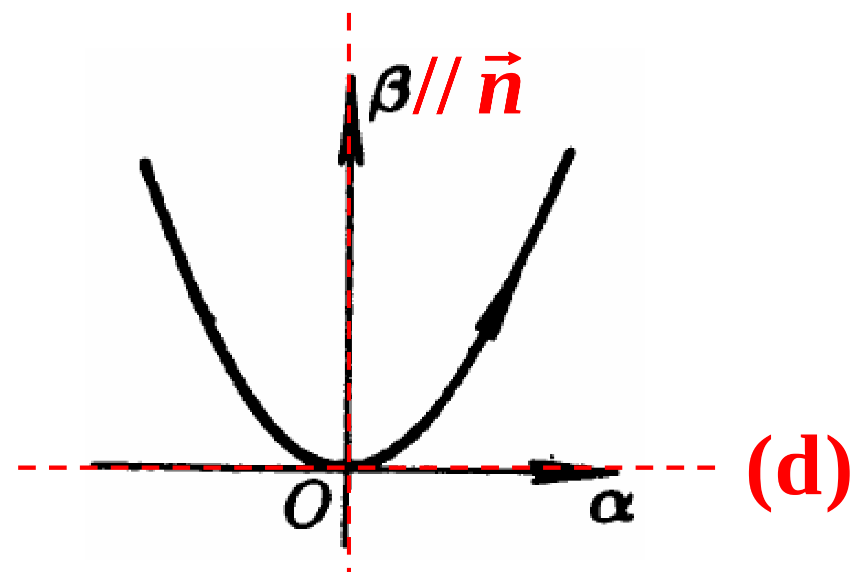
$$\vec{r}(s_0+s)-\vec{r}=$$

$$[s+\tfrac{1}{6}(\varepsilon_1-k^2)s^3]\vec{\alpha}+[\tfrac{1}{2}ks^2+\tfrac{1}{6}(\dot{k}+\varepsilon_2)s^3]\vec{\beta}+\tfrac{1}{6}(k\tau+\varepsilon_3)s^3\vec{\gamma}$$

若为法截线, 则近似方程为

$k\neq 0$时 $y=\frac{1}{2}kx^2$;

$k=0$时 $y=\frac{1}{6}\dot{k}x^3$.



考虑与曲面上一点P邻近的曲面形状.

由Euler公式,沿方向(d)的法曲率$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$.

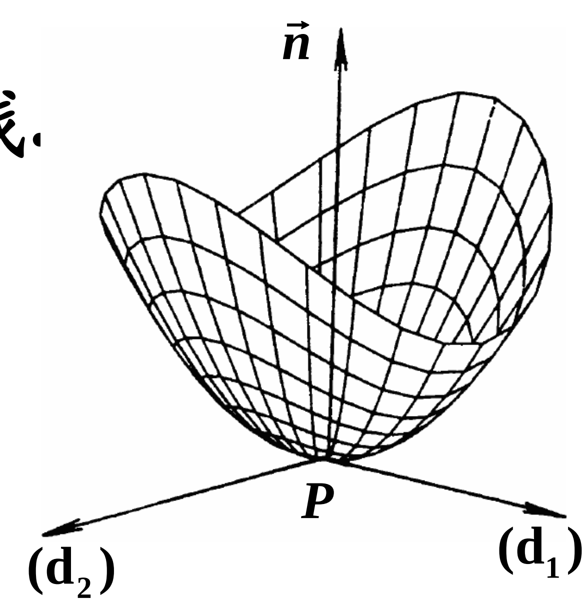
1. 当$K = k_1 k_2 > 0$时, $LN - M^2 = K(EG - F^2) > 0$,

P为椭圆点.

法截线的近似为$y = \dfrac{k_n}{2}x^2$,为一抛物线.

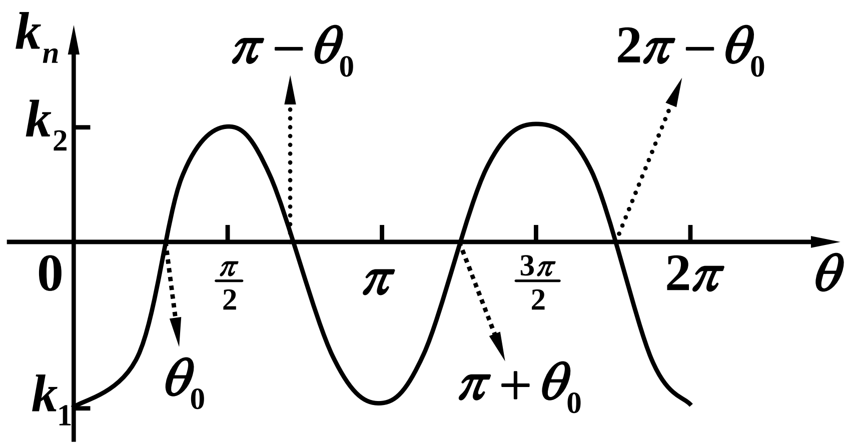
k_n的符号不变,抛物线开口方向不变.

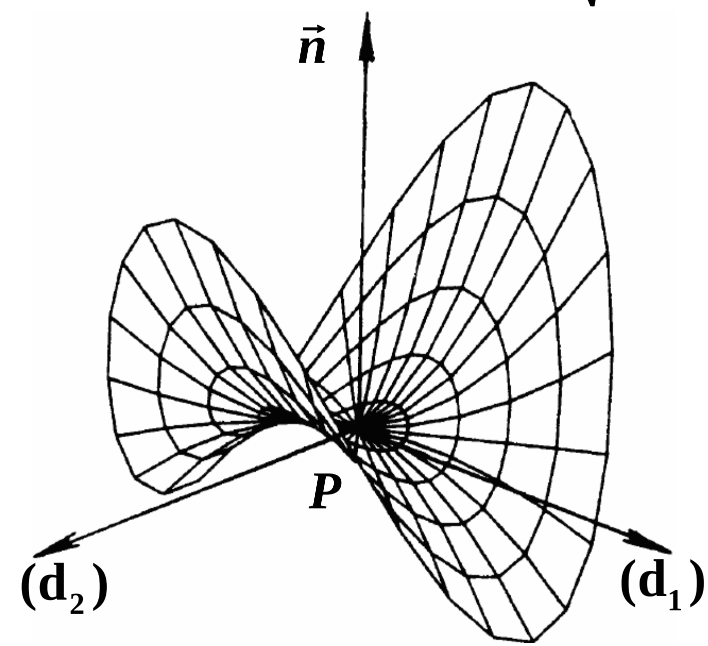
邻近曲面近似于椭圆抛物面.



2. 当$K=k_1k_2<0$时， $LN-M^2<0$， P为双曲点.

选取法向$\vec{n}$的方向使$k_1<0, k_2>0$, 令$\theta_0=\arctan\sqrt{-\frac{k_1}{k_2}}$.





θ	$[0,\theta_0)$	$(\theta_0,\pi-\theta_0)$	$(\pi-\theta_0,\pi+\theta_0)$	$(\pi+\theta_0,2\pi-\theta_0)$	$(2\pi-\theta_0,2\pi]$
法截线形状	开口向下的抛物线	开口向上的抛物线	开口向下的抛物线	开口向上的抛物线	开口向下的抛物线

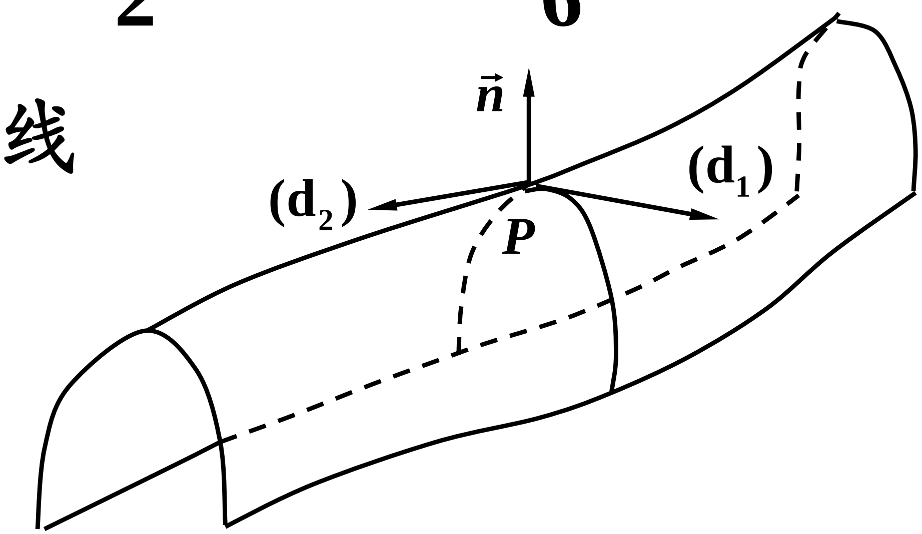
3. 当$K=k_1k_2=0$时，$LN-M^2=0$，P为抛物点或平点.

若为抛物点，选取法向$\vec{n}$的方向使$k_1<0, k_2=0$.

主方向上的法截线近似为$y=\frac{k_1}{2}x^2$和$y=\frac{\dot{k}_2}{6}x^3$，

分别为朝$\vec{n}$反向弯曲的抛物线

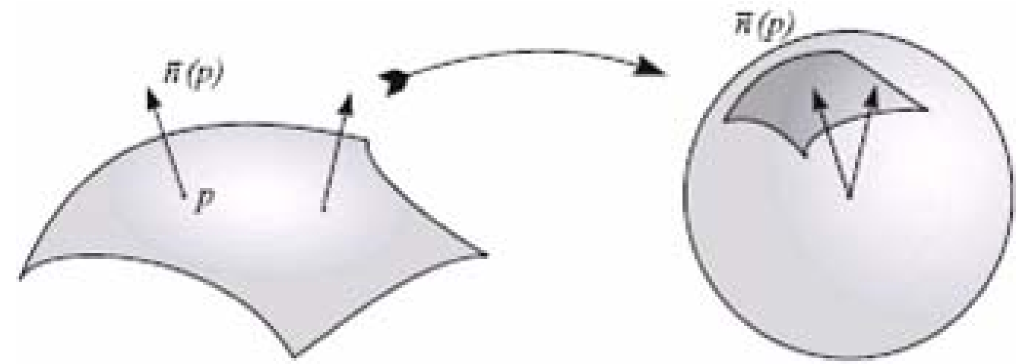
和立方抛物线.



若为平点，$k_1=k_2=0$，两条主法截线近似为

$y=\frac{\dot{k}_1}{6}x^3$和$y=\frac{\dot{k}_2}{6}x^3$，为两条立方抛物线.

八、Gauss曲率的几何意义



1. Gauss映射

$$\vec{r}(u,v) \xrightarrow{\text{Gauss映射}} \vec{n}(u,v)=\frac{\vec{r}_u(u,v)\times\vec{r}_v(u,v)}{\left|\vec{r}_u(u,v)\times\vec{r}_v(u,v)\right|}$$

称$\vec{n}(u,v)$为曲面$\vec{r}(u,v)$的球面表示.

称球面表示$\vec{n}(u,v)$的第一基本形式为原曲面$\vec{r}(u,v)$的第三基本形式. 记作

$\mathrm{III}=edu^2+2fdudv+gdv^2$, 其中 $e=\vec{n}_u^2$, $f=\vec{n}_u\vec{n}_v$, $g=\vec{n}_v^2$.

称 e, f, g 为曲面$\vec{r}(u,v)$的第三类基本量.

2. 曲面的第一、二、三类基本形式之间的联系

$$\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$$

证 等式中涉及的量 Ⅰ, Ⅱ, Ⅲ, K和 H都是表示长度和弯曲程度的量, 与曲面参数的选择无关, 因此不妨选取曲率线网为曲纹坐标网.

于是可设曲面方程为$\vec{r}(u,v)$, 且它的$F(u,v) \equiv M(u,v) \equiv 0.$

则曲面的第一、二基本形式为

$$\mathrm{I} = E\mathrm{d}u^2 + G\mathrm{d}v^2,\ \mathrm{II} = L\mathrm{d}u^2 + N\mathrm{d}v^2.$$

设k_1, k_2分别为u-曲线和v-曲线方向的主曲率,

(下面先把其他基本量都用E, G, k_1和k_2表示)

由主方向判别定理$\vec{n}_u = -k_1\vec{r}_u, \vec{n}_v = -k_2\vec{r}_v$.

$$L = -\vec{n}_u \cdot \vec{r}_u = k_1\vec{r}_u^{\,2} = k_1 E, \quad N = -\vec{n}_v \cdot \vec{r}_v = k_2\vec{r}_v^{\,2} = k_2 G,$$

$$\mathrm{II} = L\mathrm{d}u^2 + N\mathrm{d}v^2 = k_1 E\mathrm{d}u^2 + k_2 G\mathrm{d}v^2.$$

$$e = \vec{n}_u^{\,2} = (-k_1\vec{r}_u)^2 = k_1^2 E,$$

$$f = \vec{n}_u \cdot \vec{n}_v = (-k_1\vec{r}_u)(-k_2\vec{r}_v) = k_1 k_2 F = 0,$$

$$g = \vec{n}_v^{\,2} = (-k_2\vec{r}_v)^2 = k_2^2 G,$$

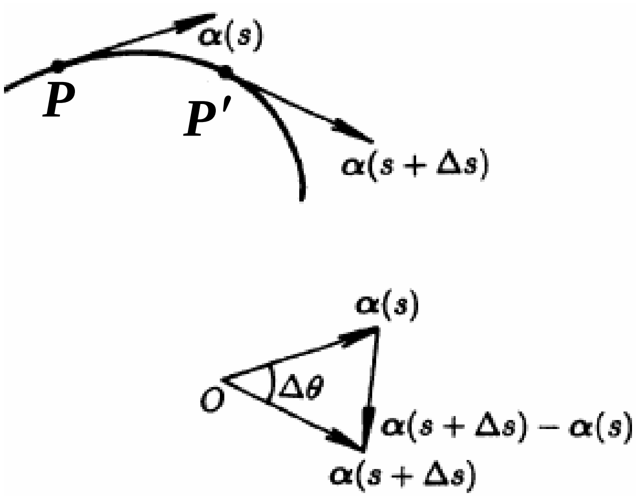
$$\mathrm{III} = e\mathrm{d}u^2 + 2f\mathrm{d}u\mathrm{d}v + g\mathrm{d}v^2 = k_1^2 E\mathrm{d}u^2 + k_2^2 G\mathrm{d}v^2.$$

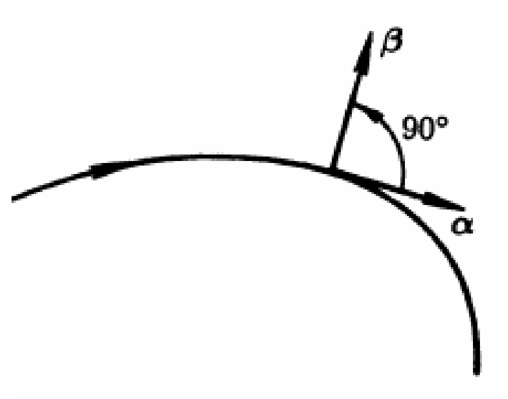
$$\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = k_1^2 E\mathrm{d}u^2 + k_2^2 G\mathrm{d}v^2$$

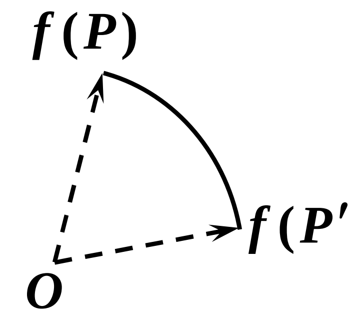
$$-(k_1 + k_2)(k_1 E\mathrm{d}u^2 + k_2 G\mathrm{d}v^2) + k_1 k_2 (E\mathrm{d}u^2 + G\mathrm{d}v^2) = 0.$$

3. Gauss曲率的几何意义 ——曲线曲率的推广

设 f 是Gauss映射, 则

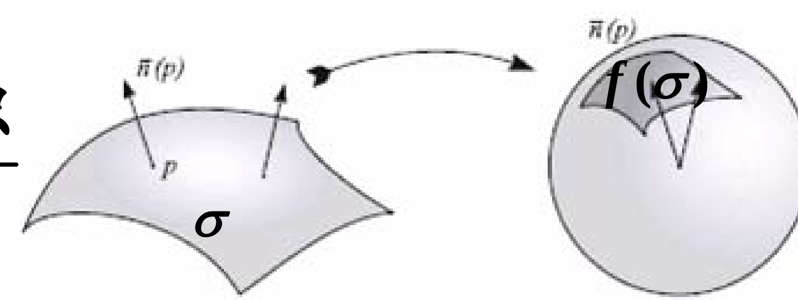






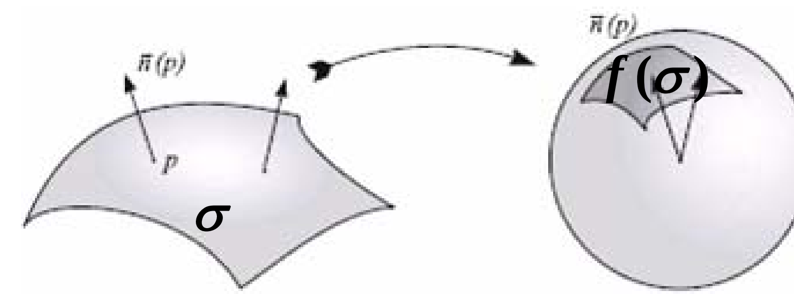
$$\text{曲线曲率} = \lim_{P'\to P} \frac{f(P)\text{与}f(P')\text{之间单位圆的弧长}}{P\text{与}P'\text{之间曲线的弧长}}$$

$$\left|\text{Gauss曲率}\right| = \left|K\right| = \lim_{\sigma\to P} \frac{f(\sigma)\text{的面积}}{\sigma\text{的面积}}$$



证 设σ在参数域中对应的区域是$D(\sigma)$，

点P的曲纹坐标为(u_0,v_0).



则$A(\sigma)=\iint\limits_{D(\sigma)}\left|\vec{r}_u\times\vec{r}_v\right|\mathrm{d}u\mathrm{d}v$,

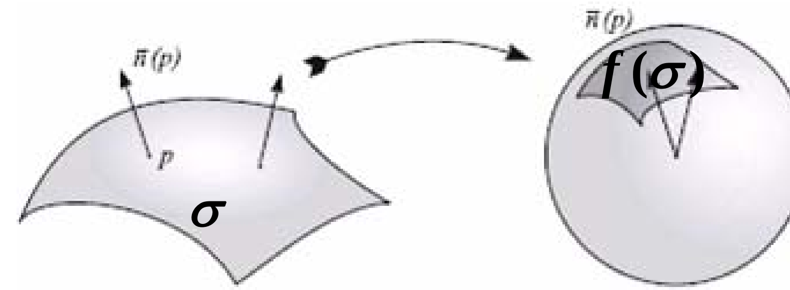
$$A(f(\sigma))=\iint\limits_{D(\sigma)}\left|\vec{n}_u\times\vec{n}_v\right|\mathrm{d}u\mathrm{d}v.$$

$\vec{n}_u\times\vec{n}_v\ //\ \vec{r}_u\times\vec{r}_v$，设 $\vec{n}_u\times\vec{n}_v=\lambda(u,v)\vec{r}_u\times\vec{r}_v$.

两边点乘$\vec{r}_u\times\vec{r}_v$得$(\vec{n}_u\times\vec{n}_v)\cdot(\vec{r}_u\times\vec{r}_v)=\lambda(u,v)(\vec{r}_u\times\vec{r}_v)^2$.

即 $\begin{vmatrix}\vec{n}_u\cdot\vec{r}_u & \vec{n}_u\cdot\vec{r}_v\\ \vec{n}_v\cdot\vec{r}_u & \vec{n}_v\cdot\vec{r}_v\end{vmatrix}=\lambda(u,v)\begin{vmatrix}\vec{r}_u\cdot\vec{r}_u & \vec{r}_u\cdot\vec{r}_v\\ \vec{r}_v\cdot\vec{r}_u & \vec{r}_v\cdot\vec{r}_v\end{vmatrix}$.

即 $\begin{vmatrix} -L & -M \\ -M & -N \end{vmatrix} = \lambda(u,v)\begin{vmatrix} E & F \\ F & G \end{vmatrix}$.



即 $\lambda(u,v) = \dfrac{LN - M^2}{EG - F^2} = K(u,v)$.

$$A(f(\sigma)) = \iint\limits_{D(\sigma)} |\lambda(u,v)| |\vec{r}_u \times \vec{r}_v| \mathrm{d}u\mathrm{d}v$$

$$= \iint\limits_{D(\sigma)} |K(u,v)| |\vec{r}_u \times \vec{r}_v| \mathrm{d}u\mathrm{d}v = |K(\xi,\eta)| \iint\limits_{D(\sigma)} |\vec{r}_u \times \vec{r}_v| \mathrm{d}u\mathrm{d}v$$

因此 $|K(\xi,\eta)| = \dfrac{A(f(\sigma))}{A(\sigma)}$，其中 $(\xi,\eta) \in D(\sigma)$.

令 $\sigma \to P$ 即得 $|K| = \lim\limits_{\sigma \to P} \dfrac{A(f(\sigma))}{A(\sigma)}$.