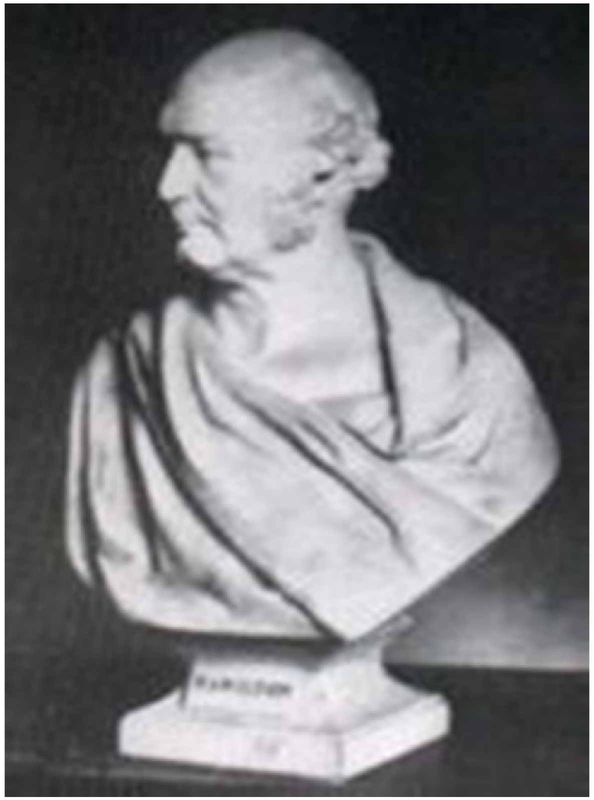
第九章

- I.　正则变换
- II.　哈密顿-雅可比方程



9.1 正则变换

正则变换的目的：通过构造**新的Hamilton函数**，该系统具有更简洁的正则形式和更多的循环坐标，即得到系统更多的首次积分，且保证正则方程的形式不变。

- 正则变换(Canonical transformation)
- 母函数的各种形式

正则　Canonical

1. 问题的提出

【思考】广义坐标间的坐标变换对哈密顿函数和正则方程的影响。

描述同一力学系统可以采用不同的广义坐标，如q_1, q_2, … , q_k和Q_1, Q_2, … , Q_k，二者之间存在着一定的变换关系 $Q_j = Q_j\left(q_1, q_2, \cdots, q_k, t\right) \quad (j = 1, 2, \cdots, k)$

上述变换是将一组旧广义坐标q_1, q_2, … , q_k所确定的位形空间中的一个点，变换到一组新广义坐标Q_1, Q_2, … , Q_k所确定的位形空间中的一个点。这种变换称为点变换。

点变换不影响Lagrange方程的结构。

在广义坐标间的坐标变换（点变换）下

$$q_\alpha \quad \Rightarrow \quad Q_\beta = f_\beta(q,t) \quad \text{满足} \quad \frac{\partial(Q_1 \cdots Q_s)}{\partial(q_1 \cdots q_s)} \neq 0$$

其逆变换　$q_\alpha = \Phi_\alpha(Q,t)$存在，也满足

$$\frac{\partial(q_1 \cdots q_s)}{\partial(Q_1 \cdots Q_s)} = \left[\frac{\partial(Q_1 \cdots Q_s)}{\partial(q_1 \cdots q_s)}\right]^{-1} \neq 0$$

Lagrange方程和Lagrange力学的理论体系不变：

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_\beta} - \frac{\partial \tilde{L}}{\partial Q_\beta} = 0 \qquad (\alpha, \beta = 1, 2, \cdots, s)$$

新旧拉格朗日函数相等：

$$L(q,\dot{q},t) = L\big(q_\alpha(Q,t), \dot{q}_\alpha(Q,\dot{Q},t), t\big) = \tilde{L}(Q,\dot{Q},t)$$

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$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha},\ H = \left[\sum_\alpha p_\alpha \dot{q}_\alpha - L\right]_{\dot{q}\to\dot{q}(p,q,t)},\quad \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha,\ \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

$$P_\beta = \frac{\partial \tilde{L}}{\partial \dot{Q}_\beta},\ \tilde{H} = \left[\sum_\beta P_\beta \dot{Q}_\beta - \tilde{L}\right]_{\dot{Q}\to\dot{Q}(P,Q,t)},\quad \frac{\partial \tilde{H}}{\partial P_\beta} = \dot{Q}_\beta,\ \frac{\partial \tilde{H}}{\partial Q_\beta} = -\dot{P}_\beta$$

坐标变换 $Q_\beta = f_\beta(q,t)$和新旧广义动量间的变换

$$P_\alpha = \frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} = \sum_\beta \frac{\partial L}{\partial \dot{q}_\beta}\frac{\partial \dot{q}_\beta}{\partial \dot{Q}_\alpha} = \sum_\beta p_\beta \frac{\partial q_\beta}{\partial Q_\alpha} = \sum_\beta p_\beta \left(\frac{\partial \Phi_\beta(Q,t)}{\partial Q_\alpha}\right)_{Q=f(q,t)} \equiv F_\alpha(p,q,t)$$

把正则方程变换为新的正则方程，新旧哈密顿量之间的关系为

$$\tilde{H} = \sum_\alpha P_\alpha \dot{Q}_\alpha - \tilde{L} = \sum_{\alpha,\beta} p_\beta \frac{\partial q_\beta}{\partial Q_\alpha}\dot{Q}_\alpha - \tilde{L} = \sum_\beta p_\beta \left(\dot{q}_\beta - \frac{\partial q_\beta}{\partial t}\right) - L = H - \sum_\beta p_\beta \frac{\partial q_\beta}{\partial t}$$

问题的提出

特例： 有心力问题，选用极坐标系，存在一个循环坐标，选用直角坐标系，则不存在循环坐标

普遍命题： 一个体系的循环坐标数目是与坐标系的选择有关系的，且对一个具体问题，总存在一种特殊的坐标选择，使得所有坐标都是循环的。

2. 正则变换的定义 (Canonical transformation)

正则变量：$q_1, q_2, \dots, q_k, p_1, p_2, \dots, p_k$

正则变量（共轭变量）：$Q_1, Q_2, \dots, Q_k, P_1, P_2, \dots, P_k$

变换关系：

$$\left.\begin{aligned} Q_j &= Q_j\left(q_1, q_2, \cdots, q_k, p_1, p_2, \cdots, p_k, t\right) \\ P_j &= P_j\left(q_1, q_2, \cdots, q_k, p_1, p_2, \cdots, p_k, t\right) \end{aligned}\right\} \qquad \text{(正则变换)}$$

对旧的正则变量，正则方程为

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \qquad \left(j = 1, 2, \cdots, k\right)$$

通过变换，旧的Hamilton函数$H=H(q_j, p_j, t)$变换成新的Hamilton函数$K=K(Q_j, P_j, t)$，且保持正则方程的形式不变，即

$$\left.\begin{aligned}\dot{Q}_j &= \frac{\partial K}{\partial P_j} \\ \dot{P}_j &= -\frac{\partial K}{\partial Q_j}\end{aligned}\right\} \qquad (j=1,2,\cdots,k)$$

变量$Q_1, Q_2, \ldots, Q_k, P_1, P_2, \ldots, P_k$仍称为正则变量或共轭变量。

3. 正则变换的条件

定理 设 P_α, Q_α, H 显含时间t，则正则变换的条件是

$$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right) + \left(K - H\right)dt = dU$$

式中dU为恰当微分，而 K 为用新变量 P_α, Q_α 表示的新哈密顿函数。

证明： 设 p_α, q_α 有变分 $\delta p_\alpha, \delta q_\alpha$ 因 $\delta t = 0$

$$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right) + \left(K - H\right)dt = dU \qquad (1)$$

变为

$$\sum_{\alpha=1}^{s}\left(p_\alpha \delta q_\alpha - P_\alpha \delta Q_\alpha\right) = \delta U \qquad (2)$$

又由(1)得

$$\sum_{\alpha=1}^{s}(p_\alpha \dot{q}_\alpha - P_\alpha \dot{Q}_\alpha) + (K - H) = \dot{U} \qquad (3)$$

对(2)和(3)分别取微商和变分，得

$$\delta\left(\sum_{\alpha=1}^{s} P_\alpha \dot{Q}_\alpha\right)-\frac{d}{dt}\left(\sum_{\alpha=1}^{s} P_\alpha \delta Q_\alpha\right)-\delta K \qquad \because \delta\dot{U}=\frac{d}{dt}\delta U$$

$$=\delta\left(\sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha\right)-\frac{d}{dt}\left(\sum_{\alpha=1}^{s} p_\alpha \delta q_\alpha\right)-\delta H \qquad (4)$$

$$\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha},\quad \dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha} \qquad \delta\dot{Q}=\frac{d}{dt}\delta Q,\ \delta\dot{q}_\alpha=\frac{d}{dt}\delta q_\alpha$$

$$\delta\left(\sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha\right)-\frac{d}{dt}\left(\sum_{\alpha=1}^{s} p_\alpha \delta q_\alpha\right)-\delta H=\sum_{\alpha=1}^{s}\left[\dot{q}_\alpha \delta p_\alpha-\dot{p}_\alpha \delta q_\alpha\right]-\delta H$$

$$=\sum_{\alpha=1}^{s}\left[\frac{\partial H}{\partial p_\alpha}\delta p_\alpha+\frac{\partial H}{\partial q_\alpha}\delta q_\alpha\right]-\delta H=\delta H-\delta H=0 \qquad (5)$$

因此 $$\delta K=\sum_{\alpha=1}^{s}\left[\frac{\partial K}{\partial P_\alpha}\delta P_\alpha+\frac{\partial K}{\partial Q_\alpha}\delta Q_\alpha\right]=\sum_{\alpha=1}^{s}\left[\dot{Q}_\alpha\delta P_\alpha-\dot{P}_\alpha\delta Q_\alpha\right]$$

即 $$\sum_{\alpha=1}^{s}\left[\left(\dot{Q}_\alpha-\frac{\partial K}{\partial P_\alpha}\right)\delta P_\alpha-\left(\dot{P}_\alpha+\frac{\partial K}{\partial Q_\alpha}\right)\delta Q_\alpha\right]=0 \quad (6)$$

得到 $$\dot{Q}_\alpha=\frac{\partial K}{\partial P_\alpha},\quad \dot{P}_\alpha=-\frac{\partial K}{\partial Q_\alpha} \quad (7)$$

即 $K=K(P,Q,t)$ 所满足的方程不改变正则方程形式

又 $$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right)+\left(K-H\right)dt = dU\left(q,Q,t\right)$$

而 $$dU\left(q,Q,t\right)=\sum_{\alpha=1}^{s}\left(\frac{\partial U}{\partial q_\alpha}dq_\alpha+\frac{\partial U}{\partial Q_\alpha}dQ_\alpha\right)+\frac{\partial U}{\partial t}dt$$

于是得 $$p_\alpha=\frac{\partial U}{\partial q_\alpha},\quad P_\alpha=-\frac{\partial U}{\partial Q_\alpha},\quad \left(K-H\right)=\frac{\partial U}{\partial t}$$

正则变换有赖于母函数 $U\left(q,Q,t\right)$ 的选取

若 P_α, Q_α, H 不显含时间t，则正则变换的条件简化为

$$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right)=dU$$

4. 母函数的各种形式

为了实现两组正则变量的变换，母函数 U 必须是包括两组变量的函数。由于 4s个两组正则变量和时间 t 通过 2s 个变换关系联系着，所以其中只有 2s+1 个变量是独立的。**母函数 U 在这 2s个变量中要求两组变量各占一半**，只含新变量或只含旧变量均不能使下式成立。

$$\left(\sum_{j=1}^{k} p_j \mathrm{d}q_j - H\left(q_j, p_j, t\right)\mathrm{d}t\right) - \left(\sum_{j=1}^{k} P_j \mathrm{d}Q_j - K\left(Q, P, t\right)\mathrm{d}t\right) = \mathrm{d}F$$

因此，母函数 F 所显含的变量在最简单的情况下有四种不同形式：

$$F_1(q, Q, t),\quad F_2(p, Q, t),\quad F_3(q, P, t),\quad F_4(p, P, t)$$

1) 母函数为$U_1(q, Q, t)$，该形式已讨论过，有关结果为

$$p_j = \frac{\partial U_1}{\partial q_j} \qquad P_j = -\frac{\partial U_1}{\partial Q_j} \qquad K = H + \frac{\partial U_1}{\partial t} \qquad (j = 1, 2, \cdots, k)$$

2) 母函数为$U_2(p, Q, t)$

应用勒让德变换，在$U_1(q, Q, t)$基础上，确定$U_2(p, Q, t)$

的变换关系 $U_1(q, Q, t) \Longrightarrow U_2(p, Q, t)$

变量以 p 代替 q；Q 保持不变，且有： $p_j = \partial U_1 / \partial q_j$，于是取

$$U_2(p, Q, t) = U_1(q, Q, t) - \sum_{j=1}^{k} q_j p_j$$

$$U_2(p,Q,t)=U_1(q,Q,t)-\sum_{j=1}^{k}q_j p_j$$

则有 $\dfrac{\partial U_2}{\partial p_j}=-q_j$ $\quad\dfrac{\partial U_2}{\partial Q_j}=\dfrac{\partial U_1}{\partial Q_j}$ 得到 $q_j=-\dfrac{\partial U_2}{\partial p_j}$ $\quad P_j=-\dfrac{\partial U_2}{\partial Q_j}$

（$\because \dfrac{\partial U_1}{\partial Q_j}=\dfrac{\partial U_2}{\partial Q_j}$，又由 $\dfrac{\partial U_1}{\partial Q_j}=-P_j$ 得到）

将式 $U_2(p,Q,t)=U_1(q,Q,t)-\sum_{j=1}^{k}q_j p_j$ 两边对 t 求导，可得

$$\frac{\partial U_2}{\partial t}=\frac{\partial U_1}{\partial t}$$

因此Hamilton函数的变换关系为

$$K=H+\frac{\partial U_2}{\partial t}$$

3) 母函数 $U_3(q, P, t)$

仍使用上述方法，此时变量以 P 代替 Q；q 保持不变，且有：$P_j = \partial U_1 / \partial Q_j$ ，于是取

$$U_3(q,P,t) = U_1(q,Q,t) + \sum_{j=1}^{K} P_j Q_j$$

且相应有如下关系成立 $\partial U_3 / \partial q_j = \partial U_1 / \partial q_j$

$$Q_j = \partial U_3 / \partial P_j$$

由此得到变换关系为

$$\left.\begin{aligned} p_j &= \frac{\partial U_3}{\partial q_j} \\ Q_j &= \frac{\partial U_3}{\partial P_j} \\ K &= H + \frac{\partial U_3}{\partial t} \end{aligned}\right\} \qquad (j = 1, 2, \cdots, k)$$

4) 母函数 $U_4(p, P, t)$

以 $U_3(q, P, t)$为旧变量的函数，此时变量以 p 代替 q；

P 保持不变，取

$$U_4(p, P, t) = U_3(q, P, t) - \sum_{j=1}^{k} q_j p_j$$

同理得到变换关系为

$$\left.\begin{aligned} q_j &= -\frac{\partial U_4}{\partial p_j} \\ Q_j &= \frac{\partial U_4}{\partial P_j} \\ K &= H + \frac{\partial U_4}{\partial t} \end{aligned}\right\} \qquad (j = 1, 2, \cdots, k)$$

例1　取母函数为 $U(q,Q,t)=\sum_{j=1}^{k} q_j Q_j$，试求由母函数生成的正则变换。

解：母函数为 $U=U(q, Q)$，属 q, Q 型，第一种母函数形式

根据

$$p_j=\frac{\partial U}{\partial q_j} \qquad P_j=-\frac{\partial U}{\partial Q_j} \qquad K=H+\frac{\partial U}{\partial t} \qquad (j=1,2,\cdots,k)$$

则

$$p_j=\frac{\partial U}{\partial q_j}=Q_j \qquad P_j=-\frac{\partial U}{\partial Q_j}=-q_j$$

则 q、p 与 Q、P 之间的关系式为

$$q_j=-P_j \qquad p_j=Q_j$$

例2 给定正则变换的母函数 $U(q,Q)=\frac{1}{2}q\sqrt{2Q-q^2}+Q\arcsin\frac{q}{\sqrt{2Q}}$

试求由母函数生成的正则变换。

解：母函数为U=U(q, Q)，属 q, Q 型，第一种母函数形式

根据 $p_j=\frac{\partial U}{\partial q_j}$ $\quad P_j=-\frac{\partial U}{\partial Q_j}$ $\quad K=H+\frac{\partial U}{\partial t}$ $\quad (j=1,2,\cdots,k)$

则

$$p=\frac{\partial U}{\partial q}$$

$$=\frac{1}{2}\sqrt{2Q-q^2}+\frac{1}{2}q\frac{1}{2}\frac{-2q}{\sqrt{2Q-q^2}}+Q\frac{\frac{1}{\sqrt{2Q}}}{\sqrt{1-q^2/2Q}}$$

$$=\frac{1}{2}\sqrt{2Q-q^2}-\frac{q^2}{2\sqrt{2Q-q^2}}+\frac{Q}{\sqrt{2Q-q^2}}$$

$$=\sqrt{2Q-q^2}$$

$$P=-\frac{\partial U}{\partial Q}$$

$$=-\frac{1}{2}q(2Q-q^2)^{-1/2}\cdot 2-\arcsin\frac{q}{\sqrt{2Q}}-\frac{-\frac{1}{2}\frac{q}{\sqrt{2Q^{3/2}}}}{\sqrt{1-q^2/2Q}}$$

$$=-\frac{1}{2}q(2Q-q^2)^{-1/2}-\arcsin\frac{q}{\sqrt{2Q}}+\frac{1}{2}q(2Q-q^2)^{-1/2}$$

$$=-\arcsin\frac{q}{\sqrt{2Q}}$$

$$p=\sqrt{2Q-q^2} \qquad P=-\arcsin\frac{q}{\sqrt{2Q}} \qquad (a)$$

$\because \quad \sin\theta=\tan\theta\cos\theta$ 　而　 $\sin\theta=-q/\sqrt{2Q}$

$$\therefore \quad \cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-(\frac{q}{\sqrt{2Q}})^2}=\sqrt{\frac{2Q-q^2}{2Q}}$$

则 $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{-\frac{q}{\sqrt{2Q}}}{\sqrt{\frac{2Q-q^2}{2Q}}}=-\dfrac{q}{\sqrt{2Q-q^2}}$ 　即　 $P=-\arctan\dfrac{q}{\sqrt{2Q-q^2}}$ 　(b)

由式(a)可解出

$$Q=\frac{1}{2}(p^2+q^2) \qquad (c)$$

将式(c)代入式(b)，得

$$P=-\arctan\frac{q}{p}$$

例3 已知 $K=H$，$[U=U(q,Q)]$，证明如下两组变换均为正则变换，并求相应的母函数

(1) $q=\sqrt{\dfrac{2P}{mk}}\sin Q$ $p=\sqrt{2mkP}\cos Q$ **(2)** $Q=\sqrt{2q}\cos p$ $P=\sqrt{2q}\sin p$

解：是否为正则变换的充分条件是要依据下式构造母函数 U

$$\left(\sum_{j=1}^{k} p_j \mathrm{d}q_j - H\left(q_j, p_j, t\right)\mathrm{d}t\right) - \left(\sum_{j=1}^{k} P_j \mathrm{d}Q_j - K\left(Q, P, t\right)\mathrm{d}t\right) = \mathrm{d}U$$

由已知条件 $K=H$ 故：$\sum_{j=1}^{k}(p_j \mathrm{d}q_j - P_j \mathrm{d}Q_j) = \mathrm{d}U$

构造 $U=U(q,Q)$ 是否存在。

（1）由于 $q=\sqrt{\frac{2P}{mk}}\sin Q \qquad p=\sqrt{2mkP}\cos Q$

因为 $\frac{p}{q}=\frac{\sqrt{2mkP}\cos Q}{\sqrt{\frac{2P}{mk}}\sin Q}=\sqrt{mk}\sqrt{mk}\cot Q$ 所以 $p=mkq\cot Q$

又因 $q^2=\frac{2P}{mk}\sin^2 Q$ 所以 $P=\frac{1}{2}mkq^2\frac{1}{\sin^2 Q}$

代入判别式 $p\mathrm{d}q-P\mathrm{d}Q=\mathrm{d}U$ 则 $mkq\cot Q\mathrm{d}q-\frac{1}{2}mkq^2\frac{1}{\sin^2 Q}\mathrm{d}Q=\mathrm{d}U_1$

即 $\mathrm{d}(\frac{1}{2}mkq^2\cot Q)=\mathrm{d}U_1$

所以 $U=\frac{1}{2}mkq^2\cot Q$ 故为正则变换

（2）由于 $Q=\sqrt{2q}\cos p \qquad P=\sqrt{2q}\sin p$ 　　则

$$\mathrm{d}Q=\mathrm{d}(\sqrt{2q}\cos p)=-\sqrt{2q}\sin p\mathrm{d}p+\frac{\sqrt{2}}{2\sqrt{q}}\cos p\mathrm{d}q$$

代入判别式

$$p\mathrm{d}q-P\mathrm{d}Q=p\mathrm{d}q-\sqrt{2q}\sin p(-\sqrt{2q}\sin p\mathrm{d}p+\frac{\sqrt{2}}{2\sqrt{q}}\cos p\mathrm{d}q)=(p-\sin p\cos p)\mathrm{d}q+2q\sin^2 p\mathrm{d}p$$

又由于 $\frac{\partial}{\partial p}(p-\sin p\cos p)=1-\cos^2 p+\sin^2 p \qquad \frac{\partial}{\partial q}(2q\sin^2 p)=2\sin^2 p$

得到 $$(p-\sin p\cos p)\mathrm{d}q+2q\sin^2 p\mathrm{d}p=\mathrm{d}U=\frac{\partial U}{\partial q}\mathrm{d}q+\frac{\partial U}{\partial p}\mathrm{d}p$$

则 $\frac{\partial U}{\partial q}=p-\sin p\cos p \qquad \frac{\partial U}{\partial p}=2q\sin^2 p$

由 $\dfrac{\partial U}{\partial q} = p - \sin p \cos p$

可得 $U = \int (p - \sin p \cos p) \mathrm{d}q + f(p) = pq - q \sin p \cos p + f(p)$

将U代入 $\partial U / \partial p = 2q \sin^2 p$ 得 $q - q\cos^2 p + q \sin^2 p + \dfrac{\partial f}{\partial p} = 2q \sin^2 p$

所以 $\partial f / \partial p = 0$ ，因此f=常数。

故母函数 U为 $U = q(p - \sin p \cos p)$

由条件 $Q = \sqrt{2q} \cos p$ 得 $p = \arccos \dfrac{Q}{\sqrt{2q}}$

所以 $U_1(q, Q) = q(p - \sin p \cos p) = q \arccos \dfrac{Q}{\sqrt{2q}} - \dfrac{Q}{2}\sqrt{2q - Q^2}$

例4 应用正则变换求解单自由度质点的线性谐振动

解：质点的质量为**m**，单自由度，取**q**为广义坐标，动能和势能为 $T = m\dot{q}^2/2$ $\quad V = kq^2/2$

则 $p = \partial T/\partial \dot{q} = m\dot{q}$ $\quad \dot{q} = p/m$

系统为保守系统，故 $H = T + V = mp^2/2 + kq^2/2$

取母函数 $$F = F_1(q,Q,t) = \frac{1}{2}\sqrt{mk}\,q^2 \cot\sqrt{\frac{k}{m}}Q$$

利用变换关系有

$$p = \frac{\partial F}{\partial q} = \sqrt{mk}\,q\cot\sqrt{\frac{k}{m}}Q \qquad P = -\frac{\partial F}{\partial Q} = \frac{1}{2}\sqrt{mk}\,q^2\csc^2\sqrt{\frac{k}{m}}Q$$

联立上式，可解得 $q=\sqrt{\frac{2P}{k}}\sin\sqrt{\frac{k}{m}}Q \qquad p=\sqrt{2mP}\cos\sqrt{\frac{k}{m}}Q$

因母函数不显含时间 ***t***，因此有 ***H=H****

将***q*** ,***p*** 代入 ***H*** 函数

$$H=\frac{1}{2m}(\sqrt{2mP}\cos\sqrt{\frac{k}{m}}Q)^2+\frac{1}{2}k(\sqrt{2mP}\cos\sqrt{\frac{k}{m}}Q)^2=\frac{1}{2m}2mP\cos^2\sqrt{\frac{k}{m}}Q+\frac{1}{2}k\frac{2P}{k}\sin^2\sqrt{\frac{k}{m}}Q=P$$

则 ***H*=P***，由此可见，经过变换后的**Hamilton** 函数更简洁，且存在循环坐标 ***Q*** 。对应新变量的正则方程为

$$\dot{Q}=\partial H^*/\partial P=1 \qquad \dot{P}=\partial H^*/\partial Q=0$$

积分上式，得 $Q=t+c_1 \qquad P=c_2$

则$H^*=P=E$即为系统的总机械能，系统的振动规律为

$$q=\sqrt{\frac{2P}{k}}\sin\sqrt{\frac{k}{m}}Q=\sqrt{\frac{2E}{k}}\sin\sqrt{\frac{k}{m}}(t+c_1)$$

由上述求解过程可以看出，正则变换后的广义坐标Q和广义动量P分别为时间t和总机械能，已不再具有原来的意义了。

作业： 9.2 9.5 9.7

正则变换的定义 (Canonical transformation)

系统有k个自由度

$H(q,p,t)$

$(q_1, q_2, \dots, q_k)$

$(p_1, p_2, \dots, p_k)$

$$\left.\begin{aligned}\dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j}\end{aligned}\right\} \quad (j=1,2,\cdots,k)$$

变换前

$K(Q,P,t)$

$(Q_1, Q_2, \dots, Q_k)$

$(P_1, P_2, \dots, P_k)$

$$\left.\begin{aligned}\dot{Q}_j &= \frac{\partial K}{\partial P_j} \\ \dot{P}_j &= -\frac{\partial K}{\partial Q_j}\end{aligned}\right\} \quad (j=1,2,\cdots,k)$$

变换后

$$\left.\begin{aligned}Q_j &= Q_j\left(q_1,q_2,\cdots,q_k,p_1,p_2,\cdots,p_k,t\right) \\ P_j &= P_j\left(q_1,q_2,\cdots,q_k,p_1,p_2,\cdots,p_k,t\right)\end{aligned}\right\}$$

<u>(正则变换)</u>

正则变换的条件

定理 设 P_α, Q_α, H 显含时间t，则正则变换的条件是

$$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right) + \left(K - H\right)dt = dU$$

式中dU为恰当微分，而 K 为用新变量 P_α, Q_α 表示的新哈密顿函数。

若 P_α, Q_α, H 不显含时间t，则正则变换的条件简化为

$$\sum_{\alpha=1}^{s}\left(p_\alpha dq_\alpha - P_\alpha dQ_\alpha\right) = dU$$

母函数的各种形式

母函数 F 所显含的变量在最简单的情况下有四种不同形式：

$F_1(q, Q, t)$,

$F_2(p, Q, t)$,

$F_3(q, P, t)$,

$F_4(p, P, t)$ 。

证明：变换 $q=-\ln\frac{Q}{\sin P}$，$p=Q ctg P$ 为一正则变换。

$$
\begin{aligned}
\text{证}: pdq-PdQ &= QctgP(-\frac{1}{Q}dQ+\frac{\cos P}{\sin P}dP)-PdQ \\
&= -ctgPdQ+Qctg^2PdP-PdQ \\
&= -ctgPdQ+QdP+Qctg^2PdP-PdQ-QdP \\
&= -ctgPdQ+Q\csc^2 PdP-d(PQ) \\
&= -d(QctgP)-d(PQ) \\
&= d(-PQ-QctgP)=dU
\end{aligned}
$$

母函数 U 不是 t 的显函数，故为正则变换。

例：用正则变换法求平面谐振子的运动.

解：$H=(p_x^2+p_y^2)/2m+m(\omega_1^2x^2+\omega_2^2y^2)/2$

选 $U_1(q,Q,t)=m(\omega_1x^2ctgQ_1/2+\omega_2y^2ctgQ_2)$

$$\begin{cases} p_x=-\dfrac{\partial U_1}{\partial x}=m\omega_1xctgQ_1 \\ P_1=\dfrac{\partial U_1}{\partial Q_1}=m\omega_1x^2\csc^2Q_1/2 \end{cases}$$

$$\begin{cases} p_y=-\dfrac{\partial U_1}{\partial y}=m\omega_2yctgQ_2 \\ P_2=\dfrac{\partial U_1}{\partial Q_2}=m\omega_2y^2\csc^2Q_2/2 \end{cases}$$

上节要点

$$H=(p_x^2+p_y^2)/2m+m(\omega_1^2x^2+\omega_2^2y^2)/2$$

选 $U_1(q,Q,t)=m(\omega_1 x^2 ctgQ_1/2+\omega_2 y^2 ctgQ_2)$

$$\begin{cases} p_x = m\omega_1 x ctgQ_1 \\ P_1 = m\omega_1 x^2 \csc^2 Q_1/2 \end{cases}, \quad \begin{cases} p_y = m\omega_2 y ctgQ_2 \\ P_2 = m\omega_2 y^2 \csc^2 Q_2/2 \end{cases},$$

$$\begin{aligned} K &= H+\frac{\partial U_1}{\partial t}=H \\ &= (m^2\omega_1^2x^2ctg^2Q_1+m^2\omega_2^2y^2ctg^2Q_2)/2m \\ &\quad +m(\omega_1^2x^2+\omega_2^2y^2)/2 \\ &= m\omega_1^2x^2(1+ctg^2Q_1)/2+m\omega_2^2y^2(1+ctg^2Q_2)/2 \\ &= m\omega_1^2x^2\csc^2Q_1/2+m\omega_2^2y^2\csc^2Q_2/2 \\ &= \omega_1P_1+\omega_2P_2 \end{aligned}$$

$$P_1 = \frac{1}{2} m\omega_1 x^2 \csc^2 Q_1, \quad P_2 = \frac{1}{2} m\omega_2 y^2 \csc^2 Q_2,$$

$$K = \omega_1 P_1 + \omega_2 P_2$$

新变量 Q_j, P_j 表示谐振子的正则方程为:

$$\begin{cases} \dot{P}_1 = -\dfrac{\partial K}{\partial Q_1} = 0 \\ \dot{Q}_1 = \dfrac{\partial K}{\partial P_1} = \omega_1 \end{cases}, \quad \Rightarrow \begin{cases} P_1 = C_1 \\ Q_1 = \omega_1 t + \delta_1 \end{cases};$$

$$\begin{cases} \dot{P}_2 = 0 \\ \dot{Q}_2 = \omega_2 \end{cases}, \quad \Rightarrow \begin{cases} P_2 = C_2 \\ Q_2 = \omega_2 t + \delta_2 \end{cases}。$$

$$P_1 = \frac{1}{2} m\omega_1 x^2 \csc^2 Q_1, \quad P_2 = \frac{1}{2} m\omega_2 y^2 \csc^2 Q_2$$

$$\begin{cases} P_1 = C_1 \\ Q_1 = \omega_1 t + \delta_1 \end{cases}; \quad \begin{cases} P_2 = C_2 \\ Q_2 = \omega_2 t + \delta_2 \end{cases}。$$

$$\because \begin{cases} P_1 = \frac{1}{2} m\omega_1 x^2 \csc^2 Q_1 = \frac{1}{2} m\omega_1 x^2 \csc^2 (\omega_1 t + \delta_1) = C_1 \\ P_2 = \frac{1}{2} m\omega_2 y^2 \csc^2 Q_2 = \frac{1}{2} m\omega_2 y^2 \csc^2 (\omega_2 t + \delta_2) = C_2 \end{cases}$$

$$\therefore \begin{cases} x = \sqrt{\frac{2C_1}{m\omega_1}} \sin(\omega_1 t + \delta_1) \\ y = \sqrt{\frac{2C_2}{m\omega_2}} \sin(\omega_2 t + \delta_2) \end{cases}$$

问题： 选择怎样的母函数，使变换后的Hamilton函数为零，这是Hamilton- Jacobi 方程要解决的问题。

1. Hamilton-Jacobi方程的建立

对于一个具有 k 个自由度的完整系统，Hamilton正则方程为

$$\dot{q}_j = \partial H / \partial p_j \qquad \dot{p}_j = -\partial H / \partial q_j \qquad (j = 1, 2, \cdots, k)$$

经过正则变换后，使 $H(q,p) \to K(Q,P)$，相应的正则方程为

$$\dot{Q}_j = \partial K / \partial P_j \qquad \dot{P}_j = -\partial K / \partial Q_j \qquad (j = 1, 2, \cdots, k)$$

如果 $K = 0$，则上式可写成 $\dot{Q}_j = 0 \qquad \dot{P}_j = 0 \qquad (j = 1, 2, \cdots, k)$

直接积分可得 $Q_j = \alpha_j \qquad P_j = \beta_j \qquad (j = 1, 2, \cdots, k)$

式中 α_j、β_j 为积分常数。

为了达到上述目的，关键在于母函数的选择。

9.2 哈密顿-雅可比方程

根据新、旧Hamilton函数K、H的关系$K= H+\partial U/\partial t$，母函数必须满足

$$H + \partial U/\partial t = 0$$

母函数 U 的形式可以有四种：

$$U_1(q, Q, t),\quad U_2(p, Q, t),\quad U_3(q, P, t),\quad U_4(p, P, t)$$

这里取$U=U_3(q, P, t)$ 为例，并用$S(q, P, t)$表示，即

$$H(q, p, t) + \frac{\partial}{\partial t} S(q, P, t) = 0$$

将$P_j=\beta_j$代入$S(q, P, t)$ 中，S则可表示成变量q_j、常数β_j和时间t的函数，

即 $$S = S(q_1, q_2, \cdots, q_k, P_1, P_2, \cdots, P_k, t) = S(q_1, q_2, \cdots, q_k, \beta_1, \beta_2, \cdots, \beta_k, t)$$

常数 β_j 可由初始条件决定。

于是，对应于母函数 $S = U_3(q, P, t)$ 形式的变换关系式可写为

$$p_j = \frac{\partial}{\partial q_j} S(q_j, \beta_j, t) \qquad Q_j = \alpha_j = \frac{\partial}{\partial \beta_j} S(q_j, \beta_j, t) \qquad (j = 1, 2, \cdots, k)$$

将 $p_j = \partial S / \partial q_j$ 代入 $H(q, p, t) + \frac{\partial}{\partial t} S(q, P, t) = 0$

就可得到 H 函数

$$\boxed{H\left(q_1, q_2, \cdots, q_k, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \cdots, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial}{\partial t} S(q_1, q_2, \cdots, q_k, \beta_1, \beta_2, \cdots, \beta_k, t) = 0}$$

Hamilton-Jacobi方程。

9.2 哈密顿-雅可比方程

$$H\left(q_1,q_2,\cdots,q_k,\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},\cdots,\frac{\partial S}{\partial q_k},t\right)+\frac{\partial}{\partial t}S\left(q_1,q_2,\cdots,q_k,\beta_1,\beta_2,\cdots,\beta_k,t\right)=0$$

该方程是关于k个变量$q_1, q_2,\ldots, q_k$和时间 t 的一阶偏微分方程，其解

$$S=S\left(q_1,q_2,\cdots,q_k,\beta_1,\beta_2,\cdots,\beta_k,t\right)$$

称为**Hamilton-Jacobi方程的全积分**。

当S 被解出后，将 S 代入

$$p_j=\frac{\partial}{\partial q_j}S\left(q_j,\beta_j,t\right) \qquad Q_j=\alpha_j=\frac{\partial}{\partial \beta_j}S\left(q_j,\beta_j,t\right) \qquad \left(j=1,2,\cdots,k\right)$$

就可得到正则方程的解

$$\left.\begin{aligned} q_j&=q_j\left(\alpha_1,\alpha_2,\cdots,\alpha_k,\beta_1,\beta_2,\cdots,\beta_k,t\right)\\ p_j&=p_j\left(\alpha_1,\alpha_2,\cdots,\alpha_k,\beta_1,\beta_2,\cdots,\beta_k,t\right)\end{aligned}\right\} \qquad \left(j=1,2,\cdots,k\right)$$

其中包含了 $2k$ 个由初始条件决定的积分常数。

2. 哈密顿主函数的意义

由于 $K=0$ ， $H(q,p,t)+\dfrac{\partial}{\partial t}S(q,P,t)=0$

$$p_j=\frac{\partial S}{\partial q_j}$$

$$\because \quad \frac{dS}{dt}=\sum\frac{\partial S}{\partial q_j}\dot{q}_j+\frac{\partial S}{\partial t}=\sum p_j\dot{q}_j-H=L$$

$\therefore \quad S=\int L\,dt$ 即是积分限不确定的哈密顿作用量,

又称为哈密顿作用函数。

$$S=\int_{t_1}^{t}L(q,\dot{q},t)\,dt=F_2(q,\eta,t)+C$$

9.2 哈密顿-雅可比方程

哈密顿主函数：沿真实运动轨迹的作用量

证明：$\dfrac{dS}{dt}=\dfrac{\partial S}{\partial t}+\sum_\alpha \dfrac{\partial S}{\partial q_\alpha}\dot{q}_\alpha$

H-J 方程 $\dfrac{\partial S}{\partial t}=-H$

另外正则变换关系 $p_\alpha=\dfrac{\partial S}{\partial q_\alpha}$

$$\Rightarrow \frac{dS}{dt}=\sum_\alpha p_\alpha \dot{q}_\alpha-H=L$$

$$S=\int L\,dt \qquad \Rightarrow S=\int_{t_0}^{t} L\,dt+S(t_0)$$

故 S 表达式与作用量相同（仅差常数）. 需要注意到是推导 H-J 方程时已经用到了正则方程，即只对真实运动成立，所以作用量的被积函数 $L(q_\alpha,\dot{q}_\alpha,t)$ 中的广义坐标和广义动量均沿真实轨迹变化. 即母函数 S 是沿真实运动轨迹的作用量.

9.2 哈密顿-雅可比方程

(1) 哈密顿主函数 S 是广义坐标和时间的函数，因此可视为场函数.

$S(q,\beta,t)=\text{Const}$ 是 S 的等值面方程，随着时间的变化，这个等值面在空间传播.

(2) 广义动量和广义能量是由 S 派生

$$p_\alpha=\frac{\partial S}{\partial q_\alpha} \qquad \frac{\partial S}{\partial t}=-H$$

$$(p_1,p_2,\cdots,p_s)\equiv \boldsymbol{p}=\nabla S\equiv\left(\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},\cdots,\frac{\partial S}{\partial q_s}\right)$$

系统在位形空间（位形点类比于“粒子”）的运动方向与 S 的等值面垂直.

因此, S的等值面可以类比于光的波前面，系统的运动可以类比为光的传播.（光的波动理论）

例1 应用Hamilton-Jacobi方法，求解单自由度质点的线性谐振动。

设系统的Hamilton函数为

$$H=\frac{p^2}{2m}+\frac{kq^2}{2}$$

解：根据

$$H\left(q_1,q_2,\cdots,q_k,\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},\cdots,\frac{\partial S}{\partial q_k},t\right)+\frac{\partial}{\partial t}S\left(q_1,q_2,\cdots,q_k,\beta_1,\beta_2,\cdots,\beta_k,t\right)=0$$

得到Hamilton-Jacobi方程

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{kq^2}{2}+\frac{\partial S}{\partial t}=0$$

设函数 **S** 的形式为 $S=S\left(q,\beta,t\right)=W\left(q\right)-\beta t$

其中，β是变换后的动量，也是积分常数。

将 S 代入Hamilton-Jacobi方程得： $\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2+\frac{kq^2}{2}-\beta=0$

则 $\frac{\partial W}{\partial q}=\sqrt{\left(\beta-\frac{kq^2}{2}\right)2m}=\sqrt{mk}\sqrt{\frac{2\beta}{k}-q^2}$　　得 $W=\sqrt{mk}\int\sqrt{\frac{2\beta}{k}-q^2}\,\mathrm{d}q$

从而 $S=\sqrt{mk}\int\sqrt{\frac{2\beta}{k}-q^2}\,\mathrm{d}q-\beta t$　　又由 $\alpha=\frac{\partial S}{\partial\beta}$

得

$$\alpha=\sqrt{mk}\int\frac{\partial}{\partial\beta}\left(\sqrt{\frac{2\beta}{k}-q^2}\right)\mathrm{d}q-\frac{\partial}{\partial\beta}(\beta t)=\sqrt{mk}\int\frac{\frac{1}{2}\frac{2}{k}}{\sqrt{\frac{2\beta}{k}-q^2}}\mathrm{d}q-t$$

$$=\sqrt{\frac{m}{k}}\int\frac{\mathrm{d}q}{\sqrt{\frac{2\beta}{k}-q^2}}-t=-\sqrt{\frac{m}{k}}\arccos q\sqrt{\frac{2\beta}{k}}-t$$

$$=-\sqrt{\frac{m}{k}}\arccos q\sqrt{\frac{k}{2\beta}}-t$$

即 $$t+\alpha=\sqrt{\frac{m}{k}}\arccos q\sqrt{\frac{k}{2\beta}}$$

设 $\omega=\sqrt{k/m}$ 则 $q=\sqrt{\frac{2\beta}{k}}\cos\omega(t+\alpha)$

另一正则变量为:

$$p=\frac{\partial S}{\partial q}=\sqrt{mk}\sqrt{\frac{2\beta}{k}-q^2}=m\omega\sqrt{\frac{2\beta}{k}}\sin\omega(t+\alpha)$$

4. 用分离变量法求哈密顿特征函数

$$T=\frac{1}{2}\left[A_1(q_1)\left(\frac{\partial W}{\partial q_1}\right)^2+\cdots+A_s(q_s)\left(\frac{\partial W}{\partial q_s}\right)^2\right]$$

$$V=V_1(q_1)+\cdots+V_s(q_s)$$

可设W的分离变量形式

$$W=W_1(q_1)+\cdots+W_s(q_s) \qquad \frac{\partial W}{\partial q_\alpha}=\frac{\partial W_\alpha}{\partial q_\alpha}=\frac{dW_\alpha}{dq_\alpha} \quad \alpha=1,2,\cdots s$$

H—J方程化为

$$H=\sum_{\alpha=1}^{s}H_\alpha=E, \quad H_\alpha=\frac{1}{2}A_\alpha(q_\alpha)\left(\frac{dW_\alpha}{dq_\alpha}\right)^2+V_\alpha(q_\alpha)=\eta_\alpha \quad \sum_{\alpha=1}^{s}\eta_\alpha=E$$

$$W = W_1(q_1) + \cdots + W_s(q_s)$$

$$W_\alpha = \int \sqrt{\frac{2(\eta_\alpha - V_\alpha)}{A_\alpha}} dq_\alpha \quad \alpha = 1, 2 \cdots s, \quad \eta_1 = E - \eta_2 - \cdots - \eta_s$$

HJ方程的完全解可表为

$$S = -Et + W_1(q_1, E - \eta_2 - \cdots - \eta_s) + \sum_{\alpha=2}^{s} W_\alpha(q_\alpha, \eta_\alpha) + C$$

正则方程的积分可表为 $\quad p_\alpha = \dfrac{dW_\alpha}{dq_\alpha} \quad \alpha = 1, 2 \cdots, s$

$$\xi_\alpha = \frac{\partial S}{\partial \eta_\alpha} = \frac{\partial W_\alpha}{\partial \eta_\alpha} + \frac{\partial W_1}{\partial \eta_\alpha} = \frac{\partial W_\alpha}{\partial \eta_\alpha} - \frac{\partial W_1}{\partial E}, \quad \alpha = 2, 3, \cdots, s$$

$$\xi_1 = -t_0 = \frac{\partial S}{\partial E} = -t + \frac{\partial W_1}{\partial E}, \quad \therefore \frac{\partial W_1}{\partial E} = t - t_0$$

【例】三维空间的谐振子。

$$H=\frac{p_1^2}{2m_1}+\frac{1}{2}m_1\omega_1^2q_1^2+\frac{p_2^2}{2m_2}+\frac{1}{2}m_2\omega_2^2q_2^2+\frac{p_3^2}{2m_3}+\frac{1}{2}m_3\omega_3^2q_3^2$$

哈密顿特征函数可设为

$$W=W_1(q_1)+W_2(q_2)+W_3(q_3)\qquad \frac{\partial W}{\partial q_\alpha}=\frac{\partial W_\alpha}{\partial q_\alpha}=\frac{dW_\alpha}{dq_\alpha}\quad \alpha=1,2,3$$

H—J方程化为

$$H=\sum_{\alpha=1}^{3}H_\alpha=E,\quad H_\alpha=\frac{1}{2m_\alpha}\left(\frac{dW_\alpha}{dq_\alpha}\right)^2+\frac{1}{2}m_\alpha\omega_\alpha^2q_\alpha^2=\eta_\alpha\qquad \sum_{\alpha=1}^{3}\eta_\alpha=E$$

$$\eta_1=E-\eta_2-\eta_3$$

$$W_1 = \int \sqrt{2m_1\left(E-\eta_2-\eta_3\right)-m_1^2\omega_1^2 q_1^2}\,dq_1$$

$$W_2 = \int \sqrt{2m_2\eta_2 - m_2^2\omega_2^2 q_2^2}\,dq_2 \qquad W_3 = \int \sqrt{2m_3\eta_3 - m_3^2\omega_3^2 q_3^2}\,dq_3$$

W_α 的相加常数均可吸收入C，主函数为

$$S = -Et + W_1\left(q_1, E-\eta_2-\eta_3\right) + W_2\left(q_2,\eta_2\right) + W_3\left(q_3,\eta_3\right) + C$$

正则方程的积分：

$$p_\alpha = \frac{dW_\alpha}{dq_\alpha} = \sqrt{2m_\alpha\eta_\alpha - m_\alpha^2\omega_\alpha^2 q_\alpha^2} \qquad \alpha = 2,3$$

$$p_1 = \frac{dW_1}{dq_1} = \sqrt{2m_1\left(E-\eta_2-\eta_3\right)-m_1^2\omega_1^2 q_1^2}$$

$$\xi_\alpha = \frac{\partial S}{\partial \eta_\alpha} = \frac{\partial W_\alpha}{\partial \eta_\alpha} + \frac{\partial W_1}{\partial \eta_\alpha} = \frac{\partial W_\alpha}{\partial \eta_\alpha} - \frac{\partial W_1}{\partial E}$$

$$= \int \frac{m_\alpha}{\sqrt{2m_\alpha \eta_\alpha - m_\alpha^2 \omega_\alpha^2 q_\alpha^2}} dq_\alpha - \int \frac{m_1}{\sqrt{2m_1\left(E-\eta_2-\eta_3\right) - m_1^2 \omega_1^2 q_1^2}} dq_1$$

$$= \frac{1}{\omega_\alpha} \arcsin \sqrt{\frac{m_\alpha \omega_\alpha^2}{2\eta_\alpha}} q_\alpha - \frac{1}{\omega_1} \arcsin \sqrt{\frac{m_1 \omega_1^2}{2\left(E-\eta_2-\eta_3\right)}} q_1 \qquad \alpha = 2,3$$

$$\xi_1 = -t_0 = \frac{\partial S}{\partial E} = -t + \frac{\partial W_1}{\partial E} = -t + \int \frac{m_1}{\sqrt{2m_1\left(E-\eta_2-\eta_3\right) - m_1^2 \omega_1^2 q_1^2}} dq_1$$

$$= -t + \frac{1}{\omega_1} \arcsin \sqrt{\frac{m_1 \omega_1^2}{2\left(E-\eta_2-\eta_3\right)}} q_1$$

$$q_1 = \sqrt{\frac{2\left(E-\eta_2-\eta_3\right)}{m_1 \omega_1^2}} \sin\left[\omega_1\left(t-t_0\right)\right] \quad p_1 = \sqrt{2m_1\left(E-\eta_2-\eta_3\right)} \cos\left[\omega_1\left(t-t_0\right)\right]$$

$$q_\alpha = \sqrt{\frac{2\eta_\alpha}{m_\alpha \omega_\alpha^2}} \sin\left[\omega_\alpha\left(t-t_0+\xi_\alpha\right)\right] \qquad p_\alpha = \sqrt{2m_\alpha \eta_\alpha} \cos\left[\omega_\alpha\left(t-t_0+\xi_\alpha\right)\right] \quad \alpha = 2,3$$

H-J方程的意义？

- 给出解正则方程的一种方法，可与其他方法互为补充。且其结果不仅包括运动规律，而且还有轨道，动量，内容丰富。
- 处理质点（组）力学问题，都用常微分方程（组），而H—J方程是偏微分方程，通常是用来处理无限多个自由度的力学体系问题的，例如波、连续介质等。
- H—J方程在量子力学的建立过程中，起了重要的作用。

用哈密顿理论解开普勒问题

例：　用哈 - 雅方程求行星绕太阳运动时轨道方程。

解：采用极坐标(r,θ)，设行星为m，太阳为M，引力常数为G.

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2),\quad U=-\frac{GMm}{r},\quad L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{GMm}{r}$$

$$p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r}\Rightarrow \dot{r}=\frac{p_r}{m};\quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}\Rightarrow \dot{\theta}=\frac{p_\theta}{mr^2}\text{。}$$

$$H=\frac{1}{2m}(p_r^2+\frac{p_\theta^2}{r^2})-\frac{GMm}{r}=E(\text{总能量})$$

设主函数：$S=-Et+W_r+W_\theta$，$p_r=\frac{dW_r}{dr}$，$p_\theta=\frac{dW_\theta}{d\theta}$，

故哈－雅方程：$H=\frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2+\left(\frac{dW_\theta}{d\theta}\right)^2\frac{1}{r^2}\right]-\frac{GMm}{r}=E$

因θ为循环坐标，$p_\theta=a_\theta$（常数）$\Rightarrow W_\theta=a_\theta\theta+c_\theta$

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2+\frac{a_\theta{}^2}{2mr^2}-\frac{GMm}{r}=E$$

$$\Rightarrow W_r=\int\sqrt{2mE+\frac{2GMm^2}{r}-\frac{a_\theta^2}{r^2}}dr,$$

$$S=-Et+a_\theta\theta+\int\sqrt{2mE+\frac{2GMm^2}{r}-\frac{a_\theta^2}{r^2}}dr$$

$$\beta=\frac{\partial S}{\partial a_\theta}=\theta+\int\frac{a_\theta d\left(1/r\right)}{\sqrt{2mE+\frac{2GMm^2}{r}-\frac{a_\theta^2}{r^2}}}$$

$$=\theta+\sin^{-1}\frac{a_\theta^2-GMm^2r}{r\sqrt{G^2M^2m^4+2mEa_\theta^2}}+c$$

初始条件：$t=0, r=r_{\min}, \theta=0$

$$\Rightarrow \beta=\sin^{-1}\frac{a_\theta^2-GMm^2r_{\min}}{r_{\min}\sqrt{G^2M^2m^4+2mEa_\theta^2}}+c$$

$$\Rightarrow \theta+\sin^{-1}\frac{a_\theta^2-GMm^2r}{r\sqrt{G^2M^2m^4+2mEa_\theta^2}}$$

$$=\sin^{-1}\frac{a_\theta^2-GMm^2r_{\min}}{r_{\min}\sqrt{G^2M^2m^4+2mEa_\theta^2}}$$

$$\frac{GMm^2r-a_\theta^2}{r\sqrt{G^2M^2m^4+2mEa_\theta^2}}$$

$$=\sin\left(\theta-\sin^{-1}\frac{a_\theta^2-GMm^2r_{\min}}{r_{\min}\sqrt{G^2M^2m^4+2mEa_\theta^2}}\right)$$

$$\frac{GMm^2 r - a_\theta^2}{r\sqrt{G^2M^2m^4 + 2mEa_\theta^2}} = \sin\left(\theta - \sin^{-1}\frac{a_\theta^2 - GMm^2 r_{\min}}{r_{\min}\sqrt{G^2M^2m^4 + 2mEa_\theta^2}}\right)$$

$$r = \frac{a_\theta^2}{GMm^2 - \sqrt{G^2M^2m^4 + 2mEa_\theta^2}\sin\left(\theta - \sin^{-1}\frac{a_\theta^2 - GMm^2 r_{\min}}{r_{\min}\sqrt{G^2M^2m^4 + 2mEa_\theta^2}}\right)}$$

当 $\sin^{-1}\frac{a_\theta^2 - GMm^2 r_{\min}}{r_{\min}\sqrt{G^2M^2m^4 + 2mEa_\theta^2}} = \frac{\pi}{2}$ 时，$r = r_{\min}$。

$$r = a_\theta^2 / [GMm^2 - \sqrt{G^2M^2m^4 + 2mEa_\theta^2}\sin(\theta - \pi/2)]$$

$$= \frac{a_\theta^2}{GMm^2 + \sqrt{G^2M^2m^4 + 2mEa_\theta^2}\cos\theta} = \frac{a_\theta^2 / GMm^2}{1 + \sqrt{1 + 2Ea_\theta^2 / GMm}\cos\theta}$$

$$= p/(1 + e\cos\theta) \qquad \left(\mathrm{p} = a_\theta^2 / GMm^2, \mathrm{e} = \sqrt{1 + 2Ea_\theta^2 / GMm}\right)$$