第三章　　刚体的定轴转动

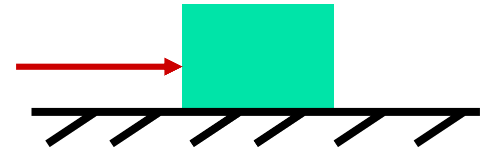
刚体(rigid　body)

——忽略由于受力而引起的形状和体积的改变的理想模型

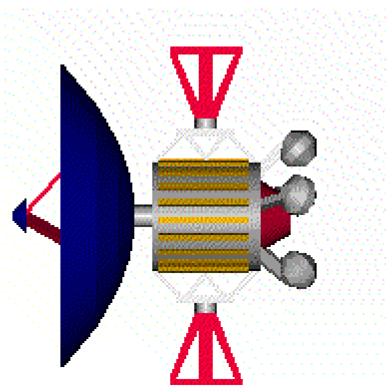
刚体研究方式——特殊质点系

一、刚体的运动

平动： 用质点运动讨论

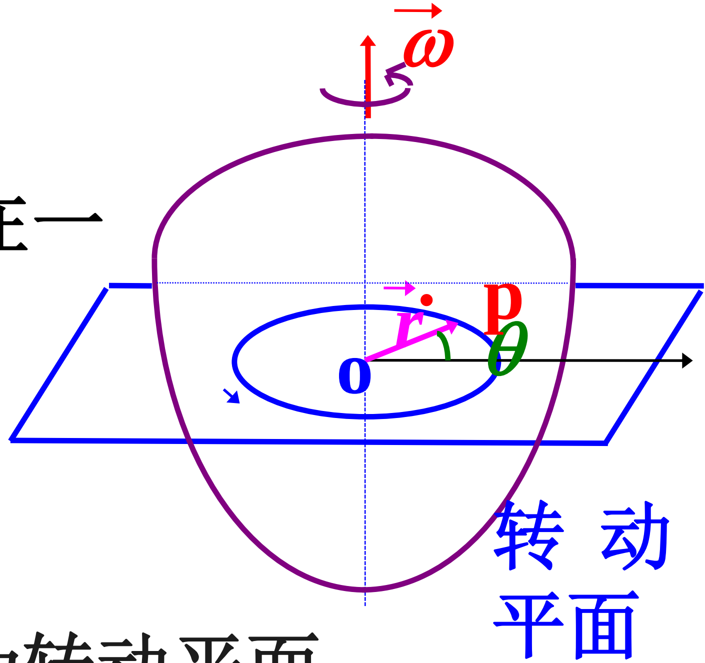


转动： 对点、对轴



既平动又转动：

刚体的定轴转动(fixed-axis rotation)



各质元均作圆周运动，其圆心都在一条固定不动的直线转轴上（定轴）

定轴转动的特点

1）每一质点均作圆周运动，圆面为转动平面；

2）任一质点运动 $\Delta\theta, \vec{\omega}, \vec{\alpha}$ 均相同，但 $\vec{v}, \vec{a}$ 不同；

整体运动		每个质元的运动
角位移	$\Delta\theta$	$v = \omega r$
角速度	$\omega = \dfrac{d\theta}{dt}$	$a_n = \omega^2 r$
角加速度	$\alpha = \dfrac{d\omega}{dt}$	$a_t = r\alpha$

$$\alpha = \frac{d\omega}{dt} = C$$

匀角加速运动

匀加速直线运动

匀角加速运动	匀加速直线运动
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2} a t^2$
$\omega^2 - \omega_0^2 = 2\Delta\theta\alpha$	$v^2 - v_0^2 = 2xa$

$$d\omega = \alpha dt$$

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt$$

$$\Rightarrow \omega - \omega_0 = \alpha t$$

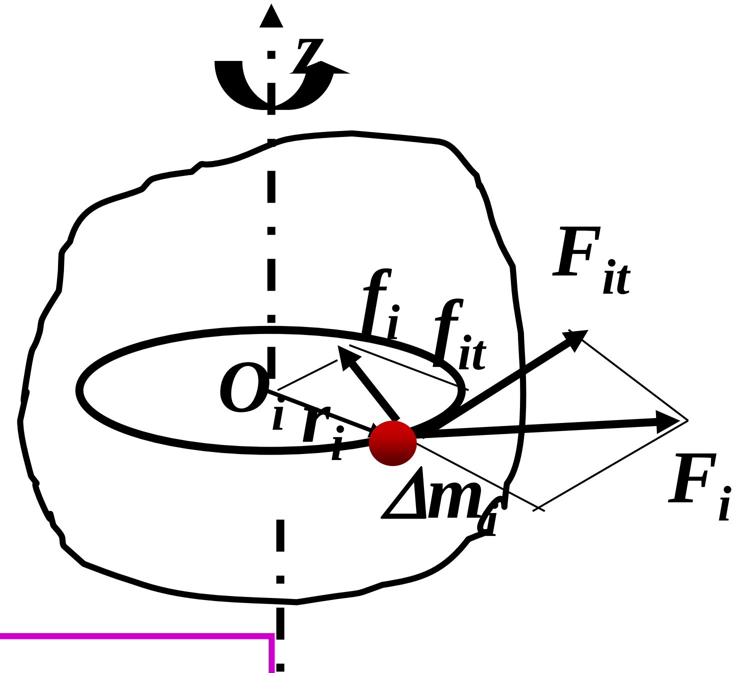
$$\omega = \frac{d\theta}{dt} = \omega_0 + \alpha t$$

$$d\theta = (\omega_0 + \alpha t)dt$$

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)dt \Rightarrow \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

二、刚体的定轴转动的转动定理

（法向力的力矩为零）



牛顿第二定律对Δm_i切向分量式为:

$$F_{it}+f_{it}=\Delta m_i a_{it}=\Delta m_i r_i \alpha$$

$$F_{it} r_i+f_{it} r_i=\Delta m_i r_i^2 \alpha$$

对所有质元求和：

一对内力的力矩之和为零

$$\sum F_{it} r_i+\sum f_{it} r_i=\sum \Delta m_i r_i^2 \alpha \longrightarrow \sum F_{it} r_i=\left(\sum \Delta m_i r_i^2\right)\alpha$$

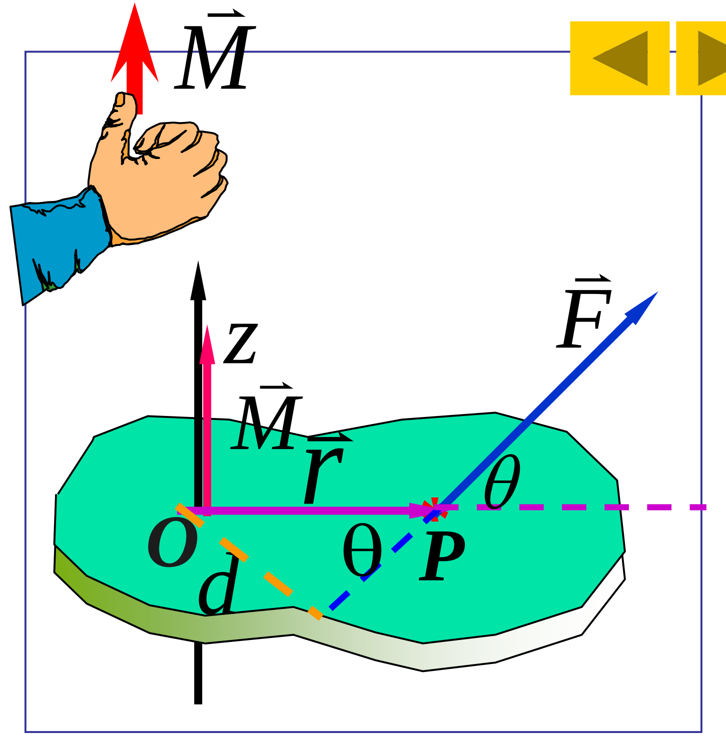
合外力矩 $M=\sum F_{it} r_i$

转动惯量 $J=\sum \Delta m_i r_i^2$

1、转动定律: $\mathbf{M=J\alpha}$ $\sim \vec{F}=m\vec{a}$

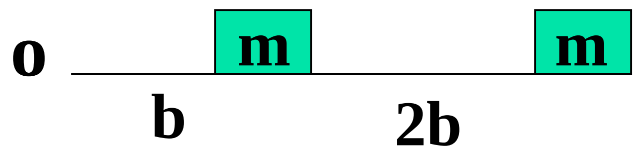
1）力矩计算 $\vec{M} = \vec{r} \times \vec{F}$

$$M = Fr\sin\theta = Fd$$



2）转动惯量计算:

质点系: $J = \sum_i m_i r_i^2$

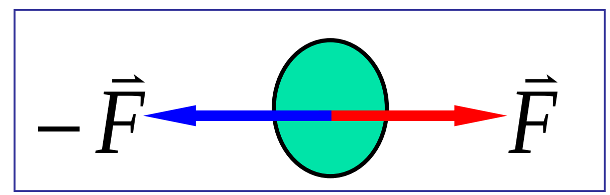


$dm = \lambda dl$

$dm = \sigma dS$

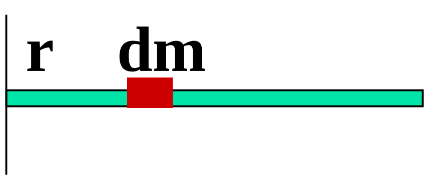
$dm = \rho dV$

J=mb²+m(3b)²

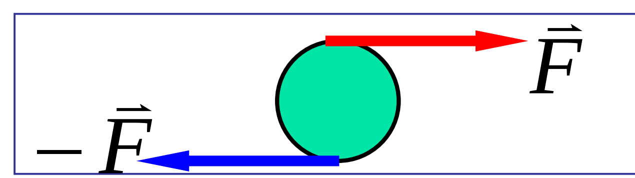


$\sum \vec{F}_i = 0, \sum \vec{M}_i = 0$

质量连续分布: $dJ = r^2 dm$



$$\Rightarrow J = \int_m r^2 dm$$

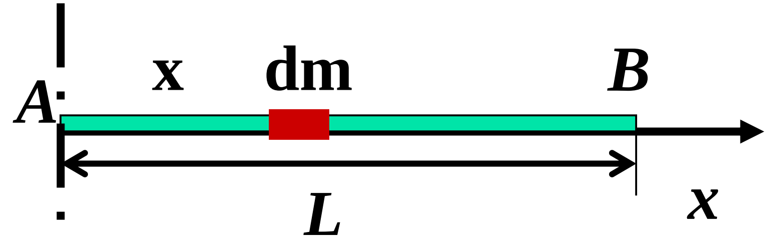


$\sum \vec{F}_i = 0, \sum \vec{M}_i \neq 0$

J 的单位：kgm^2 量纲：ML^2

例1 求m、L的均匀细棒对图中不同轴A、C的转动惯量

解： 取如图坐标

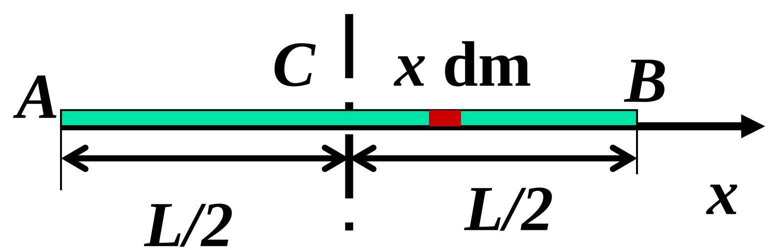


dm=λdx $\left(\lambda = \dfrac{m}{L}\right)$

$$\mathrm{d}J = x^2\,\mathrm{d}m = x^2\lambda\,\mathrm{d}x$$

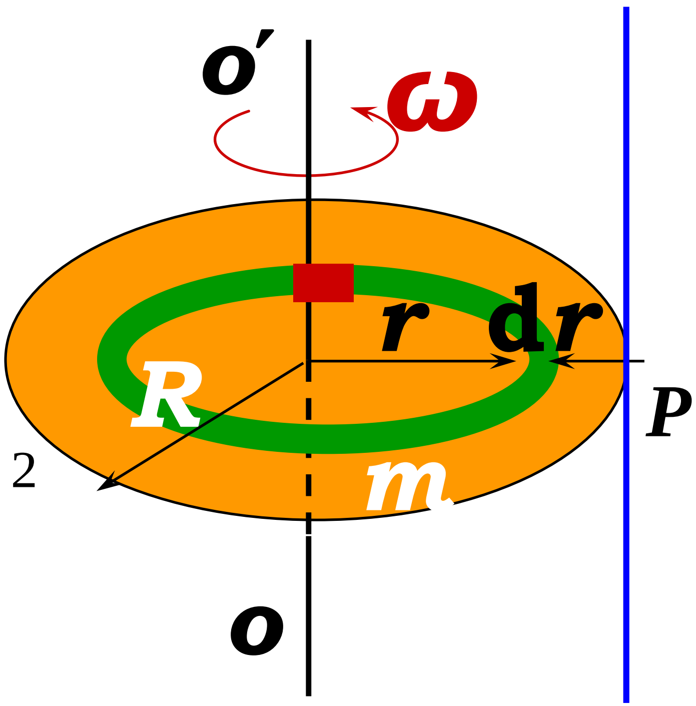
$$J_A = \int_0^L x^2\lambda\,\mathrm{d}x = \frac{m}{L}\,\frac{1}{3}\,x^3\Big|_0^L = \frac{1}{3}mL^2$$

$$J_C = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\lambda\,\mathrm{d}x = \frac{1}{12}mL^2$$



例2 求m、R 的均质圆盘绕 OO’轴旋转的转动惯量

解：设质量面密度为$\sigma = \dfrac{m}{\pi R^2}$



$$dm = \sigma dS$$

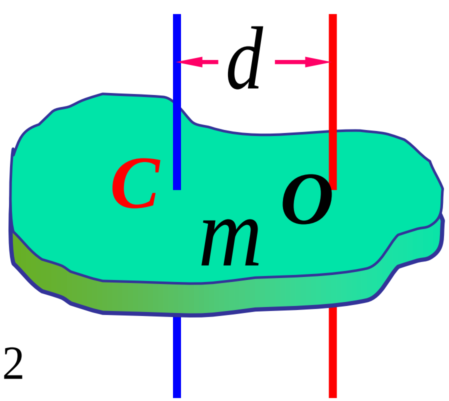
$$dJ = r^2 dm = r^2 \sigma 2\pi r dr$$

$$J = \int_0^R r^2 dm = \int_0^R \sigma 2\pi r^3 dr = \frac{1}{2} mR^2$$

平行轴定理

$$J_O = J_C + md^2$$

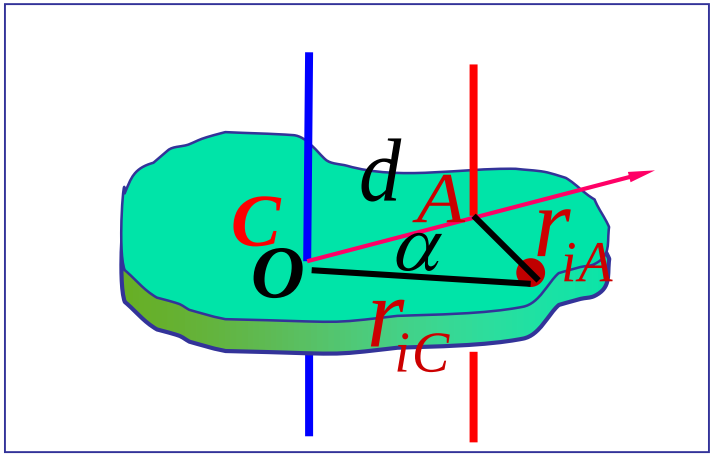
$$J_P = \frac{1}{2} mR^2 + mR^2$$



P106 表3-1 转动惯量

转动惯量的大小取决于刚体的质量、形状及转轴的位置.

$$J_A = \sum_i m_i r_{iA}^2$$



$$r_{iA}^2 = r_{iC}^2 + d^2 - 2r_{iC} d \cos\alpha$$

$$J_A = \sum_i m_i r_{iA}^2 = \sum_i m_i (r_{iC}^2 + d^2 - 2r_{iC} d \cos\alpha)$$

$$= J_c + md^2 \ - 2d \sum_i m_i x_i$$

$$\because x_C = \frac{\sum_i m_i x_i}{m} = 0$$

$$\Rightarrow J_A = J_c + md^2$$

例3、书P126 3-5

$$J=\frac{1}{2}mr^2$$

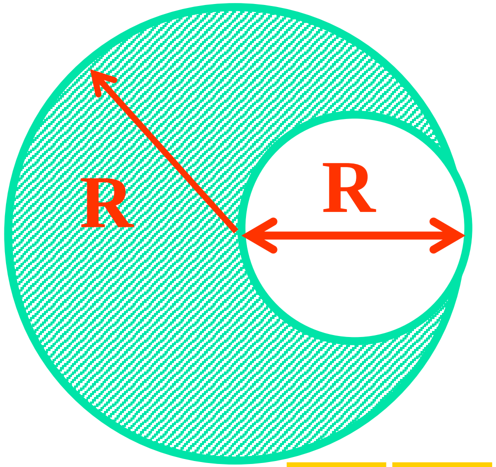
解： 采用补偿法——大圆盘减小圆盘

$$\sigma=\frac{m}{\pi(R^2-(\frac{R}{2})^2)}\qquad m_0=\sigma\pi\left(\frac{R}{2}\right)^2$$

$$J_1=\frac{1}{2}(m+m_0)R^2$$

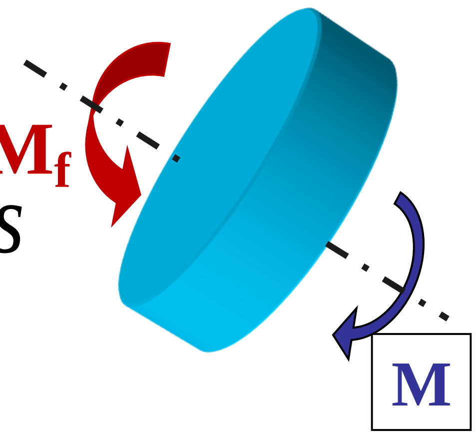
$$J_2=\frac{1}{2}m_0\left(\frac{R}{2}\right)^2+m_0\left(\frac{R}{2}\right)^2$$

$$\therefore\ J=J_1-J_2=\frac{13}{24}mR^2$$



注：实际上转动惯量均由实验测定

例1、以20N•m的恒力矩作用在有固定轴的转轮上，在10 s内该轮的转速由零增大到100rev/min。此时移去该力矩，转轮因摩擦力矩的作用经100s而停止。试推算此转轮对其固定轴的转动惯量。



解：$$M - M_f = J\alpha_1$$

$$-M_f = J\alpha_2$$

$$\omega = \frac{2\pi n}{60} rad/s$$

$$\alpha_1 = \frac{\omega_1 - 0}{\Delta t_1} = \frac{\omega_1}{\Delta t_1}$$

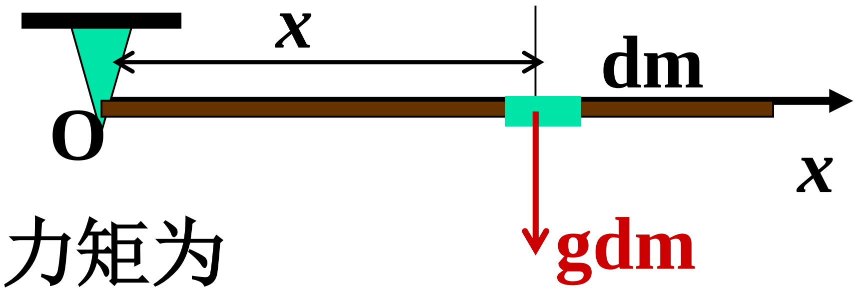
$$(\omega - \omega_0 = \alpha t)$$

$$\alpha_2 = \frac{0 - \omega_1}{\Delta t_2} = -\frac{\omega_1}{\Delta t_2}$$

$$J = \frac{M}{\frac{\omega_1}{\Delta t_1} + \frac{\omega_1}{\Delta t_2}}$$

$$= 17.3 Kg \cdot m^2$$

例2 书p128 3-15



解(1) 棒上取质元*dm*, 则重力矩为

$$dM = xgdm$$

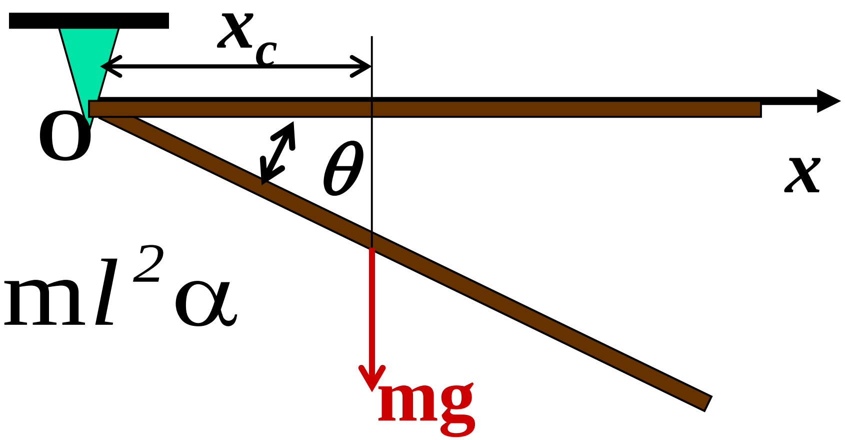
$$M = \int xgdm = g\int xdm$$

$$x_C = \frac{\sum m_i x_i}{m} \Rightarrow x_C = \frac{\int xdm}{m} \Rightarrow \int xdm = mx_C$$

$$\therefore M = mgx_C = mg\frac{1}{2}l = J\alpha = \frac{1}{3}ml^2\alpha$$

$$\Rightarrow \alpha = \frac{3g}{2l}$$

(2) $M = mg x_C$



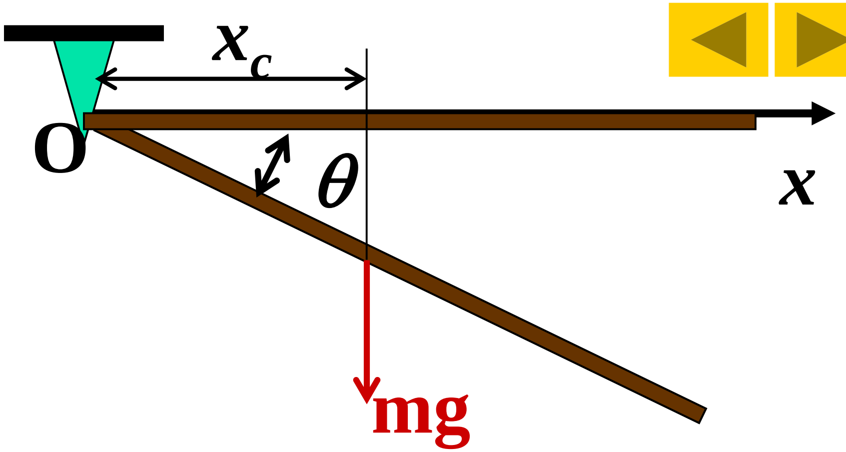
$$= mg\frac{l}{2}\cos\theta = \frac{1}{3}ml^2\alpha$$

$$\Rightarrow \alpha = \frac{3g\cos\theta}{2l} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\cdot\omega$$

$$\frac{3g}{2l}\cos\theta d\theta = \omega d\omega$$

$$\int_0^{\frac{\pi}{2}}\frac{3g}{2l}\cos\theta d\theta = \int_0^{\omega}\omega d\omega \Rightarrow \omega = \sqrt{\frac{3g}{l}}$$

(3) $$\alpha = \frac{3g\cos\theta}{2l}$$



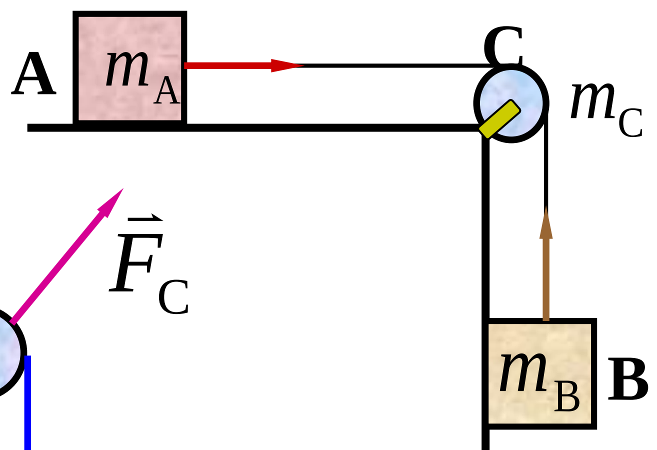
$$\alpha = \frac{3g\cos\theta}{2l} = \frac{d\omega}{dt}$$

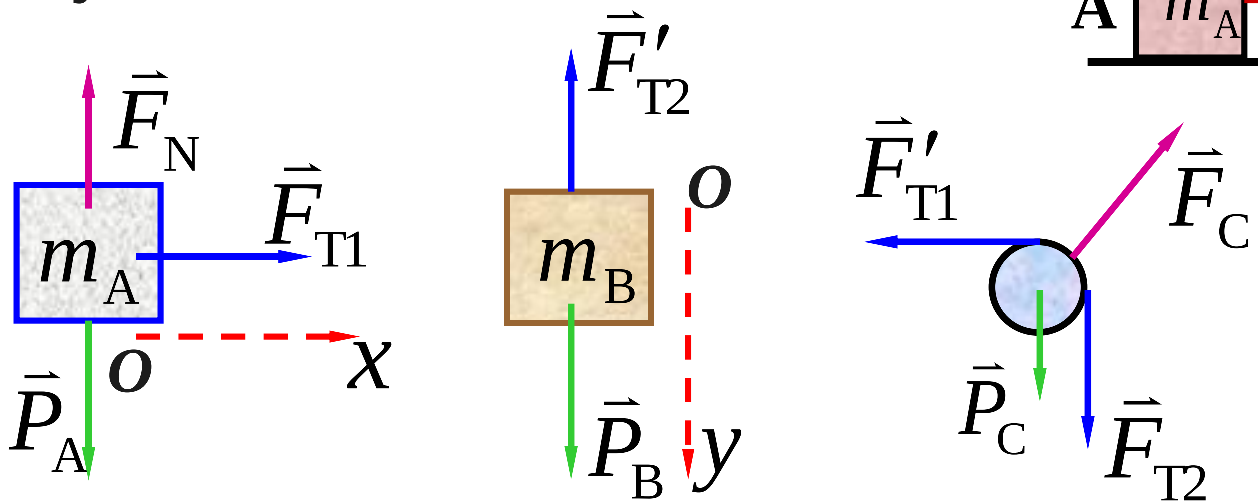
$$= \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

$$\int_0^{30^0} \frac{3g\cos\theta}{2l} d\theta = \int_0^{\omega} \omega d\omega \Rightarrow \omega = \sqrt{\frac{3g\sin 30^0}{l}}$$

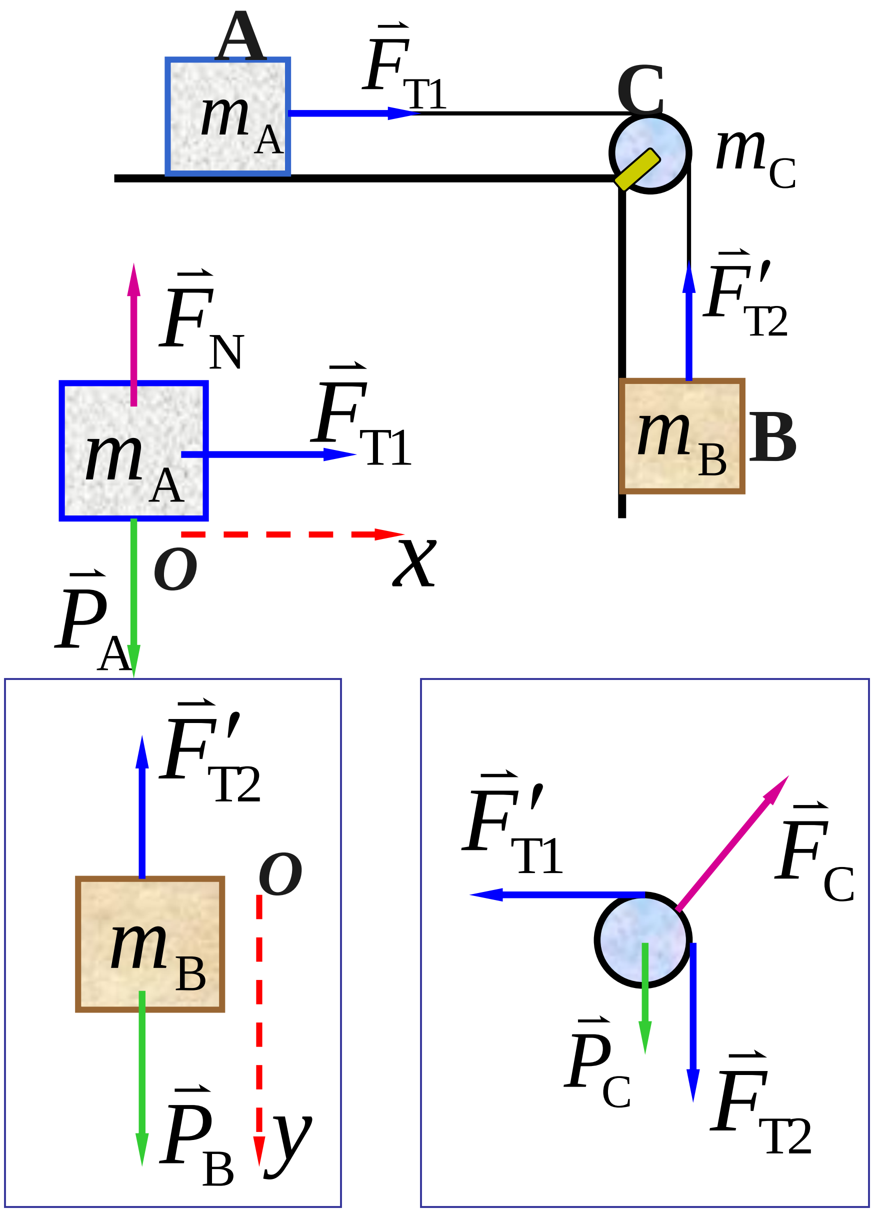
$$a_n = \omega^2 \frac{l}{2} = 3g\sin 30^0 \qquad a_t = \alpha\frac{l}{2} = \frac{3}{4} g\cos 30^0$$

例3、m_A 静止在光滑水平面上，和一质量不计的绳索相连接，绳索跨过一半径为 R、m_C 的圆柱形滑轮 C，并系在 m_B上. 滑轮与绳索间没有滑动，且滑轮与轴承间的摩擦力可略去不计. 问：1） 两物体的线加速度为多少？ 水平和竖直两段绳索的张力各为多少？（2）物体 B 从静止落下距离y 时，其速率是多少？





解（1）



$$\begin{cases} F_{T1} = m_A a \\ m_B g - F_{T2} = m_B a \\ R F_{T2} - R F_{T1} = J\alpha \\ a = R\alpha \end{cases}$$

$$\Rightarrow a = \frac{m_B g}{m_A + m_B + m_C/2}$$

（2） **B由静止出发作匀加速直线运动，下落的速率**

$$y = \frac{1}{2} a t^2 \Rightarrow t \Rightarrow v = at$$

三、刚体的平衡　$\vec{F} = 0$　$\vec{M} = 0$

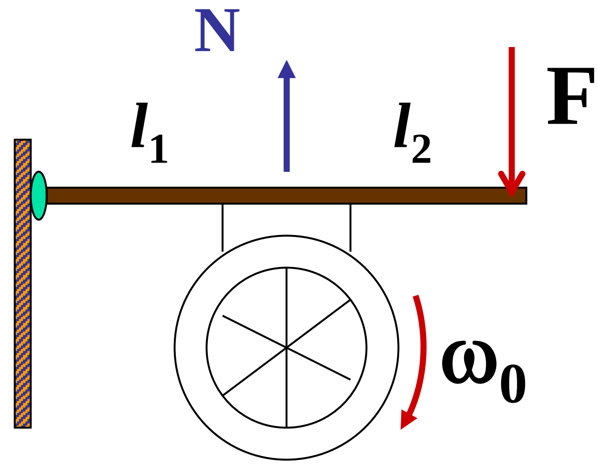
例3（书　P128　3-8)

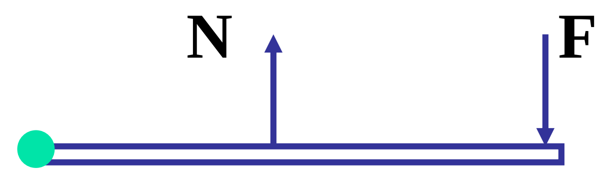
$$F(l_1 + l_2) - Nl_1 = 0$$

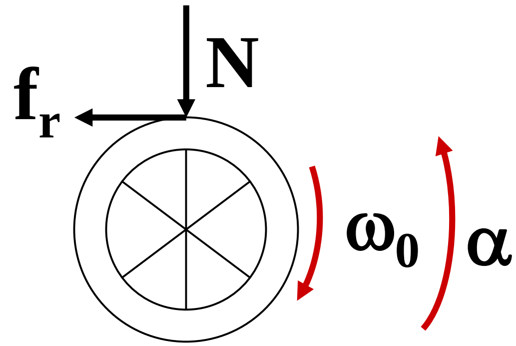
$$-f_r \frac{d}{2} = J\alpha$$

$$f_r = \mu N$$

$$\omega = \omega_0 + \alpha t = 0$$

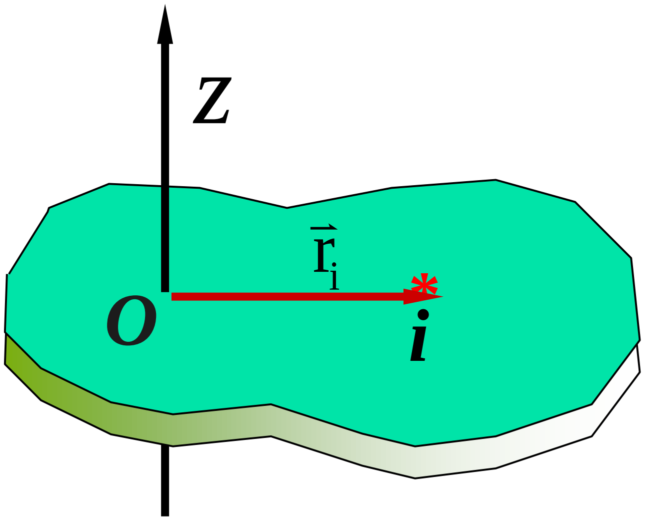






四、刚体中的功能关系

1、刚体的转动动能



对第i个质点：$E_{ki} = \frac{1}{2} m_i v_i^2$

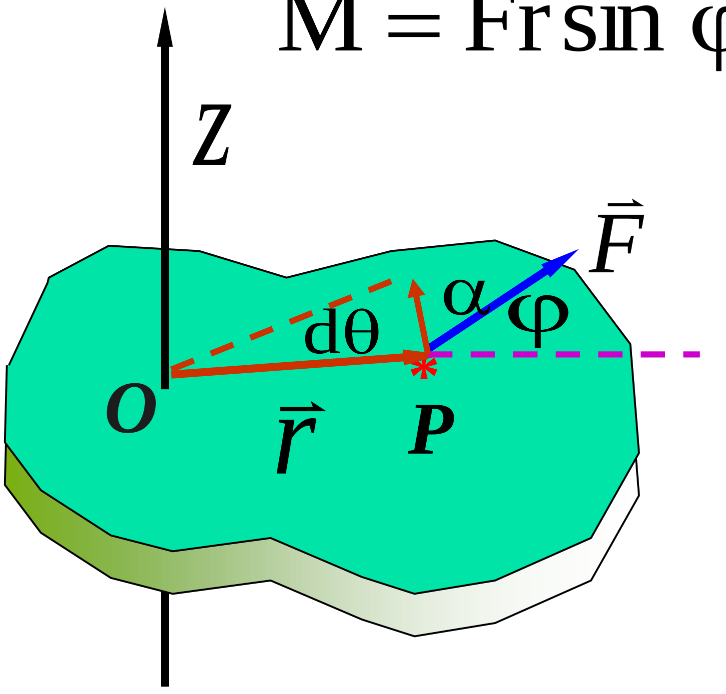
对整个刚体：

$$E_k = \sum E_{ki} = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (r_i \omega)^2$$

$$= \frac{1}{2} \omega \sum m_i r_i^2 = \frac{1}{2} J\omega^2$$

2、力矩的功

$$M = Fr\sin\varphi$$



$$\begin{aligned} dA &= \vec{F}\cdot d\vec{r} \\ &= F\cos\alpha\left|d\vec{r}\right| \\ &= F\cos\alpha\, ds \\ &= F\cos\alpha\; rd\theta \\ &= F\sin\varphi\, rd\theta \\ &= M d\theta \end{aligned} \quad \Rightarrow A = \int M d\theta$$

若刚体受到几个外力矩的作用

$$A = \sum A_i = \sum \int M_i d\theta_i$$

$$= \int (\sum_i M_i) d\theta = \int M_{合外} d\theta$$

3、刚体的重力势能

对于第i个质点： $E_{pi} = m_i g z_i$

$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

对于整个刚体：

$$E_P = \sum E_{pi} = \sum m_i g z_i$$

$$= g\sum m_i z_i = mgz_c$$

$$z_c = \frac{\sum m_i z_i}{m}$$

$$M = J\alpha = J\frac{d\omega}{dt} = J\frac{d\omega}{d\theta}\frac{d\theta}{dt} = J\omega\frac{d\omega}{d\theta}$$

$$dA = Md\theta = J\omega d\omega$$

4、刚体的动能定律、功能原理及机械能守恒

动能定理：

$$A = \int M d\theta = \int_{\omega_0}^{\omega} J\omega d\omega = \frac{1}{2}J\omega^2 - \frac{1}{2}J\omega_0^2$$

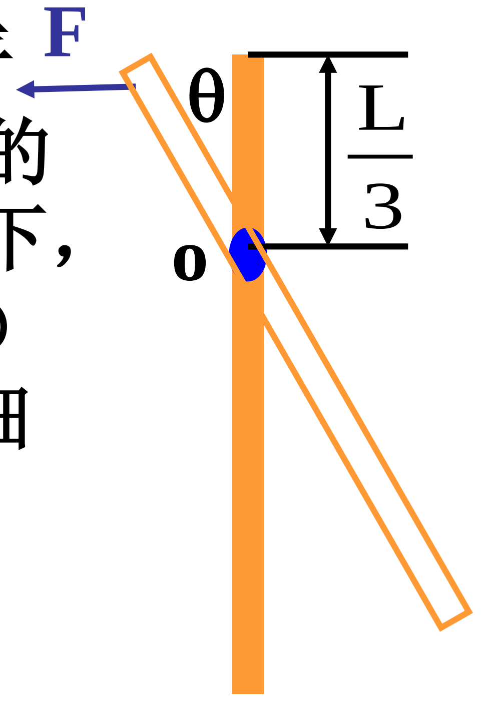
功能原理：

$$A = \frac{1}{2}J\omega^2 + mgz_c - \left(\frac{1}{2}J\omega_0^2 + mgz_{c0}\right)$$

机械能守恒：

$$\begin{cases} \text{刚体} \quad A_{\text{外}}=0\ (A_{\text{内}}=0) \\ \text{刚体＋质点} \quad A_{\text{外}}+A_{\text{非保守内力}}=0 \end{cases}$$

例1、均质细棒L、m，可绕通过o点水平轴在竖直平面内转动，如图所示。在棒的A端作用一水平恒力F，棒在F力的作用下，由静止转过角度θ（ θ =30º），求：（1）F力所作的功；2）若此时撤去F力，则细棒回到平衡位置时的角速度。



解：1） $M = F\cos\theta \dfrac{L}{3}$

$$A = \int M d\theta = \int_0^{\frac{\pi}{6}} F\frac{L}{3}\cos\theta d\theta = \frac{1}{6}FL$$

2）根据功能原理　（从竖直位置回到竖直位置）

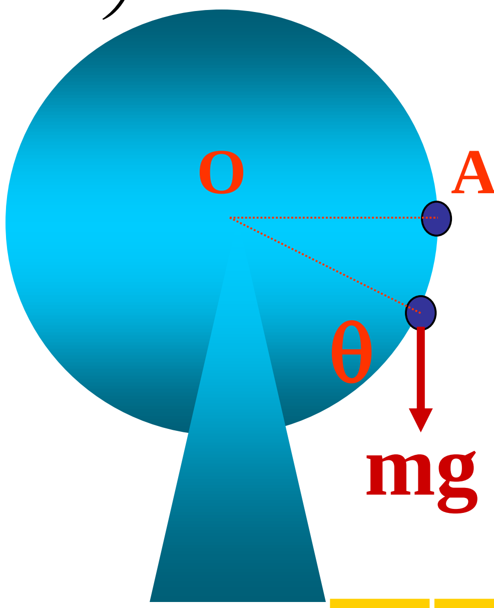
$$\left.\begin{aligned} A &= \frac{1}{6}FL = \frac{1}{2}J\omega^2 - 0 \\ J &= \frac{1}{12}mL^2 + m\left(\frac{L}{2} - \frac{L}{3}\right)^2 = \frac{1}{9}mL^2 \end{aligned}\right\} \Rightarrow \omega = \sqrt{\frac{3F}{mL}}$$

例2、m_0、R的匀质园盘可绕垂直于盘的光滑轴O在铅直平面内转动，盘点A固定着m的质点，先使OA处于水平位置，然后释放，盘由静止开始转动。当OA转过来30°时，质点的a_n、a_t为多少？

解：根据转动定律

$$mgR\cos\theta = J\alpha = \left(\frac{1}{2}m_0R^2 + mR^2\right)\alpha$$

$$a_t = \alpha R = \frac{\sqrt{3}mg}{(m_0 + 2m)}$$



地球　盘的系统：E=C

$$mgR\sin\theta = \frac{1}{2}J\omega^2$$

$$a_n = \omega^2 R = \frac{2mg}{m_0 + 2m}$$