# 《微分几何》课程电子课件

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# 第三章　外微分形式和活动标架

# §3.1 外微分形式

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# 一、格拉斯曼(Grassmann)代数

## 古典微积分的弊端1:

微积分的基本定理没有维数不变性

$$f(b)-f(a)=\int_a^b f'(x)\,\mathrm{d}x$$

$$\int_{\partial D} P\,\mathrm{d}x+Q\,\mathrm{d}y=\int_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$\int_{\partial V} P\,\mathrm{d}y\,\mathrm{d}z+Q\,\mathrm{d}z\,\mathrm{d}x+R\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\int_{\partial S} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z =$$

$$\int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

共同点：将某一几何图形上的积分用该图形的边界上的积分来表示.

这些公式彼此不同,各自冠以著名数学家的名字,不是普通的推广.

三维空间有两个公式(2维 $\leftrightarrow$ 1维, 3维 $\leftrightarrow$ 2维)

四维空间应该有三个公式

$n$ 维空间应该有 $n-1$ 个公式

$$\int_{\partial G} \omega = \int_G \mathrm{d}\omega$$

## 古典微积分的弊端2:
## 被积表达式没有坐标不变性

在$\iint_{\Omega} f(x,y)\mathrm{d}x\mathrm{d}y$中通常将$\mathrm{d}x\mathrm{d}y$理解为 $\mathrm{d}x$乘以$\mathrm{d}y$

若采用极坐标,则该积分变为$\iint_{\Omega_1} F(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta$

若令$x=x(u,v), y=y(u,v)$,则该积分变为

$$\iint_{\Omega_2} F(u,v)\left|\frac{D(x,y)}{D(u,v)}\right|\mathrm{d}u\mathrm{d}v$$

根源:未考虑$\mathrm{d}x$和$\mathrm{d}y$的方向.

为了使"$\mathrm{d}x$乘以$\mathrm{d}y$"能与坐标的选取无关,就要对乘积赋以新的意义.

## 外乘 $\wedge$ (wedge)

若将 $\mathrm{d}x\,\mathrm{d}y$ 理解为"$\mathrm{d}x\times\mathrm{d}y$",则表示以$\mathrm{d}x$和$\mathrm{d}y$为边的平行四边形的有向面积,具有坐标不变性.

定义一种新的乘法,叫作外乘,记为 $\wedge$,使得

$\vec{a}_1\wedge\vec{a}_2$ 刻画以$\vec{a}_1$与$\vec{a}_2$为边的平行四边形的有向面积;

$\vec{a}_1\wedge\vec{a}_2\wedge\vec{a}_3$ 刻画以$\vec{a}_1,\vec{a}_2,\vec{a}_3$为边的平行六面体的有向体积; ……

问题:如何定义外乘$\wedge$?

思路:特殊 $\Rightarrow$ 一般

## 考虑$\mathbb{R}^3$中的三元外乘运算

给定三个向量$\vec{a}_i = a_{i1}\vec{e}_1 + a_{i2}\vec{e}_2 + a_{i3}\vec{e}_3 \in \mathbb{R}^3, (i = 1,2,3)$

则以$\vec{a}_1,\vec{a}_2,\vec{a}_3$为棱边的平行六面体为

$$V = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3, 0 \le \lambda_1,\lambda_2,\lambda_3 \le 1\}$$

其有向体积(体积空间中的一个向量)为

$$\vec{a}_1 \wedge \vec{a}_2 \wedge \vec{a}_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$$

其中$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$为以单位体积为模长的一个体积基底.

$$\vec{a}_1\wedge\vec{a}_2\wedge\vec{a}_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$$

受行列式启发，可认为运算$\wedge$应服从以下法则：

① 重线性：设$\vec{b},\vec{c}\in\mathbb{R}^3,\beta,\gamma\in\mathbb{R}$，则

$$\vec{a}_1\wedge(\beta\vec{b}+\gamma\vec{c})\wedge\vec{a}_3=\beta\vec{a}_1\wedge\vec{b}\wedge\vec{a}_3+\gamma\vec{a}_1\wedge\vec{c}\wedge\vec{a}_3$$

(对参与运算的每个向量满足分配律和数乘运算)

② 反交换律：

在$\vec{a}_1\wedge\vec{a}_2\wedge\vec{a}_3$中任何两个向量交换之后符号相反，即

$$\vec{a}_1\wedge\vec{a}_2\wedge\vec{a}_3=-\vec{a}_2\wedge\vec{a}_1\wedge\vec{a}_3=-\vec{a}_1\wedge\vec{a}_3\wedge\vec{a}_2=-\vec{a}_3\wedge\vec{a}_2\wedge\vec{a}_1$$

## 考虑$\mathbb{R}^3$中的二元外乘运算

给定两个向量$\vec{a}_i = a_{i1}\vec{e}_1 + a_{i2}\vec{e}_2 + a_{i3}\vec{e}_3 \in \mathbb{R}^3, (i = 1, 2)$

以$\vec{a}_1, \vec{a}_2$为相邻边的平行四边形为

$$A = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2, 0 \le \lambda_1, \lambda_2 \le 1\}$$

引入外积$\vec{a}_1 \wedge \vec{a}_2$, 使其满足重线性和反交换律, 则

$$\vec{a}_1 \wedge \vec{a}_2 = (a_{11}\vec{e}_1 + a_{12}\vec{e}_2 + a_{13}\vec{e}_3) \wedge (a_{21}\vec{e}_1 + a_{22}\vec{e}_2 + a_{23}\vec{e}_3)$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \vec{e}_1 \wedge \vec{e}_2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \vec{e}_2 \wedge \vec{e}_3 + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \vec{e}_3 \wedge \vec{e}_1$$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}$刚好为$A$在各个坐标面上的投影的有向面积. (注意与叉积的异同)

## 由三维向量空间$\mathbb{R}^3$引申出来的一些空间

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$构成$V^1(=\mathbb{R}^3)$的基底 (长度空间)

$\{\vec{e}_1\wedge\vec{e}_2, \vec{e}_1\wedge\vec{e}_3, \vec{e}_2\wedge\vec{e}_3\}$构成某个向量空间$V^2$的基底

(面积空间)

$\{\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3\}$构成某个向量空间$V^3$的基底(体积空间)

$\{1\}$构成$V^0(=\mathbb{R})$的基底 (数域)

向量空间$G(\mathbb{R}^3)=V^0\oplus V^1\oplus V^2\oplus V^3$的基底为

$$\{1, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_1\wedge\vec{e}_2, \vec{e}_1\wedge\vec{e}_3, \vec{e}_2\wedge\vec{e}_3, \vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3\}$$

## 外乘和外积(外乘的运算结果)的定义

设$V$是实数域$\mathbb{R}$上的$n$维向量空间,它的一组基底是$\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_n$.

对于$p=0,1,\cdots,n$, 构造$\mathbb{R}$上新的向量空间$V^p$如下:

$V^0 \triangleq \mathbb{R}$ (基底为1) $\qquad V^1 \triangleq V$ (基底为$\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_n$)

$$V^2 \triangleq \left\{\vec{u}\wedge\vec{v} \mid \vec{u},\vec{v}\in V\right\}$$

要求其中的外乘运算“$\wedge$”满足重线性和反交换律.

称$\vec{u}\wedge\vec{v}$为$\vec{u}$与$\vec{v}$的外积.

对基底的作用: $\vec{e}_i\wedge\vec{e}_j=-\vec{e}_j\wedge\vec{e}_i$, $\quad \vec{e}_i\wedge\vec{e}_i=\vec{0}$.

若$\vec{u}=\sum\limits_{i=1}^{n}\alpha_i\vec{e}_i$, $\vec{v}=\sum\limits_{j=1}^{n}\beta_j\vec{e}_j$, 则

$$\vec{u}\wedge\vec{v}=\sum_{i,j=1}^{n}(\alpha_i\vec{e}_i)\wedge(\beta_j\vec{e}_j)=\sum_{i,j=1}^{n}\alpha_i\beta_j\vec{e}_i\wedge\vec{e}_j$$

$$=\sum_{1\le i<j\le n}\alpha_i\beta_j\vec{e}_i\wedge\vec{e}_j+\sum_{1\le i<j\le n}\alpha_j\beta_i\vec{e}_j\wedge\vec{e}_i$$

$$=\sum_{1\le i<j\le n}(\alpha_i\beta_j-\alpha_j\beta_i)\vec{e}_i\wedge\vec{e}_j.$$

可见 $V^2=\left\{\sum\limits_{1\le i<j\le n}a_{ij}\vec{e}_i\wedge\vec{e}_j\,\middle|\,a_{ij}\in\mathbb{R}\right\}$.

它是以$\vec{e}_i\wedge\vec{e}_j\,(1\le i<j\le n)$为基底的$C_n^2$维的向量空间,

称它的元素为2次外形式(2次形式, 2-形式).

相仿地可给出$V^p(1\le p\le n)$的定义,

$$V^p \triangleq \left\{\vec{u}_1\wedge\vec{u}_2\wedge\cdots\wedge\vec{u}_p \mid \vec{u}_1,\vec{u}_2,\cdots,\vec{u}_p\in V\right\}$$

要求其中的外乘运算"$\wedge$"满足重线性和反交换律.

称$\vec{u}_1\wedge\vec{u}_2\wedge\cdots\wedge\vec{u}_p$为$\vec{u}_1,\vec{u}_2,\cdots,\vec{u}_p$的外积.

$$V^p=\left\{\sum_{1\le i_1<i_2<\cdots<i_p\le n} a_{i_1i_2\cdots i_p}\vec{e}_{i_1}\wedge\vec{e}_{i_2}\wedge\cdots\wedge\vec{e}_{i_p}\ \middle|\ a_{i_1i_2\cdots i_p}\in\mathbb{R}\right\}$$

它以$\vec{e}_{i_1}\wedge\vec{e}_{i_2}\wedge\cdots\wedge\vec{e}_{i_p}(1\le i_1<i_2<\cdots<i_p\le n)$为基底.

称它的元素为$p$次外形式($p$次形式, $p$-形式).

## $V^p$与$V^q$的元素之间的外乘运算$\wedge$

设$p \geq 1, q \geq 1, p+q \leq n, \vec{x} \in V^p, \vec{y} \in V^q$,

定义$\vec{x}\wedge\vec{y}$为将$\vec{x}$和$\vec{y}$各自用基底表示后再作$\wedge$运算.

即规定基底的外乘运算满足结合律.

**例1** 设 $\vec{x} = A\vec{e}_1 + B\vec{e}_2 + C\vec{e}_3$,

$$\vec{y} = P\vec{e}_2\wedge\vec{e}_3 + Q\vec{e}_3\wedge\vec{e}_1 + R\vec{e}_1\wedge\vec{e}_2,$$

则 $\vec{x}\wedge\vec{y} = (AP + BQ + CR)\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$.

显然$\vec{x}\wedge\vec{y} \in V^{p+q}$.

规定$V^0$与$V^p$的元素之间的外乘运算$\wedge$就是数乘运算

$\forall \alpha \in V^0 = \mathbb{R}, \forall \vec{\omega} \in V^p$, 定义$\alpha\wedge\vec{\omega} = \vec{\omega}\wedge\alpha = \alpha\vec{\omega}$.

# P123 命题1(外乘的交换性质)

设 $\vec{x}\in V^p,\ \vec{y}\in V^q$, 则$\vec{x}\wedge\vec{y}=(-1)^{pq}\,\vec{y}\wedge\vec{x}.$

**证** 由线性运算性质,只需证明

$\vec{x}=\vec{e}_{i_1}\wedge\vec{e}_{i_2}\wedge\cdots\wedge\vec{e}_{i_p},\ \vec{y}=\vec{e}_{j_1}\wedge\vec{e}_{j_2}\wedge\cdots\wedge\vec{e}_{j_q}$时的情形.

此时 $\vec{x}\wedge\vec{y}=\vec{e}_{i_1}\wedge\vec{e}_{i_2}\wedge\cdots\wedge\boxed{\vec{e}_{i_p}\wedge\vec{e}_{j_1}}\wedge\vec{e}_{j_2}\wedge\cdots\wedge\vec{e}_{j_q}$

$$=-\,\vec{e}_{i_1}\wedge\vec{e}_{i_2}\wedge\cdots\wedge\vec{e}_{i_{p-1}}\wedge\boxed{\vec{e}_{j_1}\wedge\vec{e}_{i_p}}\wedge\vec{e}_{j_2}\wedge\cdots\wedge\vec{e}_{j_q}$$

$$=(-1)^p\,\vec{e}_{j_1}\wedge\vec{x}\wedge\vec{e}_{j_2}\wedge\cdots\wedge\vec{e}_{j_q}$$

$$=(-1)^{2p}\,\vec{e}_{j_1}\wedge\vec{e}_{j_2}\wedge\vec{x}\wedge\vec{e}_{j_3}\wedge\cdots\wedge\vec{e}_{j_q}$$

$$=(-1)^{qp}\,\vec{e}_{j_1}\wedge\vec{e}_{j_2}\wedge\vec{e}_{j_3}\cdots\wedge\vec{e}_{j_q}\wedge\vec{x}\ =(-1)^{pq}\,\vec{y}\wedge\vec{x}.$$

## P124 命题2 (外积的坐标表示)

设 $\vec{y}_i = \sum_{j=1}^{n} a_{ij}\vec{e}_j,\ i = 1, 2, \cdots, p$, 则

$$\vec{y}_1 \wedge \vec{y}_2 \wedge \cdots \wedge \vec{y}_p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \begin{vmatrix} a_{1i_1} & \cdots & a_{1i_p} \\ \vdots & \ddots & \vdots \\ a_{pi_1} & \cdots & a_{pi_p} \end{vmatrix} \vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge \cdots \wedge \vec{e}_{i_p}.$$

特别地, 若 $p = n$, 则

$$\vec{y}_1 \wedge \vec{y}_2 \wedge \cdots \wedge \vec{y}_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n.$$

当$p=2, n=3$时证明如下:

$$\vec{y}_1=a_{11}\vec{e}_1+a_{12}\vec{e}_2+a_{13}\vec{e}_3,\quad \vec{y}_2=a_{21}\vec{e}_1+a_{22}\vec{e}_2+a_{23}\vec{e}_3,$$

$$\vec{y}_1\wedge\vec{y}_2=(a_{11}\vec{e}_1+a_{12}\vec{e}_2+a_{13}\vec{e}_3)\wedge(a_{21}\vec{e}_1+a_{22}\vec{e}_2+a_{23}\vec{e}_3)$$

$$=\sum_{i,j=1}^{3}(a_{1i}\vec{e}_i)\wedge(a_{2j}\vec{e}_j)=\sum_{i,j=1}^{3}a_{1i}a_{2j}\vec{e}_i\wedge\vec{e}_j$$

$$=\sum_{1\le i<j\le 3}a_{1i}a_{2j}\vec{e}_i\wedge\vec{e}_j+\sum_{1\le j<i\le 3}a_{1i}a_{2j}\vec{e}_i\wedge\vec{e}_j$$

$$=\sum_{1\le i<j\le 3}a_{1i}a_{2j}\vec{e}_i\wedge\vec{e}_j-\sum_{1\le i<j\le 3}a_{1j}a_{2i}\vec{e}_i\wedge\vec{e}_j$$

$$=\sum_{1\le i<j\le 3}(a_{1i}a_{2j}-a_{1j}a_{2i})\vec{e}_i\wedge\vec{e}_j=\sum_{1\le i<j\le 3}\begin{vmatrix}a_{1i}&a_{1j}\\a_{2i}&a_{2j}\end{vmatrix}\vec{e}_i\wedge\vec{e}_j.$$

# 对于原命题的证明

$$\vec{y}_i = \sum_{j=1}^{n} a_{ij}\vec{e}_j \quad (i = 1, 2, \cdots, p)$$

$$\vec{y}_1 \wedge \vec{y}_2 \wedge \cdots \wedge \vec{y}_p = \sum_{i_1, i_2, \cdots i_p = 1}^{n} a_{1i_1} a_{2i_2} \cdots a_{pi_p} \vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge \cdots \wedge \vec{e}_{i_p}$$

表示排列$A(i_1, i_2, \cdots, i_p)$的反序数

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \left( \boxed{\sum_{A(i_1, i_2, \cdots, i_p)}} (-1)^{\boxed{[A(i_1, i_2, \cdots, i_p)]}} a_{1i_1} a_{2i_2} \cdots a_{pi_p} \right) \vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge \cdots \wedge \vec{e}_{i_p}$$

表示对于$i_1, i_2, \cdots, i_p$的任意排列求和

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \begin{vmatrix} a_{1i_1} & \cdots & a_{1i_p} \\ \vdots & \ddots & \vdots \\ a_{pi_1} & \cdots & a_{pi_p} \end{vmatrix} \vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge \cdots \wedge \vec{e}_{i_p}.$$

**例** 设2-形式 $\vec{f}=a\vec{e}_2\wedge\vec{e}_3+b\vec{e}_3\wedge\vec{e}_1+c\vec{e}_1\wedge\vec{e}_2$, $a,b,c\in\mathbb{R}$, $a\neq 0$. 试把 $\vec{f}$ 表示为两个1-形式的外积.

**解** 令 $\vec{f}=(x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3)\wedge(y_1\vec{e}_1+y_2\vec{e}_2+y_3\vec{e}_3)$

$$=(x_2y_3-x_3y_2)\vec{e}_2\wedge\vec{e}_3+(x_3y_1-x_1y_3)\vec{e}_3\wedge\vec{e}_1+(x_1y_2-x_2y_1)\vec{e}_1\wedge\vec{e}_2$$

则 $x_2y_3-x_3y_2=a$, $x_3y_1-x_1y_3=b$, $x_1y_2-x_2y_1=c$.

即$(x_1,x_2,x_3)$和$(y_1,y_2,y_3)$满足方程$ax+by+cz=0$.

它的两个线性无关的解为$(-\frac{b}{a},1,0)$和$(-\frac{c}{a},0,1)$.

而 $(-\frac{b}{a}\vec{e}_1+\vec{e}_2)\wedge(-\frac{c}{a}\vec{e}_1+\vec{e}_3)=\frac{1}{a}\vec{f}$,

故 $\vec{f}=a(-\frac{b}{a}\vec{e}_1+\vec{e}_2)\wedge(-\frac{c}{a}\vec{e}_1+\vec{e}_3)$.

# 格拉斯曼(Grassmann)代数

在$V^p(p=0,1,\cdots,n)$的基础上构造一个更大的

向量空间$G(V)=V^0\oplus V^1\oplus\cdots\oplus V^n$.

即$G(V)$的每一个元素$\vec{\omega}$都可表示为

$$\vec{\omega}=\vec{\omega}_0+\vec{\omega}_1+\cdots+\vec{\omega}_n\ (\text{其中}\,\vec{\omega}_i\in V^i)$$

并且这一表示是唯一的. (以后省掉向量符号上的箭头)

在$G(V)$内不仅具有向量空间的代数结构,而且还

带有外乘,称$G(V)$是由$V$生成的Grassmann代数.

$G(V)$的基底为……

请理解课本内容后及时独立地完成如下作业!

**3.1** 设$\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$是某个向量空间的基, 计算外积:

(1) $(x\mathrm{d}x + 7z^2\mathrm{d}y)\wedge(y\mathrm{d}x - x\mathrm{d}y + 6\mathrm{d}z)$;

(2) $(6\mathrm{d}x\wedge \mathrm{d}y + 27\mathrm{d}x\wedge\mathrm{d}z)\wedge(\mathrm{d}x + \mathrm{d}y + \mathrm{d}z)$.

**3.2** 设$V$是$n$维实向量空间, $\{e_1, e_2, \cdots, e_n\}$是它的一组基, 命 $\alpha = e_1\wedge e_2\wedge\cdots\wedge e_p$ $(0 < p < n)$. 设$V$中一个向量$v$满足$v\wedge\alpha = 0$. 求证: $v$是$e_1, e_2, \cdots, e_p$的线性组合.

**3.3** 设$V$是$n$维实向量空间, $\{e_1, e_2, \cdots, e_n\}$是它的一组基, 命$\alpha = e_1\wedge e_2 + e_3\wedge e_4 + \cdots + e_{2r-1}\wedge e_{2r}$, $2r \le n$. 求证: $\alpha^r \triangleq \underbrace{\alpha\wedge\cdots\wedge\alpha}_{r个\alpha} \neq 0, \quad \alpha^{r+1} = 0.$

## 二、外微分形式

$$\int_{\partial G}\omega=\int_G \mathrm{d}\omega$$

前面定义Grassmann代数$G(V)$时,$V$是实数域$\mathbb{R}$上的$n$维向量空间,它的一组基底是$\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_n$.

称这里的实数域$\mathbb{R}$为系数域,称$V$为模(module).

为了推广微分的定义,

将系数域$\mathbb{R}$换为定义在空间$\mathbb{R}^n$(以$x^1,\cdots,x^n$为自变量)的某开集$U$上的全体$C^\infty$类的数量函数的集合.

将模$V$取为以$\mathrm{d}x^1,\mathrm{d}x^2,\cdots,\mathrm{d}x^n$为基底的$n$维向量空间.

称以前定义的$p$-形式为$U$上的$p$次外微分形式($p$-形式).

特别地,称$U$上的1-形式为$U$上的Pfaff(普法夫)形式.

例 (1)当$n=1$时, 设数量函数的自变量为$x$, 则

0-形式是 $\omega_0 = f(x)$;

1-形式是 $\omega_1 = \varphi(x)\mathrm{d}x$. (定积分的被积表达式)

(2)当$n=2$时, 设数量函数的自变量为$(x, y)$, 则

0-形式是 $\omega_0 = f(x, y)$;

1-形式是 $\omega_1 = P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$;

(平面第二类曲线积分的被积表达式)

2-形式是 $\omega_2 = \varphi(x, y)\mathrm{d}x \wedge \mathrm{d}y$.

(二重积分的被积表达式)

(3)当$n=3$时，设数量函数的自变量为$(x,y,z)$，则

0-形式是 $\omega_0 = f(x,y,z)$;

1-形式是 $\omega_1 = P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y$

$+ R(x,y,z)\mathrm{d}z$;

(空间第二类曲线积分的被积表达式)

2-形式是 $\omega_2 = P(x,y,z)\mathrm{d}y\wedge\mathrm{d}z + Q(x,y,z)\mathrm{d}z\wedge\mathrm{d}x$

$+ R(x,y,z)\mathrm{d}x\wedge\mathrm{d}y$;

(第二类曲面积分的被积表达式)

3-形式是 $\omega_3 = \varphi(x,y,z)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$.

(三重积分的被积表达式)

设$U$是$\mathbb{R}^n$中的一个开集,

则$U$上的每一个Pfaff形式

$$\omega=\sum_{i=1}^{n}a_i\mathrm{d}x^i=a_1\mathrm{d}x^1+a_2\mathrm{d}x^2+\cdots+a_n\mathrm{d}x^n$$

(其中$a_i$是定义域为$U$的$n$元$C^\infty$类数量函数$(i=1,2,\cdots n)$)

都对应于$U$上的一个$n$元$n$维的向量场$a=(a_1,a_2,\cdots,a_n)$.

给出一组Pfaff形式$f_1,f_2,\cdots,f_p(1\le p\le n)$,

若它们对应的向量场在$U$中的每一点都是线性无关的,

则称这组Pfaff形式是线性无关的.

# P126命题3(Cartan(嘉当)引理)

给出$U$上 $p$ 个线性无关的Pfaff形式

$$f_1, f_2, \cdots, f_p (1 \le p \le n).$$

如果另有$U$上的 $p$ 个Pfaff形式$g_1, g_2, \cdots, g_p$,使得

$$f_1 \wedge g_1 + f_2 \wedge g_2 + \cdots + f_p \wedge g_p = 0,$$

则存在$U$上的$C^\infty$类的函数组$a_{ij} (i, j = 1, 2, \cdots, p)$使得

$$g_i = a_{i1} f_1 + a_{i2} f_2 + \cdots + a_{ip} f_p \ (i = 1, 2, \cdots, p),$$

并且$a_{ij} = a_{ji}$.

**证** 逐点证明. 对于$U$上的一固定点来说，当$p=n$时，

因 $f_1, f_2, \cdots, f_n$线性无关，所以可以选作空间$V$的基底.

$g_i (i=1,\cdots,n)$作为$V$中的向量，存在$a_{ij}$使得$g_i=\sum_{j=1}^{n} a_{ij} f_j$.

将等式两端都在左边外乘 $f_i$得 $f_i \wedge g_i = \sum_{j=1}^{n} a_{ij} f_i \wedge f_j$.

代入引理的条件得$0=\sum_{i=1}^{n} f_i \wedge g_i = \sum_{i,j=1}^{n} a_{ij} f_i \wedge f_j$

$$=\sum_{1\le i<j\le n} a_{ij} f_i \wedge f_j - \sum_{1\le i<j\le n} a_{ji} f_i \wedge f_j = \sum_{1\le i<j\le n} (a_{ij}-a_{ji}) f_i \wedge f_j.$$

而 $f_i \wedge f_j (1\le i<j\le n)$是$V^2$的基底，所以$a_{ij}=a_{ji}$.

当$p<n$时，因 $f_1, f_2, \cdots, f_p$线性无关,

所以可以补充$f_{p+1}, \cdots, f_n$,使其作为空间$V$的基底.

令 $g_{p+1}=g_{p+2}=\cdots=g_n=0$,

则$\sum_{i=1}^{n} f_i \wedge g_i=\sum_{i=1}^{p} f_i \wedge g_i+\sum_{i=p+1}^{n} f_i \wedge 0=0.$

根据$p=n$时的结论,存在$a_{ij}$使得$g_i=\sum_{j=1}^{n} a_{ij} f_j$,且$a_{ij}=a_{ji}$.

因为当$i>p$时 $g_i=0$，所以$a_{ij}=0=a_{ji}\ (\forall i>p)$.

从而$g_i=\sum_{j=1}^{n} a_{ij} f_j=\sum_{j=1}^{p} a_{ij} f_j+\sum_{j=p+1}^{n} 0 f_j=\sum_{j=1}^{p} a_{ij} f_j$.

## 多元微积分的基本公式(Stokes公式)：$\int_{\partial G}\omega=\int_G \mathrm{d}\omega$

$$f(x)\Big|_a^b=\int_a^b f'(x)\,\mathrm{d}x$$

$$\int_{\partial\Omega}P\,\mathrm{d}x+Q\,\mathrm{d}y=\int_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$$

$$\int_{\partial\Omega}P\,\mathrm{d}y\wedge\mathrm{d}z+Q\,\mathrm{d}z\wedge\mathrm{d}x+R\,\mathrm{d}x\wedge\mathrm{d}y$$

$$=\int_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

$$\int_{\partial S}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=$$

$$\int_S\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$$

## 外微分的引入

对于$\omega = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$,

若定义$\mathrm{d}\omega = \mathrm{d}P(x,y)\wedge\mathrm{d}x + \mathrm{d}Q(x,y)\wedge\mathrm{d}y$, 则有

$$\mathrm{d}\omega = \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y\right)\wedge\mathrm{d}x + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y\right)\wedge\mathrm{d}y$$

$$= \frac{\partial P}{\partial y}\mathrm{d}y\wedge\mathrm{d}x + \frac{\partial Q}{\partial x}\mathrm{d}x\wedge\mathrm{d}y = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$$

Green公式

$\int_{\partial\Omega} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$ 变为 $\int_{\partial\Omega}\omega = \int_{\Omega}\mathrm{d}\omega.$

# 外微分的引入(续)

对于$\omega = P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z$,

若定义$\mathrm{d}\omega = \mathrm{d}P\wedge \mathrm{d}x + \mathrm{d}Q\wedge \mathrm{d}y + \mathrm{d}R\wedge \mathrm{d}z$,

则有

$$
\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}y \\
&\quad + \left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}z \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge \mathrm{d}y
\end{aligned}
$$

Stokes公式变为$\int_{\partial\Omega}\omega = \int_{\Omega}\mathrm{d}\omega$.

# 外微分的引入(续)

对于$\omega = P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y$,

若定义$\mathrm{d}\omega = \mathrm{d}P \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}Q \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}R \wedge \mathrm{d}x \wedge \mathrm{d}y$,

则有 $$\mathrm{d}\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

Gauss公式变为$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega.$

教材P129有Stokes公式的严格叙述.

**What happens on the outside is purely a function of the change within.**

## 外微分的定义

设$\omega \in V^p$，即$\omega = \sum_{1\le i_1<i_2<\cdots<i_p\le n} a_{i_1 i_2\cdots i_p} \mathrm{d}x^{i_1}\wedge \mathrm{d}x^{i_2}\wedge\cdots\wedge \mathrm{d}x^{i_p}$.

定义$\mathrm{d}\omega$(称d为外微分算子)为

$$\mathrm{d}\omega = \sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathrm{d}a_{i_1 i_2\cdots i_p} \wedge \mathrm{d}x^{i_1}\wedge \mathrm{d}x^{i_2}\wedge\cdots\wedge \mathrm{d}x^{i_p}$$

$$= \sum_{1\le i_1<i_2<\cdots<i_p\le n} \sum_{i=1}^{n} \frac{\partial a_{i_1 i_2\cdots i_p}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^{i_1}\wedge \mathrm{d}x^{i_2}\wedge\cdots\wedge \mathrm{d}x^{i_p}$$

显然当$0\le p<n$时$\mathrm{d}\omega \in V^{p+1}$；当$\omega\in V^n$时$\mathrm{d}\omega = 0$.

设$\omega_1 \in V^p, \omega_2 \in V^q$，定义$\mathrm{d}(\omega_1+\omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$.

## 外微分的运算规则

### ①外微分是普通微分的推广

若$\omega \in V^0$，则外微分 $\mathrm{d}\omega$就是$\omega$的普通微分.

### ②线性运算性质

若$\lambda$和$\mu$是常数，$\omega, \theta \in V^p$，则 $\mathrm{d}(\lambda\omega + \mu\theta) = \lambda \mathrm{d}\omega + \mu \mathrm{d}\theta$.

例 若 $\omega = (x^2 - z^2)\mathrm{d}x + (2xy + z)\mathrm{d}z$,

$$
\begin{aligned}
\text{则 } \mathrm{d}\omega &= \mathrm{d}(x^2 - z^2) \wedge \mathrm{d}x + \mathrm{d}(2xy + z) \wedge \mathrm{d}z \\
&= (2x\mathrm{d}x - 2z\mathrm{d}z) \wedge \mathrm{d}x + (2y\mathrm{d}x + 2x\mathrm{d}y + \mathrm{d}z) \wedge \mathrm{d}z \\
&= -2z\mathrm{d}z \wedge \mathrm{d}x + 2y\mathrm{d}x \wedge \mathrm{d}z + 2x\mathrm{d}y \wedge \mathrm{d}z \\
&= 2(y + z)\mathrm{d}x \wedge \mathrm{d}z + 2x\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

## ③乘积法则(P130命题5)

设$\omega\in V^p,\theta\in V^q$，则$\mathrm{d}(\omega\wedge\theta)=\mathrm{d}\omega\wedge\theta+(-1)^p\omega\wedge\mathrm{d}\theta$.

**证** 由外微分的线性运算性质，

只需证明$\omega,\theta$为单项式的情形.

设$\omega=a(x^1,\cdots,x^n)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}$,

$\theta=b(x^1,\cdots,x^n)\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}$, 则

$$\begin{aligned}\mathrm{d}(\omega\wedge\theta)&=\mathrm{d}(ab\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q})\\&=\mathrm{d}(ab)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}\\&=(b\mathrm{d}a+a\mathrm{d}b)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}\end{aligned}$$

$$\mathrm{d}(\omega\wedge\theta)=\cdots$$

$$=(b\mathrm{d}a+a\mathrm{d}b)\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}$$

$$=b\mathrm{d}a\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}$$
$$+a\mathrm{d}b\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}$$

$$=(\mathrm{d}a\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p})\wedge(b\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q})$$
$$+\mathrm{d}b\wedge(a\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p})\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q}$$

$$=\mathrm{d}\omega\wedge\theta+(-1)^p(a\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p})\wedge(\mathrm{d}b\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_q})$$

$$=\mathrm{d}\omega\wedge\theta+(-1)^p\omega\wedge\mathrm{d}\theta$$

## ④Poincaré(庞加莱)引理(P130命题4)

设$\omega \in G(V)$, 则$\mathrm{dd}\omega \triangleq \mathrm{d}(\mathrm{d}\omega) = 0.$

**证** 由外微分的线性运算性质,只需证明$\omega$为单项式的情形.

设$\omega = f(x^1, \cdots, x^n)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$, 则

$$\mathrm{d}\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p},$$

$$\mathrm{dd}\omega = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$$

$$= \sum_{0 \le j < i \le n} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right) \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = 0.$$

请理解课本内容后及时独立地完成如下作业!

**3.4** 设$\omega=\sum\limits_{1\le i<j\le n}a_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j$, $a_{ij}+a_{ji}=0$. 求证:

$$\mathrm{d}\omega=\sum_{1\le i<j<k\le n}\left(\frac{\partial a_{ij}}{\partial x^k}+\frac{\partial a_{jk}}{\partial x^i}+\frac{\partial a_{ki}}{\partial x^j}\right)\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k.$$

**3.5** 设$\varphi=yz\mathrm{d}x+\mathrm{d}z$, $\xi=\sin z\mathrm{d}x+\cos z\mathrm{d}y$,

$\eta=\mathrm{d}y+z\mathrm{d}z$, 计算

(1) $\varphi\wedge\xi$, $\xi\wedge\eta$, $\eta\wedge\varphi$;　(2) $\mathrm{d}\varphi$, $\mathrm{d}\xi$, $\mathrm{d}\eta$.

**3.6** 设$f$和$g$是两个光滑函数, d为外微分算子, 计算

(1) $\mathrm{d}(f\mathrm{d}g+g\mathrm{d}f)$;　(2) $\mathrm{d}[(f-g)(\mathrm{d}f+\mathrm{d}g)]$;

(3) $\mathrm{d}[(f\mathrm{d}g)\wedge(g\mathrm{d}f)]$;　(4) $\mathrm{d}(g\mathrm{d}f)+\mathrm{d}(f\mathrm{d}g)$.

## 三、Frobenius(弗罗贝尼乌斯)定理

先看一个例子

考虑$\mathbb{R}^3$中的一个Pfaff方程

$$\omega = P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z = 0$$

由一阶偏微分方程的理论(略),它的完全可积条件为

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \quad (*)$$

注: 完全可积是指存在$\mathbb{R}^3$中的曲面 $f(x,y,z) =$ 常数, 使得$\mathrm{d}f = 0$与$\omega = 0$等价, 即$\exists\lambda(x,y,z)$使得 $\mathrm{d}f = \lambda\omega$.

命题 条件(*) $\Leftrightarrow$ 存在Pfaff形式$\theta$使 $\mathrm{d}\omega = \theta\wedge\omega$.

**证（必要性）**

若$P, Q, R$都为零，则命题显然成立．不妨设$R \neq 0$，则

由$\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$得$\mathrm{d}z = \frac{1}{R}(\omega - P\mathrm{d}x - Q\mathrm{d}y)$.

将$\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$求外微分得

$$\mathrm{d}\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y \wedge \frac{1}{R}(\omega - P\mathrm{d}x - Q\mathrm{d}y)$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{1}{R}(\omega - P\mathrm{d}x - Q\mathrm{d}y) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y$$

记为$\theta$

$$= \left[\boxed{\frac{1}{R}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathrm{d}x + \frac{1}{R}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y}\right] \wedge \omega$$

(*)式左边

$$+ \frac{1}{R}\left[\boxed{P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}\right]\mathrm{d}x \wedge \mathrm{d}y.$$

因此 $d\omega=\theta\wedge\omega$, 其中 $\theta=\frac{1}{R}(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z})dx+\frac{1}{R}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dy$.

**(充分性)** 存在Pfaff形式 $\theta$ 使 $d\omega=\theta\wedge\omega$,

设 $\theta=a(x,y,z)dx+b(x,y,z)dy+c(x,y,z)dz$.

$$d\omega=(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dy\wedge dz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dz\wedge dx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dx\wedge dy.$$

$$\theta\wedge\omega=(bR-cQ)dy\wedge dz+(cP-aR)dz\wedge dx+(aQ-bP)dx\wedge dy.$$

而 $dy\wedge dz, dz\wedge dx, dx\wedge dy$ 为 $V^2$ 的基底, 线性无关, 因此

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=bR-cQ,\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=cP-aR,\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=aQ-bP.$$

从中消去 $a,b,c$ 得到

$$P(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})+Q(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})+R(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})=0.$$

## 推广到高维情形

设$\mathbb{R}^{n+p}$中的坐标是$(x^1, x^2, \cdots, x^n, y^1, y^2, \cdots, y^p)$,

简记为$(x, y)$,

$U$是$\mathbb{R}^{n+p}$中的一个开集,

$$\omega^l \equiv \sum_{i=1}^{n} \psi_i^l(x, y)\mathrm{d}x^i + \sum_{k=1}^{p} \psi_{n+k}^l(x, y)\mathrm{d}y^k \ (l = 1, 2, \cdots, p)$$

是$U$上的$p$个线性无关($\det(\psi_{n+k}^l) \neq 0$)的Pfaff形式,

它们确定了$U$上的**Pfaff方程组**

$$\omega^l = 0,\ l = 1, \cdots, p.$$

如果存在$U$中的某个$p$维空间中的$n$维曲面$S$($n$元函数)

$$y^k = y^k(x^1, x^2, \cdots, x^n), k = 1, 2, \cdots, p$$

使得将$y^k(x)$代入$\omega^l(l = 1, 2, \cdots, p)$以后有$\omega^l \equiv 0$,即

$$\sum_{i=1}^{n} \psi_i^l(x, y(x))\mathrm{d}x^i + \sum_{k=1}^{p} \psi_{n+k}^l(x, y(x))\left(\sum_{i=1}^{n} \frac{\partial y^k}{\partial x^i}\mathrm{d}x^i\right) \equiv 0$$

则称曲面$S$为Pfaff方程组$\omega^l = 0(l = 1, 2, \cdots, p)$的

**积分曲面**,称$y = y(x)$为**Pfaff方程组的解**.

如果过$U$中任一点只存在唯一的积分曲面,则称该

Pfaff方程组是**完全可积**的.

**定义**

设有$U$上的$p$个Pfaff形式$\omega^l(l=1,2,\cdots,p)$，

如果存在$U$上的Pfaff形式$f_k^l(k,l=1,2,\cdots,p)$使得

$$\mathrm{d}\omega^l \equiv \sum_{k=1}^{p} f_k^l \wedge \omega^k, \ l=1,2,\cdots,p,$$

则称$\omega^l(l=1,2,\cdots,p)$满足**Frobenius条件**.

# Frobenius定理

设$\mathbb{R}^{n+p}$的坐标是$(x^1, x^2, \cdots, x^n, y^1, y^2, \cdots, y^p)$, $U$是$\mathbb{R}^{n+p}$中的一个开集, $\omega^l (l = 1, 2, \cdots, p)$是$U$上的$p$个Pfaff形式, 则Pfaff方程组

$$\omega^l \equiv \sum_{i=1}^{n} \psi_i^l(x, y)\mathrm{d}x^i + \sum_{k=1}^{p} \psi_{n+k}^l(x, y)\mathrm{d}y^k = 0$$

(其中$l = 1, 2, \cdots, p$, $\det(\psi_{n+k}^l) \neq 0$)

完全可积的充要条件是$\omega^l$满足Frobenius条件.