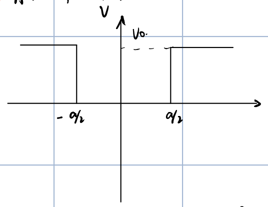
有限深势阱



$$V = \begin{cases} 0, & |x| < \frac{a}{2} \\ V_0, & \text{其他.} \end{cases}$$

只讨论 束缚态情形. $E < V_0$

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi = 0$$

$$k' = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} > 0$$

$$\psi \propto e^{\pm k'x} \quad \begin{cases} e^{-k'x} & x > \frac{a}{2} \\ e^{k'x} & x < \frac{a}{2} \end{cases}$$

势阱内部. $\psi'' + k^2\psi = 0$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \psi \sim \begin{cases} \cos kx & \text{偶宇称} \\ \sin kx & \text{奇宇称} \end{cases}$$

波函数及其一阶导数连续.

构造 $\frac{\psi'}{\psi}$ (偶宇称时)

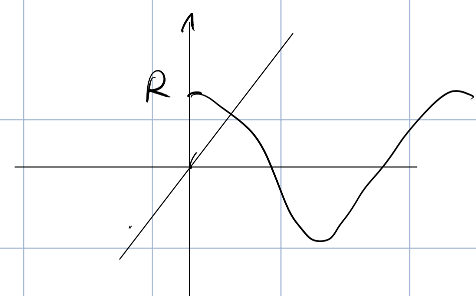
$$\frac{\psi'}{\psi} = \frac{-k\sin kx}{\cos kx} = \frac{-k' e^{-k'x}}{e^{-k'x}} \Rightarrow \tan kx\Big|_{x=\frac{a}{2}} = \frac{k'}{k}.$$

注意到 $\begin{cases} k' = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \\ k = \sqrt{\frac{2m}{\hbar^2}E} \end{cases}$ $\quad k'^2 + k^2 = \frac{2mV_0}{\hbar^2}$

定义 $k\frac{a}{2} = \xi$, $k'\frac{a}{2} = \eta$ $\quad \begin{cases} \xi\tan\xi = \eta \\ \xi^2 + \eta^2 = \frac{2m}{\hbar^2}V_0 \cdot \frac{a^2}{4} \end{cases}$

$$\xi^2(1 + \tan^2\xi) = R^2$$

$\sec^2\xi \cdot \xi^2 = R^2$ $\quad \xi > 0 \quad \frac{\xi}{\cos\xi} = R, \quad \xi = R\cos\xi.$



一维谐振子.

$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \quad \omega = \sqrt{\frac{k}{m}}$

$H = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2x^2\right)$

$\psi'' + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2x^2\right)\psi = 0 \qquad \left(\frac{2m}{\hbar^2}E - \overset{\alpha^4}{\frac{m^2\omega^2}{\hbar^2}}x^2\right)\psi$

$\alpha = \sqrt{\frac{m\omega}{\hbar}} \quad \xi = \alpha x \quad \lambda = \frac{E}{\frac{1}{2}\hbar\omega} \qquad \frac{\cancel{2m}/\cancel{\hbar^2}}{\frac{1}{2}\frac{\cancel{m\omega}}{\cancel{\hbar}}\hbar^2} = \frac{E}{\frac{1}{2}\hbar\omega}$

$\Downarrow$

$$\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0$$

$\xi \to \infty$ 时, 渐近 方程 $\psi'' - \xi^2\psi(\xi) = 0.$

$$\Rightarrow \psi = C_1 e^{\frac{1}{2}\xi^2} + C_2 e^{-\frac{1}{2}\xi^2}$$
$$= C \cdot e^{-\frac{1}{2}\xi^2} \quad (\xi \to \infty,\ \psi \to 0)$$

变换. $\psi = e^{-\frac{1}{2}\xi^2} \cdot u(\xi)$ 　常数变易

代入原方程.

$$u(\xi)'' - 2\xi u(\xi)' + (\lambda - 1) u(\xi) = 0$$

设 $u(\xi) = \sum c_n \cdot \xi^n$

只有 $\lambda = 2n+1$, $u(\xi)$ 才不发散.

$$\lambda = 2n+1 = \frac{E}{\frac{1}{2}\hbar\omega} \qquad E = \left(n + \frac{1}{2}\right)\hbar\omega$$

基态 $E_0 = \frac{1}{2}\hbar\omega$

第一激发态. $\frac{3}{2}\hbar\omega$

本征态. 厄密方程. $u'' - 2\xi u' + 2n u = 0$

$$\Rightarrow H_n = (-1)^n e^{\xi^2} \left(e^{-\xi^2}\right)^{(n)}.$$

$$\begin{cases} H_0 = 1 \\ H_1 = 2\xi \\ H_2 = 4\xi^2 - 2 \\ H_3 = 8\xi^3 - 12\xi \\ \quad \vdots \end{cases}$$

递推关系 $\begin{cases} \frac{d}{d\xi} H_n(\xi) = 2n H_{n-1}(\xi) \\ H_{n+1}(\xi) - 2\xi H_n(\xi) + 2n H_{n-1}(\xi) = 0 \end{cases}$

正交性(带权) $$\int_{-\infty}^{+\infty} H_m(\xi) H_n(\xi) e^{-\xi^2} d\xi = \sqrt{\pi}\, 2^n n!\, \delta_{mn}$$

$$\psi_n = N_n e^{-\frac{1}{2}\alpha^2 x^2} H_n(\alpha x)$$

归一化. $\int_{-\infty}^{\infty} |\psi_n|^2 dx$ 求得 $N_n = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{\frac{1}{2}}$