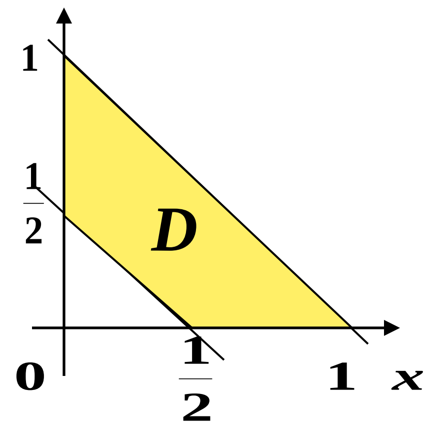
练习三十

一、选择题

(1) 设平面区域D由四条直线$x=0, y=0, x+y=\frac{1}{2}$, $x+y=1$围成,并记$I_1=\iint\limits_D [\ln(x+y)]^7 d\sigma$，$I_2=\iint\limits_D (x+y)^7 d\sigma$，$I_3=\iint\limits_D [\sin(x+y)]^7 d\sigma$，则 [C]

(A) $I_1 < I_2 < I_3$　　(B) $I_3 < I_2 < I_1$

(C) $I_1 < I_3 < I_2$　　(D) $I_3 < I_1 < I_2$



解：在D内 ,$0 < x+y < 1$,

由 $\ln(x+y)^7 < 0 < \sin(x+y)^7 < (x+y)^7$

得 $\iint\limits_D [\ln(x+y)^7] d\sigma < 0 < \iint\limits_D [\sin(x+y)^7] d\sigma < \iint\limits_D (x+y)^7 d\sigma$

即 $I_1 < I_3 < I_2$　　故选（C）

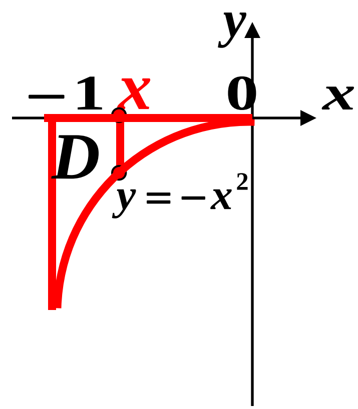
(2) 设 D由曲线$x=-\sqrt{-y}$及直线$x=-1, y=0$围成，

则$\iint\limits_D f(x,y)d\sigma=$ [**B**]

(A) $\int_{-1}^{0}dx\int_{0}^{-x^2} f(x,y)dy$; (B) $\int_{-1}^{0}dx\int_{-x^2}^{0} f(x,y)dy$;

(C) $\int_{0}^{-1}dx\int_{-x^2}^{0} f(x,y)dy$; (D) $\int_{-1}^{0}dy\int_{-\sqrt{-y}}^{0} f(x,y)dx$

解：积分区域D如图所示,



积分次序先y后x,

则所求积分$=\int_{-1}^{0}dx\int_{-x^2}^{0} f(x,y)dy$

故选（B）

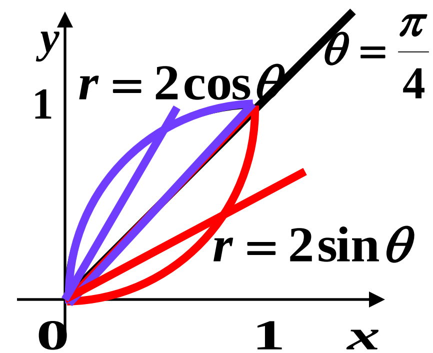
(3) $\int_0^1 dx\int_{1-\sqrt{1-x^2}}^{\sqrt{2x-x^2}} f(x^2+y^2)dy$在极坐标下的二次积分为[D]

(A) $\int_0^{\frac{\pi}{2}} d\theta\int_{2\sin\theta}^{2\cos\theta} f(\rho^2)d\rho$; (B) $\int_0^{\frac{\pi}{2}} d\theta\int_{2\cos\theta}^{2\sin\theta} f(\rho^2)d\rho$;

(C) $\int_0^{\frac{\pi}{2}} d\theta\int_0^{\sqrt{2}} f(\rho^2)d\rho$; (D) $[\int_0^{\frac{\pi}{4}} d\theta\int_0^{2\sin\theta} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta}] f(\rho^2)d\rho$

解：积分区域为D:$\begin{cases} 0<x<1 \\ 1-\sqrt{1-x^2}<y<\sqrt{2x-x^2} \end{cases}$，如图所示

$$原积分=\int_0^{\frac{\pi}{4}} d\theta\int_0^{2\sin\theta} f(\rho^2)d\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} f(\rho^2)d\rho$$



故选（D）

（4）若 $\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\cos\theta}f(\rho\cos\theta,\rho\sin\theta)\mathrm{d}\rho$，则积分区域 D 为 （C ）

（A） $x^2+y^2\le a^2\ (a>0)$;

（B） $x^2+y^2\le a^2, x\ge 0\ (a>0)$;

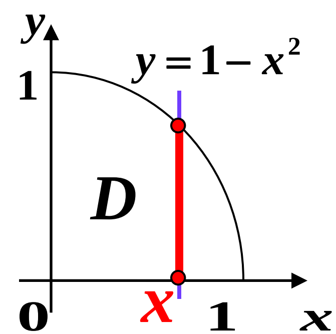
（C） $x^2+y^2\le ax\ (a>0)$; （D） $x^2+y^2\le ax\ (a<0)$.

(5) 设 $f(x,y)$是连续函数，则$\int_0^1 dy\int_0^{\sqrt{1-y}} f(x,y)dx =$ [C]

(A) $\int_0^1 dx\int_0^{\sqrt{1-x}} f(x,y)dy$;　(B) $\int_0^{\sqrt{1-y}} dx\int_0^1 f(x,y)dy$;

(C) $\int_0^1 dx\int_0^{1-x^2} f(x,y)dy$;　(D) $\int_0^1 dx\int_0^{1+x^2} f(x,y)dy$

解：积分区域为D:$\begin{cases} 0 < x < \sqrt{1-y}, \\ 0 < y < 1 \end{cases}$ 如图所示



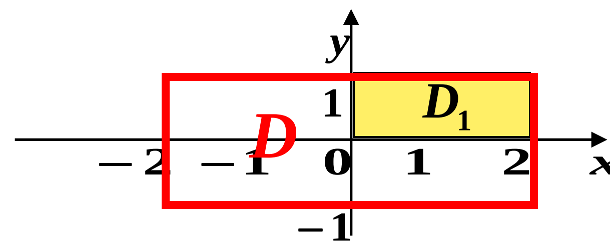
交换积分次序（先y后x），

原积分$= \int_0^1 dx\int_0^{1-x^2} f(x,y)dy$　　故选（C）

二、填空题

(1)设$D: |x| \le 2, |y| \le 1$，则$\iint\limits_D \frac{1}{1+y^2} d\sigma =$ <u>2π</u>

解：积分区域D如图所示,



D关于x轴、y轴都是对称的,

D在第一象限的部分记为D_1

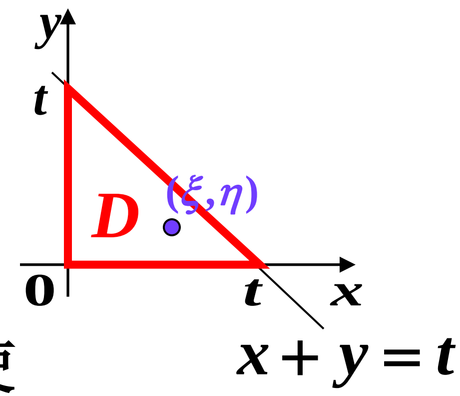
$\because$ 被积函数$\frac{1}{1+y^2}$关于x、y都是偶函数,

$$\therefore \iint\limits_D \frac{1}{1+y^2} d\sigma = 4\iint\limits_{D_1} \frac{1}{1+y^2} d\sigma = 4\int_0^2 dx \int_0^1 \frac{1}{1+y^2} dy$$

$$= 4\int_0^2 \left(\arctan y\Big|_0^1\right) dx = 4\int_0^2 \frac{\pi}{4} dx = 2\pi$$

$$(2)\lim_{t\to 0+}\frac{1}{t^2}\int_0^t dy\int_0^{t-y}\frac{dx}{1+x^4+y^4}=\underline{\quad\frac{1}{2}\quad}$$

解：积分区域D如图所示,



D的面积 $S_D=\frac{1}{2}t^2$

由积分中值定理，存在$(\xi,\eta)\in D$,使

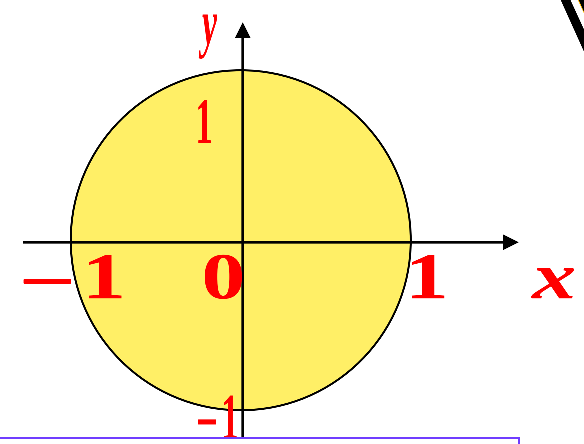
$$\int_0^t dy\int_0^{t-y}\frac{dx}{1+x^4+y^4}=\frac{1}{1+\xi^4+\eta^4}S_D=\frac{t^2}{2(1+\xi^4+\eta^4)}$$

当$t\to 0+$时，有$\xi\to 0+$，$\eta\to 0+$

$$\therefore \lim_{t\to 0+}\frac{1}{t^2}\int_0^t dy\int_0^{t-y}\frac{dx}{1+x^4+y^4}=\lim_{\substack{\xi\to 0+\\ \eta\to 0+}}\frac{1}{2(1+\xi^4+\eta^4)}=\frac{1}{2}$$

(3) $\iint_{x^2+y^2\le 1}(x+y)^2 e^{x^2+y^2}dxdy =$ __ π __

解：积分区域D如图所示，



D关于x轴、y轴都是对称的，

$$\therefore \iint_{x^2+y^2\le 1}(x+y)^2 e^{x^2+y^2}dxdy$$

$$= \iint_{x^2+y^2\le 1}(x^2+y^2)e^{x^2+y^2}dxdy + \underset{\underset{0}{\parallel}}{\boxed{\iint_{x^2+y^2\le 1}2xye^{x^2+y^2}dxdy}}$$

$$= \int_0^{2\pi}d\theta\int_0^1 \rho^2 e^{\rho^2}\rho\, d\rho$$

$$= 2\pi\cdot\frac{1}{2} = \pi$$

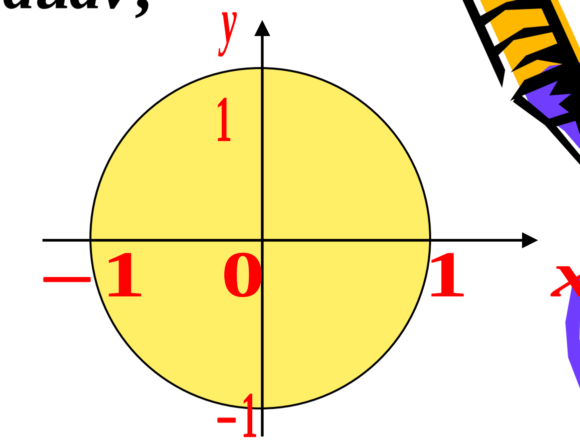
（4）设积分区域 $D: 0\le x\le 3, 0\le y\le 1$，则二重积分

$$\iint\limits_D \min(x,y)\mathrm{d}x\mathrm{d}y = \underline{\qquad\qquad\qquad\qquad}. \quad \frac{4}{3}$$

(5)若连续函数$f(x,y)$满足

$$f(x,y)=5(x^2+y^2)^{\frac{3}{2}}-\iint_{u^2+v^2\le 1} f(u,v)dudv,$$

则$\iint_{x^2+y^2\le 1} f(x,y)dxdy=$ $\underline{\quad\dfrac{2\pi}{\pi+1}\quad}$



解：积分区域D如图所示,

等式两边同时在D上求积分,

记所求积分$\iint_{x^2+y^2\le 1} f(x,y)dxdy=I$，则

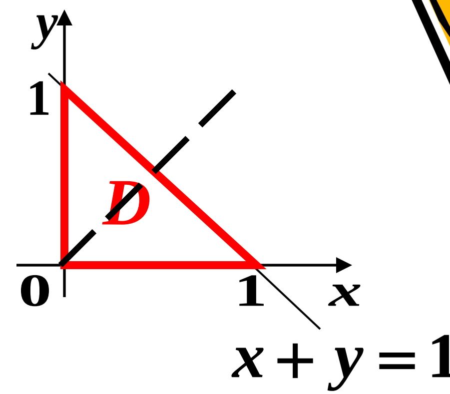
$$I=5\iint_{x^2+y^2\le 1}(x^2+y^2)^{\frac{3}{2}}dxdy-\iint_{x^2+y^2\le 1} Idxdy$$

$$(\pi+1)I=5\int_0^{2\pi}d\theta\int_0^1\rho^3\cdot\rho\, d\rho \ =2\pi$$

$$\therefore I=\frac{2\pi}{\pi+1}$$

三、 设 D 由 $x+y=1, x=0, y=0$ 围成，计算二重积分

$$\iint\limits_D \sqrt[3]{x^3-y^3}\,d\sigma$$



解：积分区域 D 如图所示(关于 $y=x$ 对称)

由于积分值与积分变量无关，故

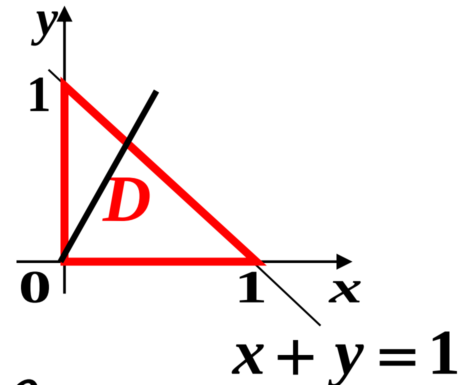
$$\iint\limits_D \sqrt[3]{x^3-y^3}\,d\sigma = \iint\limits_D \sqrt[3]{y^3-x^3}\,d\sigma$$

$$= -\iint\limits_D \sqrt[3]{x^3-y^3}\,d\sigma$$

$$\therefore \iint\limits_D \sqrt[3]{x^3-y^3}\,d\sigma = 0$$

四、求$\iint\limits_D e^{\frac{y}{x+y}}dxdy$，其中$D$由直线$x=0$，$y=0$，$x+y=1$围成。

解：积分区域D如图所示，在极坐标下



$$原积分=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}}e^{\frac{\sin\theta}{\cos\theta+\sin\theta}}\rho d\rho$$

$$=\int_0^{\frac{\pi}{2}}e^{\frac{\sin\theta}{\cos\theta+\sin\theta}}\cdot\frac{1}{2(\cos\theta+\sin\theta)^2}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{\frac{\sin\theta}{\cos\theta+\sin\theta}}d\left(\frac{\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$=\frac{1}{2}e^{\frac{\sin\theta}{\cos\theta+\sin\theta}}\Bigg|_0^{\pi/2}=\frac{1}{2}(e-1)$$

五、 计算二重积分 $\iint\limits_D \dfrac{\ln(1+x)\ln(1+y)}{1+x^2+y^2+x^2y^2}\mathrm{d}x\mathrm{d}y$ ，

其中 $D: 0 \le x \le 1, 0 \le y \le 1.$

解：$I = \iint\limits_D \dfrac{\ln(1+x)\ln(1+y)}{(1+x^2)(1+y^2)}\mathrm{d}x\mathrm{d}y$

$$= \int_0^1 \frac{\ln(1+x)}{(1+x^2)}\mathrm{d}x \int_0^1 \frac{\ln(1+y)}{(1+y^2)}\mathrm{d}y$$

$$= \left[\int_0^1 \frac{\ln(1+x)}{(1+x^2)}\mathrm{d}x\right]^2$$

令 $M=\int_0^1 \frac{\ln(1+x)}{(1+x^2)}\mathrm{d}x \qquad (x=\tan t)$

$$=\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan t)}{(1+\tan^2 t)}\cdot \sec^2 t\,\mathrm{d}t$$

$$\boxed{=\int_0^{\frac{\pi}{4}} \ln(1+\tan t)\,\mathrm{d}t} \qquad \left(t=\frac{\pi}{4}-u\right)$$

$$=\int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)\cdot(-1)\,\mathrm{d}u=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan u}{1+\tan u}\right)\mathrm{d}u$$

$$=\int_0^{\frac{\pi}{4}} \ln 2\,\mathrm{d}u-\boxed{\int_0^{\frac{\pi}{4}} \ln(1+\tan u)\mathrm{d}u}$$

$$\therefore M=\frac{\pi}{8}\ln 2 \qquad \therefore I=M^2=\frac{\pi^2}{64}\ln^2 2$$

六、计算二重积分 $\iint\limits_D |3x+4y|\,\mathrm{d}x\mathrm{d}y$，其中 $D: x^2+y^2\le 1$

解: 在极坐标系下，积分区域 $D: 0\le\theta\le 2\pi, 0\le\rho\le 1$.

$$\therefore I=\iint\limits_D |3\rho\cos\theta+4\rho\sin\theta|\,\rho\,\mathrm{d}\rho\mathrm{d}\theta$$

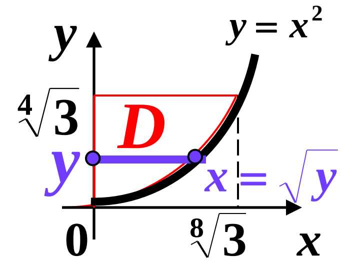
$$=\int_0^{2\pi}|3\cos\theta+4\sin\theta|\,\mathrm{d}\theta\int_0^1\rho^2\mathrm{d}\rho=\frac{5}{3}\int_0^{2\pi}\left|\frac{3}{5}\cos\theta+\frac{4}{5}\sin\theta\right|\mathrm{d}\theta$$

$$=\frac{5}{3}\int_0^{2\pi}|\sin(\theta+\theta_0)|\,\mathrm{d}\theta\qquad\left(\theta_0=\arcsin\frac{3}{5}\right)$$

$$=\frac{5}{3}\int_{\theta_0}^{2\pi+\theta_0}|\sin t|\,\mathrm{d}t\qquad（令 t=\theta+\theta_0）$$

$$=\frac{5}{3}\int_0^{2\pi}|\sin t|\,\mathrm{d}t=\frac{10}{3}\int_0^{\pi}\sin t\,\mathrm{d}t=\frac{20}{3}$$

七、 计算二次积分$\int_0^{3^{1/8}} dx \int_{x^2}^{3^{1/4}} \dfrac{xy^2}{\sqrt{1+y^4}} dy$



解：积分区域为D:$\begin{cases} 0 < x < \sqrt[8]{3}, \\ x^2 < y < \sqrt[4]{3} \end{cases}$ 如图所示

交换积分次序,先x后y，则 $D:\begin{cases} 0 < x < \sqrt{y} \\ 0 < y < \sqrt[4]{3} \end{cases}$

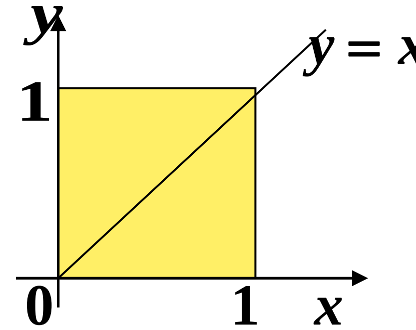
$$\text{则原积分} = \int_0^{3^{1/4}} dy \int_0^{\sqrt{y}} \frac{xy^2}{\sqrt{1+y^4}} dx$$

$$= \int_0^{\sqrt[4]{3}} \frac{y^2}{\sqrt{1+y^4}} \cdot \frac{1}{2} y\, dy$$

$$= \frac{1}{4}\sqrt{1+y^4}\Big|_0^{\sqrt[4]{3}} = \frac{1}{4}(2-1) = \frac{1}{4}$$

八、 求$\int_0^1 dx\int_0^1 \frac{\cos x}{\cos x+\cos y}dy$

解：积分区域D如图所示(关于$y=x$对称)



由于积分值与积分变量无关，

对换 x与 y，得

$$\int_0^1 dx\int_0^1 \frac{\cos x}{\cos y+\cos x}dy=\int_0^1 dy\int_0^1 \frac{\cos y}{\cos y+\cos x}dx$$

（交换积分次序）

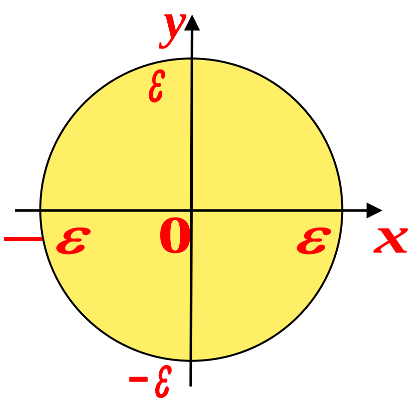
$$=\int_0^1 dx\int_0^1 \frac{\cos y}{\cos x+\cos y}dy$$

$$2\int_0^1 dx\int_0^1 \frac{\cos x}{\cos y+\cos x}dy=\int_0^1 dx\int_0^1 [\frac{\cos x+\cos y}{\cos x+\cos y}]dy=1$$

$$\therefore \int_0^1 dx\int_0^1 \frac{\cos x}{\cos y+\cos x}dy=\frac{1}{2}$$

九、试证明积分$\iint_{x^2+y^2\le\varepsilon^2}\dfrac{d\sigma}{x^2+y^2+\varepsilon^2}(\varepsilon>0)$之值与$\varepsilon$无关。

证明：$$\iint_{x^2+y^2\le\varepsilon^2}\frac{d\sigma}{x^2+y^2+\varepsilon^2}=\int_0^{2\pi}d\theta\int_0^{\varepsilon}\frac{1}{\rho^2+\varepsilon^2}\cdot\rho\,d\rho$$



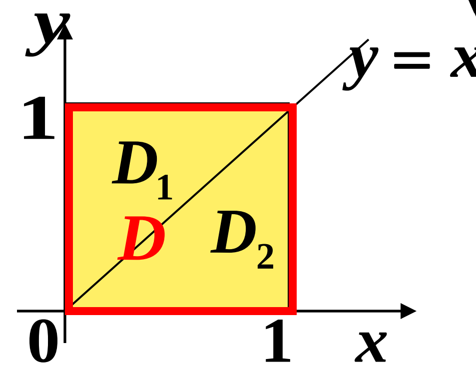
$$=2\pi\cdot\frac{1}{2}\ln(\rho^2+\varepsilon^2)\Big|_0^{\varepsilon}$$

$$=\pi\ln 2$$ （不含ε）

$\therefore$积分$\iint_{x^2+y^2\le\varepsilon^2}\dfrac{d\sigma}{x^2+y^2+\varepsilon^2}(\varepsilon>0)$之值与$\varepsilon$无关。

十、求$\int_0^1 dx\int_0^1 e^{\max(x^2,y^2)}dy$

解：积分区域D如图所示



$y=x$将D分成2个区域D_1和D_2

$$原积分=\iint\limits_{D_1} e^{\max(x^2,y^2)}dxdy+\iint\limits_{D_2} e^{\max(x^2,y^2)}dxdy$$

$$=\iint\limits_{D_1} e^{y^2}dxdy+\iint\limits_{D_2} e^{x^2}dxdy$$

$$=\int_0^1 dy\int_0^y e^{y^2}dx+\int_0^1 dx\int_0^x e^{x^2}dy$$

$$=\int_0^1 ye^{y^2}dy+\int_0^1 xe^{x^2}dx=2\int_0^1 xe^{x^2}dx$$

$$=e^{x^2}\Big|_0^1= e-1$$

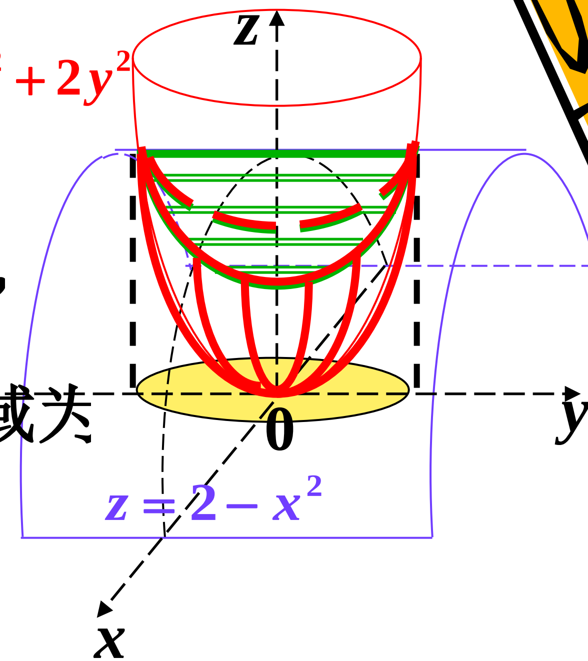
十一（1） 利用二重积分计算立体Ω：$x^2+2y^2\le z\le 2-x^2$ 的体积。

解：Ω是由曲面$z=2-x^2$（绿色）

与曲面$z=x^2+2y^2$（红色）围成，

其交线在xoy平面上的投影区域为

D：$x^2+y^2\le 1$（黄色）



$$\Omega\text{的体积}V=\iint\limits_D[(2-x^2)-(x^2+2y^2)]dxdy$$

$$=2\iint_{x^2+y^2\le 1}(1-x^2-y^2)dxdy$$

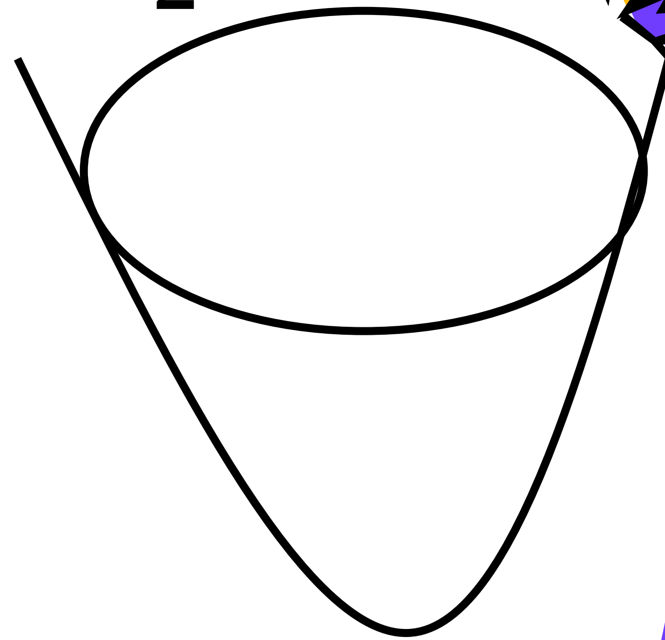
$$=2\int_0^{2\pi}d\theta\int_0^1(1-\rho^2)\rho d\rho$$

$$=2\pi\cdot[\rho^2-\frac{1}{2}\rho^4]\Big|_0^1=\pi$$

十一（2）

$$(x-\frac{1}{2})^2+(y+\frac{3}{4})^2-\frac{29}{16}\le z\le x-\frac{3}{2}y$$

解：首先，求出交线在xoy面的投影。



$$\begin{cases} z=(x-\frac{1}{2})^2+(y+\frac{3}{4})^2-\frac{29}{16} \\ 2x+2z=3y \end{cases}$$

消去z，得　$x^2+y^2=1$

所以，投影为一个圆，从而立体的投影为

$$D=\{(x,y)\,|\,x^2+y^2\le 1\}$$

故所求立体的体积为:

$$V=\iint\limits_{D}\{\frac{3y-2x}{2}-[(x-\frac{1}{2})^{2}+(y+\frac{3}{4})^{2}-\frac{29}{16}]\}dxdy$$

$$=\iint\limits_{D}\{3y-2x-x^{2}-y^{2}+1]\}dxdy$$

$$=\iint\limits_{D}\{-x^{2}-y^{2}+1\}dxdy$$

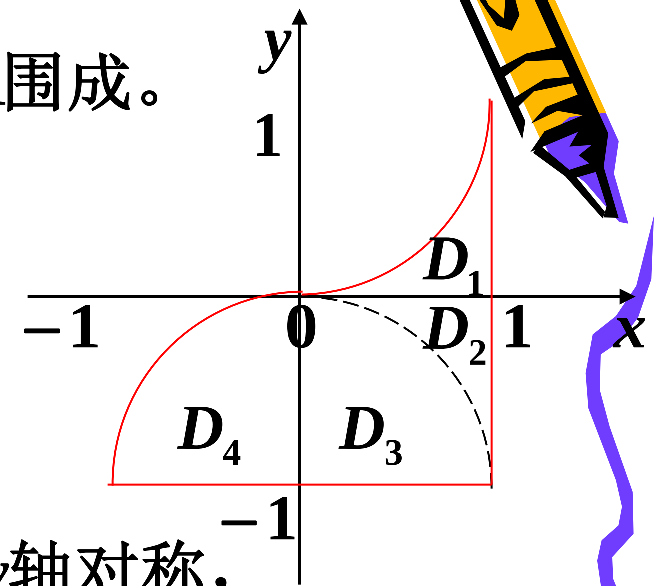
$$=\int_{0}^{2\pi}d\theta\int_{0}^{1}(1-\rho^{2})\rho d\rho$$

$$=\frac{\pi}{2}$$

十二、$f(x)$是连续函数，求$\iint\limits_{D}[xyf(x^2+y^2)-x-y]dxdy$，

其中D是由曲线$y=x^3$, $x=1$, $y=-1$围成。

解：积分区域D如图所示（红色）



用$y=-x^3(x>0)$，x轴和y轴将

D分成四个区域，则

D_1与D_2关于x轴对称，D_3与D_4关于y轴对称，

$xyf(x^2+y^2)$关于x、y均为奇函数，

$$\iint\limits_{D} xyf(x^2+y^2)dxdy = \iint\limits_{D_1+D_2} xyf(x^2+y^2)dxdy$$

$$+\iint\limits_{D_3+D_4} xyf(x^2+y^2)dxdy = 0+0=0$$

$$\therefore \iint_D [xyf(x^2+y^2) - x - y]dxdy = \iint_D -(x+y)dxdy$$

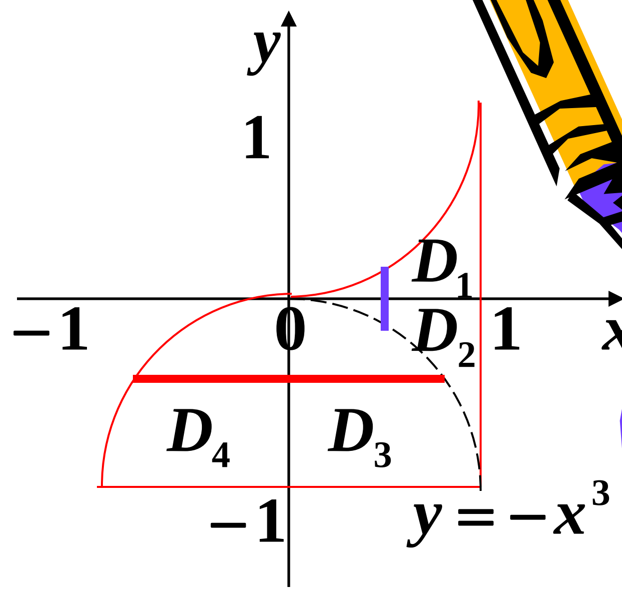
$$= -\iint_{D_1+D_2}(x+y)dxdy - \iint_{D_3+D_4}(x+y)dxdy$$

$$= -\iint_{D_1+D_2} x\,dxdy - \iint_{D_3+D_4} y\,dxdy$$

$$= -\int_0^1 xdx\int_{-x^3}^{x^3} dy - \int_{-1}^0 ydy\int_{\sqrt[3]{y}}^{-\sqrt[3]{y}} dx$$

$$= -\int_0^1 x\cdot 2x^3dx - \int_{-1}^0 y\cdot(-2\sqrt[3]{y})dy$$

$$= -\frac{2}{5}x\Big|_0^1 + \frac{6}{7}y\Big|_{-1}^0 = -\frac{2}{5} + \frac{6}{7} = \frac{16}{35}$$

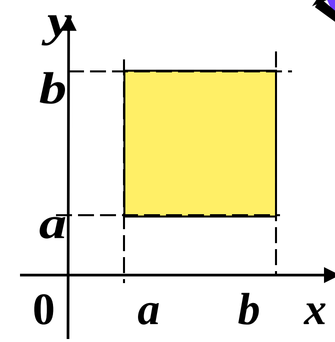


十三、设 $f(x)$在$[a，b]$上连续且恒正，利用二重积分证明:

$$\left[\int_a^b f(x)\cos kx\,dx\right]^2+\left[\int_a^b f(x)\sin kx\,dx\right]^2\le\left[\int_a^b f(x)dx\right]^2$$

证明：左式$=\left[\int_a^b f(x)\cos kx\,dx\right]\cdot\left[\int_a^b f(y)\cos ky\,dy\right]$

$$+\left[\int_a^b f(x)\sin kx\,dx\right]\cdot\left[\int_a^b f(y)\sin ky\,dy\right]$$



$$=\iint_D f(x)f(y)[\cos kx\cdot\cos ky]dxdy$$

$$+\iint_D f(x)f(y)[\sin kx\cdot\sin ky]dxdy$$

$$=\iint_D f(x)f(y)\cos k(x-y)dxdy\le\iint_D f(x)f(y)dxdy$$

$$(\cos k(x-y)\le 1)$$

$$=\int_a^b f(x)dx\cdot\int_a^b f(y)dy=\left[\int_a^b f(x)dx\right]^2=\text{右边}$$

十四、设 $f(x)$是[0,1]上单调减少的连续函数且恒正，利用二重积分证明：

$$\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx} \le \frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx} \qquad (1)$$

分析:(1)

$$\Leftrightarrow \int_0^1 xf^2(x)dx\cdot\int_0^1 f(x)dx \le \int_0^1 xf(x)dx\cdot\int_0^1 f^2(x)dx \qquad (2)$$

$$\Leftrightarrow \int_0^1 xf^2(x)dx\cdot\int_0^1 f(y)dy \le \int_0^1 xf(x)dx\cdot\int_0^1 f^2(y)dy \qquad (3)$$

$$\Leftrightarrow \iint_D xf^2(x)f(y)dxdy \le \iint_D xf(x)f^2(y)dxdy \qquad (4)$$

$$\Leftrightarrow \iint_D xf(x)f(y)[f(x)-f(y)]dxdy \le 0 \qquad (5)$$

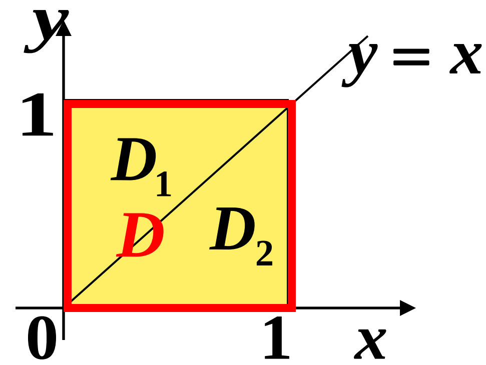
$$\|$$

$$\iint_{D_1} xf(x)f(y)[f(x)-f(y)]dxdy \quad (\ge 0)$$

$$+\iint_{D_2} xf(x)f(y)[f(x)-f(y)]dxdy \quad (\le 0)$$

只要证明（5）成立

证明：注意到D关于$y=x$对称，



$$\iint_{D_1} xf(x)f(y)[f(x)-f(y)]dxdy$$

（对换x与y）

$$=\iint_{D_2} yf(y)f(x)[f(y)-f(x)]dxdy$$

$$=-\iint_{D_2} yf(x)f(y)[f(x)-f(y)]dxdy$$

$$\therefore \iint_{D} xf(x)f(y)[f(x)-f(y)]dxdy$$

$$=\iint_{D_2} \underbrace{(x-y)}_{+}\underbrace{f(x)}_{+}\underbrace{f(y)}_{+}[\underbrace{f(x)-f(y)}_{-}]dxdy \le 0$$

$\therefore$(5)成立。

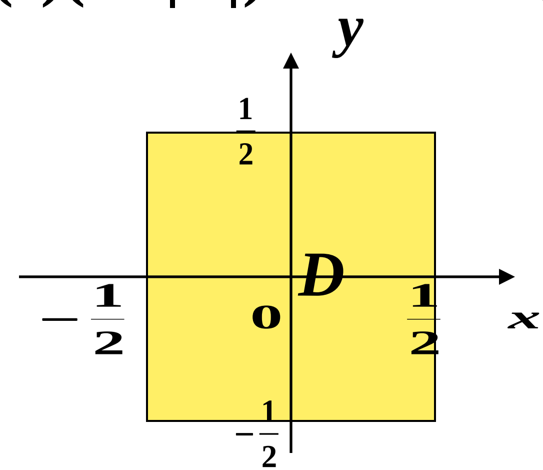
$\because (5)\Leftrightarrow(4)\Leftrightarrow(3)\Leftrightarrow(2)\Leftrightarrow(1)$

$\therefore$(1)式成立

十五、$f(x)$连续，$D:|x|\le\frac{1}{2},|y|\le\frac{1}{2}$,

试证明：$\iint\limits_D f(x-y)d\sigma=\int_{-1}^{1}f(t)(1-|t|)dt$。

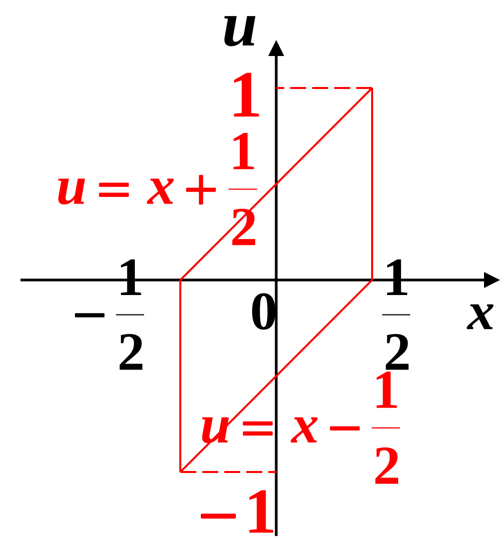
证明：积分区域D如图所示



$$\iint\limits_D f(x-y)d\sigma=\int_{-\frac{1}{2}}^{\frac{1}{2}}dx\int_{-\frac{1}{2}}^{\frac{1}{2}}f(x-y)dy$$

$$\xlongequal[-dy=du]{x-y=u}\int_{-\frac{1}{2}}^{\frac{1}{2}}dx\int_{x+\frac{1}{2}}^{x-\frac{1}{2}}-f(u)du$$

$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}dx\int_{x-\frac{1}{2}}^{x+\frac{1}{2}}f(u)du$$



$$\xlongequal{\text{交换积分次序}}\int_{-1}^{0}du\int_{-\frac{1}{2}}^{u+\frac{1}{2}}f(u)dx$$

$$+\int_{0}^{1}du\int_{u-\frac{1}{2}}^{\frac{1}{2}}f(u)dx$$

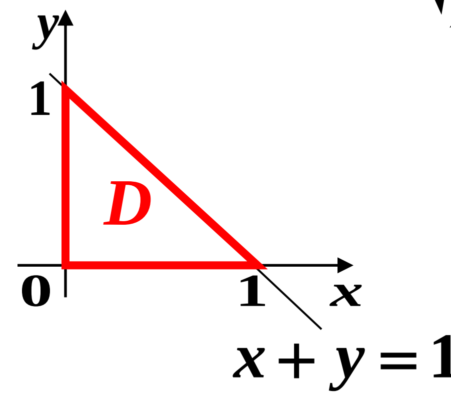
交换积分次序 $\int_{-1}^{0} du \int_{-\frac{1}{2}}^{u+\frac{1}{2}} f(u)dx + \int_{0}^{1} du \int_{u-\frac{1}{2}}^{\frac{1}{2}} f(u)dx$

$$= \int_{-1}^{0} (u+1) f(u)du + \int_{0}^{1} (1-u) f(u)du$$

$$= \int_{-1}^{1} (1-|u|) f(u)du$$

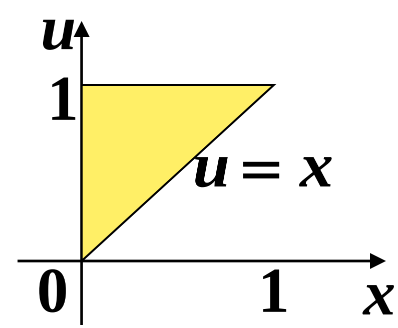
$$= \int_{-1}^{1} (1-|t|) f(t)dt$$

十六、 设 $f(x)$在[0,1]上连续，$D=\{(x,y)\mid x\geq 0, y\geq 0,$ $x+y\leq 1\}$，试证明：$\iint\limits_D f(x+y)d\sigma=\int_0^1 xf(x)dx$



证明：
$$\iint\limits_D f(x+y)d\sigma=\int_0^1 dx\int_0^{1-x} f(x+y)dy$$

$$\overset{x+y=u}{\underset{dy=du}{=\!=\!=\!=}}\int_0^1 dx\int_x^1 f(u)du$$

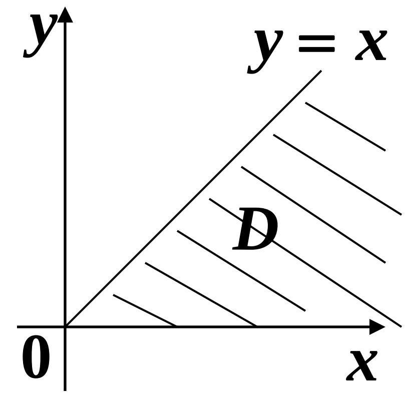


$$\overset{\text{交换积分次序}}{=\!=\!=\!=}\int_0^1 f(u)du\int_0^u dx$$

$$=\int_0^1 f(u)udu$$

$$=\int_0^1 xf(x)dx$$

十七、计算二次积分$\int_0^{+\infty}dy\int_y^{+\infty}\frac{x\,f'(x)}{x^2+y^2}dx$，其中$f'(x)$在$[0,+\infty)$上连续，在$x\to+\infty$时，曲线$y=f(x)$有水平渐近线$y=f(0)+20$.



解：在极坐标下，所求积分

$$\int_0^{+\infty}dy\int_y^{+\infty}\frac{x\,f'(x)}{x^2+y^2}dx$$

$$=\int_0^{\frac{\pi}{4}}d\theta\int_0^{+\infty}\frac{r\cos\theta\,f'(r\cos\theta)}{r^2}rdr$$

$$=\int_0^{\frac{\pi}{4}}d\theta\int_0^{+\infty}f'(r\cos\theta)d(r\cos\theta)=\int_0^{\frac{\pi}{4}}d\theta\int_0^{+\infty}f'(u)d(u)$$

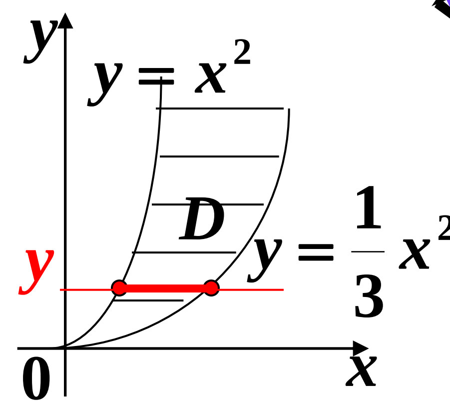
$$=\int_0^{\frac{\pi}{4}}d\theta\cdot[\lim_{u\to+\infty}f(u)-f(0)]=\int_0^{\frac{\pi}{4}}[f(0)+20-f(0)]d\theta$$

$$=\frac{\pi}{4}\cdot 20=5\pi$$

十八、 设$D=\{(x,y)\mid \frac{1}{3}x^2\le y\le x^2, x\ge 0\}$,

计算广义二重积分$\iint_D xe^{-y^2}d\sigma$

解：积分区域D如图所示



$$\iint_D xe^{-y^2}d\sigma$$

$$=\int_0^{+\infty}dy\int_{\sqrt{y}}^{\sqrt{3y}}xe^{-y^2}dx$$

$$=\int_0^{+\infty}e^{-y^2}\cdot\frac{1}{2}x^2\Big|_{\sqrt{y}}^{\sqrt{3y}}dy=\int_0^{+\infty}ye^{-y^2}dy$$

$$=-\frac{1}{2}e^{-y^2}\Big|_0^{+\infty}=-\frac{1}{2}(0-1)=\frac{1}{2}$$