eg. 已知 $[r, p_r] = i\hbar$，　$p_r^2 = -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$

$$[r, p_r^2] = p_r[r, p_r] + [r, p_r]p_r$$
$$= 2i\hbar\, p_r$$

$$[r, p_r^2] = -\hbar^2\frac{1}{r}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\underbrace{\frac{\partial}{\partial r}(r)}^{1+r\frac{\partial}{\partial r}}$$
$$= -\hbar^2\frac{1}{r}\left(\cancel{2r\frac{\partial}{\partial r}} + \cancel{r^2\frac{\partial^2}{\partial r^2}}\right) + \hbar^2\frac{1}{r}\left(\frac{2r}{r} + \cancel{3r}\frac{\partial}{\partial r} + \cancel{r^2\frac{\partial^2}{\partial r^2}}\right)$$
$$= \hbar^2\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)$$

证明. $e^L A e^{-L} = A + [L, A] + \frac{1}{2!}[L, [L, A]] + \cdots$

构造. $F(x) = e^{xL}Ae^{-xL}$

$$F'(x) = L\,F(x) - F(x)\,L$$
$$= [L, F(x)]$$
$$F''(x) = [L, F'(x)] = [L, [L, F(x)]]$$
$$\vdots$$
$$F(1) = F(0) + [L, A] + \cdots$$

eg

$2\cos^2 x + \cancel{\sin^2 x} = \cos 2x + 1$

$$\frac{4}{\sqrt{a}}\sin\frac{\pi}{a}x\,\underline{\cos^2\frac{\pi}{a}x}$$
$$= \frac{1+\cos\frac{2\pi}{a}x}{2}$$
$$\frac{2}{\sqrt{a}}\left(\sin\frac{\pi}{a} + \sin\frac{\pi}{a}x\cos\frac{2\pi}{a}x\right)$$

$$\sin a\cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\cancel{\sin\frac{\pi}{a}} + \underline{\frac{1}{2}\sin\frac{3\pi}{a} + \frac{1}{2}\sin(-\frac{\pi}{a})}$$

$$\frac{1}{\sqrt{a}}\left(\sin(\underbrace{\frac{\pi}{a}x}_{n=1}) + \underbrace{\sin\frac{3\pi}{a}x}_{n=3}\right)$$

$$\frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_3$$