⋆4.1.2 初边值条件的处理

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- 初值问题
  - 对于初始条件$u(x,0)=\phi(x)$,对Ω作两族平行线

$$x = x_j = jh, j = 0, \pm 1, \pm 2, \cdots .$$

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  - 若采取古典显格式，计算的次序可以从下一层的中间位置$x=0$开始轮流向两边扩展
  - 如果希望得到第M层上$|x|\le x_l$(l为确定的某个正整数)范围内的节点处的近似值，那么在第1层至少要算出$|x|\le x_{l+M-1}$范围内的节点的近似值，即至少要取$2(l+M)+1$个初值$u_j^0=\phi(x_j), j=0,\pm1,\cdots,\pm l+M$进行计算

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    * 对$\frac{\partial u}{\partial x}$在$(x_0, t_k)$处用向前差商逼近，在$(x_N, t_k)$处用向后差商逼近:

$$\alpha_1^k \frac{u_1^k - u_0^k}{h} + \alpha_0^k u_0^k = \alpha_2^k,$$

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    * 误差为$O(h)$.

- 为提高精度，可用中心差商来逼近$\frac{\partial u}{\partial x}$,

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- 假设方程$Lu=\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}=f$在边界上也成立，可把内点的差分格式推广到边界上。

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- 考虑古典显格式，在左边界(x_0,t_k)得

$$u_0^{k+1}=ru_1^k+(1-2r)u_0^k+ru_{-1}^k+\tau f_0^k.$$

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- 同理在右边界(x_N,t_k)得

$$u_N^{k+1}=2ru_{N-1}^k+(1-2r-\frac{2rh\beta_0^k}{\beta_1^k})u_N^k+\frac{2rh\beta_2^k}{\beta_1^k}+\tau f_N^k. \tag{34}$$

⋆3.数值计算例子

例1.考虑扩散方程的第一边值问题：

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & \text{当} 0 < x < 1, t > 0 \text{时} \\ u(x,0) = \sin \pi x, & \text{当} 0 \le x \le 1 \text{时} \\ u(0,t) = u(1,t) = 0, & \text{当} t \ge 0 \text{时} \end{cases}$$

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- 利用古典隐格式(14)和Crank-Nicolson格式(18)进行计算，显然Crank-Nicolson格式的结果较好，这是因为它的截断误差阶比前者高

例2.考虑扩散方程的第三初边值问题：

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- 它的解析解为

$$u(x,t) = 4\sum_{i=1}^{\infty} \frac{\sec\alpha_i}{3+4\alpha_i^2} e^{-4\alpha_i^2} \cos 2\alpha_i (x - \frac{1}{2}), 0 < x < 1$$

其中α_i为方程$\alpha \tan\alpha = \frac{1}{2}$的正根

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- 取$h = 0.1$,用古典显格式计算内点的值，取$r = \frac{1}{4}$即$\tau = 0.0025$.用两种方法处理边界条件

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- 用(31)，得

$$\begin{cases} \frac{u_1^k - u_0^k}{h} - u_0^k = 0 \\ \frac{u_{10}^k - u_9^k}{h} - u_{10}^k = 0 \end{cases}$$

即$u_0^k = \frac{10}{11}u_1^k, u_{10}^k = \frac{10}{9}u_9^k$.由此算出的数值解记为$u_j^{k(1)}$.

- 用式(33)和(34)得

$$u_0^{k+1} = \frac{1}{2}(u_1^k + \frac{9}{10}u_0^k),\ u_{10}^{k+1} = \frac{1}{2}(u_9^k + \frac{11}{10}u_{10}^k)$$

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- 对于$x=0.2$,表4.3给出了对不同的k得到的数值解，$u(0.2,t_k)$表示精确解

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- 显然具有2阶精度的对边界的第二种处理方法得到的结果较好

⋆4.1.3 变系数方程的差分格式

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⋆4.1.3 变系数方程的差分格式

利用积分插值法构造方程(1)在定界条件(3)和(6)下的差分格式。对区域的离散与1.1节相同

- 记$W=-a\frac{\partial u}{\partial x}$,在矩形域$\{x_{j-\frac{1}{2}}\le x\le x_{j+\frac{1}{2}}, t_k\le t\le t_{k+1}\}$上对方程(1)积分得

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}[u(x,t_{k+1})-u(x,t_k)]dx+\int_{t_k}^{t_{k+1}}[W(x_{j+\frac{1}{2}},t)-W(x_{j-\frac{1}{2}},t)]dt$$

$$+\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}d(x,t)u(x,t)dxdt=\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}f(x,t)dxdt \quad (35)$$

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$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}[u(x,t_{k+1}) - u(x,t_k)]dx + \int_{t_k}^{t_{k+1}}[W(x_{j+\frac{1}{2}},t) - W(x_{j-\frac{1}{2}},t)]dt$$

$$+\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} d(x,t)u(x,t)dxdt = \int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x,t)dxdt \quad (35)$$

- 利用W的表达形式及中矩形积分公式和积分中值定理，得

$$W(x_{j-\frac{1}{2}},t)h \approx \int_{x_{j-1}}^{x_j} W(x,t)dx = -\int_{x_{j-1}}^{x_j} a(x,t)\frac{\partial u}{\partial x}dx$$

$$\approx -\frac{\partial u}{\partial x}(x_{j-\frac{1}{2}},t)\int_{x_{j-\frac{1}{2}}}^{x_j} a(x,t)dx$$

- 记$A_j(t)=\frac{1}{h}\int_{x_{j-1}}^{x_j}a(x,t)dx$,并用中心差商代替$\frac{\partial u}{\partial x}(x_{j-\frac{1}{2}},t)$,上式为

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- 于是

$$\frac{1}{\tau}\int_{t_k}^{t_{k+1}}W(x_{j-\frac{1}{2}},t)dt\approx \theta W(x_{j-\frac{1}{2}},t_{k+1})+(1-\theta)W(x_{j-\frac{1}{2}},t_k)$$

$$\approx -\theta A_j^{k+1}\frac{u(x_j,t_{k+1})-u(x_{j-1},t_{k+1})}{h}-(1-\theta)A_j^k\frac{u(x_j,t_k)-u(x_{j-1},t_k)}{h}, \tag{36}$$

其中θ为参数，$0\le\theta\le 1$, $A_j^k=A_j(t_k)$

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- 于是

$$\frac{1}{\tau}\int_{t_k}^{t_{k+1}}W(x_{j-\frac{1}{2}},t)dt\approx \theta W(x_{j-\frac{1}{2}},t_{k+1})+(1-\theta)W(x_{j-\frac{1}{2}},t_k)$$

$$\approx -\theta A_j^{k+1}\frac{u(x_j,t_{k+1})-u(x_{j-1},t_{k+1})}{h}-(1-\theta)A_j^k\frac{u(x_j,t_k)-u(x_{j-1},t_k)}{h},\tag{36}$$

其中θ为参数，$0\le\theta\le 1, A_j^k=A_j(t_k)$

- 此外，有

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}u(x,t_k)dx\approx hu(x_j,t_k),\tag{37}$$

-

$$\int_{t_k}^{t_{k+1}}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx \int_{t_k}^{t_{k+1}} u(x_j,t)[\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dx]dt$$

$$\approx \int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$

- $$\int_{t_k}^{t_{k+1}}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx \int_{t_k}^{t_{k+1}} u(x_j,t)[\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dx]dt$$

$$\approx \int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt[\theta u(x_j,t_{k+1}) + (1-\theta)u(x_j,t_k)]$$

- 记$D_j^k = \frac{1}{\tau h}\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt$,则上式化为

$$\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx D_j^k[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]\tau h. \tag{38}$$

- $$\int_{t_k}^{t_{k+1}}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx \int_{t_k}^{t_{k+1}} u(x_j,t)[\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dx]dt$$

$$\approx \int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$

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- 再记

$$F_j^k=\frac{1}{\tau h}\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} f(x,t)dxdt. \tag{39}$$

- $$\int_{t_k}^{t_{k+1}}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx \int_{t_k}^{t_{k+1}} u(x_j,t)[\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dx]dt$$

$$\approx \int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$

- 记$D_j^k=\frac{1}{\tau h}\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)dxdt$,则上式化为

$$\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} d(x,t)u(x,t)dxdt \approx D_j^k[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]\tau h. \tag{38}$$

- 再记

$$F_j^k=\frac{1}{\tau h}\int_{t_k}^{t_{k+1}}\int_{x_{j-\frac{1}{2}}}^{x+\frac{1}{2}} f(x,t)dxdt. \tag{39}$$

- 将式(36)∼ (39)代入式(35)并整理，得

- $$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\frac{1}{h^2}[\theta A_{j+1}^{k+1}(u(x_{j+1},t_{k+1})-u(x_j,t_{k+1}))$$

$$-\theta A_j^{k+1}(u(x_j,t_{k+1})-u(x_{j-1},t_{k+1}))+(1-\theta)A_{j+1}^{k}(u(x_{j+1},t_k)-u(x_j,t_k))$$

$$-(1-\theta)A_j^{k}(u(x_j,t_k)-u(x_{j-1},t_k))]+D_j^k[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$

$$=F_j^k+R_j^k$$

- $$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\frac{1}{h^2}[\theta A_{j+1}^{k+1}(u(x_{j+1},t_{k+1})-u(x_j,t_{k+1}))$$
$$-\theta A_j^{k+1}(u(x_j,t_{k+1})-u(x_{j-1},t_{k+1}))+(1-\theta)A_{j+1}^k(u(x_{j+1},t_k)-u(x_j,t_k))$$
$$-(1-\theta)A_j^k(u(x_j,t_k)-u(x_{j-1},t_k))]+D_j^k[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$
$$=F_j^k+R_j^k$$

- R_j^k为局部截断误差。可以证明当$\theta=\frac{1}{2}$时，$R_j^k=O(\tau^2+h^2)$.当$\theta\neq\frac{1}{2}$时，$R_j^k=O(\tau+h^2)$

- $$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\frac{1}{h^2}[\theta A_{j+1}^{k+1}(u(x_{j+1},t_{k+1})-u(x_j,t_{k+1}))$$
$$-\theta A_j^{k+1}(u(x_j,t_{k+1})-u(x_{j-1},t_{k+1}))+(1-\theta)A_{j+1}^k(u(x_{j+1},t_k)-u(x_j,t_k))$$
$$-(1-\theta)A_j^k(u(x_j,t_k)-u(x_{j-1},t_k))]+D_j^k[\theta u(x_j,t_{k+1})+(1-\theta)u(x_j,t_k)]$$
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- R_j^k为局部截断误差。可以证明当$\theta=\frac{1}{2}$时，$R_j^k=O(\tau^2+h^2)$.当$\theta\neq\frac{1}{2}$时，$R_j^k=O(\tau+h^2)$
- 舍去截断误差，则有逼近方程(1)的差分格式

$$\frac{u_j^{k+1}-u_j^k}{\tau}-\frac{\theta}{h^2}[A_{j+1}^{k+1}(u_{j+1}^{k+1}-u_j^{k+1})-A_j^{k+1}(u_j^{k+1}-u_{j-1}^{k+1})]$$
$$-\frac{1-\theta}{h^2}[A_{j+1}^k(u_{j+1}^k-u_j^k)-A_j^k(u_j^k-u_{j-1}^k)]+D_j^k[\theta u_j^{k+1}+(1-\theta)u_j^k]=F_j^k,$$
$$j=1,2,\cdots,N-1,k=0,1,\cdots,M-1.$$

- 在实际计算时，需要具体算出A_j^k, D_j^k, F_j^k一般可用数值积分公式，选取的数值积分公式的误差不要低于R_j^k的阶，否则会降低解的精确度。

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- 上述用积分插值法构造差分格式的方法可推广到空间变量是高维的情形，也可推广到非均匀网格，推导过程与上面的推导完全类似。