Polynomial Ergodicity of Langevin Dynamics

Draft Draft

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Table of contents

Bibliography 1

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An overdamped Langevin dynamics on $\mathbb{R}$ is defined as the solution to the following SDE:

$$dX_t = -\nabla V(X_t)\,dt + \sqrt{2\beta^{-1}}\,dB_t, \qquad X_0 = x_0.$$

If $V(x) = \frac{x^2}{2}$, X becomes an Ornstein-Uhlenbeck process. Transforming via $f(t,x) = xe^t$ and using Itô's formula, we get

$$X_t = x_0 e^{-t} + \sqrt{2\beta^{-1}} \int_0^t e^{-(t-s)}\,dB_s.$$

Hence, X is a Gaussian process with $X_t \sim \mathrm{N}\left(x_0 e^{-t}, \beta^{-1}\left(1 - e^{-2t}\right)\right)$.

In this case, expectation with respect to $G(y) = e^{\kappa V(y)} = e^{\frac{\kappa y^2}{2}} (\kappa \in \mathbb{R})$ is given by

Taking a closer look at the numerator inside exp,

Therefore, we conclude

$$E_x[G(X_t)] < \infty \quad \Leftrightarrow \quad \kappa\beta^{-1}\left(1 - e^{-2t}\right) < 1.$$

In other words, $P_t G(x)$ is finite as long as

$$t < -\frac{1}{2} \log\left(1 - \frac{\beta}{\kappa}\right).$$

Bibliography