Polynomial Ergodicity of Langevin Dynamics

Draft Draft

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An overdamped Langevin dynamics on $\mathbb{R}$ is defined as the solution to the following SDE:

$$dX_t = -\nabla V(X_t)\,dt + \sqrt{2\beta^{-1}}\,dB_t, \qquad X_0 = x_0.$$

If $V(x) = \frac{x^2}{2}$, X becomes an Ornstein-Uhlenbeck process. Transforming via $f(t,x) = xe^t$ and using Itô's formula, we get

$$X_t = x_0 e^{-t} + \sqrt{2\beta^{-1}} \int_0^t e^{-(t-s)}\,dB_s.$$

Hence, X is a Gaussian process with $X_t \sim \mathrm{N}\left(x_0 e^{-t}, \beta^{-1}(1-e^{-2t})\right)$.

In this case, expectation with respect to $G(y) = e^{\kappa V(y)} = e^{\frac{\kappa y^2}{2}}$ ($\kappa \in \mathbb{R}$) is given by

$$\begin{aligned}
\mathrm{E}_x[G(X_t)] &= \int_{\mathbb{R}} G(y) \frac{1}{\sqrt{2\pi\beta^{-1}(1-e^{-2t})}} \exp\left(-\frac{(y-xe^{-t})^2}{2\beta^{-1}(1-e^{-2t})}\right) dy \\
&= \frac{1}{\sqrt{2\pi\beta^{-1}(1-e^{-2t})}} \int_{\mathbb{R}} \exp\left(\frac{\kappa\beta^{-1}(1-e^{-2t})y^2 - (y-xe^{-t})^2}{2\beta^{-1}(1-e^{-2t})}\right) dy.
\end{aligned}$$

Taking a closer look at the numerator inside exp,

$$\begin{aligned}
&\kappa\beta^{-1}(1-e^{-2t})y^2 - (y-xe^{-t})^2 \\
&= y^2\left(\kappa\beta^{-1}(1-e^{-2t}) - 1\right) - 2xe^{-t}y + x^2e^{-2t}.
\end{aligned}$$

Therefore, we conclude

$$\mathrm{E}_x[G(X_t)] < \infty \quad \Leftrightarrow \quad \kappa\beta^{-1}(1-e^{-2t}) < 1.$$

In other words, $P_tG(x)$ is finite as long as

$$t < -\frac{1}{2}\log\left(1 - \frac{\beta}{\kappa}\right).$$