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Resampling: The Most Important Component of a Particle Filter

Recent Developments towards Continuous Time Models

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Introduction: What Is a Particle Filer?

Particle filters, also known as Sequential Monte Carlo methods (SMCs), were invented in Kitagawa, 1993 and Gordon et al., 1993 as an simulation-based algorithm which performs filtering in non-Gaussian and non-linear state space models, overcoming the weeknesses of then-standard Kalman-based filtering methods.

In particle-based approaches, a filtering distribution is approximated by a cloud of weighted samples, hence giving rise to the term 'particle methods'. The samples are propagated to approximate the next distribution, leading to efficient sequential estimation in dynamic settings.

Resampling: The Key to Effective Particle Filtering

The *resampling step* is the key difference from sequential importance sampling methods. Particle filters incorporate a resampling step to occasionally reset the weights of the samples, while maintaining the overall distribution they represent, in order to prevent the effective number of particles participating in the estimation from becoming too small–a situation also called *weight degeneracy*.

Expanding Applications of Particle Filters

Recent developments have highlithgted the capability of particle filters as general-purpose samplers, extending their applicability beyond the traditional realm of temporal graphical models to a broader range of statistical inference problems. This versatility has earned them the alternative name 'SMC', a term reminiscent of 'MCMC'. This poster trys to be another contribution in this direction.

A Generic Particle Filter: An Algorithmic Description

Aside from sampling schemes, almost all particle filter algorithms share the following structure:

Algorithm 1 Generic Particle Filter algorithm

Sample $X_0^n \sim \mathbb{M}_0(dx_0)$
$w_0^n \leftarrow G_0(X_0^n)$
$W_0^n \leftarrow \frac{w_0^n}{\sum_{m=1}^N w_0^m}$
for all $t \leftarrow 1 : T$ **do**
    $A_t^{1:N} \sim \texttt{Resample}(W_{t-1}^{1:N})$
    Sample $X_t^n \sim M_t(X_{t-1}^{A_t^n}, dx_t)$
    $w_t^n \leftarrow G_t(X_{t-1}^{A_t^n}, X_t^n)$
    $W_t^n \leftarrow \frac{w_t^n}{\sum_{m=1}^N w_t^m}$
end for

where the distribution $\mathbb{M}_0(dx_0)$ and Markov kernels $M_1(x_0, dx_1), M_2(x_1, dx_2), \cdots$ describe system dynamics, while the potentials $G_0(x_0), G_1(x_0, x_1), G_2(x_1, x_2), \cdots$ encapsulate the change of measure from the system dynamics, such as a Bayesian update.

The resampling scheme $\texttt{Resample}(W_{t-1}^{1:N})$ can be formally defined as a mapping

$$r : \mathbb{R}_+^N \to \mathcal{P}([N]^{[N]})$$

where $\mathcal{P}([N]^{[N]})$ represents the space of probability measures on the set of resampling enumerations

$$[N]^{[N]} := \{a : [N] \to [N] \mid a \text{ is a mapping}\}, \qquad [N] := \{1, 2, \cdots, N\}$$

Results

Setting 1: Continuous Target Model

Let $\{z_t\} \subset \mathcal{L}(\Omega'; \mathbb{R}^d)$ be an Ito process on $\mathbb{R}^d$ defined by the SDE

$$z_t = b(z_t)dt + \sigma(z_t)dB_t, \qquad b_i, \sigma_{ij} \in \mathrm{Lip}_b(\mathbb{R}^d), \quad i, j \in [d].$$

Let $\Pi \in C_c(D_{\mathbb{R}^d}(\mathbb{R}_+))^*$ be Feynman-Kac measure composed of $\{z_t\}$ and a potential $V \in C_b(\mathbb{R}^d)_+$, that is,

$$\Pi(f) := \frac{1}{\mathcal{Z}} \mathrm{E}\left[f(z_{[0,\tau]}) \exp\left(-\int_0^\tau V(z_u)\, du \right) \right]$$

$$\mathcal{Z} := \mathrm{E}\left[\exp\left(-\int_0^\tau V(z_u)\, du \right) \right], \qquad f \in C_c(D_{\mathbb{R}^d}(\mathbb{R}_+); \mathbb{R}^d).$$

Setting 2: Particle Filter targeted at the Continuous Target Model

A particle filter $\{X_k^\Delta\} \subset \mathcal{L}(\Omega; M_{Nd}(\mathbb{R}))$ is defined through the natural discritization of the Feynman-Kac model Π.

Let r be a resampling scheme with N particles, and $\Delta > 0$ be discretization time step, a particle filter is a discrete stochastic process on $(\mathbb{R}^d)^N$ defined recursively as follows: for $i \in [N], k \in \mathbb{N}, A_k \overset{\text{i.i.d.}}{\sim} r(e^{-\Delta V(X_k^\Delta)})$

$$(X_k^\Delta)^i := (X_{k-1}^\Delta)^{A_k(i)} + \Delta \cdot b\left((X_{k-1}^\Delta)^{A_k(i)} \right) + \sigma\left((X_{k-1}^\Delta)^{A_k(i)} \right) (B_{k\Delta}^i - B_{(k-1)\Delta}^i)$$

Then, the convergence result is stated in terms of the càdlàg extension $\{Z_t^\Delta\}$ of $\{X_k^\Delta\}$:

$$Z_t^\Delta := X_{\lfloor \frac{t}{\Delta} \rfloor}^\Delta.$$

Resampling Intensity

Limit Theorem

Theorem

Assume the components $\{z_t\}, V : \mathbb{R}^d \to \mathbb{R}_+$ of the Feynman-Kac model Π satisfy the following three conditions:

$$b_i, \sigma_{ij} \in \mathrm{Lip}_b(\mathbb{R}^d), \qquad \inf_{x \in \mathbb{R}^d} \inf_{\substack{\theta \in \mathbb{R}^d \\ \|\theta\|_2 = 1}} \|\sigma(x)\theta\|_2 > 0 \quad V \in C_b(\mathbb{R}^d)_+$$

Additionally, let the resampling scheme r has the limiting sampling intensity measure $\bar{\iota}$ and the map

$$\bar{\iota} : \mathbb{R}_+^N \to \mathcal{M}^1([N]^{[N]} \setminus \{\mathrm{id}_{[N]}\})$$

is continuous.

Then, càdlàg extension $\{Z_t^\Delta\}_{t \in \mathbb{R}_+}$ of the associated particle filter $\{X_k^\Delta\}_{k=0}^\infty$ converges in distribution, for every decreasing sequence $\{\Delta_n\} \subset \mathbb{R}^+$, to an unique Lévy process $\{Z_t\}$, whose generator can be described as follows: for all $f \in C_c^2(\mathbb{R}^{dN}), x \in \mathbb{R}^{dN}, x^j \in \mathbb{R}^N$

$$\begin{aligned}\mathcal{L}f(x) := &\sum_{j=1}^N \sum_{k=1}^d b_k(x^j) \frac{\partial f}{\partial x_k^j}(x) + \sum_{j=1}^N \frac{1}{2} \sum_{k,l=1}^d (\sigma\sigma^\top)_{kl}(x^j) \frac{\partial^2 f}{\partial x_k^j \partial x_l^j}(x) \\ &+ \sum_{a \in [N]^{[N]} \setminus \{\mathrm{id}_{[N]}\}} \bar{\iota}(V(x))(a) \left(f(x^{a(1:N)}) - f(x^{1:N}) \right)\end{aligned}$$

where $\mathcal{L}$ is the generator of the Ito diffusion $\{z_t\}$. Moreover, for every bounded continuous function $\mathcal{V} \in C_b(\mathbb{R}_+ \times \mathbb{R}^{dN})_+$ and $f \in C_b(\mathbb{R}^{dN})$,

$$\begin{aligned}\lim_{n \to \infty} \mathrm{E}&\left[f(X_{\lfloor \tau/\Delta_n \rfloor}^{\Delta_n}) \exp\left(-\sum_{k=0}^{\lfloor \tau/\Delta_n \rfloor - 1} \Delta_n \mathcal{V}(k\Delta_n, X_k^{\Delta_n}) \right) \right] \\ &= \mathrm{E}\left[f(Z_\tau) \exp\left(-\int_0^\tau \mathcal{V}(u, Z_u)\, du \right) \right].\end{aligned}$$

Numerical Experiment

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Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 1: Table caption

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Conclusions

- Pellentesque eget orci eros. Fusce ultricies, tellus et pellentesque fringilla, ante massa luctus libero, quis tristique purus urna nec nibh. Phasellus fermentum rutrum elementum. Nam quis justo lectus.
- Vestibulum sem ante, hendrerit a gravida ac, blandit quis magna.

Forthcoming Research

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References

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