Numerical Recipes for Astrophysics
Solutions hand-in assignment-2

Luther Algra - s1633376

May 8, 2019

Abstract

The current document contains the solutions for the second hand-in assignment of Numerical Recipes. Each main question 1, 2, 3, ..., 7 is given its own section and contains a subsection for each sub-question (1.a, 1.b, ..., 1.f). A main question always ends with a final subsection that contains two segments of code. The first segment contains the full code of the program that executes the sub-questions. The second segment contains the shared modules (if any) used by the sub-question. A sub-question itself always starts with a short summary of the question that needs to be answered followed by an explanation of how the problem is solved. Next, the code and its output are provided. Finally the output is discussed if relevant.

1 - Normally distributed pseudo-random numbers

Question 1.a

Problem

Write a random number generator that returns a random floating-point number between 0 and 1. At minimum, use some combination of an MWC and a 64-bit XOR-shift. Plot a sequential of random numbers against each other in a scatter plot (x_{i+1} vs x_i) for the first 1000 numbers generated. Also plot the value of the random numbers for the first 1000 numbers vs the index of the random number, this mean the x-axis has a value from 0 through 999 and the y-axis 0 through 1). Finally, have your code generate 1,000,000 random numbers and plot the result of binning these in 20 bins 0.05 wide.

Solution

bleh

Question 1.b

Problem

Now use the Box-Muller method to generate 1000 normally-distributed random numbers. To check if they are following the expected Gaussian distribution, make a histogram (scaled appropriate) with the corresponding true probability distribution (normalized to integrate to 1) as line. This plot should contain the interval of -5 σ until 5σ from the theoretical probability distribution. Indicate the theoretical 1σ, 2σ, 3σ and 4σ interval with a line. For this plot, use $\mu = 3$ and $\sigma = 2.4$ and choose bins that are appropriate.

Solution

The Box-Muller method allows two i.i.d uniform variables to be transformed to two i.i.d Gaussian distributed variables. It thereby overcomes the problem that there lacks a closed form for the CDF of a Gaussian distribution. The method overcomes the problem by transforming the joined CDF of the two random Gaussian variables to polar coordinates. This transformation makes it possible to find the CDFs for the polar coordinates of the random Gaussian variables. The CDFs can be used to convert the two uniform distributed variables to the polar coordinates of the Gaussian distributed variables. The polar coordinates can then finally be transformed back to Cartesian coordinates to find the transformation between the two uniform random variables and two gaussian random variables.

Let $X, Y \sim G(\mu, \sigma^2)$ be two i.i.d Gaussian distributed random variables. Their joined CDF is then given by,

$$P(X \leq x_1, Y \leq y_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} G(x|\mu, \sigma^2) G(y|\mu, \sigma^2) dx dy \tag{1}$$

Transforming to polar coordinates by substituting $(x-\mu) = r\cos(\theta)$ and $(y-\mu) = r\sin(\theta)$ yields,

$$\begin{aligned}
P(R \leq r_1, \Theta \leq \theta_1) &= \int_0^{r_1} \int_0^{\theta_1} G(r\cos(\theta)\sigma + \mu|\mu, \sigma^2) G(r\sin(\theta)\sigma + \mu|\mu, \sigma^2) r dr d\theta \\
&= \frac{1}{2\pi\sigma^2} \int_0^{r_1} \int_0^{\theta_1} r e^{-\frac{1}{2}\left[\left(\frac{r\cos(\theta)}{\sigma}\right)^2 + \left(\frac{r\sin(\theta)}{\sigma}\right)^2\right]} dr d\theta \\
&= \frac{1}{2\pi\sigma^2} \int_0^{r_1} \int_0^{\theta_1} r e^{-\frac{r^2}{2\sigma^2}} dr d\theta
\end{aligned}$$

The CDF's for the polar coordinates are now given by,

$$P(R \leq r_1) = \frac{1}{\sigma^2} \int_0^{r_1} r e^{-\frac{r^2}{2\sigma^2}} dr = \int_0^{r_1} \frac{d}{dr}\left(-e^{-\frac{r^2}{2\sigma^2}}\right) dr = 1 - e^{-\frac{r_1^2}{2\sigma^2}} \tag{2}$$

$$P(\Theta \leq \theta_1) = \frac{1}{2\pi}\left[-e^{-\frac{r^2}{2\sigma^2}}\right]_0^{\infty} \int_0^{\theta_1} d\theta = \frac{\theta_1}{2\pi} \tag{3}$$

Let $U_1, U_2 \sim U(0,1)$ be two i.i.d uniform variables. From the transformation law of probability we then must have that,

$$P(R \leq r_1) = P(U_1 \leq u_1) \rightarrow 1 - e^{-\frac{r_1^2}{2\sigma^2}} = \int_0^{u1} du_1 = u_1 \tag{4}$$

$$P(\Theta \leq \theta) = P(U_2 \leq u_2) \rightarrow \frac{\theta_1}{2\pi} = \int_0^{u2} du_2 = u_2 \tag{5}$$

The transformation from the two uniform distributed variables to the polar coordinates of the Gaussian distributed variables then becomes,

$$r_1 = \sqrt{-2\sigma^2 \ln(1 - u_1)} \tag{6}$$
$$\theta_1 = 2\pi u_2 \tag{7}$$

Finally converting back to carthesian c oordinates then yields the transformation from two i.i.d uniform distributed variables to two i.i.d gaussian distributed variables;

$$x_1 = r\cos(\theta) + \mu = \sqrt{-2\sigma^2 \ln(1 - u_1)}\cos(2\pi u_2) + \mu$$
$$y_1 = r\sin(\theta) + \mu = \sqrt{-2\sigma^2 \ln(1 - u_1)}\sin(2\pi u_2) + \mu$$