CS4423 - Networks

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7. Growing Graphs

Lecture 23: Preferential Attachment

In the random graph models we have studied so far, a network is generated on a given number of nodes, and edges are randomly added or modified. Here we introduce and study preferential attachment (https://en.wikipedia.org/wiki/Preferential_attachment) models, where a network is grown by adding one node at a time, plus some random edges. It turns out that, under suitable circumstances, such a network has a power law degree distribution.

In [1]:
```
import numpy as np
import pandas as pd
import networkx as nx
import matplotlib.pyplot as plt
```

The Molloy-Reed Criterion

Graphs with a given degree sequence can be analyzed. In particular, a criterion for the existence of a giant component can be formulated in terms of the first and second moments of the degree distribution:

$$\langle k \rangle = \sum_k k p_k, \qquad \langle k^2 \rangle = \sum_k k^2 p_k.$$

Theorem (Molloy-Reed, 1995/1998). Suppose $\underline{p} = (p_0, p_1, \ldots)$ is a degree distribution, i.e., $\sum_k p_k = 1$ and consider the ensemble of all graphs on n nodes with degree distribution $\underline{p}$. Define

$$Q(\underline{p}) := \sum_k k(k-2)p_k = \langle k^2 \rangle - 2\langle k \rangle.$$

Then: * if $Q(\underline{p}) > 0$ almost every graph in the ensemble contains a giant component, * if $Q(\underline{p}) < 0$ all components are small.

- For example, in an ER random graph (models $G(n,m)$ or $G(n,p)$) the moments are related as $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$, whence $Q(\underline{p}) = \langle k \rangle^2 - \langle k \rangle > 0$ if and only if $\langle k \rangle > 1$, as shown previously.

- In a network with power law degree distribution $p_k \simeq ck^{-\gamma}$ for $2 \leq \gamma \leq 3$, the second moment $\langle k^2 \rangle$ diverges (with $n \to \infty$) and thus $Q(\underline{p}) > 0$: such networks have giant components.

- **Characteristic Path Length.** Scale free networks with $2 \leq \gamma \leq 3$ are **ultrasmall**: $L \sim \ln \ln n$ (as opposed to $L \sim \ln n$ in a small world network.)

- **Clustering Coefficient.** It can be shown that in a scale free network with $2 \leq \gamma \leq 3$ the clustering coefficient is $C \approx n^{\alpha}$, where $\alpha = \frac{7-3\gamma}{\gamma-1}$.

Citation Networks.

Citation networks are **directed graphs**. In a citation network, the nodes are **scientific publications**, and the directed arcs represent **citations** between papers (where the citing paper points at the cited paper).

- In addition to the network structure, the nodes in a citation network have a natural order, corresponding to their **time of publication**.
- With respect to that order, citations (almost) always **point backwards** in time.
- As a consequence, a citation network is a DAG (**directed acyclic graph**, contains no loops or directed cycles).

Example. The Book's website has a data set of citations between the 1655 articles that appeared from 1978 to 2004 in Scientometrics (https://en.wikipedia.org/wiki/Scientometrics_(journal)), a journal devoted to the field of bibliometrics and the "Science of Science". To load it into this notebook, read the file as an edge list for a directed graph (`Digraph`), insisting that the nodes are of type `int` . Then copy the edges into an empty directed graph on the nodes $1, 2, \ldots, 1655$, to have the nodes in their natural order.

In [2]:
```
E = nx.read_edgelist("scientometrics.net", create_using=nx.DiGraph, nodetype
=int)
G = nx.DiGraph()
G.add_nodes_from(range(1, 1656))
G.add_edges_from(E.edges())
```

Let's check the order and the size of the resulting graph.

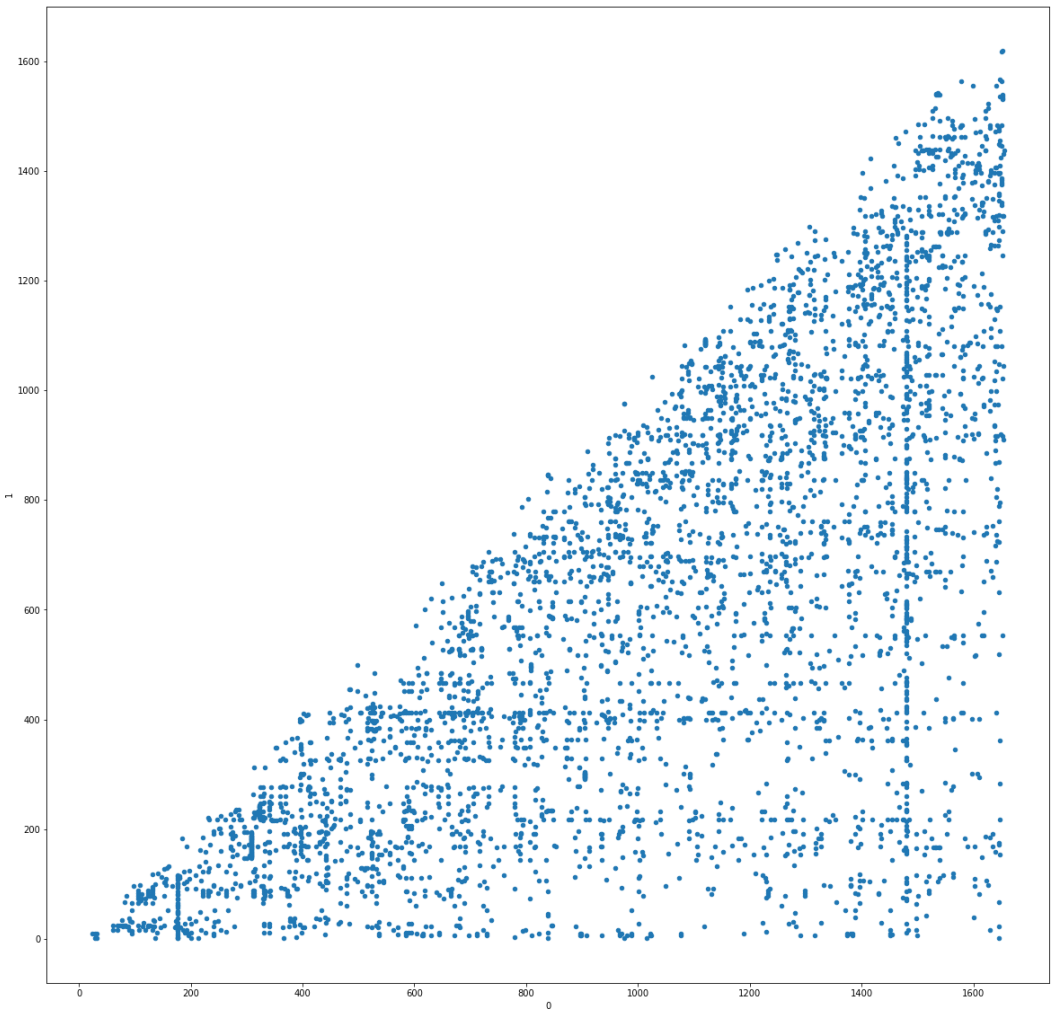
In [3]:
```
n, m = G.order(), G.size()
n, m
```

Out[3]:
```
(1655, 4123)
```

A scatter plot of the adjacency matrix reveals the DAG nature of the graph: all its nonzero entries lie below the line $i = j$.

In [4]:
```
pd.DataFrame(list(G.edges())).plot.scatter(x = 0, y = 1, figsize = (20, 20))
```

Out[4]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7f2b9d78e4e0>
```



What are the highest degrees? Inwards and outwards?

In [5]:
```
max_in = max(dict(G.in_degree()).values())
max_in
```

Out[5]:
```
71
```

In [6]:
```
max_out = max(dict(G.out_degree()).values())
max_out
```

Out[6]:
```
156
```

Next, load the time line.

In [7]:
```
years = np.loadtxt("scientometrics_paper_year.txt", dtype="int")
years[0]
```

Out[7]:
```
array([   1, 1978])
```

Add the year of publication as attribute to each node.

In [8]:
```
for date in years:
    G.nodes[date[0]]['year'] = date[1]
```

How many publications per year? (There may be articles without `year` attribute!)

In [9]:
```
counts = {}
for x in G.nodes():
    year = G.nodes[x].get('year')
    counts[year] = counts.get(year, 0) + 1

print(counts)
```

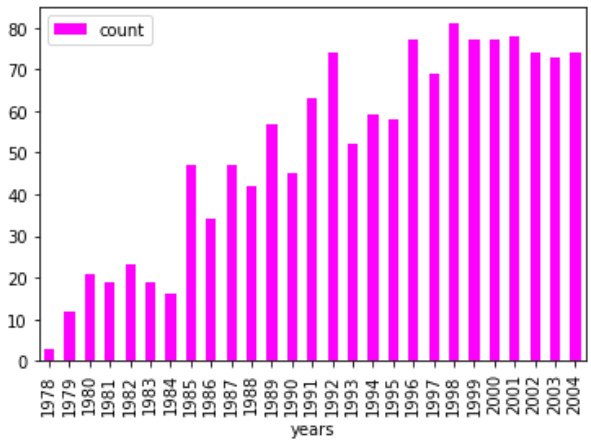
```
{1978: 3, None: 284, 1979: 12, 1980: 21, 1981: 19, 1982: 23, 1983: 19, 1984:
16, 1985: 47, 1986: 34, 1987: 47, 1988: 42, 1989: 57, 1990: 45, 1991: 63, 199
2: 74, 1993: 52, 1994: 59, 1995: 58, 1996: 77, 1997: 69, 1998: 81, 1999: 77,
2000: 77, 2001: 78, 2002: 74, 2003: 73, 2004: 74}
```

Remove the count of undated articles from the dictionary.

In [10]:
```
undated = counts.pop(None, 0)
```

In [11]:
```
data = {}
data['years'] = list(counts.keys())
data['count'] = list(counts.values())
pd.DataFrame(data).plot.bar(x = 'years', y = 'count', rot=90, colormap='spri
ng')
```

Out[11]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7f2b9d429c50>
```



Identify node(s) of maximal citation count (in-degree) and show its development over time ...

In [12]:
```
hi = [x for x in G if G.in_degree(x) == max_in]
print(hi)
```

```
[412]
```

In [13]:
```
hi = hi[0]
citing = [x for x in G if hi in G[x]]
print(citing)
```

```
[448, 502, 505, 520, 524, 527, 528, 541, 543, 580, 585, 590, 594, 603, 611, 6
44, 654, 661, 662, 666, 671, 680, 681, 691, 695, 703, 705, 713, 720, 722, 732
, 779, 787, 793, 806, 823, 830, 836, 849, 878, 886, 902, 913, 1003, 1010, 102
2, 1023, 1034, 1044, 1127, 1130, 1133, 1149, 1153, 1155, 1181, 1183, 1195, 12
00, 1221, 1244, 1266, 1314, 1333, 1396, 1462, 1480, 1510, 1578, 1582, 1641]
```

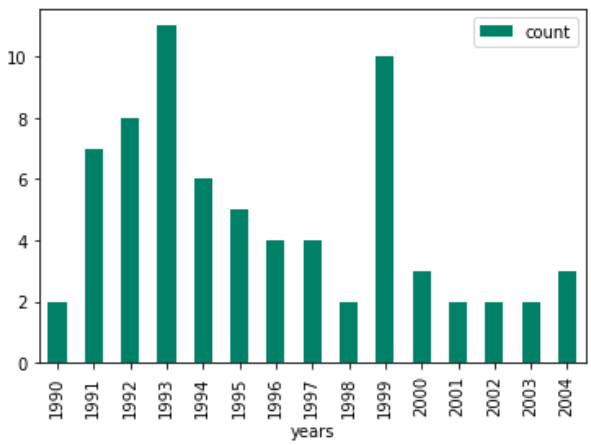
In [14]:
```
citing_years = [G.nodes[x].get('year') for x in G if hi in G[x]]
print(citing_years)
```

```
[1990, 1990, 1991, 1991, 1991, 1991, 1991, 1991, 1991, 1992, 1992, 1992, 1992
, 1992, 1992, 1992, 1992, 1993, 1993, 1993, 1993, 1993, 1993, 1993, 1993, 199
3, 1993, 1993, 1994, 1994, 1994, 1994, 1994, 1994, 1995, 1995, 1995, 1995, 19
95, 1996, 1996, 1996, 1996, 1997, 1997, 1997, 1997, 1998, 1998, 1999, 1999, 1
999, 1999, 1999, 1999, 1999, 1999, 1999, 2000, 2000, 2000, 2001, 2001,
2002, 2002, 2003, 2003, 2004, 2004, 2004]
```

In [15]:
```
years = {}
for year in citing_years:
    years[year] = years.get(year, 0) + 1
```

In [16]:
```
data = {}
data['years'] = list(years.keys())
data['count'] = list(years.values())
pd.DataFrame(data).plot.bar(x='years', y='count', colormap='summer')
```

Out[16]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7f2b9d30e710>
```

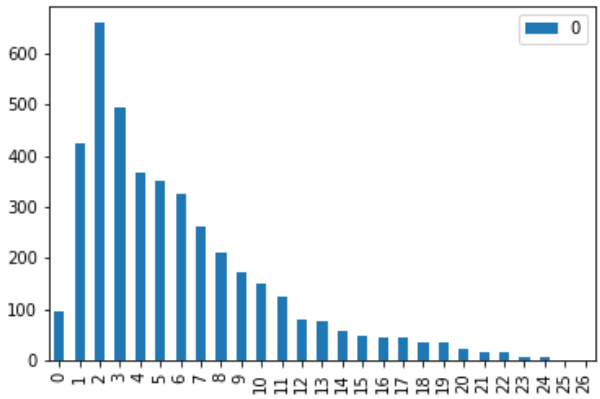


How far back do citations go?

In [17]:
```
span = {}
for e in G.edges():
    years = [G.nodes[n].get('year') for n in e]
    if None not in years:
        s = years[0] - years[1]
        span[s] = span.get(s, 0) + 1
```

In [18]:
```
all = [x[1] for x in sorted(span.items())]
pd.DataFrame(all).plot.bar()
```

Out[18]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7f2b9d2c5048>
```



Go back in time. Recover the state of the citation network at the end of 2003.

In [19]:
```
nodes0 = [x for x in G if G.nodes[x].get('year', 0) < 2004]
nodes1 = [x for x in G if G.nodes[x].get('year', 0) == 2004]
G0 = G.subgraph(nodes0)
G0.order(), G0.size()
```

Out[19]:
```
(1581, 3787)
```

Now analyse the references in the articles published in 2004 with respect to the in-degree of the cited article in the 2003 network:

- add citation counter as attribute to nodes in `G0`
- then loop over all nodes and determine citations per in_degree ...

In [20]:
```
for x in G0:
    G0.nodes[x]['citations'] = 0

for x in nodes1:
    for y in G[x]:
        if y in G0:
            G0.nodes[y]['citations'] += 1
```

In [21]:
```
max_in0 = max(dict(G0.in_degree()).values())
cd = [0 for d in range(max_in0+1)]
hist = [0 for d in range(max_in0+1)]
for x in G0:
    d = G0.in_degree(x)
    hist[d] += 1
    cd[d] += G0.nodes[x]['citations']

print(cd)
print(hist)
```

```
[95, 37, 30, 28, 25, 18, 22, 14, 22, 6, 4, 7, 1, 3, 0, 1, 1, 0, 1, 6, 0, 0, 1
, 0, 0, 2, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3]
[598, 304, 203, 139, 87, 52, 46, 36, 35, 10, 12, 8, 8, 9, 5, 5, 2, 3, 6, 3, 0
, 0, 3, 0, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]
```

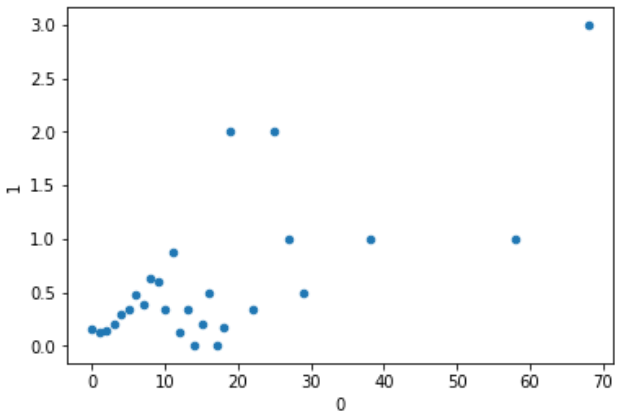
In [22]:
```
cd = [(i, cd[i]/v) for i, v in enumerate(hist) if v != 0]
```

In [23]:
```
print(cd)
```

```
[(0, 0.1588628762541806), (1, 0.12171052631578948), (2, 0.1477832512315271),
(3, 0.2014388489208633), (4, 0.28735632183908044), (5, 0.34615384615384615),
(6, 0.4782608695652174), (7, 0.3888888888888889), (8, 0.6285714285714286), (9
, 0.6), (10, 0.3333333333333333), (11, 0.875), (12, 0.125), (13, 0.3333333333
333333), (14, 0.0), (15, 0.2), (16, 0.5), (17, 0.0), (18, 0.16666666666666666
), (19, 2.0), (22, 0.3333333333333333), (25, 2.0), (27, 1.0), (29, 0.5), (38,
1.0), (58, 1.0), (68, 3.0)]
```

In [24]:
```
pd.DataFrame(cd).plot.scatter(x = 0, y = 1)
```

Out[24]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7f2b9d1a4fd0>
```



Apparently, the number of citations that an article receives is proportional to the number of citations it already has ... the rich get richer.

Linear Preferential Attachment.

The tendency of new nodes to attach themselves to the most popular nodes, with a probabilty that is proportional to a node's popularity, has many names, and has been discussed (in a biological context) as far back as 1925. These days, the phenomenon is known as **linear preferential attachment**, after Barabási-Albert, who rediscoverd it in their analysis of a portion of the WWW in 1999.

Definition (BA-Model). Suppose n, a, b are integers with $0 < b \leq a \ll n$. An (n, a, b)-BA model is a (simple) graph on n nodes, constructed as follows. 1. start with a complete graph on a nodes $\{1, 2, \ldots, a\}$ (at time $t = 0$) 2. for $t = 1, \ldots, n - a - 1$: * add new node $x = a + t$ * and b links to old nodes i with probability

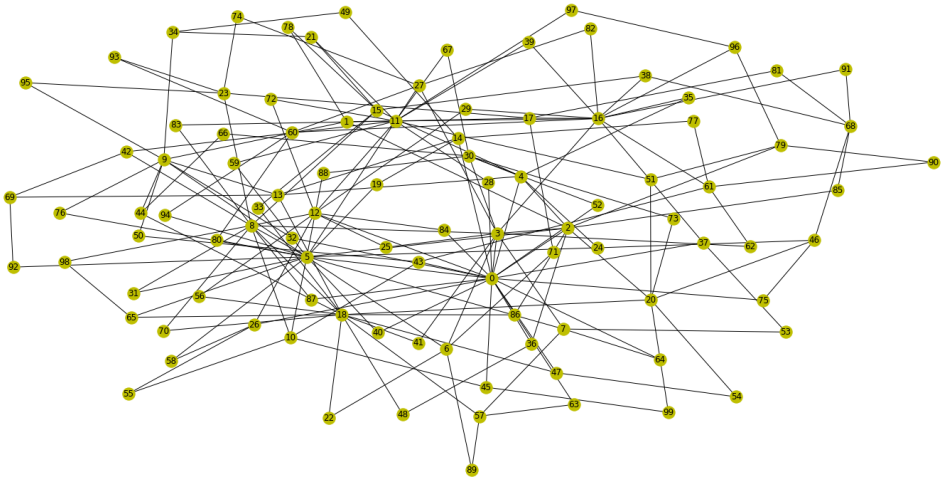
$$p_{x \to i} = \frac{k_{i,t-1}}{2m_{t-1}},$$

where $k_{i,t}$ is the degree of node i at time t and m_t is the number of edges at time t.

In `networkx` , a BA-model can be generated with the function `nx.barabasi_albert_graph(n, b)` (where $a = b$ in terms of our parameters, and the initial graph is an **empty** graph on the a nodes $\{0, 1, \ldots, a - 1\}$).

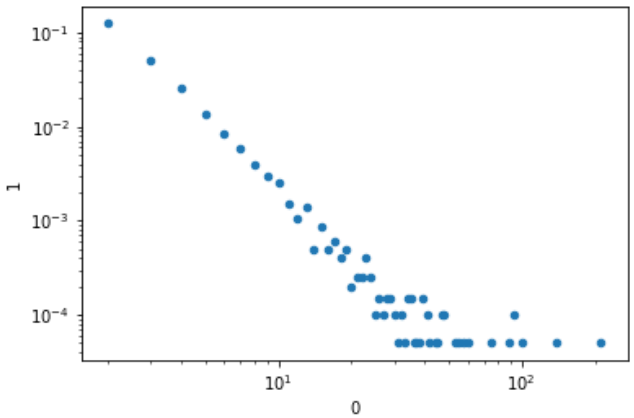
In [25]:
```
B = nx.barabasi_albert_graph(100, 2)
```

In [26]:
```
plt.figure(figsize=(20,10))
nx.draw(B, with_labels=True, node_color='y')
```



In [27]:
```
B = nx.barabasi_albert_graph(5000, 2)
hist = nx.degree_histogram(B)
m2 = 2*B.size()
data = [(i, v/m2) for i, v in enumerate(hist) if v > 0]
pd.DataFrame(data).plot.scatter(x = 0, y = 1, loglog=True)
```

Out[27]: <matplotlib.axes._subplots.AxesSubplot at 0x7f2b9cc139b0>



Fact. An (n, a, b)-BA model (for sufficiently large n) has a power law degree distribution $p_k \simeq ck^{-\gamma}$ with $\gamma = 3$.

More precisely,

$$p_k = \frac{2b(b+1)}{k(k+1)(k+2)} \simeq 2b(b+1)k^{-3}.$$

But ...

In [28]:
```
nx.average_clustering(B)
```

Out[28]:
```
0.0064337352997739005
```

In [ ]:
```

```