CS4423 - Networks

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3. Random Networks

Lecture 13: Properties of Random Graphs.

In [1]:
```
import numpy as np
import pandas as pd
import networkx as nx
import matplotlib.pyplot as plt
```

Probability Distributions

Denote by G_n the set of *all* graphs on the n points $X = \{1, \dots, n\}$. Regard the ER models A and B as **probability distributions**:

Notation: $N = \binom{n}{2}$, the maximal number of edges of a graph $G \in G_n$.

$m(G)$: the number of edges of a graph G.

$G(n, m)$:

$$P(G) = \begin{cases} \binom{N}{m}^{-1}, & \text{if } m(G) = m, \\ 0, & \text{else.} \end{cases}$$

$G(n, p)$:

$$P(G) = p^m (1-p)^{N-m},$$

where $m = m(G)$.

Expected Values

In $G(n, m)$:

- the expected **size** is
$$\bar{m} = m,$$
as every graph G in $G(n, m)$ has exactly m edges.
- the expected **average degree** is
$$\bar{k} = \frac{2m}{n},$$
as every graph has average degree $2m/n$.

Other properties of $G(n, m)$ are less straightforward, it is easier to work with the $G(n, p)$. However, in the limit (as n grows larger) the differences between the two models can be neglected.

In $G(n, p)$, with $N = \binom{n}{2}$:

- the **expected size** is
$$\bar{m} = pN,$$
- and the **variance** is
$$\sigma_m^2 = Np(1-p);$$
- the expected **average degree** is
$$\bar{k} = p(n-1).$$
- and the **standard deviation** is
$$\sigma_k = \sqrt{p(1-p)n}$$

In particular, the **relative standard deviation** (or the **coefficient of variation**) of the size of a random model B graph is
$$\frac{\sigma_m}{\bar{m}} = \sqrt{\frac{1-p}{pN}} = \sqrt{\frac{2(1-p)}{pn(n-1)}} = \sqrt{\frac{2}{n\bar{k}} - \frac{2}{n(n-1)}},$$
a quantity that converges to 0 as $n \to \infty$ if $p(n-1) = \bar{k}$, the average node degree, is kept constant.

In that sense, for large graphs, the fluctuations in the size of random graphs in model B can be neglected.

Degree distribution

Definition. The **degree distribution** $p \colon \mathbb{N}_0 \to \mathbb{R}$, $k \mapsto p_k$ of a graph G is defined as
$$p_k = \frac{n_k}{n},$$
where, for $k \geq 0$, n_k is the number of nodes of degree k in G.

This definition can be extended to ensembles of graphs with n nodes (like the random graphs $G(n, m)$ and $G(n, p)$), by setting
$$p_k = \bar{n}_k / n,$$
where $\bar{n}_k$ denotes the expected value of the random variable n_k over the ensemble of graphs.

- The degree distribution in a random graph $G(n, p)$ is a **binomial distribution**:
$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k} = \mathrm{Bin}(n-1, p, k)$$

- In the limit $n \to \infty$, with $\bar{k} = p(n-1)$ kept constant, the binomial distribution $\mathrm{Bin}(n-1, p, k)$ is well approximated by the **Poisson distribution**:
$$p_k = e^{-\lambda} \frac{\lambda^k}{k!} = \mathrm{Pois}(\lambda, k),$$
where $\lambda = p(n-1)$.

In [2]:
```
import math
math.factorial(16)
```
Out[2]: 20922789888000

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k}$$

In [3]:
```
def binomial(n, k):
    prd, top, bot = 1, n, 1
    for i in range(k):
        prd = (prd * top) // bot
        top, bot = top - 1, bot + 1
    return prd
```

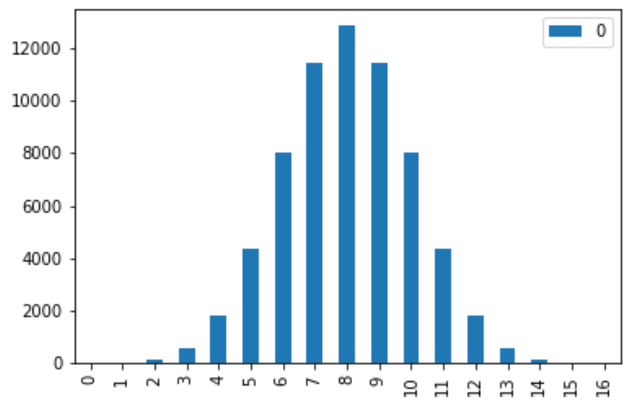
In [4]:
```
l = [binomial(16, k) for k in range(17)]
print(l)
```

```
[1, 16, 120, 560, 1820, 4368, 8008, 11440, 12870, 11440, 8008, 4368, 1820, 560
, 120, 16, 1]
```

In [5]:
```
df = pd.DataFrame(l)
```

In [6]:
```
df.plot.bar()
```

Out[6]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7fe5e44228d0>
```



For n larger than k, Stirlings formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

can be used to approximate a binomial coefficient as follows:

$$\binom{n}{k} = \frac{n \cdot (n-1) \dots (n-k+1)}{k!} \approx \frac{(n-k/2)^k}{k^k e^{-k}\sqrt{2\pi k}} = \frac{(n/k-0.5)^k e^k}{\sqrt{2\pi k}}$$

In [7]:
```
from math import exp, sqrt, pi, log
def binom_approx(n, k):
    return (n/k - 0.5)**k * exp(k) / sqrt(2 * pi * k)
```

In [8]:
```
n = binomial(100, 2)
k = 50
print(binomial(n, k))
```

```
137831378168286642607524453056135424869947114595898731212427379458736583264470
8111045663868872218997025383324264236036878
```

In [9]:
```
print(binom_approx(n, k))
print(binomial(n, k) / 10**120)
```

```
1.3739175898110523e+120
1.3783137816828663
```

Phase Transitions

Point of view: for the random graph $G(n, p)$, suppose that $p = p(n)$ is a function of n, the number of nodes, and study the ensemble of graphs $G(n, p(n))$, as $n \to \infty$.

Then, to say that *almost every graph has property Q* means that the probability of a graph in the ensemble to have property Q tends to 1, as $n \to \infty$.

Theorem (Appearance of Subgraphs). Let F be a connected graph with a nodes and b edges.

- If $p(n)/n^{-a/b} \to 0$ then almost every graph in the ensemble $G(n, p(n))$ does not contain a copy of F.
- If $p(n)/n^{-a/b} \to \infty$ then almost every graph in the ensemble $G(n, p(n))$ does contain a copy of F.
- If $p(n) = cn^{-a/b}$ then, as $n \to \infty$, the number n_F of F-subgraphs in G has distribution $\mathrm{Pois}(\lambda, r)$, where $\lambda = c^b/|\mathrm{Aut(F)}|$, with $|\mathrm{Aut}(F)|$ being the number of *automorphisms* of F.

For example:

- Trees with a nodes appear when $p(n) = cn^{-a/(a-1)}$.
- Cycles of order a appear when $p(n) = cn^{-1}$.
- Complete subgraphs of order a appear when $p(n) = cn^{-2/(a-1)}$.

Numbers of

- triads: $3\binom{n}{3}p^2 = \frac{1}{2}n(n-1)(n-2)p^2$,
- triangles: $\binom{n}{3}p^3 = \frac{1}{6}n(n-1)(n-2)p^3$.

The Giant Connected Component

Definition (Giant Component). A connected component of a graph G is called a giant component if its number of nodes increases with the order n of G as some positive power of n.

Suppose $p(n) = c(n-1)^{-1}$ for some positive constant c. (Then $\bar{k} = c$ remains fixed as $n \to \infty$.)

Theorem (Erdös-Rényi).

- If $c < 1$ the graph contains many small components, orders bounded by $O(\ln n)$.
- If $c = 1$ the graph has large components of order $O(n^{2/3})$.
- If $c > 1$ there is a unique *giant component* of order $O(n)$.

Moreover, $p(n) = \frac{1}{n}\ln n$ is the threshold probability for G to be connected. (This corresponds to $m = \frac{1}{2}n \ln n$ in model A.)

Exercises

1. Design an experiment with random graphs of suitable degree and size to verify the predicted numbers of triads and triangles above.

```
In [ ]:
```