CS4423 - Networks

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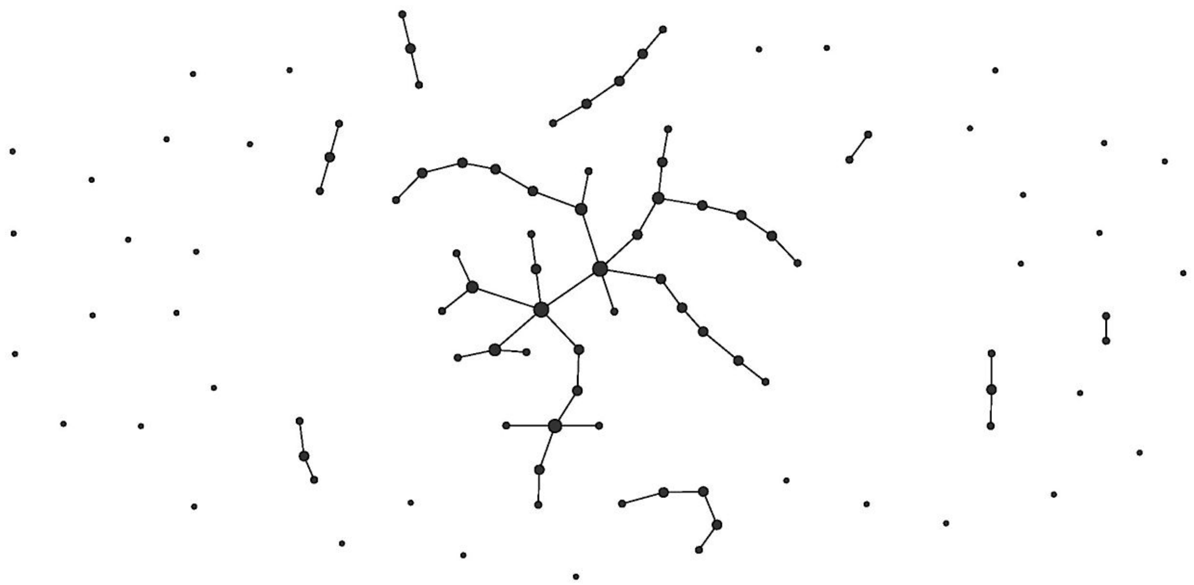
3. Random Networks

Lecture 11: Erdös-Rényi Random Graph Models.

A random graph (https://en.wikipedia.org/wiki/Random_graph) is a mathematical model of a family of networks, where certain parameters (like the number of nodes and edges) have fixed values, but other aspects (like the actual edges) are randomly assigned. The simplest example of a random graph is in fact the network $G(n, m)$ with fixed numbers n of nodes and m of edges, placed randomly between the vertices. Although a random graph is not a specific object, many of its properties can be described precisely, in the form of expected values, or probability distributions.



Randomly Selected Edges

Let us denote by $G(n, m)$ a network with n nodes and m chosen edges, chosen uniformly at random (out of the possible $\binom{n}{2}$). Equivalently, one can choose uniformly at random one network in the **set** $G(n, m)$ of **all** networks on a given set of n nodes with **exactly** m edges.

Definition (ER Model A: Uniform Random Graphs). Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq m \leq N$. The model $G(n, m)$ consists of the ensemble of graphs G with n nodes $X = \{1, 2, \ldots, n\}$, and m randomly selected edges, chosen uniformly from the N possible edges.

One could think of $G(n, m)$ as a probability distribution $P \colon G(n, m) \to \mathbb{R}$, that assigns to each network $G \in G(n, m)$ the same probability

$$P(G) = \binom{N}{m}^{-1},$$

where $N = \binom{n}{2}$.

For example ...

71 3.1 Erdős and Rényi (ER) Models

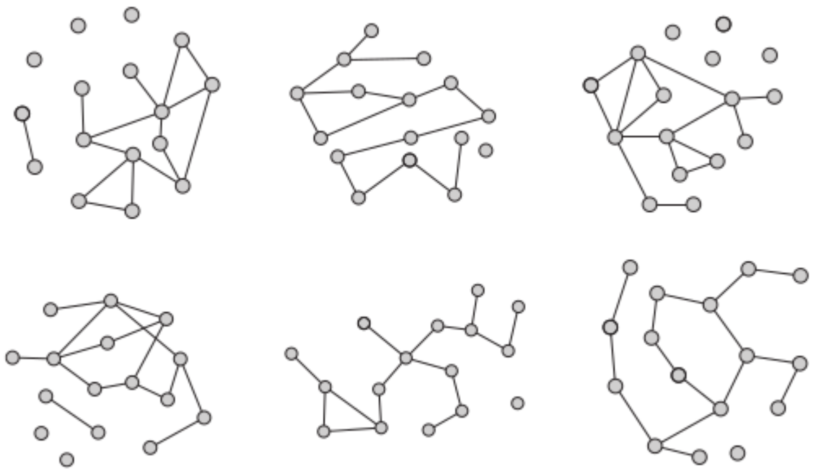


Fig. 3.1 Six different realisations of model A with $N = 16$ and $K = 15$.

In order to make some random graphs, we first need to import the standard libraries ...

In [1]:
```
import numpy as np
import pandas as pd
import networkx as nx
import matplotlib.pyplot as plt
```

... and the `random` module, for a function `choice` that selects a random element from a list.

In [2]:
```
import random
```

Picking edges at random involves picking several (2 to be precise) nodes from the list of nodes of a graph, avoiding repetition. A simple algorithm for selection without repetition can be based on the following function `pick` , which works on a set `elements` and a subset `chosen` of already chosen elements, as follows:

1. pick a random element `x` from the set `elements`
2. if `x` is in the subset `chosen` go back to step 1, else return `x` .

In [3]:
```
def pick(elements, chosen = []):
    while True:
        x = random.choice(elements)
        if x not in chosen:
            return x
```

Note. This algorithm (and its implementation here) has no explict termination condition. Hence, under unfortunate circumstances, it may run for a very long time, or indeed forever ...

In [4]:
```
lll = [2, 3, 5, 7, 11, 13, 17, 19]
pick(lll, [2,3,5,7,11,13])
```

Out[4]: 19

To select l random elements, simply `pick` l times, while keeping track of already selected elements in a list `chosen` .

In [5]:
```
def pick_elements(elements, count):
    chosen = []
    for i in range(count):
        chosen.append(pick(elements, chosen))
    return chosen
```

In [6]:
```
lll = [2, 3, 5, 7, 11, 13, 17, 19]
pick_elements(lll, 3)
```

Out[6]: [17, 19, 13]

Note. Suppose the vertex set X has n elements and that k elements

$$(x_0, x_1, \ldots, x_{k-1})$$

have already been chosen. Then the next element x_k is chosen with probability $\frac{1}{n-k}$ (from the $n-k$ remaining elements at this stage:

Clearly, x_0 is chosen with probability $\frac{1}{n}$ from the n elements. Next, in a first draw x_1 is chosen with probability $\frac{1}{n}$. With the same probability $\frac{1}{n}$, this first draw produces element x_0 again, in which case a second draw has to be carried out, where x_1 has another chance of $\frac{1}{n}$ to be drawn. And x_0 too, calling for a third draw, and so on. In total, x_1's chances of being drawn are

$$P(x_1) = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots = \frac{1}{n} \sum_{l \geq 0} \left(\frac{1}{n}\right)^l = \frac{1}{n} \frac{1}{1 - \frac{1}{n}} = \frac{1}{n-1}.$$

Similarly, x_k's chances of being drawn are

$$P(x_k) = \frac{1}{n} + \frac{k}{n^2} + \frac{k^2}{n^3} + \cdots = \frac{1}{n} \sum_{l \geq 0} \left(\frac{k}{n}\right)^l = \frac{1}{n} \frac{1}{1 - \frac{k}{n}} = \frac{1}{n-k}.$$

To pick a random edge means to pick 2 elements from the node set of G. Again, one can use a list of already chosen edges to avoid repetition.

In [7]:
```
def pick_edge(G, chosen):
    while True:
        edge = pick_elements(list(G.nodes()), 2)
        edge.sort()
        if edge not in chosen:
            return edge
```

In [8]:
```
G = nx.Graph()
G.add_nodes_from(range(16))
```

In [9]:
```
list(G.nodes)
```

Out[9]:
```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]
```

In [10]:
```
pick_edge(G, [[11,14]])
```

Out[10]:
```
[8, 13]
```

To pick m edges, without repetition, simply apply `pick_edge` m times.

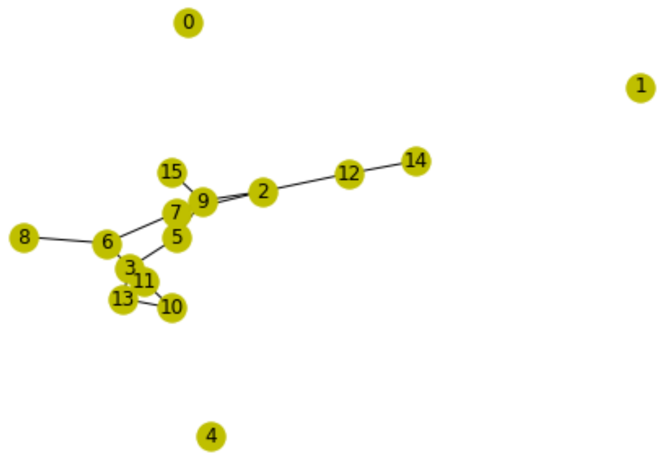
In [11]:
```
def pick_edges(G, m):
    chosen = []
    for i in range(m):
        chosen.append(pick_edge(G, chosen))
    return chosen
```

Now we have all the building blocks for a function `random_graph_A` that takes the order n and the size m of a random ER graph of type A as arguments, and constructs such a graph.

In [12]:
```
def random_graph_A(n, m):
    """construct a random type A graph
    with n nodes and m = links"""
    G = nx.Graph()
    G.add_nodes_from(range(n))
    G.add_edges_from(pick_edges(G, m))
    return G
```

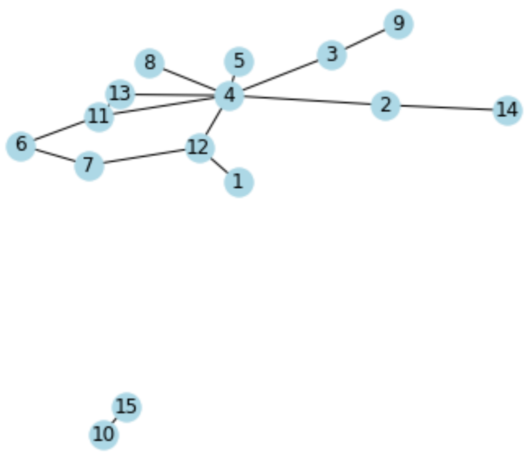
Now we can construct and draw a random graph on 16 vertices with 15 edges.

In [13]:
```
G = random_graph_A(16, 15)
nx.draw(G, with_labels = True, node_color = 'y')
```



The `networkx` version of this random graph constructor is called `gnm_random_graph` and should produce the same random graphs with the same probability.

In [14]:
```
G = nx.gnm_random_graph(16, 15)
nx.draw(G, with_labels = True, node_color = 'lightblue')
```



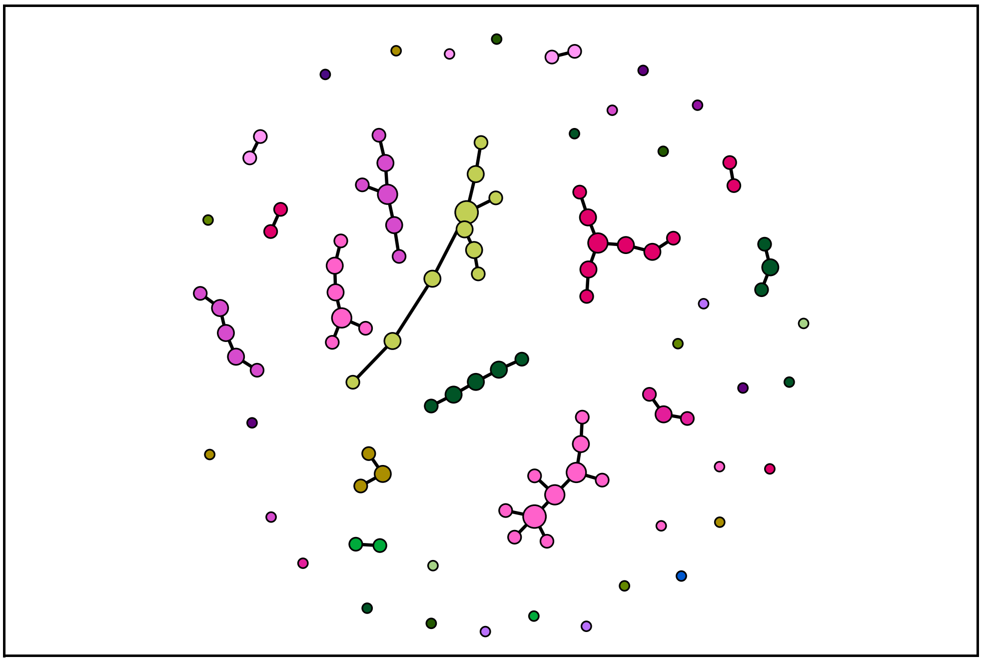
In [15]:
```
list(G.nodes())
```

Out[15]: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]

An Animated Random Graph

In [16]:
```
%%html
<script src="https://d3js.org/d3.v3.min.js"></script>
```

In [17]:
```
%%html
<div id="random">
<div id="count"></div>
  <link rel="stylesheet" href="random.css">
  <script src="random.js"></script>
</div>
```



100 nodes, 46 components, 10 maxSize, 54 links.

Definition (ER Model B: Binomial Random Graphs). Let $n \geq 1$, let $N = \binom{n}{2}$ and let $0 \leq p \leq 1$. The model $G(n, p)$ consists of the ensemble of graphs G with n nodes $X = \{1, 2, \dots, n\}$, and each of the N possible edges chosen with probability p.

The probability P_G of a particular graph $G = (X, E)$ with $X = \{1, 2, \dots, n\}$ and $m = |E|$ edges in the $G(n, p)$ model is

$$P_G = p^m (1 - p)^{N-m}.$$

For example ...

72 Random Graphs

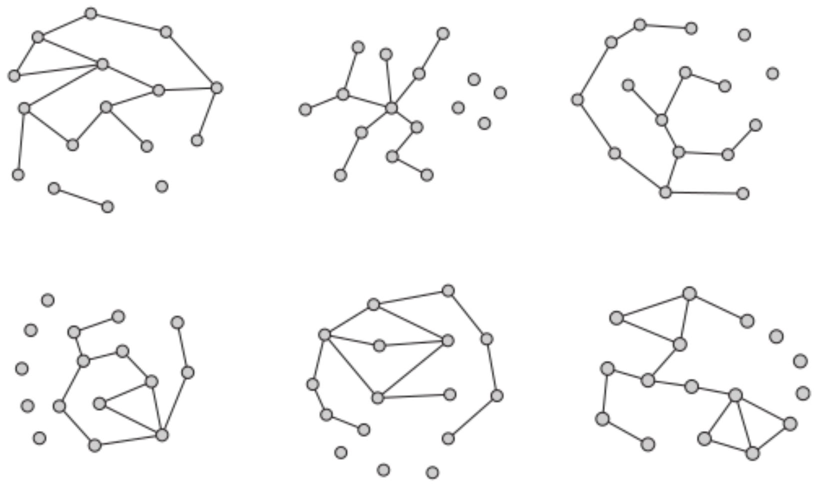


Fig. 3.2 Six different realisations of model B, with $N = 16$ and $p = 0.125$.

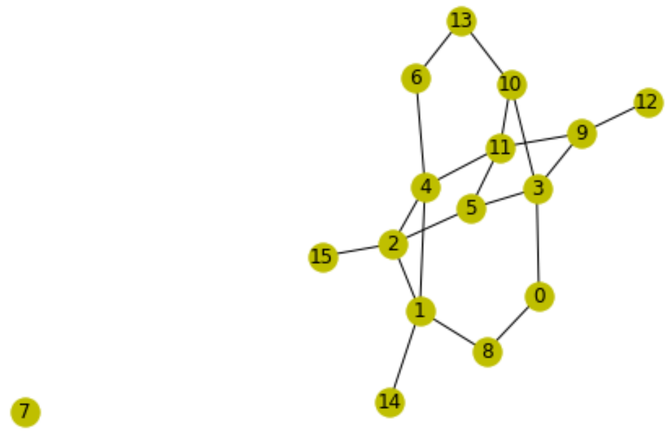
Such a random graph is easy to generate programmatically, using python's basic random number generator `random.random()` which returns a random number in the (half-open) interval $[0, 1)$, every time it is called. If this number is less then p, we include the edge, if not we don't.

In [18]:
```
def random_graph_B(n, p):
    """construct a random type B graph
    with n nodes and edge probability p"""
    G = nx.Graph()
    G.add_nodes_from(range(n))
    for a in range(n):
        for b in range(a):
            if random.random() < p:
                G.add_edge(a, b)
    return G
```

In [19]:
```
15/120
```

Out[19]:
```
0.125
```

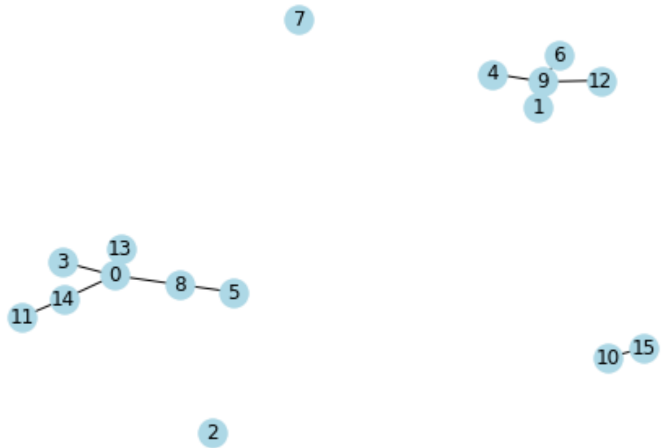
In [20]:
```
G = random_graph_B(16, 0.125)
nx.draw(G, with_labels=True, node_color = 'y')
```



In [21]:
```
G.number_of_nodes(), G.number_of_edges()
```

Out[21]: (16, 20)

In [22]:
```
G = nx.gnp_random_graph(16,0.125)
nx.draw(G, with_labels=True, node_color='lightblue')
```



In [23]:
```
G.number_of_nodes(), G.number_of_edges()
```

Out[23]: (16, 11)

In [ ]: