GE2262 Business Statistics
Topic 2 Basic Probability

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Outline

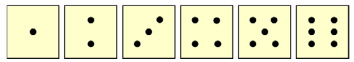
- Basic Probability Concepts
- Conditional Probability
- Counting Rules

Part 1

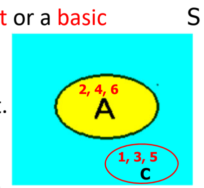
- Basic Probability Concepts
- Conditional Probability
- Counting Rules

Experiment, Sample Space, Event

- Random Experiment
  - Is a process which results in ONE of a number of possible outcomes.
  - Random means we don't know the result of the experiment beforehand
    - Throw a die – gives one of the six possible outcomes, we don't know which number shows up before we throw a die



- Sample Space (S)
  - Is the set of all possible outcomes of an experiment
    - Sample space of throwing a die is {1,2,3,4,5,6}
    - Each of the possible outcome in S is called a simple event or a basic outcome. Example: {1}, {2}, ...



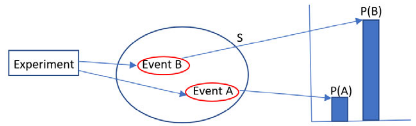
- Event
  - Is a collection of some possible outcomes of the experiment.
  - Is a subset of the sample space
    - Examples: A={2, 4, 6}, C={1,3,5}
  - An event occurs when any one of the outcomes in the event occurs.
    - Example: when a number 4 shows up in throwing a die, event *A* is said to occur.

Mutually Exclusive and Collectively Exhaustive

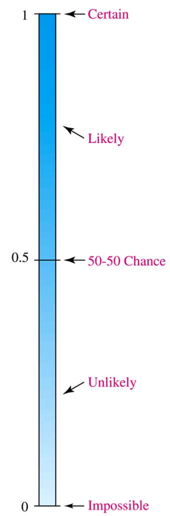
- Events are said to be mutually exclusive if one and only one of them can occur at a time.
  - Throw a die example: A={2, 4, 6}, B={1,2,3}, C={1,3,5}
  - Events A and C are mutually exclusive
  - Events A and B, Events B and C are not mutually exclusive
- A list of events is said to be collectively exhaustive if it includes every possible outcome of the experiment.
  - Events A, B, C are collectively exhaustive
  - $\{\underbrace{2,4,6},\underbrace{1,2,3},\underbrace{1,3,5}\} = \{1, 2, 3, 4, 5, 6\}$

Probability of an Event

- A probability, which is a numerical value, is assigned to each event to denote the chance that the event will occur



- Probability value is between 0 and 1, inclusive $0 \le P(\text{event}) \le 1$
- When P(event) = 0, that event has no chance of occurring
  - The event is called Impossible event
  - Example: probability of obtaining number 7 in throwing a die = 0
- When P(event) = 1, that event is sure to occur
  - The event is called Certain event
  - Example:
    - P(S)=1
    - The probability of obtaining either 1, or 2, or 3, or 4, or 5, or 6 in throwing a die =1 (the event comprises the sample space)



Three Methods to Find Probability

- A priori classical probability method
  - Calculate the probability **objectively** based on prior or theoretical knowledge of the process
- Empirical method (relative frequency method)
  - Calculate the probability **objectively** based on observed data
- Subjective method
  - Determine probability based on a person's experiences, opinions, and analysis of a particular situation

A priori / Theoretical Method

- Assume the outcomes are equally likely to occur.
- Use counting techniques to count the number of possible outcomes in the sample space and the event
- The probability of event A is:

$$P(A) = \frac{\text{No of possible outcomes in A}}{\text{No of possible outcomes in S}} = \frac{n(A)}{n}$$

- Throw a die example

  S={1, 2, 3, 4, 5, 6}

  A={2, 4, 6}, B={1,2,3}, C={1,3,5}

$$P(\mathrm{A}) = \frac{3}{6} = \frac{1}{2}, P(\mathrm{B}) = \frac{3}{6} = \frac{1}{2}, P(\mathrm{C}) = \frac{3}{6} = \frac{1}{2}$$

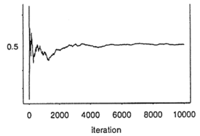
Empirical Method (Relative Frequency Concept of Probability)

- Repeat the experiment *n* times under the same condition.
- The empirical probability of an event is determined by the number of times the event occurred (relative frequency)

$$P(A) = \frac{\text{number of times the event occurred}}{n}$$

- Example: tossing a fair coin 100 times, 58 heads are obtained.

$$P(H) = \frac{58}{100} = 0.58$$



- Example: tossing a biased coin 100 times, 20 heads are obtained $P(H) = \frac{20}{100} = 0.2$

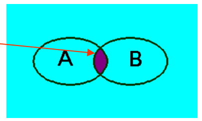
Subjective Method

- Determine the probability based on a person's experiences, opinions, and analysis of a particular situation
  - It may differ from person to person
  - It is useful in situations when a priori or empirical probability cannot be computed
- Example:
  - Manager A assigns a 60% probability of success to its **new** ad campaign
  - Manager B is less optimistic and assigns a 40% of success to the **new** ad campaign

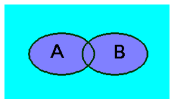
Union and Intersection

- Given events A and B in a sample space, the intersection of A and B (denoted by A AND B , AB, $A \cap B$) is the event that both A and B occur

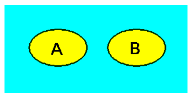
$(A \cap B)$



- The union of A and B (denoted by A OR B , $A \cup B$) is the event that either one or both events occur (whole purple area)



- If A and B are **mutually exclusive**, one and only one of them can occur at a time, they cannot both occur and their circles do not overlap

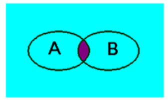


Union and Intersection

Throw a die example :

Event A={ 2, 4, 6}, Event B={1, 2 ,3}, S={1, 2, 3, 4, 5, 6}

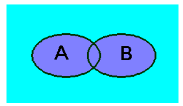
Intersection



$$\text{A and B} = A \cap B = \{2\}$$

$$P(A \text{ and B}) = P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}$$

Union



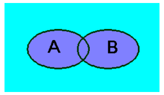
$$A \text{ or B} = A \cup B = \{1,2,3,4,6\}$$

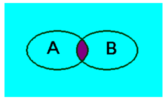
$$P(A \text{ or B}) = P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{5}{6}$$

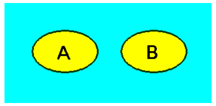
General Addition Rule

- If A and B are not mutually exclusive events, the probability of either event A or event B occurs is defined as

  P(A or B) = P(A ∪ B)= P(A) + P(B) – P(A and B) ← Addition rule
- If A and B are mutually exclusive events, P(A and B) =0, the addition rule is simplified as : P(A or B) = P(A) + P(B)







Throw a die example: Event A={ 2, 4, 6}, Event B={1, 2 ,3}, S={1, 2, 3, 4, 5, 6}

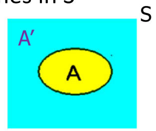
A and B $(A \cap B) = \{2\}$

$$P(A) = \frac{3}{6}, P(B) = \frac{3}{6}, \text{P(A and B)} = P(A \cap B) = \frac{1}{6}$$

$$\text{P(A or B)} = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$$

Complement Rule

- Complement of an event A (denoted A’) is the set of outcomes in S but not in A
- A and A’ are mutually exclusive and collectively exhaustive
- P(A or A’) = P(A ∪ A’)= P(A) + P(A’) =1 => P(A’) =1- P(A)



- The complement rule provides a way to calculate a probability based on the probability of its complement
- Example: toss two coins and count the number of heads. What is the probability that at least one head occurs?

Outcome				
Event	0 heads	1 head	1 head	2 heads

P(no head) = ¼

P(at least one head) = 1-P(no head) = 1- ¼ = ¾

Part 2

- Basic Probability Concepts
- Conditional Probability
- Counting Rules

Conditional Probability

- The conditional probability of event A given event B occurs, denoted by P(A|B), is defined as

  $$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \quad \text{with } P(B) > 0$$
- Similarly, the conditional probability of event B given event A occurs, denoted by P(B|A), is defined as

  $$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{with } P(A) > 0$$
- Example:
  - In a room, there are 5 men and 3 women (one is called Jane). If we choose 1 representative, P(Jane chosen) = 1/8 .
  - Suppose a representative is chosen and is known to be a women, P(Jane chosen) = 1/3 .
  - A = {Jane Chosen}
  - B = {representative is woman}
  - P(Jane chosen| representative is woman)= $P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{1/8}{3/8} = 1/3$

Market Basket Analysis Example

Suppose a supermarket has the following five transactions:

Customer	*Items*
1	Bread, milk
2	Bread, diaper, beer, eggs
3	Milk, diaper, beer, coke
4	Bread, milk, diaper, beer
5	Bread, milk, diaper, coke

$$P(diaper) = \frac{\text{No of transactions containing diaper}}{\text{No of transactions}} = \frac{4}{5} = 0.8$$

$$P(\text{diaper and beer}) = \frac{\text{No of transactions containing both diaper and beer}}{\text{No of transactions}} = \frac{3}{5} = 0.6 \text{ (support)}$$

$$P(\text{beer} \mid \text{diaper}) = \frac{\text{No of transactions containing both diaper and beer}}{\text{No of transactions containing diaper}} = \frac{3}{4} = 0.75 \text{ (confidence)}$$

or

$$P(\text{beer} \mid \text{diaper}) = \frac{P(\text{diaper and beer})}{P(diaper)} = \frac{0.6}{0.8} = 0.75$$

Market Basket Analysis Example

Customer	*Items*
1	Bread, milk
2	Bread, diaper, beer, eggs
3	Milk, diaper, beer, coke
4	Bread, milk, diaper, beer
5	Bread, milk, diaper, coke

Association Report

Relations	Expected Confidence (%)	Confidence (%)	Support (%)	Lift	Transaction Count	Rule	Left Hand of Rule	Right Hand of Rule	Rule Item 1	Rule Item 2	Rule Item 3	Rule Index
2	60.00	100.00	20.00	1.67	1.00	Eggs ==> Beer	Eggs	Beer	Eggs	=======>	Beer	1
2	20.00	33.33	20.00	1.67	1.00	Beer ==> Eggs	Beer	Eggs	Beer	=======>	Eggs	2
2	60.00	75.00	60.00	1.25	3.00	Diaper ==> Beer	Diaper	Beer	Diaper	=======>	Beer	3
2	80.00	100.00	20.00	1.25	1.00	Eggs ==> Bread	Eggs	Bread	Eggs	=======>	Bread	10
2	40.00	50.00	40.00	1.25	2.00	Milk ==> Coke	Milk	Coke	Milk	=======>	Coke	5
2	40.00	50.00	40.00	1.25	2.00	Diaper ==> Coke	Diaper	Coke	Diaper	=======>	Coke	7
2	80.00	100.00	60.00	1.25	3.00	Beer ==> Diaper	Beer	Diaper	Beer	=======>	Diaper	4
2	80.00	100.00	40.00	1.25	2.00	Coke ==> Diaper	Coke	Diaper	Coke	=======>	Diaper	8
2	80.00	100.00	20.00	1.25	1.00	Eggs ==> Diaper	Eggs	Diaper	Eggs	=======>	Diaper	9
2	80.00	100.00	40.00	1.25	2.00	Coke ==> Milk	Coke	Milk	Coke	=======>	Milk	6
2	80.00	75.00	60.00	0.94	3.00	Milk ==> Bread	Milk	Bread	Milk	=======>	Bread	13
2	80.00	75.00	60.00	0.94	3.00	Diaper ==> Bread	Diaper	Bread	Diaper	=======>	Bread	15
2	80.00	75.00	60.00	0.94	3.00	Milk ==> Diaper	Milk	Diaper	Milk	=======>	Diaper	11
2	80.00	75.00	60.00	0.94	3.00	Bread ==> Diaper	Bread	Diaper	Bread	=======>	Diaper	16
2	80.00	75.00	60.00	0.94	3.00	Diaper ==> Milk	Diaper	Milk	Diaper	=======>	Milk	12
2	80.00	75.00	60.00	0.94	3.00	Bread ==> Milk	Bread	Milk	Bread	=======>	Milk	14
2	60.00	50.00	40.00	0.83	2.00	Milk ==> Beer	Milk	Beer	Milk	=======>	Beer	17
2	60.00	50.00	40.00	0.83	2.00	Bread ==> Beer	Bread	Beer	Bread	=======>	Beer	18
2	60.00	50.00	20.00	0.83	1.00	Coke ==> Beer	Coke	Beer	Coke	=======>	Beer	19
2	80.00	66.67	40.00	0.83	2.00	Beer ==> Bread	Beer	Bread	Beer	=======>	Bread	21
2	40.00	33.33	20.00	0.83	1.00	Beer ==> Coke	Beer	Coke	Beer	=======>	Coke	22
2	80.00	66.67	40.00	0.83	2.00	Beer ==> Milk	Beer	Milk	Beer	=======>	Milk	20
2	80.00	50.00	20.00	0.63	1.00	Coke ==> Bread	Coke	Bread	Coke	=======>	Bread	23

Analysts are interested in cases with high support (joint probability) or high confidence (conditional probability) or both.

Multiplication Rule and Statistical Independence

- P(A|B), P(B|A) are called Conditional probability

  $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ $\qquad$ $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$

- P(A), P(B) are called Marginal probability - probability of only 1 event occurring
- P(A and B) is called Joint probability - probability of 2 or more events occurring together
- Multiplication rule
  - P(A and B) = P(A|B)P(B) = P(B|A)P(A)
- Statistical independence
  - Two events, A and B, are independent if the occurrence of event A does not affect the probability of occurrence of event B, or vice versa
    - P(A|B) = P(A), or
    - P(B|A) =P(B), or
    - P(A and B) = P(A)P(B)

Purchase Example – Calculate Marginal Probability

- A sample of 1,000 households were asked about their intentions to purchase a new product sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased the product. The results are:

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- What is the probability of selecting a household that planned to purchase a new product in the next 12 months?
  - P(planned to purchase) = 250/1000 =0.25
- What is the probability of selecting a household that actually purchased the product in the next 12 months?
  - P(actually purchased) = 300/1000 =0.3

Purchase Example – Calculate Joint Probability

- A sample of 1,000 households were asked about their intentions to purchase a new product sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased the product. The results are:

Planned to Purchase	Actually Purchased: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- What is the probability of selecting a household that planned to purchase a new product and actually purchased?

P(planned to purchase and actually purchased)

$= \frac{\text{No of households that planned to purchase and actually purchased}}{\text{Total number of households}} = \frac{200}{1000} = 0.2$

- What is the probability of selecting a household that planned to purchase a new product and actually did not purchase?

P(planned to purchase and actually did not purchase)

$= \frac{\text{No of households that planned to purchase and actually did not purchase}}{\text{Total number of households}} = \frac{50}{1000} = 0.05$

Purchase Example – Addition Rule

- A sample of 1,000 households were asked about their intentions to purchase a new product sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased the product. The results are:

Planned to Purchase	Actually Purchased: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

P(planned to purchase) = 250/1000 =0.25

P(actually purchased) = 300/1000 =0.3

P(planned to purchase and actually purchased) $= \frac{200}{1000} = 0.2$

P(Planned to purchase or actually purchased)

= P(Planned to purchase) + P(Actually purchased) –

P(Planned to purchase and actually purchased)

= 0.25 + 0.30 – 0.20 = 0.35

Purchase Example – Conditional Probability and Statistical Independence

- A sample of 1,000 households were asked about their intentions to purchase a new product sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased the product. The results are:

Planned to Purchase	Actually Purchased: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

P(planned to purchase) = 250/1000 =0.25

P(actually purchase) = 300/1000 =0.3

P(planned to purchase and actually purchased) $= \frac{200}{1000} = 0.2$

P(planned to purchase)* P(actually purchase) = 0.25*0.3 = 0.075

P(Actually purchased | Planned to purchase)

$= \frac{\text{P(Planned to purchase and actually purchased)}}{\text{P(Planned to purchase)}} = \frac{0.2}{0.25} = 0.80$

As (1) P(actually purchased | planned to purchase) $\neq$ P(actually purchased),

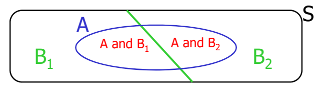
(2) P(planned to purchase and actually purchased) $\neq$ P(planned to purchase) * P(actually purchased),

"Planned to purchase" and "Actually purchase" are not statistically independent.

Purchase Example –Law of Total Probability

- If B_1 and B_2 is a partition of the sample space S, then for any event A,

  P(A) = P(A and B_1) + P(A and B_2)



- A sample of 1,000 households were asked about their intentions to purchase a new product sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased the product. The results are:

Planned to Purchase	Actually Purchased: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

Let A = planned to purchase, B_1 = actually purchase, B_2= did not actually purchase

P(planned to purchase and actually purchased) $= 0.2$

P(planned to purchase and actually did not purchase) $= 0.05$

P(Planned to purchase)

= P(Planned to purchase and actually purchased)

+P(Planned to purchase and did not actually purchase)

= 0.2 + 0.05 = 0.25

Bayes Theorem

Worker Example

- A company is considering changing its starting business hour from 8am to 7:30am. The company has 1200 workers, including 450 office and 750 production workers. A census shows that 370 production workers favor the change, and a total of 715 office and production workers favor the change. Is worker type and favor change independent?

Worker type	Favor change?		Total
	Favor	Not favor	
Office	?	?	450
Production	370	?	750
Total	715	?	1200

Worker Example

- Is worker type and favor change independent?

Worker type	Favor change?		Total
	Favor	Not favor	
Office	345	105	450
Production	370	380	750
Total	715	485	1200

P(favor change) = $\frac{715}{1200}$ = 0.596

P(favor change| office worker)= $\frac{P(office\ worker\ and\ favor)}{P(office\ worker)} = \frac{345/1200}{450/1200}$ = 0.767

P(favor change | production worker)

= $\frac{P(production\ worker\ and\ favor)}{P(production\ worker)} = \frac{370/1200}{750/1200}$ = 0.4933

$P(Production\ workers\ and\ Favor\ change)$

$= \frac{370}{1200} = 0.3083$

$P(Production\ workers) \times P(Favor\ change)$

$= \frac{750}{1200} \times \frac{715}{1200} = 0.3724$

As (1) P(favor change| production worker) ≠ P(favor change)

(2) P(Production workers and Favor change) ≠ P(Production workers)×P(Favor change),

"Worker type" and "favor change" are not statistically independent.

Part 3

- Basic Probability Concepts
- Conditional Probability
- Counting Rules

Counting Rule 1

- For a sample space with a large number of possible outcomes, counting rules can be used to compute probabilities
- Counting rule 1:
  - If any one of *k* different mutually exclusive and collectively exhaustive events can occur on each of *n* trials, the number of possible outcomes is equal to

    k^n
  - Example: If you roll a fair die 3 times then there are 6^3 = 216 possible outcomes

Counting Rule 2

- If there are k_1 events on the first trial, k_2 events on the second trial, … and k_n events on the n^{th} trial, the number of possible outcomes is

  $(k_1)(k_2)\cdots(k_n)$

- Example: You want to go to a park, eat at a restaurant, and see a movie on a holiday. There are 3 parks, 4 restaurants, and 6 movie choices. How many different possible choices are there for you?
  - Answer: (3)(4)(6) = 72 different choices

Counting Rule 3

- The number of ways that n items can be arranged in order is $n! = (n)(n-1)\cdots(1)$

  where $n!$ is called n factorial

- Example: You have five books to put on a bookshelf. How many different ways can these books be placed on the shelf?
  - Answer: 5! = (5)(4)(3)(2)(1) = 120 different possibilities

Counting Rule 4

- Permutations: The number of ways of arranging x objects selected from n objects in order is

$$
\begin{aligned}
{}_nP_x &= n(n-1)\ldots(n-x+1) \\
&= \frac{n(n-1)\ldots(n-x+1)(n-x)(n-x-1)\ldots1}{(n-x)(n-x-1)\ldots1} \\
&= \frac{n!}{(n-x)!}
\end{aligned}
$$

- Example: You have five books and are going to put three on a bookshelf. How many different ways can the books be ordered on the bookshelf?
  - Answer: ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$ different ways

Counting Rule 5

- Combinations: The number of ways of selecting x objects from n objects, irrespective of order, is

$$ {}_nC_x = \frac{n!}{x!(n-x)!} $$

- Note that ${}_nC_x(x!) = {}_nP_x$

- Example: You have five books and are going to select three to read. How many different combinations are there, ignoring the order in which they are selected?
  - Answer: ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{120}{(6)(2)} = 10$ different combinations