GE2262 Business Statistics
Topic 6 Hypothesis Testing for Population Mean

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Outline

- General Steps in Hypothesis Testing
- Hypothesis Testing for One Population Mean
- Examples

Reference

Levine, D.M., Krehbiel, T.C. and Berenson, M.L., *Business Statistics: A First Course*, Pearson Education Ltd, Chapter 9

Part One

- General Steps in Hypothesis Testing
- Hypothesis Testing for One Population Mean
- Examples

Inferential Statistics

- Inferential Statistics (Topics 5 – 7)
  - to infer, conclude, and make decisions about a large group (population) from a small group (sample).
- Estimation
  - Estimate the unknown population parameter
  - Examples
    - we want to estimate the mean waiting time of bank service, …
    - We want to estimate the proportion of customers being satisfied with bank service
- Hypothesis Testing
  - Test whether a hypothesis (claim or statement) about the population parameter holds or not
  - Example: suppose a bank manager claims that (1) the mean waiting time for their service is no more than 10 mins and (2) the proportion of customers being satisfied with their service is at least 0.9. We want to estimate whether the manager's claims hold or not

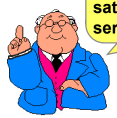
Measure	Population parameter	Sample statistic $\bar{X}$	Lecture
Mean	μ	$\hat{p}$	Topic 5 (estimation) Topic 6 (hypothesis testing)
Proportion	p		Topic 7 (estimation and hypothesis testing)

What is a Hypothesis?

- A hypothesis is a claim or statement about the population parameter rather than a sample statistic

Population mean: I claim the mean waiting time for our service is no more than 10 mins!

Population proportion: I claim the proportion of customers being satisfied with our service is at least 0.9!



- Two Types of Hypothesis
  - NULL HYPOTHESIS (H_0): A maintained hypothesis that is held to be true until sufficient evidence to the contrary is obtained
    - H_0 : established, to be protected, "Mr X is innocent"
  - ALTERNATIVE HYPOTHESIS (H_1 or H_a): A hypothesis against which the null hypothesis is tested and which will be held to be true if the null is held false
    - H_1 : felt to be correct, to challenge H_0, "Mr X is guilty"
- The goal of hypothesis testing is to see if there is enough evidence to reject the null hypothesis. If there is not enough evidence, then we fail to reject the null hypothesis.

Three Different Sets of Hypothesis

Lower-tail test	Upper-tail test	Two-tail test
$H_0: \mu = \mu_0$ or $H_0: \mu \geq \mu_0$	$H_0: \mu = \mu_0$ or $H_0: \mu \leq \mu_0$	$H_0: \mu = \mu_0$
$H_1: \mu < \mu_0$	$H_1: \mu > \mu_0$	$H_1: \mu \neq \mu_0$

- Hypothesis Always about a population parameter (μ), rather than a sample statistic ($\bar{X}$)
- The null hypothesis, H_0
  - States the status quo
  - Always assumed to be true at start
  - Represent the current belief in a situation
  - Always contains the "=" , or "≤",or "≥"sign
- The alternative hypothesis, H_1
  - The opposite of the null hypothesis
  - Challenges the status quo
  - Is generally the hypothesis that the researcher is trying to prove
  - Never contains the "=" , or "≤",or "≥"sign

Two types of Decisions and Errors

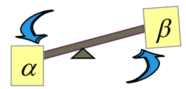
- Two decisions -- at the end of the test, one of two decisions will be made:
  - Do not reject H_0
  - Reject H_0

Decision	The Truth: H_0 True	The Truth: H_0 False
Do not reject H_0	Correct decision	Type II Error P (Type II Error)= β
Reject H_0	Type I Error P (Type I Error)= α	Correct decision

- Two Types of Error
  - Type I Error
    - Reject a true null hypothesis (reject Ho when Ho is true)
    - Probability of Type I error is denoted α
      - $\alpha = P(Reject\ H_0 | H_0\ true)$
      - Also called level of significance of the test
      - Set by researcher in advance
  - Type II Error
    - Fail to reject a false null hypothesis (do not reject Ho when Ho is false)
    - Probability of Type II error is denoted β
    - $\beta = P(Do\ not\ reject\ H_0 |\ H_0\ false)$

Two types of Decisions and Errors

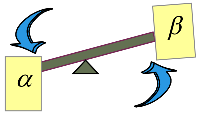
Decision	The Truth: H_0 True	The Truth: H_0 False
Do not reject H_0	Correct decision	Type II Error P (Type II Error)= β
Reject H_0	Type I Error P (Type I Error)= α	Correct decision



- Type I and Type II errors have inverse relationship for a fixed sample size
  - If Type I error probability (α) increases, then Type II error probability (β) decreases and vice versa
  - We cannot decrease both
- A criminal trial
  - H_0: innocent, H_1: guilty
  - Type I error : convicting an innocent person
  - Type II error : let a guilty person go free
  - The cost of convicting an innocent person (Type I error) is high
    → need to choose very small α
- Choose smaller Type I Error when the cost of rejecting the null hypothesis is high

Two types of Decisions and Errors

Decision	The Truth	
	H_0 True	H_0 False
Do not reject H_0	Correct decision	Type II Error P (Type II Error)= β
Reject H_0	Type I Error P (Type I Error)= α	Correct decision



- Ways to reduce the probability of making a Type II error
  - By increasing α
    - This is preferred only if the cost of committing Type II error is higher than that of Type I error
  - By increasing the sample size for the test.
    - This is preferred if there are sufficient resources to do so

How to Set Significance Level?

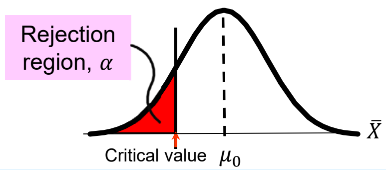
- Example:
  - H_0 : Mr X is innocent
  - H_1 : Mr X is guilty
- α= P(Type I error)
  - = P(conclude Mr X is guilty | he is innocent)
- Parking offence: α can be large, say 0.2
- Speeding case: α can be moderate, say 0.1
- Murder case: α must be small, say 0.00001
- Usual α values: 0.01, 0.05, 0.1

Critical Value Approach to Hypothesis Testing

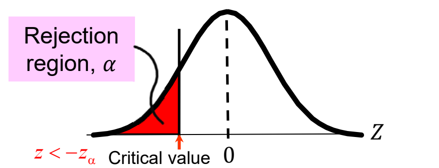
- Collect sample data, convert sample statistic ($\bar{X}$) to test statistic (Z or t)
- Obtain critical value(s) for a specified α from Z or t table
- Set up the decision rule to identify the rejection region
  - If the test statistic falls in the rejection region, reject H_0
  - Otherwise, do not reject H_0

Critical Value Approach to Hypothesis Testing

- For **lower-tail** test: $H_0: \mu \geq \mu_0$ against $H_1: \mu < \mu_0$

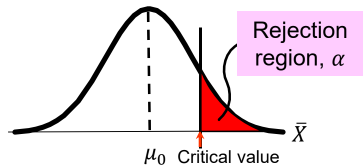


$\bar{X}$ **must be significantly smaller than** μ_0 **to reject** H_0

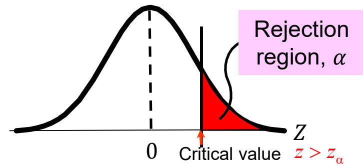


Z **must be significantly smaller than 0 to reject** H_0

- For **upper-tail** test: $H_0: \mu \leq \mu_0$ against $H_1: \mu > \mu_0$



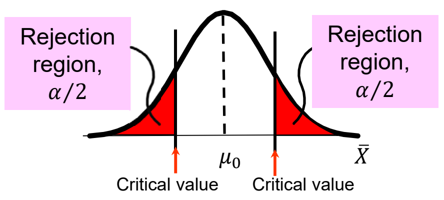
$\bar{X}$ **must be significantly larger than** μ_0 **to reject** H_0



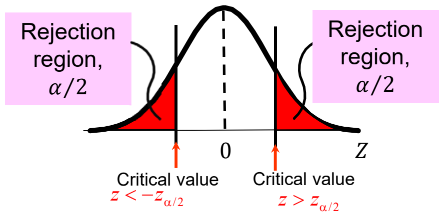
Z **must be significantly larger than 0 to reject** H_0

Critical Value Approach to Hypothesis Testing

- For **two-tail** test: $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$



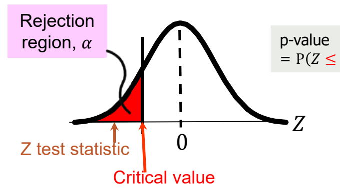
$\bar{X}$ **must be significantly different from** μ_0 **to reject** H_0



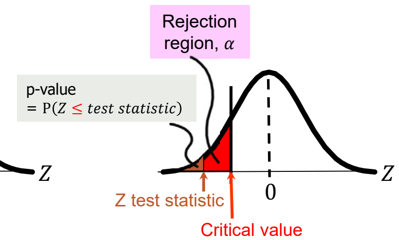
Z **must be significantly different from 0 to reject** H_0

p-value Approach to Hypothesis Testing

- Convert the test statistic (Z *or* t) to p-value
  - The p-value is the probability of obtaining a test statistic as extreme or more extreme ($\leq$ or $\geq$) than the observed test statistic value given H_0 is true
- Compare the p-value with the level of significance α
  - If p-value < α, reject H_0
  - Otherwise, do not reject H_0



Reject H_0 if Z test statistic falls in rejection region



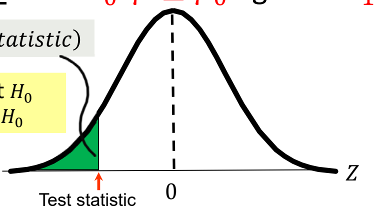
Reject H_0 if p-value < α

p-value Approach to Hypothesis Testing

- For **lower-tail** test: $H_0: \mu \geq \mu_0$ against $H_1: \mu < \mu_0$

p-value = $P(Z \leq test\ statistic)$

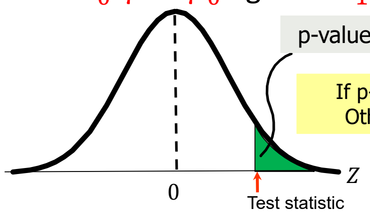
If p-value $< \alpha$, then reject H_0
Otherwise, do not reject H_0



- For **upper-tail** test: $H_0: \mu \leq \mu_0$ against $H_1: \mu > \mu_0$

p-value = $P(Z \geq test\ statistic)$

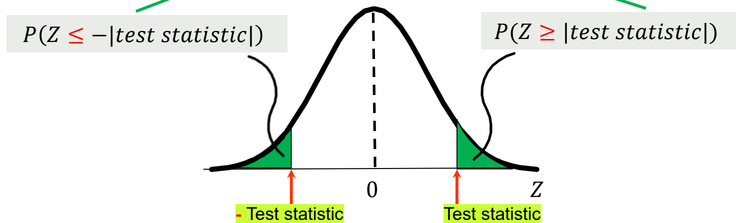
If p-value $< \alpha$, then reject H_0
Otherwise, do not reject H_0



p-value Approach to Hypothesis Testing

- For **two-tail** test: $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$

p-value = $P(Z \leq -|test\ statistic|) + P(Z \geq |test\ statistic|)$



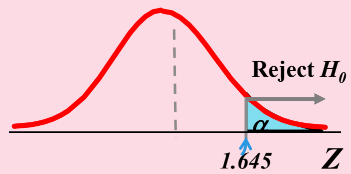
If p-value $< \alpha$, then reject H_0
Otherwise, do not reject H_0

General Steps in Hypothesis Testing

Example: A bank manager wants to test whether the mean waiting time for providing bank service is 10 mins or less at 5% significance level. (Assume waiting time is normally distributed and population standard deviation is known)

1. State the H_0	$H_0 : \mu \leq 10$
2. State the H_1	$H_1 : \mu > 10$
3. Choose α	$\alpha = .05$
4. Choose n	$n = 40$ days
5. Determine test statistic	Z *test*

General Steps in Hypothesis Testing

6. Determine critical value(s) and rejection region based on α	
7. Collect sample data	Record waiting time for 40 days
8. Compute test statistic and p-value assuming that H_0 is true	Z=2, p-value = .0228
9. Make statistical decision	Reject null hypothesis
10. Express conclusion	The mean waiting time for providing bank service is more than 10 mins

Five-Step Hypothesis Testing Procedure

Step 1: State the null and alternative hypotheses

Step 2: Determine the test statistic (Z or t)

Step 3: Determine the rejection region based on the significance level

Step 4: Compute the value of the test statistic

Step 5: Make statistical decision (Do not reject H_0 , Reject H_0) and give a conclusion in terms of the original problem

Part Two

- General Steps in Hypothesis Testing
- Hypothesis Testing for One Population Mean
- Examples

100(1−α)%Confidence Interval Estimation for Population Mean μ

Population distribution	Sample size n	σ known	σ unknown
Normal	Large (n≥30)	$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$\bar{X} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$ use z value if no t value for large n in the table
	Small (n<30)	$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$\bar{X} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$
Not normal	Large (n≥30) Due to central limit theorem	$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$\bar{X} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$ use z value if no t value for large n in the table

Case 1: σ known, (Population normal or n large), use Z

Case 2: σ unknown, (Population normal or n large), use *t* (use *Z* if you cannot find *t* value for large *n* in the *t* table)

Hypothesis Testing for Population Mean μ

Population distribution	Sample size n	σ known	σ unknown
Normal	Large (n≥30)	$Z=\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ reject H_0 if Lower tail: $z<-z_\alpha$ Upper tail: $z>z_\alpha$ Two tail: $z<-z_{\alpha/2}$ or $z>z_{\alpha/2}$	$t=\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ reject H_0 if Lower tail: $t<-t_\alpha$ Upper tail: $t>t_\alpha$ Two tail: $t<-t_{\alpha/2}$ or $t>t_{\alpha/2}$
	Small (n<30)		
Not normal	Large (n≥30)	$Z=\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ reject H_0 if Lower tail: $z<-z_\alpha$ Upper tail: $z>z_\alpha$ Two tail: $z<-z_{\alpha/2}$ or $z>z_{\alpha/2}$	$t=\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ reject H_0 if Lower tail: $t<-t_\alpha$ Upper tail: $t>t_\alpha$ Two tail: $t<-t_{\alpha/2}$ or $t>t_{\alpha/2}$

Case 1: σ known, (Population normal or n large), use Z

Case 2: σ unknown, (Population normal or n large), use *t* (use *Z* if you cannot find *t* value for large *n* in the *t* table)

Case 1: σ Known, Population Normal or *n* large, use Z (Lower Tail Test)

$H_0: \mu = \mu_0$ (or $H_0: \mu \ge \mu_0$)

$H_1: \mu < \mu_0$

Test statistic is $\bar{X}$

If $\bar{X} \ge \mu_0$, accept H_0

If $\bar{X} < \mu_0$ slightly, do not reject H_0

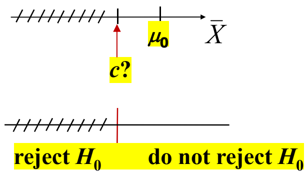
If $\bar{X} < \mu_0$ substantially, reject H_0

c is a critical value such that

$\bar{X} \ge c$, do not reject H_0

$\bar{X} < c$, reject H_0

How to find c? Use α



Case 1: σ Known, Population Normal or *n* large, use Z (Lower Tail Test)

If H_0 is true (ie $\mu=\mu_0$), $\bar{X} \sim N(\mu_0, \sigma^2/n)$

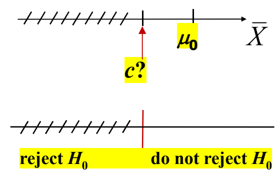
P(reject H_0 | H_0 is true)=α

$$P\left[\bar{X} < c \;\middle|\; \bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)\right] = \alpha$$

$$P\left[\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < \frac{c-\mu_0}{\sigma/\sqrt{n}}\right] = \alpha \quad \Rightarrow \quad P\left[Z < \frac{c-\mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

$$\frac{c-\mu_0}{\sigma/\sqrt{n}} = -z_\alpha \Rightarrow c = \mu_0 - z_\alpha(\sigma/\sqrt{n})$$

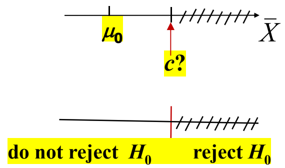
Reject H_0 if $\bar{X} < \mu_0 - z_\alpha(\sigma/\sqrt{n})$, or if $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < -z_\alpha$



Case 1: σ Known, Population Normal or *n* large, use Z (Upper Tail Test)

$H_0 : \mu = \mu_0$ $(or\ H_0 : \mu \le \mu_0\ )$

$H_1 : \mu > \mu_0$



Test statistic is $\overline{X}$

If $\overline{X} \le \mu_0$, do not reject H_0

If $\overline{X} > \mu_0$ slightly, do not reject H_0

If $\overline{X} > \mu_0$ substantially, reject H_0

c is a critical value such that

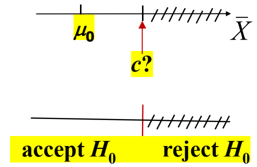
$\overline{X} \le c$, do not reject H_0

$\overline{X} > c$, reject H_0

How to find c? Use α

Case 1: σ Known, Population Normal or *n* large, use Z (Upper Tail Test)

If H_0 is true (ie $\mu=\mu_0$), $\overline{X} \sim N(\mu_0, \sigma^2/n)$



P(reject H_0 | H_0 is true)$=\alpha$

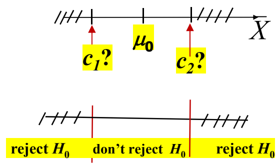
$$P\left[\overline{X} > c \ \middle|\ \overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)\right] = \alpha$$

$$P\left[\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{c-\mu_0}{\sigma/\sqrt{n}}\right] = \alpha \Rightarrow P\left[Z > \frac{c-\mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

$$\frac{c-\mu_0}{\sigma/\sqrt{n}} = z_\alpha \Rightarrow c = \mu_0 + z_\alpha(\sigma/\sqrt{n})$$

Reject H_0 if $\overline{X} > \mu_0 + z_\alpha(\sigma/\sqrt{n})$, or if $Z = \dfrac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha$

Case 1: σ Known, Population Normal or *n* large, use Z (Two Tail Test)



$H_0 : \mu = \mu_0$

$H_1 : \mu \ne \mu_0$

Test statistic is $\overline{X}$

If $\overline{X} = \mu_0$, do not reject H_0

If $\overline{X} < \mu_0$ slightly or $\overline{X} > \mu_0$ slightly, do not reject H_0

If $\overline{X} < \mu_0$ substantially or $\overline{X} > \mu_0$ substantially, reject H_0

c_1 and c_2 are critical values such that

$c_1 \le \overline{X} \le c_2$, do not reject H_0

$\overline{X} < c_1$ or $\overline{X} > c_2$, reject H_0

How to find c_1 and c_2? Use α

Case 1: σ Known, Population Normal or *n* large, use Z (Two Tail Test)

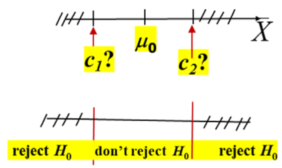
If H_0 is true (ie $\mu=\mu_0$), $\overline{X} \sim N(\mu_0, \sigma^2/n)$

P(reject H_0 | H_0 is true)$=\dfrac{\alpha}{2}$ on both lower tail and upper tail

On lower tail, $P\left[\overline{X} < c_1 \ \middle|\ \overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)\right] = \dfrac{\alpha}{2}$

$$P\left[\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} < \frac{c_1-\mu_0}{\sigma/\sqrt{n}}\right] = \frac{\alpha}{2} \Rightarrow P\left[Z < \frac{c_1-\mu_0}{\sigma/\sqrt{n}}\right] = \frac{\alpha}{2}$$

$$\frac{c_1-\mu_0}{\sigma/\sqrt{n}} = -z_{\alpha/2} \Rightarrow c_1 = \mu_0 - z_{\alpha/2}(\sigma/\sqrt{n})$$



Case 1: σ Known, Population Normal or *n* large, use Z (Two Tail Test)

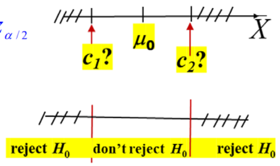
On upper tail, $P\left[\bar{X} > c_2 \;\middle|\; \bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)\right] = \frac{\alpha}{2}$

$$P\left[\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{c_2-\mu_0}{\sigma/\sqrt{n}}\right] = \frac{\alpha}{2} \quad \Rightarrow \quad P\left[Z > \frac{c_2-\mu_0}{\sigma/\sqrt{n}}\right] = \frac{\alpha}{2}$$

$$\frac{c_2-\mu_0}{\sigma/\sqrt{n}} = z_{\alpha/2} \Rightarrow c_2 = \mu_0 + z_{\alpha/2}(\sigma/\sqrt{n})$$

Reject H_0 if $\bar{X} < \mu_0 - z_{\alpha/2}(\sigma/\sqrt{n})$ or $\bar{X} > \mu_0 + z_{\alpha/2}(\sigma/\sqrt{n})$;

or if $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2}$ or $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2}$

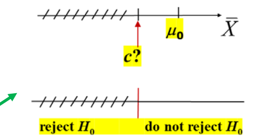


Case 1 Summary: σ Known, Population Normal or *n* large, use Z

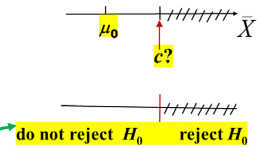
If H_0 is true (ie $\mu=\mu_0$), $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

Reject H_0 if

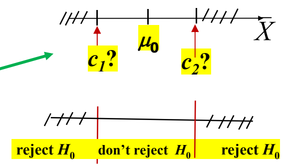
Lower tail: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < -z_\alpha$



Upper tail: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha$



Two tail: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2}$ or $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2}$

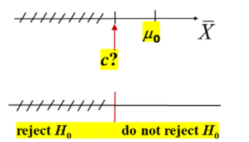


Case 2: σ Unknown, Population Normal or *n* large, use *t*

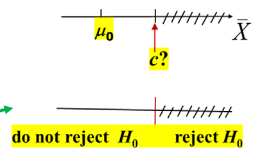
If H_0 is true (ie $\mu=\mu_0$), $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} \sim t_{n-1}$

Reject H_0 if

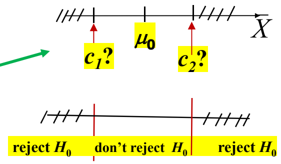
Lower tail: $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} < -t_\alpha$



Upper tail: $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} > t_\alpha$



Two tail: $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} < -t_{\alpha/2}$ or $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} > t_{\alpha/2}$



Part Three

- General Steps in Hypothesis Testing
- Hypothesis Testing for One Population Mean
- Examples

Example 1: σ Known, Population Normal (Case 1, use Z), Two Tail Test

- A company sets up the filling machine for cereal. Each cereal box should contain 368 g of cereals.
- The company has specified that the weight of the cereal box is normally distributed and the standard deviation of the weight of cereal box is 15 g.
- A random sample of 25 boxes of cereals gave a mean weight of 364.5 g
- Test if the population mean weight of the cereal l equal to 368 g at 5% level of significance

Example 1: σ Known, Population Normal (Case 1, use Z), Two Tail Test

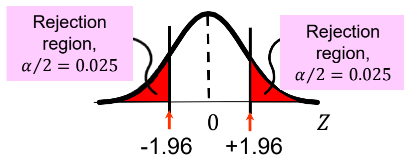
$H_0: \mu = 368$ (Step 1)
$H_1: \mu \neq 368$

Steps 2-3:

At $\alpha = 0.05$
$n = 25$

Critical Value = ± 1.96

Reject H_0 if $Z < -1.96$ or $Z > +1.96$



Step 4:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{364.5 - 368}{15/\sqrt{25}}$$
$$= -1.17$$

Step 5:

As z= -1.17 is greater than the critical value (-1.96), do not reject H_0 at $\alpha = 0.05$

There is no evidence that the true mean weight is not equal to 368 g

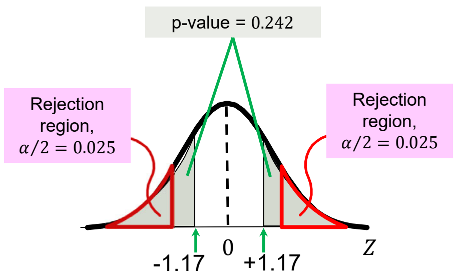
Example 1: σ Known, Population Normal (Case 1, use Z), Two Tail Test

$H_0: \mu = 368$
$H_1: \mu \neq 368$

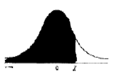
$$Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = \frac{364.5-368}{15/\sqrt{25}} = -1.17$$

p-value
$= P(Z \leq -1.17) + P(Z \geq 1.17)$
$= 2 \times P(Z \leq -1.17)$
$= 2 \times 0.121$
$= 0.242$

As p-value > α, do not reject H_0
There is no evidence that the true mean weight is not 368 g



The Cumulative Standardized Normal Distribution (Continued)
Entry represents area under the cumulative standardized normal distribution from -∞ to Z



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170

Example 2: σ Known, Population Normal (Case 1, use Z), Lower Tail Test

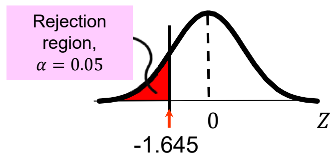
- The company received complaints from customers that the amount of cereal is less than the specified 368 g. Is there evidence that the mean weight of cereal box is less than 368 g?

H_0: $\mu \geq 368$
H_1: $\mu < 368$

At $\alpha = 0.05$
$n = 25$

Critical Value = -1.645

Reject H_0 if $Z < -1.645$



$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{364.5 - 368}{15/\sqrt{25}}$$
$$= -1.17$$

As z= -1.17 is greater than the critical value (-1.645), do not reject H_0 at $\alpha = 0.05$

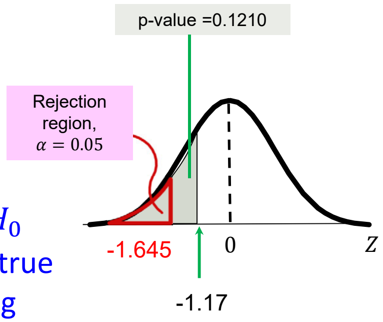
There is no evidence that the true mean weight is less than 368 g

Example 2: σ Known, Population Normal (Case 1, use Z), Lower Tail Test

$H_0: \mu \geq 368$
$H_1: \mu < 368$

$$Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = \frac{364.5-368}{15/\sqrt{25}} = -1.17$$

p-value
$= P(Z \leq -1.17)$
$= 0.1210$



As p-value > α, do not reject H_0
There is no evidence that the true mean weight is less than 368 g

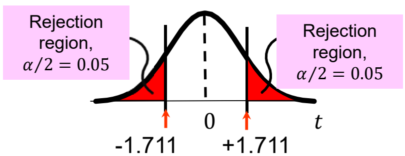
Example 3: σ Unknown, Population Normal (Case 2, use *t*), Two Tail Test

- In addition to cereals, the company also sets up the filling machine for milk. The company also specifies the population distribution of the volume of milk bottle is normal.
- Each bottle should contain 1 L of milk
- A random sample of 25 bottles are selected, giving an average 1.03 L and standard deviation 0.06 L
- At 10% level of significance, test to see if the filling machine is working properly

Example 3: σ Unknown, Population Normal (Case 2, use *t*), Two Tail Test

$H_0: \mu = 1$
$H_1: \mu \neq 1$

At $\alpha = 0.10$
$n = 25, df = 24$
Critical Value = ± 1.7109
Reject H_0 if
$t < -1.7109$ or $t > +1.7109$

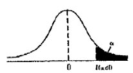


$$t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} = \frac{1.03-1}{0.06/\sqrt{25}}$$
$$= 2.5$$

As t= 2.5 is greater than the critical value (1.7109), reject H_0 at $\alpha = 0.1$

There is evidence that the true mean amount is not 1 L

Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
26	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707
28	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633
29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564

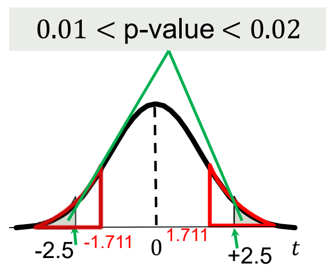
Example 3: σ Unknown, Population Normal (Case 2, use *t*), Two Tail Test

$H_0: \mu = 1$
$H_1: \mu \neq 1$

$$t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} = \frac{1.03-1}{0.06/\sqrt{25}} = 2.5$$

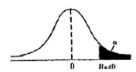
$\alpha = 0.10, \quad n = 25 \quad , df = 24$

p-value
$= P(t \leq -2.5) + P(t \geq 2.5)$
$= 2 \times P(t \geq 2.5)$
$= 2 \times (0.005, 0.01)$
$= (0.01, 0.02)$



As p-value < α, H_0 is rejected
There is evidence that the true mean amount is not 1 L
Using Excel "T.DIST" function, the p-value is found to be 0.019654

Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
26	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707
28	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633
29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564

Example 4: σ Unknown, Population Normal (Case 2, use *t*), Upper Tail Test

- In the last example, we found that the mean amount of milk is not 1 L
- Now, test to see if the mean amount is more than 1 L at 10% level of significance

$H_0: \mu \le 1$
$H_1: \mu > 1$

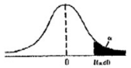
At $\alpha = 0.10$
$n = 25, df = 24$
Critical Value = 1.3178
Reject H_0 if $t > 1.3178$

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{1.03 - 1}{0.06/\sqrt{25}}$$
$$= 2.5$$

As t= 2.5 is greater than the critical value (1.3178), reject H_0 at $\alpha = 0.1$

There is evidence that the true mean amount is more than 1 L

Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
26	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707
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29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564

Example 4: σ Unknown, Population Normal (Case 2, use *t*), Upper Tail Test

$H_0: \mu \le 1$
$H_1: \mu > 1$

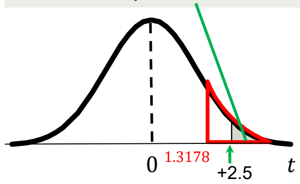
$\alpha = 0.10, \quad n = 25 \quad , df = 24$

p-value
$= P(t \ge 2.5)$
$= (0.005, 0.01)$

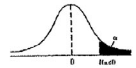
As p-value < α, H_0 is rejected
There is evidence that the true mean amount is more than 1 L
Using Excel "T.DIST" function, the p-value is found to be 0.009827

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{1.03 - 1}{0.06/\sqrt{25}} = 2.5$$

0.005 < p-value < 0.01



Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
26	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707
28	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633
29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564

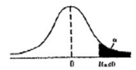
Example 5: σ Unknown, Population Distribution Unknown , *n* small, Two Tail Test

- Besides direct selling to the consumers, the milk is used to make processed cheese
- It is known that excess water will change the freezing point of the milk
- The freezing point of natural milk is distributed with a mean of -0.545 °C
- 14 randomly selected bottles of milk shows a mean -0.550 °C and standard deviation 0.016 °C
- At 5% level of significance, is the milk containing excess water? What assumption(s) is(are) required for performing the hypothesis testing?

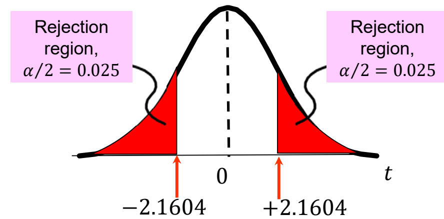
Example 5: σ Unknown, Population Distribution Unknown , *n* small, Two Tail Test

- Step 1: Define hypotheses
  - $H_0: \mu = -0.545$
  - $H_1: \mu \ne -0.545$
- Steps 2-3: determine test statistic and rejection region
  - Population distribution: Unknown
  - σ: unknown
  - Sample size: 14
  - **Any assumption needed? Yes, Normal population**
  - Distribution to be used: t
    - Significance level: 0.05
    - Degrees of freedom: 13
    - Critical value(s): ± 2.1604
    - Decision rule: Reject H_0 if t < -2.1604 or t > +2.1604

Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



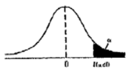
Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768



Example 5: σ Unknown, Population Distribution Unknown , n small, Two Tail Test

- Step 4: Compute test statistic
  - Test statistic = $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} = \frac{-0.550-(-0.545)}{0.016/\sqrt{14}} = -1.17$
  - p-value = (0.20, 0.50)
- Step 5: Make statistical decision and conclusion
  - Decision: At α = 0.05, do not reject H_0
  - Conclusion: There is insufficient evidence that the mean freezing point of the milk is not -0.545 °C

Critical Values of t
For a particular number of degrees of freedom, entry represents the critical value of t corresponding to a specified upper-tail area (α)



Degrees of Freedom	Upper-Tail Areas 0.25	0.10	0.05	0.025	0.01	0.005
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768

Using Excel "T.DIST" function, the p-value is found to be 0.263

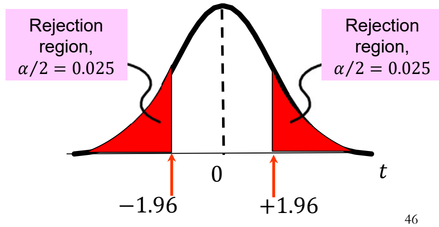
Example 6: σ Unknown, Population Distribution Unknown , n large (Case 2, use t), Two Tail Test

- What would happened if the sample size is 144 rather than 14? Assumed the sample mean and standard deviation remain unchanged
- Step 1: Define hypotheses

  $H_0: \mu = -0.545$

  $H_1: \mu \neq -0.545$
- Steps 2-3: Determine test statistic and rejection region
  - Population distribution: Unknown
  - σ: unknown
  - Sample size: 144
  - Any assumption needed? No
  - Distribution to be used: t
    - Significance level: 0.05
    - Degrees of freedom: 143 ≈ ∞
    - Critical value(s): $\pm$1.96
    - Decision rule: Reject H_0 if t < -1.96 or t > +1.96

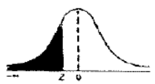
Degrees of Freedom	0.25	0.10	0.05	0.025	0.01	0.005
100	0.6770	1.2901	1.6602	1.9840	2.3642	2.6259
110	0.6767	1.2893	1.6588	1.9818	2.3607	2.6213
120	0.6765	1.2886	1.6577	1.9799	2.3578	2.6174
∞	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758



Example 6: σ Unknown, Population Distribution Unknown , n large (Case 2, use t), Two Tail Test

- Step 4: Compute test statistic
  - Test statistic =t= $\frac{\bar{X}-\mu_0}{s/\sqrt{n}} = \frac{-0.550-(-0.545)}{0.016/\sqrt{144}} = -3.75$
  - p-value = 0.00009*2=0.00018 < 0.01
- Step 5: Make statistical decision and conclusion
  - Decision: At α = 0.05, reject H_0
  - Conclusion: There is sufficient evidence that the mean freezing point of the milk is not -0.545 °C

The Cumulative Standardized Normal Distribution
Entry represents area under the cumulative standardized normal distribution from -∞ to Z



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-6.0	0.000000001									
-5.5	0.000000019									
-5.0	0.000000287									
-4.5	0.000003398									
-4.0	0.000031671									
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008

Connection of Two Tail Tests to Confidence Intervals

- If the hypothesized mean μ_0 is in the CI → do not reject H_0
- If the hypothesized mean μ_0 is not in the CI → reject H_0

Example	μ_0	α	P-value	Decision	$100(1-\alpha)$% Confidence Interval (CI)
1	368	0.05	0.242	Do not reject H_0	$364.5 \pm 1.96\frac{15}{\sqrt{25}} = [358.62, 370.38]$
3	1	0.1	0.01-0.02	Reject H_0	$1.03 \pm 1.7109\frac{0.06}{\sqrt{25}} = [1.0095, 1.0505]$
5	-0.545	0.05	0.2-0.5	Do not reject H_0	$-0.55 \pm 2.1604\frac{0.016}{\sqrt{14}}$ $= [-0.5592, -0.5408]$
6	-0.545	0.05	0.00018	Reject H_0	$-0.55 \pm 1.96\frac{0.016}{\sqrt{144}}$ $= [-0.5526, -0.5474]$