§ 6.2 抽样分布

确定统计量的分布—— 抽样分布, 是数理统计的基本问题之一. 采用求随机向量的函数的分布的方法可得到抽样分布. 由于样本容量一般不止 2 或 3 (甚至还可能是随机的), 故计算往往很复杂, 有时还需要特殊技巧或特殊工具.

由于正态总体是最常见的总体, 故本节介绍的几个抽样分布均对正态总体而言.

一、数理统计中常用分布

（1）正态分布

若 $X_1, X_2, \cdots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu_i, \sigma_i^2)$

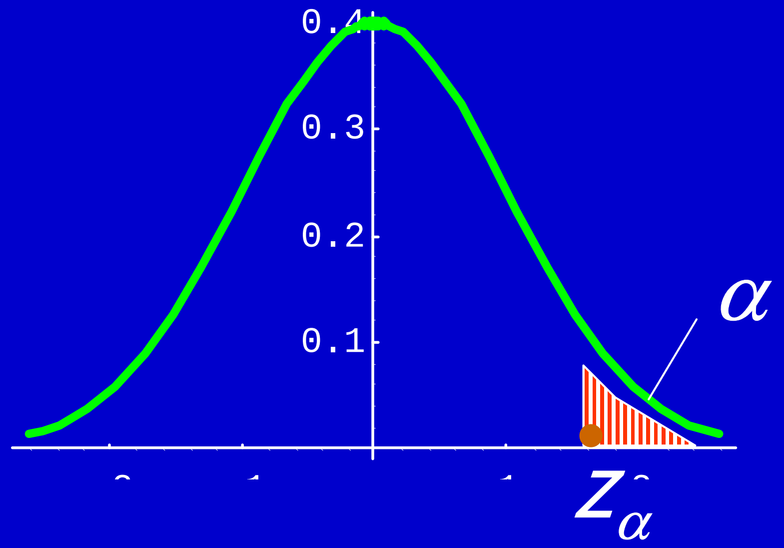
则 $$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

特别地，

若 $X_1, X_2, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \quad X_i \sim N(\mu, \sigma^2)$

则 $$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

标准正态分布的上α 分位点 z_{α}



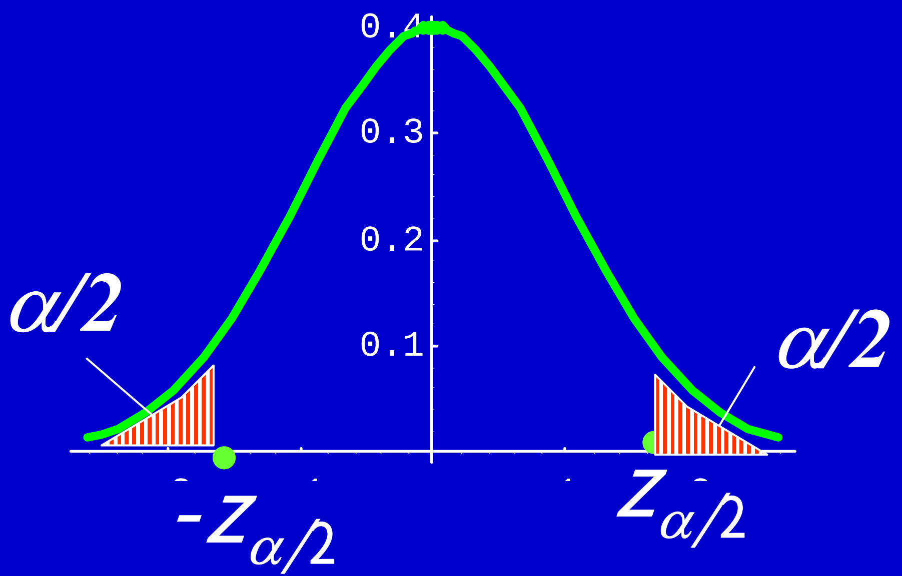
$$P\{X > z_{\alpha}\} = \alpha$$

$$z_{0.05} = 1.645$$

$$z_{0.025} = 1.96$$

常用数字

$$z_{0.005} = 2.575$$



$$-z_{\alpha/2} = z_{1-\alpha/2}$$

(2) $\chi^2(n)$分布(n为自由度)

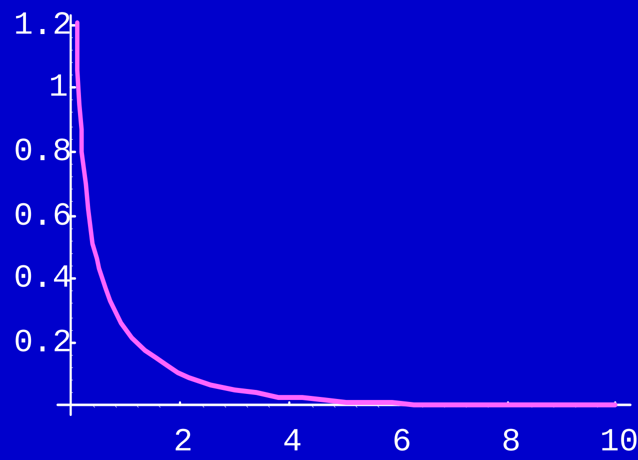
定义 设 $X_1, X_2, \cdots, X_n$相互独立,

且都服从标准正态分布$N(0,1)$,则

$$\sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

$n = 1$ 时,其密度函数为

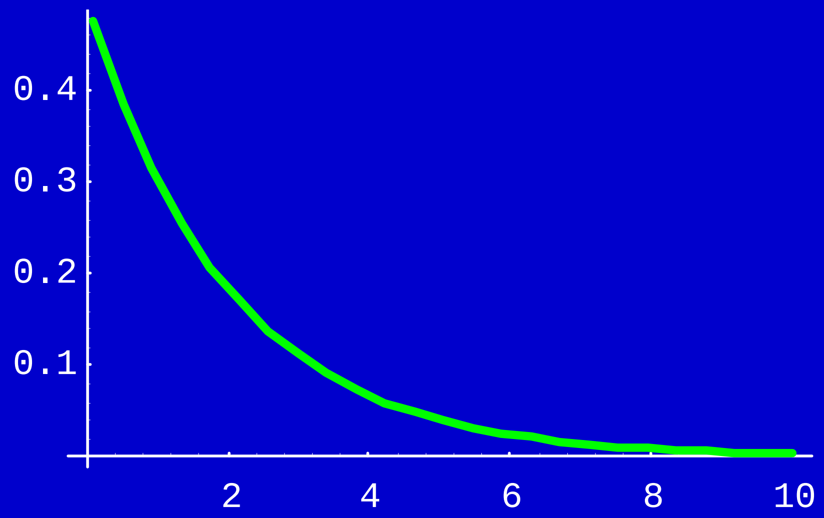
$$f(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{-\frac{x}{2}}, & x>0\\ 0, & x\le 0\end{cases}$$



$n = 2$ 时，其密度函数为

$$f(x) = \begin{cases} \dfrac{1}{2} e^{-\frac{x}{2}}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

为参数为1/2的指数分布.



一般地,自由度为 n 的 $\chi^2(n)$ 的密度函数为

$$f(x)=\begin{cases}\dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}e^{-\frac{x}{2}}x^{\frac{n}{2}-1}, & x>0\\ 0, & x\le 0\end{cases}$$

其中,

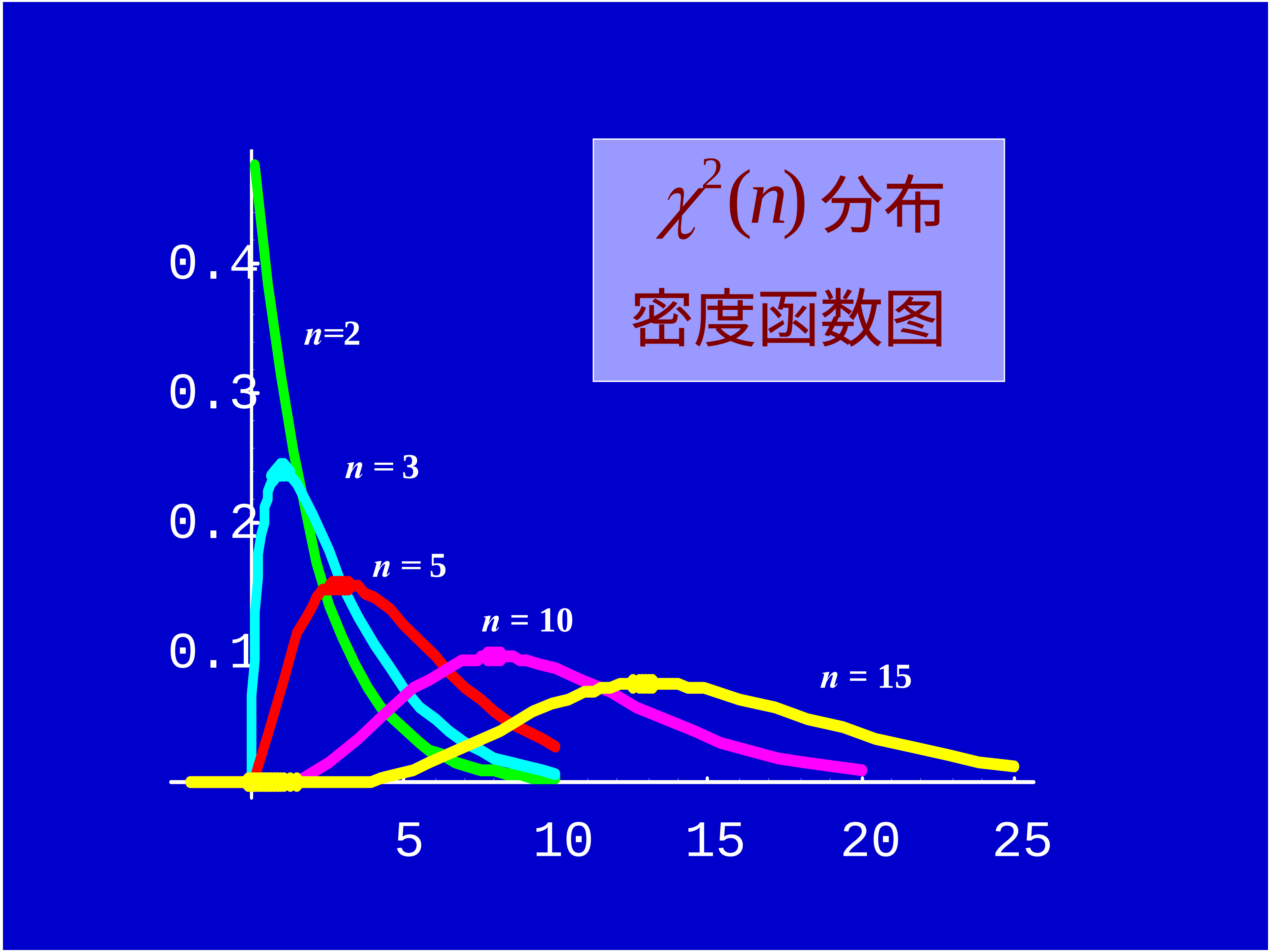
$$\Gamma(x)=\int_0^{+\infty}t^{x-1}e^{-t}dt$$

在$x>0$时收敛，称为Γ 函数，具有性质

$$\Gamma(x+1)=x\Gamma(x),$$

$$\Gamma(1)=1,\ \Gamma(1/2)=\sqrt{\pi}$$

$$\Gamma(n+1)=n!\ \ (n\in N)$$


$\chi^2(n)$ 分布
密度函数图
0.4
0.3
0.2
0.1
n=2
n = 3
n = 5
n = 10
n = 15
5
10
15
20
25

$\chi^2(n)$ 分布的性质

1° $E\left(\chi^2(n)\right) = n,\ D\left(\chi^2(n)\right) = 2n$

2° 若$X_1 \sim \chi^2(n_1), X_2 \sim \chi^2(n_2), X_1, X_2$相互独立，

则 $X_1 + X_2 \sim \chi^2(n_1 + n_2)$

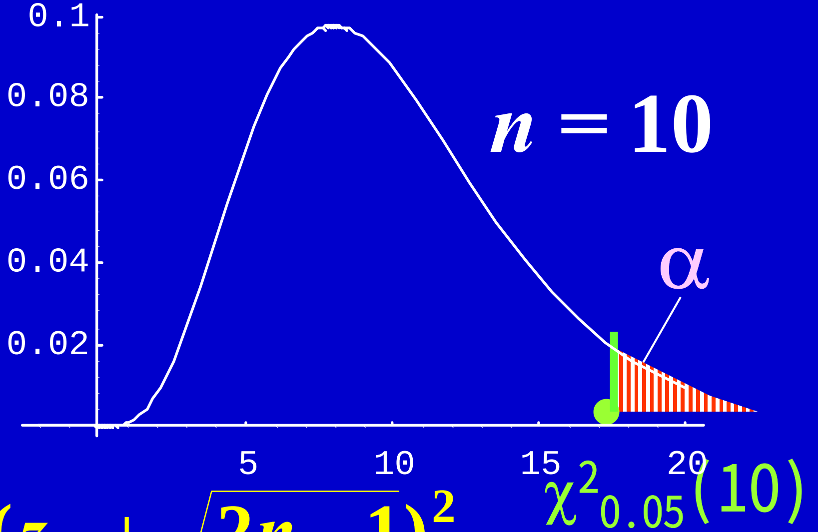
3° $n \to \infty$ 时，$\chi^2(n) \to$ 正态分布

4° $\chi^2(n)$ 分布的上 α 分位点有表可查

例如

$\chi^2_{0.05}(10) = 18.307$

$P\left\{\chi^2(10) > 18.307\right\} = 0.05$



当n充分大时，$\chi^2_\alpha(n) \approx \frac{1}{2}(z_\alpha + \sqrt{2n-1})^2$

证 1°设 $\chi^2(n)=\sum_{i=1}^{n} X_i^2 \quad X_i \sim N(0,1)\ \ i=1,2,\cdots,n$

$X_1, X_2, \cdots, X_n$ 相互独立,

则 $E(X_i)=0,\ D(X_i)=1,\ E(X_i^2)=1$

$$E\left(\chi^2(n)\right)=E\left(\sum_{i=1}^{n} X_i^2\right)=n$$

$$E(X_i^4)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}}\mathrm{d}x=3$$

$$D(X_i^2)=E(X_i^4)-E^2(X_i^2)=2$$

$$D\left(\chi^2(n)\right)=D\left(\sum_{i=1}^{n} X_i^2\right)=2n$$

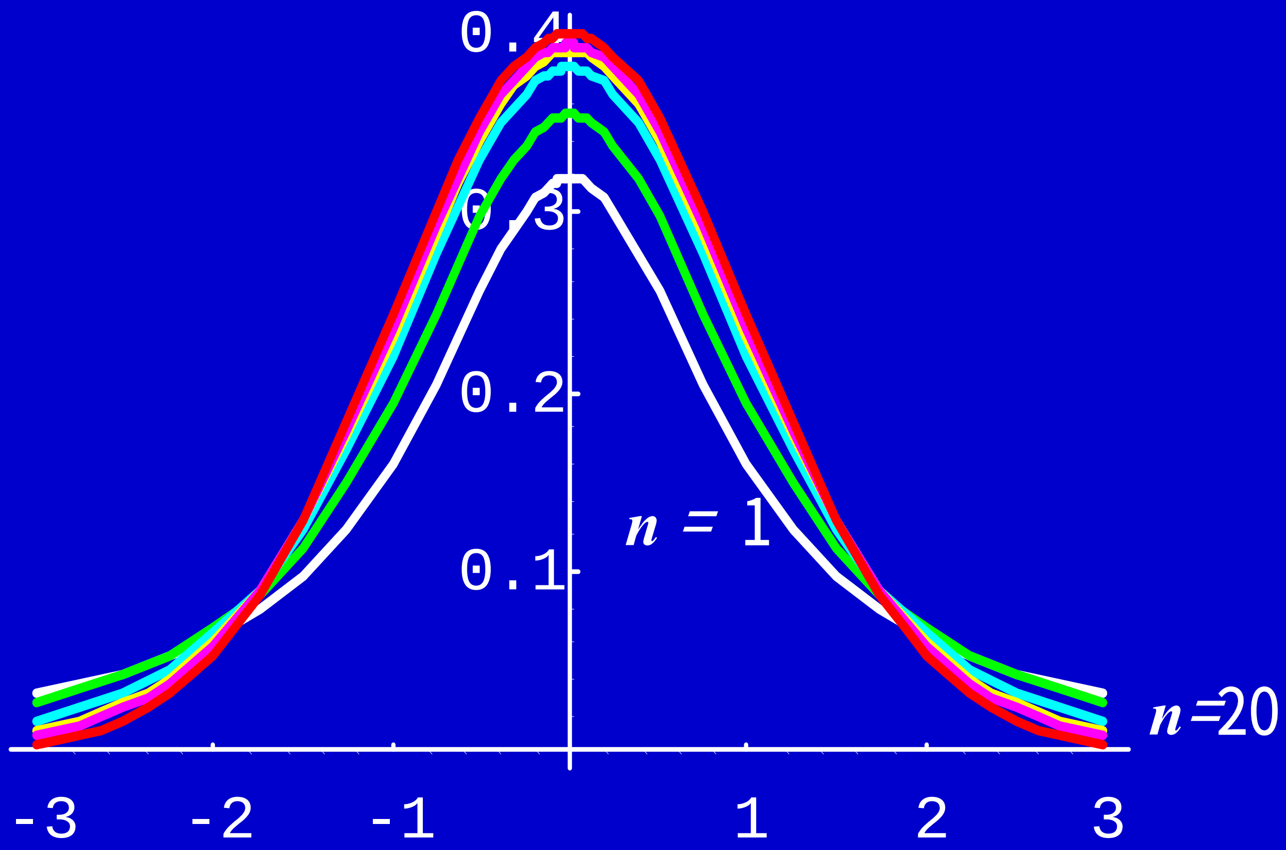
(3) t 分布（Student 分布）

定义 设$X \sim N(0,1)$，$Y \sim \chi^2(n)$, X，Y 相互独立，

$$T = \frac{X}{\sqrt{Y/n}}$$

则T所服从的分布称为自由度为 n 的t分布，其密度函数为

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad -\infty < t < +\infty$$



t 分布的图形(红色的是标准正态分布)

t 分布的性质

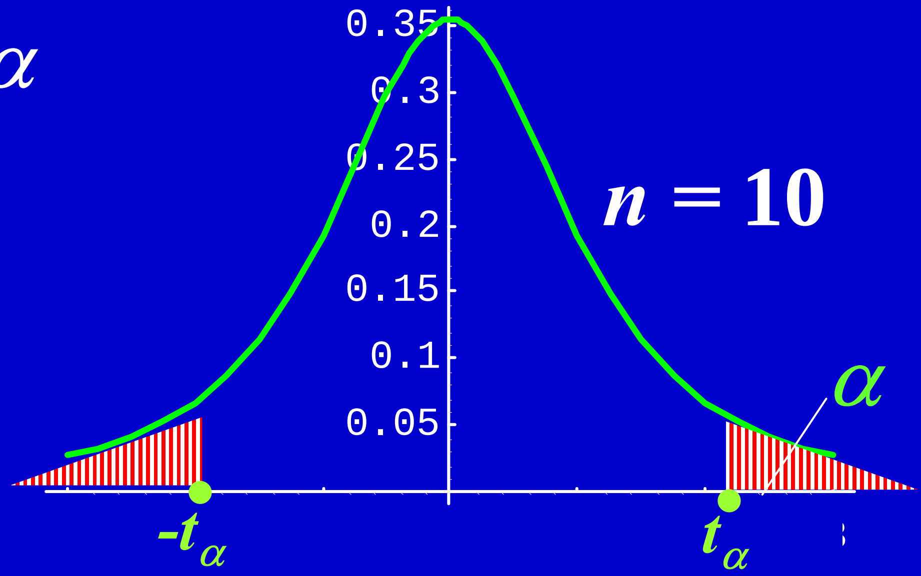
1° $f_n(t)$是偶函数,

$$n \to \infty, f_n(t) \to \varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

2° t 分布的上α分位数 t_α与双测 α分位数 $t_{\alpha/2}$ 有表可查

$$P\{T > t_\alpha\} = \alpha$$

$$-t_\alpha = t_{1-\alpha}$$



$$n = 10$$

$$P\{T > 1.8125\} = 0.05 \;\Rightarrow\; t_{0.05}(10) = 1.8125$$

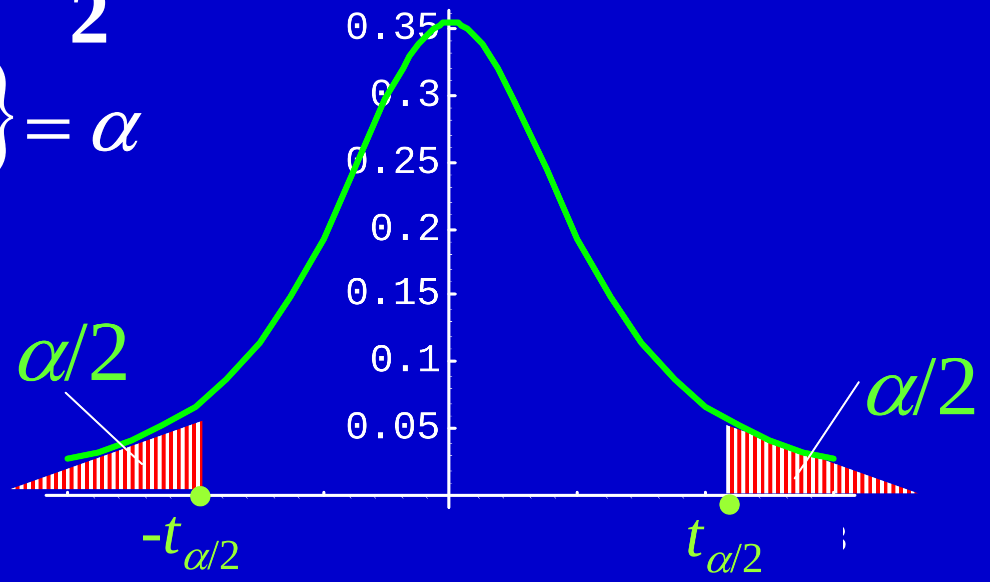
$$P\{T < -1.8125\} = 0.05$$

$$P\{T > -1.8125\} = 0.95$$

$$\Rightarrow t_{0.95}(10) = -1.8125$$

$$P\{T > t_{\alpha/2}\} = \frac{\alpha}{2}$$

$$P\left\{|T| > t_{\alpha/2}\right\} = \alpha$$



$$P\{T > 2.2281\} = 0.025$$

$$P\{|T| > 2.2281\} = 0.05$$

$$\Rightarrow t_{0.025}(10) = 2.2281$$

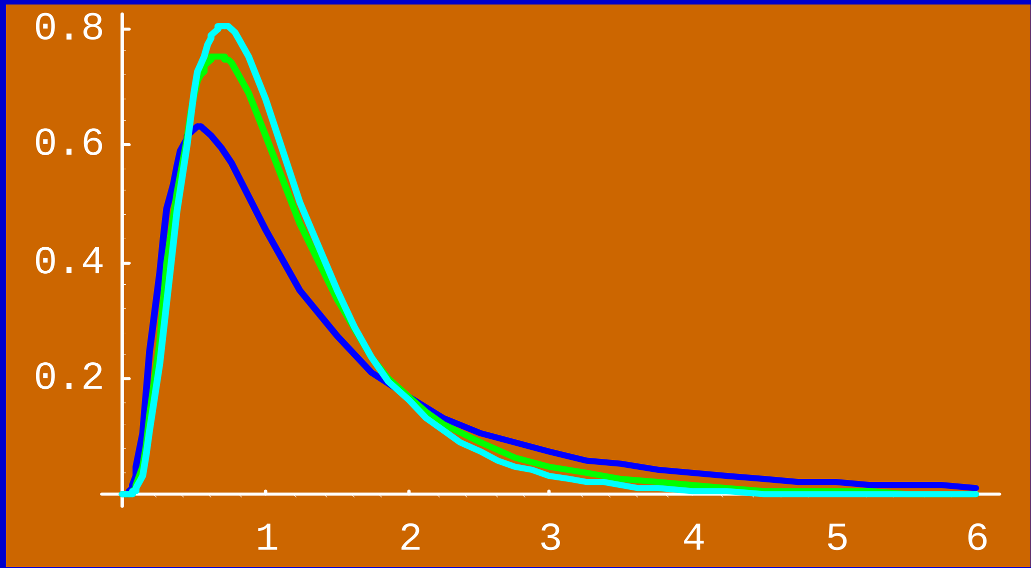
(4) F 分布

定义 设 $X \sim \chi^2(n), Y \sim \chi^2(m), X, Y$相互独立，

令
$$F = \frac{X/n}{Y/m}$$

则F所服从的分布称为第一自由度为n，第二自由度为m 的F分布，其密度函数为

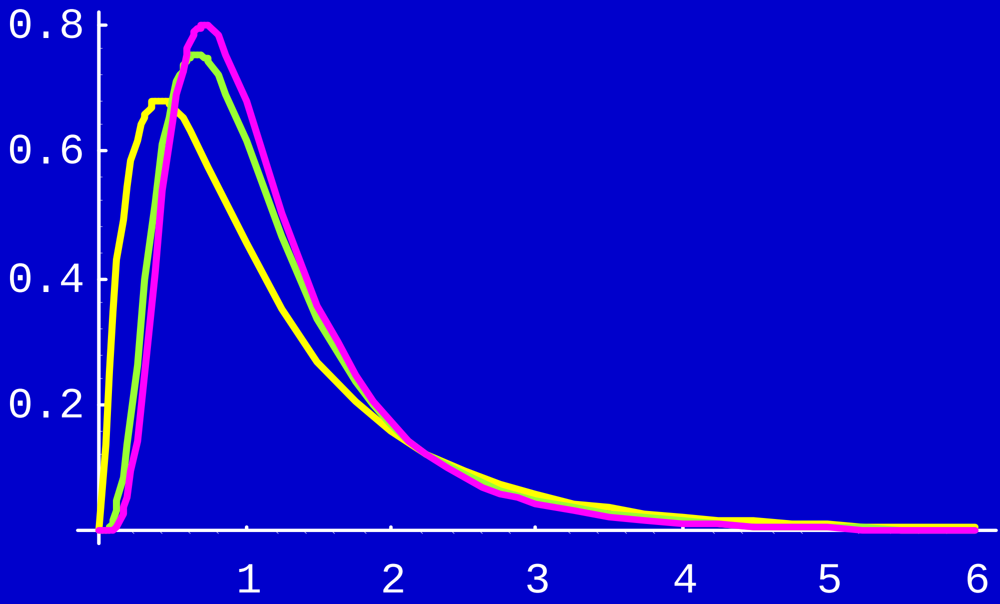
$$f(t,n,m)=\begin{cases}\dfrac{\Gamma\left(\dfrac{n+m}{2}\right)}{\Gamma\left(\dfrac{n}{2}\right)\Gamma\left(\dfrac{m}{2}\right)}\left(\dfrac{n}{m}\right)^{\frac{n}{2}} t^{\frac{n}{2}-1}\left(1+\dfrac{n}{m}t\right)^{-\frac{n+m}{2}}, & t>0 \\ 0, & t\le 0\end{cases}$$



$m = 10, n = 4$

$m = 10, n = 10$

$m = 10, n = 15$



$m = 4, n = 10$

$m = 10, n = 10$

$m = 15, n = 10$

F 分布的性质

1° 若$F \sim F(n,m)$，则$\dfrac{1}{F} \sim F(m,n)$

2° $F(n,m)$的上α分位数$F_\alpha(n,m)$有表可查：

$$P\{F > F_\alpha(n,m)\} = \alpha$$

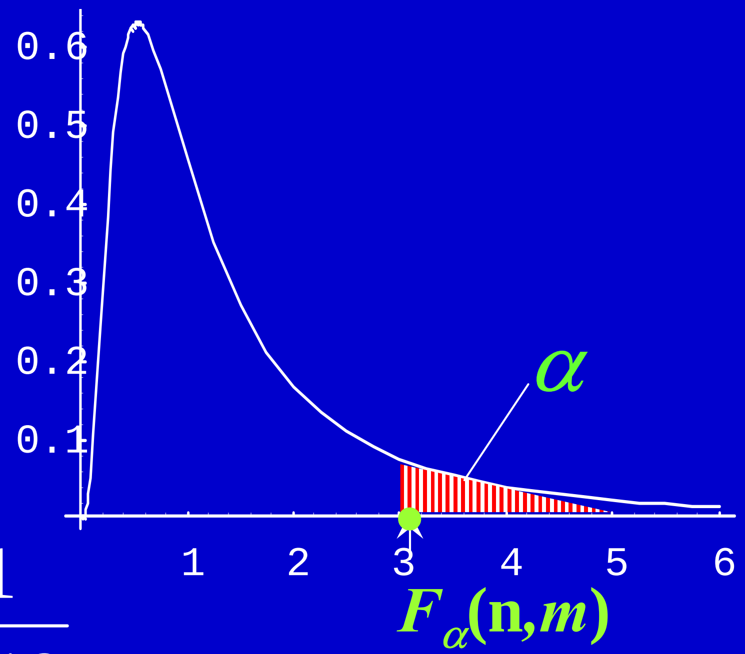
例如 $F_{0.05}(4,5) = 5.19$

但 $F_{0.95}(5,4) = ?$

事实上，

$$F_{1-\alpha}(n,m) = \frac{1}{F_\alpha(m,n)}$$

故 $F_{0.95}(5,4) = \dfrac{1}{F_{0.05}(4,5)} = \dfrac{1}{5.19}$



例1　证明 $F_{1-\alpha}(n,m)=\dfrac{1}{F_{\alpha}(m,n)}$

证　$P\{F \ge F_{1-\alpha}(n,m)\} = P\left\{\dfrac{1}{F} \le \dfrac{1}{F_{1-\alpha}(n,m)}\right\}$

$$= 1 - P\left\{\frac{1}{F} \ge \frac{1}{F_{1-\alpha}(n,m)}\right\} = 1-\alpha$$

故　$P\left\{\dfrac{1}{F} \ge \dfrac{1}{F_{1-\alpha}(n,m)}\right\} = \alpha$　由于 $\dfrac{1}{F} \sim F(m,n)$

因而 $\dfrac{1}{F_{1-\alpha}(n,m)} = F_{\alpha}(m,n)$

例2 证明：$[t_{1-\frac{\alpha}{2}}(n)]^2 = F_\alpha(1,n)$

证 设 $X \sim t(n)$, $X = \dfrac{G}{\sqrt{\dfrac{\chi^2(n)}{n}}}$, $G \sim N(0,1)$

令 $$Y = X^2 = \frac{G^2}{\dfrac{\chi^2(n)}{n}} = \frac{\dfrac{\chi^2(1)}{1}}{\dfrac{\chi^2(n)}{n}} \sim F(1,n)$$

因而 $$P\{|X| > |t_{1-\frac{\alpha}{2}}(n)|\} = P\{|X| > t_{\frac{\alpha}{2}}(n)\} = \alpha$$

$$= P\{X^2 > t_{\frac{\alpha}{2}}^2(n)\} = P\{Y > t_{1-\frac{\alpha}{2}}^2(n)\}$$

即 $$t_{1-\frac{\alpha}{2}}^2(n) = F_\alpha(1,n)$$

二、抽样分布定理

（Ⅰ）一个正态总体

设 $X \sim N(\mu,\sigma^2)$ $E(X)=\mu,$ $D(X)=\sigma^2$

总体的样本为($X_1, X_2, \cdots, X_n$)，则

$$\overline{X} \sim N(\mu,\frac{\sigma^2}{n}) \quad \Rightarrow \quad \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\boxed{\frac{(n-1)S^2}{\sigma^2}=\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 \sim \chi^2(n-1)}$$

$\frac{(n-1)S^2}{\sigma^2}$ 与 $\overline{X}$ 相互独立 (1)

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \div \frac{S}{\sigma} = \boxed{\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)} \quad (2)$$

（II）两个正态总体

设 $X_1, X_2, \cdots, X_n$ 是来自正态总体$X \sim N(\mu_1, \sigma_1^2)$ 的一个简单随机样本

$Y_1, Y_2, \cdots, Y_m$ 是来自正态总体 $Y \sim N(\mu_2, \sigma_2^2)$ 的一个简单随机样本

它们相互独立.

令 $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$　　$\overline{Y} = \dfrac{1}{m}\sum_{j=1}^{m} Y_j$

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad S_2^2 = \frac{1}{m-1}\sum_{j=1}^{m}(Y_j - \overline{Y})^2$$

则

$$\frac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2(n-1) \qquad \frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2(m-1)$$

$$\Rightarrow \boxed{\frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2} \sim F(n-1, m-1)} \cdots\cdots\cdots \textbf{(3)}$$

若 $\sigma_1 = \sigma_2$ 则 $\dfrac{S_1^2}{S_2^2} \sim F(n-1, m-1)$

设 $X_1, X_2, \cdots, X_n$ 是来自正态总体 $X \sim N(\mu_1, \boxed{\sigma^2})$ 的一个简单随机样本

$Y_1, Y_2, \cdots, Y_m$ 是来自正态总体 $Y \sim N(\mu_2, \boxed{\sigma^2})$ 的一个简单随机样本，它们相互独立.

则 $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu_1, \frac{\sigma^2}{n})$　$\overline{Y} = \frac{1}{m}\sum_{j=1}^{m} Y_j \sim N(\mu_2, \frac{\sigma^2}{m})$

$$\longrightarrow \overline{X} - \overline{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma^2}{n} + \frac{\sigma^2}{m})$$

$$\longrightarrow \frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}}} \sim N(0,1)$$

$$\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2(n-1) \qquad \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(m-1)$$

$$\Longrightarrow \quad \frac{(n-1)S_1^2}{\sigma^2} + \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(n+m-2)$$

$\overline{X} - \overline{Y}$ 与 $\dfrac{(n-1)S_1^2}{\sigma^2} + \dfrac{(m-1)S_2^2}{\sigma^2}$ 相互独立

$$\Rightarrow \quad \frac{\dfrac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma^2}{n}+\dfrac{\sigma^2}{m}}}}{\sqrt{\dfrac{\dfrac{(n-1)S_1^2}{\sigma^2}+\dfrac{(m-1)S_2^2}{\sigma^2}}{n+m-2}}}$$

$$= \boxed{\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{1}{n}+\dfrac{1}{m}}\sqrt{\dfrac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}} \sim t(n+m-2)}$$

$$\cdots\cdots\cdots\cdots \textbf{(4)}$$

例3 设总体 $X \sim N(72,100)$,为使样本均值大于70 的概率不小于 90% ,则样本容量

$n = \underline{\quad 42 \quad}$.

解 设样本容量为 n ，则 $\overline{X} \sim N(72,\dfrac{100}{n})$

故 $P\{\overline{X} > 70\} = 1 - P\{\overline{X} \le 70\} = \Phi\left(0.2\sqrt{n}\right)$

令 $\Phi\left(0.2\sqrt{n}\right) \ge 0.9$ 查表得 $0.2\sqrt{n} \ge 1.29$

即 $n \ge 41.6025$ 所以取 $n = 42$

例4 从正态总体$X \sim N(\mu, \sigma^2)$中，抽取了$n = 20$的样本$X_1, X_2, \cdots, X_{20}$

(1) 求 $P\left\{0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \le 1.76\sigma^2\right\}$

(2) 求 $P\left\{0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i - \mu\right)^2 \le 1.76\sigma^2\right\}$

解 (1) $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

即 $\dfrac{19S^2}{\sigma^2} = \dfrac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \sim \chi^2(19)$

故 $$P\left\{0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \le 1.76\sigma^2\right\}$$

$$= P\left\{7.4 \le \frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \le 35.2\right\}$$

$$= P\left\{\frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \ge 7.4\right\} - P\left\{\frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \ge 35.2\right\}$$

查表

$$= 0.99 - 0.01 = 0.98$$

(P.386)

(2) $\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(20)$

故 $P\left\{0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}(X_i-\mu)^2 \le 1.76\sigma^2\right\}$

$$= P\left\{7.4 \le \sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \le 35.2\right\}$$

$$= P\left\{\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \ge 7.4\right\} - P\left\{\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \ge 35.2\right\}$$

$$= 0.995 - 0.025 = 0.97$$

例5 设X与Y相互独立， $X \sim N(0,16)$, $Y \sim N(0,9)$，$X_1, X_2, \dots, X_9$ 与 $Y_1, Y_2, \dots, Y_{16}$ 分别是取自 X 与 Y 的简单随机样本, 求统计量

$$\frac{X_1 + X_2 + \cdots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \cdots + Y_{16}^2}}$$

所服从的分布

解 $X_1 + X_2 + \cdots + X_9 \sim N(0, 9 \times 16)$

$$\frac{1}{3 \times 4}(X_1 + X_2 + \cdots + X_9) \sim N(0,1)$$

$$\frac{1}{3}Y_i \sim N(0,1) \quad , i = 1,2,\cdots,16$$

$$\sum_{i=1}^{16}\left(\frac{1}{3}Y_i\right)^2 \sim \chi^2(16)$$

从而 $$\frac{X_1 + X_2 + \cdots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \cdots + Y_{16}^2}}$$

$$= \frac{\frac{1}{3\times 4}(X_1 + X_2 + \cdots + X_9)}{\sqrt{\frac{\sum_{i=1}^{16}\left(\frac{1}{3}Y_i\right)^2}{16}}} \sim t(16)$$

例6 设总体 $X\sim N(0,1)$，$X_1, X_2, \cdots, X_6$ 为总体 X 的样本，$Y=(X_1+X_2+X_3)^2+(X_4+X_5+X_6)^2$ 试确定常数 c 使 cY 服从 χ^2 分布.

解 $X_1+X_2+X_3\sim N(0,3)$， $X_4+X_5+X_6\sim N(0,3)$

$$\frac{1}{\sqrt{3}}(X_1+X_2+X_3), \frac{1}{\sqrt{3}}(X_4+X_5+X_6)\sim N(0,1)$$

故 $$\left[\frac{1}{\sqrt{3}}(X_1+X_2+X_3)\right]^2+\left[\frac{1}{\sqrt{3}}(X_4+X_5+X_6)\right]^2$$

$$=\frac{1}{3}Y\sim\chi^2(2)$$

因此 $$c=\frac{1}{3}$$

例7 设 $X_1, X_2, \cdots, X_n$ 是来自正态总体 $N(\mu, \sigma^2)$ 的简单随机样本, $\overline{X}$ 是样本均值,

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2, \qquad S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

$$S_3^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \mu)^2, \qquad S_4^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2,$$

则服从自由度为 $n-1$ 的 t 分布的随机变量为:

(A) $\dfrac{\overline{X}-\mu}{S_1}\sqrt{n-1}$　　(B) $\dfrac{\overline{X}-\mu}{S_2}\sqrt{n-1}$

(C) $\dfrac{\overline{X}-\mu}{S_3}\sqrt{n}$　　(D) $\dfrac{\overline{X}-\mu}{S_4}\sqrt{n}$

解 $\dfrac{\overline{X}-\mu}{\dfrac{\sigma}{\sqrt{n}}} \sim N(0,1)$ $\dfrac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2 \sim \chi^2(n-1)$

$$\frac{\dfrac{\overline{X}-\mu}{\dfrac{\sigma}{\sqrt{n}}}}{\sqrt{\dfrac{\dfrac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}}} = \frac{\sqrt{n(n-1)}(\overline{X}-\mu)}{\sqrt{\sum_{i=1}^{n}(X_i-\overline{X})^2}} \sim t(n-1)$$

故应选(B)