高数二期末考试试题

一. 填空题

1. 以$y_1=\cos 2x, y_2=\sin 2x$为特解的阶数最低的常系数齐次线性微分方程是(A)

(A)$y''+4y=0$　　(B)$y''-4y=0$

(C)$y''-2y'-4y=0$　　(D)$y''-2y'+3y=0$

2. zox坐标面上的直线$x=z-1$绕oz轴旋转而成的圆锥面的方程是(B)

(A)$x^2+y^2=z-1$　　(B)$x^2+y^2=(z-1)^2$

(C)$z^2=x^2+y^2+1$　　(D) $(x+1)^2=y^2+z^2$

3. 设$z=x\ln\left(x+y^2\right)$,则$\left.\dfrac{\partial z}{\partial x}\right|_{(1,1)}=$(D)

(A)$\dfrac{1}{2}$　　(B)$1+\ln 2$　　(C)$\ln 2$　　(D)$\dfrac{1}{2}+\ln 2$

4. D为平面区域$x^2+y^2\leqslant 4$,利用二重积分的性质，$\iint_D\left(x^2+4y^2+9\right)\mathrm{d}x\,\mathrm{d}y$的最佳估值区间为(C)

(A) $[9\pi,\ 25\pi]$　　(B) $[36\pi,\ 52\pi]$　　(C) $[36\pi,\ 100\pi]$　　(D) $[36\pi,\ 116\pi]$

5. Ω为球体: $x^2+y^2+z^2\leqslant 1$,则$\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,\mathrm{d}v=$(D)

(A)$\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 r^3\sin\varphi\,\mathrm{d}r$　　(B)$\iiint\limits_{\Omega}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(C)$\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 r^3\sin\theta\,\mathrm{d}r$　　(D)$\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 r^3\sin\varphi\,\mathrm{d}r$

6. 设曲线τ的方程为$\begin{cases}x^2+y^2+z^2=9\\ x+y+z=0\end{cases}$,则$\int_{\tau}\left(x^2+y^2+z^2\right)\mathrm{d}s=$(C)

(A)108π　　(B)216π　　(C)54π　　(D)36π

7. L为平面闭区域D的正向边界，则$\int_L\left(x\mathrm{e}^y+x-2y\right)\mathrm{d}x+\left(x\mathrm{e}^y+x-2y\right)\mathrm{d}y=$(AB)

(A)$\iint\limits_{\mathrm{D}}\left(\mathrm{e}^y-x\mathrm{e}^y+3\right)\mathrm{d}x\,\mathrm{d}y$　　(B)$\iint\limits_{\mathrm{D}}\left(\mathrm{e}^y-x\mathrm{e}^y+3\right)\mathrm{d}x\,\mathrm{d}y$

(C)$\iint\limits_{\mathrm{D}}\left(\mathrm{e}^y-x\mathrm{e}^y+2\right)\mathrm{d}x\,\mathrm{d}y$　　(D)$\iint\limits_{\mathrm{D}}\left(x\mathrm{e}^y+\mathrm{e}^y-1\right)\mathrm{d}x\,\mathrm{d}y$

8. 在下列级数中，发散的是(B)

(A) $\sum_{\mathrm{n}=1}^{\infty}\dfrac{1}{\sqrt{\mathrm{n}^3}}$　　(B) $0.01+\sqrt{0.01}+\sqrt[3]{0.01}+\cdots$

(C)$\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots$　　(C)$\dfrac{3}{5}-\left(\dfrac{3}{5}\right)^2+\left(\dfrac{3}{5}\right)^3-\left(\dfrac{3}{5}\right)^4+\cdots$

二. 填空题

1. 微分方程$x''+6x'+5x=\mathrm{e}^{2t}$的一个待定特解$\tilde{x}$的形式是$\tilde{x}=$ <u>$a\mathrm{e}^{2t}$</u>.

2. 过点$(3,0,-1)$且与平面$3x-7y+5z-12=0$平行的平面方程为 <u>$3x-7y+5z-4=0$</u>.

3. 设$z=\arctan(x-y)$,则$\mathrm{d}z|_{(1,1)}=$ <u>$\mathrm{d}x-\mathrm{d}y$</u>.

4. $\iint\limits_{\mathrm{D}}x^2y^2\,\mathrm{d}x\,\mathrm{d}y=$ <u>$\dfrac{1}{9}$</u>,其中$\mathrm{D}=\{0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1\}$.

5．交换二次积分的积分次序后，$\int_0^2 \mathrm{d}y \int_{-2}^{-y} f(x,y)\mathrm{d}x = \underline{\int_{-2}^{0} \mathrm{d}x \int_0^{-x} f(x,y)\mathrm{d}y}$.

6．曲线积分$\int_L (x\mathrm{e}^{2y}+1)\mathrm{d}x+(x^2\mathrm{e}^{2y}-2x)\mathrm{d}y = \underline{\ 4\ }$,$L$为$x$轴上从0到2的一段.

7．Σ为圆锥面$z=1-\sqrt{x^2+y^2}$与平面$z=0$围成区域的表面，取外侧，则

$$\iint_\Sigma (x-yz)\mathrm{d}y\,\mathrm{d}z+(y+2z)\mathrm{d}z\,\mathrm{d}x+(2z+1)\mathrm{d}x\,\mathrm{d}y = \underline{\frac{4}{3}\pi}\ .$$

8．已知$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^n = -\ln(1+x), x\in(-1,1]$，则$\sum_{n=1}^{\infty}\frac{x^n}{n}$的和函数是$\underline{-\ln(1-x)}$.

三. 综合题

1．求过点$(1,1,2)$,且与直线$\begin{cases} x-2y+4z-7=0 \\ x+5y-2z+1=0 \end{cases}$垂直的平面方程.

$$\begin{vmatrix} x & y & z \\ 1 & -2 & 4 \\ 1 & 5 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 5 & -2 \end{vmatrix} x - \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} y + \begin{vmatrix} 1 & -2 \\ 1 & 5 \end{vmatrix} z$$

$=-16x+6y+7z$

不妨设平面法向量为$\vec{n}=(-16,6,7)$

那么平面方程为$-16(x-1)+6(y-1)+7(z-2)=0$

即$-16x+6y+7z-4=0$

2．设$z=f(x^2-y^2,\ \mathrm{e}^{xy})$，且$f$具有一阶连续偏导，求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 2xf_1' + y\mathrm{e}^{xy}f_2'$$

$$\frac{\partial z}{\partial y} = -2yf_1' + x\mathrm{e}^{xy}f_2'$$

3．计算二重积分$\iint_D \sin\left(\sqrt{x^2+y^2}\right)\mathrm{d}x\,\mathrm{d}y$,其中$\mathrm{D}=\{(x,y)|\pi^2\leqslant x^2+y^2\leqslant 4\pi^2\}$

$$\iint_D \sin\left(\sqrt{x^2+y^2}\right)\mathrm{d}x\,\mathrm{d}y = \iint_D \rho\sin\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta\int_\pi^{2\pi}\rho\sin\rho\,\mathrm{d}\rho$$

$$=\int_0^{2\pi}[-\rho\cos\rho+\sin\rho]_\pi^{2\pi}\mathrm{d}\theta = -6\pi^2$$

4．试求由圆锥面$z=\sqrt{x^2+y^2}$及旋转抛物面$z=x^2+y^2$所围立体的体积.

设所围区域为Ω，投影到xoy面区域为D

$$\iiint_{\Omega} \mathrm{d}v = \iint_{D} \mathrm{d}x\,\mathrm{d}y \int_{x^2+y^2}^{\sqrt{x^2+y^2}} \mathrm{d}z = \iint_{D} \sqrt{x^2+y^2} - x^2 - y^2 \mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{D} \rho^2 - \rho^3 \mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho^2 - \rho^3 \mathrm{d}\rho = \frac{\pi}{6}$$

5. 计算$\oiint_{\Sigma} x\,\mathrm{d}S$，其中$\Sigma$是平面$x+y+z=1$.

错题

6. 利用高斯公式计算$\oiint_{\Sigma} x^3\mathrm{d}y\,\mathrm{d}z + y^3\mathrm{d}z\,\mathrm{d}x + z^3\mathrm{d}x\,\mathrm{d}y$ ，

其中Σ是球面$x^2+y^2+z^2=1$,取外侧.

$$\oiint_{\Sigma} x^3\mathrm{d}y\,\mathrm{d}z + y^3\mathrm{d}z\,\mathrm{d}x + z^3\mathrm{d}x\,\mathrm{d}y = 3\iiint_{\Omega} x^2+y^2+z^2\mathrm{d}v$$

$$= 3\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1 r^4\mathrm{d}r = \frac{12}{5}\pi$$

7. 判断级数$\sum_{n=1}^{\infty}\frac{2^n}{n(n+1)}$的敛散性.

$\lim_{n\to\infty}\frac{2^n}{n(n+1)} = +\infty$，级数发散

8. 求幂级数$\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}\ (|x|<1)$的和函数.

设和函数为$s(x)$

$$s(x) = \sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\int_0^x x^{2n}\mathrm{d}x$$

$$= \int_0^x\sum_{n=1}^{\infty}x^{2n}\mathrm{d}x = \int_0^x\frac{x^2}{1-x^2}\mathrm{d}x = -x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

附录(常用公式)

1.偏导数定义

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}=f_x\left(x_0,y_0\right)=\lim_{\Delta x\to 0}\frac{f\left(x_0+\Delta x,y_0\right)-f\left(x_0,y_0\right)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}=f_y\left(x_0,y_0\right)=\lim_{\Delta y\to 0}\frac{f\left(x_0,y_0+\Delta y\right)-f\left(x_0,y_0\right)}{\Delta y}$$

2.二阶偏导数

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{yx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}(x,y)$$

3.全微分

若$z=f(x,y)$在点(x,y)可微，则

$$\mathrm{d}z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$$

4.多元复合函数求导法则

$u=\varphi(t)$, $v=\psi(t)$

$z=f(u,v)=f[\varphi(t),\ \psi(t)]$

若u,v在t点可导，$z=f(u,v)$在对应点处具有连续偏导数，则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}$$

$u=\varphi(x,y)$, $v=\psi(x,y)$

$z=f(u,v)=f[\varphi(x,y),\ \psi(x,y)]$

若u,v在(x,y)具有对x及y的偏导数，$z=f(u,v)$在对应点处具有连续偏导数，则

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}y}$$

5.隐函数求导公式

$$F(x,y)=0 \quad \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

6.方向导数与梯度

$$\left.\frac{\partial f}{\partial l}\right|_{x_0,y_0}=f_x\left(x_0,y_0\right)\cos\alpha+f_y\left(x_0,y_0\right)\cos\beta$$

$$\mathbf{grad}f\left(x_0,y_0\right)=\boldsymbol{\nabla}f\left(x_0,y_0\right)=f_x\left(x_0,y_0\right)\boldsymbol{i}+f\left(x_0,y_0\right)\boldsymbol{j}$$

7.二重积分的性质

$$m\leqslant f(x,y)\leqslant M\Rightarrow\iint_D m\,\mathrm{d}\sigma\leqslant\iint_D f(x,y)\mathrm{d}\sigma\leqslant\iint_D M\,\mathrm{d}\sigma$$

$$D=D_1+D_2\Rightarrow\iint_D f(x,y)=\iint_{D_1}f(x,y)\mathrm{d}\sigma+\iint_{D_2}f(x,y)\mathrm{d}\sigma$$

8.三重积分

$$\iiint_\Omega f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_a^b\left\{\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)}f(x,y)\mathrm{d}z\right]\mathrm{d}y\right\}\mathrm{d}x$$

9.弧线积分

$$\begin{cases}x=\varphi(t)\\y=\psi(t)\end{cases},\ \alpha\leqslant t\leqslant\beta$$

$$\int_L f(x,y)\mathrm{d}s=\int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,\mathrm{d}t$$

10.坐标曲线积分

$$\begin{cases}x=\varphi(t)\\y=\psi(t)\end{cases},\ \alpha\leqslant t\leqslant\beta$$

$$\int_L P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=\int_\alpha^\beta\{P[\varphi(t),\psi(t)]\varphi'(t)+Q[\varphi(t),\psi(t)]\psi'(t)\}\mathrm{d}t$$

11.格林公式

设闭区域D有分段光滑的曲线L围成，

若函数$P(x,y)$和$Q(x,y)$在D上具有一阶连续偏导数，则有

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\oint_L P\,\mathrm{d}x+Q\,\mathrm{d}y$$

其中L是D的正向边界线

12.第一类曲面积分

$$\iint_S f(x,y,z)\mathrm{d}S=\iint_{D_{xy}}f[x,y,z(x,y)]\sqrt{1+{z_x}^2+{z_y}^2}\,\mathrm{d}x\,\mathrm{d}y$$

13．第二类曲面积分

曲面上侧

$$\iint_{\Sigma^+} R(x,y,z)\mathrm{d}x\,\mathrm{d}y = \iint_{D_{xy}} R[x,y,z(x,y)]\mathrm{d}x\,\mathrm{d}y$$

曲面下侧

$$\iint_{\Sigma^-} R(x,y,z)\mathrm{d}x\,\mathrm{d}y = -\iint_{D_{xy}} R[x,y,z(x,y)]\mathrm{d}x\,\mathrm{d}y$$

14．高斯公式

设空间闭区域Ω是由分片光滑的闭曲面Σ（外侧）所围成，

函数$P(x,y,z)$，$Q(x,y,z)$，$R(x,y,z)$在Ω上具有一阶连续偏导数，则有

$$\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}v = \oiint_{\Sigma} P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y$$

或

$$\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}v = \oiint_{\Sigma}(P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$