高等数学(二)

第九章 多元函数微分法习题课

一、基本概念

二、多元函数微分法

三、多元函数微分法的应用

四、例题分析

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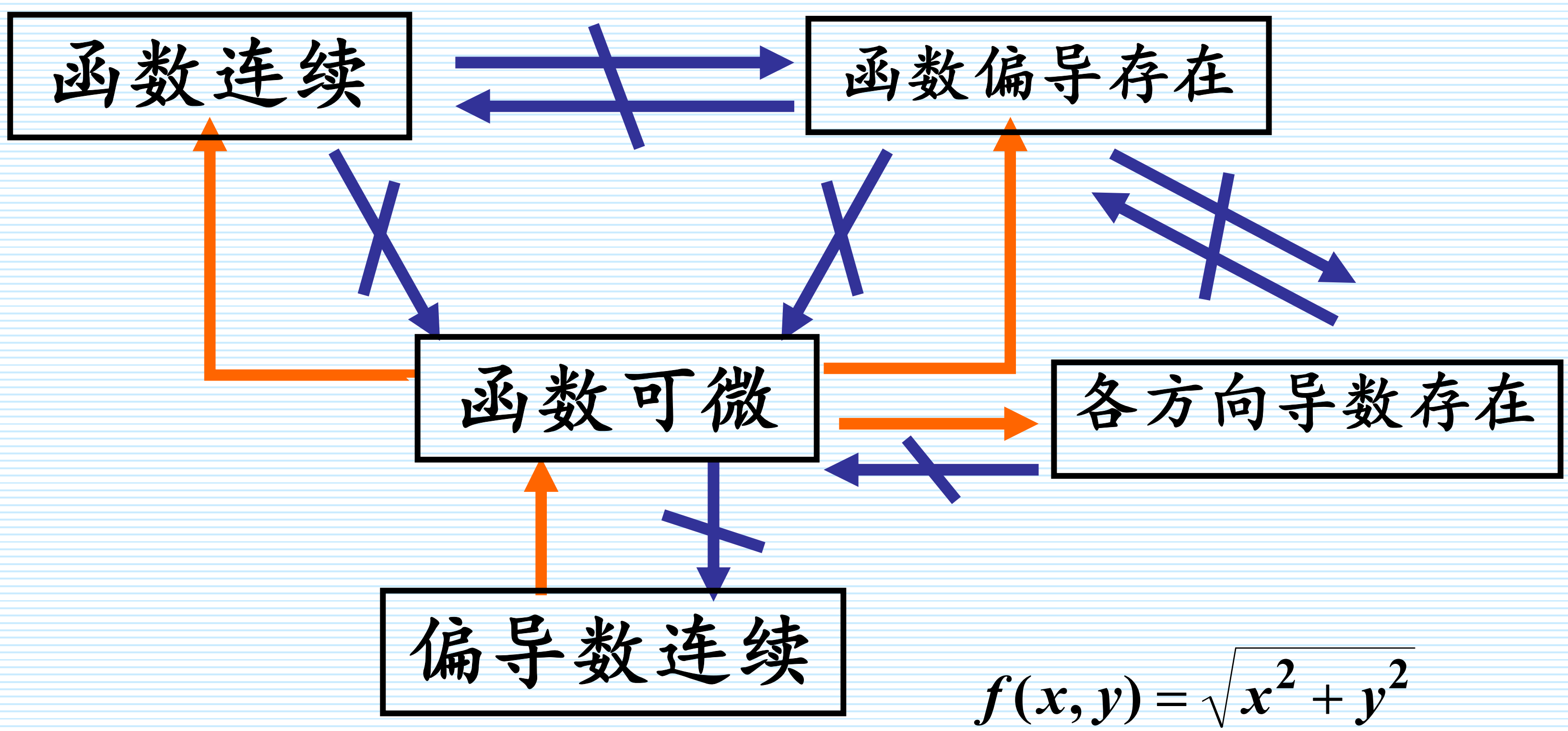
一、基本概念

1. 多元函数的定义、极限 、连续

- 定义域及对应规律
- 判断极限不存在及求极限的方法
- 函数的连续性及其性质

2. 几个基本概念的关系

多元函数连续、可导、可微的关系



$$f(x,y)=\sqrt{x^2+y^2}$$

各方向导存在 $\not\rightleftarrows$ 偏导存在

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

一些反例，讨论$f(x,y)$在(0,0)处的情况。

1.连续 $\not\Longrightarrow$ 偏导∃ $\quad f(x,y)=\sqrt{x^2+y^2}$

2.偏导∃ $\not\Longrightarrow$ 连续 $\quad f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

3.连续 $\not\Longrightarrow$ 可微∃ $\quad f(x,y)=\sqrt{x^2+y^2}$

4.偏导∃ $\not\Longrightarrow$ 可微 $\quad f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

5.可微 $\not\Longrightarrow$ 偏导连续 $\quad f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

思考与练习

1. 讨论二重极限 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x+y}$ 时, 下列方法是否正确?

解法1　原式 $=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{1}{\frac{1}{y}+\frac{1}{x}}=0$

解法2　令 $y=kx$, 原式 $=\lim\limits_{x\to 0}x\dfrac{k}{1+k}=0$

解法3　令 $x=r\cos\theta\,,y=r\sin\theta,$

$$原式=\lim_{r\to 0}\frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta}=0$$

分析:

解法1 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x+y}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{1}{\frac{1}{y}+\frac{1}{x}}=0$

此法第一步排除了沿坐标轴趋于原点的情况,第二步未考虑分母变化的所有情况,例如,$y=\frac{x}{x-1}$时,$\frac{1}{y}+\frac{1}{x}=1$,此时极限为 1 .

解法2 令 $y=kx$, 原式$=\lim\limits_{x\to 0}x\dfrac{k}{1+k}=0$

此法排除了沿曲线趋于原点的情况.例如 $y=x^2-x$时

$$\lim_{\substack{x\to 0\\ y=x^2-x}}\frac{xy}{x+y}=\lim_{x\to 0}\frac{x^3-x^2}{x^2}=-1$$

解~~法~~3 令 $x = r\cos\theta, y = r\sin\theta,$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x+y} = \lim_{r\to 0} \frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta} = 0$$

此法忽略了 θ 的任意性，当 $r \to 0, \theta \to -\frac{\pi}{4}$ 时

$$\frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta} = \frac{r\cos\theta\sin\theta}{\sqrt{2}\sin(\frac{\pi}{4}+\theta)}$$ 极限不存在！

由以上分析可见，三种解法都不对，因为都不能保证自变量在定义域内以任意方式趋于原点．同时还可看到，本题极限实际上不存在．

特别要注意，在某些情况下可以利用极坐标求极限，但要注意在定义域内 r, θ 的变化应该是任意的．

2. 证明:

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0\quad, & x^2+y^2=0\end{cases}$$

在点(0,0) 处连续且偏导数存在，但不可微.

提示: 利用 $2xy\le x^2+y^2$, 知

$$|f(x,y)|\le\frac{1}{2}\sqrt{x^2+y^2}$$

$$\therefore\quad \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=0=f(0,0)$$

故f 在 (0,0) 连续;

又因 $f(x,0)=f(0,y)=0$, 所以 $f_x(0,0)=f_y(0,0)=0$

$$\text{可微} \Leftrightarrow \Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + o(\rho),$$

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0 \quad , & x^2+y^2 = 0 \end{cases}$$

$$\Delta f\big|_{(0,0)} = \frac{(\Delta x)(\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

当 $\Delta x \to 0, \Delta y \to 0$ 时,

$$\frac{\Delta f - (\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \frac{(\Delta x)(\Delta y)}{(\Delta x)^2+(\Delta y)^2}$$

$$\not\longrightarrow 0 \quad (\text{只要令}\Delta y = k\Delta x)$$

所以 $f(x,y)$ 在点 $(0,0)$ 不可微！

二、多元函数微分法

1. 求偏导基本方法

2. 分析复合结构$\begin{cases}\text{显示结构}\\ \text{隐式结构}\end{cases}$ (画变量关系图)

自变量个数 = 变量总个数 – 方程总个数

自变量与因变量由所求对象判定

3. 正确使用求导法则

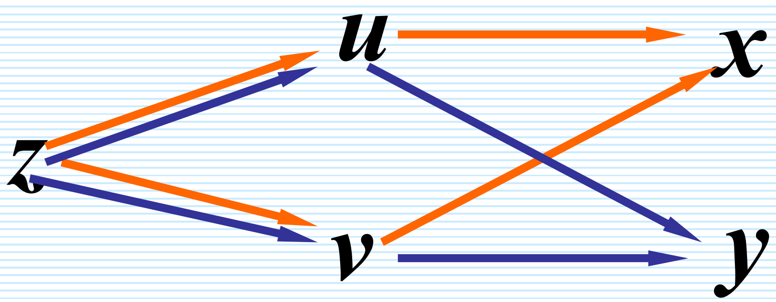
"分段用乘,分叉用加,单路全导,叉路偏导"

注意正确使用求导符号

4. 利用一阶微分形式不变性

“分段用乘,分叉用加,单路全导,叉路偏导”

$$z = f(u,v), u = \varphi(x,y), v = \psi(x,y)$$



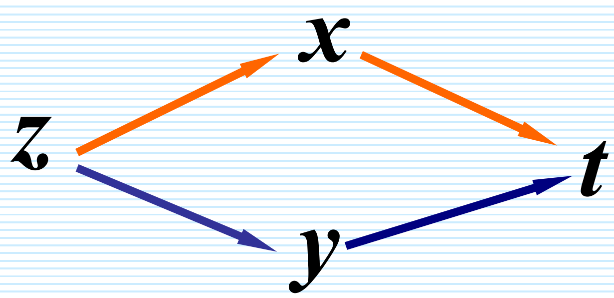
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$z = f(x,y),$$

$$x = \varphi(t),$$

$$y = \psi(t)$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

例2. 设 $z = x f(x+y)$, $F(x,y,z)=0$, 其中 f 与 F 分别具有一阶导数或偏导数, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. (99 考研)

解法1　方程两边对 x 求导, 得

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}x} = f + x f' \cdot \left(1 + \dfrac{\mathrm{d}y}{\mathrm{d}x}\right) \\ F_1' + F_2' \dfrac{\mathrm{d}y}{\mathrm{d}x} + F_3' \dfrac{\mathrm{d}z}{\mathrm{d}x} = 0 \end{cases} \longrightarrow \begin{cases} -x f' \dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\mathrm{d}z}{\mathrm{d}x} = f + x f' \\ F_2' \dfrac{\mathrm{d}y}{\mathrm{d}x} + F_3' \dfrac{\mathrm{d}z}{\mathrm{d}x} = -F_1' \end{cases}$$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\begin{vmatrix} -x f' & f + x f' \\ F_2' & -F_1' \end{vmatrix}}{\begin{vmatrix} -x f' & 1 \\ F_2' & F_3' \end{vmatrix}} = \frac{x F_1' f' - x F_2' f' - f F_2'}{-x f' F_3' - F_2'}$$

练习题

1. 设函数 f 二阶连续可微, 求下列函数的二阶偏导数 $\dfrac{\partial^2 z}{\partial x \partial y}$.

(1) $z = x f(\dfrac{y^2}{x})$

(2) $z = f(x, \dfrac{y^2}{x})$

解答提示:

(1) $z = x f(\frac{y^2}{x})$:
$$\frac{\partial z}{\partial y} = x f'(\frac{y^2}{x}) \cdot \frac{2y}{x} = 2y f'$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2y f'' \cdot (-\frac{y^2}{x^2}) = -\frac{2y^3}{x^2} f''$$

(2) $z = f(x, \frac{y^2}{x})$:
$$\frac{\partial z}{\partial y} = \frac{2y}{x} f_2'$$

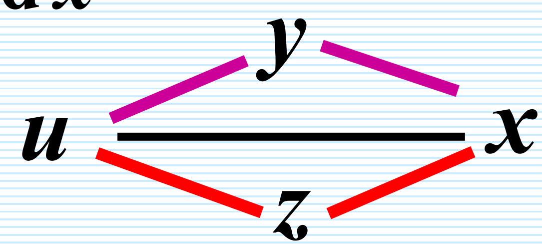
$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{2y}{x^2} f_2' + \frac{2y}{x} (f_{21}'' - \frac{y^2}{x^2} f_{22}'')$$

2. 设 $u=f(x,y,z)$有连续的一阶偏导数，又函数

$y=y(x)$及 $z=z(x)$ 分别由下两式确定

$$e^{xy}-xy=2,\qquad e^{x}=\int_{0}^{x-z}\frac{\sin t}{t}dt$$ 求 $\frac{\mathrm{d}u}{\mathrm{d}x}$.（2001考研）

解：$$\frac{\mathrm{d}u}{\mathrm{d}x}=f_1'+f_2'\cdot\frac{dy}{dx}+f_3'\cdot\frac{dz}{dx}$$



由$e^{xy}-xy=2$, $\frac{dy}{dx}=-\frac{y}{x}$

由$e^{x}=\int_{0}^{x-z}\frac{\sin t}{t}dt$, $e^{x}=\frac{\sin(x-z)}{x-z}(1-\frac{dz}{dx})$, $\frac{dz}{dx}=1-\frac{e^{x}(x-z)}{\sin(x-z)}$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=f_1'-\frac{y}{x}f_2'+[1-\frac{e^{x}(x-z)}{\sin(x-z)}]f_3'$$

三、多元函数微分法的应用

1. 在几何中的应用

求曲线在切线及法平面 (关键: 抓住切线切向量)

求曲面的切平面及法线 (关键: 抓住切平面法向量)

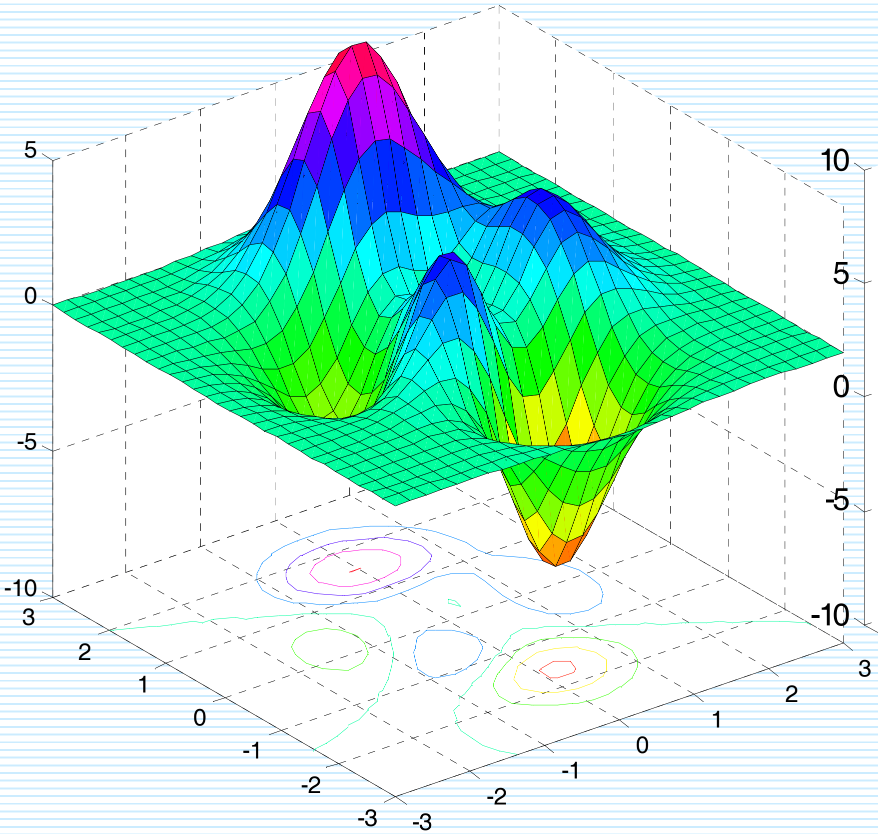
2. 极值与最值问题

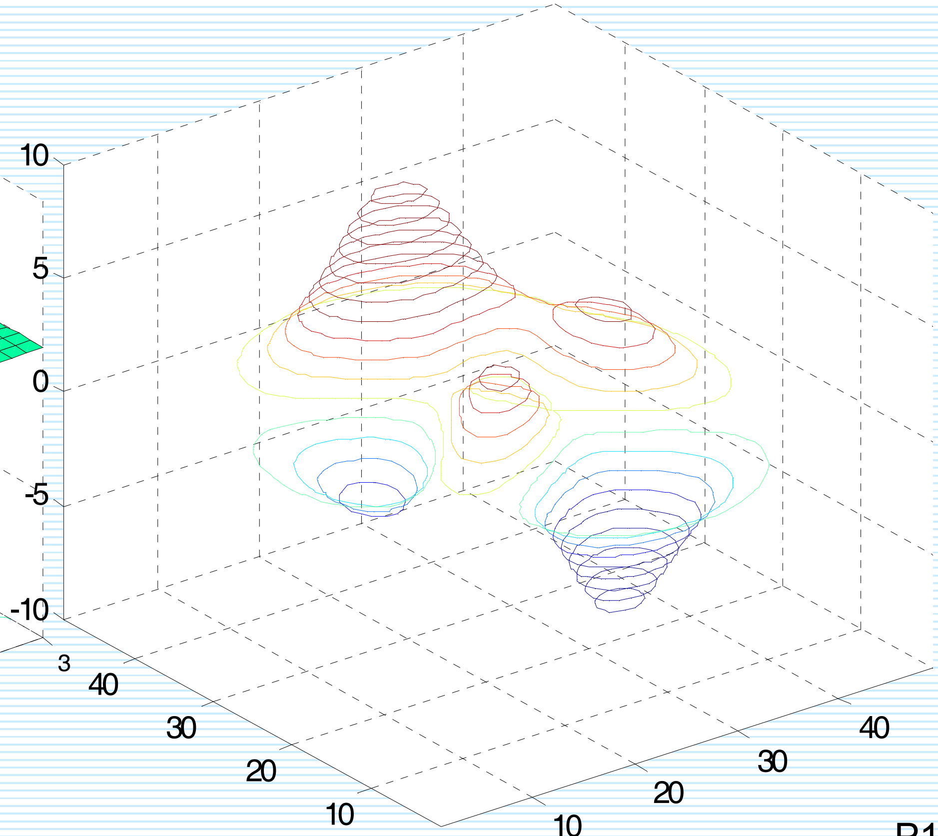
- 极值的必要条件与充分条件
- 求条件极值的方法 (消元法, 拉格朗日乘数法)
- 求解最值问题
- 最小二乘法

曲面图形及等高线

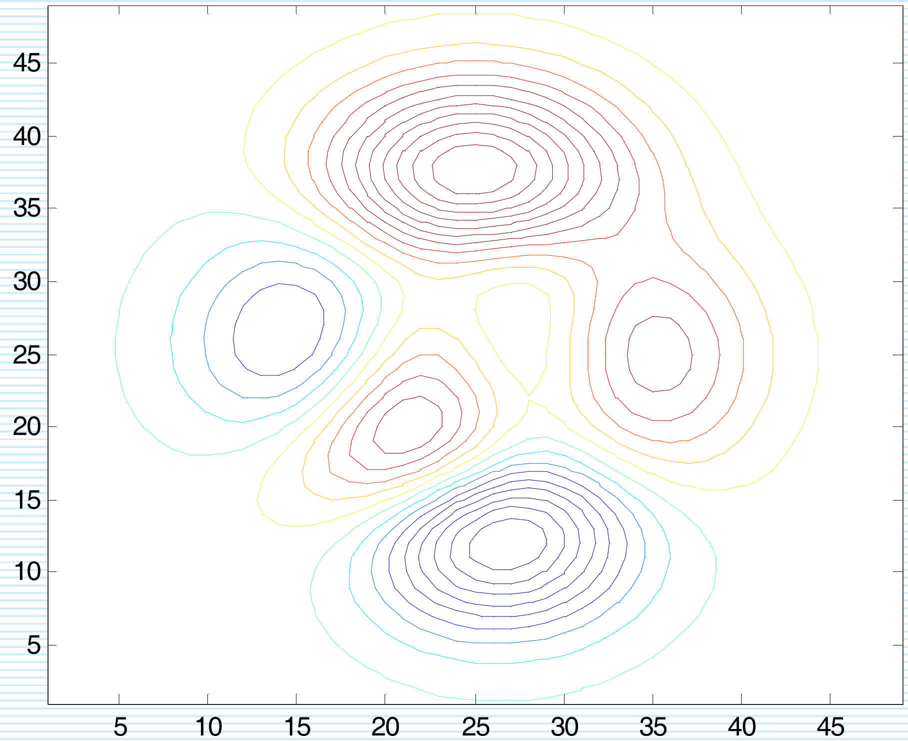
$$z = f(x,y) = 3(1-x)^2 e^{-x^2-(y^2+1)^2}$$

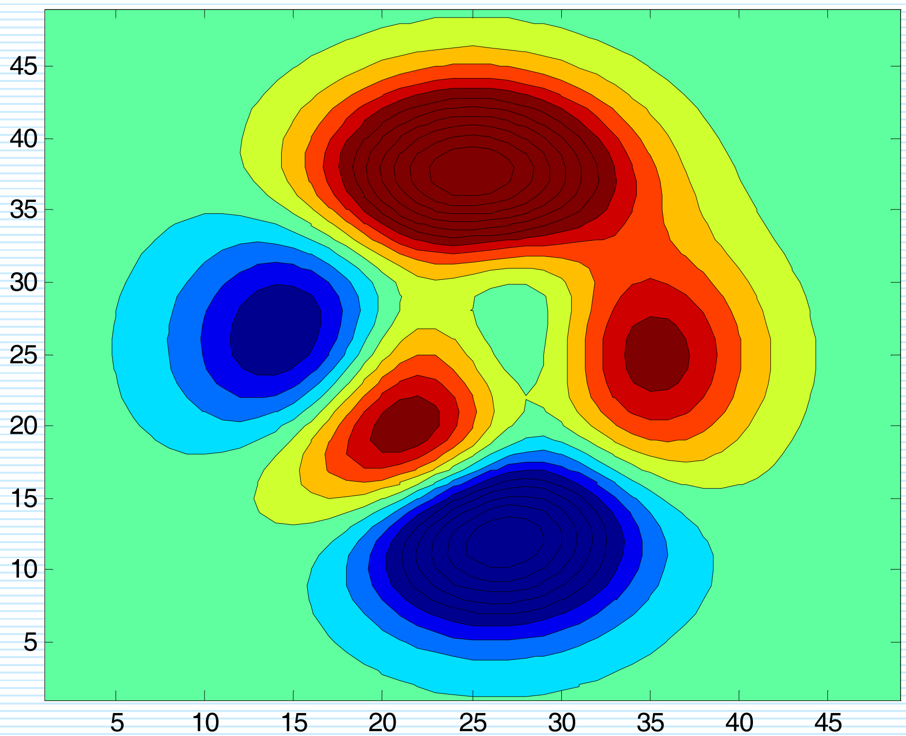
$$-10\left(\frac{x}{5} - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$





$$z = f(x, y) = 3(1-x)^2 e^{-x^2-(y^2+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$





四、例题分析

例1 $z=(1+xy)^y$，求 $\left.\frac{\partial z}{\partial y}\right|_{(1,1)}$

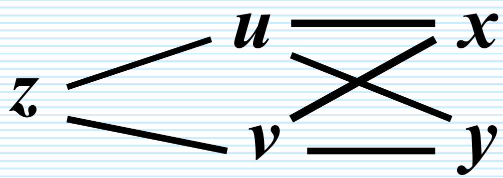
解 $z=e^{y\ln(1+xy)}$

$$\frac{\partial z}{\partial y}=e^{y\ln(1+xy)}\cdot[\ln(1+xy)+y\cdot\frac{x}{1+xy}]$$

$$=(1+xy)^y\cdot[\ln(1+xy)+y\cdot\frac{x}{1+xy}]$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)}=2\cdot(\ln 2+\frac{1}{2})=2\ln 2+1$$

例2 设 $x = e^u \cos v,\ y = e^u \sin v,\ z = uv$，试求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.



解 $$x^2 + y^2 = e^{2u}$$

$$u = \frac{1}{2}\ln(x^2 + y^2)$$

$$\tan v = \frac{y}{x}, \qquad v = \arctan\frac{y}{x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = v\cdot\frac{x}{x^2+y^2} + u\cdot\frac{-\dfrac{y}{x^2}}{1+\dfrac{y^2}{x^2}}$$

$$= \frac{x\arctan\dfrac{y}{x} - y\ln(x^2+y^2)}{x^2+y^2}$$

设$x=e^u\cos v, y=e^u\sin v, z=uv$,试求$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$.

$$\left.\begin{aligned}1&=e^u\cos v\frac{\partial u}{\partial x}-e^u\sin v\frac{\partial v}{\partial x}\\0&=e^u\sin v\frac{\partial u}{\partial x}+e^u\cos v\frac{\partial v}{\partial x}\end{aligned}\right.\Longrightarrow\begin{aligned}\frac{\partial u}{\partial x}&=e^{-u}\cos v\\\frac{\partial v}{\partial x}&=-e^{-u}\sin v\end{aligned}$$

$$\left.\begin{aligned}0&=e^u\cos v\frac{\partial u}{\partial y}-e^u\sin v\frac{\partial v}{\partial y}\\1&=e^u\sin v\frac{\partial u}{\partial y}+e^u\cos v\frac{\partial v}{\partial y}\end{aligned}\right.\Longrightarrow\begin{aligned}\frac{\partial u}{\partial y}&=e^{-u}\sin v\\\frac{\partial v}{\partial y}&=e^{-u}\cos v\end{aligned}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=(v\cos v-u\sin v)e^{-u}$$

例3 求曲线 $x^2+y^2+z^2=6$, $x+y+z=0$ 在点 $(1,-2,1)$ 处的切线及法平面方程.

解 将所给方程的两边对 x 求导并移项，得

$$\begin{cases} y\dfrac{dy}{dx}+z\dfrac{dz}{dx}=-x \\ \dfrac{dy}{dx}+\dfrac{dz}{dx}=-1 \end{cases} \Rightarrow \quad \frac{dy}{dx}=\frac{z-x}{y-z}, \quad \frac{dz}{dx}=\frac{x-y}{y-z},$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(1,-2,1)} = 0, \qquad \left.\frac{dz}{dx}\right|_{(1,-2,1)} = -1,$$

由此得切向量 $\vec{T} = (1, 0, -1)$,

所求切线方程为 $\dfrac{x-1}{1} = \dfrac{y+2}{0} = \dfrac{z-1}{-1}$,

法平面方程为 $(x-1) + 0 \cdot (y+2) - (z-1) = 0$,

$$\Rightarrow \quad x - z = 0$$

例3'求曲线 $\begin{cases} x^2+y^2+z^2-3x=0 \\ 2x-3y+5z-4=0 \end{cases}$ 在点(1,1,1) 的切线与法平面.

解: 点 (1,1,1) 处两曲面的法向量为

$$\vec{n}_1=(2x-3,2y,2z)\big|_{(1,1,1)}=(-1,2,2)$$

$$\vec{n}_2=(2,-3,5)$$

因此切线的方向向量为 $\vec{l}=\vec{n}_1\times\vec{n}_2=(16,9,-1)$

由此得切线: $$\frac{x-1}{16}=\frac{y-1}{9}=\frac{z-1}{-1}$$

法平面: $16(x-1)+9(y-1)-(z-1)=0$

即 $$16x+9y-z-24=0$$

例4 求旋转抛物面$z = x^2 + y^2 - 1$在点$(2,1,4)$处的切平面及法线方程.

$f(x,y) = x^2 + y^2 - 1,$

解 令$F(x,y,z) = x^2 + y^2 - 1 - z,$

$$\vec{n}\big|_{(2,1,4)} = \{2x, 2y, -1\}\big|_{(2,1,4)} = \{4, 2, -1\},$$

切平面方程为 $4(x-2) + 2(y-1) - (z-4) = 0,$

$$\Rightarrow \quad 4x + 2y - z - 6 = 0,$$

法线方程为 $\dfrac{x-2}{4} = \dfrac{y-1}{2} = \dfrac{z-4}{-1}.$

例5 求旋转抛物面 $z = x^2 + y^2$ 与平面 $x + y - 2z = 2$ 之间的最短距离.

解 设 $P(x, y, z)$ 为抛物面 $z = x^2 + y^2$ 上任一点,则 P 到平面 $x + y - 2z - 2 = 0$ 的距离为 d,

$$d = \frac{1}{\sqrt{6}}\left|x + y - 2z - 2\right|.$$

分析: 本题变为求一点 $P(x, y, z)$, 使得 x, y, z 满足 $x^2 + y^2 - z = 0$ 且使 $d = \frac{1}{\sqrt{6}}\left|x + y - 2z - 2\right|$

(即 $d^2 = \frac{1}{6}(x + y - 2z - 2)^2$) 最小.

令 $F(x,y,z)=\frac{1}{6}(x+y-2z-2)^2+\lambda(z-x^2-y^2)$, 得

$$\begin{cases} F'_x=\frac{1}{3}(x+y-2z-2)-2\lambda x=0, & (1) \\ F'_y=\frac{1}{3}(x+y-2z-2)-2\lambda y=0, & (2) \\ F'_z=\frac{1}{3}(x+y-2z-2)(-2)+z=0, & (3) \\ z=x^2+y^2, & (4) \end{cases}$$

解此方程组得　$x=\frac{1}{4}, y=\frac{1}{4}, z=\frac{1}{8}.$

即得唯一驻点 $(\frac{1}{4},\frac{1}{4},\frac{1}{8})$,

根据题意距离的最小值一定存在，且有唯一驻点，故必在 $(\frac{1}{4},\frac{1}{4},\frac{1}{8})$ 处取得最小值.

$$d_{\min}=\frac{1}{\sqrt{6}}\left|\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-2\right|=\frac{7}{4\sqrt{6}}.$$

已知函数 $z=u(x,y)e^{ax+by}$，且 $\dfrac{\partial^2 u}{\partial x\partial y}=0$，确定常数 a 和 b，

使函数 $z=z(x,y)$ 满足方程 $\dfrac{\partial^2 z}{\partial x\partial y}-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}+z=0$；

解：$\dfrac{\partial z}{\partial x}=e^{ax+by}\left[\dfrac{\partial u}{\partial x}+au(x+y)\right]$，$\dfrac{\partial z}{\partial y}=e^{ax+by}\left[\dfrac{\partial u}{\partial y}+bu(x+y)\right]$，

$$\frac{\partial^2 z}{\partial x\partial y}=e^{ax+by}\left[b\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}+abu(x,y)\right]$$

$$\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}+z=e^{ax+by}\left[(b-1)\frac{\partial u}{\partial x}+(a-1)\frac{\partial u}{\partial y}+(ab-a-b+1)u(x,y)\right]$$

若使 $\dfrac{\partial^2 z}{\partial x\partial y}-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}+z=0$，只有

$(b-1)\dfrac{\partial u}{\partial x}+(a-1)\dfrac{\partial u}{\partial y}+(ab-a-b+1)u(x,y)=0$，即 $a=b=1$