第一章习题及解答

1.1 给定三个矢量 $\boldsymbol{A}$ 、$\boldsymbol{B}$ 和 $\boldsymbol{C}$ 如下：

$$\boldsymbol{A}=\boldsymbol{e}_x+\boldsymbol{e}_y2-\boldsymbol{e}_z3$$

$$\boldsymbol{B}=-\boldsymbol{e}_y4+\boldsymbol{e}_z$$

$$\boldsymbol{C}=\boldsymbol{e}_x5-\boldsymbol{e}_z2$$

求：（1）$\boldsymbol{a}_A$；（2）$|\boldsymbol{A}-\boldsymbol{B}|$；（3）$\boldsymbol{A}\bullet\boldsymbol{B}$；（4）$\theta_{AB}$；（5）$\boldsymbol{A}$ 在 $\boldsymbol{B}$ 上的分量；（6）$\boldsymbol{A}\times\boldsymbol{C}$；（7）$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C})$ 和 $(\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C}$；（8）$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$ 和 $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$。

解 （1）$\boldsymbol{a}_A=\dfrac{\boldsymbol{A}}{|\boldsymbol{A}|}=\dfrac{\boldsymbol{e}_x+\boldsymbol{e}_y2-\boldsymbol{e}_z3}{\sqrt{1^2+2^2+(-3)^2}}=\boldsymbol{e}_x\dfrac{1}{\sqrt{14}}+\boldsymbol{e}_y\dfrac{2}{\sqrt{14}}-\boldsymbol{e}_z\dfrac{3}{\sqrt{14}}$

（2）$|\boldsymbol{A}-\boldsymbol{B}|=|(\boldsymbol{e}_x+\boldsymbol{e}_y2-\boldsymbol{e}_z3)-(-\boldsymbol{e}_y4+\boldsymbol{e}_z)|=|\boldsymbol{e}_x+\boldsymbol{e}_y6-\boldsymbol{e}_z4|=\sqrt{53}$

（3）$\boldsymbol{A}\bullet\boldsymbol{B}=(\boldsymbol{e}_x+\boldsymbol{e}_y2-\boldsymbol{e}_z3)\bullet(-\boldsymbol{e}_y4+\boldsymbol{e}_z)=-11$

（4）由 $\cos\theta_{AB}=\dfrac{\boldsymbol{A}\bullet\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}=\dfrac{-11}{\sqrt{14}\times\sqrt{17}}=-\dfrac{11}{\sqrt{238}}$，得 $\theta_{AB}=\cos^{-1}(-\dfrac{11}{\sqrt{238}})=135.5^\circ$

（5）$\boldsymbol{A}$ 在 $\boldsymbol{B}$ 上的分量 $A_B=|\boldsymbol{A}|\cos\theta_{AB}=\dfrac{\boldsymbol{A}\bullet\boldsymbol{B}}{|\boldsymbol{B}|}=-\dfrac{11}{\sqrt{17}}$

（6）$\boldsymbol{A}\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ 1 & 2 & -3\\ 5 & 0 & -2\end{vmatrix}=-\boldsymbol{e}_x4-\boldsymbol{e}_y13-\boldsymbol{e}_z10$

（7）由于 $\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ 0 & -4 & 1\\ 5 & 0 & -2\end{vmatrix}=\boldsymbol{e}_x8+\boldsymbol{e}_y5+\boldsymbol{e}_z20$

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ 1 & 2 & -3\\ 0 & -4 & 1\end{vmatrix}=-\boldsymbol{e}_x10-\boldsymbol{e}_y1-\boldsymbol{e}_z4$$

所以 $\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{e}_x+\boldsymbol{e}_y2-\boldsymbol{e}_z3)\bullet(\boldsymbol{e}_x8+\boldsymbol{e}_y5+\boldsymbol{e}_z20)=-42$

$$(\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C}=(-\boldsymbol{e}_x10-\boldsymbol{e}_y1-\boldsymbol{e}_z4)\bullet(\boldsymbol{e}_x5-\boldsymbol{e}_z2)=-42$$

（8）$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ -10 & -1 & -4\\ 5 & 0 & -2\end{vmatrix}=\boldsymbol{e}_x2-\boldsymbol{e}_y40+\boldsymbol{e}_z5$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ 1 & 2 & -3\\ 8 & 5 & 20\end{vmatrix}=\boldsymbol{e}_x 55-\boldsymbol{e}_y 44-\boldsymbol{e}_z 11$$

1.2 三角形的三个顶点为$P_1(0,1,-2)$、$P_2(4,1,-3)$和$P_3(6,2,5)$。

（1）判断$\Delta P_1P_2P_3$是否为一直角三角形；

（2）求三角形的面积。

解 （1）三个顶点$P_1(0,1,-2)$、$P_2(4,1,-3)$和$P_3(6,2,5)$的位置矢量分别为

$$\boldsymbol{r}_1=\boldsymbol{e}_y-\boldsymbol{e}_z 2,\quad \boldsymbol{r}_2=\boldsymbol{e}_x 4+\boldsymbol{e}_y-\boldsymbol{e}_z 3,\quad \boldsymbol{r}_3=\boldsymbol{e}_x 6+\boldsymbol{e}_y 2+\boldsymbol{e}_z 5$$

则
$$\boldsymbol{R}_{12}=\boldsymbol{r}_2-\boldsymbol{r}_1=\boldsymbol{e}_x 4-\boldsymbol{e}_z,\qquad \boldsymbol{R}_{23}=\boldsymbol{r}_3-\boldsymbol{r}_2=\boldsymbol{e}_x 2+\boldsymbol{e}_y+\boldsymbol{e}_z 8,$$
$$\boldsymbol{R}_{31}=\boldsymbol{r}_1-\boldsymbol{r}_3=-\boldsymbol{e}_x 6-\boldsymbol{e}_y-\boldsymbol{e}_z 7$$

由此可见

$$\boldsymbol{R}_{12}\bullet\boldsymbol{R}_{23}=(\boldsymbol{e}_x 4-\boldsymbol{e}_z)\bullet(\boldsymbol{e}_x 2+\boldsymbol{e}_y+\boldsymbol{e}_z 8)=0$$

故$\Delta P_1P_2P_3$为一直角三角形。

（2）三角形的面积 $S=\frac{1}{2}\left|\boldsymbol{R}_{12}\times\boldsymbol{R}_{23}\right|=\frac{1}{2}\left|\boldsymbol{R}_{12}\right|\times\left|\boldsymbol{R}_{23}\right|=\frac{1}{2}\sqrt{17}\times\sqrt{69}=17.13$

1.3 求$P'(-3,1,4)$点到$P(2,-2,3)$点的距离矢量$\boldsymbol{R}$及$\boldsymbol{R}$的方向。

解 $\boldsymbol{r}_{P'}=-\boldsymbol{e}_x 3+\boldsymbol{e}_y+\boldsymbol{e}_z 4$，$\boldsymbol{r}_P=\boldsymbol{e}_x 2-\boldsymbol{e}_y 2+\boldsymbol{e}_z 3$，

则
$$\boldsymbol{R}_{P'P}=\boldsymbol{r}_P-\boldsymbol{r}_{P'}=\boldsymbol{e}_x 5-\boldsymbol{e}_y 3-\boldsymbol{e}_z$$

且$\boldsymbol{R}_{P'P}$与x、y、z轴的夹角分别为

$$\phi_x=\cos^{-1}\left(\frac{\boldsymbol{e}_x\bullet\boldsymbol{R}_{P'P}}{\left|\boldsymbol{R}_{P'P}\right|}\right)=\cos^{-1}\left(\frac{5}{\sqrt{35}}\right)=32.31^\circ$$

$$\phi_y=\cos^{-1}\left(\frac{\boldsymbol{e}_y\bullet\boldsymbol{R}_{P'P}}{\left|\boldsymbol{R}_{P'P}\right|}\right)=\cos^{-1}\left(\frac{-3}{\sqrt{35}}\right)=120.47^\circ$$

$$\phi_z=\cos^{-1}\left(\frac{\boldsymbol{e}_z\bullet\boldsymbol{R}_{P'P}}{\left|\boldsymbol{R}_{P'P}\right|}\right)=\cos^{-1}\left(-\frac{1}{\sqrt{35}}\right)=99.73^\circ$$

1.4 给定两矢量$\boldsymbol{A}=\boldsymbol{e}_x 2+\boldsymbol{e}_y 3-\boldsymbol{e}_z 4$和$\boldsymbol{B}=\boldsymbol{e}_x 4-\boldsymbol{e}_y 5+\boldsymbol{e}_z 6$，求它们之间的夹角和$\boldsymbol{A}$在$\boldsymbol{B}$上的分量。

解 $\boldsymbol{A}$与$\boldsymbol{B}$之间的夹角为 $\theta_{AB}=\cos^{-1}\left(\frac{\boldsymbol{A}\bullet\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{-31}{\sqrt{29}\times\sqrt{77}}\right)=131^\circ$

$\boldsymbol{A}$在$\boldsymbol{B}$上的分量为 $A_B=\boldsymbol{A}\bullet\frac{\boldsymbol{B}}{|\boldsymbol{B}|}=\frac{-31}{\sqrt{77}}=-3.532$

1.5 给定两矢量$\boldsymbol{A}=\boldsymbol{e}_x 2+\boldsymbol{e}_y 3-\boldsymbol{e}_z 4$和$\boldsymbol{B}=-\boldsymbol{e}_x 6-\boldsymbol{e}_y 4+\boldsymbol{e}_z$，求$\boldsymbol{A}\times\boldsymbol{B}$在$\boldsymbol{C}=\boldsymbol{e}_x-\boldsymbol{e}_y+\boldsymbol{e}_z$上的分量。

解 $\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ 2 & 3 & -4\\ -6 & -4 & 1\end{vmatrix}=-\boldsymbol{e}_x13+\boldsymbol{e}_y22+\boldsymbol{e}_z10$

所以 $\boldsymbol{A}\times\boldsymbol{B}$ 在 $\boldsymbol{C}$ 上的分量为 $(\boldsymbol{A}\times\boldsymbol{B})_C=\dfrac{(\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C}}{|\boldsymbol{C}|}=-\dfrac{25}{\sqrt{3}}=-14.43$

1.6 证明：如果 $\boldsymbol{A}\bullet\boldsymbol{B}=\boldsymbol{A}\bullet\boldsymbol{C}$ 和 $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$，则 $\boldsymbol{B}=\boldsymbol{C}$；

解 由 $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$，则有 $\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{C})$，即

$$(\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{A}-(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{B}=(\boldsymbol{A}\bullet\boldsymbol{C})\boldsymbol{A}-(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{C}$$

由于 $\boldsymbol{A}\bullet\boldsymbol{B}=\boldsymbol{A}\bullet\boldsymbol{C}$，于是得到 $(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{B}=(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{C}$

故 $$\boldsymbol{B}=\boldsymbol{C}$$

1.7 如果给定一未知矢量与一已知矢量的标量积和矢量积，那么便可以确定该未知矢量。设 $\boldsymbol{A}$ 为一已知矢量，$p=\boldsymbol{A}\bullet\boldsymbol{X}$ 而 $\boldsymbol{P}=\boldsymbol{A}\times\boldsymbol{X}$，$p$ 和 $\boldsymbol{P}$ 已知，试求 $\boldsymbol{X}$。

解 由 $\boldsymbol{P}=\boldsymbol{A}\times\boldsymbol{X}$，有

$$\boldsymbol{A}\times\boldsymbol{P}=\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{X})=(\boldsymbol{A}\bullet\boldsymbol{X})\boldsymbol{A}-(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{X}=p\boldsymbol{A}-(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{X}$$

故得 $$\boldsymbol{X}=\frac{p\boldsymbol{A}-\boldsymbol{A}\times\boldsymbol{P}}{\boldsymbol{A}\bullet\boldsymbol{A}}$$

1.8 在圆柱坐标中，一点的位置由 $(4,\dfrac{2\pi}{3},3)$ 定出，求该点在：（1）直角坐标中的坐标；（2）球坐标中的坐标。

解 （1）在直角坐标系中 $x=4\cos(2\pi/3)=-2$、$y=4\sin(2\pi/3)=2\sqrt{3}$、$z=3$

故该点的直角坐标为 $(-2,2\sqrt{3},3)$。

（2）在球坐标系中 $r=\sqrt{4^2+3^2}=5$、$\theta=\tan^{-1}(4/3)=53.1^\circ$、$\phi=2\pi/3=120^\circ$

故该点的球坐标为 $(5,53.1^\circ,120^\circ)$

1.9 用球坐标表示的场 $\boldsymbol{E}=\boldsymbol{e}_r\dfrac{25}{r^2}$，

（1）求在直角坐标中点 $(-3,4,-5)$ 处的 $|\boldsymbol{E}|$ 和 E_x；

（2）求在直角坐标中点 $(-3,4,-5)$ 处 $\boldsymbol{E}$ 与矢量 $\boldsymbol{B}=\boldsymbol{e}_x2-\boldsymbol{e}_y2+\boldsymbol{e}_z$ 构成的夹角。

解 （1）在直角坐标中点 $(-3,4,-5)$ 处，$r^2=(-3)^2+4^2+(-5)^2=50$，故

$$|\boldsymbol{E}|=\left|\boldsymbol{e}_r\frac{25}{r^2}\right|=\frac{1}{2}$$

$$E_x=\boldsymbol{e}_x\bullet\boldsymbol{E}=|\boldsymbol{E}|\cos\theta_{rx}=\frac{1}{2}\times\frac{-3}{5\sqrt{2}}=-\frac{3\sqrt{2}}{20}$$

（2）在直角坐标中点 $(-3,4,-5)$ 处，$\boldsymbol{r}=-\boldsymbol{e}_x3+\boldsymbol{e}_y4-\boldsymbol{e}_z5$，所以

$$\boldsymbol{E}=\frac{25}{r^2}=\frac{25\boldsymbol{r}}{r^3}=\frac{-\boldsymbol{e}_x3+\boldsymbol{e}_y4-\boldsymbol{e}_z5}{10\sqrt{2}}$$

故 $\boldsymbol{E}$ 与 $\boldsymbol{B}$ 构成的夹角为 $\theta_{EB}=\cos^{-1}(\dfrac{\boldsymbol{E}\bullet\boldsymbol{B}}{|\boldsymbol{E}|\bullet|\boldsymbol{B}|})=\cos^{-1}(-\dfrac{19/(10\sqrt{2})}{3/2})=153.6^\circ$

1.10 球坐标中两个点(r_1,θ_1,ϕ_1)和(r_2,θ_2,ϕ_2)定出两个位置矢量$\boldsymbol{R}_1$和$\boldsymbol{R}_2$。证明$\boldsymbol{R}_1$和$\boldsymbol{R}_2$间夹角的余弦为

$$\cos\gamma=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$

解 由 $\boldsymbol{R}_1=\boldsymbol{e}_x r_1\sin\theta_1\cos\phi_1+\boldsymbol{e}_y r_1\sin\theta_1\sin\phi_1+\boldsymbol{e}_z r_1\cos\theta_1$

$\boldsymbol{R}_2=\boldsymbol{e}_x r_2\sin\theta_2\cos\phi_2+\boldsymbol{e}_y r_2\sin\theta_2\sin\phi_2+\boldsymbol{e}_z r_2\cos\theta_2$

得到 $\cos\gamma=\dfrac{\boldsymbol{R}_1\bullet\boldsymbol{R}_2}{|\boldsymbol{R}_1||\boldsymbol{R}_2|}=$

$\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2+\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2+\cos\theta_1\cos\theta_2=$

$\sin\theta_1\sin\theta_2(\cos\phi_1\cos\phi_2+_1\sin\phi_1\sin\phi_2)+\cos\theta_1\cos\theta_2=$

$\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\cos\theta_1\cos\theta_2$

1.11 一球面S的半径为5，球心在原点上，计算：$\oint_S(\boldsymbol{e}_r 3\sin\theta)\bullet\mathrm{d}\boldsymbol{S}$的值。

解 $\oint_S(\boldsymbol{e}_r 3\sin\theta)\bullet\mathrm{d}\boldsymbol{S}=\oint_S(\boldsymbol{e}_r 3\sin\theta)\bullet\boldsymbol{e}_r\,\mathrm{d}S=\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}3\sin\theta\times5^2\sin\theta\,\mathrm{d}\theta=75\pi^2$

1.12 在由$r=5$、$z=0$和$z=4$围成的圆柱形区域，对矢量$\boldsymbol{A}=\boldsymbol{e}_r r^2+\boldsymbol{e}_z 2z$验证散度定理。

解 在圆柱坐标系中 $\nabla\bullet\boldsymbol{A}=\dfrac{1}{r}\dfrac{\partial}{\partial r}(rr^2)+\dfrac{\partial}{\partial z}(2z)=3r+2$

所以 $\int_\tau\nabla\bullet\boldsymbol{A}\,\mathrm{d}\tau=\int_0^4\mathrm{d}z\int_0^{2\pi}\mathrm{d}\phi\int_0^5(3r+2)r\,\mathrm{d}r=1200\pi$

又 $\oint_S\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{S}=\oint_S(\boldsymbol{e}_r r^2+\boldsymbol{e}_z 2z)\bullet(\boldsymbol{e}_r\,\mathrm{d}S_r+\boldsymbol{e}_\phi\,\mathrm{d}S_\phi+\boldsymbol{e}_z\,\mathrm{d}S_z)=$

$$\int_0^4\int_0^{2\pi}5^2\times5\,\mathrm{d}\phi\,\mathrm{d}z+\int_0^5\int_0^{2\pi}2\times4r\,\mathrm{d}r\,\mathrm{d}\phi=1200\pi$$

故有 $\int_\tau\nabla\bullet\boldsymbol{A}\,\mathrm{d}\tau=1200\pi=\oint_S\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{S}$

1.13 求（1）矢量$\boldsymbol{A}=\boldsymbol{e}_x x^2+\boldsymbol{e}_y x^2y^2+\boldsymbol{e}_z 24x^2y^2z^3$的散度；（2）求$\nabla\bullet\boldsymbol{A}$对中心在原点的一个单位立方体的积分；（3）求$\boldsymbol{A}$对此立方体表面的积分，验证散度定理。

解 （1）$\nabla\bullet\boldsymbol{A}=\dfrac{\partial(x^2)}{\partial x}+\dfrac{\partial(x^2y^2)}{\partial y}+\dfrac{\partial(24x^2y^2z^3)}{\partial z}=2x+2x^2y+72x^2y^2z^2$

（2）$\nabla\bullet\boldsymbol{A}$对中心在原点的一个单位立方体的积分为

$$\int_\tau\nabla\bullet\boldsymbol{A}\,\mathrm{d}\tau=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(2x+2x^2y+72x^2y^2z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{24}$$

（3）$\boldsymbol{A}$对此立方体表面的积分

$$\oint_S\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{S}=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\frac{1}{2})^2\,\mathrm{d}y\,\mathrm{d}z-\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(-\frac{1}{2})^2\,\mathrm{d}y\,\mathrm{d}z+$$

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} 2x^2(\frac{1}{2})^2 \,\mathrm{d}\,x\,\mathrm{d}z - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} 2x^2(-\frac{1}{2})^2 \,\mathrm{d}\,x\,\mathrm{d}z +$$

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} 24x^2y^2(\frac{1}{2})^3 \,\mathrm{d}\,x\,\mathrm{d}y - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} 24x^2y^2(-\frac{1}{2})^3 \,\mathrm{d}\,x\,\mathrm{d}y = \frac{1}{24}$$

故有
$$\int_\tau \nabla\bullet\boldsymbol{A}\,\mathrm{d}\,\tau = \frac{1}{24} = \oint_S \boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{S}$$

1.14 计算矢量$\boldsymbol{r}$对一个球心在原点、半径为a的球表面的积分，并求$\nabla\bullet\boldsymbol{r}$对球体积的积分。

解
$$\oint_S \boldsymbol{r}\bullet\mathrm{d}\,\boldsymbol{S} = \oint_S \boldsymbol{r}\bullet\boldsymbol{e}_r\,\mathrm{d}\,S = \int_0^{2\pi}\mathrm{d}\,\phi\int_0^{\pi} aa^2\sin\theta\,\mathrm{d}\,\theta = 4\pi a^3$$

又在球坐标系中，$\nabla\bullet\boldsymbol{r} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 r) = 3$，所以

$$\int_\tau \nabla\bullet\boldsymbol{r}\,\mathrm{d}\,\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3r^2\sin\theta\,\mathrm{d}\,r\,\mathrm{d}\,\theta\,\mathrm{d}\phi = 4\pi a^3$$

1.15 求矢量$\boldsymbol{A} = \boldsymbol{e}_x x + \boldsymbol{e}_y x^2 + \boldsymbol{e}_z y^2 z$沿$xy$平面上的一个边长为$2$的正方形回路的线积分，此正方形的两边分别与$x$轴和$y$轴相重合。再求$\nabla\times\boldsymbol{A}$对此回路所包围的曲面积分，验证斯托克斯定理。

解
$$\oint_C \boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{l} = \int_0^2 x\,\mathrm{d}\,x - \int_0^2 x\,\mathrm{d}\,x + \int_0^2 2^2\,\mathrm{d}\,y - \int_0^2 0\,\mathrm{d}\,y = 8$$

又
$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & x^2 & y^2 z \end{vmatrix} = \boldsymbol{e}_x 2yz + \boldsymbol{e}_z 2x$$

所以
$$\int_S \nabla\times\boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{S} = \int_0^2\int_0^2 (\boldsymbol{e}_x 2yz + \boldsymbol{e}_z 2x)\bullet\boldsymbol{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = 8$$

故有
$$\oint_C \boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{l} = 8 = \int_S \nabla\times\boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{S}$$

1.16 求矢量$\boldsymbol{A} = \boldsymbol{e}_x x + \boldsymbol{e}_y xy^2$沿圆周$x^2 + y^2 = a^2$的线积分，再计算$\nabla\times\boldsymbol{A}$对此圆面积的积分。

解
$$\oint_C \boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{l} = \oint_C x\,\mathrm{d}\,x + xy^2\,\mathrm{d}\,y = \int_0^{2\pi}(-a^2\cos\phi\sin\phi + a^4\cos^2\phi\sin^2\phi)\,\mathrm{d}\,\phi = \frac{\pi a^4}{4}$$

$$\int_S \nabla\times\boldsymbol{A}\bullet\mathrm{d}\,\boldsymbol{S} = \int_S \boldsymbol{e}_z(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\bullet\boldsymbol{e}_z\,\mathrm{d}\,S = \int_S y^2\,\mathrm{d}\,S = \int_0^a\int_0^{2\pi} r^2\sin^2\phi r\,\mathrm{d}\,\phi\,\mathrm{d}\,r = \frac{\pi a^4}{4}$$

1.17 证明：（1）$\nabla\bullet\boldsymbol{R} = 3$；（2）$\nabla\times\boldsymbol{R} = \boldsymbol{0}$；（3）$\nabla(\boldsymbol{A}\bullet\boldsymbol{R}) = \boldsymbol{A}$。其中$\boldsymbol{R} = \boldsymbol{e}_x x + \boldsymbol{e}_y y + \boldsymbol{e}_z z$，$\boldsymbol{A}$为一常矢量。

解 （1）$\nabla \bullet \boldsymbol{R} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

（2）
$$\nabla \times \boldsymbol{R} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & y \end{vmatrix} = \boldsymbol{0}$$

（3）设 $\boldsymbol{A} = \boldsymbol{e}_x A_x + \boldsymbol{e}_y A_y + \boldsymbol{e}_z A_z$，则 $\boldsymbol{A} \bullet \boldsymbol{R} = A_x x + A_y y + A_z z$，故

$$\nabla(\boldsymbol{A} \bullet \boldsymbol{R}) = \boldsymbol{e}_x \frac{\partial}{\partial x}(A_x x + A_y y + A_z z) + \boldsymbol{e}_y \frac{\partial}{\partial y}(A_x x + A_y y + A_z z) +$$
$$\boldsymbol{e}_z \frac{\partial}{\partial z}(A_x x + A_y y + A_z z) = \boldsymbol{e}_x A_x + \boldsymbol{e}_y A_y + \boldsymbol{e}_z A_z = \boldsymbol{A}$$

1.18 一径向矢量场 $\boldsymbol{F} = \boldsymbol{e}_r f(r)$ 表示，如果 $\nabla \bullet \boldsymbol{F} = 0$，那么函数 $f(r)$ 会有什么特点呢？

解 在圆柱坐标系中，由 $\nabla \bullet \boldsymbol{F} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}[rf(r)] = 0$

可得到

$$f(r) = \frac{C}{r} \qquad C \text{ 为任意常数。}$$

在球坐标系中，由 $\nabla \bullet \boldsymbol{F} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}[r^2 f(r)] = 0$

可得到 $f(r) = \frac{C}{r^2}$

1.19 给定矢量函数 $\boldsymbol{E} = \boldsymbol{e}_x y + \boldsymbol{e}_y x$，试求从点 $P_1(2,1,-1)$ 到点 $P_2(8,2,-1)$ 的线积分 $\int \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l}$：（1）沿抛物线 $x = y^2$；（2）沿连接该两点的直线。这个 $\boldsymbol{E}$ 是保守场吗？

解 （1）$\int_C \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l} = \int_C E_x \,\mathrm{d}x + E_y \,\mathrm{d}y = \int_C y \,\mathrm{d}x + x \,\mathrm{d}y =$

$$\int_1^2 y \,\mathrm{d}(2y^2) + 2y^2 \,\mathrm{d}y = \int_1^2 6y^2 \,\mathrm{d}y = 14$$

（2）连接点 $P_1(2,1,-1)$ 到点 $P_2(8,2,-1)$ 直线方程为

$$\frac{x-2}{y-1} = \frac{x-8}{y-2} \qquad \text{即} \qquad x - 6y + 4 = 0$$

故 $\int_C \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l} = \int_C E_x \,\mathrm{d}x + E_y \,\mathrm{d}y = \int_1^2 y \,\mathrm{d}(6y-4) + (6y-4)\,\mathrm{d}y = \int_1^2 (12y-4)\,\mathrm{d}y = 14$

由此可见积分与路径无关，故是保守场。

1.20 求标量函数 $\Psi = x^2 yz$ 的梯度及 Ψ 在一个指定方向的方向导数，此方向由单位矢量 $\boldsymbol{e}_x \frac{3}{\sqrt{50}} + \boldsymbol{e}_y \frac{4}{\sqrt{50}} + \boldsymbol{e}_z \frac{5}{\sqrt{50}}$ 定出；求 $(2,3,1)$ 点的方向导数值。

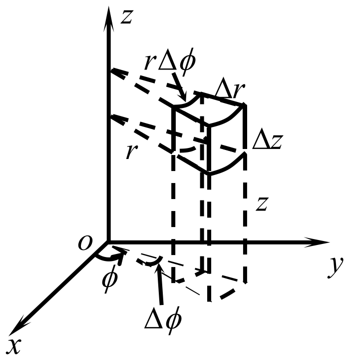
解 $$\nabla\Psi=\boldsymbol{e}_x\frac{\partial}{\partial x}(x^2yz)+\boldsymbol{e}_y\frac{\partial}{\partial y}(x^2yz)+\boldsymbol{e}_z\frac{\partial}{\partial z}(x^2yz)=$$
$$\boldsymbol{e}_x2xyz+\boldsymbol{e}_yx^2z+\boldsymbol{e}_zx^2y$$

故沿方向 $\boldsymbol{e}_l=\boldsymbol{e}_x\frac{3}{\sqrt{50}}+\boldsymbol{e}_y\frac{4}{\sqrt{50}}+\boldsymbol{e}_z\frac{5}{\sqrt{50}}$ 的方向导数为

$$\frac{\partial\Psi}{\partial l}=\nabla\Psi\bullet\boldsymbol{e}_l=\frac{6xyz}{\sqrt{50}}+\frac{4x^2z}{\sqrt{50}}+\frac{5x^2y}{\sqrt{50}}$$

点 $(2,3,1)$ 处沿 $\boldsymbol{e}_l$ 的方向导数值为

$$\frac{\partial\Psi}{\partial l}=\frac{36}{\sqrt{50}}+\frac{16}{\sqrt{50}}+\frac{60}{\sqrt{50}}=\frac{112}{\sqrt{50}}$$



题 1.21 图

1.21 试采用与推导直角坐标中 $\nabla\bullet\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ 相似的方法推导圆柱坐标下的公式

$$\nabla\bullet\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{\partial A_\phi}{r\partial\phi}+\frac{\partial A_z}{\partial z}\text{。}$$

解 在圆柱坐标中，取小体积元如题 1.21 图所示。矢量场 $\boldsymbol{A}$ 沿 $\boldsymbol{e}_r$ 方向穿出该六面体的表面的通量为

$$\Psi_r=\int_{\phi}^{\phi+\Delta\phi}\int_{z}^{z+\Delta z}A_r\big|_{r+\Delta r}(r+\Delta r)\,\mathrm{d}r\,\mathrm{d}\phi-\int_{\phi}^{\phi+\Delta\phi}\int_{z}^{z+\Delta z}A_r\big|_r r\,\mathrm{d}r\,\mathrm{d}\phi\approx$$
$$[(r+\Delta r)A_r(r+\Delta r,\phi,z)-rA_r(r,\phi,z)]\Delta\phi\Delta z\approx\frac{\partial(rA_r)}{\partial r}\Delta r\Delta\phi\Delta z=\frac{1}{r}\frac{\partial(rA_r)}{\partial r}\Delta\tau$$

同理

$$\Psi_\phi=\int_{r}^{r+\Delta r}\int_{z}^{z+\Delta z}A_\phi\big|_{\phi+\Delta\phi}\mathrm{d}r\,\mathrm{d}z-\int_{r}^{r+\Delta r}\int_{z}^{z+\Delta z}A_\phi\big|_\phi\mathrm{d}r\,\mathrm{d}z\approx$$
$$[A_\phi(r,\phi+\Delta\phi,z)-A_\phi(r,\phi,z)]\Delta r\Delta z\approx\frac{\partial A_\phi}{\partial\phi}\Delta r\Delta\phi\Delta z=\frac{\partial A_\phi}{r\partial\phi}\Delta\tau$$

$$\Psi_z=\int_{r}^{r+\Delta r}\int_{\phi}^{\phi+\Delta\phi}A_z\big|_{z+\Delta z}r\,\mathrm{d}r\,\mathrm{d}\phi-\int_{r}^{r+\Delta r}\int_{\phi}^{\phi+\Delta\phi}A_z\big|_z r\,\mathrm{d}r\,\mathrm{d}\phi\approx$$
$$[A_z(r,\phi,z+\Delta z)-A_z(r,\phi,z)]r\Delta r\Delta\phi\Delta z\approx\frac{\partial A_z}{\partial z}r\Delta r\Delta\phi\Delta z=\frac{\partial A_z}{\partial z}\Delta\tau$$

因此，矢量场 $\boldsymbol{A}$ 穿出该六面体的表面的通量为

$$\Psi=\Psi_r+\Psi_\phi+\Psi_z\approx[\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{\partial A_\phi}{r\partial\phi}+\frac{\partial A_z}{\partial z}]\Delta\tau$$

故得到圆柱坐标下的散度表达式 $\nabla\cdot\boldsymbol{A}=\lim_{\Delta\tau\to0}\frac{\Psi}{\Delta\tau}=\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{\partial A_\phi}{r\partial\phi}+\frac{\partial A_z}{\partial z}$

1.22 方程 $u=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ 给出一椭球族。求椭球表面上任意点的单位法向矢量。

解　由于
$$\nabla u = \boldsymbol{e}_x \frac{2x}{a^2} + \boldsymbol{e}_y \frac{2y}{b^2} + \boldsymbol{e}_z \frac{2z}{c^2}$$
$$\left|\nabla u\right| = 2\sqrt{(\frac{x}{a^2})^2 + (\frac{y}{b^2})^2 + (\frac{z}{c^2})^2}$$
故椭球表面上任意点的单位法向矢量为
$$\boldsymbol{n} = \frac{\nabla u}{\left|\nabla u\right|} = (\boldsymbol{e}_x \frac{x}{a^2} + \boldsymbol{e}_y \frac{y}{b^2} + \boldsymbol{e}_z \frac{z}{c^2}) \bigg/ \sqrt{(\frac{x}{a^2})^2 + (\frac{y}{b^2})^2 + (\frac{z}{c^2})^2}$$

1.23　现有三个矢量 $\boldsymbol{A}$、$\boldsymbol{B}$、$\boldsymbol{C}$ 为
$$\boldsymbol{A} = \boldsymbol{e}_r \sin\theta\cos\phi + \boldsymbol{e}_\theta \cos\theta\cos\phi - \boldsymbol{e}_\phi \sin\phi$$
$$\boldsymbol{B} = \boldsymbol{e}_r z^2 \sin\phi + \boldsymbol{e}_\phi z^2 \cos\phi + \boldsymbol{e}_z 2rz\sin\phi$$
$$\boldsymbol{C} = \boldsymbol{e}_x (3y^2 - 2x) + \boldsymbol{e}_y x^2 + \boldsymbol{e}_z 2z$$
（1）哪些矢量可以由一个标量函数的梯度表示？哪些矢量可以由一个矢量函数的旋度表示？

（2）求出这些矢量的源分布。

解（1）在球坐标系中
$$\nabla \bullet \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} =$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi) =$$
$$\frac{2}{r}\sin\theta\cos\phi + \frac{\cos\phi}{r\sin\theta} - \frac{2\sin\theta\cos\phi}{r} - \frac{\cos\phi}{r\sin\theta} = 0$$
$$\nabla\times\boldsymbol{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} =$$
$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \end{vmatrix} = 0$$
故矢量 $\boldsymbol{A}$ 既可以由一个标量函数的梯度表示，也可以由一个矢量函数的旋度表示；

在圆柱坐标系中
$$\nabla \bullet \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} =$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rz^2\sin\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(z^2\cos\phi) + \frac{\partial}{\partial z}(2rz\sin\phi) =$$
$$\frac{z^2\sin\phi}{r} - \frac{z^2\sin\phi}{r} + 2r\sin\phi = 2r\sin\phi$$

$$\nabla\times\boldsymbol{B}=\frac{1}{r}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ B_r & rB_\theta & B_z\end{vmatrix}=\frac{1}{r}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ z^2\sin\phi & rz^2\cos\phi & 2rz\sin\phi\end{vmatrix}=0$$

故矢量 $\boldsymbol{B}$ 可以由一个标量函数的梯度表示；

直角在坐标系中

$$\nabla\bullet\boldsymbol{C}=\frac{\partial C_x}{\partial x}+\frac{\partial C_y}{\partial y}+\frac{\partial C_z}{\partial z}=$$

$$\frac{\partial}{\partial x}(3y^2-2x)+\frac{\partial}{\partial y}(x^2)+\frac{\partial}{\partial z}(2z)=0$$

$$\nabla\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3y^2-2x & x^2 & 2z\end{vmatrix}=\boldsymbol{e}_z(2x-6y)$$

故矢量 $\boldsymbol{C}$ 可以由一个矢量函数的旋度表示。

（2）这些矢量的源分布为

$\nabla\bullet\boldsymbol{A}=0$，$\nabla\times\boldsymbol{A}=0$；

$\nabla\bullet\boldsymbol{B}=2r\sin\phi$，$\nabla\times\boldsymbol{B}=0$；

$\nabla\bullet\boldsymbol{C}=0$，$\nabla\times\boldsymbol{C}=\boldsymbol{e}_z(2x-6y)$

1.24 利用直角坐标，证明

$$\nabla\bullet(f\boldsymbol{A})=f\nabla\bullet\boldsymbol{A}+\boldsymbol{A}\bullet\nabla f$$

解 在直角坐标中

$$f\nabla\bullet\boldsymbol{A}+\boldsymbol{A}\bullet\nabla f=f(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z})+(A_x\frac{\partial f}{\partial x}+A_y\frac{\partial f}{\partial y}+A_z\frac{\partial f}{\partial z})=$$

$$(f\frac{\partial A_x}{\partial x}+A_x\frac{\partial f}{\partial x})+(f\frac{\partial A_y}{\partial y}+A_y\frac{\partial f}{\partial y})+(f\frac{\partial A_z}{\partial z}+A_z\frac{\partial f}{\partial z})=$$

$$\frac{\partial}{\partial x}(fA_x)+\frac{\partial}{\partial y}(fA_y)+\frac{\partial}{\partial z}(fA_z)=\nabla\bullet(f\boldsymbol{A})$$

1.25 证明

$$\nabla\bullet(\boldsymbol{A}\times\boldsymbol{H})=\boldsymbol{H}\bullet\nabla\times\boldsymbol{A}-\boldsymbol{A}\bullet\nabla\times\boldsymbol{H}$$

解 根据 ∇ 算子的微分运算性质，有

$$\nabla\bullet(\boldsymbol{A}\times\boldsymbol{H})=\nabla_A\bullet(\boldsymbol{A}\times\boldsymbol{H})+\nabla_H\bullet(\boldsymbol{A}\times\boldsymbol{H})$$

式中 ∇_A 表示只对矢量 $\boldsymbol{A}$ 作微分运算，∇_H 表示只对矢量 $\boldsymbol{H}$ 作微分运算。

由 $\boldsymbol{a}\bullet(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\bullet(\boldsymbol{a}\times\boldsymbol{b})$，可得

$$\nabla_A\bullet(\boldsymbol{A}\times\boldsymbol{H})=\boldsymbol{H}\bullet(\nabla_A\times\boldsymbol{A})=\boldsymbol{H}\bullet(\nabla\times\boldsymbol{A})$$

同理 $$\nabla_H\bullet(\boldsymbol{A}\times\boldsymbol{H})=-\boldsymbol{A}\bullet(\nabla_H\times\boldsymbol{H})=-\boldsymbol{A}\bullet(\nabla\times\boldsymbol{H})$$

故有 $$\nabla\bullet(\boldsymbol{A}\times\boldsymbol{H})=\boldsymbol{H}\bullet\nabla\times\boldsymbol{A}-\boldsymbol{A}\bullet\nabla\times\boldsymbol{H}$$

1.26 利用直角坐标，证明

$$\nabla\times(f\boldsymbol{G})=f\nabla\times\boldsymbol{G}+\nabla f\times\boldsymbol{G}$$

解 在直角坐标中

$$f\nabla\times\boldsymbol{G}=f[\boldsymbol{e}_x(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z})+\boldsymbol{e}_y(\frac{\partial G_x}{\partial z}-\frac{\partial G_z}{\partial x})+\boldsymbol{e}_z(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y})]$$

$$\nabla f\times\boldsymbol{G}=[\boldsymbol{e}_x(G_z\frac{\partial f}{\partial y}-G_y\frac{\partial f}{\partial z})+\boldsymbol{e}_y(G_x\frac{\partial f}{\partial z}-G_z\frac{\partial f}{\partial x})+\boldsymbol{e}_z(G_y\frac{\partial f}{\partial x}-G_x\frac{\partial f}{\partial y})]$$

所以

$$\begin{aligned}f\nabla\times\boldsymbol{G}+\nabla f\times\boldsymbol{G}=&\boldsymbol{e}_x[(G_z\frac{\partial f}{\partial y}+f\frac{\partial G_z}{\partial y})-(G_y\frac{\partial f}{\partial z}+f\frac{\partial G_y}{\partial z})]+\\&\boldsymbol{e}_y[(G_x\frac{\partial f}{\partial z}+f\frac{\partial G_x}{\partial z})-(G_z\frac{\partial f}{\partial x}+f\frac{\partial G_z}{\partial x})]+\\&\boldsymbol{e}_z[(G_y\frac{\partial f}{\partial x}+f\frac{\partial G_y}{\partial x})-(G_x\frac{\partial f}{\partial y}+f\frac{\partial G_x}{\partial y})]=\\&\boldsymbol{e}_x[\frac{\partial(fG_z)}{\partial y}-\frac{\partial(fG_y)}{\partial z}]+\boldsymbol{e}_y[\frac{\partial(fG_x)}{\partial z}-\frac{\partial(fG_z)}{\partial x}]+\\&\boldsymbol{e}_z[\frac{\partial(fG_y)}{\partial x}-\frac{\partial(fG_x)}{\partial y}]=\nabla\times(f\boldsymbol{G})\end{aligned}$$

1.27 利用散度定理及斯托克斯定理可以在更普遍的意义下证明 $\nabla\times(\nabla u)=0$ 及 $\nabla\bullet(\nabla\times\boldsymbol{A})=0$，试证明之。

解 （1）对于任意闭合曲线 C 为边界的任意曲面 S，由斯托克斯定理有

$$\int_S(\nabla\times\nabla u)\bullet\mathrm{d}\boldsymbol{S}=\oint_C\nabla u\bullet\mathrm{d}\boldsymbol{l}=\oint_C\frac{\partial u}{\partial l}\mathrm{d}l=\oint_C\mathrm{d}u=0$$

由于曲面 S 是任意的，故有

$$\nabla\times(\nabla u)=0$$

（2）对于任意闭合曲面 S 为边界的体积 τ，由散度定理有

$$\int_\tau\nabla\bullet(\nabla\times\boldsymbol{A})\mathrm{d}\tau=\oint_S(\nabla\times\boldsymbol{A})\bullet\mathrm{d}\boldsymbol{S}=\int_{S_1}(\nabla\times\boldsymbol{A})\bullet\mathrm{d}\boldsymbol{S}+\int_{S_2}(\nabla\times\boldsymbol{A})\bullet\mathrm{d}\boldsymbol{S}$$

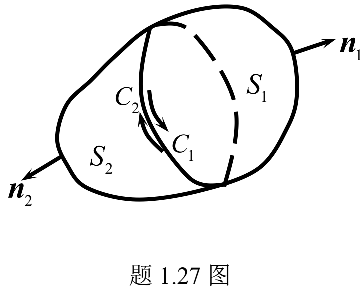
其中 S_1 和 S_2 如题 1.27 图所示。由斯托克斯定理，有

$$\int_{S_1}(\nabla\times\boldsymbol{A})\bullet\mathrm{d}\boldsymbol{S}=\oint_{C_1}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l},\qquad\int_{S_2}(\nabla\times\boldsymbol{A})\bullet\mathrm{d}\boldsymbol{S}=\oint_{C_2}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}$$

由题 1.27 图可知 C_1 和 C_2 是方向相反的同一回路，则有 $\oint_{C_1}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}=-\oint_{C_2}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}$

所以得到 $\int_\tau\nabla\bullet(\nabla\times\boldsymbol{A})\mathrm{d}\tau=\oint_{C_1}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}+\oint_{C_2}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}=-\oint_{C_2}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}+\oint_{C_2}\boldsymbol{A}\bullet\mathrm{d}\boldsymbol{l}=0$

由于体积 τ 是任意的，故有 $\nabla\bullet(\nabla\times\boldsymbol{A})=0$



题 1.27 图