拉梅系数与哈米尔顿算子

整理: 仲 弘毅 来源: 肖老师 教材[*注,]

设 u_1,u_2,u_3 分别为正交曲线坐标轴, $\vec{e}_{u_1},\vec{e}_{u_2},\vec{e}_{u_3}$ 分别为各坐标轴上的单位矢量。

$$\vec{A} = A_1 \vec{e}_{u_1} + A_2 \vec{e}_{u_2} + A_2 \vec{e}_{u_3}$$

由 $dl_1 = h_1 du_1$,则标量场 Φ 在 \overline{l}_1 方向上方向导数为:

$$\frac{\partial \Phi}{\partial l_1} = \frac{1}{h_1} \frac{\partial \Phi}{\partial u_1}$$

梯度:

$$egin{align}
abla & \Phi = rac{\partial \Phi}{\partial l_1} ec{e}_{u_1} + rac{\partial \Phi}{\partial l_2} ec{e}_{u_2} + rac{\partial \Phi}{\partial l_3} ec{e}_{u_3} \ & = rac{1}{h_1} rac{\partial \Phi}{\partial u_1} ec{e}_{u_1} + rac{1}{h_2} rac{\partial \Phi}{\partial u_2} ec{e}_{u_2} + rac{1}{h_3} rac{\partial \Phi}{\partial u_3} ec{e}_{u_3} \ \end{split}$$

散度:

$$abla ullet ec{A} = rac{1}{h_1 h_2 h_3} igg[rac{\partial}{\partial u_1} (A_1 h_2 h_3) + rac{\partial}{\partial u_2} (A_2 h_1 h_3) + rac{\partial}{\partial u_3} (A_3 h_1 h_2) igg]$$

旋度:

$$abla imes ec{A} = rac{1}{h_1 h_2 h_3} egin{bmatrix} h_1 ec{e}_{u_1} & h_2 ec{e}_{u_2} & h_3 ec{e}_{u_3} \ rac{\partial}{\partial u_1} & rac{\partial}{\partial u_2} & rac{\partial}{\partial u_3} \ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{bmatrix}$$

标量拉普拉斯:

$$abla^2 \Phi =
abla ullet (
abla \Phi)$$

矢量拉普拉斯:

矢量公式
$$[*\dot{z}_2]$$
: $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ 即: $\nabla^2 \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$

一. 直角坐标系

1) 基本公式

单位矢量 \vec{e}_x \vec{e}_y \vec{e}_z

线段元
$$d\vec{l} = dx\vec{e}_x + dy\vec{e}_y + dz\vec{e}_z$$

面积元 $d\vec{S} = dy dz \vec{e}_x + dz dx \vec{e}_y + dx dy \vec{e}_z$

体积元 dV = dx dy dz

2) 算子

$$\nabla u = \frac{\partial u}{\partial x} \vec{e}_x + \frac{\partial u}{\partial y} \vec{e}_y + \frac{\partial u}{\partial z} \vec{e}_z$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \vec{e}_z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 \vec{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) \vec{e}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) \vec{e}_y$$

$$+ \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \vec{e}_z$$

二. 圆柱坐标系

1) 基本公式

圆柱坐标系与直角坐标系的坐标变换

$$x = r \cos \alpha, \ y = r \sin \alpha, \ z = z$$
 $r = \sqrt{x^2 + y^2}, \ \alpha = \arctan \frac{y}{x}, \ z = z$

单位矢量 \vec{e}_r \vec{e}_o \vec{e}_z

线段元 $dl = dr\vec{e}_r + r d\alpha\vec{e}_\alpha + dz\vec{e}_z$

面积元 $d\vec{S} = r d\alpha dz \vec{e}_r + dr dz \vec{e}_\alpha + r dr d\alpha \vec{e}_z$

体积元 $dV = r dr d\alpha dz$

2) 算子公式

$$abla u = rac{\partial u}{\partial r}ec{e}_r + rac{1}{r}rac{\partial u}{\partial lpha}ec{e}_lpha + rac{\partial u}{\partial z}ec{e}_z$$

$$abla oldsymbol{ec{A}} = rac{1}{r}rac{\partial}{\partial r}(rA_r) + rac{1}{r}rac{\partial A_lpha}{\partial lpha} + rac{\partial A_z}{\partial z}$$

$$abla imes ec{A} = \left(rac{1}{r}rac{\partial A_z}{\partial lpha} - rac{\partial A_lpha}{\partial z}
ight)ec{e}_r + \left(rac{\partial A_r}{\partial z} - rac{\partial A_z}{\partial r}
ight)ec{e}_lpha + rac{1}{r}igg[rac{\partial}{\partial r}\left(rA_lpha
ight) - rac{\partial A_r}{\partial lpha}igg]$$

$$abla^2 u = rac{1}{r} rac{\partial}{\partial r} \left(r rac{\partial u}{\partial r}
ight) + rac{1}{r^2} rac{\partial^2 u}{\partial lpha^2} + rac{\partial^2 u}{\partial z^2}$$

$$egin{aligned}
abla^2ec{A} = \left(
abla^2A_r - rac{2}{r^2}rac{\partial A_r}{\partial lpha} - rac{A_r}{r^2}
ight)ec{e}_r + \left(
abla^2A_lpha + rac{2}{r^2}rac{\partial A_r}{\partial lpha} - rac{A_lpha}{r^2}
ight)ec{e}_lpha \ & + (
abla^2A_z)ec{e}_z \end{aligned}$$

三. 球坐标系

1) 基本公式

球坐标系与直角坐标系变换

$$x = r\sin\theta\cos\alpha$$
, $y = r\sin\theta\sin\alpha$, $z = r\cos\theta$

$$r = \sqrt{x^2 + y^2 + z^2}$$

单位矢量 \vec{e}_r \vec{e}_θ \vec{e}_α

线积分 $d\vec{l} = dr\vec{e}_r + rd\theta\vec{e}_\theta + r\sin\theta d\alpha\vec{e}_\alpha$

面积分 $d\vec{S} = r^2 \sin\theta d\theta d\alpha \vec{e}_r + r \sin\theta dr d\alpha \vec{e}_\theta + r dr d\theta \vec{e}_\alpha$

体积分 $dV = r^2 \sin\theta d\alpha dr$

2) 算子公式

$$\nabla u = \frac{\partial u}{\partial r} \vec{e}_r + \frac{1}{\partial r} \frac{\partial u}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \alpha} \vec{e}_\alpha$$

$$\nabla \bullet \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\alpha}{\partial \alpha}$$

$$\nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\alpha) - \frac{\partial A_\theta}{\partial \alpha} \right] \vec{e}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \alpha} - \frac{\partial}{\partial r} (r A_\alpha) \right] \vec{e}_\theta$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \vec{e}_\alpha$$

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \alpha^2}$$

$$\begin{split} \nabla^2 \vec{A} &= \left[\nabla^2 A_r - \frac{2}{r^2} \Big(A_r + \cot \theta A_\theta + \frac{1}{\sin \theta} \frac{\partial A_\alpha}{\partial \theta} + \frac{\partial A_\theta}{\partial \theta} \Big) \right] \vec{e}_r \\ &+ \left[\nabla^2 A_\theta - \frac{1}{r^2} \Big(\frac{1}{\sin^2 \theta} A_\theta - 2 \frac{\partial A_r}{\partial \theta} + 2 \frac{\cos \theta}{\sin^2 \theta} \frac{\partial A_\alpha}{\partial \alpha} \Big) \right] \vec{e}_\theta \\ &+ \left[\nabla^2 A_\alpha - \frac{1}{r^2} \Big(\frac{1}{\sin^2 \theta} A_\alpha - \frac{2}{\sin \theta} \frac{\partial A_r}{\partial \alpha} - 2 \frac{\cos \theta}{\sin^2 \theta} \frac{\partial A_\theta}{\partial \alpha} \Big) \right] \vec{e}_\alpha \end{split}$$

示例,以圆柱坐标系为例:

圆柱坐标系下,线段元 $d\vec{l} = dr\vec{e}_r + rd\alpha\vec{e}_\alpha + dz\vec{e}_z$

$$\parallel \parallel h_1 = 1 \ , \ h_2 = r \ , \ h_3 = 1 \ ; \ u_1 = r \ , \ u_2 = \alpha \ , \ u_3 = z \ .$$

$$egin{aligned}
abla u &= rac{1}{h_1}rac{\partial u}{\partial u_1}ec{e}_{u_1} + rac{1}{h_2}rac{\partial u}{\partial u_2}ec{e}_{u_2} + rac{1}{h_3}rac{\partial u}{\partial u_3}ec{e}_{u_3} \ &= rac{1}{1}rac{\partial u}{\partial r}ec{e}_r + rac{1}{r}rac{\partial u}{\partial lpha}ec{e}_lpha + rac{1}{1}rac{\partial u}{\partial z}ec{e}_z \end{aligned}$$

$$egin{aligned}
abla ullet ec{A} &= rac{1}{h_1 h_2 h_3} iggl[rac{\partial}{\partial u_1} (A_1 h_2 h_3) + rac{\partial}{\partial u_2} (A_2 h_1 h_3) + rac{\partial}{\partial u_3} (A_3 h_1 h_2) iggr] \ &= rac{1}{1 \cdot r \cdot 1} iggl[rac{\partial}{\partial r} (A_r \cdot r \cdot 1) + rac{\partial}{\partial lpha} (A_lpha 1 \cdot 1) + rac{\partial}{\partial z} (Az \cdot 1 \cdot r) iggr] \end{aligned}$$

$$\begin{split} \nabla\times\vec{A} &= \frac{1}{h_1h_2h_3} \begin{vmatrix} h_1\vec{e}_{u_i} & h_2\vec{e}_{u_2} & h_3\vec{e}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \\ &= \frac{1}{1\cdot r\cdot 1} \begin{vmatrix} \vec{e}_r & r\vec{e}_{\alpha} & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial z} \\ A_r & rA_{\alpha} & A_z \end{vmatrix} \\ &= \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_{\alpha} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\alpha) - \frac{\partial A_r}{\partial \alpha}\right] \\ \nabla^2 u &= \nabla \bullet \left(\nabla u\right) = \nabla \bullet \left(\frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}\vec{e}_{u_i} + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}\vec{e}_{u_z} + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}\vec{e}_{u_3}\right) \\ &= \nabla \bullet \left(\frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \alpha}\vec{e}_{\alpha} + \frac{\partial u}{\partial z}\vec{e}_z\right) \\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_1h_3\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right] \quad \left[*\dot{\Xi}: \quad \vec{A} = \nabla u\right] \\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}r\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial z^2}\right] \\ \nabla^2 \vec{A} &= \nabla^2 A_r \vec{e}_r + \nabla^2 A_\alpha \vec{e}_\alpha + \nabla^2 A_z \vec{e}_z \\ &= \left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_r}{\partial \alpha} - \frac{A_r}{r^2}\right)\vec{e}_r + \left(\nabla^2 A_\alpha + \frac{2}{r^2}\frac{\partial A_r}{\partial \alpha} - \frac{A_\alpha}{r^2}\right)\vec{e}_\alpha + \left(\nabla^2 A_z\right)\vec{e}_z \end{aligned}$$

后记:本文档只是根据老师讲的和编者自己理解所写,若有错误或者读者有更好的想法欢迎与编者交流。

[*注:]: 工程电磁场(第2版) 王泽忠 全玉生 卢斌先 编著

[*注₂]: 教材 P_{24} 5. ∇ 算子的常用运算式——(15)

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