拉梅系数与哈米尔顿算子

整理：仲 弘毅 来源：肖老师 教材[*注1]

设u_1,u_2,u_3分别为正交曲线坐标轴，$\vec{e}_{u_1},\vec{e}_{u_2},\vec{e}_{u_3}$分别为各坐标轴上的单位矢量。

$$\vec{A}=A_1\vec{e}_{u_1}+A_2\vec{e}_{u_2}+A_2\vec{e}_{u_3}$$

由$\mathrm{d}l_1=h_1\mathrm{d}u_1$,则标量场$\varPhi$在$\vec{l}_1$方向上方向导数为:

$$\frac{\partial\varPhi}{\partial l_1}=\frac{1}{h_1}\frac{\partial\varPhi}{\partial u_1}$$

梯度：

$$\begin{aligned}\nabla\varPhi&=\frac{\partial\varPhi}{\partial l_1}\vec{e}_{u_1}+\frac{\partial\varPhi}{\partial l_2}\vec{e}_{u_2}+\frac{\partial\varPhi}{\partial l_3}\vec{e}_{u_3}\\&=\frac{1}{h_1}\frac{\partial\varPhi}{\partial u_1}\vec{e}_{u_1}+\frac{1}{h_2}\frac{\partial\varPhi}{\partial u_2}\vec{e}_{u_2}+\frac{1}{h_3}\frac{\partial\varPhi}{\partial u_3}\vec{e}_{u_3}\end{aligned}$$

散度：

$$\nabla\bullet\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_1h_3)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

旋度：

$$\nabla\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\vec{e}_{u_1}&h_2\vec{e}_{u_2}&h_3\vec{e}_{u_3}\\\frac{\partial}{\partial u_1}&\frac{\partial}{\partial u_2}&\frac{\partial}{\partial u_3}\\h_1A_1&h_2A_2&h_3A_3\end{vmatrix}$$

标量拉普拉斯：

$$\nabla^2\Phi=\nabla\bullet(\nabla\Phi)$$

矢量拉普拉斯：

$$矢量公式^{[*注_2]}：\nabla\times\left(\nabla\times\vec{A}\right)=\nabla\left(\nabla\bullet\vec{A}\right)-\nabla^2\vec{A}$$

$$即：\nabla^2\vec{A}=\nabla\left(\nabla\bullet\vec{A}\right)-\nabla\times\left(\nabla\times\vec{A}\right)$$

一．直角坐标系

1）基本公式

单位矢量 $\vec{e}_x\ \vec{e}_y\ \vec{e}_z$

线段元 $\mathrm{d}\vec{l}=\mathrm{d}x\vec{e}_x+\mathrm{d}y\vec{e}_y+\mathrm{d}z\vec{e}_z$

面积元 $\mathrm{d}\vec{\mathrm{S}}=\mathrm{d}y\,\mathrm{d}z\vec{e}_x+\mathrm{d}z\,\mathrm{d}x\vec{e}_y+\mathrm{d}x\,\mathrm{d}y\vec{e}_z$

体积元 $\mathrm{d}V=\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

2）算子

$$\nabla u=\frac{\partial u}{\partial x}\vec{e}_x+\frac{\partial u}{\partial y}\vec{e}_y+\frac{\partial u}{\partial z}\vec{e}_z$$

$$\nabla\cdot\vec{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{e}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\vec{e}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\vec{e}_z$$

$$\nabla^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}$$

$$\nabla^2\vec{A}=\left(\frac{\partial^2A_x}{\partial x^2}+\frac{\partial^2A_x}{\partial y^2}+\frac{\partial^2A_x}{\partial z^2}\right)\vec{e}_x+\left(\frac{\partial^2A_y}{\partial x^2}+\frac{\partial^2A_y}{\partial y^2}+\frac{\partial^2A_y}{\partial z^2}\right)\vec{e}_y$$
$$+\left(\frac{\partial^2A_z}{\partial x^2}+\frac{\partial^2A_z}{\partial y^2}+\frac{\partial^2A_z}{\partial z^2}\right)\vec{e}_z$$

二. 圆柱坐标系

1）基本公式

圆柱坐标系与直角坐标系的坐标变换

$$x=r\cos\alpha,\ y=r\sin\alpha,\ z=z$$

$$r=\sqrt{x^2+y^2},\ \alpha=\arctan\frac{y}{x},\ z=z$$

单位矢量$\vec{e}_r$ $\vec{e}_\alpha$ $\vec{e}_z$

线段元 $\mathrm{d}l=\mathrm{d}r\vec{e}_r+r\,\mathrm{d}\alpha\vec{e}_\alpha+\mathrm{d}z\vec{e}_z$

面积元 $\mathrm{d}\vec{\mathrm{S}}=r\,\mathrm{d}\alpha\,\mathrm{d}z\vec{e}_r+\mathrm{d}r\,\mathrm{d}z\vec{e}_\alpha+r\,\mathrm{d}r\,\mathrm{d}\alpha\vec{e}_z$

体积元 $\mathrm{d}V=r\,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}z$

2）算子公式

$$\nabla u=\frac{\partial u}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial u}{\partial\alpha}\vec{e}_\alpha+\frac{\partial u}{\partial z}\vec{e}_z$$

$$\nabla\bullet\vec{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A}=\left(\frac{1}{r}\frac{\partial A_z}{\partial\alpha}-\frac{\partial A_\alpha}{\partial z}\right)\vec{e}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\vec{e}_\alpha+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\alpha)-\frac{\partial A_r}{\partial\alpha}\right]$$

$$\nabla^2u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u}{\partial\alpha^2}+\frac{\partial^2u}{\partial z^2}$$

$$\nabla^2\vec{A}=\left(\nabla^2A_r-\frac{2}{r^2}\frac{\partial A_r}{\partial\alpha}-\frac{A_r}{r^2}\right)\vec{e}_r+\left(\nabla^2A_\alpha+\frac{2}{r^2}\frac{\partial A_r}{\partial\alpha}-\frac{A_\alpha}{r^2}\right)\vec{e}_\alpha+(\nabla^2A_z)\vec{e}_z$$

三. 球坐标系

1）基本公式

球坐标系与直角坐标系变换

$$x=r\sin\theta\cos\alpha,\ y=r\sin\theta\sin\alpha,\ z=r\cos\theta$$

$$r=\sqrt{x^2+y^2+z^2}$$

单位矢量$\vec{e}_r$ $\vec{e}_\theta$ $\vec{e}_\alpha$

线积分 $\mathrm{d}\vec{l}=\mathrm{d}r\vec{e}_r+r\,\mathrm{d}\theta\vec{e}_\theta+r\sin\theta\,\mathrm{d}\alpha\vec{e}_\alpha$

面积分 $\mathrm{d}\vec{\mathrm{S}}=r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\alpha\vec{e}_r+r\sin\theta\,\mathrm{d}r\,\mathrm{d}\alpha\vec{e}_\theta+r\,\mathrm{d}r\,\mathrm{d}\theta\vec{e}_\alpha$

体积分 $\mathrm{d}V=r^2\sin\theta\,\mathrm{d}\alpha\,\mathrm{d}r$

2）算子公式

$$\nabla u=\frac{\partial u}{\partial r}\vec{e}_r+\frac{1}{\partial r}\frac{\partial u}{\partial \theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial u}{\partial \alpha}\vec{e}_\alpha$$

$$\nabla\bullet\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\alpha}{\partial\alpha}$$

$$\nabla\times\vec{A}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\alpha)-\frac{\partial A_\theta}{\partial\alpha}\right]\vec{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\alpha}-\frac{\partial}{\partial r}(rA_\alpha)\right]\vec{e}_\theta$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\vec{e}_\alpha$$

$$\nabla^2u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\alpha^2}$$

$$\nabla^2\vec{A}=\left[\nabla^2A_r-\frac{2}{r^2}\left(A_r+\cot\theta A_\theta+\frac{1}{\sin\theta}\frac{\partial A_\alpha}{\partial\theta}+\frac{\partial A_\theta}{\partial\theta}\right)\right]\vec{e}_r$$
$$+\left[\nabla^2A_\theta-\frac{1}{r^2}\left(\frac{1}{\sin^2\theta}A_\theta-2\frac{\partial A_r}{\partial\theta}+2\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\alpha}{\partial\alpha}\right)\right]\vec{e}_\theta$$
$$+\left[\nabla^2A_\alpha-\frac{1}{r^2}\left(\frac{1}{\sin^2\theta}A_\alpha-\frac{2}{\sin\theta}\frac{\partial A_r}{\partial\alpha}-2\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\theta}{\partial\alpha}\right)\right]\vec{e}_\alpha$$

示例，以圆柱坐标系为例：

圆柱坐标系下，线段元$\mathrm{d}\vec{l}=\mathrm{d}r\vec{e}_r+r\,\mathrm{d}\alpha\vec{e}_\alpha+\mathrm{d}z\vec{e}_z$

则 $h_1=1$, $h_2=r$, $h_3=1$ ； $u_1=r$, $u_2=\alpha$, $u_3=z$.

$$\nabla u=\frac{1}{h_1}\frac{\partial u}{\partial u_1}\vec{e}_{u_1}+\frac{1}{h_2}\frac{\partial u}{\partial u_2}\vec{e}_{u_2}+\frac{1}{h_3}\frac{\partial u}{\partial u_3}\vec{e}_{u_3}$$
$$=\frac{1}{1}\frac{\partial u}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial u}{\partial\alpha}\vec{e}_\alpha+\frac{1}{1}\frac{\partial u}{\partial z}\vec{e}_z$$

$$\nabla \bullet \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$
$$= \frac{1}{1 \cdot r \cdot 1}\left[\frac{\partial}{\partial r}(A_r \cdot r \cdot 1) + \frac{\partial}{\partial \alpha}(A_\alpha 1 \cdot 1) + \frac{\partial}{\partial z}(Az \cdot 1 \cdot r)\right]$$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \vec{e}_{u_1} & h_2 \vec{e}_{u_2} & h_3 \vec{e}_{u_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$
$$= \frac{1}{1 \cdot r \cdot 1}\begin{vmatrix} \vec{e}_r & r\vec{e}_\alpha & \vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \alpha} & \dfrac{\partial}{\partial z} \\ A_r & rA_\alpha & A_z \end{vmatrix}$$
$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\alpha + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\alpha) - \frac{\partial A_r}{\partial \alpha}\right]$$

$$\nabla^2 u = \nabla \bullet (\nabla u) = \nabla \bullet \left(\frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}\vec{e}_{u_1} + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}\vec{e}_{u_2} + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}\vec{e}_{u_3}\right)$$
$$= \nabla \bullet \left(\frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \alpha}\vec{e}_\alpha + \frac{\partial u}{\partial z}\vec{e}_z\right)$$
$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right] \quad \left[\text{*注：} \vec{A} = \nabla u\right]$$
$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} r\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\nabla^2 \vec{A} = \nabla^2 A_r \vec{e}_r + \nabla^2 A_\alpha \vec{e}_\alpha + \nabla^2 A_z \vec{e}_z$$
$$= \left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_r}{\partial \alpha} - \frac{A_r}{r^2}\right)\vec{e}_r + \left(\nabla^2 A_\alpha + \frac{2}{r^2}\frac{\partial A_r}{\partial \alpha} - \frac{A_\alpha}{r^2}\right)\vec{e}_\alpha + (\nabla^2 A_z)\vec{e}_z$$

后记：本文档只是根据老师讲的和编者自己理解所写，若有错误或者读者有更好的想法欢迎与编者交流。

[*注$_1$]：工程电磁场(第2版) 王泽忠 全玉生 卢斌先 编著

[*注$_2$]：教材P$_{24}$ 5.∇算子的常用运算式——(15)

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