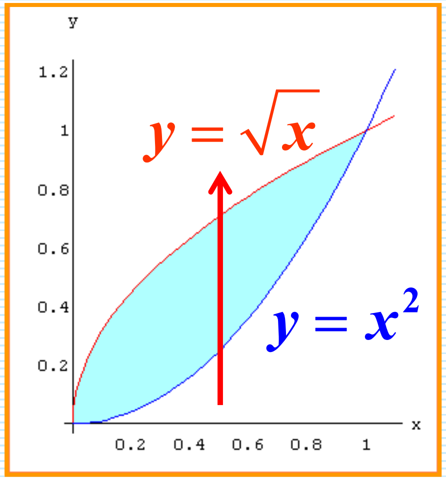
第十章
重积分习题选解

习题10–2

2. 画出积分区域，并计算下列二重积分：

(1) $\iint\limits_D x\sqrt{y}\,d\sigma$，其中 D 是由两条抛物线

$y=\sqrt{x},\ y=x^2$ 所围成的闭区域；



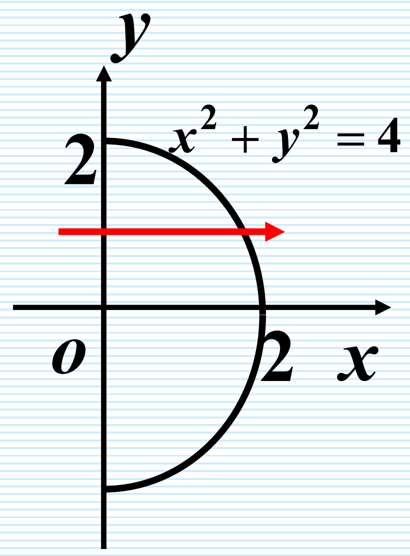
解　积分区域如图，并且

$$D=\{(x,y)\,|\,0\le x\le 1,\ x^2\le y\le\sqrt{x}\}.$$

$$\iint\limits_D x\sqrt{y}\,d\sigma=\int_0^1 dx\int_{x^2}^{\sqrt{x}} x\sqrt{y}\,dy=\int_0^1 x\left[\frac{2}{3}y^{\frac{3}{2}}\right]_{x^2}^{\sqrt{x}}dx$$

$$=\int_0^1\left(\frac{2}{3}x^{\frac{7}{4}}-\frac{2}{3}x^4\right)dx=\frac{6}{55}$$

(2) $\iint\limits_D xy^2 d\sigma$，其中 D 是由圆周 $x^2+y^2=4$ 及 y 轴所围成的右半闭区域;



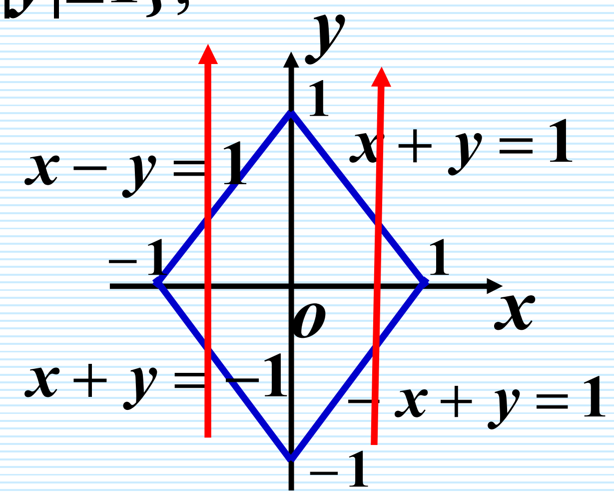
解 积分区域图如，并且

$D=\{(x,y)|\ -2\le y\le 2,\ 0\le x\le\sqrt{4-y^2}\}.$

$$\iint\limits_D xy^2 d\sigma = \int_{-2}^{2} dy\int_0^{\sqrt{4-y^2}} xy^2 dx = \int_{-2}^{2}[\frac{1}{2}x^2y^2]_0^{\sqrt{4-y^2}}dy$$

$$= \int_{-2}^{2}(2y^2-\frac{1}{2}y^4)dy = [\frac{2}{3}y^3-\frac{1}{10}y^5]_{-2}^{2} = \frac{64}{15}$$

(3) $\iint\limits_{D} e^{x+y} d\sigma$，其中 $D=\{(x,y) \mid |x|+|y| \le 1\}$;



解 积分区域如图，并且

$$D=\{(x,y) \mid -1\le x\le 0, -x-1\le y\le x+1\}$$

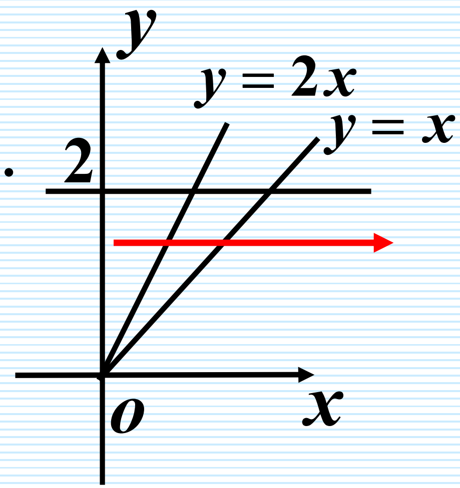
$$\cup\{(x,y) \mid 0\le x\le 1, x-1\le y\le -x+1\}.$$

$$\iint\limits_{D} e^{x+y} d\sigma = \int_{-1}^{0} e^{x} dx \int_{-x-1}^{x+1} e^{y} dy + \int_{0}^{1} e^{x} dx \int_{x-1}^{-x+1} e^{y} dy$$

$$= \int_{-1}^{0} (e^{2x+1} - e^{-1}) dx + \int_{0}^{1} (e - e^{2x-1}) dx$$

$$= e - e^{-1}$$

(4) $\iint\limits_D (x^2+y^2-x)d\sigma$，其中 D 是由直线 $y=2, y=x$ 及 $y=2x$ 轴所围成的闭区域.



解 积分区域图如，并且

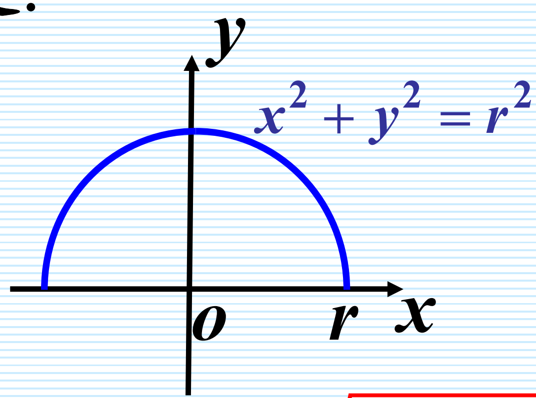
$D=\{(x,y)|\ 0\le y\le 2,\ \frac{1}{2}y\le x\le y\}$.

$$\iint\limits_D (x^2+y^2-x)d\sigma = \int_0^2 dy\int_{\frac{y}{2}}^{y}(x^2+y^2-x)dx$$

$$=\int_0^2\left[\frac{1}{3}x^3+y^2x-\frac{1}{2}x^2\right]_{\frac{y}{2}}^{y}dy=\int_0^2\left(\frac{19}{24}y^3-\frac{3}{8}y^2\right)dy=\frac{13}{6}$$

4. 化二重积分 $I=\iint\limits_D f(x,y)d\sigma$ 为二次积分(分别列出对两个变量先后次序不同的两个二次积分), 其中积分区域 D 是:

(2) 由 x 轴及半圆周 $x^2+y^2=r^2(y\geq 0)$ 所围成的闭区域;



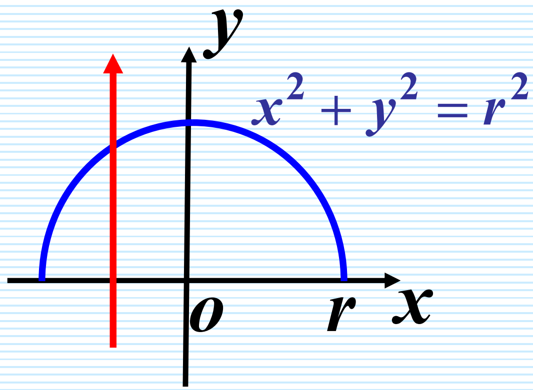
解 积分区域如图所示, 并且

$y=\sqrt{r^2-x^2}$

$D=\{(x,y)|\},-r\leq x\leq r,\ 0\leq y\leq\sqrt{r^2-x^2}\}$ $\quad x=\pm\sqrt{r^2-y^2}$

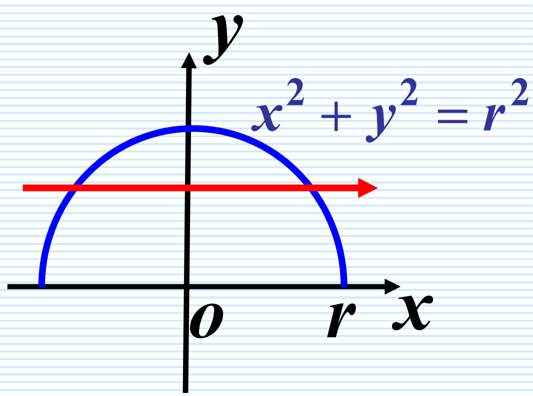
或 $D=\{(x,y)|\ 0\leq y\leq r,\ -\sqrt{r^2-y^2}\leq x\leq\sqrt{r^2-y^2}\}$,

$$I=\int_{-r}^{r}dx\int_{0}^{\sqrt{r^2-x^2}}f(x,y)dy$$



$$y=\sqrt{r^2-x^2}$$

$$I=\int_{0}^{r}dy\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}f(x,y)dx$$



$$x=\pm\sqrt{r^2-y^2}$$

5. 设 $f(x, y)$在 D 上连续，其中 D 是由直线 $y=x$、$y=a$ 及 $x=b(b>a)$围成的闭区域，

证明：$\int_a^b dx\int_a^x f(x,y)dy=\int_a^b dy\int_y^b f(x,y)dx$

证明 积分区域如图所示，

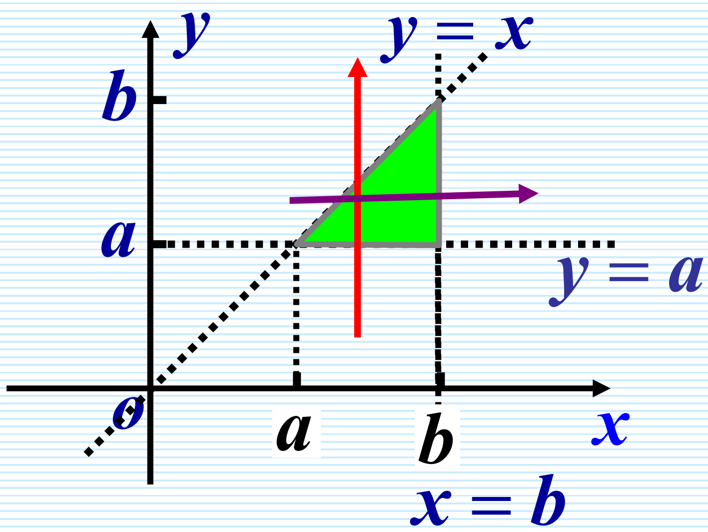
$D=\{(x,y)|a\le x\le b,\ a\le y\le x\}$，

或 $D=\{(x,y)|a\le y\le b,\ y\le x\le b\}$.

$$\iint_D f(x,y)d\sigma=\int_a^b dx\int_a^x f(x,y)dy$$

$$\iint_D f(x,y)d\sigma=\int_a^b dy\int_y^b f(x,y)dx$$

$$\int_a^b dx\int_a^x f(x,y)dy=\int_a^b dy\int_y^b f(x,y)dx$$

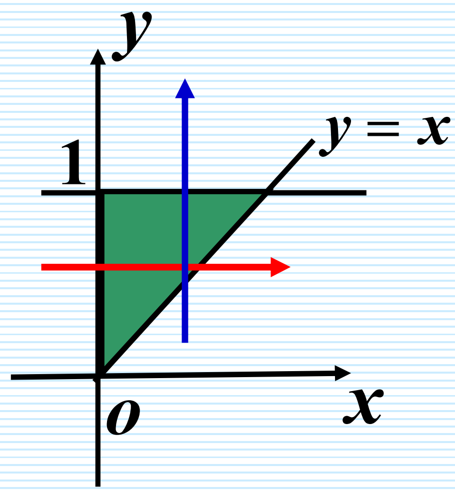


6. 改换下列二次积分的积分次序:

$$(1)\int_0^1 dy\int_0^y f(x,y)dx;$$

解 由根据积分限可得积分区域

$D=\{(x,y)|0\le y\le 1,\ 0\le x\le y\}$，如图



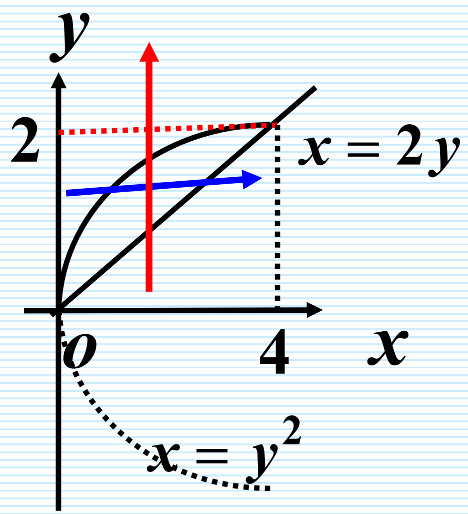
或 $D=\{(x,y)|0\le x\le 1,\ x\le y\le 1\}$,

$$\int_0^1 dy\int_0^y f(x,y)dx=\int_0^1 dx\int_x^1 f(x,y)dy$$

$$(2)\int_0^2 dy\int_{y^2}^{2y} f(x,y)dx;$$

解 由根据积分限可得积分区域

$D=\{(x,y)|0\le y\le 2,\ y^2\le x\le 2y\}$，如图.



或 $D=\{(x,y)|0\le x\le 4,\ \dfrac{x}{2}\le y\le \sqrt{x}\}$,

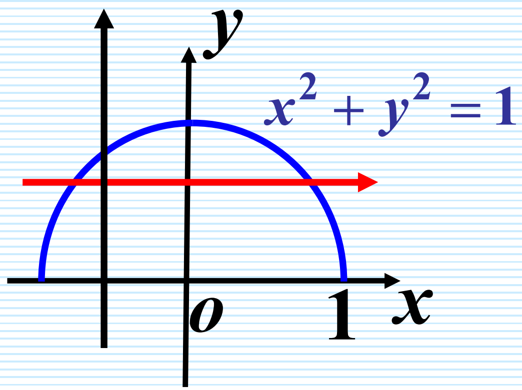
$$\int_0^2 dy\int_{y^2}^{2y} f(x,y)dx = \int_0^4 dx\int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)dy$$

$$(3)\int_0^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx;$$

解 由根据积分限可得积分区域

$$D=\{(x,y)\,|\,0\le y\le 1,\ -\sqrt{1-y^2}\le x\le\sqrt{1-y^2}\}$$

$$D=\{(x,y)\,|\,-1\le x\le 1,\ 0\le y\le\sqrt{1-x^2}\}$$



$$x=\pm\sqrt{1-y^2}$$

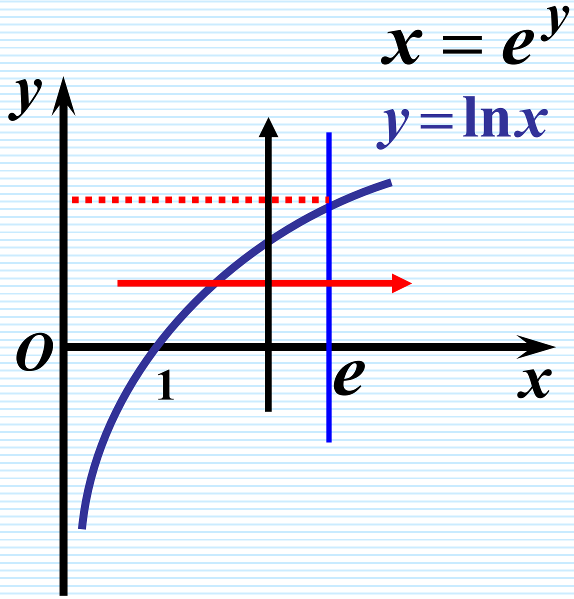
$$\int_0^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx$$

$$=\int_{-1}^1 dx\int_0^{\sqrt{1-x^2}} f(x,y)dy$$

$$(5)\int_1^e dx\int_0^{\ln x} f(x,y)dy;$$

解 由根据积分限可得积分区域

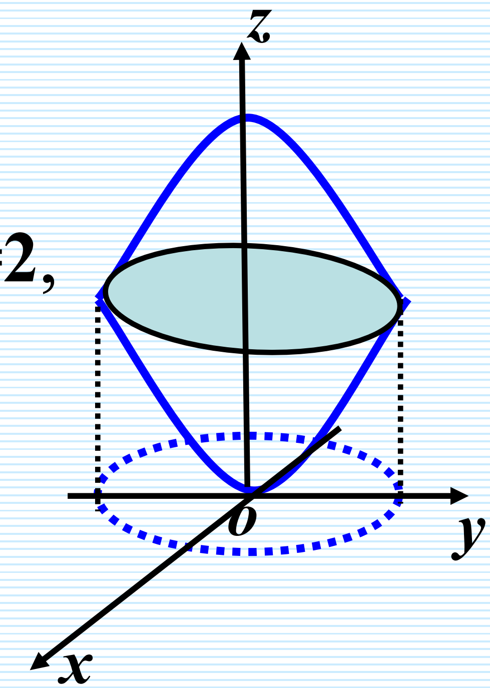
$D=\{(x,y)|1\le x\le e,\ 0\le y\le \ln x\}$，如图.



$D=\{(x,y)|0\le y\le 1,\ e^y\le x\le e\}$,

$$\int_1^e dx\int_0^{\ln x} f(x,y)dy$$
$$=\int_0^1 dy\int_{e^y}^{e} f(x,y)dx$$

10. 求由曲面 $z=x^2+2y^2$ 及 $z=6-2x^2-y^2$ 所围成的立体的体积.



解 由 $\begin{cases} z = x^2 + 2y^2 \\ z = 6 - 2x^2 - y^2 \end{cases}$ 消去 z, 得 $x^2+y^2=2$,

$$D_{xy}: x^2 + y^2 \le 2$$

$$V = \iint_D [(6-2x^2-y^2)-(x^2+2y^2)]d\sigma$$

$$= \iint_D (6-3x^2-3y^2)d\sigma$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (6-3\rho^2)\rho d\rho$$

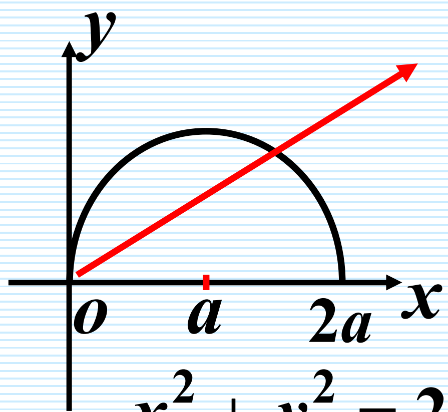
$$= 6\pi$$

$$= 12\int_0^{\sqrt{2}} dx \int_0^{\sqrt{2-x^2}} (2-x^2-y^2)dy$$

$$= 8\int_0^{\sqrt{2}} \sqrt{(2-x^2)^3}dx = 6\pi$$

13. 把下列积分化为极坐标形式, 并计算积分值:

$$(1)\int_0^{2a}dx\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dy;$$



解 积分区域D如图所示.

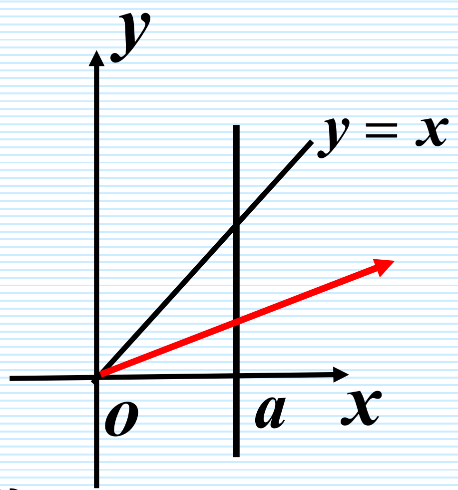
$x^2+y^2=2ax,$

$$D=\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{2},\ 0\le\rho\le 2a\cos\theta\}\qquad y\ge 0$$

$$\int_0^{2a}dx\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dy=\iint_D\rho^2\cdot\rho d\rho d\theta$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\rho^2\cdot\rho d\rho=4a^4\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta=\frac{3}{4}\pi a^4$$

$$(2)\int_0^a dx\int_0^x \sqrt{x^2+y^2}\,dy;$$



解 积分区域D如图所示.

$$D=\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{4},\ 0\le\rho\le a\sec\theta\}$$

$$x=a$$

$$\rho\cos\theta=a$$

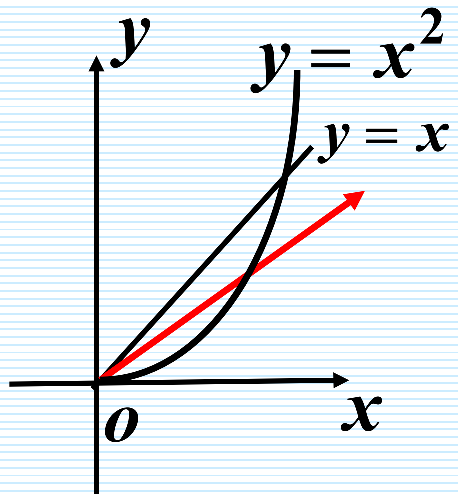
$$\rho=a\sec\theta$$

$$\int_0^a dx\int_0^x \sqrt{x^2+y^2}\,dy=\iint_D \rho\cdot\rho\, d\rho\, d\theta$$

$$=\int_0^{\frac{\pi}{4}} d\theta\int_0^{a\sec\theta}\rho\cdot\rho\, d\rho=\frac{a^3}{3}\int_0^{\frac{\pi}{4}}\sec^3\theta\, d\theta$$

$$=\frac{a^3}{6}[\sqrt{2}+\ln(\sqrt{2}+1)]$$

(3) $\int_0^1 dx \int_{x^2}^{x} (x^2+y^2)^{-\frac{1}{2}} dy;$



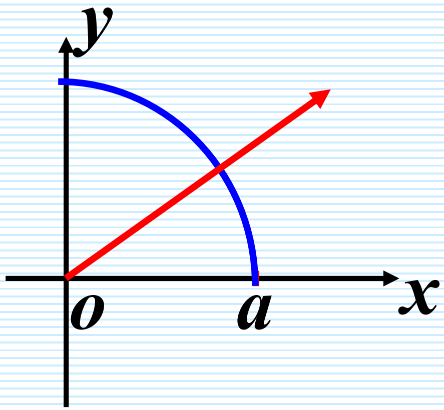
解 积分区域D如图所示.

$$D=\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{4},$$
$$0\le\rho\le\sec\theta\ \tan\theta\}$$

$$y=x^2$$
$$\rho\sin\theta=\rho^2\cos^2\theta$$
$$\rho=\sec\theta\ \tan\theta$$

$$\int_0^1 dx\int_{x^2}^{x}(x^2+y^2)^{-\frac{1}{2}}dy$$
$$=\iint_D \rho^{-\frac{1}{2}}\cdot\rho d\rho d\theta$$
$$=\int_0^{\frac{\pi}{4}}d\theta\int_0^{\sec\theta\tan\theta}\rho^{-\frac{1}{2}}\cdot\rho d\rho=\int_0^{\frac{\pi}{4}}\sec\theta\tan\theta d\theta=\sqrt{2}-1$$

(4) $\int_0^a dy \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dx$.



解 积分区域D如图所示.

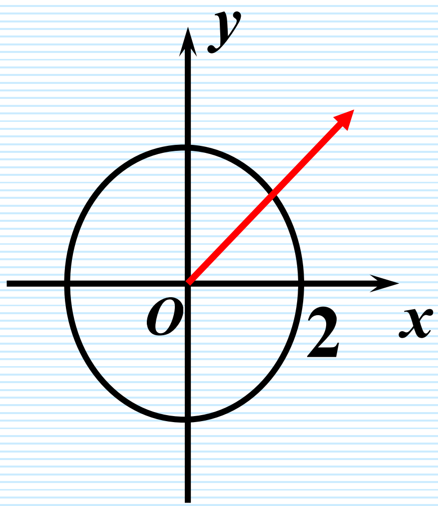
$$D=\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{2},\ 0\le\rho\le a\}$$

$$\int_0^a dy \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dx=\iint_D \rho^2\cdot\rho d\rho d\theta$$

$$=\int_0^{\frac{\pi}{2}} d\theta \int_0^a \rho^2\cdot\rho d\rho=\frac{\pi}{8}a^4$$

14. 利用极坐标计算下列各题:

(1) $\iint\limits_D e^{x^2+y^2}d\sigma$，其中 D 是由圆周 $x^2+y^2=4$ 所围成的闭区域;



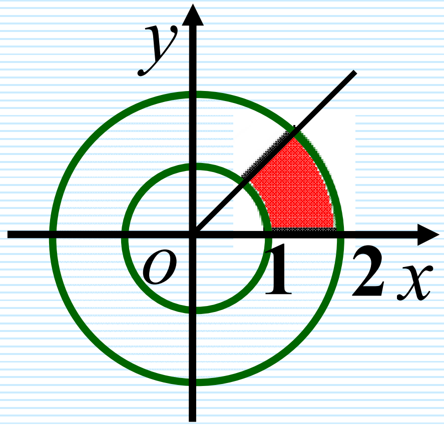
解 在极坐标下

$D=\{(\rho,\theta)|0\le\theta\le2\pi, 0\le\rho\le2\}$,

$$\iint\limits_D e^{x^2+y^2}d\sigma=\iint\limits_D e^{\rho^2}\rho d\rho d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^2 e^{\rho^2}\rho d\rho=2\pi\cdot\frac{1}{2}(e^4-1)=\pi(e^4-1)$$

(3) $\iint\limits_{D} \arctan\frac{y}{x}d\sigma$，其中 D 是由圆周 $x^2+y^2=4$，$x^2+y^2=1$ 及直线 $y=0$，$y=x$ 所围成的第一象限内的闭区域.



解 在极坐标下

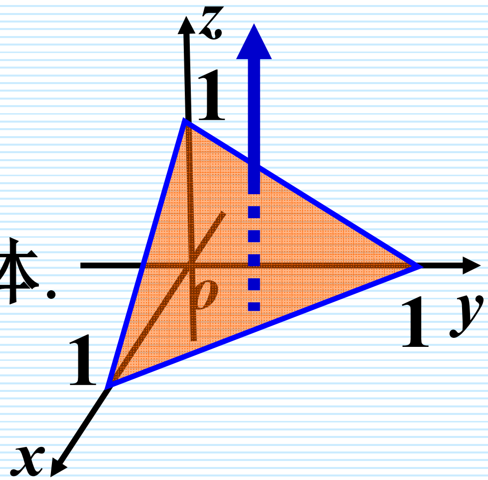
$$D=\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{4},\ 1\le\rho\le 2\},$$

$$\iint\limits_{D}\arctan\frac{y}{x}d\sigma=\iint\limits_{D}\arctan(\tan\theta)\cdot\rho d\rho d\theta=\iint\limits_{D}\theta\cdot\rho d\rho d\theta$$

$$=\int_0^{\frac{\pi}{4}}d\theta\int_1^2\theta\cdot\rho d\rho=\int_0^{\frac{\pi}{4}}\theta d\theta\int_1^2\rho d\rho=\frac{3\pi^3}{64}$$

习题10–3

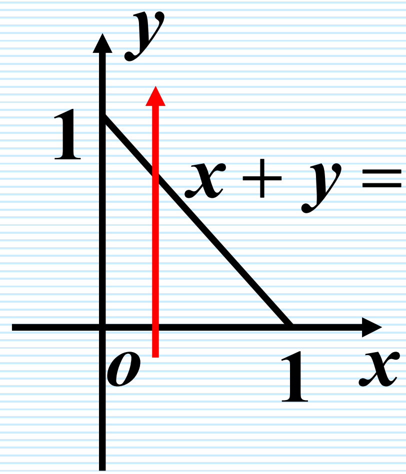
5. 计算$\iiint\limits_{\Omega}\dfrac{dxdydz}{(1+x+y+z)^3}$，其中 Ω 为



平面 $x=0, y=0, z=0, x+y+z=1$ 所围成的四面体.

解 积分区域可表示为

$\Omega=\{(x, y, z)|\ 0\le z\le 1-x-y,\ 0\le y\le 1-x,\ 0\le x\le 1\}$,



$$\iiint\limits_{\Omega}\frac{dxdydz}{(1+x+y+z)^3}$$

$$=\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{1}{(1+x+y+z)^3}dz$$

$$=\frac{1}{2}(\ln 2-\frac{5}{8})$$

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz$$

$$= \int_0^1 dx \int_0^{1-x} [\frac{1}{-2(1+x+y+z)^2}]_0^{1-x-y} dy$$

$$= \int_0^1 dx \int_0^{1-x} [\frac{1}{2(1+x+y)^2} - \frac{1}{8}] dy$$

$$= \int_0^1 [\frac{1}{-2(1+x+y)} - \frac{1}{8} y]_0^{1-x} dx = \int_0^1 [\frac{1}{2(1+x)} - \frac{3}{8} + \frac{1}{8} x] dx$$

$$= [\frac{1}{2} \ln(1+x) - \frac{3}{8} x + \frac{1}{16} x^2]_0^1 = \frac{1}{2}(\ln 2 - \frac{5}{8})$$

6. 计算$\iiint\limits_{\Omega} xyz\,dxdydz$, 其中$\Omega$为球面$x^2+y^2+z^2=1$及三个坐标面所围成的在第一卦限内的闭区域.

解 $\Omega=\{(x,y,z)\,|\,0\le z\le\sqrt{1-x^2-y^2},0\le y\le\sqrt{1-x^2},0\le x\le 1\}$

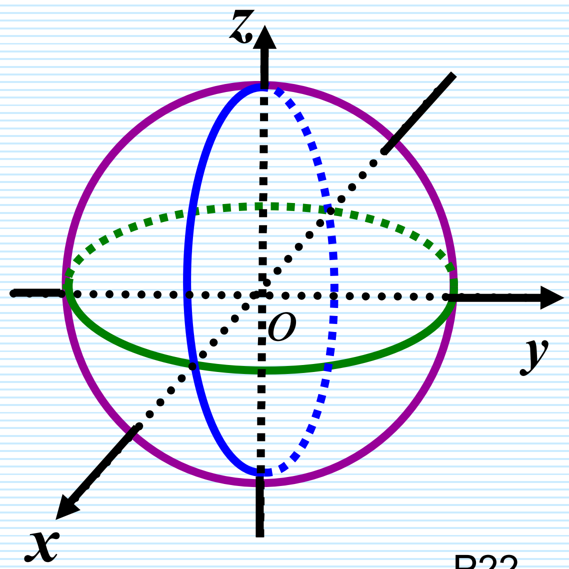
$$\iiint\limits_{\Omega} xyz\,dxdydz$$

$$=\int_0^1 dx\int_0^{\sqrt{1-x^2}}dy\int_0^{\sqrt{1-x^2-y^2}}xyz\,dz$$

$$=\int_0^1 dx\int_0^{\sqrt{1-x^2}}\frac{1}{2}xy(1-x^2-y^2)dy$$

$$=\int_0^1\frac{1}{8}x(1-x^2)^2dx=\frac{1}{48}$$

$z=\sqrt{1-x^2-y^2}$



习题10－3，9. 利用柱面坐标计算下列三重积分:

(1) $\iiint\limits_{\Omega} z\,dv$，其中 Ω 是由曲面 $z=\sqrt{2-x^2-y^2}$ 及 $z=x^2+y^2$ 所围成的闭区域;

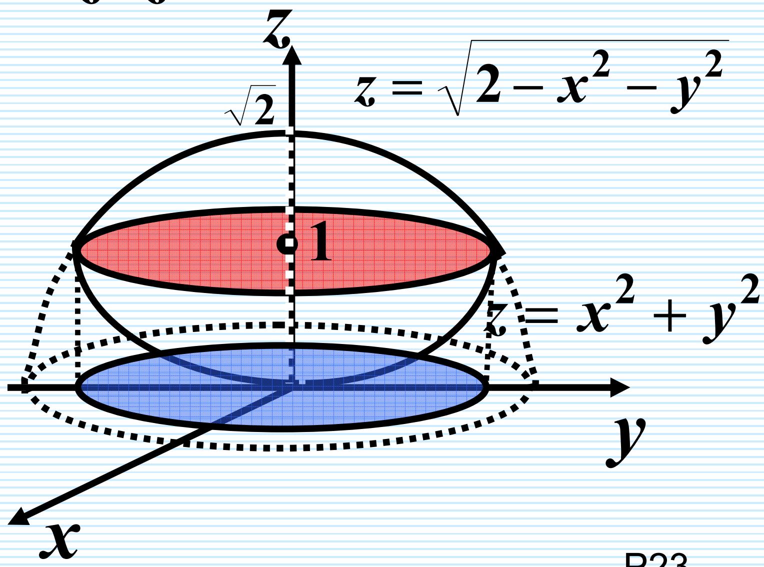
解　Ω：$0\le\theta\le 2\pi,\ 0\le\rho\le 1,\ \rho^2\le z\le\sqrt{2-\rho^2}$，

$$\iiint\limits_{\Omega} z\,dv=\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho\int_{\rho^2}^{\sqrt{2-\rho^2}}z\,dz$$

$$=2\pi\int_0^1\frac{1}{2}\rho(2-\rho^2-\rho^4)\,d\rho$$

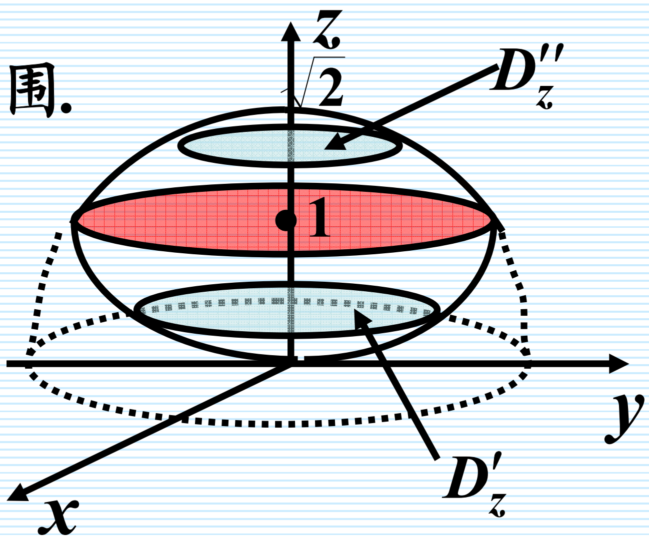
$$=\pi\int_0^1(2\rho-\rho^3-\rho^5)\,d\rho$$

$$=\frac{7}{12}\pi$$



另解　计算 $\iiint\limits_{\Omega} z\,dxdydz$,

Ω为由$z=\sqrt{2-x^2-y^2}$与$z=x^2+y^2$所围.



解：$\Omega=\Omega_1+\Omega_2$

$\Omega_1: z=x^2+y^2$与$z=1$所围

$\Omega_2: z=\sqrt{2-x^2-y^2}$与$z=1$所围

$D'_z: x^2+y^2\le z, (0\le z\le 1), D''_z: x^2+y^2\le 2-z^2, (1\le z\le\sqrt{2})$

$$\iiint\limits_{\Omega_1} z\,dxdydz=\int_0^1 dz\iint\limits_{D'_z} z\,dxdy=\int_0^1 z\pi\cdot z\,dz=\frac{\pi}{3}$$

$$\iiint\limits_{\Omega_2} z\,dxdydz=\int_1^{\sqrt{2}} dz\iint\limits_{D''_z} z\,dxdy=\int_1^{\sqrt{2}} z\pi\cdot(2-z^2)\,dz=\frac{\pi}{4}$$

$$\iiint\limits_{\Omega} z\,dxdydz=\iiint\limits_{\Omega_1} z\,dxdydz+\iiint\limits_{\Omega_2} z\,dxdydz=\frac{7\pi}{12}$$

习题10－3，9. 利用柱面坐标计算下列三重积分:

(2) $\iiint\limits_{\Omega}(x^2+y^2)dv$，其中 Ω 是由曲面 $x^2+y^2=2z$ 及平面 $z=2$ 所围成的闭区域.

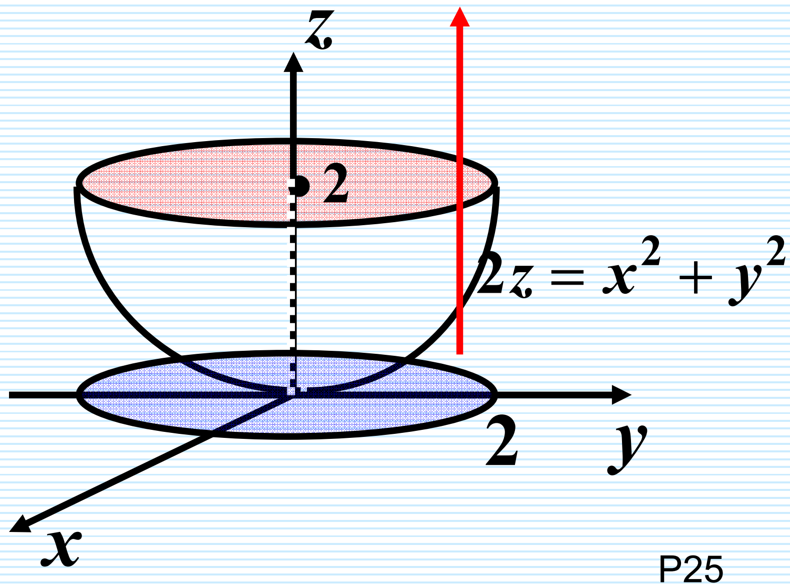
解 Ω：$0\le\theta\le2\pi,\ 0\le\rho\le2,\ \dfrac{\rho^2}{2}\le z\le 2,$

$$\iiint\limits_{\Omega}(x^2+y^2)dv=\iiint\limits_{\Omega}\rho^2\cdot\rho d\rho d\theta dz$$

$$=\int_0^{2\pi}d\theta\int_0^2\rho^3d\rho\int_{\frac{1}{2}\rho^2}^2dz$$

$$=\int_0^{2\pi}d\theta\int_0^2(2\rho^3-\frac{1}{2}\rho^5)d\rho$$

$$=\int_0^{2\pi}\frac{8}{3}d\theta=\frac{16}{3}\pi$$



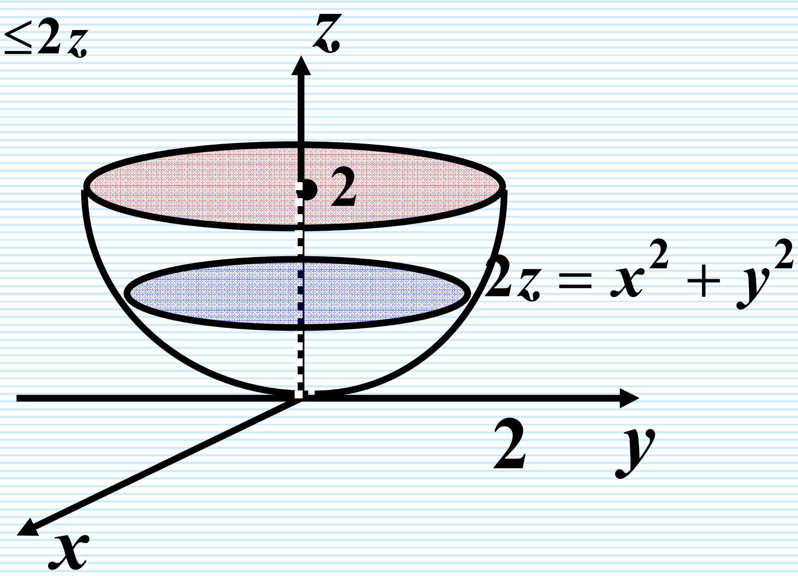
(2) $\iiint\limits_{\Omega}(x^2+y^2)dv$, 其中 Ω 是由曲面 $x^2+y^2=2z$ 及平面 $z=2$ 所围成的闭区域.

解 $\Omega: x^2+y^2\le 2z, 0\le z\le 2$, 用截面法

$$\iiint\limits_{\Omega}(x^2+y^2)dv=\int_0^2 dz\iint\limits_{x^2+y^2\le 2z}(x^2+y^2)dxdy$$

$$=\int_0^2 dz\int_0^{2\pi}d\theta\int_0^{\sqrt{2z}}\rho^3 d\rho$$

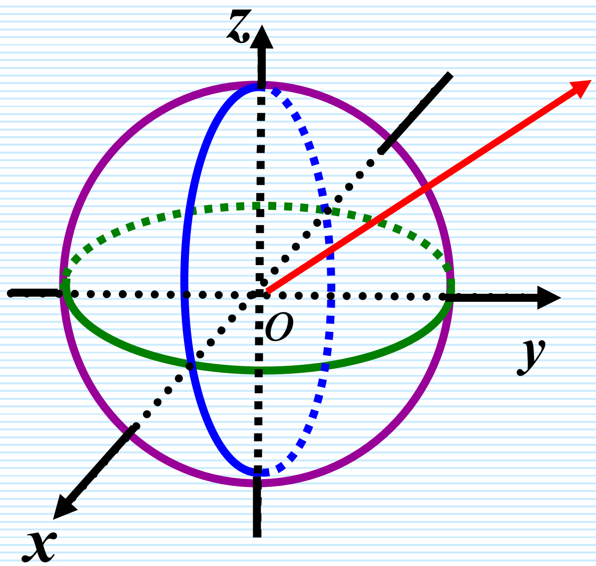
$$=2\pi\int_0^2 z^2 dz=\frac{16}{3}\pi$$



10. 利用球面坐标计算下列三重积分:

(1) $\iiint\limits_{\Omega}(x^2+y^2+z^2)dv$，其中 Ω 是由球面 $x^2+y^2+z^2=1$ 所围成的闭区域.

解 Ω: $0\le\theta\le 2\pi,\ 0\le\varphi\le\pi,\ 0\le r\le 1$,



$$\iiint\limits_{\Omega}(x^2+y^2+z^2)dv$$

$$=\iiint\limits_{\Omega} r^4\cdot\sin\varphi\, dr\, d\varphi\, d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_0^1 r^4 dr=\frac{4}{5}\pi$$

(2) $\iiint\limits_{\Omega} z dv$，其中闭区域 Ω 由不等式

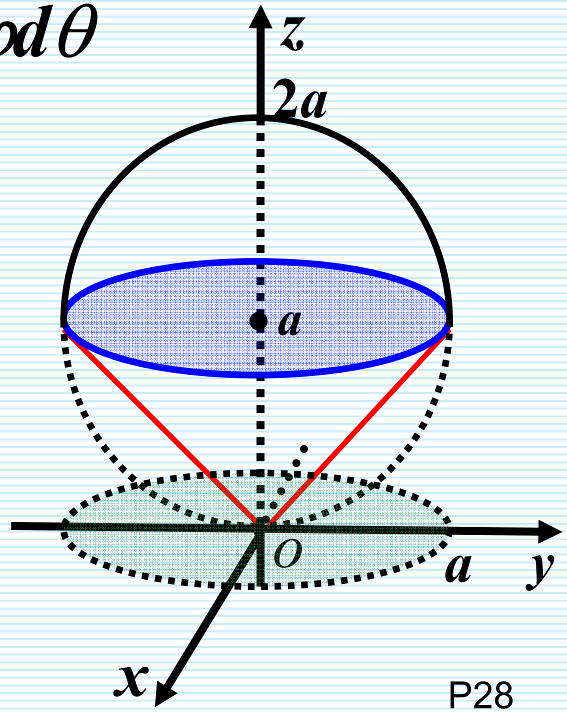
$x^2+y^2+(z-a)^2\le a^2,\ x^2+y^2\le z^2$ 所确定.

解　Ω ：$0\le\theta\le 2\pi,\ 0\le\varphi\le\dfrac{\pi}{4},\ 0\le r\le 2a\cos\varphi,$

$$\iiint\limits_{\Omega} z dv = \iiint\limits_{\Omega} r\cos\varphi\cdot r^2\sin\varphi drd\varphi d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{4}}\sin\varphi\cos\varphi\cdot\frac{1}{4}(2a\cos\varphi)^4 d\varphi$$

$$= 8\pi a^4\int_0^{\frac{\pi}{4}}\sin\varphi\cos^5\varphi d\varphi = \frac{7}{6}\pi a^4$$



11. 选用适当的坐标计算下列三重积分:

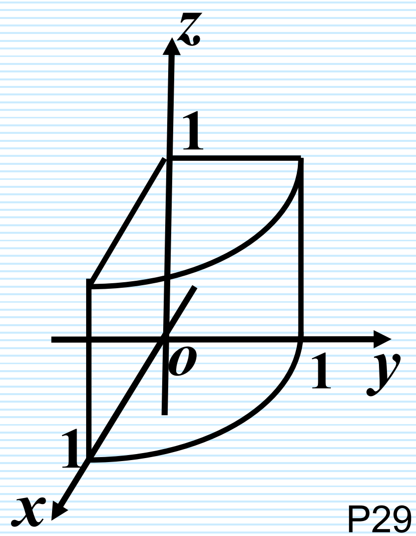
(1) $\iiint\limits_{\Omega} xy\,dv$, 其中 Ω 为柱面 $x^2+y^2=1$ 及平面 $z=1$, $z=0$, $x=0$, $y=0$ 所围成的在第一卦限内的闭区域;

解

$$\Omega:\ 0\le\theta\le\frac{\pi}{2},\ 0\le\rho\le 1,\ 0\le z\le 1,$$

$$\iiint\limits_{\Omega} xy\,dv=\iiint\limits_{\Omega}\rho\cos\theta\cdot\rho\sin\theta\cdot\rho\,d\rho\,d\theta\,dz$$

$$=\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^1\rho^3\,d\rho\int_0^1 dz=\frac{1}{8}$$



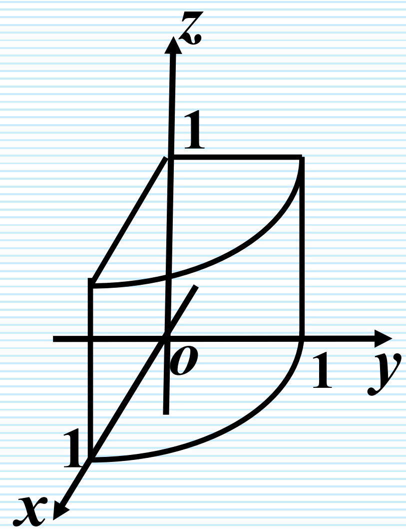
别解: 用直角坐标计算

$$\iiint_{\Omega} xy\,dv = \int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} y\,dy \int_0^1 dz$$

$$= \int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} y\,dy$$

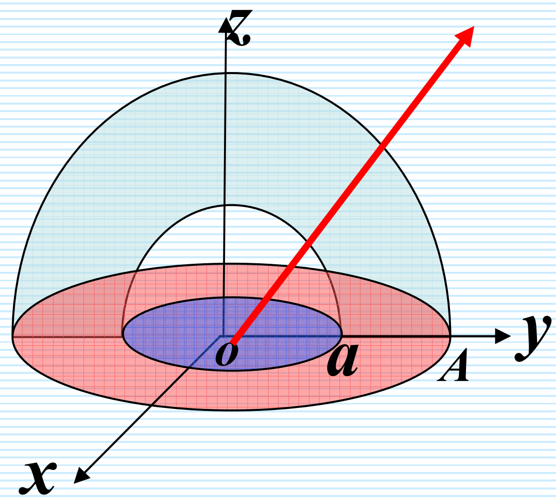
$$= \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2}\right) dx$$

$$= \left[\frac{x^2}{4} - \frac{x^4}{8}\right]_0^1 = \frac{1}{8}$$



(4) $\iiint\limits_{\Omega}(x^2+y^2)dv$，其中闭区域 Ω 由不等式

$0 < a \le \sqrt{x^2+y^2+z^2} \le A,\ z \ge 0$ 所确定.



解 Ω： $0 \le \theta \le 2\pi, 0 \le \varphi \le \dfrac{\pi}{2}, a \le r \le A,$

$$\iiint\limits_{\Omega}(x^2+y^2)dv$$

$$= \iiint\limits_{\Omega}(r^2\sin^2\varphi\cos^2\varphi + r^2\sin^2\varphi\sin^2\theta)r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin^3\varphi\, d\varphi\int_a^A r^4 dr = \frac{4\pi}{15}(A^5 - a^5)$$