时间序列作业1

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应用统计1701

1. （调和平均序列）设a,b是常数，随机变量U在$(-\pi,\pi)$内均匀分布，则 $X_t=b\cos(at+U), t\in T$是平稳序列。

   数学期望

$$\begin{aligned}
E(X_t) &= bE[\cos(at+U)] \\
&= b\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(at+U)dU \\
&= \frac{b}{2\pi}\cdot\sin(at+U)\Big|_{-\pi}^{\pi} \\
&= \frac{b}{2\pi}[\sin(\pi+at)-\sin(-\pi+at)] \\
&= 0
\end{aligned}$$

   协方差

$$\begin{aligned}
Cov(X_t,X_s) &= b^2E\left[\cos(as+U)\cos(at+U)\right]-E(X_t)\cdot E(X_s) \\
&= b^2E\left\{\left[\cos(at+as+2U)+\cos(at-as)\right]/2\right\}-0 \\
&= b^2\cdot\left[\int_{-\pi}^{\pi}\frac{\cos(at+as+2U)}{2}\cdot\frac{1}{2\pi}dU+\frac{\cos(at-as)}{2}\right] \\
&= \frac{b^2}{2}\cos a(t-s)
\end{aligned}$$

   由于X_t的数学期望为常数0，其协方差仅与时间的差值有关，所以X_t为平稳序列。

2. 设a,b是常数，随机变量$U_1,U_2,\cdots$独立同分布且都在$(0,2\pi)$上均匀分布，证明：$X_t=b\cos(at+U_t), t\in T$是独立的$WN(0,\frac{b^2}{2})$。

   数学期望

$$\begin{aligned}
E(X_t) &= bE[\cos(at+U_t)] \\
&= b\int_{0}^{2\pi}\frac{1}{2\pi}\cos(at+U_t)dU_t \\
&= \frac{b}{2\pi}\cdot\sin(at+U_t)\Big|_{0}^{2\pi} \\
&= \frac{b}{2\pi}[\sin(2\pi+at)-\sin(at)] \\
&= 0
\end{aligned}$$

   自相关函数

$$
\begin{aligned}
R(X_t, X_s) &= b^2 E\left[\cos(as+U_t)\cos(at+U_t)\right] \\
&= b^2 E\left\{\left[\cos(at+as+2U_t)+\cos(at-as)\right]/2\right\} \\
&= b^2 \cdot \left[\int_0^{2\pi} \frac{\cos(at+as+2U_t)}{2} \cdot \frac{1}{2\pi} dU_t + \frac{\cos(at-as)}{2}\right] \\
&= \frac{b^2}{2}\cos a(t-s)
\end{aligned}
$$

3. 考察如下四个模型平稳性。

对1阶自回归模型AR(1)

$$
\begin{gathered}
X_t = \varphi X_{t-1} + \varepsilon_t \\
E(X_t^2) = \varphi^2 E(X_{t-1}^2) + E(\varepsilon_t^2) + 2E(X_{t-1}\varepsilon_t)
\end{gathered}
$$

由于X_t仅与ε_t相关，$\therefore E(X_{t-1}\varepsilon_t) = 0$。如果该模型稳定，则有$E(X_t^2) = E(X_{t-1}^2)$，$\therefore$

$$
\gamma_0 = \sigma_X^2 = \frac{\sigma_\varepsilon^2}{1-\varphi^2}
$$

在平稳条件下，该方差是一非负的常数，从而$|\varphi| < 1$。

对AR(2)模型

$$
\begin{gathered}
X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t \\
E(X_t^2) = \varphi_1 E(X_t X_{t-1}) + \varphi_2 E(X_t X_{t-2}) + E(X_t \varepsilon_t)
\end{gathered}
$$

即

$$
\gamma_0 = \varphi_1\gamma_1 + \varphi_2\gamma_2 + E(X_t\varepsilon_t)
$$

又由于

$$
\begin{gathered}
E(X_t\varepsilon_t) = \varphi_1 E(X_{t-1}\varepsilon_t) + \varphi_2 E(X_{t-2}\varepsilon_t) + E(\varepsilon^2) = \sigma_\varepsilon^2 \\
\gamma_0 = \varphi_1\gamma_1 + \varphi_2\gamma_2 + \sigma_\varepsilon^2
\end{gathered}
$$

同理

$$
\begin{gathered}
\gamma_1 = \varphi_1\gamma_0 + \varphi_2\gamma_1 \\
\gamma_2 = \varphi_1\gamma_1 + \varphi_2\gamma_0
\end{gathered}
$$

$\therefore$

$$
\gamma_0 = \frac{(1-\varphi_2)\sigma_\varepsilon^2}{(1+\varphi_2)(1-\varphi_1-\varphi_2)(1+\varphi_1-\varphi_2)}
$$

由平稳性的定义，该方差必须是正数，即

$$
\varphi_1 + \varphi_2 < 1, \varphi_2 - \varphi_1 < 1, |\varphi_2| < 1
$$

（1）$x_t = 0.8x_{t-1} + \varepsilon_t$

因为0.8<1，所以该模型是平稳的。

（2）$x_t = -1.1x_{t-1} + \varepsilon_t$

因为|-1.1|>1，所以该模型不是平稳的。

（3）$x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$

因为$\varphi_1+\varphi_2=0.5<1,\varphi_2-\varphi_1=-1.5<1,|\varphi_2|=0.5<1$，所以该模型是平稳的。

（4）$x_t=x_{t-1}+0.5x_{t-2}+\varepsilon_t$

因为$\varphi_1+\varphi_2=1.5>1$，所以该模型不是平稳的。

4. 已知AR(2)模型为$(1-0.5B)(1-0.3B)X_t=\varepsilon_t,\varepsilon_t\sim WN(0,\sigma_\varepsilon^2)$，请回答以下问题：(1)期望$E(X_t)$；(2)方差$Var(X_t)$；(3)自相关函数$\rho_k,k=1,2,3$

$$(1-0.5B)(1-0.3B)X_t=(1-0.8B+0.15B^2)=\varepsilon_t$$

$\therefore\varphi_1=0.8,\varphi_2=-0.15$

$\therefore X_t=0.8X_{t-1}-0.15X_{t-2}+\varepsilon_t$

因为$\varphi_1+\varphi_2=0.65<1,\varphi_2-\varphi_1=-0.95<1,|\varphi_2|=0.15<1$，所以该模型是平稳的。

$E(X_t)=0.8E(X_{t-1})-0.15E(X_{t-2})$

因为该模型是平稳的，所以$E(X_t)=E(X_{t-1})=E(X_{t-2})$。

$E(X_t)=0$

$Var(X_t)=E(X_t^2)\gamma_0=\dfrac{(1-\varphi_2)\sigma_\varepsilon^2}{(1+\varphi_2)(1-\varphi_1-\varphi_2)(1+\varphi_1-\varphi_2)}=1.98\sigma_\varepsilon^2$

$\rho_1=\dfrac{\varphi_1}{1-\varphi_2}=0.70$

$\rho_2=\varphi_1\rho_1+\varphi_2=0.41$

$\rho_3=\varphi_1\rho_2+\varphi_2\rho_1=0.223$

5. 考虑美国从1947年第1季度到2011年第3季度的季度实际GNP，该数据存放于文件 g-GNPC96.txt中，数据已做了季节调整，以2005年GNP为基础进行了通胀调整，以10亿美元为单位（billions of chained 2005 dollars）。假设x_t代表GNP增长率的时间序列数据。

(a)通过ar，应用AIC准则，可以为x_t识别一个AR(4)模型。拟合这个模型，拟合的模型充分吗？为什么？

```
Coefficients:
         ar1     ar2      ar3      ar4  intercept
      0.3369  0.1513  -0.1010  -0.0887     0.0078
s.e.  0.0619  0.0652   0.0651   0.0619     0.0008

sigma^2 estimated as 8.368e-05:  log likelihood = 844.9,  aic = -1677.8
```

$x_t=0.0078+0.3369x_{t-1}+0.1513x_{t-2}-0.1010x_{t-3}-0.0887_{t-4}+a_t$

```
Box-Ljung test

data:  mm1$residuals
X-squared = 15.229, df = 20, p-value = 0.7632
```

残差序列服从自由度为20的卡方分布，$p-value = 0.7632 > 0.05$，说明模型很不充分。

(b)数据x_t的样本PACF识别的是AR(3)时间序列模型，拟合这个模型，拟合这个模型，拟合的模型充分吗？为什么？

```
Coefficients:
         ar1     ar2      ar3  intercept
      0.3485  0.1386  -0.1317     0.0078
s.e.  0.0616  0.0648   0.0617     0.0009

sigma^2 estimated as 8.436e-05:  log likelihood = 843.88,  aic = -1677.76
```

$x_t = 0.0078 + 0.3485x_{t-1} + 0.1386x_{t-2} - 0.1317x_{t-3} + a_t$

```
Box-Ljung test

data:  mm2$residuals
X-squared = 18.596, df = 20, p-value = 0.5482
```

残差序列服从自由度为20的卡方分布，$p-value = 0.5482 > 0.05$，说明模型很不充分。

6. 已知$X_t = a_t - 0.7a_{t-1} + 0.4a_{t-2}$，求$\rho_k$。

当$t > s$时，$E(a_t X_s) = 0$。

- 当$k = 0$时，$E(X_t X_{t-k}) = \gamma_k = E(X_k X_t) = \gamma_0$。

$$\begin{aligned} E(X_t X_t) &= E[(a_t - 0.7a_{t-1} + 0.4a_{t-2})(a_t - 0.7a_{t-1} + 0.4a_{t-2})] \\ &= E(a_t^2 + 0.49a_{t-1}^2 + 0.16a_{t-2}^2) \\ &= 1.65\sigma_a^2 \end{aligned}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = 1$$

- 当$k = 1$时，$E(X_t X_{t-1}) = \gamma_1$。

$$\begin{aligned} E(X_t X_{t-1}) &= E[(a_t - 0.7a_{t-1} + 0.4a_{t-2})X_{t-1}] \\ &= E[(-0.7a_{t-1} + 0.4a_{t-2})(a_{t-1} - 0.7a_{t-2} + 0.4a_{t-3})] \\ &= -0.7\sigma_a^2 - 0.28\sigma_a^2 \\ &= -0.98\sigma_a^2 \end{aligned}$$

$$\rho_k = \rho_1 = \frac{\gamma_1}{\gamma_0} = -\frac{0.98}{1.65} = 0.594$$

- 当$k = 2$时，$E(X_t X_{t-2}) = \gamma_2$。

$$\begin{aligned} E(X_t X_{t-2}) &= E[(a_t - 0.7a_{t-1} + 0.4a_{t-2})X_{t-2}] \\ &= E[0.4a_{t-2}(a_{t-2} - 0.7a_{t-3} + 0.4_{t-4})] \\ &= 0.4\sigma_a^2 \end{aligned}$$

$$\rho_k = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.4}{1.65} = 0.242$$

- 当$k > 2$时，$E(X_t)E_{t-k} = \gamma_k$。

$$\begin{aligned} E(X_t X_{t-k}) &= E[(a_t - 0.7a_{t-1} + 0.4a_{t-2})X_{t-k}] \\ &= 0 \end{aligned}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = 0$$

综上

$$\begin{cases} \rho_k = 1 & k = 1 \\ \rho_k = 0.594 & k = 1 \\ \rho_k = 0.242 & k = 2 \\ \rho_k = 0 & k > 2 \end{cases}$$