

Emerging Markets and Inflation

Lecture 5. Introduction to Inflation

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Lecture 5. Introduction to Inflation

- Lecture is dedicated to the phenomenon of inflation
- Define and explain economics
- Use it to explain and understand mathematics behind

1. Inflation Economics
2. Inflation-Linked Products and Markets
3. Bootstrapping Inflation Curve

Lecture 5. Introduction to Inflation

1. Inflation Economics

- Definition of Inflation
- Economic Theories of Inflation
- Inflation Measures

2. Inflation-Linked Products and Markets

3. Bootstrapping Inflation Curve

4. Seasonality Effect

- This will be a rare occasion of a deeper than usual economical study before diving into mathematics of the phenomenon
- Mathematical model of Inflation is fully founded on economics

Definition of Inflation

- Barron's *Economics* [[Wessels, 2006](#)]: "*A rise in general level of prices*"
- Excellent book by Comley "*Inflation Matters*" [[Comley, 2015](#)] goes further
- Conceptually inflation is often perceived synonymously to *cost of living*, but we will see how all currently used inflation measures are not very well fit for that

Definition of Inflation (continued)

- Dictionary: *a persistent increase in the level of consumer prices or a persistent decline in the purchasing power of money*
- The above seems to be different if we compare the *purchasing power of money* to *the general level of prices*. Or is it?
- Noticeable time evolution for these definitions!
- Compare to dictionary from 1970's: *an increase in the amount of fiduciary (paper or token) money issued beyond what is justified by the country's tangible resources*
- Most definitions converge to easiest and common description of inflation referring to a measuring Index of some kind, but more on that later

Economic Theories of Inflation

Let us follow already mentioned [[Comley, 2015](#)] and excellent book by Deacon, Derry and Mirfenderski, *Inflation-indexed Securities* [[Deacon, 2004](#)] for an overview of main economic theories of inflation

Money Supply Theory

- One of the oldest definitions connects it to Inflation
- Simple picture of the world with limited supply of goods → money inflow sparks prices increase through bidding
- Good example: Europe after discovery of Americas sustained sharp prices raise due to inflow of gold and silver into the world of still rather limited supply of goods and services

Money Supply Theory (continued)

- Copernicus was the first one connecting prices to money supply:
 P (Prices) \propto M (Money Supply)
- In 1848, John Stewart Mill's *Quantity Theory of Money* expands the above introducing *Equation of Exchange* in stable economy:

$$MV = PQ \quad (1)$$

For a given period of time, Money Supply (M) scaled by time it is being used (V) is equal to Average prices (P) scaled by total value of goods (Q) generated during that time

Money Supply Theory (continued)

- See some evidence of Eq. (1) in history on Figure 1:

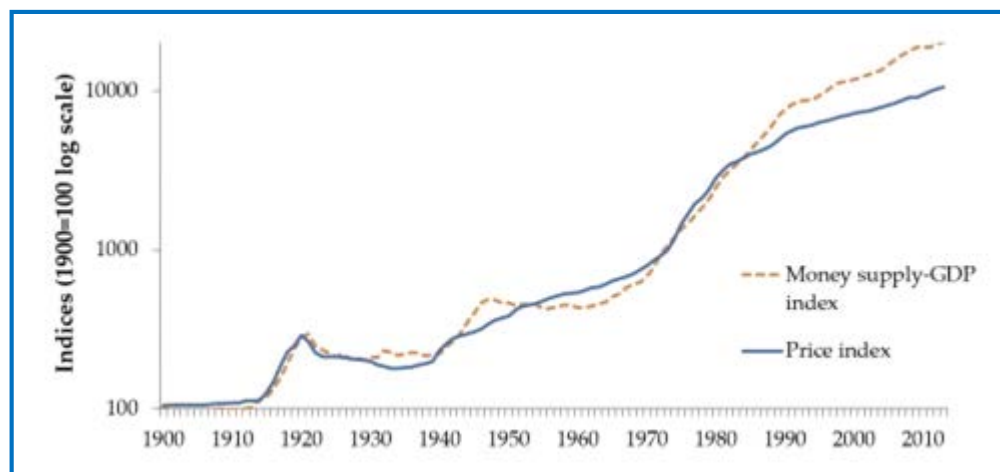


Figure 1. The link between UK prices and money supply (Comley, 2015)

- But there are issues: curves often diverge
- One explanation: money inflow going first into people's savings rather than economy directly. See last years on the chart
- Money supply diversion into assets, not retail prices

Keynesian Theory

- John Maynard Keynes proposes inflation theory as a combo of
 - Demand-pull factors
 - Inflation already built into the system
 - Cost-push factors:

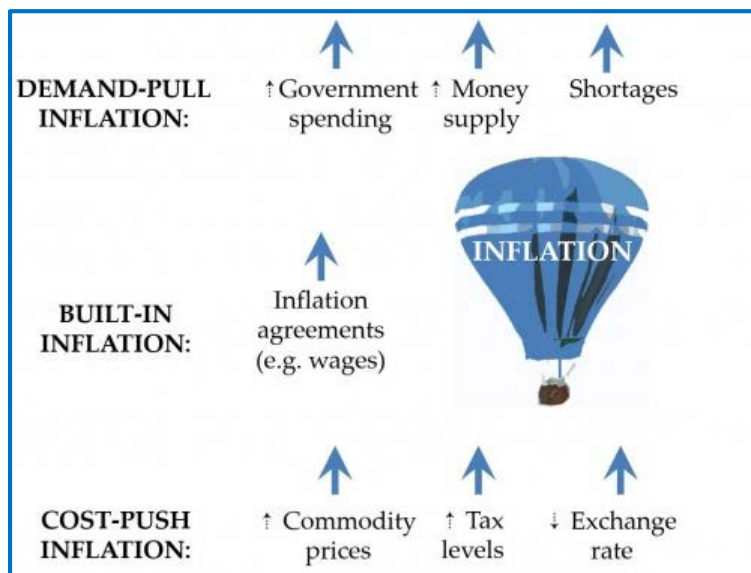


Figure 2. Keynesian view of inflation (Comley, 2015)

Malthusian Theory

- Rev. (Thomas) Robert Malthus proposed that *long term prices increase is mostly due to population increase*
- Suggests population's rate of growth outpaces commodities supply

Joined picture

- Three theories are not really competing
- Rather are addressing different time horizons with explanations

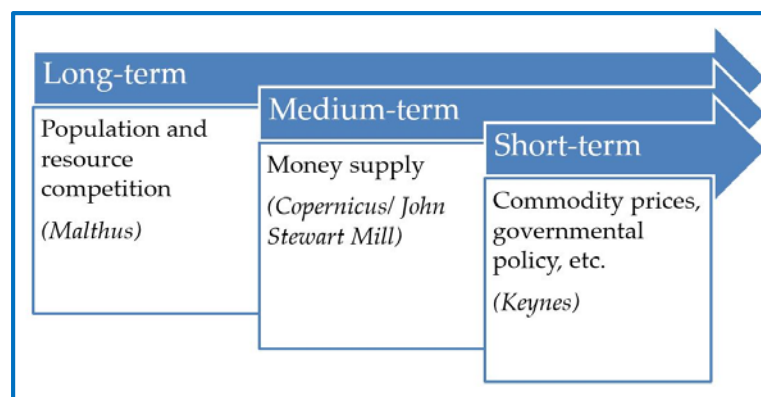


Figure 3. Theories of inflation and their impact (Comley, 2015)

- Define Inflation via level of prices, but how do we measure?
- *Consumer Price Indices (CPIs) are economic indicators constructed to measure the changes over time in the prices of consumer goods and services acquired, used or paid for by households*
[[J.P.Morgan, 2009](#)]
- Usually CPI refers to well-defined weighted basket of goods and services



Measures (continued)

- CPI is rather young measure and started as a *Harmonized Index of Consumer Prices* (HICP)
- HICP was created to align inflation measures in pre-union Europe
- Countries still mostly stick to their original measures using HICP for comparison only
- Few other indices to mention:
 - ❑ RPI - Retail Prices Index
 - ❑ PPI - Producer Prices Index
 - ❑ GDP Deflator for actual calculated GDP

Measures Calculations

- Most use *Laspeyres approach*: compare the cost of fixed basket of goods in two periods based upon the quantities set at the start of period one

$$CPI_L(t_o, t) = \frac{\sum_i P_i(t) \cdot Q_i(t_o)}{\sum_i P_i(t_o) \cdot Q_i(t_o)}$$

Here $P_i(t)$ - price of item i at time t

$Q_i(t)$ is quantity consumed of item i at time 0 (base period)

- Alternative is *Paasche* that fixes the basket in the second period instead:

$$CPI_P(t_o, t) = \frac{\sum_i P_i(t) \cdot Q_i(t)}{\sum_i P_i(t_o) \cdot Q_i(t)}$$

Measures Calculations (continued)

- Using same basket at inception or at maturity is not optimal!
- Law of demand: *item's price increase leads to decrease in demand!*
- Thus if consumer substitutes high weighted item with increased price by a cheaper one with lower weight →
 - ❑ Laspeyres overestimates the inflation
 - ❑ Paasche underestimates it
 - ❑ *Fisher* index was designed to compensate for that:

$$CPI_F = \sqrt{CPI_L \cdot CPI_P}$$

Inflation Measurement Issues

■ Index Complications:

- ☐ Multitude of indices could be confusing
- ☐ Prices and quantities contributing to Index must be reliably and continuously observable
- ☐ Substitution proxies are often used but their choices are tricky

■ Chaining

- ☐ Long time monitoring
- ☐ Rebalance approaches
- ☐ Substitutions and discontinuity

Inflation Measurement Issues (continued)

■ New items addition:

- ❑ Development of new products and services is hard to account for
- ❑ Changes in quality is hard to incorporate
- ❑ Theory of *Hedonic Pricing*: mathematical model for prices of items depending on their features. Backfires estimating price decline based on features while street prices goes up

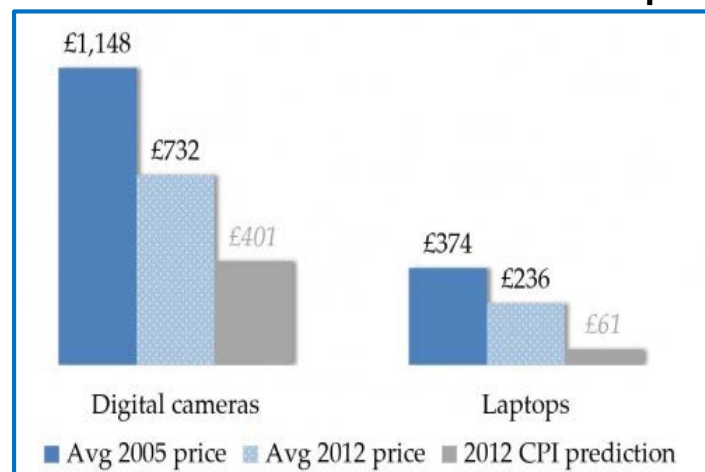


Figure 5. Average prices in "Which?" 2005 vs 2012 compared with hedonic CPI prediction (Comley, 2015)

Lecture 5. Introduction to Inflation

1. Inflation Economics
2. Inflation-Linked Products and Markets
 - Securities
 - Breakeven Inflation
 - Futures and Zero Coupon Swaps
 - Term-on-term Swaps
 - Inflation-linked derivatives market
3. Bootstrapping Inflation Curve

Inflation-Linked Products and Markets

- Inflation present throughout financial markets whenever future cash payments are involved
- Thus only natural for the market to develop products and instruments helping participants to mitigate the risk

Securities

- Naturally Inflation-linked with each cash flow linked to inflation
- Protecting investors against inflation
- First in 1742 when State of Massachusetts issues bills of public credit linked to cost of silver on London exchange
- But price of silver grows faster than general level of prices → so law was passed and inflation-linked weighted commodities basket was introduced in 1747

Securities (continued)

- Inflation-indexed or *Capital Indexed Bond* (CIB) with notional and coupon adjusted by inflation:

$$CF(t_i) = \begin{cases} (P + C_i) \cdot InflAdj(t_i) & , t_i = T \\ C_i \cdot InflAdj(t_i) & , t_i \neq T \end{cases}$$
$$InflAdj(t_i) \stackrel{def}{=} \frac{CPI(t_i - lag)}{CPI(t_o - lag)} \quad (2)$$

T is bond's Maturity

P is Notional or Principal of the bond

C is coupon

lag is period it takes to collect and publish inflation data. Usually is 1-3mth.
Lag distorts protection quality, especially just before bond's maturity leaving principal repayment without protection

Securities (continued)

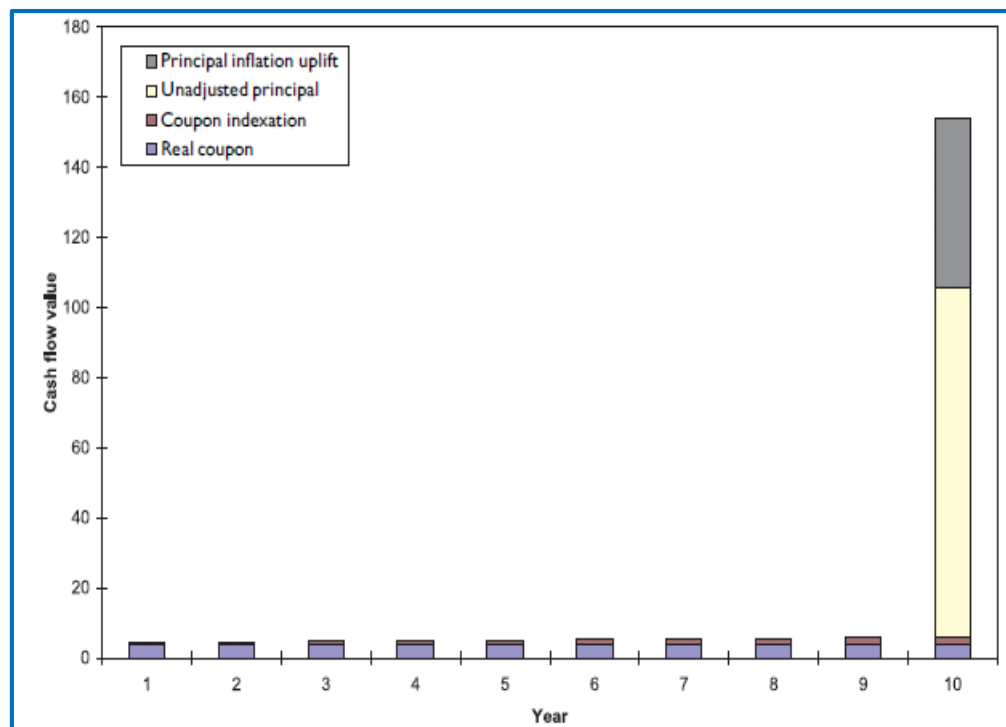


Figure 6: Capital Indexed Bond (CIB) cash flow structure (Deacon, 2004)

Securities (continued)

- Let us concentrate on CIB and look at yield to maturity in details:
 - Recall “regular” bond’s yield definition. We call it **Nominal** here and express in exponential terms:

$$PV_N(t, T) = P \cdot e^{-y_N(T-t)} + \sum_{t_i=t_1}^T C_i e^{-y_N(t_i-t)}$$

- Use similar expression for CIB to define **Real** yield-to-maturity

$$\begin{aligned} PV_{CIB}(t, T) &= P \cdot InflAdj(T) \cdot e^{-y_{CIB}(T-t)} + \sum_{t_i=t_1}^T C_i \cdot InflAdj(t_i) \cdot e^{-y_{CIB}(t_i-t)} \\ &\equiv P \cdot e^{-y_R(T-t)} + \sum_{t_i=t_1}^T C_i \cdot e^{-y_R(t_i-t)} \end{aligned}$$

Securities (continued)

HW1: Derive relationship between *Nominal* and *Real* yield of a simple semi-annual B/360 bond. Discuss definition of a break-even inflation based on Inflation Adjustment differential

- Continue via Eq. (2) separating Nominal and CIB parts:

$$CF_N(t_i) \stackrel{\text{def}}{=} \begin{cases} (P + C_i) & , t_i = T \\ C_i, & , t_i \neq T \end{cases}$$
$$CF_{CIB}(t_i) = CF_N(t_i) \cdot \text{InflAdj}(t_i)$$

- Generalize for a bond's Dirty Price:

$$\begin{aligned} DP_N(t) &= \sum_i CF_N(t_i) \cdot Z_N(t, t_i) \\ DP_{CIB}(t) &= \sum_i CF_{CIB}(t_i) \cdot Z_N(t, t_i) \\ &\equiv \sum_i CF_N(t_i) \cdot \text{InflAdj}(t_i) \cdot Z_N(t, t_i) \end{aligned} \quad (3)$$

Securities (continued)

- Define now *Real* discounting and *Real* yield via Eq. (3):

$$\begin{aligned} DP_{CIB}(t) &\stackrel{def}{=} \sum_i CF_N(t_i) \cdot Z_R(t, t_i) \\ Z_R(t, t_i) &\stackrel{def}{=} InflAdj(t_i) \cdot Z_N(t, t_i) \end{aligned} \quad (4)$$

Break-Even Inflation: level of inflation required for an inflation-linked bond (CIB) to be of equal value to its *Nominal* equivalent

→ A bit theoretical definition since it assumes existence of such bonds of course

Break-Even Inflation (continued)

- Decompose *nominal* and *real* yields more carefully:

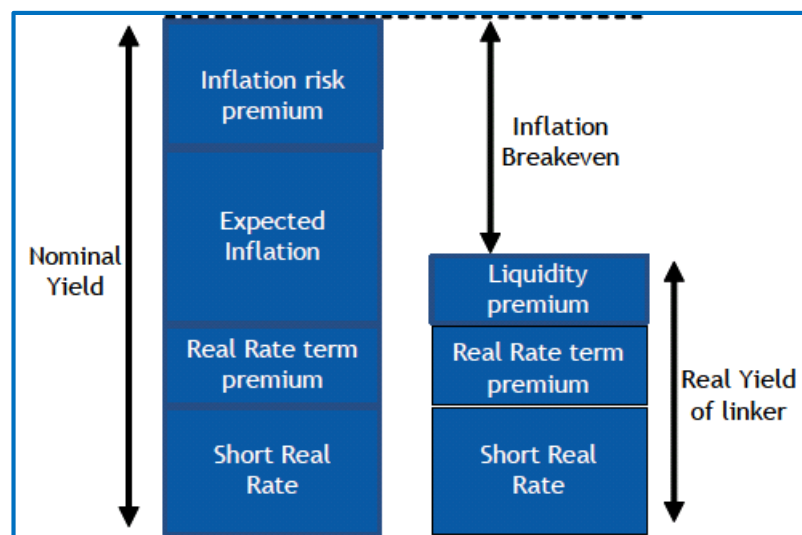


Figure 7: What is Breakeven Inflation? (J.P.Morgan, Inflation-Linked Markets, 2009)

- *Inflation risk premium*: risk of increased inflation in the long end suggesting breakeven inflation overstating the *Expected Inflation*
- *Liquidity premium* accounts for a lesser liquidity of inflation-linked bonds compared to their nominal counterparts

Breakeven Inflation

- The Fisher Equation via re-writing Eq. (4) with annualized yields and newly defined *breakeven inflation* bi

$$\begin{aligned} Z_B(t, T) &= (1 + y)^{-\tau} \\ (1 + y_R)^{-\tau} &= (1 + bi)^{\tau} \cdot (1 + y_N)^{-\tau} \\ (1 + y_N) &= (1 + y_R) \cdot (1 + bi) \end{aligned} \tag{5}$$

- More details on differences between expected and breakeven inflation using [[Kerkhof, 2005](#)]:

- ❑ Compounding effect due to stochastic nature is underestimating inflation if ignored. Effect is increasing in high vols and rates!

$$\mathbb{E}[(1 + bi(t, T))^{\tau}] \geq (1 + \mathbb{E}[bi(t, T)])^{\tau}$$

- ❑ Inflation Convexity in classical sense of second order derivative
- ❑ Inflation Risk Premium due to certainty of the offered protection

Breakeven Inflation (continued)

- Add all above mentioned components to the *Fisher Equation*:

$$(1 + y_N) = (1 + y_R) \cdot (1 + E[bi]) \cdot (1 + Cxty) \cdot (1 + RiskPremium)$$

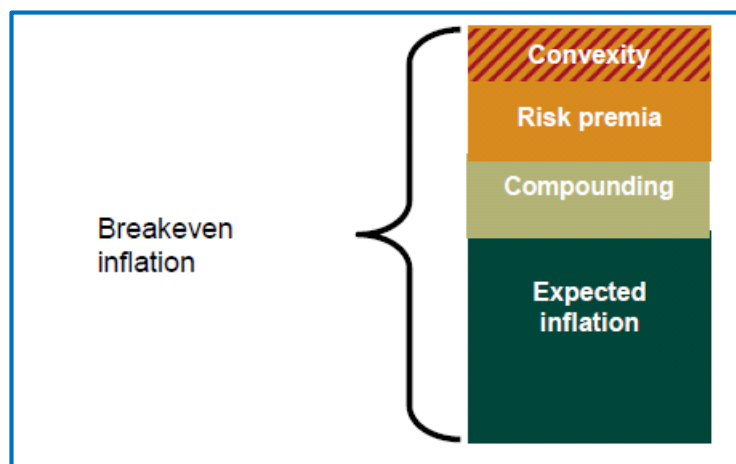


Figure 8: Components of breakeven inflation ([Kerkhof, 2005](#))

- Similar to our studies of IR instruments in Lecture 3, we start here by looking at instruments that are building inflation trading:

Futures and Zero Coupon Swaps

- Inflation Futures indexing US CPI are trading on CME since 2004:
 - Speculate on the outright level of inflation
 - Hedge future levels of inflation
 - Mitigate counterparty risk
 - Resemble Eurodollar futures:

$$\Phi = 1 - \frac{1}{\delta} \left(\frac{CPI_T}{CPI_{T-\delta}} - 1 \right)$$

with δ for the 3mth time interval

Zero Coupon Swaps

- One of the most liquidly traded inflation derivative
- Single cash flow exchange at maturity:
 - ❑ Swap payer receives annualized Fixed Rate coupon ZC at maturity
 - ❑ Swap Receiver gets the accrued inflation:

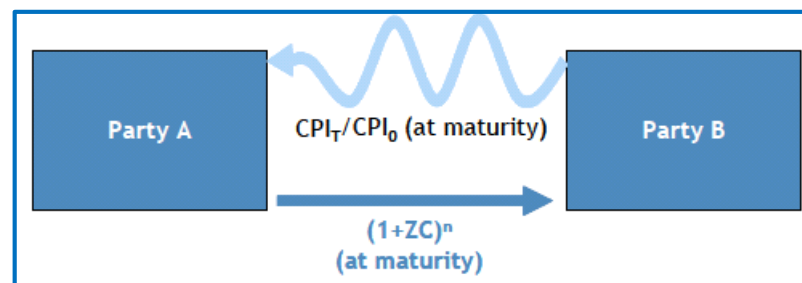


Figure 9: ZC Inflation Swap cashflows (J.P.Morgan, Inflation-Linked Markets, 2009)

Zero Coupon Swaps (continued)

- Accurate expression for PV of a Floating leg in Nominal risk-neutral economy:

$$PV_{ZCIS}(t, 0, T) = \mathbf{E}_n \left[\frac{CPI_T}{CPI_o} \cdot \exp \left(- \int_t^T n(x) dx \right) \middle| F_t \right] \quad (6)$$

With $n(x)$ for stochastic nominal interest rates,

time of inception 0 and maturity T , with t as running time $0 \leq t \leq T$

Zero Coupon Swaps (continued)

- Transform Eq.(6) now using Eq.(4) and definition of *Real* rates

$$\begin{aligned} \mathbf{E}_n \left[CPI_T \cdot \exp \left(- \int_t^T n(x) dx \right) \middle| F_t \right] &\equiv CPI_t \cdot Z_R(t, T) \\ PV_{ZCIS}(t, 0, T) &= \frac{CPI_t}{CPI_o} \cdot Z_R(t, T) \end{aligned} \quad (7)$$

- Note model-independent expression for price of zero coupon inflation swap

Term-on-term Inflation-linked Swaps

- Generalize Zero coupon to multi-coupon:

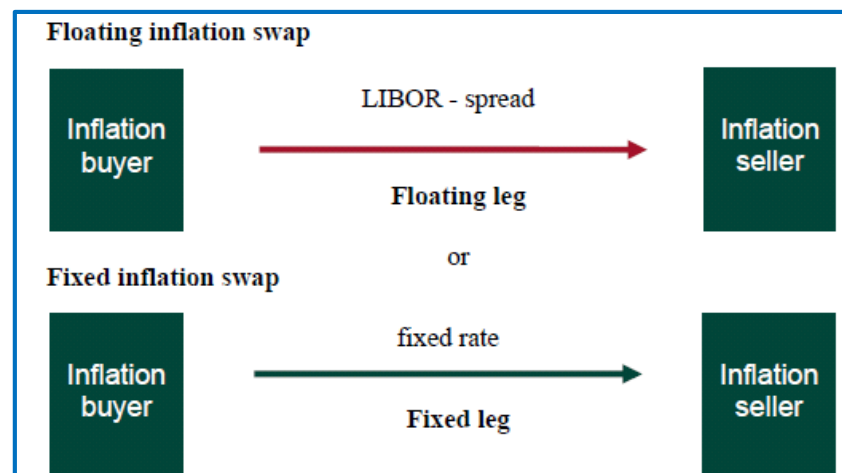


Figure 10: Fixed and Floating Inflation swaps (Kerkhof, 2005)

- Flavor when inflation adjustment is applied from inception resembles CIB bond and is not very interesting now
- So, we do not look at swaps with base date inflation adjustment
- Look at structures that reset inflation base every coupon instead

Term-on-term Swaps (continued)

■ Year-on-year variety:

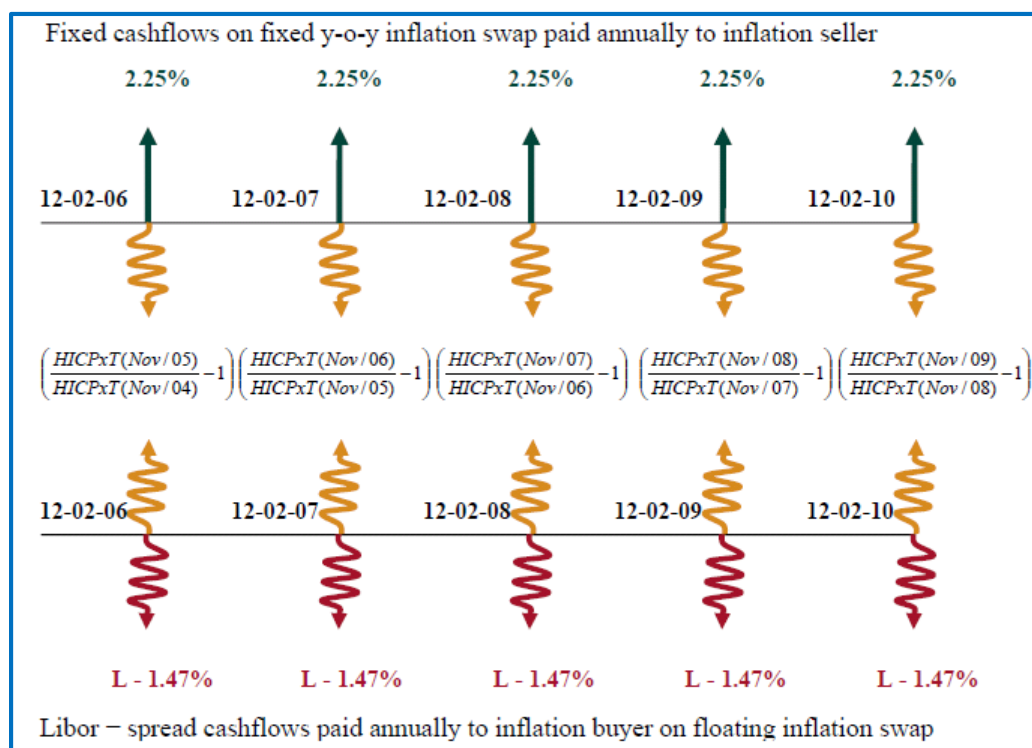


Figure 11: Fixed and Floating Inflation swaps (Kerkhof, 2005)

Term-on-term Swaps (continued)

- Let's repeat the valuation analysis as in Eq. (6) and (7) looking at one cash flow only:

$$PV_{YoYIS}(t, 0, T_{j-1}, T_j) = \mathbf{E}_n \left[\left(\frac{CPI_{T_j}}{CPI_{T_{j-1}}} - 1 \right) \cdot \exp \left(- \int_t^{T_j} n(x) dx \right) \middle| F_t \right] \quad (8)$$

- More complex than before, but still can try to simplify re-arranging terms:

$$\begin{aligned} PV_{YoYIS}(t, 0, T_{j-1}, T_j) &= \mathbf{E}_n \left[e^{-\int_t^{T_{j-1}} n(x) dx} \cdot \mathbf{E}_n \left\{ e^{-\int_{T_{j-1}}^{T_j} n(x) dx} \cdot \left(\frac{CPI_{T_j}}{CPI_{T_{j-1}}} - 1 \right) \middle| F_{T_{j-1}} \right\} \middle| F_t \right] \\ &= \mathbf{E}_n \left[\{ Z_R(T_{j-1}, T_j) - Z_N(T_{j-1}, T_j) \} \cdot \exp \left(- \int_t^{T_{j-1}} n(x) dx \right) \middle| F_t \right] \end{aligned}$$

- Not much simpler → need a short rate model to proceed. Will come back to that later

Other Inflation Linear Products

- Leave many out of our scope, but still mention for an overview:

- **Inflation Forwards:**

- ☐ More common in Emerging Markets and Latin America
- ☐ Allows extracting medium and longer term inflation
- ☐ Cover in more details next week

- **Inflation Asset Swap:**

- ☐ Combination of Inflation-linked bond and inflation-linked swap
- ☐ Isolates credit and liquidity component

Other Inflation Volatility Products

■ Inflation Cap and Floor:

- ☐ Most common inflation vol product
- ☐ Stems from inflation-linked bonds having embedded Floor at final redemption: long term and way out of the money, but still...

■ Inflation Swaption:

- ☐ Option to enter into an inflation-linked swap

■ LPI Swap:

- ☐ *Limited Price Index* swap flooring inflation adjustment at pre-agreed level

■ Inflation Spread Option:

- ☐ Pays spread between 2 inflation indices
- ☐ Could be single Currency (Index) or multi

Inflation-Linked Derivatives Markets

- Who are its participants and what are their interests?
- Started to appear in 1990s in UK
- Is formed by natural Long's and Short's
- Supply (Issuers or Inflation Payers):
 - ❑ Linked to entities receiving inflation connected cash flows
 - ❑ **Real Estate** contracts (rents) structured in Real terms → issue Inflation-protected debt to hedge
 - ❑ **Sovereigns or Governments**: receive tax flows that are usually inflation adjusted, so also issue inflation-linked debt to diversify inflation-linked revenues
 - ❑ **Utility companies** own big portion of corporate issuance of the inflation-linked debt

Inflation-Linked Derivatives Markets (continued)

- Demand (Investors or Inflation Receivers):
 - Linked to entities making inflation connected payments
 - **Pension Funds**: exposed to future inflation-adjusted payouts, so invest in inflation-protected debt to hedge. Most of pension schemes are inflation connected. Examples are Cost of Living in US or LPI's in UK
 - **Insurance companies**: similar are involved in selling inflation dedicated protection

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1. Inflation Economics
2. Inflation-Linked Products and Markets
3. Bootstrapping Inflation Curve
 - Breakeven Inflation Curve
 - Inflation Curve via mean-Reverting Process
 - Forward Inflation via mean-Reverting Process

Bootstrapping Inflation Curve

- Same as everywhere else and especially Interest Rates derivatives, pricing Inflation-Linked products requires a curve
- Same as with *Nominal* curves and products, curve must be bootstrapped from market instruments
- Will allow us to extract future inflation index values via forward inflation instantaneous rates

Breakeven Inflation Curve

- Use directly observable breakeven inflation rates:

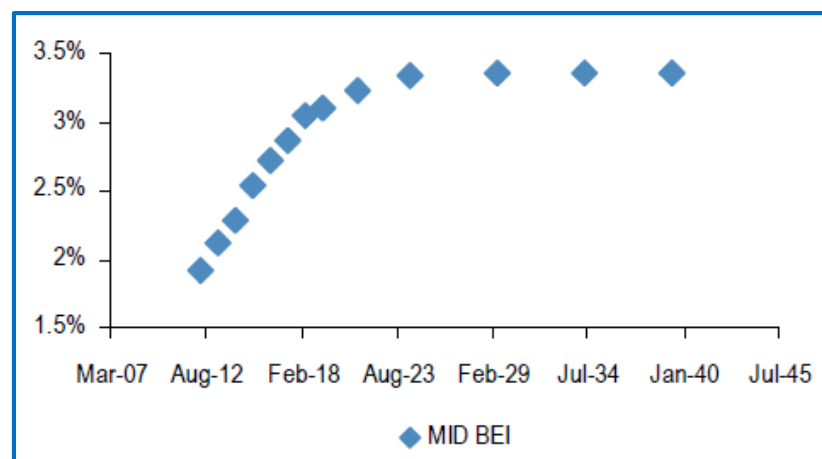


Figure 12: UK RPI Zero Coupon Curve (J.P.Morgan, Inflation-Linked Markets, 2009)

- Only available from 2 years and out
- More info needed to shape the front end of the curve
- Use economist's predictions and other subjective measures

Inflation Curve via Mean-Reverting process

- Bootstrap inflation curve from inflation-adjusted bonds
- Assume mean-reverting process for *Real* rates to use later in forward inflation extraction
- Approximate via exponential rates in Eq.(7):

$$Z_R(0, T) = \mathbf{E} \left[\exp \left\{ - \int_0^T r(s) ds \right\} \right] \quad (9)$$

Inflation Curve via Mean-Reverting process (continued)

- Select mean-reverting O-U process for *real* rates:

$$dr(t) = \lambda(t)[\Theta(t) - r(t)]dt + \sigma(t)dW_t$$

With $r(t)$ being short instantaneous real rates,

Mean reversion rate $\lambda(t)$ and

Mean reversion level $\Theta(t)$

- Simplifying it for constants collapsing once again to Vasicek:

$$dr(t) = \lambda_r \cdot [\theta_r - r(t)]dt + \sigma_r \cdot dW_t$$

Inflation Curve via Mean-Reverting process (continued)

- And integrate as usual to arrive at standard answer

$$r(t) = \theta + e^{-\lambda_r t} (r_o - \theta_r) + \sigma_r \int_0^t e^{-\lambda_r (t-s)} dW_s$$

- Next is to integrate for the zero coupon bond expression
- Can say that $\int_0^t r(s) ds$ is normally distributed with drift μ and variance var

$$\begin{aligned} \mu &= \int_0^t [\theta_r + e^{-\lambda_r s} (r_o - \theta_r)] ds = r_o h(t, \lambda_r) + \lambda_r t - \lambda_r h(t, \lambda_r) \\ h(t, \lambda) &\stackrel{def}{=} \frac{1 - e^{-\lambda t}}{\lambda} \\ var &= \left[\sigma_r \int_0^t \int_0^t e^{-\lambda_r (s-u)} ds du dW_u \right]^2 = \frac{\sigma_r^2}{\lambda_r^2} [t - 2h(t, \lambda_r) + h(t, 2\lambda_r)] \end{aligned} \tag{10}$$

Inflation Curve via Mean-Reverting process (continued)

- Then price of Inflation-indexed Zero coupon bond from Eq.(9) simply is:

$$Z_R(0, T) = \exp\left\{-\mu + \frac{var}{2}\right\} \quad (11)$$

- Now we can decompose N coupon-paying bonds into set of zero coupons in order to apply Eq.(11)
- But need to upgrade to a piece-wise constant mean reversion

$$\Theta(t) = \begin{cases} \theta_1, & 0 < t \leq T_1 \\ \theta_2, & T_1 < t \leq T_2 \\ \dots & \\ \theta_N, & T_{N-1} < t \end{cases}$$

Inflation Curve via Mean-Reverting process (continued)

- This will lead us to a component-based expression for the drift from Eq.(10)

$$\mu = r_o h(t, \lambda_r) + \theta_N \left[\frac{1}{\lambda_r} \left(e^{\lambda_r(t_{N-1}-t)} - 1 \right) + t - t_{N-1} \right] + \sum_{i=1}^{N-1} \theta_i \left[\frac{1}{\lambda_r} \left(e^{\lambda_r(t_{i-1}-t)} - e^{\lambda_r(t_i-t)} \right) + t_i - t_{i-1} \right]$$

- Starting in order of increasing maturity
- With collection of market or historically estimated parameters
- Making assumption for the final redemption floor value in bonds
- Dirty Price of the i -th bond with M_i cash flows could be written:

$$DP_i = \frac{CPI_{Spot}}{CPI_{Base}} \left\{ \sum_{j=1}^{M_i} Z_r(0, T_{ij}) \cdot CF_{ij} + Z_r(0, T_{iM_i}) \right\} + Z_r(0, T_{iM_i}) Floor$$

Forward Inflation via Mean-Reverting process

- Let us now go back to term-on-term inflation swap and use our mean-reverting process and bootstrapped curve
- Rewrite Eq.(8) for forward inflation term with time period δ separating deterministic part from Convexity Adjustment under the T -forward measure:

$$\begin{aligned} FwdInfl_{\delta}(t, T) &= \mathbf{E}^T \left[\left(\frac{CPI_T}{CPI_{T-\delta}} - 1 \right) \middle| F_t \right] \\ &= \frac{CPI_T}{CPI_{T-\delta}} \cdot e^{ConvAdj} - 1 \end{aligned} \tag{12}$$

- We also carefully separate Forward and Spot CPI similar to forward and short rate and write Eq.(3) in these new notations:

$$\begin{aligned} FCPI(t, T) &= \mathbf{E}^T [CPI(T) | F_t] \\ &= CPI(t) \frac{Z_R(t, T)}{Z_N(t, T)} \end{aligned} \tag{13}$$

Forward Inflation via Mean-Reverting process (continued)

- Assume BGM-like dynamics for forward LIBOR and CPI:

$$\begin{aligned}\frac{dFL(t, T - \delta, T)}{FL(t, T_{i-1}, T_i)} &= \sigma_L dW^T \\ \frac{dFCPI(t, T)}{FCPI(t, T - \sigma)} &= \sigma_T dW^T\end{aligned}$$

- On the other hand expanding on Eq. (12) we can write:

$$\frac{dFCPI(t, T)}{FCPI(t, T)} = \frac{dCPI(t)}{CPI(t)} + \frac{dZ_r(t, T)}{Z_r(t, T)} + \frac{dZ_n(t, T)}{Z_n(t, T)}$$

- Or simply for the vols part:

$$(\sigma_T)^2 = (\sigma_c)^2 + (\sigma_r)^2 + (\sigma_n)^2 + 2\rho_{r,c}\sigma_r\sigma_c - 2\rho_{n,c}\sigma_n\sigma_c - 2\rho_{n,r}\sigma_n\sigma_r$$

Using σ_c for vol of the spot CPI and collection of correlations ρ

Forward Inflation via Mean-Reverting process (continued)

- Then explicitly recognizing CPI processes at inception and maturity of the period as separate and correlated dynamics and
- Defining $T_i = T - \delta$ and $T_f = T$ we can write connection between Brownian's

$$dW^{T_i} = dW^{T_f} - \frac{\delta L}{1 + \delta L} \sigma_L dt$$

- And then naturally

$$\frac{dFCPI(t, T_f)}{FCPI(t, T_f)} \propto \sigma_f dW^{T_f}$$

$$\frac{dFCPI(t, T_i)}{FCPI(t, T_i)} \propto \sigma_i dW^{T_i} = \sigma_i dW^{T_f} - \gamma dt; \quad \gamma \stackrel{def}{=} \frac{\delta L}{1 + \delta L} \rho_{L, T_i} \sigma_L \sigma_i$$

Where L is forward rate of our time period and ρ_{L, T_i} is correlation between $FCPI(t, T_i)$ and L

Forward Inflation via Mean-Reverting process (continued)

- Then for ratio of CPI indices at start and end of the period

$$\frac{FCPI(t, T_f)}{FCPI(s, T_i)} = \frac{FCPI(0, T_f) \cdot \exp\left\{-\frac{1}{2}\sigma_f^2 t + \sigma_f W^{T_f}(t)\right\}}{FCPI(0, T_i) \cdot \exp\left\{-\frac{1}{2}\sigma_i^2 s + \sigma_i W^{T_i}(s) - \int_0^s \gamma dx\right\}} \quad (14)$$

- Re-write the above with some substitutions:

$$\frac{FCPI(T_f, T_f)}{FCPI(T_i, T_i)} = \frac{FCPI(0, T_f) \cdot \exp\left\{-\frac{1}{2}\sigma_f^2 T_f + \sigma_f W^{T_f}(T_f)\right\}}{FCPI(0, T_i) \cdot \exp\left\{-\frac{1}{2}\sigma_i^2 T_i + \sigma_i W^{T_i}(T_i) - \int_0^{T_i} \gamma dx\right\}}$$

Forward Inflation via Mean-Reverting process (continued)

- And finally for the inflation term combining all of the above and leaving some tedious calculations out
- Convexity is derived starting from

$$FwdInfl_{\delta}(t, T) = \mathbf{E}^T \left[\frac{FCPI(T_f, T_f)}{FCPI(T_i, T_i)} \middle| F_t \right] - 1$$

- And arriving at

$$ConvAdj = \left[\rho_{L, T_i} \sigma_L \sigma_i \frac{\delta L}{1 + \delta L} + \sigma_i (\sigma_i - \sigma_f \rho_{T_f, T_i}) \right] (T_i - t)$$

Where we introduce ρ_{T_f, T_i} as correlation between $FCPI(t, T_f)$ and $FCPI(t, T_i)$

HW2: Estimate levels of this Convexity Adjustment for ranges of vols, correlations and maturities. Suggest validity limits where convexity could be safely ignored

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