Emerging Markets and Inflation

Lecture 4. Brazil

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Yury Blyakhman

Agenda for Today

Lecture 4. Brazil

- Lecture fully dedicated to Brazil markets
- Variety of special onshore and offshore products
- Special features of FX and Rates products and connections
- FX Convertibility

1. **Benchmarks and Day Count Conventions**
2. **Foreign Exchange (FX) products**
3. **Linear Interest Rate products**
4. **Interest Rate Options**

Agenda for Today

Lecture 4. Brazil

- Most special and peculiar Emerging Markets country
- Financial markets, conventions, products
- Special attention due to sheer size of the market
- Constrains due to capital controls and taxation

1. **Benchmarks and Day Count Conventions**
   - SELIC Rate
   - CDI Rate
   - FX PTAX
2. Foreign Exchange (FX) products
3. Linear Interest Rate products
4. Interest Rate Options

Benchmarks and DCC. SELIC

- **Introduce benchmarks and Day Count Conventions**
- **Mostly follow [JPMorgan, 2004] and use throughout the lecture**

SELIC Rate:

- **Central bank controlled repo rate**
- **Overnight borrowing and lending benchmark rate to banks**
- **Used for Govie's FRN overnight accrual**
- **Traded daily on Business / 252 convention:**

$$\beta_{SELIC}(t,T)=\prod_{i=1}^{N}\left(1+SELIC_i\right)^{1/252}$$

SELIC Rate (continue):

$$\beta_{SELIC}(t,T)=\prod_{i=1}^{N}\left(1+SELIC_i\right)^{1/252}$$

- β is a SELIC-based bank account

- *N* is number of Brazil business days between *t* and *T*

- Note Holidays' contribution to any financial calculation!
- Addition or removal of public holidays:
  - Creates MTM and risk impact on any financial derivative
  - Propagates into fundamental Capital requirements!

CDI Rate:

$$\beta(t,T) = \prod_{i=1}^{N} \left(1 + CDI_i\right)^{1/252} \qquad (1)$$

- No subscript for a main bank account β
- Uncollateralized interbank overnight rate, priced off SELIC
- Interestingly and surprisingly, historically trades below SELIC:
  - Reflecting tight interbank credit constraints (tighter than CB!)
  - Serving as main interbank benchmark rate

PTAX Rate:

- FX Fixing rate
- Weighted Average of intraday FX commercial transactions reported
- Main reference FX rate for settling:
  - Onshore and Offshore OTC trading
  - Exchange (BM&F) based trading
- ASK side only is used

Agenda for Today. FX

Lecture 4. Brazil

Lecture fully dedicated to Brazil markets:

1. Benchmarks and Day Count Conventions
2. **Foreign Exchange (FX) products**
   - ❑ **Onshore and Offshore FX**
   - ❑ **FX Futures**
   - ❑ **FX Convertibility**
3. Linear Interest Rate products
4. Interest Rate Options

FX Products. Onshore and Offshore

- **Follow [JPMorgan, 2011]**
- **Brazilian Real (BRL) is non-deliverable**
- **USD/BRL spot exchange rate Is traded in domestic OTC, but is registered on local BM&F exchange**
- **Brazil Central Bank (BCB) buys and sells USD providing liquidity and controlling volatility**

Onshore and Offshore FX:

- **Cash settled FX derivatives traded onshore and offshore**
- **Note switch between Asset and Denominated in cases of Onshore vs. Offshore markets: settling in different currencies!**

Onshore and Offshore FX (continued):

- **Onshore, Quantity in Asset=BRL**

$$FV^{BRL}(T) = N^{BRL} \cdot \left[1 - \frac{K}{F(T)}\right]$$

- **Onshore, Quantity in Denominated=USD**

$$FV^{BRL}(T) = N^{USD} \cdot [F(T) - K] \qquad (2)$$

- **Offshore, Quantity in Asset=USD**

$$FV^{USD}(T) = N^{USD} \cdot \left[1 - \frac{K}{F(T)}\right] \qquad (3)$$

- **Keep definition of FX Spot rate of BRL per USD**

FX Future:

- Onshore market
- FX Spot rate as underlying
- Settles on PTAX
- Virtually risk-free with daily margining
- Long dated trading + high rates + high vols → sizable Convexity Adjustment compared to FX Forwards as discussed in Lecture 2
- One of special Brazil features: locally traded USD rate! Different from US LIBOR
- USD local trading is still cash-settled in BRL of course

FX Future (continued):

$$\Phi_T = S \cdot \frac{\left[1 + r_{CDI}(t,T)\right]^{BD/252}}{\left[1 + r_{USD}(t,T) \cdot \frac{(T-t)}{360}\right] \times \left[1 + CDI(T-1,T)\right]} \quad (4)$$

- *S* is FX Spot
- *BD* is number of business days between *t* and *T*
- r_{USD} is onshore USD rate in Act/360 conventions
- r_{CDI} is effective term CDI rate defined via Eq.[1] as:

$$\left[1 + r_{CDI}(t_1 = t, t_N = T)\right]^{BD/252} \stackrel{def}{=} \prod_{i=1}^{N} (1 + CDI_i)^{1/252}$$

- Overnight correction term *[1+CDI(T-1,T)]*. Will discuss later in more details. Comes from different settlement rules of BRL and USD

FX Future (continued):

- **Futures standardization in Exchange trading +**
- **Better risk protection +**
- **Wider access of external investors to FX Future than to local FX Spot**
- **As a result, a phenomenon of front FX Future being much more liquid and attractive than FX Spot!**
- **Massive position to front FX Future *vs*. FX Spot creates a differential position between the two ➔**
- **Phenomenon of Casado**

FX Convertibility:

- **Onshore and Offshore FX markets trading NDFs. Differences?**
- **Onshore used by Offshore players for hedging [Lipscomb, 2005]**
- **Offshore players trying to mange their risk onshore**
- **Offshore may have perception of lower risk:**
  - **Wider range of counterparties**
  - **Generally stricter (or different) regulations**
  - **Expectation of lower transaction cost than onshore**
- **Result: diverging On- and Off-shore prices leading to *Convertibility***

FX Convertibility (continued):

- **Taken to an extreme with Offshore market being driven by Onshore**
- **Convertibility evolves from a phenomenon to Tradable Asset**
- **Write Forward FX expression for Onshore and Offshore:**

$$
\boxed{
\begin{aligned}
F^{OnShr}(T) &= S^{OnShr}\cdot\frac{Z_{\$}^{OnShr}(t,T)}{Z_{BRL}^{OnShr}(t,T)} \\
F^{OffShr}(T) &= S^{OffShr}\cdot\frac{Z_{\$}^{OffShr}(t,T)}{Z_{BRL}^{OffShr}(t,T)}
\end{aligned}
}
\qquad (5)
$$

- $S^{X\text{-}Shr}$ and $F^{X\text{-}Shr}$ are FX Spot and Forward respectively

- $Z^{X\text{-}Shr}$ is CCY Discount Factor

FX Convertibility (continued):

- Combining Eq.[4] and [5] we can write for each of Discount Facotrs:

$$
\begin{aligned}
Z_{\$}^{OffShr} &= \exp\left[-r_{US\ LIBOR}(t,T)\cdot(T-t)\right] \\
Z_{\$}^{OnShr} &= \frac{1}{1+r_{\$}^{OnShr}(t,T)\cdot\dfrac{(T-t)}{360}} \\
Z_{BRL}^{OnShr} &= \left[1+r_{CDI}(t,T)\right]^{-BD(t,T)/252}
\end{aligned}
\qquad (6)
$$

- Define Convertibility as On- vs. Off- Shore FX Forward differential:

$$
C(t,T) \overset{def}{=} \frac{F^{OffShr}(T)}{F^{OnShr}(T)}
$$

- Or same in terms of Convertibility rate:

$$
\begin{aligned}
C(t,T) &= 1+c_{A360}(t,T)\cdot\frac{T-t}{360} \\
&= \left[1+c_{B252}\right]^{BD(t,T)/252}
\end{aligned}
$$

FX Convertibility (continued):

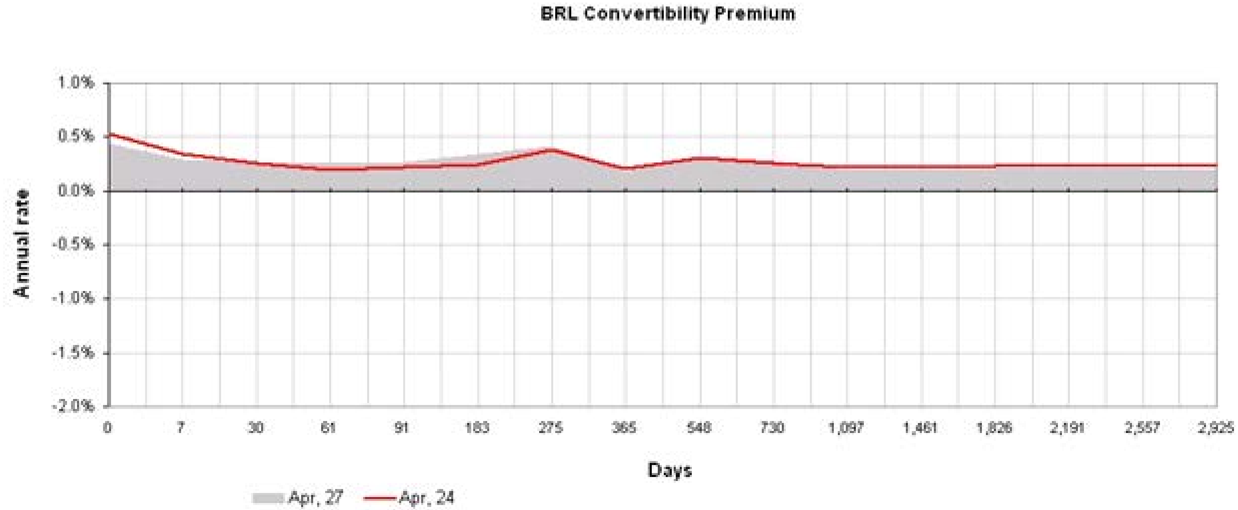

Figure 1. Future Cash Flow and Collateral example

- **Assume same FX Spot to define Offshore FX via tradable Convertibility:**

$$F^{OffShr}(T) = F^{OnShr}(T) \cdot C(t,T)$$

$$\frac{Z_{\$}^{OffShr}(t,T)}{Z_{BRL}^{OffShr}(t,T)} = \frac{Z_{\$}^{OnShr}(t,T)}{Z_{BRL}^{OnShr}(t,T)} \cdot C(t,T)$$

$$Z_{BRL}^{OffShr} \overset{def}{=} Z_{BRL}^{OnShr}(t,T) \cdot \frac{Z_{\$}^{OffShr}(t,T)}{Z_{\$}^{OnShr}(t,T)} \cdot \frac{1}{C(t,T)}$$

Emerging Markets and Inflation

FX Convertibility (continued):

- Or same expressed via exponential rates for better visualization

$$
\begin{aligned}
Z_{BRL}^{OffShr} &= e^{-r_{CDI}\cdot\tau}\cdot\frac{e^{-r_{\$}^{OffShr}\cdot\tau}}{e^{-r_{\$}^{OnShr}\cdot\tau}}\cdot\frac{1}{e^{c\cdot\tau}} \\
&= \exp\left[-\left(r_{CDI}+r_{\$}^{OffShr}-r_{\$}^{OnShr}+c\right)\cdot\tau\right] \\
r_{BRL}^{OffShr} &\equiv r_{CDI}-\left(r_{\$}^{OnShr}-r_{\$}^{OffShr}\right)+c
\end{aligned}
$$

- Offshore FX implied BRL rate is thus equal to:
  - Onshore CDI rate
  - Less US Rates differential (always positive! As will be shown later!)
  - And FX Convertibility

FX Convertibility (continued):

- **Very important methodologically, but also practically!**
- **Defines formation of the offshore FX market and Hedging approach!**
- **Controls Interpolation rules:**
  - **Assume defined FX Forward benchmarks for 1Y and 2Y BRLUSD FX**
  - **How do we come up with 18m point?**
    1. **Interpolate directly in FX Forwards space**
    2. **Interpolate USD and BRL FX Implied curves separately, then combine**
    3. **Interpolate 3 rates per conventions + Convertibility, then combine!**

  - **More on that later once we consider onshore rates interpolation**

Agenda for Today. Rates

Lecture 4. Brazil

1. Benchmarks and Day Count Conventions
2. Foreign Exchange (FX) products
3. **Linear Interest Rate products**
   - **IR Futures**
   - **CDI Swap**
4. Interest Rate Options

- **Consider Rates products linked to benchmarks considered in Sec 1**

IR Products

- **Main peculiarity: onshore rates products are FX dependent**

IR Futures:

- **Summarize in a table:**

Name	Underlying	Category	Benchmark Reference
DI Future	Capitalized interbank deposit rates between trading day and Expiry	IR	CDI
DDI Future	Ratio between capitalized CDI and USD/BRL exchange rate variation up to futures expiry	FX / IR	CDI and USD/BRL PTAX
OC Future	Capitalized repo rate (SELIC) between trading day and Expiry	IR	SELIC
DCO Future	Ratio between capitalized SELIC and USD/BRL exchange rate variation up to futures expiry	FX / IR	SELIC and USD/BRL PTAX

Table 1. Interest Rates Futures trading on Brazil BM&F exchange.

DI Future

- **Most liquid one and is CDI Rate based**
- **Market participants express views on overnight lending CDI**
- **As always: daily margin for participants to place on Exchange**
- **Effectively swaps compounded floating O/N rate for a Fixed one:**
- **Traded as purely discount instrument, so the price PU is:**

$$PU_{DI} = \frac{100{,}000}{\left[1 + r_{CDI}(t,T)\right]^{BD(t,T)/252}} \tag{7}$$

- **Collection of PU's across maturities uniquely defines Zero Curve**
- **Note: no convexity as Futures are effectively on a Zero Coupon bond in a market with daily compounding rates**

DI Future (continued)

- Recall zero curve bootstrapping complexities discussed in Lecture 3
- Specifically on possible Interpolation approaches!
- Brazil market is very specific on using Flat Forward
- Flat Forward Interpolation in rates space for $T_1 < S < T_2$ between two market traded DI Futures:

$$PU_{T_1} = \frac{100{,}000}{\left[1 + r_{CDI}(t, T_1)\right]^{BD(t,T_1)/252}}$$

$$PU_{T_2} = \frac{100{,}000}{\left[1 + r_{CDI}(t, T_2)\right]^{BD(t,T_2)/252}} \equiv PU_{T_1} \cdot \frac{1}{\left[1 + f_{CDI}(T_1, T_2)\right]^{BD(T_1,T_2)/252}} \quad \textbf{(8)}$$

here $f_{CDI}(T_1, T_2)$ is Forward CDI Rate between T_1 and T_2, then

$$PU_S = PU_{T_1} \cdot \frac{1}{\left[1 + f_{CDI}(T_1, T_2)\right]^{BD(T_1,S)/252}}$$

DDI Future

- USD referencing DDI Future or Cupom Cambial
- Expect some FX dependency (recall FX/IR category in Table 1)
- Difference between effective CDI rate and FX PTAX variation
- Serves as source and effective hedge against onshore USD rate
- Traded also as discount instrument via onshore USD rates:

$$\boxed{PU_{DDI} = \frac{100{,}000}{1 + r_{\$}(t,T) \cdot \frac{T-t}{360}}}$$

Homework

- **Let us now go back to FX Forwards pricing and consider Convertibility based interpolation compared to other simpler methods**

HW1: **Compare pricing of the 18m USD-BRL off-shore NDF via 3 approaches using market data provided:**

1. **Direct interpolation in the forward FX space**
2. **Two curves interpolation: US Libor and FX implied BRL off-shore**
3. **Convertibility approach interpolating 4 curves at once. Use Linear interp for Onshore USD and Convertibility, use market defined Flat Forward for DI rate**

DDI Future continued…

- **That is where complexity starts!**
- **Simple no-arbitrage must uniquely connect DI and DDI Futures via FX Futures**
- **Complexity comes from different settlement rules as already noted in Eq.(4)**
- **Different settlement rules between Rates and FX, but also BRL and USD**
- **From Eq.(4) connecting *PU* prices of DI and DDI Futures, FX Future price *Φ(T)* and TC_{T-1} is FX PTAX fixing prior to expiry:**

$$PU_{DDI}(T) = PU_{DI}(T) \cdot \frac{\Phi(T)}{TC_{T-1}}$$

DDI Future (continued)

- $TC_{T\text{-}1}$ is prior expiry FX PTAX fixing, so is not a good indicator of current market and current USD rates!

- Thus a name of **Dirty FX Coupon**

- And dedicated FRA called FRC or **Clean FX Coupon** to overcome is defined as differential between Short and Long DDI Future:

- **Now can deduce bootstrapping of the Onshore USD Curve from:**
  - **Traded local DI and DDI Futures**
  - **FX Futures and**
  - **FRC contracts**
- **Follow [Fabozzi, 2002] first to discuss why onshore USD rate is different from USD LIBOR**
- **No-arbitrage domestic (onshore) rate expression via:**
  - **Offshore US LIBOR**
  - **Some spread and**
  - **FX devaluation we already used in Lecture 3**

Onshore USD Curve [Fabozzi, 2002] (continued)

- Onshore rate via Offshore, spread *s* and FX devaluation $\hat{e}_t$

$$\boxed{\left(r_\$^{OffShr} + s\right)\cdot\tau = \frac{\left(1 + r_\$^{OnShr}\right)^\tau}{\left(1 + \hat{e}_t\right)} - 1} \qquad (9)$$

  - Right side: interest earned investing locally, hedge currency risk
  - Left side: cost of funding in USD (assuming LIBOR funding contrary to Differential Discounting discussion in Lecture 3)

- Break in equality means arbitrage and is equivalent to free money flowing in or out
- Domestic rate side is higher ➔ money starts flowing in
- And the other way around
- Eq.(9) also offers clear path for local policy actions to balance money flows across borders via capital controls

Onshore USD Curve [Fabozzi, 2002] (continued)

- **Easy and most usual capital control is to add tax on earnings**
- **Thus restricting capital flow into the country**
- **Amend Eq.(9) to reflect that:**

$$\left(r_{\$}^{OffShr}+s\right)\cdot\tau=\frac{\left[\left(1+r_{\$}^{OnShr}\right)^{\tau}-1\right]\times\left(1-tax\right)+1}{\left(1+\hat{e}_{t}\right)}-1$$

- **Anything else? ➔**

HW2: **More economics than math, but suggest effects of upfront fees and minimum tenor requirements as yet another measures of capital control on on-shore USD rates in Brazil**

CDI Swap

- OTC version of DI Future called Pre-DI or CDI Swap

- Combine Eq.(1) and (7) to write for Future Value of unit legs:

$$fv^{FIX}\left(t_o, t_N = T\right) = \left(1+K\right)^{BD(t_o,T)/252}$$
$$fv^{FLT}\left(t_o, t_N = T\right) = \prod_{i=0}^{N}\left(1+CDI_i\right)^{1/252} \qquad (10)$$

- PV converges to simple discounting:

$$pv^{FLT} = \mathbf{E}\left[\frac{fv^{FLT}\left(t_o, T\right)}{\beta\left(t_o, T\right)}\right]$$
$$= Z_{BRL}^{OnShr}\left(t, T\right) \cdot \prod_{i=0}^{N}\left(1+CDI_i\right)^{1/252} \qquad (11)$$

- Spot starting Floating leg must price to par (unity)

Percentage CDI Swap

- Market uses a scaling factor instead of the additive Spread to adjust the Floating payment when needed
- Reason is clear after looking at historical levels of rates
- Leverage factor κ (quoted in percentage terms) or percentage accumulation of the overnight rate

$$fv^{FLT}\left(t_o, t_N = T\right) = \prod_{i=0}^{N} \left\{ \left[\left(1 + CDI_i\right)^{1/252} - 1 \right] \cdot \kappa + 1 \right\} \qquad (12)$$

- No simple discounting as in Eq.(11), so need Convexity Adjustment:

$$pv^{FLT}\left(t_o, t_N = T\right) = Cxty \cdot Z\left(t, T\right) \cdot \prod_{i=0}^{N} \left\{ \left[\left(1 + CDI_i\right)^{1/252} - 1 \right] \cdot \kappa + 1 \right\}$$

- Use same Discount Factor notation as in Eq.(11)

Percentage CDI Swap (continued)

- **Approximate Eq.(12) via short *exp* rate ex subscript for simplicity and replace starting time t_o with 0:**

$$fv^{FLT}(0,T)=\prod_{i=0}^{N}\left\{\left[(1+CDI_i)^{1/252}-1\right]\cdot\kappa+1\right\}$$
$$\approx \exp\left(\kappa\int_0^T r(s)ds\right)$$

- **Expectation for Present Value:**

$$pv^{FLT}(0,T)=\mathbf{E}\left[\exp\left(\kappa\int_0^T r(s)ds\right)\cdot\exp\left(-\int_0^T r(s)ds\right)\right]$$
$$=\mathbf{E}\left[\exp\left(-(1-\kappa)\int_0^T r(s)ds\right)\right]$$

Percentage CDI Swap (continued)

- **As per Lecture 2 Convexity Primer, try simple model. Vasicek:**

$$\boxed{dr_t = (\theta - \mu \cdot r_t)dt + \sigma \cdot dW_t}$$

- here μ is constant mean reversion and

- σ is constant volatility

- **Integrate Vasicek for closed form of zero-coupon bond price (more on that later)**

$$\boxed{pv(0,T) = \exp\left[(1-\kappa)\cdot r_o \cdot B_T + (1-\kappa)\frac{\theta}{\mu}(B_T - T) - (1-\kappa)^2 \frac{\sigma^2}{2\mu^2}\left(B_T - T + \frac{\mu}{2}B_T^2\right)\right]}$$

- here $B_T \overset{def}{=} \frac{1}{\mu}\left(1 - e^{-\mu T}\right)$

Percentage CDI Swap (continued)

- **No-Convexity term ignores expectation:**

$$pv_{NoCxty}(0,T) = \exp\left[(1-\kappa)\cdot r_o \cdot B_T + (1-\kappa)\frac{\theta}{\mu}(B_T - T) - (1-\kappa)\frac{\sigma^2}{2\mu^2}\left(B_T - T + \frac{\mu}{2}B_T^2 \right)\right]$$

- **And Convexity expression is a ratio of the two:**

$$Cxty \overset{def}{=} \frac{pv(0,T)}{pv_{NoCxty}(0,T)} = \exp\left[\kappa(1-\kappa)\frac{\sigma^2}{2\mu^2}\left(B_T - T + \frac{\mu}{2}B_T^2 \right)\right]$$

- **Simplify more for μ=*0***

$$Cxty \approx \lim_{\mu\to 0}\left\{ \exp\left[\kappa(1-\kappa)\frac{\sigma^2}{2\mu^2}\left(B_T - T + \frac{\mu}{2}B_T^2 \right)\right]\right\}$$
$$= \exp\left[-\frac{1}{6}\kappa(1-\kappa)\sigma^2 T^3 \right]$$

Percentage CDI Swap (continued)

HW3: **Estimate level of Percentage CDI Swap convexity adjustment for ranges of κ, σ and T. Suggest validity limits where convexity could be safely ignored and where it should be accounted.**

Lecture 4. Brazil

1. Benchmarks and Day Count Conventions
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4. **Interest Rate Options**
   - **CDI Swaption and Option on DI Future**
   - **CDI Cap**
   - **IDI Option**

- **Assume general familiarity with Options pricing theory and IR Options**

CDI Swaption, Option on DI Future

- **Future Value of Fixed Rate payer CDI swap as per Eq.(10)**

$$fv(t_o, t_N = T) = \prod_{i=0}^{N} (1 + CDI_i)^{1/252} - (1+K)^{BD(t_o,T)/252}$$

- **Rewrite with effective CDI rate *r* and PV as in Eq.(11)**

$$pv(t,T) = \left[(1 + r(t,T))^{BD(t,T)/252} - (1+K)^{BD(t,T)/252} \right] \cdot Z(t,T)$$

- **Option to enter on swap starting date *t***

$$\pi^{CDI\ Swptn}(t) = \max\left[(1 + r(t,T))^{BD(t,T)/252} - (1+K)^{BD(t,T)/252}, 0 \right] \cdot Z(t,T)$$

CDI Swaption (continued)

- **Discounting as per Eq.(6)**

$$\pi_o^{CDI\ Swptn} = \max\left[\left(1+r(t,T)\right)^{BD(t,T)/252} - (1+K)^{BD(t,T)/252}, 0\right] \cdot Z(0,T)$$
$$\pi_o^{CDI\ Swptn} \equiv Z(0,T) \cdot \max(FLT - FIX, 0)$$

- **As per market conventions quote via implied Black-Sholes vol σ_{FLT}:**

$$\pi_o^{CDI\ Swptn} \equiv Z(0,T) \cdot BS(FLT, FIX, t, T, \sigma_{FLT})$$

- **Also, per same conventions it must be expressed via rate vol as follows:**

$$\frac{dr}{r} = \mu dt + \sigma dW_{CDI}$$
$$\frac{dFLT}{FLT} = \mu_{FLT} dt + \sigma_{FLT} dW_{FLT}$$

CDI Swaption (continued)

- **Continue from before**

$$
\begin{aligned}
dFLT &= \frac{dFLT}{dr}dr \\
&= \frac{BD(t,T)}{252}(1+r)^{BD(t,T)/252-1}dr \\
&= \frac{BD(t,T)}{252}\frac{FLT}{(1+r)}dr
\end{aligned}
$$

$$
\frac{dFLT}{FLT} = \frac{BD(t,T)}{252}\frac{r}{(1+r)}\frac{dr}{r} \quad \Rightarrow \quad \sigma_{FLT} = \frac{BD(t,T)}{252}\frac{r}{(1+r)}\sigma
$$

- **Market strongly is required to confirm every swaption on price (premium), rate vol and price vol as per above!**
- **Anything wrong with these assumptions?**
- **Only flexibility left is in Vol smile or Vol-per-strike approach**

CDI Cap

- **Cap on a CDI rate as single Caplet similarly to a regular IR Cap**
- **Accumulation starting at *t* and ending at *T*:**

$$\pi\left(t_o = t, t_N = T\right) = \max\left[\prod_{i=0}^{N}\left(1 + CDI_i\right)^{1/252} - \left(1 + K\right)^{BD(t,T)/252}, 0\right]$$

- **Approximate overnight accumulation via integral over short rate *r(s)* again as** $e^r \approx 1 + CDI$ **to re-write Eq.(1) for the bank account:**

$$\beta(t,T) = \exp\left[\int_t^T r(s)ds\right] \propto \prod_{i=0}^{N}\left(1 + CDI_i\right)^{1/252}$$

- **Then Cap's pay-off reduces to**

$$\pi_T = \max\left[\beta(t,T) - FIX, 0\right]$$

CDI Cap (continued)

- **PV of that pay-off we can write via short rate stochastic Discount Factor:**

$$\pi_t = \max\left[1 - D(t,T)\cdot FIX, 0\right]$$
$$D(t,T) \stackrel{def}{=} \exp\left(-\int_t^T r(s)ds\right)$$

- **Try deriving *D(t,T)* from dynamics of a Zero Coupon bond. Note drift explicitly expressed via the short rate**

$$\frac{dZ(t,T)}{Z(t,T)} = r(t)dt + \sigma_Z(t,T)dW_t \qquad \textbf{(13)}$$

- **And solve it:**

$$d\ln[Z(t,T)] = \left(r(s) - \frac{\sigma_Z^{\ 2}(t,T)}{2}\right)dt + \sigma_Z(t,T)dW_t$$
$$Z(t,T) = Z(0,T)\cdot\exp\left[\int_0^t\left(r(s) - \frac{\sigma_Z^{\ 2}(s,T)}{2}\right)ds\right]\cdot\exp\left[\int_0^t \sigma_Z(s,T)dW_s\right]$$

CDI Cap (continued)

- **Invert and add a unit Zero Coupon bond:**

$$\frac{Z(T,T)}{Z(t,T)} = \exp\left[\int_0^T \left(r(s) - \frac{\sigma_Z{}^2(s,T)}{2} \right) ds - \int_0^t \left(r(s) - \frac{\sigma_Z{}^2(s,T)}{2} \right) ds \right] \cdot \exp\left[\int_0^T \sigma_Z(s,T) dW_s - \int_0^t \sigma_Z(s,T) dW_s \right]$$

$$= \exp\left[\int_t^T r(s) ds \right] \cdot \exp\left[\int_t^T \left(-\frac{\sigma_Z{}^2(s,T)}{2} \right) ds \right] \cdot \exp\left[\int_t^T \sigma_Z(s,T) dW_s \right]$$

- **To extract Stochastic Discount Factor expression:**

$$D(t,T) = \exp\left[-\int_t^T r(s) ds \right] = Z(t,T) \cdot \exp\left[\int_t^T \left(-\frac{\sigma_Z{}^2(s,T)}{2} \right) ds \right] \cdot \exp\left[\int_t^T \sigma_Z(s,T) dW_s \right] \quad \textbf{(14)}$$

CDI Cap. HJM for short CDI rate

- **Introduce forward rate *f(t,T)* and use martingale *Z(t,T)*/β*(t)***

$$df\left(t,T\right)=\mu_f\left(t,T\right)dt+\sigma_f\left(t,T\right)dW_t$$

$$Z\left(t,T\right)=\exp\left[-\int_t^T f\left(t,s\right)ds\right]$$

$$\mu_f\left(t,T\right)=\sigma_f\left(t,T\right)\int_t^T\sigma_f\left(t,s\right)ds \qquad (15)$$

- **Drift condition above derived equating drift of *Z(t,T)* from Eq.(13) and (15):**

$$r\left(t\right)-\int_t^T\mu_f\left(t,s\right)ds+\frac{1}{2}\left\|\int_t^T\sigma_f\left(t,s\right)ds\right\|^2=r\left(t\right)$$

CDI Cap. HJM for short CDI rate

- **Simplest form of Hull-White vol for extended Vasicek**

$$\sigma_f(t,T) = \sigma(t)\exp\left[-\int_t^T \lambda(x)dx\right]$$

- **Or even easier for λ=*Const*:**

$$\sigma_f(t,T) = \sigma(t)e^{-\lambda(T-t)}$$

CDI Cap. HJM for short CDI rate

- **Derive Zero Coupon Bond vol σ_Z via Ito and Eq.(13) and (15):**

$$f(t,T) = -\frac{\partial \ln[Z(t,T)]}{\partial T}$$

$$df(t,T) = \sigma_Z \frac{\partial \sigma_Z}{\partial T} dt - \frac{\partial \sigma_Z}{\partial T} dW_t$$

- **To arrive at that we write**

$$\sigma_Z(t,T) = -\int_t^T \sigma_f(t,s) ds$$

- **That for constant vol converges to**

$$\sigma_Z(t,T) = -\sigma \int_t^T e^{-\lambda(s-t)} ds$$
$$= \frac{\sigma}{\lambda}\left(e^{-\lambda(T-t)} - 1\right)$$

CDI Cap. HJM for short CDI rate (continued)

- **Back to Eq.(14) to integrate σ_Z**

$$\exp\left[-\int_t^T \frac{\sigma_Z^2(s,T)}{2}ds\right] = \exp\left[-\frac{1}{2}\int_t^T \frac{\sigma^2}{\lambda^2}\left(e^{-\lambda(T-s)}-1\right)^2 ds\right] \equiv \exp\left[-\frac{1}{2}\Sigma_Z^2\right]$$

- **Where**

$$\begin{aligned}\Sigma_Z^2 &\equiv \int_t^T \frac{\sigma^2}{\lambda^2}\left(e^{-\lambda(T-s)}-1\right)^2 ds \\ &= \frac{\sigma^2}{\lambda^2}\left\{\frac{e^{-2\lambda T}}{2\lambda}\left(e^{2\lambda T}-e^{2\lambda t}\right)-\frac{2e^{-\lambda T}}{\lambda}\left(e^{\lambda T}-e^{\lambda t}\right)+(T-t)\right\}\end{aligned} \quad (16)$$

- **Combine all of the above re-writing Eq.(14) for our stochastic DF:**

$$D(t,T)=\exp\left[-\int_t^T r(s)ds\right]=Z(t,T)\cdot\exp\left[-\frac{1}{2}\Sigma_Z^2\right]\cdot\exp\left[\int_t^T \sigma_Z(s,T)dW_s\right] \quad (17)$$

CDI Cap. Final Cap price

- **Once again for Option's price with Expectation:**

$$\pi_o = [1 - D(0,T) \cdot FIX]^+$$
$$\equiv \int_{-\infty}^{\infty} \max[1 - D_T \cdot FIX, 0] \cdot \phi(D_T) dD_T$$

- **Take log of Eq.(17):**

$$\ln(D_T) = \left(\ln(Z(0,T)) - \frac{1}{2}\Sigma_Z^2 \right) + \int_0^T \sigma_Z(s,T) dW_s$$

to see it is Normally distributed with mean

$$m_D \equiv \left(\ln(Z(0,T)) - \frac{1}{2}\Sigma_Z{}^2 \right) \qquad \textbf{(18)}$$

and variance Σ_Z^2

CDI Cap. Final Cap price (continued)

- **Define new variable** $x \equiv \dfrac{\ln(D_T) - m_D}{\Sigma_Z}$

- **Now** $x \propto N(0,1)$ **and** $D(0,T) = \exp(x \cdot \Sigma_Z + m_D)$**, so we can write**

$$\boxed{\pi_o = \int_{-\infty}^{\infty} \max\left[1 - e^{x \cdot \Sigma_Z + m_D} \cdot FIX, 0\right] \cdot \varphi(x) dx,}$$

where $\phi(x) \equiv \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

- **Thus we can write similar to Black-Sholes with *m_D* and Σ_Z^2 defined in Eq.(16) and (18)**

$$\boxed{\begin{array}{l} \pi_o = \mathrm{N}(d_1) - e^{m_D + \Sigma_Z^2/2} \mathrm{N}(d_2) \cdot FIX \\ d_1 \equiv -\dfrac{\ln(FIX) + m_D}{\Sigma_Z} \qquad d_2 \equiv d_1 - \Sigma_Z \end{array}}$$

CDI Cap. Final Cap price (continued)

- **Once again Brazil IR Option via Black-Sholes:**

$$\pi_o^{CAP} = BS\left[CALL, FLT(t,T), FIX(t,T), T, \frac{\Sigma_Z(t,T,\sigma,\lambda)}{\sqrt{T-t}}\right]\cdot Z(0,T)$$

IDI Option

- **IDI Index: average overnight interbank deposit rate index**

$$IDI_t = IDI_{t-1}\cdot\left[\left(1+CDI_{t-1}\right)^{1/252}\right]$$

- **Compare that to Eq.(1) to see that IDI is identical to a bank account with some initial reset**

IDI Option (continued)

- **Cash settled IDI Call Option pay-off is:**

$$\pi_T^{IDI\ CALL} = \max\left[IDI_T - K, 0\right]$$

is a simplified expression for Today starting CDI Cap:

$$\begin{aligned}\pi_o^{IDI\ CALL} &= IDI_o \cdot \max\left[FLT(0,T) - FIX(0,T), 0\right] \cdot Z(0,T) \\ &= IDI_o \cdot \max\left[1 - FIX(0,T) \cdot Z(0,T), 0\right] \\ &= IDI_o \cdot BS\left[CALL, 1, FIX(0,T) \cdot Z(0,T), T, \frac{\Sigma_Z(T,\sigma,\lambda)}{\sqrt{T}}\right]\end{aligned}$$

where

$$\Sigma_Z(T,\sigma,\lambda) = \frac{\sigma^2}{\lambda^2}\left\{\frac{e^{-2\lambda T}}{2\lambda}\left(e^{2\lambda T} - 1\right) - \frac{2e^{-\lambda T}}{\lambda}\left(e^{\lambda T} - 1\right) + T\right\}$$

IDI Option. Last comment

- **HJM vol σ we used in derivation could be connected to CDI rate vol σ_{CDI} from earlier via simple relationship:**

$$\boxed{\sigma = \sigma_{CDI} \cdot f\left(T-1, T\right)}$$

where $f_{CDI}(T\text{-}1, T)$ is instantaneous forward CDI rate

References

[JPMorgan, 2004] *Guide to Brazilian Local Markets.* J.P.Morgan. 2004

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[Lipscomb, 2005] Lipscomb, (2005). *An Overview of Non-Deliverable Foreign Exchange Forward Markets*. Federal Reserve Bank of New York

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1. **Compare pricing of the 18m USD-BRL off-shore NDF via 3 approaches using market data provided:**

Today	23-Nov-15			
FX Spot	3.75			
Multiplier	10,000			
FX Market Data				
Tenor	FX Fwd Pts	USD Disc Factor	Convertibility Rate, Linear Act/360	
1Y	4,400.0	0.9935	0.96%	
2Y	9,050.0	0.9810	0.40%	
Brazil Onshore Dates and Rates				
	Relevant Dates	BRL Onshore Rate, Bus/252	USD Onshore Rate, Linear Act/360	Num BRL Bus Days
	1-Apr-17	15.33%	4.03%	341
	1-Jul-17	15.43%	4.02%	402
18M date =>	23-May-17			375

3. **Estimate level of Percentage CDI Swap convexity adjustment for ranges of κ, σ and T via few study cases provided. Suggest validity limits where convexity could be safely ignored and where it should be accounted for properly.**

Start with level of rates at 15%, consider cases of log-normal rate vol of 10% and 30%, consider κ from 50% to 150%, and consider maturities from 1y to 10y. Discuss adjustment for swap with a notional of $1mm.

Hint: note that Vasicek model of short rates we considered requires normal volatility. Discuss how to convert log-normal vol provided to normal vol needed in our formulae