MTH 9831. Solutions to Quiz 6.

Let $(B(t))_{t\geq 0}$ be a standard Brownian motion under $\mathbb{P}$, $(\mathcal{F}(t))_{t\geq 0}$ be a the filtration for this Brownian motion, and $(\Theta(t))_{t\geq 0}$, be a stochastic process adapted to this filtration and such that $\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \Theta^2(t)\,dt\right)\right) < \infty$.

(1) (4 points) Let

$$Z(t) = \exp\left(-\int_0^t \Theta(s)\,dB(s) - \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right), \quad 0 \leq t \leq T$$

and

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P} \quad \forall A \in \mathcal{F}(T).$$

Show that $1/Z(t)$, $0 \leq t \leq T$, is a martingale under $\tilde{\mathbb{P}}$ and state clearly with respect to which filtration. Hint: rewrite $1/Z(t)$ in terms of the appropriately chosen process $\tilde{B}(t)$ (get rid of $B(t)$) and compute its Itô differential.

Solution. We write

$$\begin{aligned}
\frac{1}{Z(t)} &= \exp\left(\int_0^t \Theta(s)\,dB(s) + \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right) \\
&= \exp\left(\int_0^t \Theta(s)\,[d\tilde{B}(s) - \Theta(s)\,ds] + \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right) \\
&= \exp\left(\int_0^t \Theta(s)\,d\tilde{B}(s) - \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right).
\end{aligned}$$

Note that $1/Z(t)$ is of the form $e^{X(t)}$ where $X(t)$ is an Itô process. Therefore,

$$\begin{aligned}
d\left(\frac{1}{Z(t)}\right) &= d(e^{X(t)}) = e^{X(t)}\left(dX(t) + \frac{1}{2}d[X]_t\right) \\
&= \frac{1}{Z(t)}\left(\Theta(t)dB(t) + \frac{1}{2}\Theta^2(t)\,dt\right) + \frac{1}{2}\Theta^2(t)dt = \frac{\Theta(t)}{Z(t)}\left(dB(t) + \Theta(t)\,dt\right),
\end{aligned}$$

Denoting $B(t) + \int_0^t \Theta(s)\,ds$ by $\tilde{B}(t)$ we get

$$d(1/Z(t)) = (1/Z(t))\Theta(t)d\tilde{B}(t).$$

We conclude that $1/Z(t)$ is a martingale under $\tilde{\mathbb{P}}$.

(2) (2 points) Which process do we call a generalized geometric Brownian motion? Give two answers: (a) in the differential form; (b) in the closed form (as an exponential).

Solution. Let $(\alpha(t))_{t\geq 0}$ and $(\sigma(t)))_{t\geq 0}$ be $\mathcal{F}(t)$-adapted processes. Generalized GBM with the starting point $x > 0$ is defined either by the equation

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t), \quad t \geq 0, \quad S(0) = x,$$

or, equivalently, by the formula

$$S(t) = x\exp\left(\int_0^t (\alpha(s) - \sigma^2(s)/2)\,ds + \int_0^t \sigma(s)\,dB(s)\right).$$

(3) (4 points) Consider a model with a unique risk-neutral measure $\tilde{\mathbb{P}}$ and constant interest rate $r \geq 0$. A chooser option gives its owner the right at time t_0 to choose either the call or the put (same strike K and expiration $T > t_0$). What is the time t_0 value of the chooser option? What is the time 0 price of the chooser option (you may assume that prices of all vanilla options are given)?

Solution. The time t_0 value of the chooser option is

$$\begin{aligned}\max\{C(t_0), P(t_0)\} &= C(t_0) + \max\{0, P(t_0) - C(t_0)\} \\ &= C(t_0) + \max\{0, e^{-r(T-t_0)}K - S(t_0)\},\end{aligned}$$

where $C(t_0)$ and $P(t_0)$ are the time t_0 values of the call and put mentioned above. In the last equation we used the put-call parity. Therefore the time 0 price of the chooser option is

$$\begin{aligned}\tilde{E}(D(T)V(T)) &= \tilde{E}(\tilde{E}(D(T)V(T) \,|\, \mathcal{F}(t_0))) \\ &= \tilde{E}\left(e^{-rt_0}\left[C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right]\right) \\ &= \tilde{E}\left[e^{-rt_0}C(t_0)\right] + \tilde{E}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] = C(0) + P^*(0),\end{aligned}$$

where $C(0)$ is the time 0 price of a call with expiration T and strike K, and $P^*(0)$ is the time 0 price of a put with expiration t_0 and strike $e^{-r(T-t_0)}K$.

Remark. Note, we could also write

$$\begin{aligned}\max\{C(t_0), P(t_0)\} &= P(t_0) + \max\{0, C(t_0) - P(t_0)\} \\ &= P(t_0) + \max\{0, S(t_0) - Ke^{-r(T-t_0)}\}\end{aligned}$$

and arrive at a different answer. The two answers are equaivalent by the put-call parity.