November 28, 2018

MTH 9831. Solutions to Quiz 9.

(1) (4 points) Assume BSM model with $r = 0$. Find explicitly the portfolio value process $X(t)$, $0 \le t \le T$, such that

$$X(T) = \frac{1}{T}\int_0^T S(t)\,dt - S(T).$$

Solution. Let $X(t) = \gamma(t)S(t) + (X(t) - \gamma(t)S(t))$ be a self-financing portfolio, where $\gamma(t)$ is a differentiable process. Since $r = 0$, we have (using the self-financing condition)

$$dX(t) = \gamma(t)dS(t).$$

On the other hand,

$$d(\gamma(t)S(t)) = \gamma(t)dS(t) + S(t)d\gamma(t) + d[S,\gamma](t).$$

Since $\gamma(t)$ is assumed to be regular, $d[S,\gamma](t) = 0$. Therefore,

$$dX(t) = d(\gamma(t)S(t)) - S(t)d\gamma(t).$$

Integrating from 0 to T we get

$$X(T) = X(0) - \gamma(0)S(0) + \gamma(T)S(T) - \frac{1}{T}\int_0^T TS(t)\gamma'(t)dt.$$

To match the required value $X(T)$ we need

$$X(0) = \gamma(0)S(0), \quad \gamma(T) = -1,$$
$$-TS(t)\gamma'(t) = S(t) \quad \Rightarrow \quad \gamma'(t) = -\frac{1}{T}.$$

Integrating the last equation from t to T we get

$$\gamma(T) - \gamma(t) = \frac{T-t}{T}.$$

Using the condition $\gamma(T) = -1$ we conclude that

$$\gamma(t) = -1 + \frac{T-t}{T}.$$

The replicating strategy is to start with $X(0) = \gamma(0)S(0) = 0$ capital and gradually (see $\gamma(t)$) increase a short position in stock to a full share at time T.

(2) (8 points) Assume BSM model with $r \ge 0$. (In case you need this: for $\mu \in \mathbb{R}$, $m > 0$, $X(t) = \mu t + B(t)$ and $\tau_m = \min\{t \ge 0 : X(t) = m\}$ we have that for all $\lambda > 0$, $\mathbb{E}(e^{-\lambda\tau_m}) = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}$.)

(a) Find the price of a perpetual American call with strike K if the stock pays no dividends.

(b) (Basic calculus exercise) Now suppose that the stock pays dividends at a constant rate $a > 0$. The price $v(x)$ of an American call option with strike K when $S = x$ can be computed as in Exercise 8.5. The answer is

$$v(x) = \begin{cases} \frac{1}{\gamma}\left(\frac{K}{1-1/\gamma}\right)^{1-\gamma} x^\gamma, & \text{if } x < L^*; \\ x - K, & \text{if } x \geq L^*, \end{cases}.$$

where

$$\gamma = \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right)^2 + 2r} - \frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right), \quad L^* = \frac{K}{1-1/\gamma}.$$

Find the limit of the price and the limiting exercise strategy as $a \to 0$. Does your answer in (a) agree with these limiting results?

Solution. (a) It is fairy intuitive that it is never optimal to exercise this option and that its price should be equal to the current stock price. But let's give a calculation. The same reasoning as for an American put option allows us to say that we should try to find an optimal level $L_* > K$ and exercise at the first time τ_{L_*} the stock price hits this level. Fixing a level $L > K$ and applying the pricing formula we see that the time 0 value of the option (if we are to exercise at $\tau_L = \inf\{t \geq 0 \mid S(t) \geq L\}$) is

$$v_L(x) = \begin{cases} x - K, & \text{if } x \geq L; \\ \tilde{E}\left(e^{-r\tau_L}(L-K)\right) = (L-K)\tilde{E}\left(e^{-r\tau_L}\right), & \text{if } x \leq L. \end{cases}$$

Note that if $S(0) = x \geq L$ then we exercise immediately, and the value is $x - K$. Suppose now that $S(0) = x < L$. Since

$$S(t) = L \quad \Leftrightarrow \quad x e^{\sigma\left(\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t + \tilde{B}(t)\right)} = L,$$

the hitting time τ_L is the same as the hitting time of the level $m = \frac{1}{\sigma}\ln\frac{L}{x} > 0$ by a Brownian motion with the constant drift $\mu = \frac{r}{\sigma} - \frac{\sigma}{2}$. Then we apply the formula for the Laplace transform and get that

$$v_L(x) = \begin{cases} x - K, & \text{if } x \geq L; \\ (L-K)\frac{x}{L} = \left(1 - \frac{K}{L}\right)x, & \text{if } x \leq L. \end{cases}$$

Maximizing over L, we get that $L_* = \infty$ and $v_{L_*}(x) = x$.

(b) We rewrite the formula for γ in a slightly more convenient form:

$$\gamma = \frac{1}{\sigma}\sqrt{\left(\frac{1}{2} - \frac{r-a}{\sigma^2}\right)^2 + 2r} + \left(\frac{1}{2} - \frac{r-a}{\sigma^2}\right)$$

and recognize that γ decreases to 1 as a decreases to 0. This can be seen from the formula for γ, since γ is a continuously differentiable function of a on $[0, \infty)$, $\gamma(0) = 1$, and $\gamma'(a) > 0$ for all $a \geq 0$. (Intuitively, if the formula for L^* is correct then it should be true that $\gamma > 1$ for all $a > 0$, since L^* should be positive and also L^* should decrease when a increases, so γ should be an increasing function of a.) From this observation we conclude that

$$\lim_{a \to 0+} L^* = \lim_{\gamma \to 1+} L^* = \infty,$$

that is in the absence of dividends it is never optimal to exercise the option. To compute the limit of the price as $a \to 0$ we should take the limit of $v(x)$ as $\gamma \to 1$. This is a basic calculus exercise. For each $x > 0$

$$\begin{aligned} \lim_{\gamma \to 1+} v(x) &= \lim_{\gamma \to 1+} \frac{1}{\gamma} x^\gamma K^{1-\gamma} \cdot \lim_{\gamma \to 1+}\left(1 - \frac{1}{\gamma}\right)^{\gamma - 1} \\ &\overset{y := \gamma - 1}{=} x \lim_{y \to 0+}\left(\frac{y}{1+y}\right)^y = x \lim_{y \to 0+} y^y = x e^{\lim_{y \to 0+} y \ln y} = x. \end{aligned}$$

This matches the answer we got in part (a).