MTH 9831 Assignment 5 (10/11 - 10/17).

Read Lecture 5. Additional references for this material are:

1. S. Shreve, Stochastic Calculus for Finance II, Sections 4.5 and 4.6.
2. A. Etheridge, A Course in Financial Calculus, Section 4.5.

(1) (Lévy's characterization of BM and Itô's formula) Let $(B_1(t), B_2(t), \ldots, B_n(t))$ be the n-dimensional Brownian motion (extend the 2-dimensional definition in an obvious way), $n \geq 2$. Define

$$R(t) = \sum_{i=1}^{n} B_i^2(t); \quad dW(t) = \sum_{i=1}^{n} \frac{B_i(t)}{\sqrt{R(t)}}\, dB_i(t).$$

The process $R(t)$, $t \geq 0$ is called the squared Bessel process of dimension n. What is the distribution of $R(t)$? The process $\sqrt{R(t)}$ (which represents the distance from the origin of the Brownian particle in n-dimensions) is called Bessel process of dimension (or order) n.

(a) Show that $W(t)$, $t \geq 0$, is a standard Brownian motion.

(b) Show that $R(t)$, $t \geq 0$, satisfies the SDE

$$dR(t) = 2\sqrt{R(t)}\, dW(t) + n dt.$$

Look at problem (12) in the handout on Itô's formula: for which parameters the CIR model gives you exactly the squared Bessel process?

(c) Show that $X(t) := \sqrt{R(t)}$, $t \geq 0$, satisfies the SDE

$$dX(t) = dW(t) + \frac{n-1}{2X(t)} dt.$$

(2) (Lévy's characterization of BM and computation of cross-variation) Exercise 4.13. Write your solution in the matrix form, i.e. as $dW(t) := A^{-1}(t) dB(t)$, where $W(t)$ and $B(t)$ are 2-dimensional vectors and $A^{-1}(t)$ is a 2×2 matrix. Essentially your goal in this problem is to find $A^{-1}(t)$ and check that it works using Lévy's characterization of BM (2 dimensions). Then look at Exercise 4.16 and check that your approach, in fact, works for all dimensions. State clearly what $A(t)$ is (for all dimensions) and under which conditions on the covariance matrix $C(t)$ the matrix $A^{-1}(t)$ is well-defined. Review from your LA class how to find A^{-1} if you know C.

(3) (Lévy's characterization of BM and computation of cross-variation) Exercise 4.15 (it complements 4.13).

(4) (Itô's product rule) Exercise 4.10. Remark: It takes more time to read this problem than to really do it. My advice is to start reading and solving directly (i) and (ii) and eventually read the rest. Self-financing is an important concept, and here you see how it is expressed in mathematical terms for a continuous time model.

(5) (Lévy's characterization of BM and Ito's product rule) Exercise 4.19.

(6) (Stop-loss start-gain paradox) Exercise 4.21. I expect practical explanations for part (a) and a brief, justified by a few words answer to part (b). Hint: to help to wrap your mind around this paradox you might want to take a look at equation (4.10.45).