September 26, 2018

MTH 9831. Solutions to Quiz 2.

(1) (3 points) Let $(B(t))_{t\geq 0}$ be a standard Brownian motion. What is the probability that $\min_{t\leq 4} B(t) > -2$?

Solution. Using the reflection principle and scaling properties of Brownian motion we get

$$P\left(\min_{t\leq 4} B(t) > -2\right) = P\left(-\min_{t\leq 4} B(t) < 2\right) = P\left(\max_{t\leq 4}(-B(t)) < 2\right)$$
$$= 1 - 2P(B(4) \geq 2) = 1 - 2P(Z \geq 1) = 2N(1) - 1,$$

where Z is a standard normal random variable and $N(x)$ is its CDF.

(2) (3 points) Let $Y(t) = 2B(t) + t$, $t \geq 0$. Find **all** $\sigma \in \mathbb{R}$ such that the process $e^{\sigma Y(t)}$ is a martingale (with respect to the natural filtration of the Brownian motion).

Solution. For every $s \in \mathbb{R}$ the process $e^{sB(t)-s^2t/2}$, $t \geq 0$, is a martingale with respect to the above mentioned filtration. The process which we want to be a martingale is $e^{2\sigma B(t)+\sigma t}$, $t \geq 0$. Therefore, $(2\sigma)^2/2 = -\sigma$. This gives $\sigma = 0$ or $\sigma = -1/2$.

(3) (4 points) Let $u(t, x)$ be the solution to the terminal value problem

$$u_t + \frac{1}{2}u_{xx} = 0, \quad (t, x) \in [0, 1) \times \mathbb{R}, \quad u(1, x) = e^{-|x|}, \quad x \in \mathbb{R}.$$

Write

(a) the stochastic representation of $u(t, x)$ (in the form of expectation);

(b) the explicit integral formula for $u(t, x)$.

Solution.

(a) $u(t, x) = E(e^{-|B(1-t)|} \mid B(0) = x) = E(e^{-|x+\tilde{B}(1-t)|} \mid \tilde{B}(0) = 0)$, where $\tilde{B}$ is a standard Brownian motion.

(b) $u(t, x) = E(e^{-|x+\sqrt{1-t}\,Z|}) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-|x+y\sqrt{1-t}|} e^{-y^2/2}\, dy.$