MTH 9831. Solutions to Quiz 10.

(1) (3 points) Let $Q(t)$, $t \geq 0$, be a compound Poisson process, i.e. $Q(t) = \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a Poisson process with intensity λ, $(Y_i)_{i\geq 1}$ are i.i.d. random variables independent of the process N. Set $\beta := E(Y_1)$, $\sigma^2 := \mathrm{Var}(Y_1)$, $M_{Y_1}(u) := E(e^{uY_1})$. Calculate the moment generating function of $Q(t)$ and use it to find the variance of $Q(t)$.

Solution. I shall use the tower property of conditional expectation, independence of $N(t)$ and all Y_i, and the fact that $M_{N(t)}(u) = e^{\lambda t(e^u-1)}$.

$$\begin{aligned}
M_{Q(t)}(u) &= E\left[E\left[e^{uQ(t)}\middle|N(t)\right]\right] = E\left[E\left[\prod_{i=1}^{N(t)} e^{uY_i}\middle|N(t)\right]\right] \\
&= E\left[\prod_{i=1}^{N(t)} E\left[e^{uY_i}\middle|N(t)\right]\right] = E\left[\prod_{i=1}^{N(t)} E\left[e^{uY_i}\right]\right] \\
&= E\left[(M_{Y_1}(u))^{N(t)}\right] = M_{N(t)}(\ln M_{Y_1}(u)) = e^{\lambda t(M_{Y_1}(u)-1)}.
\end{aligned}$$

Computing the first and second derivatives of $M_{Q(t)}(u)$ at $u = 0$ we get

$M'_{Q(t)}(u) = \lambda t M'_{Y_1}(u) M_{Q(t)}(u) \quad \Rightarrow \quad E[Q(t)] = M'_{Q(t)}(0) = \lambda t\beta.$
$M''_{Q(t)}(u) = (\lambda t M'_{Y_1}(u))^2 M_{Q(t)}(u) + \lambda t M''_{Y_1}(u) M_{Q(t)}(u) \quad \Rightarrow \quad E\left[Q^2(t)\right] = \lambda^2 t^2 \beta^2 + \lambda t E\left[Y_1^2\right].$

Conclusion: $\mathrm{Var}(Q(t)) = \lambda t(\beta^2 + \sigma^2)$.

(2) (4 points) Compute $\int_0^t M(s)\, dM(s)$, where M is a compensated Poisson process.

Solution. Recall that when the integrator in Itô's formula is a jump process and the integrand is not continuous the value of the integral depends on whether we use left-continuous or right-continuous version of the integrand. More precisely, in our case

$$\int_0^t (M(s) - M(s-))\, dM(s) = \int_0^t \Delta N(s)\, dM(s) = \int_0^t \Delta N(s)\, dN(s) = \sum_{0<s\leq t} (\Delta N(s))^2 = N(t).$$

Now all we need to do now is to compute

$$\int_0^t M(s-)\, dM(s).$$

The first guess would be that the answer has to do with $M^2(t)$. We apply Itô's formula to $M^2(t)$, i.e. we take $X(t) = M(t) = N(t) - \lambda t$ (so that $X^c(t) = -\lambda t$ and $[X^c, X^c](t) \equiv$

0) and $f(x) = x^2$. We get

$$
\begin{aligned}
M^2(t) &= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\le t} (M^2(s) - M^2(s-)) \\
&= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\le t} ((M(s-) + \Delta N(s))^2 - M^2(s-)) \\
&= 2\int_0^t M(s-)\,dM^c(s) + 2\sum_{0<s\le t} M(s-)\Delta N(s) + \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)\,dM(s) + \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)\,dM(s) + \sum_{0<s\le t} \Delta N(s) = 2\int_0^t M(s-)\,dM(s) + N(t).
\end{aligned}
$$

Therefore,

$$
\int_0^t M(s-)\,dM(s) = \frac{1}{2}\,(M^2(t) - N(t)) \quad \text{and} \quad \int_0^t M(s)\,dM(s) = \frac{1}{2}\,(M^2(t) + N(t)).
$$

Alternatively, one can compute the desired integral directly.

$$
\begin{aligned}
M^2(t) &= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\le t} (M^2(s) - M^2(s-)) \\
&= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\le t} ((M(s-) + \Delta N(s))^2 - M^2(s-)) \\
&= 2\int_0^t M(s)\,dM^c(s) + 2\sum_{0<s\le t} M(s-)\Delta N(s) + \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)\,dM^c(s) + 2\sum_{0<s\le t} (M(s) - \Delta N(s))\Delta N(s) + \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)\,dM^c(s) + 2\sum_{0<s\le t} M(s)\Delta N(s) - \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)\,dM(s) - \sum_{0<s\le t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)\,dM(s) - \sum_{0<s\le t} \Delta N(s) = 2\int_0^t M(s)\,dM(s) - N(t).
\end{aligned}
$$

This method, of course, gives the same result.

(3) (4 points) Let $N(t)$, $t \ge 0$, be a Poisson process with intensity λ. Show that the process $Z(t) = e^{-\lambda\sigma t}(1+\sigma)^{N(t)}$, $t \ge 0$ is a martingale in two different ways: (i) by the definition; (ii) by using Itô-Doeblin formula.

Solution. (i) Z is a positive process and has finite expectation. We only need to check the martingale property. Denote by $(\mathcal{F}(t))_{t\ge 0}$ the filtration for our Poisson process. Then using the fact that for $0 \le s \le t$ the increment $N(t) - N(s)$ is independent from $\mathcal{F}(s)$ we get

$$
\begin{aligned}
E\left[Z(t) \mid \mathcal{F}(s)\right] &= e^{-\lambda\sigma t}(1+\sigma)^{N(s)} E\left[(1+\sigma)^{N(t)-N(s)} | \,\mathcal{F}(s)\right] \\
&= e^{-\lambda\sigma t}(1+\sigma)^{N(s)} M_{N(t)-N(s)}(\ln(1+\sigma)) \\
&= e^{-\lambda\sigma t}(1+\sigma)^{N(s)} e^{\sigma\lambda(t-s)} = Z(s).
\end{aligned}
$$

Compare with (ii) and convince yourself that it is easier to use the definition than Itô's formula.[1]

(ii) Set $X(t) = -\lambda\sigma t + (\ln(1+\sigma))N(t)$. Then $X^c(t) = -\lambda\sigma t$, $[X^c, X^c](t) \equiv 0$, and

$$X(t) - X(t-) = (\ln(1+\sigma))\Delta N(t).$$

We shall apply Itô's formula to $f(X(t))$, where $f(x) = e^x$.

$$\begin{aligned}
Z(t) = e^{X(t)} &= 1 + \int_0^t e^{X(s)}\, dX^c(s) + \sum_{0<s\le t} (e^{X(s)} - e^{X(s-)}) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}(e^{X(s)-X(s-)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}(e^{(\ln(1+\sigma))\Delta N(s)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}((1+\sigma)^{\Delta N(s)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}\sigma\Delta N(s) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sigma \int_0^t e^{X(s-)} dN(s) \\
&= 1 + \int_0^t e^{X(s-)}\, d(X^c(s) + \sigma N(s)) = 1 + \sigma \int_0^t e^{X(s-)}\, dM(s),
\end{aligned}$$

where $M(s) = N(s) - \lambda t$, $t \ge 0$, is a martingale, since the intensity of N is equal to λ. We conclude that Z is a martingale (as a constant plus an integral of a left-continuous adapted process with respect to a martingale jump process).

[1] Let's not forget our basic definitions and keep using them.