October 10, 2018

MTH 9831. Solutions to Quiz 3.

Let $B(t)$, $t \geq 0$, be a standard Brownian motion.

(1) (2 points) Let $S(t) = S(0)e^{\mu t + \sigma B(t)}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $S(0) > 0$. Find $d(\ln S(t)) - \dfrac{dS(t)}{S(t)}$.

Solution. Let $f(t,x) = S(0)e^{\mu t + \sigma x}$. Then $S(t) = f(t, B(t))$, $f_t(t,x) = \mu f(t,x)$, $f_x(t,x) = \sigma f(t,x)$, $f_{xx}(t,x) = \sigma^2 f(t,x)$, and

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) + \frac{1}{2}\sigma^2 S(t)\,dt;$$
$$\frac{dS(t)}{S(t)} = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma\, dB(t).$$

On the other hand, $\ln S(t) = \ln S(0) + \mu t + \sigma B(t)$, so $d \ln S(t) = \mu dt + \sigma dB(t)$. Thus,

$$d(\ln S(t)) - \frac{dS(t)}{S(t)} = -\frac{\sigma^2}{2}\, dt.$$

(2) (3 points) Let $X(t) = \exp\left(6\int_0^t u\, dB(u) - 6t^3\right)$. Compute $dX(t)$ and find $E(X(t))$.

Solution. Let $f(x) = e^x$ and $I(t) = 6\int_0^t u\, dB(u) - 6t^3$. Then $I(t)$, $t \geq 0$, is an Itô process with $\Delta(t) = 6t$ and $\Theta(t) = -18t^2$. We have that $X(t) = f(I(t))$ and, thus,

$$dX(t) = f_x(I(t))\, dI(t) + \frac{1}{2} f_{xx}(I(t))\, d[I]_t.$$

Substituting $f_x(x) = f(x)$, $f_{xx}(x) = f(x)$, $dI(t) = 6t\, dB(t) - 18t^2 dt$, and $d[I]_t = 36t^2\, dt$ we get after cancellations

$$dX(t) = 6tX(t)\, dB(t) = 6te^{6\int_0^t u\, dB(u) - 6t^3}\, dB(t).$$

We conclude that $X(t)$, $t \geq 0$, is a martingale. Therefore $E(X(t)) = E(X(0)) = 1$.

(3) (5=2+3 points) Let

$$X(t) = \int_0^t s\, dB(s) \quad \text{and} \quad Y(t) = \int_0^t B(s)\, ds.$$

Find the distribution of (a) $X(t)$; (b) $Y(t)$.

Solution. (a) The integrand is a deterministic function. Therefore, by Tool C $X(t)$ has a normal distribution with mean 0 and variance $\int_0^t u^2\,du = t^3/3$.

(b) Since $d(tB(t)) = t\,dB(t) + B(t)\,dt$, integrating and rearranging the terms we get

$$\int_0^t B(s)\,ds = tB(t) - \int_0^t s\,dB(s) = \int_0^t t\,dB(s) - \int_0^t s\,dB(s) = \int_0^t (t-s)\,dB(s).$$

By the same Tool C the last integral has a normal distribution with mean 0 and variance

$$\int_0^t (t-s)^2\,ds \overset{u=t-s}{=} \int_0^t u^2\,du = \frac{t^3}{3}.$$