Example: how to use the optional stopping theorem.

References below are to A. Etheridge's textbook "Financial Calculus".

3.18. Let T_a and T_b denote the first hitting times of levels a and b respectively of a $\mathbb{P}$-Brownian motion, $\{W_t\}_{t\geq 0}$, but now W_0 is not necessarily 0. Prove that if $a < x < b$ then

$$\mathbb{P}(T_a < T_b \mid W_0 = x) = \frac{b-x}{b-a}.$$

Solution. I shall prove the formula for $x = 0$ and then show how to get the formula for $a < x < b$.

Let $x = 0$ ($a < 0 < b$) and $T = T_a \wedge T_b$. Brownian motion is a martingale. Consider the stopped martingale $\{W_{T\wedge t}\}_{t\geq 0}$. Notice that

$$|W_{T\wedge t}| \leq \max\{-a, b\} \quad \text{for all } t \geq 0. \tag{1}$$

Also $P(T < \infty) = 1$. Indeed,

$$P(T < \infty) \geq P(T_a < \infty) \overset{(*)}{=} \lim_{\theta\to 0+} E\exp(-\theta T_a) = \lim_{\theta\to 0+} \exp(-|a|\sqrt{2\theta}) = 1.$$

See Proposition 3.4.9 and lecture notes for an explanation of (*). The same is, of course, true if we use T_b instead of T_a.

An application of the Optional Stopping Theorem with $\tau_1 = 0$ and $\tau_2 = T\wedge t$ gives $EW_{T\wedge t} = EW_0 = 0$. On the other hand,

$$EW_{T\wedge t} = aP(T\wedge t = T_a) + bP(T\wedge t = T_b) + E(W_{T\wedge t}; T > t). \tag{2}$$

The last term can be estimated using (1) and the fact that $T < \infty$ a.s.[1]:

$$\begin{aligned}|E(W_{T\wedge t}; T > t)| &\leq E(|W_{T\wedge t}|; T > t)\\ &\leq \max\{-a, b\}P(T > t) \to 0 \quad \text{as } t \to \infty.\end{aligned} \tag{3}$$

Since T is a.s. finite, $T \wedge t \to T$ as $t \to \infty$ a.s.. By the continuity of the Brownian motion $W_{T\wedge t} \to W_T$ as $t \to \infty$ a.s.. From the bound (1) and the Bounded Convergence Theorem[2] we conclude that $0 = EW_{T\wedge t} \to EW_T$ as $t \to \infty$. Letting $t \to \infty$ in (2) and using (3) we obtain

$$0 = EW_T = aP(T_a < T_b) + bP(T_a > T_b).$$

Taking into account that $P(T_a < T_b) + P(T_a > T_b) = 1$, we get

$$P(T_a < T_b)(= P(T_a < T_b|W_0 = 0)) = \frac{b}{b-a}.$$

Let now $a < x < b$. Then

$$P(T_a < T_b|W_0 = x) = P(T_{a-x} < T_{b-x}|W_0 = 0) = \frac{b-x}{b-x-(a-x)} = \frac{b-x}{b-a}.$$

□

[1] $\{T = \infty\} = \cup_{n\in\mathbb{N}}\{T > n\}$ and by the continuity of probability measure, $0 = \mathbb{P}(T = \infty) = \lim_{n\to\infty} \mathbb{P}(T > n)$ (n can be replaced with t by monotonicity).

[2] $|W_{T\wedge t}| \leq \max\{-a, b\}$ for all $t \geq 0$.