MTH 9831. LECTURE 11

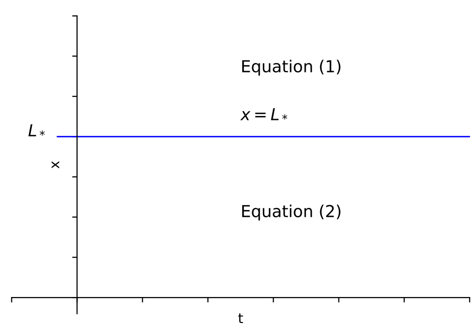
ELENA KOSYGINA

ABSTRACT. After a brief discussion of a finite expiration American put we shall turn to processes with jumps.

1. Two words about a finite expiration American put.
2. Poisson process (see Lecture 6 of the summer probability course).
3. Compound Poisson process.
4. Jump processes as integrators.
5. Quadratic and cross variation of jump process.
6. Itô-Doeblin formula for jump process ($d = 1$).

1. Finite expiration American put.

Return for a moment to a perpetual American put option. Its price $v(x)$ when $S(0) = x$, does not depend on t and the optimal exercise time is the hitting time τ_{L_*} of level L_*, where L_* is given explicitly and depends only on K, σ, and r.



Continuation set: $\mathcal{C} = \{(t,x) : t \geq 0, x \geq 0, v(x) > (K-x)_+\}$, where $v(x)$ satisfies the equation

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2x^2v''(x) = 0. \tag{1}$$

Stopping set: $\mathcal{S} = \{(t,x) : t \geq 0, x \geq 0, v(x) = (K-x)_+\}$, where $v(x)$ statisfies the equation

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2x^2v''(x) = rK. \tag{2}$$

The inclusion of the time variable in the description of $\mathcal{S}$ and $\mathcal{C}$ is superficial, as everything is determined by the value of x. But for the finite expiration American put option, the price $v(t,x)$ and both $\mathcal{S}$ and $\mathcal{C}$, will depend on t. The equations (1) and (2) will be replaced by parabolic equations, and L_* will depend on the time to expiration.

We shall continue to work within BSM framework and now consider an American put with strike K and expiration T.

Definition 1.1. *Let $t \in [0,T]$ and $x \geq 0$ be given and $S(t) = x$. For each $u \in [t,T]$ denote by $\mathcal{F}_u^{(t)}$ the σ-algebra generated by the price process $(S(v))_{t \leq v \leq u}$ and let $\mathcal{T}_{t,T}$ be the set of all stopping times for the filtration $\{\mathcal{F}_u^{(t)}\}_{t \leq u \leq T}$ taking values in $[t,T] \cup \{\infty\}$. The time t price oan American put option with strike K and expiration T is defined to be*

$$v(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}}^{t,x}\left[e^{-r(\tau-t)}(K - S(\tau))\mathbb{1}_{\{\tau<\infty\}}\right].$$

***Analytical characterization of** $v(t,x)$.* It is not difficult to believe (by the analogy with the perpetual American put) that the following statement holds.

Theorem 1.2. *Let $v(t,x)$ be the time t price of the American put with strike K and expiration T. Then $v(t,x)$ satisfies the linear complementary conditions.*

$$v(t,x) \geq (K-x)_+, \quad t \in [0,T],\ x \geq 0 \tag{3}$$

$$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0, \quad t \in [0,T],\ x \geq 0 \tag{4}$$

(5) *for each $t \in [0,T]$ and $x \geq 0$, equality holds in either* (3) *or* (4).

We shall not give a proof (see Section 8.4.1 of the textbook for more details). The idea is that the holder of the put option has to wait until the stock price falls to a certain level below K. This level L, will now depend on the time left to expiration, i.e. $L = L(T-t)$. Clearly,

$$\lim_{T\to\infty} L(T) = L_* \quad \text{and} \quad L(0) = K.$$

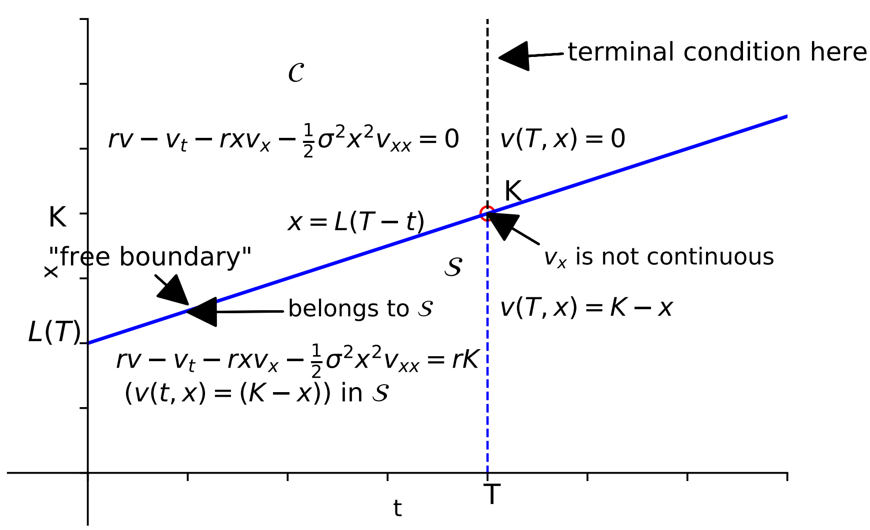
It is also plausible that $L(T-t)$ should be non-decreasing in t: the less time is left, the higher is the level at which the owner should be willling to exercise. No formula for $L(T-t)$ is known.

Let us define a stopping set

$$\mathcal{S} = \{(t,x) : t \in [0,T],\ x \geq 0,\ v(t,x) = (K-x)_+\},$$

and a continuation set

$$\mathcal{C} = \{(t,x) : t \in [0,T],\ x \geq 0,\ v(t,x) > (K-x)_+\}.$$



Smooth pasting conditions are:

(6) $\quad v(t, L(T-t)_+) = v(t, L(T-t)_-) \quad$ (v is continuous across $x = L(T-t)$)

(7) $\quad v_x(t, L(T-t)_+) = v_x(t, L(T-t)_-) = -1 \quad$ (v_x is continuous across $x = L(T-t)$ for $0 \le t < T$)

We note that v_t and v_{xx} are not continuous across the free boundary.

Theorem 1.3. *There is a unique bounded function $v(t,x)$ on $t \in [0,T]$, $x \ge 0$, and a curve $x = L(T-t)$, $t \in [0,T]$, that satisfy the equations*

$$\begin{cases} rv - v_t - rxv_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = 0, & x \ge L(T-t) \\ v(t,x) = (K - x), & 0 \le x \le L(T-t)], \end{cases}$$

the smooth pasting conditions (6) *and* (7)*, the terminal conditions $L(0) = K$ and $v(T,x) = (K-x)_+$, and the asymptotic condition*

$$\lim_{x\to\infty} v(t,x) = 0.$$

Probabilistic characterization of $v(t,x)$***.*** We shall only state the main result. See Section 8.4.2 for more details.

Theorem 1.4. *Let $(S(u))_{t\le u\le T}$ be the stock price process starting at $S(t) = x$ with the stopping set*

$$\mathcal{S} := \{(t,x) : t \in [0,T], x \ge 0, v(t,x) = (K-x)_+\}.$$

Let $\tau_* = \min\{u \in [t,T] : (u, S(u)) \in S\}$, where $\tau_* = \infty$ if $(u, S(u))$ does not enter $\mathcal{S}$ for any $u \in [t,T]$. Then $(e^{-ru}v(u, S(u)))_{t\le u\le T}$ is a supermartingale under $\widetilde{\mathbb{P}}$ relative to $(\mathcal{F}_u^{(t)})_{t\le u\le T}$ and the stopped process $\left(e^{-r(u\wedge\tau_*)}v(u\wedge\tau_*, S(u\wedge\tau_*))\right)_{t\le u\le T}$ is a martingale.

2. Poisson process.

Review refresher Lecture 6 or textbook section 11.2.

3. Compound Poisson process.

Definition 3.1. *Let $(N(t))_{t\ge0}$ be a Poisson process and $Y_1, Y_2, \ldots$ be i.i.d. random variables independent of $(N(t))_{t\ge0}$. The compound Poisson process $(Q(t))_{t\ge0}$ is defined as follows:*

$$Q(t) = 0, \quad \textit{if } N(t) = 0; \quad Q(t) = \sum_{i=1}^{N(t)} Y_i, \quad \textit{if } N(t) \ge 1.$$

A compound Poisson process has stationary and independent increments, but the distribution of $Q(t+s) - Q(t)$ is not Poisson, it depends on the distribution of Y_1. The following theorem generalizes Lemma 4 of the summer Lecture 6 to compound Poisson processes.

Theorem 3.2. *Let $(Q(t))_{t\ge0}$ be a compound Poisson process and assume that all quantities below are well-defined. Set $\beta = \mathbb{E}Y_1$; $\sigma^2 = VarY_1$, and $M_{Y_1}(u) = \mathbb{E}(e^{uY_1})$. Then*

(i) $\mathbb{E}(Q(t)) = \beta\lambda t$;
(ii) $Var(Q(t)) = (\sigma^2 + \beta^2)\lambda t = \lambda t(\mathbb{E}Y_1)^2$;
(iii) $M_{Q(t)}(u) = \exp(\lambda t(\mu_Y(u) - 1))$;
(iv) *The compensated compound Poisson process $Q(t) - \beta\lambda t$, $t \ge 0$ is a martingale (w.r.t its natural filtration).*

The proof is done by conditioning on the value of $N(t)$ and is left as an exercise.
Theorems 6 and 7 from the summer Lecture 6 have the following generalizations.

Theorem 3.3. *Let $y_1, ..., y_M \in \mathbb{R} \setminus \{0\}$ and $p_1, \dots, p_M > 0$ such that $\sum_{i=1}^M p_i = 1$. Let $\lambda > 0$ be given and $N_1(t), ..., N_M(t)$ be independent Poisson processes with intensities $\lambda p_1, ..., \lambda p_M$ respectively. Then $Q(t) := \sum_{m=1}^M y_m N_m(t)$, $t \geq 0$, is a compound Poisson process whose jump size distribution is given by $P(Y_1 = y_i) = p_i$ and the times of jumps coincide with those of $N(t) := \sum_{m=1}^M N_m(t)$.*

In other words, the process Q has the same law as $\bar{Q}(t) := \sum_{i=1}^{N(t)} \bar{Y}_i$, $t \geq 0$, where $\bar{Y}_1, \bar{Y}_2, \dots$ are i.i.d. and $P(\bar{Y}_1 = y_m) = p_m$, $m = 1, 2, \dots, M$. For a proof see Theorem 11.3.3 of the textbook.

Theorem 3.4. *Let $y_1, ..., y_M \in \mathbb{R} \setminus \{0\}$ and $p_1, \dots, p_M > 0$ such that $\sum_{i=1}^M p_i = 1$. Let $Y_1, Y_2, \dots$ be i.i.d. and $P(\bar{Y}_1 = y_m) = p_m$, $m = 1, 2, \dots, M$, and $N(t)$ be a Poisson process with intensity λ. Define*

$$Q(t) := \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

For $m = 1, 2, \dots, M$, denote by $N_m(t)$ the number of jumps of Q of size y_m up to time t inclusively. Then,

$$N(t) = \sum_{m=1}^M N_m(t) \quad \text{and} \quad Q(t) = \sum_{m=1}^M y_m N_m(y).$$

Moreover, the processes $N_1, \dots, N_M$ are independent Poisson processes with intensities $\lambda p_1, \dots, \lambda p_M$ respectively.

See Corollary 11.3.4 of the textbook.

4. Jump processes as integrators

Definition 4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}(t))_{t \geq 0}$, be a filtration on it. We say that*

(i) *Brownian motion B is a Brownian motion relative to $(\mathcal{F}(t))_{t\geq 0}$ if $B(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$, $B(t) - B(s)$ is independent of $\mathcal{F}(s)$.*
(i) *Poisson process N is a Poisson process relative to $(\mathcal{F}(t))_{t\geq 0}$ if $N(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$, $N(t) - N(s)$ is independent of $\mathcal{F}(s)$.*
(ii) *Compound Poisson process Q is a compound Poisson process relative to $(\mathcal{F}(t))_{t\geq 0}$ if $Q(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$, $Q(t) - Q(s)$ is independent of $\mathcal{F}(s)$.*

Definition 4.2. *A jump process with starting point $X(0)$ is a process of the form*

$$X(t) = X(0) + I(t) + R(t) + J(t),$$

where $X(0)$ is a constant, I is an Itô integral, $I(t) = \int_0^t \Gamma(s)dB(s)$, $\Gamma(\cdot)$ is adapted to $\mathcal{F}(\cdot)$, R is a Riemann integral, $R(t) = \int_0^t \theta(s)ds$, $\theta(\cdot)$ is adapted to $\mathcal{F}(\cdot)$, and J is a pure jump process[1] *such that $\mathcal{J}(0) = 0$. We shall assume that J is right continuous with left limits, i.e.*

$$\mathcal{J}(t) = \lim_{s \downarrow t} \mathcal{J}(s) \quad \text{and} \quad \mathcal{J}(t_-) = \lim_{s \uparrow t} \mathcal{J}(s),$$

that there is no jump at time 0, and that in any finite time interval $(0, T]$, there are only finitely many jumps.

The process $X^c(t) := X(0) + I(t) + R(t)$ is called a continuous parr of $X(t)$. It is just an Itô process. $J(t)$ is a pure jump part of $X(t)$. We shall denote by $\delta\mathcal{J}(t) = \mathcal{J}(t) - \mathcal{J}(t_-)$ the jump size at time t. Then

$$\Delta X(t) = X(t) - X(t-) = J(t) - J(t-) = \Delta J(t).$$

Note that when there is no jump at time t we have $X(0-) = X(0)$ and $\Delta X(t) = 0$.

[1] this means that the process is constant between jumps.

Definition 4.3. *Let X be a jump process and Φ be a process adapted to $(\mathcal{F}(t))_{t\geq 0}$. Then*

$$\int_0^t \Phi(s)dX(s) := \int_0^t \Phi(s)\Gamma(s)dB(s) + \int_0^t \Phi(s)\theta(s)ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s),$$

or, in the differential form,

$$\Phi(t)dX(t) = \underbrace{\Phi(t)\Gamma(t)dB(t) + \Phi(t)\theta(t)dt}_{\Phi(t)dX^c(t)} + \Phi(t)dJ(t).$$

Example 4.4. Let $\Phi(t) = \Delta N(t)$ and $X(t) = N(t) - \lambda t$.

$$\begin{aligned}\int_0^t \Phi(s)dX(s) &= -\lambda \int_0^t \Delta N(s)ds + \int_0^t \Delta N(s)dN(s)\\ &= 0 + \sum_{0<s\leq t} (\Delta N(s))^2 = N(t).\end{aligned}$$

The integrator $X(t)$ is a martingale but the integral is not a martingale!

Theorem 4.5. *Assume that $X(t)$ is a martingale and $\Phi(t)$ is left-continuous and adapted, and*

$$\mathbb{E}\int_0^t \Gamma^2(s)\Phi^2(s)ds < \infty, \quad \forall t \geq 0$$

Then $\int_0^t \Phi(s)dX(s)$, $t \geq 0$, is a martingale with respect to $(\mathcal{F}(t))_{t\geq 0}$.

Remark 4.6. Left-continuity is not really necessary, but it is easy to check. Left-continuity can be replaced by predictability: $\Phi(t)$ is $\mathcal{F}(t-)$-measurable, where $\mathcal{F}(t-)$ is the σ-algebra generated by $\cup_{s<t}\mathcal{F}(s)$.

5. Quadratic and cross variation of jump processes.

Let Π be a finite partition of $[0,t]$: $0 = t_0 < t_1 < \cdots < t_n = t$ and

$$Q_\Pi(x) := \sum_{i=0}^{n-1} (X(t_{j+1}) - X(t_j))^2.$$

If this quantity has (an L^2) limit for all $t \geq 0$ as $\|\Pi\| \to 0$, then the limiting process is denoted $[X,X](t)$, $t \geq 0$, and is called a quadratic variation process. We state without a proof the following fact.

Theorem 5.1. *Let $X_i(t)$, $t \geq 0$, $i = 1,2$, be jump processes as defined above. Then*

$$\begin{aligned}[X_i, X_j](t) &= [X_i^c, X_j^c](t) + [J_i, J_j](t)\\ &= \int_0^t \Gamma_i(s)\Gamma_j(s)ds + \sum_{0<s\leq t} \Delta J_i(s)\Delta J_j(s) \quad (i,j = 1,2).\end{aligned}$$

The important new developments are:

- the quadratic variation of a pure jump process on $(0,t]$ is the sum of the squares of jumps up to time t inclusively.
- the cross-variation of a continuous process X_i^c and a pure jump process J_i (subject to all conditions we imposed on it) is zero $(i,j = 1,2)$.

Corollary 5.2. *Let X be a jump process and Φ be a left-continuous adapted process. Set $Y(t) = \int_0^t \Phi(s)dX(s) + Y(0)$, where $Y(0)$ is a constant. Then*

$$\begin{aligned}[Y,Y](t) &= \int_0^t \Phi^2(s)d[X,X](s) \\ &= \int_0^t \Phi^2(s)\Gamma^2(s)ds + \sum_{0<s\le t} \Phi^2(s)(\Delta J(s))^2.\end{aligned}$$

6. Itô-Doeblin formula for jump processes.

Theorem 6.1. *Let $X(t)$ be a jump process and $f \in C^2(\mathbb{R})$. Then*

$$\begin{aligned}f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))d[X^c,X^c](s) \\ + \sum_{0<s\le t}(f(X(s)) - f(X(s-))).\end{aligned}$$

Read the proof of Theorem 11.5.1 in the textbook.

Remark 6.2. $\int_0^t f'(X(s))dX^c(s)$ can be replaced with $\int_0^t f'(X(s-))dX^c(s)$ and $\int_0^t f''(X(s))d[X^c,X^c](s)$ with $\int_0^t f''(X(s-))d[X^c,X^c](s)$.

Example 6.3 (Geometric Poisson process)**.**

$$S(t) = S(0)e^{-\lambda\sigma t}(1+\sigma)^{N(t)}, \quad t \ge 0,$$

where $\sigma > -1$ is a constant. If $\sigma > 0$ then the process jumps up and moves down between the jumps; if $\sigma \in [-1, 0)$, the process jumps down and moves up between the jumps; if $\sigma = 0$, then $S(t) \equiv S(0)$.

Geometric Poisson process is a martingale. It satisfies $dS(t) = \sigma S(t-)dM(t)$, where $M(t) = N(t) - \lambda t$. Let us derive this formula. We write

$S(t) = S(0)f(X(t))$, where $f(x) = f'(x) = e^x$ and $X(t) = -\lambda\sigma t + N(t)\ln(1+\sigma)$ ($X(t)$ has no Itô part!).

Then

$$dX(t) = -\lambda\sigma dt + \ln(1+\delta)dN(t)$$

and applying Itô's formula we get

$$S(t) = S(0) - \lambda\sigma\int_0^t S(u)du + \sum_{0<u\le t}(S(u) - S(u-)).$$

Looking at one term of a jump sum we find that

$$\begin{aligned}S(u) - S(u-) &= S(0)e^{-\lambda\sigma u}((1+\sigma)^{N(u)} - (1+\sigma)^{N(u-)}) \\ &= S(0)e^{-\lambda\sigma u}(1+\sigma)^{N(u-)}((1+\sigma)^{N(u)-N(u-)} - 1) \\ &= S(u-)\sigma(N(u) - N(u-)) = S(u-)\sigma\Delta N(u).\end{aligned}$$

Summing over all jumps up to time t we get

$$\sum_{0<u\le t}(S(u) - S(u-)) = \sigma\sum_{0<u\le t} S(u-)\Delta N(u) = \sigma\int_0^t S(u-)dN(u).$$

Putting everything together we conclude that

$$\begin{aligned}S(t) &= S(0) - \lambda\sigma\int_0^t S(u-)du + \sigma\int_0^t S(u-)dN(u); \\ &= S(0) + \sigma\int_0^t S(u-)dM(u), \quad \text{or} \quad dS(t) = \sigma S(t-)dM(t)\end{aligned}$$

as claimed.

Example 6.4 (Doléans-Dade exponential)**.** Let $X(t)$ be a jump process. The Doléans-Dade exponential of $X(t)$ is

$$Z(t) := e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0<s\le t} (1 + \Delta X(s)). \tag{8}$$

This is a generalization of previous example:

$$\begin{aligned}
&X(t) = \sigma M(t) = -\sigma\lambda t + \sigma N(t) = X^c(t) + J(t);\\
&(1 + \Delta X(s)) = (1 + \Delta J(s)) = (1 + \sigma \Delta N(s));\\
&\prod_{0<s\le t} (1 + \Delta X(s)) = \prod_{0<s\le t} (1 + \sigma \Delta N(s)) = (1+\sigma)^{N(t)};\\
&\frac{S(t)}{S(0)} = e^{-\lambda\sigma t}(1+\sigma)^{N(t)}.
\end{aligned}$$

We shall show that if Z satisfies

$$dZ(t) = Z(t-)dX(t) \tag{9}$$

then it is given by (8). Indeed, we have

$$Z(t) = \underbrace{Z(0) + \int_0^t Z(s-)dX^c(s)}_{Z^c(t)} + \underbrace{\sum_{0<s\le t} Z(s-)\Delta X(s)}_{J_z(t)}.$$

Applying Itô's formula for jump processes we get

$$\begin{aligned}
\log Z(t) &= \log Z(0) + \int_0^t \frac{1}{Z(s-)} dZ^c(s) - \frac{1}{2}\int_0^t \frac{1}{Z^2(s-)} d[Z^c, Z^c](s) + \sum_{0<s\le t} (\log Z(s) - \log Z(s-))\\
&= \log Z(0) + \int_0^t dX^c(s) - \frac{1}{2}\int_0^t d[X^c, X^c](s) + \sum_{0<s\le t} \log(1 + \Delta X(s)),
\end{aligned}$$

since $\log Z(s) - \log Z(s-) = \log \frac{Z(s)}{Z(s-)} = \log(1 + \Delta X(s))$. Therefore,

$$\log Z(t) = \log Z(0) + X^c(t) - \frac{1}{2}[X^c, X^c](t) + \sum_{0<\le t} \log(1 + \Delta X(s)).$$

Exponentiating, we see that

$$Z(t) = Z(0)e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0<s\le t} (1 + \Delta X(s)).$$

If $X(t)$ is a martingale, then by theorem Theorem 4.5

$$Z(t) = Z(0) + \int_0^t Z(s-)dX(t)$$

is also a martingale since the integrand is left-continuous.

Example 6.5 (Merton's jump diffusion model)**.** Let $X(t) = \mu t + \sigma B(t) + Q(t)$, where $Q(t)$ is a compound Poisson process $Q(t) = \sum_{i=1}^{N(t)} Y_i$. The stock price dynamics is

$$S(t) = S(0) + \int_0^t S(u)dX^c(u) + \sum_{0<u\le t} S(u-)\Delta X(u).$$