question 7

expectation

$$E(\int_0^t B^2(s)ds) = \int_0^t E(B^2(s))ds$$

we have

$$E(B^2(s)) = s$$

then

$$E(\int_0^t B^2(s)ds) = \int_0^t E(B^2(s))ds = \int_0^t sds = \frac{t^2}{2}$$

variance

$$Var = E((\int_0^t B^2(s)ds - \frac{t^2}{2})^2) = E((\int_0^t B^2(s)ds)^2 + (\frac{t^2}{2})^2 - 2(\frac{t^2}{2}\int_0^t B^2(s)ds))$$

$$Var = E((\int_0^t B^2(s)ds)^2) - \frac{t^4}{4}$$

$$E((\int_0^t B^2(s)ds)^2) = E(\int_0^t \int_0^t B^2(s)B^2(w)dsdw)$$

then

$$E((\int_0^t B^2(s)ds)^2) = \int_0^t \int_0^t E(B^2(s)B^2(w))dsdw$$

when $w > s$

$$E[B^2(s)B^2(w)] = E[B^2(s)(B^2(w) - B^2(s)) + B^4(s)]$$

we know $B^2(s)$ and $(B^2(w) - B^2(s))$ are independent,then

$$E[B^2(s)B^2(w)] = E[B^2(s)]E[(B^2(w) - B^2(s))] + E[B^4(s)]$$

we also know the kurtosis for normal distribution is 3
then

$$E(B^4(s)) = 3(E(B^2(s)))^2 = 3s^2$$

so

$$E[B^2(s)B^2(w)] = s(w-s) + 3s^2 = sw + 2s^2$$

similarly,when $s > w$,we have

$$E[B^2(s)B^2(w)] = w(s-w) + 3w^2 = sw + 2w^2$$

$$\int_0^t \int_0^t E(B^2(s)B^2(w))dsdw$$

$$= \int_0^t (\int_0^w (sw + 2s^2)ds + \int_w^t (sw + 2w^2)ds)dw$$

$$= \int_0^t (-\frac{4w^3}{3} + \frac{wt^2}{2} + 2w^2t)dw$$

$$= \frac{7t^4}{12}$$

then

$$Var = \frac{7t^4}{12} - \frac{t^4}{4} = \frac{t^4}{3}$$