MTH 9831 Assignment 6 (10/18/2018 - 10/24/2018).

Read Lecture 6. Additional references for this material are:

1. S. Shreve, Stochastic Calculus for Finance II, Sections 5.2 and 5.3.
2. A. Etheridge, A Course in Financial Calculus, Section 4.5 and 4.6.

(1) (Interpretation of $N(d_+(T,x))$ using Girsanov's theorem) Exercise 5.3.

(2) (Binomial Representation Theorem) Consider a binomial model with parameters u, d, r; $d < 1+r < u$. Let $\tilde{p}$ be the risk-neutral probability of the stock to go up in any one period (stock movements are assumed to be independent) and S_n be the stock price at time n. We know that the discounted stock price process $(\tilde{S}_n)_{n\geq 0}$, where $\tilde{S}_n = (1+r)^{-n}S_n$, is a martingale (with respect to its natural filtration and the risk-neutral measure). We also know that the discrete stochastic integral of a bounded predictable sequence with respect to $(\tilde{S}_n)_{n\geq 0}$ is again a martingale (see refresher lecture 5, Theorem 1.19). The converse of this statement also holds and is known as the Binomial Representation Theorem: *Let $(\tilde{V}_n)_{n\geq 0}$ be a martingale with respect to the natural filtration of $(\tilde{S}_n)_{n\geq 0}$. Then there is a predictable process $(H_n)_{n\geq 1}$ such that*

$$\tilde{V}_n = \tilde{V}_0 + \sum_{i=1}^{n} H_i(\tilde{S}_i - \tilde{S}_{i-1}).$$

Check that the process defined by

$$H_n(\omega_1\omega_2\ldots\omega_{n-1}\omega_n) = \frac{V_n(\omega_1\omega_2\ldots\omega_{n-1}u) - V_n(\omega_1\omega_2\ldots\omega_{n-1}d)}{S_n(\omega_1\omega_2\ldots\omega_{n-1}u) - S_n(\omega_1\omega_2\ldots\omega_{n-1}d)}$$

satisfies the requirements. Here $(\omega_1\omega_2\ldots\omega_n)$ represents a path on the n-step binomial tree, $\omega_i \in \{u,d\}$.

The significance of this representation is that if we think of $\tilde{V}_n$ as the discounted value of a contingent claim at time n then the process $(H_n)_{n\geq 1}$ is the hedging strategy (H_n is the number of shares of the underlying stock in the replicating portfolio to be held over the n-th period). So this proves that every contingent claim can be hedged and tells you exactly how (in this model). Compare with the Martingale Representation Theorem from lecture 6.

(3) (Martingale Representation Theorem) Exercise 5.5.

(4) (Every strictly positive price process is a generalized GBM) Exercise 5.8.

(5) (Finding the risk-neutral distribution from call prices across all strikes) Exercise 5.9.

(6) (Chooser option) Exercise 5.10.

(7) (Hedging a cash flow) Exercise 5.11.