Lim, sup, and max.

Below a_n, b_n $n \in \mathbb{N}$, are sequences of real numbers. Recall one of the definitions of lim sup:

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}(\sup_{k\geq n} a_k).$$

It always exists (can be $\pm\infty$), since the sequence $s_n := \sup_{k\geq n} a_k$ is monotone non-increasing.

Exercise 1. (4 points) Suppose that

$$\limsup_{n\to\infty} a_n = L \quad \text{and} \quad \liminf_{n\to\infty} a_n = -L \quad (\text{i.e. } \limsup_{n\to\infty}(-a_n) = L).$$

Show that $L \geq 0$ and $\limsup\limits_{n\to\infty} |a_n| = \limsup\limits_{n\to\infty}(a_n \vee (-a_n)) = L$. Redo Exercise 3 from Lecture 2.

Solution. By the definiton of lim sup, $\forall \epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\forall n \geq N$

$$a_n < L + \epsilon \quad \text{and} \quad -a_n < L + \epsilon.$$

Combining the last two inequalities we get that $\forall n \geq N$, $|a_n| < L + \epsilon$. In particular, $L \geq 0$ (if L were negative we could choose $\epsilon = |L|/2$ and get a contradiction). Unsing again the definition of lim sup we conclude that $\limsup_{n\to\infty} |a_n| \leq L$. On the other hand,

$$L = \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} |a_n|.$$

This gives the desired equality.

Exercise 2. (3 points) Is it always the case that

$$\limsup_{n\to\infty}(a_n \vee b_n) = (\limsup_{n\to\infty} a_n) \vee (\limsup_{n\to\infty} b_n)?$$

Give a proof or a counterexample.

Solution. One way the inequality is obvious: since

$$\limsup_{n\to\infty}(a_n \vee b_n) \geq \limsup_{n\to\infty} a_n$$

and also

$$\limsup_{n\to\infty}(a_n \vee b_n) \geq \limsup_{n\to\infty} b_n,$$

we get

$$\limsup_{n\to\infty}(a_n \vee b_n) \geq \limsup_{n\to\infty} a_n \vee \limsup_{n\to\infty} b_n.$$

The opposite inequality follows from the definition. Let $\limsup_{n\to\infty} a_n = a$ and $\limsup_{n\to\infty} b_n = b$. If either a or b is infinite then there is nothing to prove. So we assume that $a, b \in \mathbb{R}$. By the definiton, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$a_n \leq a + \epsilon \quad \text{and} \quad b_n \leq b + \epsilon.$$

This implies that $\forall n \geq N$, $a_n \vee b_n \leq a \vee b + \epsilon$. Therefore, again by the definiton we conclude that $\limsup_{n\to\infty}(a_n \vee b_n) \leq a \vee b$, and we are done.

Note that this proof will work not only for two sequences but for any **finite** number of them.

Exercise 3. (3 points) Is it always the case that

$$\limsup_{n\to\infty} \sup_{m\in\mathbb{N}} a(m,n) = \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} a(m,n)?$$

Give a proof or a counterexample.

Solution. Now we have infinitely many sequences: for each $m \in \mathbb{N}$ we have a sequence $a(m,n)$, $n \in \mathbb{N}$. The statement is false in general. Looking back at the proof in the previous exercise, we see that the first part will work all the same, that is

$$\limsup_{n\to\infty} \sup_{m\in\mathbb{N}} a(m,n) \geq \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} a(m,n).$$

The problem is with the opposite inequality: the number N depends on m (for $m = 1$ it is N_1, for $m = 2$ it is N_2, and so on). $N = N(m)$ can become larger and larger with m so that there is no single N which will work for all m. When we have only two or finitely many sequences (i.e. choices of m) this problem does not arise as a finite set of positive integers always has a finite maximum which we can call N.

For a counterexample, take $a(m,n) = (m \wedge n)/n$. Then

$$\limsup_{n\to\infty} \sup_{m\in\mathbb{N}} a(m,n) = 1 > 0 = \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} a(m,n).$$