MTH 9831. LECTURE 7

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ABSTRACT. In this lecture we shall work in the setting of multidimensional market model introduced in lecture 6.

1. Existence of the risk-neutral measure.
   - Market price of risk equations.
   - Definition of arbitrage.
   - The first fundamental theorem of asset pricing.
2. Uniqueness of the risk-neutral measure.
   - Complete market.
   - The second fundamental theorem of asset pricing.
3. Change of Numeraire.
   - Stochastic representation of assets.
   - Change of risk-neutral measure.
   - Example: zero-coupon bond as numéraire.

1. EXISTENCE OF THE RISK-NEUTRAL MEASURE.

Recall the multidimensional market model. It consists of m stocks (driven by d independent Brownian motions with the associated filtration $(\mathcal{F}(t))_{0\le t\le T}$) and a money market account (MMA).

- MMA: interest rate $(R(t))_{0\le t\le T}$, a non-negative adapted process. The discounting factor $D(t)=e^{-\int_0^t R(u)\,du}$, $0\le t\le T$, satisfies $D'(t)=-R(t)D(t)$, or $dD(t)=-R(t)D(t)dt$, and is a regular process.
- m stocks with the price vector $S(t)=(S_1(t),S_2(t),\dots,S_m(t))^T$, $0\le t\le T$. We assume that for some adapted vector process $\alpha(t)=(\alpha_1(t),\alpha_2(t),\dots,\alpha_m(t))^T$, $0\le t\le T$, and an $m\times d$ matrix process $\sigma(t)=\|\sigma_{ij}(t)\|_{i=1,2,\dots,m;j=1,2,\dots,d}$ such that

$$\sigma_i^2(t):=\sum_{j=1}^d \sigma_{ij}^2(t)>0 \ \text{ for all } 0\le t\le T \text{ (a.s.) and } i=1,2,\dots,m,$$

  the price vector satisfies

$$dS_i(t)=\alpha_i(t)S_i(t)\,dt+S_i(t)\sum_{j=1}^d \sigma_{ij}(t)\,dB_j(t),\quad m=1,2,\dots,m. \tag{1.1}$$

We have shown that each of $(S_i(t))_{0\le t\le T}$, $i=1,2,\dots,m$, is a generalized geometric Brownian motion with the drift process $(\alpha_i(t))_{0\le t\le T}$ and the volatility process $(\sigma_i(t))_{0\le t\le T}$.

Definition 1.1. *A probability measure $\widetilde{P}$ is said to be* ***risk-neutral*** *if*

(i) *$\widetilde{P}$ is equivalent to P and*

(ii) *under $\widetilde{P}$ each of the processes $(D(t)S_i(t))_{0\le t\le T}$, $i=1,2,\dots,m$, is an $\mathcal{F}(t)$-martingale.*

Using Itô-Doeblin formula, from the equations for $dD(t)$ and $dS_i(t)$ we obtain

$$d(D(t)S_i(t))=D(t)S_i(t)\left((\alpha_i(t)-R(t))\,dt+\sum_{i=1}^d \sigma_{ij}(t)\,dB_j(t)\right). \tag{1.2}$$

But we would like to have

$$d(D(t)S_i(t)) = D(t)S_i(t)\sum_{i=1}^{d}\sigma_{ij}(t)\underbrace{(\Theta_j(t)\,dt + dB_j(t))}_{=d\widetilde{B}_j(t)} \tag{1.3}$$

for some process $\Theta(t) = (\Theta_1(t), \Theta_2(t), \ldots, \Theta_d(t))^T$, $0 \le t \le T$. This would allow us to use Girsanov's theorem and obtain a risk-neutral measure. Comparing (1.2) and (1.3) we arrive at the following system of equations (**market price of risk equations**):

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t), \quad i = 1, 2, \ldots, m. \tag{1.4}$$

This is a linear system of m equations in d unknown processes $(\Theta_i(t))_{0\le t\le T}$, $i = 1, 2, \ldots, d$. The bottom line is that if this system has no solutions then there is an arbitrage "built" into the model, and the model should not be used for pricing. Let us first define arbitrage.

Definition 1.2. *An* ***arbitrage*** *is a portfolio value process $X(t)$ satisfying the following conditions:*

(i) $X(0) = 0$;
(ii) $P(X(T) \ge 0) = 1$;
(iii) $P(X(T) > 0) > 0$.

Example 1.3. Let $m = 2$, $d = 1$, and all parameters be constant. Then the market price of risk equations are

$$\begin{cases} \alpha_1 - r = \sigma_1\Theta_1 \\ \alpha_2 - r = \sigma_2\Theta_1. \end{cases}$$

This system is consistent if and only if

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}.$$

The common value is then Θ_1. Let us construct an arbitrage explicitly when the last equality does not hold. Set

$$\mu := \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2}.$$

Suppose that $\mu \ne 0$. Intuition tells us that if, say, $\mu > 0$ we should buy the first asset and sell the second one by borrowing/depositing money from/to the MMA as necessary. The question is: what is the "winning" proportion of the assets at each time t? The text provides one such "winning" portfolio, $\Delta_1(t) = (S_1(t)\sigma_1)^{-1}$ and $\Delta_2(t) = (S_2(t)\sigma_2)^{-1}$, and shows that it works (p. 229). Before proceeding with the check as in the textbook, let us see how one can arrive at this answer. The idea is to choose $\Delta_1(t)$ and $\Delta_2(t)$ in such a way that the "noise term" $dB(t)$ in the dynamics cancels out. This will allow us to synthesize a riskless asset with the rate higher than r. We set $X(t) = \Delta_1(t)S_1(t)+\Delta_2(t)S_2(t)+\Gamma(t)M(t)$ and just in the case of a single stock model get (under an appropriate self-financing condition)

$$dX(t) = \Delta_1(t)dS_1(t) + \Delta_2(t)dS_2(t) + r(X(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t))dt.$$

From the above and the equations for the stock prices we find that

$$\begin{aligned} d(D(t)X(t)) = D(t)&(\sigma_1\Delta_1(t)S_1(t) + \sigma_2\Delta_2(t)S_2(t))dB(t) \\ &+ D(t)((\alpha_1 - r)\Delta_1(t)S_1(t) + (\alpha_2 - r)\Delta_2(t)S_2(t))dt. \end{aligned} \tag{1.5}$$

Setting the coefficient in front of $dB(t)$ to zero we arrive at the equation which gives us the "winning" proportions

$$\sigma_1\Delta_1(t)S_1(t) = -\sigma_2\Delta_2(t)S_2(t) \quad \text{or} \quad \frac{\Delta_1(t)}{\Delta_2(t)} = -\frac{\sigma_2 S_2(t)}{\sigma_1 S_1(t)}. \tag{1.6}$$

Notice that the textbook choice satisfies the above equation. Then

$$d(D(t)X(t)) = D(t)((\alpha_1 - r)\Delta_1(t)S_1(t) + (\alpha_2 - r)\Delta_2(t)S_2(t))dt = \mu\sigma_1 D(t)\Delta_1(t)S_1(t)dt,$$

where $\sigma_1,\ D(t),\ S_1(t) > 0$. If $\mu > 0$ then we should choose $\Delta_1(t) > 0$ to ensure a sure win (as opposed to sure loss), and if $\mu < 0$ we should choose $\Delta_1(t) < 0$. In either case the proportion of each asset is determined by (1.6).

Now assume that $\mu > 0$ and make the textbook choice of $\Delta_1(t)$ and $\Delta_2(t)$. The initial price of the portfolio is $1/\sigma_1 - 1/\sigma_2$. If this is positive we borrow this amount from the MMA (remember the self-financing condition), if this is negative we deposit $1/\sigma_2 - 1/\sigma_1$ to the MMA. Therefore, at time 0 we have $X(0) = 0$ (no cost to us). With our choice of $\Delta_1(t)$ and $\Delta_2(t)$ we get

$$d(D(t)X(t)) = \mu D(t)dt.$$

The right-hand side is positive and is also non-random. This portfolio will make money faster than at rate r. We synthesized a second money market account with an additional return. All we have to do is to borrow at rate r from the old MMA and earn more by making deposits to the new one.

Exercise 1. Let $X(t) = \Delta_1(t)S_1(t) + \Delta_2(t)S_2(t) + \Gamma(t)M(t)$. Spell out the appropriate self-financing condition and derive (1.5).

Theorem 1.4 (First fundamental theorem of asset pricing)**.** *If a market model has a risk-neutral measure, then it does not admit arbitrage.*

For the proof we shall need the following lemma.

Lemma 1.5. *Let $\widetilde{P}$ be a risk-neutral measure and $X(t)$ be the value of a portfolio at time t. Under $\widetilde{P}$ the process $(D(t)X(t))_{0\le t\le T}$ is a martingale.*

Recall that we have already proved (see Section 2 of Lecture 6) such a result for a model with one stock and an MMA.

Exercise 2. Use Itô's product rule and equations for portfolio and stock dynamics to show that

$$d(D(t)X(t)) = \sum_{i=1}^{m} \Delta_i(t)\, d(D(t)S_i(t)). \tag{1.7}$$

Use the definition of a risk-neutral measure to infer the statement of the above lemma.[1]

Proof of Theorem 1.4. Let $(X(t))_0 \le t \le T$ be an arbitrary portfolio value process.[2] If there is a risk-neutral measure $\widetilde{P}$ then by Lemma 1.5 $(D(t)X(t))_{0\le t\le T}$ is a $\widetilde{P}$-martingale. In particular, if $X(0) = 0$ then

$$\widetilde{E}(D(T)X(T)) = \widetilde{E}(D(0)X(0)) = X(0) = 0. \tag{1.8}$$

We shall assume that $(X(t))_{0\le t\le T}$ satisfies (ii) of Definition 1.2 and show that then (iii) must fail.

Let $P(X(T) \ge 0) = 1$. Since $\widetilde{P} \sim P$,

$$\widetilde{P}(X(T) \ge 0) = 1 \quad \Leftrightarrow \quad \widetilde{P}(X(T) \ge 0) = 1.$$

If, in addition, $\widetilde{P}(X(T) > 0) > 0$ then ($D(T)$ is always positive) $\widetilde{P}(D(T)X(T) > 0) > 0$ and $\widetilde{E}(D(T)x(T)) > 0$. The last inequality contradicts (1.8). Therefore, $\widetilde{P}(X(T) > 0) = 0$, and (iii) of Definition 1.2 is not satisfied. $\square$

[1] Reference: p. 230 of the text.

[2] We always assume that the process is self-financing.

2. Uniqueness of the risk-neutral measure.

How many risk-neutral measures can there be for a given model. Example 1.3 shows that there might be none, and in such case we constructed an arbitrage. An overall conclusion was that models without a risk-neutral measure should be avoided. In the same example we can have a situation when the process $(\Theta(t))_{0\le t\le T}$ is not unique. Different Θ processes can lead to different risk-neutral measures. So there can be infinitely many risk-neutral measures for the same model. Then how does one even approach pricing if different risk-neutral measures could give different prices? The key to the problem is hedging. Any derivative security which can be hedged will have a unique price, so that the prices computed under different risk-neutral measures for this derivative security will be the same.

Definition 2.1. *A market model is* ***complete*** *if every derivative security can be hedged (i.e. replicated).*

Assume that in our multidimensional stock model the filtration $(\mathcal{F})_{0\le t\le T}$ is generated by $(B(t))_{0\le t\le T}$ so that the Martingale Representation Theorem is applicable. Suppose that $(\Theta(t))_{0\le t\le T}$ is a solution to the market price of risk equations and $\widetilde{P}$ is the corresponding risk-neutral measure. Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable which represents a payoff of some derivative security. We would like to hedge a short position in this security. Define $(V(t))_{0\le t\le T}$ by

$$V(t)=\frac{1}{D(t)}\,\widetilde{E}(D(T)V(T)\,|\,\mathcal{F}(t)),\quad 0\le t\le T.$$

Then $(D(t)V(t))_{0\le t\le T}$ is a martingale under $\widetilde{P}$. By the Martingale Representation Theorem there is a process $(\widetilde{\Gamma}(t))_{0\le t\le T}$, $\widetilde{\Gamma}(t)=(\widetilde{\Gamma}_1(t),\widetilde{\Gamma}_2(t),\dots,\widetilde{\Gamma}_d(t))$, such that

$$D(t)V(t)=D(0)V(0)+\int_0^t\sum_{j=1}^d\widetilde{\Gamma}_j(u)\,d\widetilde{B}_j(u),\quad 0\le t\le T.$$

Consider a portfolio value process with the initial value $X(0)$. By (1.7) we have

$$\begin{aligned} d(D(t)X(t)) &= \sum_{i=1}^m \Delta_i(t)\,d(D(t)S_i(t)) \\ &\overset{(1.3)}{=} \sum_{j=1}^d\sum_{i=1}^m \Delta_i(t)D(t)S_i(t)\sigma_{ij}(t)\,d\widetilde{B}(t), \end{aligned}$$

or, in the integral form,

$$D(t)X(t)=X(0)+\int_0^t\sum_{j=1}^d\sum_{i=1}^m\Delta_i(u)D(u)S_i(u)\sigma_{ij}(u)\,d\widetilde{B}(u).$$

Therefore, to hedge a short position we need $X(0)=V(0)$ and $(\Delta_1(t),\Delta_2(t),\dots,\Delta_m(t))_{0\le t\le T}$ which satisfy

$$\widetilde{\Gamma}_j(t)=\sum_{i=1}^m\Delta_i(t)D(t)S_i(t)\delta_{ij}(t),$$

or

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)}=M(t)\widetilde{\Gamma}_j(t)=\sum_{i=1}^m\Delta_i(t)S_i(t)\delta_{ij}(t),\quad j=1,2,\dots,d. \tag{2.1}$$

We have d equations for m unknown processes. Compare (2.1) with (1.4). If we write (1.4) as

$$Ax=b,\quad \text{where } A=\sigma(t),\ x=\Theta(t),\ b=\begin{pmatrix}\alpha_1(t)-R(t)\\ \alpha_2(t)-R(t)\\ \dots\\ \alpha_m(t)-R(t)\end{pmatrix}$$

then (2.1) is of the form $A^T y = c$ where

$$y = \begin{pmatrix} \Delta_1(t)S_1(t) \\ \Delta_2(t)S_2(t) \\ \dots \\ \Delta_m(t)S_m(t) \end{pmatrix}, \quad c = M(t)(\widetilde{\Gamma}(t))^T.$$

You should know from the linear algebra course that if $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$ then $A^T y = c$ has at least one solution for every $c \in \mathbb{R}^d$ (look up the rank-nullity theorem or see Appendix C of the text). This fact plays the key role in the proof of the following theorem.

Theorem 2.2 (Second fundamental theorem of asset pricing)**.** *Suppose that a market model has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.*

The proof is omitted (see pp. 233-234 of the text for a sketch).

Exercise 3. Show that if a derivative security can be hedged then its price under any risk-neutral measure is the same. Hint: use the same idea as in the first part of the proof of Theorem 2.2 in the text.

3. Change of Numéraire.

Definition 3.1. *A **numéraire** is the unit of account in which other assets ore denominated.*

Usually the numéraire is the domestic currency. Other possible numéraires are foreign currency, domestic MMA, foreign MMA, zero-coupon bond maturing at time T. In fact, any positively priced asset or derivative can serve as a numéraire.

We shall continue to work with the multidimensional market model with filtration $(\mathcal{F}(t))_{t\geq 0}$ generated by the driving d-dimensional Brownian motion B. In addition, we assume that there is a unique risk-neutral measure $\widetilde{P}$. Recall that under $\widetilde{P}$ the discounted stock prices $(D(t)S_i(t))_{0\leq t\leq T}$ are $\mathcal{F}(t)$-martingales. Notice that $D(t)S_i(t) = S_i(t)/M(t)$, so we can say that the value of the i-th asset denominated in shares of the MMA is a martingale under $\widetilde{P}$. We can also express this by stating that the measure $\widetilde{P}$ is risk-neutral for the MMA numéraire.

When one changes the numérare, the value of the asset in the units of the new numéraire is no longer martingale under $\widetilde{P}$. Therefore, one needs to change the risk-neutral measure in order to maintain the risk neutrality. This is done in the next two statements.

Consider any positively priced asset which pays no dividends[3].

Theorem 3.2. *Let N be a strictly positive price process for a non-dividend-paying asset in the multidimensional market model. Then there exists a (row) vector volatility process $\nu(t) = (\nu_1(t), \nu_2(t), \dots, \nu_d(t))$ such that*

$$dN(t) = R(t)N(t)\,dt + N(t)\nu(t)\,d\widetilde{B}(t). \tag{3.1}$$

This equation is equivalent to the equations

$$d(D(t)N(t)) = D(t)N(t)\nu(t)\,d\widetilde{B}(t); \tag{3.2}$$

$$D(t)N(t) = N(0)\exp\left(\int_0^t \nu(u)\,d\widetilde{B}(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\right); \tag{3.3}$$

$$N(t) = N(0)\exp\left(\int_0^t \nu(u)\,d\widetilde{B}(u) + \int_0^t \left(R(u) - \frac{1}{2}\|\nu(u)\|^2\right)du\right). \tag{3.4}$$

[3]Currency pays a dividend in the form of an interest which can be considered as invested in the MMA. Recall that our original risk-neutral measure corresponds to the MMA as a numéraire.

Proof. Under $\widetilde{P}$, every discounted price process $D(t)N(t)$ is a martingale. By the Martingale Representation Theorem, there is an adapted d-dimensional (row) vector process $\widetilde{\Gamma}(t) = (\widetilde{\Gamma}_1(t), \widetilde{\Gamma}_2(t), \dots, \widetilde{\Gamma}_d(t))$ such that

$$d(D(t)N(t)) = \sum_{j=1}^{d} \widetilde{\Gamma}_j(t)\, d\widetilde{B}_j(t).$$

Since we have assumed that $N(t) > 0$, we can define

$$\nu_j(t) = \frac{\widetilde{\Gamma}_j(t)}{D(t)N(t)}, \quad j = 1, 2, \dots, d$$

and get

$$d(D(t)N(t)) = D(t)N(t) \sum_{j=1}^{d} \widetilde{\nu}_j(t)\, d\widetilde{B}_j(t) = D(t)N(t)\nu(t)\, d\widetilde{B}(t),$$

which is (3.2). The latter is equivalent to (3.1) by the Itô product formula and the fact that $dD(t) = -R(t)D(t)\, dt$. The rest of the statement is a basic exercise in stochastic calculus. $\square$

Exercise 4. Complete the proof of the above theorem. Hint: write the equation for $\ln(D(t)N(t))$ and integrate to get (3.3). To go the other way, apply Itô-Doeblin formula for Itô processes.

Take a careful look at the expression in (3.3) and think about the process $Z(t)$ in Girsanov's formula. If $D(t)N(t)/N(0)$ is to be $Z(t)$ then what is the drift process that appears in Girsanov's formula? Right, $-\nu(t)$.

Why do we care about this particular change of measure? What will it do? We shall see in the next theorem that under this measure all stock price processes denominated in the units of $N(t)$, i.e. $S_i^{(N)}(t) = S_i(t)/N(t)$, $i = 1, 2, \dots, m$, are going to be martingales. In other words, this new measure will be the risk-neutral measure for $N(t)$ as a numéraire.

Define $Z(t) = D(t)N(t)/N(0)$, $0 \le t \le T$, and set

$$\widetilde{P}^{(N)}(A) = \int_A Z\, d\widetilde{P} = \frac{1}{N(0)} \int_A D(T)N(T)\, d\widetilde{P} \quad \text{for all } A \in \mathcal{F} = \mathcal{F}(T).$$

Then by the definition of $Z(t)$ (see(3.3)) and Girsanov's theorem the process

$$\widetilde{B}^{(N)}(t) = \widetilde{B}(t) - \int_0^t \nu(u)\, du, \quad 0 \le t \le T,$$

is a standard d-dimensional Brownian motion under $\widetilde{P}^{(N)}$.

Theorem 3.3. *Let $S(t)$ and $N(t)$ be the prices of two assets denominated in a common currency, and let $\sigma(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_d(t))$ and $\nu(t) = (\nu_1(t), \nu_2(t), \dots, \nu_d(t))$ denote their respective volatility (row) vector processes, i.e.*

$$d(D(t)S(t)) = D(t)S(t)\sigma(t)\, d\widetilde{B}(t), \quad d(D(t)N(t)) = D(t)N(t)\nu(t)\, d\widetilde{B}(t).$$

Take $N(t)$ as the numéraire, so that the price of $S(t)$ becomes $S^{(N)}(t) = S(t)/N(t)$. Under the measure $\widetilde{P}^{(N)}$, the process $(S^{(N)}(t))_{0\le t\le T}$ is a martingale. Moreover,

$$dS^{(N)}(t) = S^{(N)}(t)(\sigma(t) - \nu(t))\, d\widetilde{B}^{(N)}(t), \quad 0 \le t \le T. \tag{3.5}$$

Before proving this theorem let us consider some simple special cases and just check that we get reasonable and familiar conclusions.

- If $S(t) := N(t)$ then $S^{(N)}(t) \equiv 1$, and its volatility vector $\sigma(t)$ is the zero vector.
- If $N(t) := M(t)$ then $D(t)N(t) \equiv 1$, and its volatility vector $\nu(t)$ is the zero vector. In this case $S^{(N)}(t) = D(t)S(t)$, that is we are in our standard setting.

Remark 3.4 (Remark 9.2.5 on p. 380 of the text and Exercise 9.1)**.** The above theorem is a special case of a more general result. Whenever M_1 and M_2 are $\mathcal{F}(t)$-martingales under a measure P, $M_2(0) = 1$, and M_2 is strictly positive, then $(M_1(t)/M_2(t))_{0\le t\le T}$ is a martingale under the measure $P^{(M_2)}$ defined by

$$P^{(M_2)}(A) = \int_A M_2(T)\, dP \quad \text{for all } A \in \mathcal{F}(T).$$

Proof of Theorem 3.3. Solving the equations for $D(t)S(t)$ and $D(t)N(t)$ we get

$$D(t)S(t) = S(0)\exp\left(\int_0^t \sigma(u)\, d\widetilde{B}(u) - \frac{1}{2}\int_0^t \|\sigma(u)\|^2\, du\right),$$
$$D(t)N(t) = \exp\left(\int_0^t \nu(u)\, d\widetilde{B}(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\, du\right),$$

and, hence,

$$S^{(N)}(t) = \frac{S(0)}{N(0)}\exp\left(\int_0^t (\sigma(u)-\nu(u))\, d\widetilde{B}(u) - \frac{1}{2}\int_0^t (\|\sigma(u)\|^2 - \|\nu(u)\|^2)\, du\right).$$

$S^{(N)}$ is a constant times the exponential function of an Itô process, or symbolically $S^{(N)}(t) = Ce^{X(t)}$. Applying Itô-Doeblin formula for functions of Itô processes we get that

$$\begin{aligned}
dS^{(N)}(t) &= S^{(N)}(t)dX(t) + \frac{1}{2}S^{(N)}(t)d[X]_t \\
&= S^{(N)}(t)((\sigma(t)-\nu(t))\, d\widetilde{B}(t) - \frac{1}{2}(\|\sigma(t)\|^2 - \|\nu(t)\|^2)\, dt) + \frac{1}{2}S^{(N)}(t)\|\sigma(t)-\nu(t)\|^2\, dt \\
&= S^{(N)}(t)(\sigma(t)-\nu(t))\, d\widetilde{B}(t) - (\nu(t)\cdot\sigma(t) - \|\nu(t)\|^2)\, dt \\
&= S^{(N)}(t)(\sigma(t)-\nu(t))(d\widetilde{B}(t) - \nu(t)\, dt) = S^{(N)}(t)(\sigma(t)-\nu(t))d\widetilde{B}^{(N)}(t).
\end{aligned}$$

Since $\widetilde{B}^{(N)}$ is a standard d-dimensional Brownian motion under $\widetilde{P}^{(N)}$, the process $S^{(N)}$ is a martingale under this measure. □

Example 3.5 (Zero-Coupon bond as Numéraire)**.** Let $B(t,T)$ be the time t value of a zero-coupon bond with maturity T. Since it is an asset, $(D(t)B(t,T))_{0\le t\le T}$ is a martingale under the risk-neutral measure $\widetilde{P}$. By Theorem 3.2 there is a vector-valued volatility process $\sigma^*(t,T)$ such that

$$d(D(t)B(t,T)) = \sigma^*(t,T)D(t)B(t,T)\, d\widetilde{B}(t).$$

Definition 3.6. *Let T be a fixed maturity date. The T-forward measure $\widetilde{P}^T$ is defined by*

$$\widetilde{P}^T(A) = \frac{1}{B(0,T)}\int_A D(T)\underbrace{B(T,T)}_{=1}\, d\widetilde{P} \quad \textit{for all } A \in \mathcal{F} = \mathcal{F}(T).$$

The T-forward measure corresponds to taking the zero-coupon bond as a numéraire, i.e. setting $N(t) := B(t,T)$. In particular, $N(T) = B(T,T) = 1$. According to Theorem 3.3, the process

$$\widetilde{B}^T(t) = \widetilde{B}(t) - \int_0^t \sigma^*(u,T)\, du$$

is a Brownian motion under $\widetilde{P}^T$. Moreover, under the T-forward measure, all assets denominated in units of zero-coupon bond maturing at time T are martingales. To repeat, *T-forward prices are martingales under the T-forward measure $\widetilde{P}^T$.* The reason for introducing the T-forward measure is that the risk-neutral pricing formula might become simpler under $\widetilde{P}^T$. Consider a contract with payoff $V(T)$ at time T. On the one hand, under $\widetilde{P}$

$$V(t) = \frac{1}{D(t)}\widetilde{E}(D(T)V(T)\,|\,\mathcal{F}(t)). \tag{3.6}$$

It is often the case that $D(T)$ and $V(T)$ are correlated, especially when the derivative security depends on the interest rate. On the other hand, choosing $B(t,T)$ as a numéraire and switching from $\widetilde{P}$ to the T-forward measure we find by Lemma 5.5 of Lecture 5 (where $Y = V(T)$, t is replaced with T, s is replaced with t, and $Z(t) = D(t)B(t,T)/B(0,T)$) that

$$\widetilde{E}^T(V(T)\,|\,\mathcal{F}(t)) = \frac{1}{D(t)B(t,T)}\,\widetilde{E}(D(T)V(T)\,|\,\mathcal{F}(t)) \overset{(3.6)}{=} \frac{1}{B(t,T)}\,V(t).$$

This gives us the formula

$$V(t) = B(t,T)\widetilde{E}^T(V(T)\,|\,\mathcal{F}(t)).$$

This looks much better than (3.6) provided that we can find a simple model for the evolution of assets under the T-forward measure.[4]

[4]Further reading: Sections 9.4.1 and 9.4.3 of the textbook.