# MTH 9831. LECTURE 4

ELENA KOSYGINA

Abstract. This lecture is devoted to Itô-Doeblin formula. It is also known as "the chain rule of stochastic calculus". A separate handout on applications will be also posted.

1. Itô-Doeblin formula for functions of Brownian motion.
2. Itô-Doeblin formula for functions which depend on time and Brownian motion.
3. Itô-Doeblin formula for Itô processes.
4. Some applications:
   (i) Itô's integral for deterministic integrands.
   (ii) Computation of expectations and variances of some Itô processes.
   (iii) Lévy's characterization of Brownian motion.

## 1. Itô-Doeblin formula for $f(B(t))$.

The idea in a nutshell is this. If $f$ and $g$ are smooth functions on $[0,T]$ then as $t\to 0$ we can write

$$f(g(t))-f(g(0))=f'(g(0))(g(t)-g(0))+\frac{1}{2}f''(g(0))(g(t)-g(0))^2+o((g(t)-g(0))^2). \tag{1.1}$$

Here the notation $o(h(t))$ as $t\to 0$ means that $\lim_{t\to 0} o(h(t))/h(t)=0$. To put it simply: the function $o(h(t))$ is small in comparison with $h(t)$ for small $t$. Since $g$ is smooth, as $t\to 0$ we have $g(t)-g(0)=t(g'(0)+o(1))$ and $(g(t)-g(0))^2=t^2(g'(0)+o(1))^2$. The last term is of order $t^2$. Thus, when we divide by $t$ and take the limit as $t\to 0$ we arrive at the well-known chain rule

$$\frac{d}{dt}f(g(t))\Big|_{t=0}=f'(g(0))g'(0).$$

This can also be expressed in a differential form and at an arbitrary point $t$ instead of 0 as

$$df(g(t))=f'(g(t))\,dg(t)=f'(g(t))g'(t)\,dt$$

or in an integrated form

$$f(g(T))-f(g(0))=\int_0^T f'(g(t))\,dg(t).$$

We want to set $g(t)=B(t)$. But $B(t)$ is not smooth. We see that $(B(t)-B(0))^2=(B(t))^2$ is not small in comparison with $t$. Indeed, $(B(t))^2/t\overset{\mathrm{d}}{=}Z^2$, where $Z$ is a standard normal random variable, no matter how small $t>0$ we choose. In particular, $E(Z^2)=1$.[1] Thus, we expect that when we divide (1.1) by $t$ and attempt to take a limit (in some sense) as $t\to 0$ the term with the second derivative will not go away. We also see that we might not get a meaningful expression if we try to take a pointwise limit as $t\to 0$. Thus we shall try to use an integrated form, take the limit in $L^2$, and hope to get for all $T>0$

$$f(B(T))-f(B(0))=\int_0^T f'(B(t))dB(t)+\frac{1}{2}\int_0^T f''(B(t))\underbrace{dB(t)dB(t)}_{=dt}.$$

The new "chain rule" is called Itô-Doeblin formula for Brownian motion. Similarly, we can also consider $f(t,B(t))$ for smooth $f$. Finally, we shall replace $B(t)$ with a more general class of processes (Itô processes).

[1] $Z^2$ is Gamma distributed with parameters $(1/2,1/2)$.

**Theorem 1.1.** *Let $f \in C^2(\mathbb{R})$ and $\int_0^T E(f'(B(t)))^2\,dt < \infty$ for all $T > 0$. Then for every $T \geq 0$*

$$f(B(T)) - f(B(0)) = \int_0^T f'(B(t))\,dB(t) + \frac{1}{2}\int_0^T f''(B(t))\,dt \quad a.s.. \tag{1.2}$$

**Remark 1.2.** The square integrability condition in the above theorem can be removed (if one extends the notion of the stochastic integral accordingly). Moreover, the $C^2$ regularity can also be relaxed: see Exercise 4.20 of the text.

*Sketch of a proof.* Assume first that $f'$ and $f''$ are bounded, i.e. there is a constant $M$ such that $|f'(x)| \leq M$ and $|f''(x)| \leq M$ for all $x \in \mathbb{R}$. Let $\Pi$ be a partition of $[0,T]$, $0 = t_0 < t_1 < \dots < t_n = T$. Then

(1.3)

$$\begin{aligned} f(B(T)) - f(B(0)) &= \sum_{i=1}^n (f(B(t_i)) - f(B(t_{i-1}))) \\ &= \sum_{i=1}^n f'(B(t_{i-1}))(B(t_i) - B(t_{i-1})) + \frac{1}{2}\sum_{i=1}^n f''(\xi_{i-1})(B(t_i) - B(t_{i-1}))^2 =: \mathrm{I} + \mathrm{II}, \end{aligned}$$

where $\xi_{i-1}$ is a point between $B(t_{i-1})$ and $B(t_i)$. Since $B(t)$ is continuous on $[t_{i-1}, t_i]$, there is a point $t^*_{i-1} \in [t_{i-1}, t_i]$ such that $B(t^*_{i-1}) = \xi_{i-1}$, $i = 1, 2, \dots, n$. As we take the limit as $\|\Pi\| \to 0$, the term I will converge in $L^2$ to $\int_0^T f'(B(t))\,dB(t)$.

Consider now the term II.

$$\mathrm{II} = \frac{1}{2}\sum_{i=1}^n f''(B(t^*_{i-1}))(B(t_i) - B(t_{i-1}))^2 = \frac{1}{2}\sum_{i=1}^n (f''(B(t_{i-1})) + \varepsilon_{i-1})(B(t_i) - B(t_{i-1}))^2,$$

where $\varepsilon_{i-1} = f''(B(t^*_{i-1})) - f''(B(t_{i-1}))$. First, we show that

$$\frac{1}{2}\sum_{i=1}^n f''(B(t_{i-1}))(B(t_i) - B(t_{i-1}))^2 \xrightarrow{L^2} \frac{1}{2}\int_0^T f''(B(t))\,dt.$$

This calculation is similar to the one done in the computation of the quadratic variation of Brownian motion.

$$\begin{aligned} &E\left(\sum_{i=1}^n f''(B(t_{i-1}))(B(t_i) - B(t_{i-1}))^2 - \int_0^T f''(B(t))\,dt\right)^2 \\ &\leq 2E\left(\sum_{i=1}^n f''(B(t_{i-1}))((B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1}))\right)^2 \\ &+ 2E\left(\sum_{i=1}^n f''(B(t_{i-1}))(t_i - t_{i-1}) - \int_0^T f''(B(t))\,dt\right)^2 =: 2J_1 + 2J_2. \end{aligned}$$

Using the boundedness of $f''$ and the same method as in the proof of Theorem 1.5 from Lecture 3 we can show that as $\|\Pi\| \to 0$ (see Exercise 4.14 of the text)

$$J_1 \leq M^2 \sum_{i=1}^n E\left((B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1})\right)^2 \xrightarrow{L^2} 0.$$

As to $J_2$, the function $f''(B(t))$ is continuous and bounded so we get the convergence of the Riemann sum to the ordinary integral both a.s. and in $L^2$.

Finally, we consider the terms with $\varepsilon_i$. As $f''(B(t))$ is continuous on $[0,T]$, it is uniformly continuous on $[0,T]$, and, therefore, $\max_{i\in\{1,2,\dots,n\}} |\varepsilon_{i-1}| \to 0$ as $\|\Pi\| \to 0$ a.s.. Moreover, $\max_{i\in\{1,2,\dots,n\}} |\varepsilon_{i-1}| \leq$

$2M$ as we assumed that $f''$ were bounded. Therefore,

$$\left|\sum_{i=1}^{n}\varepsilon_{i-1}(B(t_i)-B(t_{i-1}))^2\right| \leq \max_{i\in\{1,2,\ldots,n\}}|\varepsilon_{i-1}|\left(\sum_{i=1}^{n}(B(t_i)-B(t_{i-1}))^2-T\right)+T\max_{i\in\{1,2,\ldots,n\}}|\varepsilon_{i-1}|$$
$$\leq 2M\left|\sum_{i=1}^{n}(B(t_i)-B(t_{i-1}))^2-T\right|+T\max_{i\in\{1,2,\ldots,n\}}|\varepsilon_{i-1}|.$$

The first term converges to 0 in $L^2$ by the computation of the quadratic variation of Brownian motion, the second term converges to 0 in $L^2$ by the dominated convergence theorem.

We have shown that when $f'$ and $f''$ are bounded, the right hand side of (1.3) converges in $L^2$ as $\|\Pi\|\to 0$ to the right hand side of (1.2). The left hand side of (1.3) does not depend on the partition, therefore it is equal to the above limit a.s.. This proves (1.2).

To remove the assumed bounds on the derivatives, we have to use some localization technique. Define $\tau_n = \inf\{t\geq 0 : |B(t)| = n\}$. Then $\tau_n$ is a stopping time. The idea is to check that the above reasoning works if we replace $B(t)$ everywhere with $B(t\wedge\tau_n)$. Note that $|B(t\wedge\tau_n)|\leq n$ and continuous functions $f'$ and $f''$ are bounded on bounded intervals. Then we have to take a limit as $n\to\infty$. $\square$

**Example 1.3.** Let $f(x) = x^2$. Then $f'(x) = 2x$, $f''(x) = 2$, and

$$B^2(T) - B^2(0) = 2\int_0^T B(t)\,dB(t) + \int_0^T 1\,dt.$$

Or in the differential form,

$$dB^2(t) = 2B(t)\,dB(t) + dt.$$

This is the already familiar formula from the last lecture.

**Example 1.4** (Computation of the moment generating function of $B(t)$)**.** Let $f(x) = e^{\theta x}$, $\theta\in\mathbb{R}$. Then $f'(x) = \theta e^{\theta x}$, $f''(x) = \theta^2 e^{\theta x}$. Itô-Doeblin formula in the differential form reads

$$de^{\theta B(t)} = \theta e^{\theta B(t)}\,dB(t) + \frac{1}{2}\,\theta^2 e^{\theta B(t)}\,dt,$$

and in the integral form we have

$$e^{\theta B(t)} - e^{\theta B(0)} = \theta\int_0^t e^{\theta B(s)}\,dB(s) + \frac{1}{2}\,\theta^2\int_0^t e^{\theta B(s)}\,ds.$$

Let $m(t) = E(e^{\theta B(t)})$. Taking the expectations in the above formula we get

$$m(t) - 1 = \frac{1}{2}\,\theta^2\int_0^t m(s)\,ds \quad\Longleftrightarrow\quad m'(t) = \frac{1}{2}\theta^2 m(t),\ m(0) = 1.$$

Solving the differential equation we conclude that $m(t) = E(e^{\theta B(t)}) = e^{\theta^2 t/2}$.

## 2. Itô-Doeblin formula for $f(t, B(t))$.

In many applications we are interested in quantities $f(t, B(t))$ where $f(t, x)$ is a smooth function of $t$ and $x$. Itô-Doeblin formula can be extended to this case.

**Theorem 2.1.** *Let $f\in C^{1,2}([0,\infty)\times\mathbb{R})$. Then for all $T\geq 0$*

$$f(T, B(T)) - f(0, B(0)) = \int_0^T f_x(t, B(t))\,dB(t) + \int_0^T\left(f_t(t, B(t)) + \frac{1}{2}f_{xx}(t, B(t))\right)dt.$$

The proof is omitted.[2] It is based on the same ideas as for the case when $f$ depends only on $x$.

[2] See the sketch of the proof of Theorem 4.4.1 and Exercise 4.14 in the text.

**Example 2.2** (Geometric Brownian motion). Let $S(t) = Ae^{\mu t+\sigma B(t)}$, where $A > 0$, $\mu$, and $\sigma$ are constants. Then $S(t) = f(t, B(t))$ for $f(t, x) = Ae^{\mu t+\sigma x}$. Calculating $f_t(t, x) = \mu f(t, x)$, $f_x(t, x) = \sigma f(t, x)$, $f_{xx}(t, x) = \sigma^2 f(t, x)$ and applying the above theorem we get

$$S(t) - S(0) = f(t, B(t)) - f(0, B(0)) = \int_0^T \sigma f(s, B(s))\, dB(s) + \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) f(s, B(s))\, ds.$$

Since $f(s, B(s)) = S(s)$, we conclude that

$$S(t) - S(0) = \int_0^t \sigma S(s)\, dB(s) + \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) S(s)\, ds,$$

or, in the differential form,

$$dS(t) = \sigma S(t) dB(t) + \left(\mu + \frac{1}{2}\sigma^2\right) S(t)\, dt.$$

Setting $\nu = \mu + \sigma^2/2$ we can say that the process $S(t) = Ae^{\sigma B(t)+(\nu-\sigma^2/2)t}$ solves the stochastic differential equation (SDE)

$$dS(t) = \nu S(t)\, dt + \sigma S(t)\, dB(t) \text{ with the initial condition } S(0) = A.$$

This topic will be discussed in more detail later.

## 3. Itô-Doeblin formula for Itô processes.

We shall consider one more generalization of Itô-Doeblin formula.

**Definition 3.1.** *Let $(B(t))_{t\geq 0}$, be a standard Brownian motion and $(\mathcal{F}(t))_{t\geq 0}$ be an associated filtration. An Itô process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(s)\, dB(s) + \int_0^t \Theta(s)\, ds,$$

*where $X(0)$ is a (non-random) constant and $(\Delta(t))_{t\geq 0}$, $(\Theta(t))_{t\geq 0}$ are adapted stochastic processes such that for all $t \geq 0$*

$$E\int_0^t \Delta^2(s)\, ds < \infty \quad \textit{and} \quad \int_0^t |\Theta(s)|\, ds < \infty \quad \textit{a.s.}.$$

Brownian motion, geometric Brownian motion, a stochastic integral $\int_0^t \Delta(s)\, dB(s)$ are examples of Itô processes.

**Theorem 3.2.** *Let $(X(t))_{t\geq 0}$ be an Itô process. Then for every $t \geq 0$*

$$[X]_t = \int_0^t \Delta^2(s)\, ds.$$

Note that the answer is the same as in the case of the stochastic integral when $\Theta(t) \equiv 0$. In other words, only the stochastic integral $I(t) = \int_0^t \Delta(s)\, dB(s)$ contributes to the quadratic variation: $[X]_t = [I]_t$. The proof is omitted but here is a "calculation" based on "stochastic multiplication rules" which we introduced to capture the quadratic variation.

Write $X(t)$ in the differential form $dX(t) = \Delta(t)\, dB(t0 + \Theta(t)\, dt$. Then

$$dX(t)dX(t) = \Delta^2(t) \underbrace{dB(t)dB(t)}_{=dt} + 2\Theta(t)\Delta(t) \underbrace{dB(t)dt}_{=0} + \Theta^2(t) \underbrace{dtdt}_{=0} = \Delta^2(t)dt.$$

**Definition 3.3** (Stochastic integral with respect to an Itô process). *Let $(X(t))_{t\geq 0}$ be an Itô process and $(\Gamma(t))_{t\geq 0}$ be an adapted[3] process such that for all $t \geq 0$*

$$E\int_0^t \Gamma^2(s)\Delta^2(s)\, ds < \infty \quad \textit{and} \quad \int_0^t |\Gamma(s)\Theta(s)|\, ds < \infty \ \textit{a.s.}.$$

[3]to the same filtration $(\mathcal{F}(t))_{t\geq 0}$ which appears in the definition of an Itô process.

*Define*

$$\int_0^t \Gamma(s)\,dX(s) = \int_0^t \Gamma(s)\Delta(s)\,dB(s) + \int_0^t \Gamma(s)\Theta(s)\,ds.$$

Notice that the stochastic integral with respect to an Itô process need not be a martingale.

**Theorem 3.4** (Itô-Doeblin formula for Itô processes)**.** *Let $(X(t))_{t\geq 0}$ be an Itô process and $f \in C^{1,2}([0,\infty)\times\mathbb{R})$. Then for every $T \geq 0$*

$$\begin{aligned} f(T,X(T)) - f(0,X(0)) &= \int_0^T f_t(t,X(t))\,dt + \int_0^T f_x(t,X(t))\,dX(t) + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\,d[X]_t \\ &= \int_0^T f_t(t,X(t)) + \Theta(t) f_x(t,X(t)) + \frac{1}{2}\Delta^2(t) f_{xx}(t,X(t))\,dt + \int_0^T f_x(t,X(t))\Delta(t)\,dB(t). \end{aligned}$$

The proof is omitted. Note that it is easy to "derive" the differential form of this this formula by applying stochastic "multiplication rules".

$$\begin{aligned} df(t,X(t)) &= f_t(t,X(t))dt + f_x(t,X(t)) \underbrace{dX(t)}_{=\Delta(t)dB(t)+\Theta(t)dt} + \frac{1}{2} f_{xx}(t,X(t)) \underbrace{dX(t)dX(t)}_{\Delta^2(t)dt} \\ &= \left( f_t(t,X(t)) + \Theta(t) f_x(t,X(t)) + \frac{1}{2}\Delta^2(t) f_{xx}(t,X(t)) \right) dt + f_x(t,X(t))\Delta(t)\,dB(t). \end{aligned}$$

## 4. Applications.

### 4.1. Itô's integral for deterministic integrands.

**Theorem 4.1.** *Let $(B(t))_{t\geq 0}$ be a standard Brownian motion, $\Delta(t)$ be a* ***non-random*** *function such that $\int_0^T \Delta^2(t)\,dt < \infty$. Then for every $t \in [0,T]$*

$$\int_0^t \Delta(s)\,dB(s) \quad \textit{is } \textbf{\textit{normal}} \textit{ with mean } 0 \textit{ and variance } \int_0^t \Delta^2(s)\,ds.$$

*Proof.* Let $I(t) = \int_0^t \Delta(s)\,dB(s)$. We already know from the properties of Itô integral that $E(I(t)) = 0$ and $\mathrm{Var}(I(t)) = \int_0^t \Delta^2(s)\,ds$ (Itô isometry combined with the fact that $\Delta$ is a non-random function). The main point is to prove that $I(t)$ is a normal random variable. The idea is to show that the MGF of $I(t)$ has the required form and deduce normality.

Let $f(x) = e^{\lambda x}$ and apply Itô-Doeblin formula for Itô processes to $f(I(t))$:

$$de^{\lambda I(t)} = \lambda e^{\lambda I(t)} dI(t) + \frac{\lambda^2}{2} w^{\lambda I(t)} d[I]_t = \lambda e^{\lambda I(t)} \Delta(t) dB(t) + \frac{\lambda^2}{2} e^{\lambda I(t)} \Delta^2(t)\,dt. \tag{4.1}$$

Rewriting (4.1) in the integral form and taking expectations we get that

$$\begin{aligned} e^{\lambda I(t)} - e^{\lambda I(0)} &= \theta \int_0^t e^{\lambda I(s)} \Delta(s)\,dB(s) + \frac{\lambda^2}{2}\int_0^t e^{\lambda I(s)}\Delta^2(s)\,ds; \\ E(e^{\lambda I(t)}) &= 1 + \frac{\lambda^2}{2}\int_0^t E(e^{\lambda I(s)})\Delta^2(s)\,ds. \end{aligned}$$

We used the fact that $\Delta$ is non-random when we took $\Delta^2(s)$ out of the expectation in the last integral. Setting $M(t,\lambda) := Ee^{\lambda I(t)}$ we get the ordinary differential equation $M_t = (\lambda^2\Delta^2(t)/2)M$ with the initial condition $M(0,\lambda) = 1$. Solving it, we find that

$$M(t,\lambda) = \exp\left(\frac{\lambda^2}{2}\int_0^t \Delta^2(s)\,ds\right).$$

This is the MGF of a normal random variable with mean 0 and variance $\int_0^t \Delta^2(s)\,ds$. Since the MGF characterizes the distribution, we conclude that $I(t)$ is normal with the stated parameters. $\square$

**Example 4.2.** Show that $\int_0^{\pi/2} B(t)\cos t\,dt$ is normal and find its parameters.

The given example is not in the form of a stochastic integral. We shall first convert it to a stochastic integral form. The computation below derives a version of a useful integration by parts formula. Applying Itô-Doeblin formula to $f(t,B(t))$ where $f(t,x)=x\sin t$ we get

$$d(B(t)\sin t)=B(t)\cos t\,dt+\sin t\,dB(t).$$

Integrating from 0 to $\pi/2$ and rearranging the terms we obtain

$$\int_0^{\pi/2} B(t)\cos t\,dt = B(t)\sin t\Big|_0^{\pi/2}-\int_0^{\pi/2}\sin t\,dB(t)=B\left(\frac{\pi}{2}\right)-\int_0^{\pi/2}\sin t\,dB(t)$$
$$=\int_0^{\pi/2}dB(t)-\int_0^{\pi/2}\sin t\,dB(t)=\int_0^{\pi/2}(1-\sin t)\,dB(t).$$

By the previous theorem, the last stochastic integral has a normal distribution with mean 0 and variance

$$\int_0^{\pi/2}(1-\sin t)^2\,dt=\frac{3\pi}{4}-2\approx 0.3562$$

4.2. **Lévy's characterization of Brownian motion.** We have already stated this remarkable result in Lecture 2. We start with a somewhat different formulation of the same result and then discuss the equivalence of the two statements. Recall that the quadratic variation of a stochastic process $(M(t))_{t\ge 0}$ is a stochastic process $([M]_t)_{t\ge0}$ such that for every $T\ge 0$ and every sequence of partitions $\Pi$, $0=t_0<\cdots<t_n=T$, with $\|\Pi\|\to 0$

$$\lim_{\|\Pi\|\to 0} E\left(\sum_{i=1}^n (M(t_i)-M(t_{i-1}))^2-[M]_T\right)^2=0.$$

**Theorem 4.3** (Lévy's characterization of Brownian motion)**.** *Let $M:=(M(t))_{t\ge0}$ be a martingale with respect to some filtration $(\mathcal{F}(t))_{t\ge0}$. Assume that $M(0)=0$, $M$ has continuous paths, and $[M]_t\equiv t$, $t\ge 0$. Then $(M(t))_{t\ge0}$ is a standard Brownian motion.*

At the first glance, it is not clear how one might even hope to argue that $M$ has independent increments or that the increments are normally distributed. The conditions of the theorem seem to provide no clue about these characteristics. Yet, the stochastic calculus machinery is able to convert the above conditions into the language of joint distributions.

First, a few preliminaries. We have defined the stochastic integral with respect to Brownian motion. Exactly the same approach can be used to define the stochastic integral with respect to any martingale with continuous paths, i.e. to define $\int_0^t \Delta(s)\,dM(s)$. Moreover, the following analog of Itô-Doeblin formula holds (under appropriate integrability conditions): let $f\in C^{1,2}([0,\infty)\times\mathbb{R})$, then a.s.

$$f(t,M(t))-f(0,M(0))=\int_0^t f_t(s,M(s))\,ds+\int_0^t f_x(s,M(s))\,dM(s)+\frac12\int_0^t f_{xx}(s,M(s))\,d[M]_s. \quad (4.2)$$

In our particular case $[M]_t\equiv t$, so the last integral is just a regular integral.[4]

*Sketch of a proof of Theorem 4.3.* We are given that $M(0)=0$ and that $M$ has continuous paths. All we need to check is that the increments are independent and normally distributed with the correct variance.

[4]In the general case when $[M]_t$ is a stochastic process, we use the fact that every realization of the quadratic variation process is a non-decreasing function of $t$. The last integral can be defined in the same way as we define the integral with respect to the CDF of some random variable, $\int_0^t f(s)dF_X(s):=\int_0^t f(s)d\mu_X(s)$. We identify the appropriate measure (distribution of $X$ in the above formula: $\mu_X((a,b])=F_X(b)-F_X(a)$) and integrate with respect to that measure. That is we set $\mu((a,b]):=[M]_b-[M]_a$ for all $a\le b$ and use $d\mu(s)$ instead of $d[M]_s$. The difference here is that, since the quadratic variation does not have to converge to 1 as $t\to\infty$, the corresponding measure need not be a probability measure. For example, when $[M]_t\equiv t$, the corresponding measure is the Lebesgue measure on $[0,\infty)$, and we get a regular integral.

We shall show first that $M(t) - M(s) \sim N(0, t-s)$, $0 \le s \le t$. The idea is again to use the MGF. Let $Y(t,\lambda) = e^{\lambda M(t) - \lambda^2 t/2}$. Then $Y(t,\lambda) = f(t, M(t))$, where $f(t,x) = e^{\lambda x - \lambda^2 t/2}$. By (4.2),

$$Y(t,\lambda) - 1 = -\frac{\lambda^2}{2}\int_0^t Y(s,\lambda)\,ds + \lambda\int_0^t Y(s,\lambda)\,dM(s) + \frac{\lambda^2}{2}\int_0^t Y(t,\lambda)\,\underbrace{d[M]_s}_{=ds} = \lambda\int_0^t Y(s,\lambda)\,dM(s).$$

From the properties of the stochastic integral we conclude that $Y(t,\lambda)$ is an $\mathcal{F}(t)$-martingale. Therefore, for all $0 \le s \le t$

$$(4.3)\quad E(Y(t,\lambda)\,|\,\mathcal{F}(s)) = Y(s,\lambda) \iff E\left(\frac{Y(t,\lambda)}{Y(s,\lambda)}\,\middle|\,\mathcal{F}(s)\right) = 1$$
$$\iff E\left(e^{\lambda(M(t)-M(s)) - \lambda^2(t-s)/2}\,|\,\mathcal{F}(s)\right) = 1 \iff E\left(e^{\lambda(M(t)-M(s))}\,|\,\mathcal{F}(s)\right) = e^{\lambda^2(t-s)/2}.$$

Taking the expectations of both sides in the last expression we get that the MGF of $M(t) - M(s)$ is equal to $\exp(\lambda^2(t-s)/2)$ and conclude that $M(t) - M(s) \sim N(0, t-s)$.

Finally we show the independence of increments. Assume that $m = 2$ and consider $M(t_2) - M(t_1)$ and $M(t_1) - M(t_0)$, $t_0 = 0$. The case of general $m$ can be treated in exactly the same way, it is just bulkier to write. By repeated conditioning we get

$$\begin{aligned} E\left(e^{\lambda_2(M(t_2)-M(t_1))+\lambda_1(M(t_1)-M(t_0))}\right) &= E\left(E\left(e^{\lambda_2(M(t_2)-M(t_1))+\lambda_1(M(t_1)-M(t_0))}\,|\,\mathcal{F}(t_1)\right)\right) \\ &= E\left(Ee^{\lambda_1(M(t_1)-M(t_0))}\left(e^{\lambda_2(M(t_2)-M(t_1))}\,|\,\mathcal{F}(t_1)\right)\right) \\ &\overset{(4.3)}{=} E\left(e^{\lambda_1(M(t_1)-M(t_0))}e^{\lambda_2^2(t_2-t_1)/2}\right) = e^{\lambda_2^2(t_2-t_1)/2}e^{\lambda_1^2(t_1-t_0)/2}. \end{aligned}$$

This implies that $M(t_2) - M(t_1)$ and $M(t_1) - M(t_0)$ are independent. □

**Example 4.4.** Let $f(t)$ be an arbitrary random or non-random integrand which satisfies conditions (i)-(iii) of Section 4, Lecture 3, so that $I(t) := \int_0^t f(s)\,dB(s)$, $t \ge 0$, is well-defined. Assume, in addition, that $f$ takes only values 1 and $-1$. Then $I(t)$, $t \ge 0$, is a Brownian motion. Indeed $I(t)$, $t \ge 0$, is a martingale relative to the filtration of Brownian motion and $[I]_t = \int_0^t f^2(s)\,ds = t$ for all $t \ge 0$.

See also Exercise 4.19 of the textbook.

**Remark 4.5.** Let us now return to the equivalence of two formulations of Theorem 4.3. In the first formulation we assumed that $M^2(t) - t$ is an $\mathcal{F}(t)$-martingale, in the second we replaced this with the requirement that $[M]_t \equiv t$. We claim that these conditions (provided that all other conditions of Theorem 4.3 are satisfied) are equivalent. Consider the process $(M^2(t))_{t\ge 0}$. It is a convex function of a martingale $(M(t))_{t\ge 0}$. Therefore, it is a submartingale (see Theorem 1.15 and Exercise 5 of refresher lecture 5). Moreover, by Itô formula mentioned above

$$M^2(t) = \underbrace{M^2(0) + 2\int_0^t M(s)\,dM(s)}_{\text{martingale}} + \underbrace{[M]_t}_{\text{non-decreasing process}}.$$

We conclude that $M^2(t) - [M]_t$, $t \ge 0$, is an $\mathcal{F}(t)$-martingale. Thus, if $[M]_t \equiv t$ then $M^2(t) - t$, $t \ge 0$, is an $\mathcal{F}(t)$-martingale. The reverse implication is based on the so-called Doob-Meyer decomposition, which states that (under conditions) every continuous submartingale can be uniquely written as a sum of a martingale and a non-decreasing process $A(t)$ with $A(0) = 0$.[5] In our case the submartingale is $(M^2(t))_{t\ge 0}$. The decomposition we obtained above is the Doob-Meyer decomposition and $[M]_0 = 0$. If we know that $M^2(t) - t$, $t \ge 0$, is an $\mathcal{F}(t)$-martingale, then by the uniqueness of the decomposition we conclude that $[M]_t \equiv t$.

[5]Doob's decomposition for discrete time is very simple, see, for example Proposition 2.4.6 in A. Etheridge, Financial Calculus. The precise statement in the continuous time setting is much more demanding and will not be given.

This gives us another view of the quadratic variation of a martingale: the quadratic variation of a martingale $M$ is such a process $[M]_t$, $t \geq 0$, that $M^2(t) - [M]_t$, $t \geq 0$, is again a martingale (with respect to the same filtration).