(5)

Proof. Recalling that the moment generating functions for $B(t)$ and $Q(t)$ are $\mathbb{E}e^{u_1 B(t)} = e^{\frac{1}{2}u_1^2 t}$ and $\mathbb{E}e^{u_2 Q(t)} = e^{\lambda t(\varphi(u_2)-1)}$, where $Q(t) = \sum_{i=1}^{N(t)} Y_i$ and $\varphi(u) = \mathbb{E}e^{uY_1}$.

Thus, to prove $B(t)$ and $Q(t)$ are independent, we only need to show that for all values of u_1, u_2

$$\mathbb{E}e^{u_1 B(t)+u_2 Q(t)} = \mathbb{E}e^{u_1 B(t)}\mathbb{E}e^{u_2 Q(t)} = e^{\frac{1}{2}u_1^2 t+\lambda t(\varphi(u_2)-1)}$$

Rewriting the above equations, we get

$$\mathbb{E}e^{u_1 B(t)-\frac{1}{2}u_1^2 t+u_2 Q(t)-\lambda t(\varphi(u_2)-1)} = 1$$

Let $X(t) \triangleq u_1 B(t) - \frac{1}{2}u_1^2 t + u_2 Q(t) - \lambda t(\varphi(u_2)-1)$, we will show that $e^{X(t)}$ is a martingale.

Using Itô's formula, we have

$$\begin{aligned} e^{X(t)} &= 1 + \int_0^t e^{X(s-)}(u_1 dW(s) - \frac{1}{2}u_1^2 ds - \lambda(\varphi(u_2)-1)ds) + \frac{1}{2}\int_0^t e^{X(s-)}u_1^2 ds + \sum_{0<s\le t}(e^{X(s)} - e^{X(s-)}) \\ &= 1 + \int_0^t e^{X(s-)}(u_1 dW(s) - \lambda(\varphi(u_2)-1)ds) + \sum_{0<s\le t}(e^{X(s)} - e^{X(s-)}) \end{aligned}$$

Note that

$$e^{X(s)} - e^{X(s-)} = e^{X(s-)}(e^{X(s)-X(s-)} - 1) = e^{X(s-)}(e^{u_2 Y_{N(s)}} - 1)\Delta N(s)$$

It follows that

$$\sum_{0<s\le t}(e^{X(s)} - e^{X(s-)}) = \sum_{0<s\le t} e^{X(s-)}(e^{u_2 Y_{N(s)}} - 1)\Delta N(s) = \int_0^t e^{X(s-)} d[\sum_{i=1}^{N(s)}(e^{u_2 Y_i} - 1)]$$

Plugging this into the former equation, we have

$$\begin{aligned} e^{X(t)} &= 1 + u_1\int_0^t e^{X(s-)}dW(s) + \int_0^t e^{X(s-)}d[\sum_{i=1}^{N(s)}(e^{u_2 Y_i} - 1) - \lambda(\varphi(u_2)-1)s] \\ &= 1 + u_1\int_0^t e^{X(s-)}dW(s) + \int_0^t e^{X(s-)}d[\sum_{i=1}^{N(s)}(e^{u_2 Y_i} - 1) - \mathbb{E}\sum_{i=1}^{N(s)}(e^{u_2 Y_i} - 1)] \end{aligned}$$

Let $Z(t) = \sum_{i=1}^{N(t)}(e^{u_2 Y_i} - 1) - \mathbb{E}[\sum_{i=1}^{N(t)}(e^{u_2 Y_i} - 1)]$, we claim that $Z(t)$ is martingale.

$$\mathbb{E}(Z(t)|\mathcal{F}_s) = \sum_{i=1}^{N(s)}(e^{u_2 Y_i}-1) - \mathbb{E}[\sum_{i=1}^{N(s)}(e^{u_2 Y_i}-1)] + \mathbb{E}[\sum_{i=N(s)+1}^{N(t)}(e^{u_2 Y_i}-1)|\mathcal{F}_s] - \mathbb{E}[\sum_{i=N(s)+1}^{N(t)}(e^{u_2 Y_i}-1)]$$

Since $N(t)$ has independent increments and $Y_i, i = 1, 2, \cdots$ are mutually independent, we have

$$\mathbb{E}[\sum_{i=N(s)+1}^{N(t)}(e^{u_2 Y_i} - 1)|\mathcal{F}_s] = \mathbb{E}[\sum_{i=N(s)+1}^{N(t)}(e^{u_2 Y_i} - 1)]$$

which gives

$$\mathbb{E}(Z(t)|\mathcal{F}_s) = Z(s)$$

i.e. $Z(t)$ is a martingale.

In summary, we have

$$e^{X(t)} = 1 + u_1 \int_0^t e^{X(s-)} dW(s) + \int_0^t e^{X(s-)} dZs)$$

Since $e^{X(s-)}$ are left continuous and $W(s), Z(s)$ are martingales, according to theorem 4.5 from lecture 11, we conclude that $e^{X(t)}$ is a martingale, and it folows that

$$\mathbb{E}e^{X(t)} = e^{X(0)} = 1$$

i.e. $\mathbb{E}e^{u_1 B(t) + u_2 Q(t)} = \mathbb{E}e^{u_1 B(t)} \cdot \mathbb{E}e^{u_2 Q(t)}$, thus we conclude that $B(t)$ and $Q(t)$ are independent.

$\square$

(6)

The time 0 cost C of a European call option with strike price K and expiration t is its price at time 0,

$$C = \mathbb{E}[e^{-rt}(S(t) - K)^+] = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n)\mathbb{E}[e^{-rt}(S(t) - K)^+ | N(t) = n]$$

Note that $S(t) = S^*(t)e^{Q(t)} = S(0)e^{\sigma B(t) + \mu t + \sum_{i=1}^{N(t)} Y_i}$, then according to independence, we have

$$\mathbb{E}[e^{-rt}(S(t) - K)^+ | N(t) = n] = \mathbb{E}[e^{-rt}(S(0)e^{\sigma B(t) + \mu t + \sum_{i=1}^{n} Y_i} - K)^+]$$

Since $Y_i \sim N(\mu_0, \sigma_0^2)$ and $B(t), Y_i, \forall i$ are independent,

$$\sigma B(t) + \sum_{i=1}^{n} Y_i \overset{d.}{=} n\mu_0 + \sqrt{\sigma^2 + \frac{n\sigma_0^2}{t}}\sqrt{t}Z$$

where $\overset{d.}{=}$ means "equally distributed" and $Z \sim N(0, 1)$.

For convenience, let $f(\lambda) = \lambda(\mathbb{E}e^{Y_1} - 1)$, then

$$\begin{aligned}
C &= \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) \cdot \mathbb{E}[e^{-rt}(S(0)e^{(r - f(\lambda) - \frac{1}{2}(\sigma^2 + \frac{n\sigma_0^2}{t}))t + \sqrt{\sigma^2 + \frac{n\sigma_0^2}{t}}\sqrt{t}Z + n(\mu_0 + \frac{1}{2}\sigma_0^2)} - K)^+] \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n)e^{n(\mu_0 + \frac{1}{2}\sigma_0^2)} \cdot \mathbb{E}[e^{-rt}(S(0)e^{(r - f(\lambda) - \frac{1}{2}(\sigma^2 + \frac{n\sigma_0^2}{t}))t + \sqrt{\sigma^2 + \frac{n\sigma_0^2}{t}}\sqrt{t}Z} - Ke^{-n(\mu_0 + \frac{1}{2}\sigma_0^2)})^+] \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n)e^{n(\mu_0 + \frac{1}{2}\sigma_0^2)} \cdot C(S(0), Ke^{-n(\mu_0 + \frac{1}{2}\sigma_0^2)}, t, f(\lambda), \sqrt{\sigma^2 + \frac{n\sigma_0^2}{t}})
\end{aligned}$$

where $C(S, K, T, q, \sigma)$ is the price of a plain vanilla European call option with spot price $S(0)$, strike price K, maturity T, continuous dividend rate q and volatility σ.