MTH 9831. LECTURE 9

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Abstract. After considering a few more examples related to connections with PDEs we turn to exotic options (barrier and Asian).

1. Further examples of connections with PDEs:
   Pricing of a zero-coupon bond when the interest rate is random.
2. Pricing of barrier options:
   (a) a probabilistic approach;
   (b) a PDE approach.
3. Pricing of Asian options. New ideas: augmentation of the state space and *reduction of dimension using the change of numeraire. (* next lecture)
4. Fubini's theorem for conditional expectations.

1. Further examples of connections with PDEs

Assume that the interest rate under $\widetilde{\mathbb{P}}$ satisfies

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\widetilde{B}(t). \tag{1}$$

Example 1.1 (Vasiček, Hull-White, CIR models)**.**

- (Vasiček model) It assumes that the rate is driven by a mean-reverting OU process, namely,
$$dR(t) = (a - bR(t))\,dt + \sigma\,d\widetilde{B}(t),$$
where $a, b, \sigma > 0$ are constants.
- (Hull-White model) This is a generalization of Vasiček model which allows the coefficients to very in time:
$$dR(t) = (a(t) - b(t)R(t))\,dt + \sigma(t)\,d\widetilde{B}(t),$$
where $a(t), b(t), \sigma(t)$ are positive non-random functions.
- (Cox-Ingersoll-Ross model) While the previous two models allow the rate to become negative with positive probability, CIR model ensures that the rate process is always non-negative. The SDE for $R(t)$ looks as follows:
$$dR(t) = (a - bR(t))\,dt + \sigma\sqrt{R(t)}\,d\widetilde{B}(t),$$
where $a, b, \sigma > 0$ are constants.

These models are examples of so-called <u>one factor</u> <u>short rate</u> models. For examples of two-factor models and more general HJM framework, see Chapter 10 of Shreve, Vol. II.

We have the discount process

$$dD(t) = -R(t)D(t)dt, \quad D(0) = 1;$$
$$D(t) = \exp\left(-\int_0^t R(s)ds\right);$$

the MMA price

$$M(t) = \frac{1}{D(t)} = \exp\left(\int_0^t R(s)ds\right),$$
$$dM(t) = R(t)M(t)dt = \frac{R(t)}{D(t)}dt, \quad M(0) = 1.$$

Recall that $B(t,T)$ is the time t price of a unit zero coupon bond maturing at T (pays \$1 at T).

$$B(t,T) = \frac{1}{D(t)}\widetilde{\mathbb{E}}(D(T)\cdot 1|\mathcal{F}(t));$$
$$B(t,T) = \widetilde{\mathbb{E}}[e^{-\int_t^T R(s)ds}|\mathcal{F}(t)].$$

Yield over the time interval $[t,T]$ is

$$Y(t,T) := -\frac{1}{T-t}\ln B(t,T);$$
$$B(t,T) = e^{-Y(t,T)(T-t)},$$

where $Y(t,T)$ can be thought of as the constant rate over $[t,T]$ which is consistent with the price $B(t,T)$. Since $R(t)$ is a solution of an SDE, it is a Markov process, and we would like to say that for some non-random function $f(t,y)$

$$B(t,T) = f(t,R(t)). \tag{2}$$

Rigorously speaking, $e^{-\int_t^T R(s)ds}$ is not of the form $h(R(T))$, so our definition of a Markov process does not allow us to conclude (2). But heuristically the only way $e^{-\int_t^T R(s)ds}$ depends on the path $R(s)$ for $0 \le s \le t$ is only through its value at $s = t$. To find the PDE for f, we use the fact that $D(t)B(t,T) = e^{-\int_0^t R(s)ds}f(t,R(t))$ is a $\widetilde{\mathbb{P}}$-martingale.

$$d(e^{-\int_0^r R(s)ds}f(t,R(t))) = e^{-\int_0^t R(s)ds}(-R(t)fdt + f_tdt + f_y(\beta dt + \gamma d\widetilde{B}(t)) + \frac{1}{2}f_{yy}\gamma^2 dt)$$
$$= e^{-\int_0^t R(s)ds}((-Rf + f_t + \beta f_y + \frac{1}{2}\gamma^2 f_{yy})dt + \gamma f_y d\widetilde{B}(t));\ f = f(t,R(t)).$$

Therefore, the corresponding PDE is

$$f_t(t,y) + \beta(t,y)f_y(t,y) + \frac{1}{2}\gamma^2(t,y)f_{yy}(t,y) = yf(t,y), \tag{3}$$

or, setting $\mathcal{A}_t := \beta(t,y)\frac{\partial}{\partial y} + \gamma(t,y)\frac{\partial^2}{\partial y^2}$ we get

$$f_t(t,y) + \mathcal{A}_t f(t,y) = yf(t,y);$$
$$f(T,y) = 1.$$

See Example 6.5.1 for a derivation of an explicit formula for the Hull-While model and Example 6.5.2 for CIR model.

Key idea for finding a solution: these models are affine yield models, which means that

$$f(t,y) = e^{-yC_1(t,T) - C_2(t,T)}$$

for some $C_1(t,T)$ and $C_2(t,T)$ (to be determined). The name comes from the fact that the yield $Y(t,T)$ can be assumed to be of the form $y\frac{C_1(t,T)}{T-t} + \frac{C_2(T-t)}{T-t}$, which is an affine function of y.
Example 6.5.3 shows that the price $C(t,y)$ of a call option on a bond with expiration $0 \le T_1 \le T$ requires to solve the same PDE (3), i.e. $C_t + \beta C_y + \frac{1}{2}\gamma^2 C_{yy} = yC$ but with a different terminal condition

$$C(T_1,y) = (f(T_1,y) - K)_+.$$

2. Pricing of barrier options

Standard barrier options can be either puts or calls, and each of these comes in 4 "flavors": up-and-out, down-and-out, up-and-in, and down-and-in.

Tools for pricing barrier options:

- joint distribution of BM with a drift and its running maximum (see lecture 6);
- stopped martingales;
- risk-neutral pricing formula;
- use of Itô's formula to determine whether a given function of a diffusion process is a martingale.

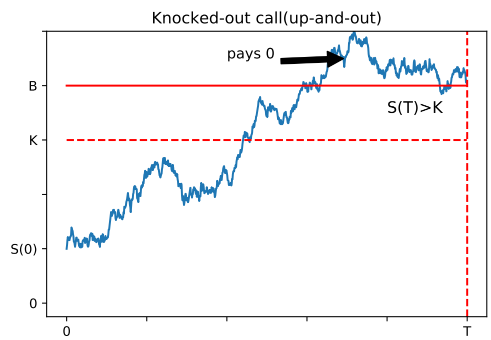

Figure 1. Knocked-out call

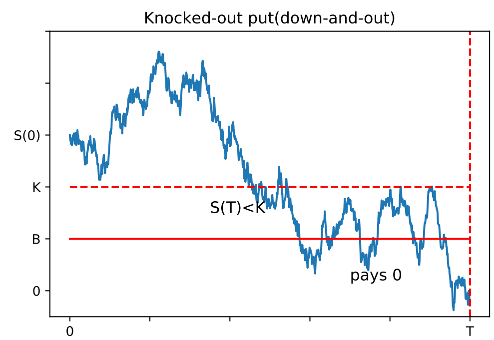

Figure 2. Knocked-out put

Framework: Black-Scholes-Merton model. Assume that under the risk-neutral measure the stock price satisfies

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{B}(t). \tag{4}$$

Options: we shall consider an up-and-out call with strike K, barrier $B > K$ and expiration T. The payoff of this option is the same as the payoff of a call option provided that the stock price stays below the barrier b for all times between 0 and T. If the stock price reaches the barrier at some time before T, we say that it was knocked out. In this case the option has payoff 0. We assume that $S(0) \leq B$.[1]

Note that the payoffs of an up-and-in call option and an up-and-out call option with the same strike and expiration add up to the payoff of a standard call option. Therefore, the price of an up-and-in call option can be determined from the prices of the other two options.

Put options and other "flavors" are dealt with in a similar way. The most natural barrier puts are down-and-out and down-and-in put options.

Goals: develop two methods of pricing of an up-and-out call option.

(a) Use a probabilistic approach to set up an integral which gives the price.
(b) Write down the PDE and boundary conditions satisfied by the call price.

(a) We shall start with a probabilistic approach. Recall that $S(t) = S(0)e^{\sigma\widetilde{B}(t)+(r-\frac{\sigma^2}{2})t} = S(0)e^{\sigma\hat{B}(t)}$, where $\hat{B}(t) = \widetilde{B}(t) + \alpha t$, and $\alpha = \frac{r-\sigma^2/2}{\sigma}$.

Define $\hat{B}^*(t) = \max_{0\leq s\leq t}\hat{B}(s)$. Since e^x is an increasing function and $\sigma > 0$, we can write

$$\max_{0\leq t\leq T} S(t) = S(0)e^{\sigma\hat{B}^*(T)}.$$

[1] Otherwise the price is 0.

The payoff of the option is

$$\begin{aligned} V(T) &= (S(0)e^{\sigma \hat{B}(T)} - K)_+ \mathbb{1}_{\{S(0)e^{\sigma \hat{B}^*(T)} < B\}} \\ &= (S(0)e^{\sigma \hat{B}(T)} - K) \mathbb{1}_{\{S(0)e^{\sigma \hat{B}(T)} \geq K,\, S(0)e^{\sigma \hat{B}^*(T)} < B\}} \\ &= (S(0)e^{\sigma \hat{B}(T)} - K) \mathbb{1}_{\left\{\hat{B}(T) \geq \underbrace{\frac{1}{\sigma}\log\frac{K}{S(0)}}_{k\in\mathbb{R}};\, \hat{B}^*(T) < \underbrace{\frac{1}{\sigma}\log\frac{B}{S(0)}}_{b>0}\right\}}. \end{aligned}$$

Risk-neutral pricing formula tells us that the value of this option at time $t \in [0, T]$ is equal to

$$V(t) = \widetilde{\mathbb{E}}\left(e^{-r(T-t)}V(T)|\mathcal{F}(t)\right),\ 0 \leq t \leq T.$$

By the tower property, $e^{-rt}V(t)$ is a $\widetilde{\mathbb{P}}$-martingale.

To compute $V(0)$ we write

$$V(0) = e^{-rT}\widetilde{\mathbb{E}}\left((S(0)e^{\sigma\hat{B}(T)} - K)\mathbb{1}_{\{\hat{B}(T)\geq k,\, \hat{B}^*(T)<b\}}\right).$$

The joint distribution of $(\hat{B}(T), \hat{B}^*(T))$ was computed in lecture 6. We have (under $\widetilde{\mathbb{P}}$, as it is our reference measure here)

$$\hat{f}(x, a) = \begin{cases} \frac{2(2a-x)}{T\sqrt{2\pi T}}\, e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2}, & \text{if } x \leq a \text{ and } a \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(Variable x corresponds to $\hat{B}(T)$, and a corresponds to $\hat{B}^*(T)$)
Therefore,

$$V(0) = e^{-rT}\int_k^b \left(\int_{x_+}^b (S(0)e^{\sigma x} - K)\frac{2(2a-x)}{T\sqrt{2\pi T}} e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2} da\right) dx.$$

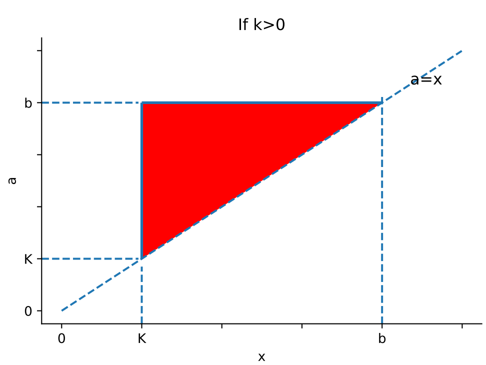

FIGURE 3. If k>0

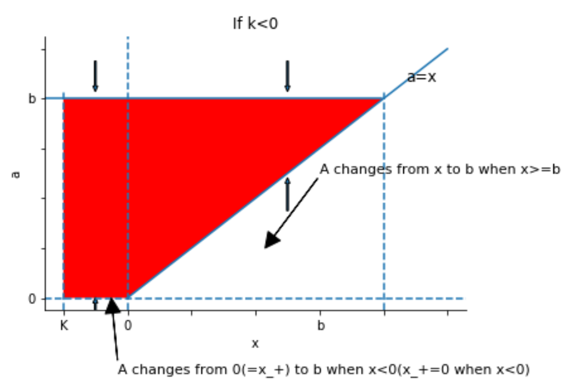

FIGURE 4. If k<0

This integral can be computed in terms of $N(x)$. This gives a closed form formula. For details, see Shreve, Vol. II (pp. 304-308, Section 7.3.3).

(b) Let us now move to the PDE description.

Theorem 2.1. *Let $v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to t and $S(t) = x$. Then $v(t, x)$ satisfies the BSM PDE*

$$v_t + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

in the rectangle $\{(t,x): 0 \le t < T,\, 0 \le x \le B\}$ *and satisfies boundary conditions*

$$v(t,0) = 0, \quad 0 \le t \le T; \tag{5}$$

$$v(t,B) = 0, \quad 0 \le t < T; \tag{6}$$

$$v(T,x) = (x-K)_+, \quad 0 \le x \le B. \tag{7}$$

Recall that $V(t) = \widetilde{\mathbb{E}}(e^{-r(T-t)}V(T)|\mathcal{F}(t))$. It is clear that $V(t)$ can not be represented as $v(t,S(t))$, since $V(t)$ "remembers" whether it has been knocked out or not, and $v(t,S(t))$ "does not remember" anything that happened prior to t.

Define $\rho = \inf\{t \ge 0 : S(t) = B\}$. Then ρ is the knock-out time (since upon reaching B, $S(t)$ will exceed B within any positive additional time increment with probability 1). Since ρ is the first passage time, it is a stopping time (namely $\{\rho \le t\} \in \mathcal{F}(t)$ for each $t \ge 0$).

We have that $(e^{-rt}V(t))_{0\le t\le T}$ is a $\widetilde{\mathbb{P}}$-martingale. We know that a martingale, stopped at a stopping time is again a martingale. Thus, $(e^{-r(t\wedge\rho)} \vee (t\wedge\rho))_{0\le t\le T}$ is a $\widetilde{\mathbb{P}}$-martingale ($t\wedge\rho := \min\{t,\rho\}$).

Lemma 2.2. *For* $0 \le t \le \rho$*,* $V(t)$ *is representable as* $v(t,S(t))$*. In particular,* $(e^{-r(t\wedge\rho)}v(t\wedge\rho, S(t\wedge\rho)))_{0\le t\le T}$ *is a* $\widetilde{\mathbb{P}}$*-martingale.*

Explanation for Lemma 2.2. On the event $\{\rho > t\}$, the option is alive, so the value of the up-and-out call is the same as for the regular call option.

Recall that for the regular call option, we had

$$V(t) = \widetilde{\mathbb{E}}(e^{-r(T-t)} \underbrace{h(S(T))}_{(S(T)-K)_+} |\mathcal{F}(t)) \underset{\text{Markov property}}{=} v(t,S(t))$$

for some function $v(t,x)$.

Since $e^{-rt}V(t)$ is a martingale, $e^{-rt}v(t,S(t))$ is a martingale. The point here is that the same is true up to the random time ρ, or for all $t \le \rho$, after that we stop our martingale, and a martingale stopped at ρ is still a martingale. □

Sketch of proof of Theorem 2.1. By Lemma 2.2, $V(t) = v(t,S(t))$ for some v and $0 \le t \le \rho$. Moreover, $e^{-rt}v(t,S(t))$ is a martingale up to time ρ. Therefore, for $0 \le t \le \rho$,

$$\begin{aligned} d(e^{-rt}v(t,S(t))) =& e^{-rt}((-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))S(t)r + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S^2(t))dt \\ &+ v_x(t,S(t))\sigma S(t)d\widetilde{B}(t)), \end{aligned}$$

and setting the dt term to zero, we get for $0 \le t \le \rho$ that

$$-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))S(t)r + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S^2(t) = 0.$$

Since $(t,S(t))$, $0 \le t \le \rho$, can reach any point in $(0,T)\times[0,B]$, the BSM PDE should hold for all $(t,x) \in [0,T)\times[0,B]$, i.e.

$$v_t(t,x) + v_x(t,x)xr + \frac{1}{2}v_{xx}\sigma^2 x^2 = rv(t,x), \quad (t,x) \in [0,T)\times[0,B].$$

The boundary conditions simply satisfy the conditions set forth by the option. □

Remark 2.3. See Shreve II, p. 303-4, for problems with the Δ-hedging strategy when the option is nearing the barrier close to the maturity time and what can be done about this in practice.

3. Pricing of Asian options

Let again $S(t)$ satisfy (4) under $\widetilde{\mathbb{P}}$. The payoff of an Asian option with strike K and expiration T is given by

$$V(T) = \left(\frac{1}{T}\int_0^T S(u)du - K\right)_+;$$

$$V(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}\left(\left(\frac{1}{T}\int_0^T S(u)du - K\right)_+ \Big| \mathcal{F}(t)\right).$$

Clearly, the price depends on the whole path $S(u)$, $0 \le u \le t$, so we can not write $V(t)$ as a function of just t and $S(t)$, not even up to some stopping time as we did in the case of barrier options. A new idea is needed.

Idea: augmentation of the state space. To calculate $V(t)$, we definitely need to know $\int_0^t S(u)du$, since

$$\int_0^T S(u)du = \int_0^t S(u)du + \int_t^T S(u)du$$

and $\int_0^t S(u)du$ is $\mathcal{F}(t)$-measurable, so it can be taken out of the conditional expectation. We shall introduce an auxiliary process $Y := (Y(t))_{t\in[0,T]}$ defined by

$$Y(t) = \int_0^t S(u)\,du,$$

or, in the differential form, $dY(t) = S(t)dt$. Y is a regular process. Consider a 2-dimensional process $(S(u), Y(u))_{t\le u\le T}$. This 2-dimensional process is a Markov process which we can describe by the following system of equations:

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)d\widetilde{B}(u), \quad S(t) = x;\\ dY(u) &= S(u)du, \quad Y(t) = y. \end{aligned}$$

The generator of this process is given by

$$\mathcal{A}v(x,y) = rxv_x + xv_y + \frac{1}{2}\sigma^2x^2v_{xx}.$$

The expression for the time t value is

$$V(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}\left(\left(\frac{1}{T}Y(T) - K\right)_+ \Big| \mathcal{F}(t)\right),$$

and $V(t)$ can be written as $v(t, S(t), Y(t))$ for some function $v(t,x,y)$.

Theorem 3.1. *Let $v(t,x,y)$ be the time t value of the Asian call option when $S(t) = x, Y(t) = y$. Then $v(t,x,y)$ satisfies the PDE*

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y) \tag{8}$$

in $[0,T)\times[0,\infty)\times\mathbb{R}$ and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)_+, \quad 0 \le t < T,\ y\in\mathbb{R}; \tag{9}$$

$$\lim_{y\to-\infty} v(t,x,y) = 0, \quad 0 \le t \le T,\ x \ge 0; \tag{10}$$

$$v(T,x,y) = \left(\frac{y}{T} - K\right)_+, \quad x \ge 0, y \in\mathbb{R}. \tag{11}$$

Remark 3.2. There are are several issues that need to be addressed.

(1) Why are we looking also at $y<0$? We know that $Y(t)=\int_0^t S(u)du \geq 0$, so how do we end up considering $y<0$?
(2) Usual existence & uniqueness questions for the above problem.

We shall start with (2): existence is not a problem but for uniqueness in an unbounded domain $(t,x,y) \in [0,T) \times [0,\infty) \times \mathbb{R}$, we need additional conditions at ∞: what happens if $x \to \infty$ and $y \to \pm\infty$? We omit this discussion, since later we shall be able to reduce the dimension and write a simpler PDE with corresponding boundary conditions.

Turn now to (1). The function $Y(t)=\int_0^t S(u)du$ is only one specific solution of $dY(u)=S(u)du$, $0 \leq u \leq T$. Namely, the one which satisfies the condition $Y(0)=0$. Yet we shall have to solve this equation for $u \in [t,T]$ $(t \leq T)$ subject to the condition $Y(t)=y$. So we get

$$Y(u) = y + \int_t^u S(s)ds, \quad t \leq u \leq T.$$

Notice that while

$$S(t)=0 \quad \Rightarrow \quad S(u)=0 \quad \text{for all } u \in [t,T] \quad \text{and} \quad Y(t)=y \text{ for all } u \in [t,T],$$

we do not have the same property for the Y process: if $Y(t)=0$, then $Y(u)=\int_t^u S(s)ds$ need not be 0, so we can not determine the value of $v(t,x,0)$, and we can not provide a boundary condition for $y=0$. But, at least mathematically, there is no problem considering $y \in \mathbb{R}$, and that is what we do.

Idea of a proof of Theorem 3.1. Since Y is a regular process, $d[Y]_t$ and $d[Y,S]_t$ are equal to 0 a.s.. Recall that $e^{-rt}V(t)=e^{-rt}v(t,S(t),Y(t))$ is $\widetilde{\mathbb{P}}$-martingale. Compute

$$d(e^{-rt}v(t,S(t),Y(t))) = e^{-rt}(-rv+v_t+rSv_x+Sv_y+\frac{1}{2}\sigma^2S^2v_{xx})dt + e^{-rt}\sigma S v_x d\widetilde{B}(t).$$

Setting the dt term to zero, we get

$$v_t(t,S(t),Y(t)) + rS(t)v_x(t,S(t),Y(t)) + S(t)v_y(t,S(t),Y(t)) + \frac{1}{2}\sigma^2S^2(t)v_{xx}(t,S(t),Y(t)) = rv(t,S(t),Y(t)).$$

This equation has to hold at all points (x,y) which can be possibly hit by S and Y process. Since we did not restrict ourselves to $y>0$, the process $Y(t)$ can hit any point in $\mathbb{R}$, and $S(t)$ can hit any $x \geq 0$, so we arrive at PDE (8).

Now we turn to boundary conditions. If $S(t)=0$, then $S(u)=0$ for all $u \in [t,T]$, and $Y(t)=y$ for all $u \in [t,T]$. Thus the value of an Asian call at time t is

$$e^{-r(T-t)}\left(\frac{y}{T}-K\right)_+ .$$

Thus, we get (9).

If $S(t)=x$, $Y(t)=y$ and we let $y \to -\infty$, then it is very unlikely that $(\frac{Y(T)}{T}-K)_+ > 0$. In other words,

$$\lim_{y \to -\infty} v(t,x,y) = 0.$$

This is (10).

Finally, (11) is just the payoff of the option. □

Remark 3.3. We have

$$d(e^{-rt}v(t,S(t),Y(t))) = \sigma e^{-rt}S(t)v_x(t,S(t),Y(t))d\widetilde{B}(t).$$

On the other hand, the discounted value of the portfolio that has $\Delta(t)$ shares of this stock is given by (see Lecture 5)

$$d(e^{-rt}X(t)) = e^{-rt}\sigma S(t)\Delta(t)d\widetilde{B}(t).$$

Thus, to hedge a short position in the Asian call, we should have $\Delta(t)=v_x(t,S(t,Y(t)))$.

In the next lecture we shall perform a change of numeraire. The new numeraire will be the stock price. This will allow us to reduce our 2-dimensional problem to 1-dimensional.

4. Fubini's theorem for conditional expectations

In conclusion, I shall state and prove one useful fact about conditional expectations. This is a corollary of Fubini's theorem.

Lemma 4.1 (Fubini's theorem for conditional expectations)**.** *Let $X(t,\omega)$, $0\le t\le T$, be a stochastic process adapted to some filtration $\mathcal{F}(t)$, $0\le t\le T$, and such that either $X(t,\omega)\ge 0$ on $[0,T]\times\Omega$ or*

$$E\int_0^T |X(s)|\,ds<\infty.$$

Then for every $t\in[0,T]$

$$E\left[\int_t^T X(s)\,ds\,\middle|\,\mathcal{F}(t)\right]=\int_t^T E[X(s)|\mathcal{F}(t)]\,ds \quad a.s..$$

Proof. This is a direct consequence of Fubini's theorem (Jacod and Protter, Probability Essentials, p. 67) and the definition of conditional expectation. Fix $t\in[0,T]$ and set

$$Y(\omega)=E\left[\int_t^T X(s,\omega)\,ds\middle|\mathcal{F}(t)\right],\quad Z(s,\omega)=E[X(s,\omega)|\mathcal{F}(t)].$$

Since t is fixed, we can drop it from the notation for simplicity. The definition of conditional expectation says that for every $A\in\mathcal{F}(t)$

$$\int_A Y(\omega)\,dP=\int_A\left(\int_t^T X(s,\omega)\,ds\right)dP.$$

By Fubini's theorem,

$$\int_A\int_t^T X(s,\omega)\,ds\,dP=\int_t^T\int_A X(s,\omega)\,dPds.$$

By the definition of conditional expectation,

$$\int_A X(s,\omega)\,dP=\int_A Z(s,\omega)\,dP.$$

Again by Fubini's theorem,

$$\int_t^T\int_A Z(s,\omega)\,dPds=\int_A\int_t^T Z(s,\omega)\,ds\,dP.$$

Putting all pieces together we obtain that for every $A\in\mathcal{F}(t)$

$$\int_A Y(\omega)\,dP=\int_A\left(\int_t^T Z(s,\omega)\,ds\right)dP.$$

Since $Y(\omega)$ and $\int_t^T Z(s,\omega)\,ds$ are $\mathcal{F}(t)$-measurable and the above equality of integrals holds for all $A\in\mathcal{F}(t)$, we conclude that

$$Y=\int_t^T Z(s,\omega)\,ds\quad\text{a.s.},$$

which is exactly what we need. □