MTH 9831. Solutions to Quiz 4.

(1) (8 points) Let $(B_1(t), B_2(t))$, $t \geq 0$, be a standard 2-dimensional Brownian motion. Define

$$Y(t) = B_1^2(t)+B_2^2(t);\ \ W(0) = 0,\ dW(t) = \frac{B_1(t)}{\sqrt{Y(t)}}\, dB_1(t)+\frac{B_2(t)}{\sqrt{Y(t)}}\, dB_2(t).$$

(a) Show that $W(t)$, $t \geq 0$, is a standard Brownian motion.

(b) Show that $Y(t)$, $t \geq 0$, satisfies the SDE

$$dY(t) = 2dt + 2\sqrt{Y(t)}\, dW(t).$$

(c) Let $Y(0) = y > 0$ and Y be a solution of the SDE in part (b). Which ODE should $f(x)$ satisfy for the process $(f(Y(t)))_{t\geq 0}$ to be a (local) martingale?

Solution. (a) Let $(\mathcal{F}(t))_{t\geq 0}$ be the natural filtration of our 2-dimensional standard BM. We shall use Lévy's characterization of BM ($d = 1$). First of all, by the properties of Itô integral[1], the process

$$W(t) := \int_0^t \frac{B_1(s)}{\sqrt{Y(s)}}\, dB_1(s) + \int_0^t \frac{B_2(s)}{\sqrt{Y(s)}}\, dB_2(s), \quad t \geq 0,$$

starts from 0, has continuous paths[2] and is a martingale with respect to $(\mathcal{F}(t))_{t\geq 0}$. Next, its quadratic variation is

$$\begin{aligned} d[W]_t &= \frac{B_1^2(t)}{Y(t)}\, d[B_1]_t + \frac{2B_1(t)B_2(t)}{Y(t)} d[B_1, B_2]_t + \frac{B_2^2(t)}{Y(t)}\, d[B_2]_t \\ &= \frac{B_1^2(t) + B_2^2(t)}{Y(t)}\, dt = dt. \end{aligned}$$

We used the fact that for independent Brownian motions B_1 and B_2 the cross-variation process is identically 0. Thus, the conditions of Lévy's characterization are satisfied, and W is a standard BM.

[1] The integrands $B_i(t)/\sqrt{Y(t)}$ are adapted and bounded in absolute value by 1 for all $t \geq 0$ (as they are simply cos and sin of the angle between the "tip" of the 2-dimensional BM vector and the positive direction of the first coordinate axis.

[2] Lévy's theorem as stated in Theorem 4.3 of lecture 4, i.e. requiring $[M]_t \equiv t$, holds without the additional assumption of continuity of $M(t)$ in t. This is due to the fact that continuity of the quadratic variation process implies continuity of paths. Yet in our very first version of Lévy's theorem, Theorem 3.4 from lecture 3, where we ask$M^2(t)-t,\ t \geq 0$, to be a martingale, continuity of paths is an essential condition. Example: let $M(t) := N(t) - t$ be a compensated Poisson process with intensity 1. Then $M(0) = 0$, it is a martingale, and $M^2(t) - t$ is also a martingale. But, clearly, $M(t)$ is not a Brownian motion.

(b) Applying 2-dimensional Itô's formula to the function $f(B_1(t), B_2(t))$ where $f(x,y) = x^2 + y^2$, we get

$$\begin{aligned} dY(t) &= 2B_1(t)\,dB_1(t) + 2B_2(t)\,dB_2(t) + \frac{1}{2}\,(2\,dt + 2d[B_1, B_2] + 2\,dt) \\ &= 2\sqrt{Y(t)}\left(2\frac{B_1(t)}{\sqrt{Y(t)}}\,dB_1(t) + 2\frac{B_2(t)}{\sqrt{Y(t)}}\,dB_2(t)\right) + 2dt \\ &= 2\sqrt{Y(t)}\,dW(t) + 2\,dt. \end{aligned}$$

(c) Applying Itô's formula we get

$$\begin{aligned} df(Y(t)) &= f'(Y(t))\,dY(t) + \frac{1}{2}\,f''(Y(t))\,d[Y]_t \\ &= f'(Y(t))(2\sqrt{Y(t)}\,dW(t) + 2\,dt) + \frac{1}{2}\,f''(Y(t))(4Y(t))\,dt \\ &= 2f'(Y(t))\sqrt{Y(t)}\,dW(t) + 2(f'(Y(t)) + f''(Y(t))Y(t))\,dt. \end{aligned}$$

Equating the dt term to zero we get that f should satisfy the equation $xf''(x) + f'(x) = 0$ for all $x > 0$ (Y is positive for all positive t).[3]

(2) (2 points) Suppose that $(B_1(t), B_2(t))$, $t \geq 0$, is a 2-dimensional correlated Brownian motion with a constant correlation coefficient ρ and standard BM marginals. Suppose that I know that the process $(W_1(t), W_2(t))$ defined by $W_1(t) = w_{11}B_1(t) + w_{12}B_2(t)$ and $W_2(t) = w_{21}B_1(t) + w_{22}B_2(t)$ is a standard 2-dimensional Brownian motion. Find ρ.

Solution. Here one does not really need any stochastic calculus. If the process W is a standard 2-dimensional BM then the correlation between $W_1(1)$ and $W_2(1)$ has to be equal to 0. Computing this correlation we get the equation:

$$w_{11}w_{21} + w_{12}w_{22} + \rho(w_{21}w_{12} + w_{11}w_{22}) = 0.$$

We conclude that if there is such a ρ then it has to be equal to

$$-\frac{w_{11}w_{21} + w_{12}w_{22}}{w_{21}w_{12} + w_{11}w_{22}}.$$

There are many other ways to compute the correlation. They lead to seemingly different expressions for ρ. The truth is that for W to be a

[3] This type of ODE, namely, $x^2 f''(x) + \alpha x f'(x) + \beta f(x) = 0$, $x > 0$, often arises in applications. The 2-dimensional vector space of its solutions is spanned by functions x^{λ_1} and x^{λ_2} for some constants λ_1 and λ_2 which depend on α and β. It is convenient to include $\ln x$ as well as constant functions in the set of possible solutions. In part (c) the solution space is spanned by 1 and $\ln x$, for example. When λ_i are complex, one can use the usual exponential interpretation of $x^{a+bi} = x^a e^{ib\ln x} = x^a(\cos(b\ln x) + i\sin(b\ln x))$ and take the real and imaginary parts as a basis for solutions.

standard Brownian motion, w_{ij} should satisfy certain conditions. Under these conditions all these different formulas are equivalent.

Note that if $W = AB$, where $A = \|w_{ij}\|$ is a given matrix, then computing covariance matrices (by scaling we can set $t = 1$ or simply compute and then cancel t on both sides) we get the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} A^T. \tag{1}$$

The left hand side is non-degenerate, so A has to be non-degenerate, and

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = (A^T A)^{-1}.$$

This tells us that the problem is indeed well-posed, i.e. ρ is uniquely defined by A (as long as AB is a standard Brownian motion).

We can also solve the reverse problem: given $\rho \in (-1, 1)$, describe all A such that W is a standard 2-dimensional Brownian motion. The solution can be obtained, for example, as follows. Write

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1-\rho^2} & 0 \\ \rho & 1 \end{bmatrix} =: RR^T$$

and set $U = AR$. Then (1) becomes $I = UU^T$, which means that U is an orthogonal matrix. Given any orthogonal matrix U we can define $A = UR^{-1}$. Every solution A of the problem is obtained in this way from some orthogonal matrix U.