MTH 9831 Homework 9 - Group5

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(1)

Proof. Recall that

$$\hat{W}(t) = \alpha t + \tilde{W}(t)$$

where $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$ is a constant and $\tilde{W}(t)$ is a standard Brownian motion, since Brownian motion $\tilde{W}(t)$ has independent increment, it follows that $\hat{W}(t)$ also has independent increment, i.e.

$$\{\hat{W}(u) - \hat{W}(t)\}_{u \geq t} \text{ is independent with } \mathcal{F}(t) \tag{1}$$

We have

$$S(T) = S(0)e^{\sigma\hat{W}(T)} = S(0)e^{\sigma\tilde{W}(t)} \cdot e^{\sigma(\hat{W}(T)-\hat{W}(t))} = S(t) \cdot e^{\sigma(\hat{W}(T)-\hat{W}(t))}$$

Also, since

$$Y(T) = \max_{0\leq u\leq T} S(u) = S(0)e^{\sigma\hat{M}(T)}$$

where $\hat{M}(T) = \max_{0\leq u\leq T}\hat{W}(u)$, note that

$$\begin{aligned}\hat{M}(T) &= \max\{\max_{0\leq u\leq t}\hat{W}(u), \max_{t\leq u\leq T}\hat{W}(u)\}\\ &= \max\{\max_{0\leq u\leq t}\hat{W}(u), \hat{W}(t) + \max_{t\leq u\leq T}[\hat{W}(u) - \hat{W(t)}]\}\end{aligned}$$

it follows that

$$\begin{aligned}Y(T) &= S(0)e^{\sigma\max\{\max_{0\leq u\leq t}\hat{W}(u), \hat{W}(t)+\max_{t\leq u\leq T}[\hat{W}(u)-\hat{W(t)}]\}}\\ &= \max\{Y(t), S(t)e^{\sigma\max_{t\leq u\leq T}[\hat{W}(u)-\hat{W(t)}]}\}\end{aligned}$$

the last equality holds since function $f(x) = e^x$ is a monotone increasing function.

In summary, we have for any measurable function $f(x, y)$,

$$f(S(T), Y(T)) = f(S(t)\cdot e^{\sigma(\hat{W}(T)-\hat{W}(t))}, \max\{Y(t), S(t)e^{\sigma\max_{t\leq u\leq T}[\hat{W}(u)-\hat{W(t)}]}) \tag{2}$$

where $\hat{M}(t, T) = \max_{t\leq u\leq T}[\hat{W}(u) - \hat{W}(t)]$.

According to equation (2), we can see that $f(S(T), Y(T))$ is a function of $S(t)$, $Y(t)$, $\hat{W}(T) - \hat{W}(t)$ and $\hat{M}(t,T) \triangleq \max_{t \le u \le T}[\hat{W}(u) - \hat{W}(t)]$. Thus, define $\tilde{f}$ as

$$\tilde{f}(S(t), Y(t), \hat{W}(T) - \hat{W}(t), \hat{M}(t,T)) \triangleq f(S(T), Y(T))$$

According to (1), we know that $\hat{W}(T) - \hat{W}(t)$ and $\hat{M}(t,T))$ are independent with $\mathcal{F}(t)$, let

$$g(s,y) = \mathbb{E}\tilde{f}(s, y, \hat{W}(T) - \hat{W}(t), \hat{M}(t,T))$$

then according to Independence Lemma 2.3.4,

$$\mathbb{E}[f(S(T), Y(T))|\mathcal{F}(t)] = \mathbb{E}[\tilde{f}(S(t), Y(t), \hat{W}(T) - \hat{W}(t), \hat{M}(t,T))|\mathcal{F}(t)] = g(S(t), Y(t))$$

thus we have proven that the pair of processes $(S(t), Y(t))$ is a Markov process. □

(2)

(ii)

Proof.

$$\begin{aligned}\sum_{j=1}^{m}(Y(t_j) - Y(t_{j-1}))^2 &\le \max_{1 \le j \le m}(Y(t_j) - Y(t_{j-1})) \cdot \sum_{j=1}^{m}(Y(t_j) - Y(t_{j-1})) \\ &= \max_{1 \le j \le m}(Y(t_j) - Y(t_{j-1})) \cdot (Y(T) - Y(0))\end{aligned}$$

Since $Y(t)$ is monotone increasing in t,

$$Y(t_j) - Y(t_{j-1}) \ge 0, \forall\ 1 \le j \le m$$

thus for all $1 \le j \le m$,

$$Y(t_j) - Y(t_{j-1}) \le \sum_{j=1}^{m}(Y(t_j) - Y(t_{j-1})) = Y(T) - Y(0) \tag{3}$$

Plugging (3) into the above inequality, it follows that

$$\sum_{j=1}^{m}(Y(t_j) - Y(t_{j-1}))^2 \le (Y(T) - Y(0))^2$$

□

Next, we prove that the left hand side converges to 0 in L^2, i.e. that

$$\lim_{||\Pi|| \to 0} \mathbb{E}[(\sum_{j=1}^{m}(Y(t_j) - Y(t_{j-1}))^2)^2] = 0$$

Since

$$\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2 \leq (Y(T)-Y(0))^2$$

it follows that

$$(\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2)^2 \leq (Y(T)-Y(0))^4$$

and according to the property of GBM,

$$\mathbb{E}(Y(T)-Y(0))^4 < \infty$$

By Domiant Covergence Theorem (Hereinafter referred to as DCT), we have

$$\lim_{||\Pi||\to 0}\mathbb{E}[(\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2)^2] = \mathbb{E}[(\lim_{||\Pi||\to 0}\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2)^2]$$

Using part (i), we know that

$$\lim_{||\Pi||\to 0}\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2 = 0 \ a.s.$$

so

$$\lim_{||\Pi||\to 0}\mathbb{E}[(\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))^2)^2] = \mathbb{E}[0^2] = 0$$

(iii) We first prove that for each $T > 0$ the cross-variation $[S, Y]_T$ exists in a.s. and is equal to 0.

Proof.

$$\begin{aligned} 0 \leq |\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))| &\leq \sum_{j=1}^{m}|(S(t_j)-S(t_{j-1}))|(Y(t_j)-Y(t_{j-1})) \\ &\leq \max_{1\leq j\leq m}|S(t_j)-S(t_{j-1})|\cdot(Y(T)-Y(0)) \end{aligned}$$

Let $||\Pi|| \to 0$ and take the limit on both sides of the inequality, then since $S(t)$ is continuous a.s., $\max_{1\leq j\leq m}|S(t_j)-S(t_{j-1})|$ has limit zero as $||\Pi|| = \max_{q\leq j\leq m}(t_j - t_{j-1})$ goes to zero.

Thus, we conclude that

$$\lim_{||\Pi||\to 0}\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1})) = 0 \ a.s.$$

i.e. $[S, Y]_T = 0$ a.s. □

Next, we turn to prove that $[S, Y]_T$ exists in L^2 sense and is equal to 0.

Proof.

$$|\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))| \leq \sum_{j=1}^{m}|S(t_j)-S(t_{j-1})|(Y(t_j)-Y(t_{j-1}))$$

Since

$$|S(t_j)-S(t_{j-1})| \leq |S(t_j)|+|S(t_{j-1})| = S(t_j)+S(t_{j-1}) \leq 2Y(T)$$

We have

$$|\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))| \leq 2Y(T)\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1})) = 2Y(T)(Y(T)-Y(0))$$

so

$$[\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))]^2 \leq 4Y(T)^2\cdot(Y(T)-Y(0))^2 \tag{4}$$

Since $\mathbb{E}[Y(T)^2\cdot(Y(T)-Y(0))^2] \leq \infty$, use (4), DCT and $[S,Y]_T = 0$ $a.s.$, we have

$$\lim_{||\Pi||\to 0}\mathbb{E}[\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))]^2 = \mathbb{E}[\lim_{||\Pi||\to 0}\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))(Y(t_j)-Y(t_{j-1}))]^2 = 0$$

i.e. $[S,Y]_T = 0$ in L^2 sense. □

(3)

(i)

$$\begin{aligned}\tilde{\mathbb{E}}[\int_0^T S(u)du|\mathcal{F}(t)] &= \tilde{\mathbb{E}}[\int_0^t S(u)du + \int_t^T S(u)du|\mathcal{F}(t)] \\ &= \int_0^t S(u)du + \tilde{\mathbb{E}}[\int_t^T s(u)du|\mathcal{F}(t)]\end{aligned}$$

Since $\tilde{\mathbb{E}}[\int_0^T S(u)du] < \infty$, then according to Fubini's theorem for conditional expectations, we have

$$\tilde{\mathbb{E}}[\int_0^T S(u)du|\mathcal{F}(t)] = \int_0^t S(u)du + \int_t^T \tilde{\mathbb{E}}[S(u)|\mathcal{F}(t)]du$$

Since under $\tilde{P}$, $e^{-ru}S(u)$ is a martingale, we have

$$\int_t^T \tilde{\mathbb{E}}[S(u)|\mathcal{F}(t)]du = \int_t^T e^{ru}\tilde{\mathbb{E}}[e^{-ru}S(u)|\mathcal{F}(t)]du = \int_t^T e^{ru}S(t)du = \frac{S(t)}{r}(e^{r(T-t)}-1)$$

Suppose we have $S(t) = x \geq 0$ and $\int_t^T S(u)du = y \geq 0$, we conclude that

$$v(t,x,y) = \frac{e^{-r(T-t)}}{T}(\frac{x}{r}(e^{r(T-t)}-1)+y)$$

(ii)

$$
\begin{aligned}
v_t &= \frac{r}{T}e^{-r(T-t)}(\frac{x}{r}(e^{r(T-t)}-1)+y)-\frac{x}{T} = rv - \frac{x}{T} \\
v_x &= \frac{1}{rT}e^{-r(T-t)}(e^{r(T-t)}-1) \\
v_y &= \frac{1}{T}e^{-r(T-t)} \\
v_{xx} &= 0
\end{aligned}
$$

Recall Black-Scholes-Merton equation,

$$
v_t + rxv_x + xv_y + \frac{1}{2}\sigma^2 v_{xx} = rv
$$

plugging the expression for v_t, v_x, v_y and v_{xx} derived above, we have

$$
v_t + rxv_x + xv_y + \frac{1}{2}\sigma^2 v_{xx} = rv - \frac{x}{T} + \frac{x}{T}e^{-r(T-t)}(e^{r(T-t)}-1) + \frac{x}{T}e^{-r(T-t)} = rv
$$

Also,

$$
\begin{aligned}
v(t,0,y) &= \frac{1}{T}e^{-r(T-t)}y = e^{-r(T-t)}(\frac{y}{T}-0)^+ \\
v(T,x,y) &= \frac{y}{T} = (\frac{y}{T}-0)^+
\end{aligned}
$$

Thus, we have verifies that the function $v(t,x,y)$ satisfies required equations.

(iii) According to (ii),

$$
\Delta(t) = v_x = \frac{1}{rT}(1-e^{-r(T-t)})
$$

which is not random.

(iv)

Proof. Denote the value of this portfolio at time t as $X(t)$, then after dt, the value of this portfolio changes and those changes consist of two parts:

1. The change in the cash position $= r(X(t)-\Delta(t)S(t))dt - S(t)d\Delta(t)$
2. The change in the value of the underlying asset $= (\Delta(t)+d\Delta(t))(S(t)+dS(t)) - \Delta(t)S(t)$

Ignore the higher order term $d\Delta(t)dS(t)$, and sum the above two parts up, we have

$$
dX(t) = r(X(t)-\Delta(t)S(t))dt + \Delta(t)dS(t)
$$

rearrange it, we have

$$
\begin{aligned}
& dX(t) - rX(t)dt = \Delta(t)(dS(t) - rS(t)dt) \\
\iff & d[e^{-rt}X(t)] = \Delta(t)d[e^{-rt}S(t)]
\end{aligned}
$$

Integrate on both sides,

$$e^{-rT}X(T) - X(0) = \Delta(t)e^{-rt}S(t)|_0^T - \int_0^T e^{-rt}S(t)d\Delta(t) \tag{5}$$

since

$$d\Delta(t) = -\frac{1}{T}e^{-r(T-t)}$$

equation (5) changes to

$$e^{-rT}X(T) - X(0) = -\Delta(0)S(0) + \frac{1}{T}e^{-rT}\int_0^T S(t)dt$$

since

$$X(0) = v(0, S(0), 0) = \frac{1}{rT}(1 - e^{-rT})S(0) = \Delta(0)S(0)$$

we conclude that

$$X(T) = \frac{1}{T}\int_0^T S(t)dt$$

$\square$