November 14, 2018

MTH 9831. Solutions to Quiz 7.

(1) (2 points) (Finding the generator of a diffusion) Let $(X_1(t), X_2(t))$ solve the following system of SDE

$$dX_1(t) = X_1(t)\,dt + \sqrt{X_2(t)}X_1(t)\,dB_1(t)$$
$$dX_2(t) = (1 - X_2(t))\,dt + 2\sqrt{X_2(t)}\,dB_2(t),$$

where $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion with correlated components, $d[B_1, B_2](t) = dt/2$. Write down the corresponding generator $\mathcal{A}$.

Solution. Since we have a 2-dimensional process, our generator will act on functions of 2 variables. Let the variables be x_1 and x_2. All we need to do is to compute the coefficients of $\mathcal{A}$. The drifts of X_1 and X_2 determine the coefficients in front of the first derivatives, and from the quadratic and cross variation we can determine the diffusion matrix. We get

$$\mathcal{A} = x_1\frac{\partial}{\partial x_1} + (1 - x_2)\frac{\partial}{\partial x_2} + \frac{x_2x_1^2}{2}\frac{\partial^2}{\partial x_1^2} + x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + 2x_2\frac{\partial^2}{\partial x_2^2}.$$

(2) (9 points) (Solving an elementary non-homogeneous PDE) Let $g(t,x)$ be a solution to the following terminal value problem:

$$g_t(t,x) + xg_x(t,x) + 2g_{xx}(t,x) + x = 0, \quad g(3,x) = x^2,\ (t,x) \in [0,3) \times \mathbb{R}.$$

(a) Write a stochastic representation for $g(t,x)$. Do not forget to include the corresponding SDE and the starting point of a stochastic process.

(b) Solve the SDE from part (a) explicitly.

(c) Substitute the solution of the SDE found in (b) into the formula of part (a) and find $g(t,x)$.

Solution. (a) We use the result of problem 1 from HW8 and write

$$g(t,x) = E_{t,x}[(X(3))^2] + \int_t^3 E_{t,x}(X(u))\,du, \quad (t,x) \in [0,3) \times \mathbb{R},$$

where $X(u)$, $t \le u \le T$, solves the SDE

$$dX(u) = X(u)du + 2dB(u),\ X(t) = x.$$

(b) This is an OU process. Using e^{-u} as an integrating factor we get

$$d(e^{-u}X(u)) = -e^{-u}X(u)\,du + e^{-u}\underbrace{(X(u)du + 2dB(u))}_{dX(u)} = 2e^{-u}dB(u).$$

Integrating from t to u and substituting $X(t) = x$ we get that

$$X(u) = xe^{u-t} + 2e^u\int_t^u e^{-s}\,dB(s),\ t \le u \le 3.$$

Since the integrand is deterministic, $X(u)$ is normally distributed.

(c) Substituting $X(u)$ into the formula from part (a) we get that

$$g(t,x) = E_{t,x}[(X(3))^2] + \int_t^3 E_{t,x}(X(u))\,du$$

where $X(3)$ is normally distributied with mean xe^{3-t} and variance

$$4e^6 \int_t^3 e^{-2s}\,ds = 2(e^{2(3-t)} - 1),$$

and $E_{t,x}(X(u)) = xe^{u-t}$. We conclude that

$$\begin{aligned} g(t,x) &= \underbrace{(xe^{3-t})^2 + 2(e^{2(3-t)} - 1)}_{\text{the second moment of } X(3)} + xe^{-t}\int_t^3 e^u\,du \\ &= x^2 e^{2(3-t)} + x(e^{3-t} - 1) + 2(e^{2(3-t)} - 1). \end{aligned}$$

Remark. This problem can be solved differently. One could first notice that the function $u(t,x) = -x$ solves the inhomogeneous PDE. This reduces the problem to solving a homogeneous equation. Namely, let $v(t,x) = g(t,x) - u(t,x) = g(t,x) + x$. Then v solves

$$v_t(t,x) + xv_x(t,x) + 2v_{xx}(t,x) = 0, \quad v(3,x) = x^2 + x,\ (t,x) \in [0,3) \times \mathbb{R}.$$

One needs only a basic FK formula to find v. This is another standard approach but sometimes such a "guess" as "$u(t,x) = -x$ is a solution" is not available.

Surely, there are other possible solutions which do not use the FK formula at all. Yet I wanted to see a solution which involved FK in some way. This was one of the purposes of the quiz.