Probability and Stochastic Processes in Finance I (MTH 9831).

Solutions to the Midterm Examination.

Problem 1. Compute the indicated parameters of the following conditional normal distributions:

(a) $(B(3), B(6))$ given that $B(1) = x$ (parameters: mean vector and covariance matrix).

(b) $B(3)$ given that $B(1) = x$ and $B(6) = z$ (parameters: mean and variance).

Solution. (a) Using the Markov property of BM and stationarity of increments we note that this conditional distribution is the same as that for the unconditional vector $(x + B(2), x + B(5)) = (x, x) + (B(2), B(5))$. The last has mean vector (x, x) and the covariance matrix

$$C = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

(b) Again by the Markov property and stationarity of increments the problem boils down to finding the distribution of $B(2)$ given that $B(5) = z - x$ and then shifting by x.

Using Gram-Schmidt decomposition, i.e. writing

$$B(2) = \lambda B(5) + W,$$

where W is independent of $B(5)$ and computing first $\mathrm{Cov}(B(2), B(5))$ and then $\mathrm{Var}(B(2))$ we determine that $\lambda = 2/5$ and

$$2 = \lambda^2 \cdot 5 + \mathrm{Var}(W) \quad \Rightarrow \quad \mathrm{Var}(W) = 2 - \frac{4 \cdot 5}{25} = \frac{6}{5}.$$

Thus,

$$B(2) = \frac{2}{5} B(5) + \sqrt{\frac{6}{5}} Z,$$

where $Z \sim \mathcal{N}(0, 1)$ is independent of $B(5)$.

Therefore, conditional on $B(1) = x$ and $B(6) = z$, the distribution of $B(3)$ is normal with mean $x + 2(z - x)/5 = 3x/5 + 2z/5$ and variance $6/5$.

Alternatively, one can write the basic formula for the conditional density:

$$f_{(B(3) \mid B(1), B(6))}(y \mid x, z) = \frac{p(2, x, y)\, p(3, y, z)}{p(5, x, z)},$$

where $p(t, x, y) = (2\pi t)^{-1/2} \exp(-(y - x)^2/(2t))$. Manipulating only the square root terms one easily gets the variance as $(2 \cdot 3)/5 = 6/5$. For the expectation (instead of doing calculations with the exponents) it is still much more preferable to draw a line through $(1, B(1)) = (1, x)$ and $(6, B(6)) = (6, z)$ and find the coordinate for $t = 3$:

$x + \dfrac{z-x}{6-1} \cdot (3-1) = 3x/5 + 2z/5$. This gives the conditional expectation of $B(3)$ given $B(1) = x$ and $B(6) = z$.

Problem 2. Assume Black-Scholes model with a constant interest rate and no dividends. A perpetual American cash-or-nothing option can be exercised at any time $t \geq 0$ (no expiration). If exercised at time t its payoff is

$$\begin{cases} 1, & \text{if } S(t) \leq K; \\ 0, & \text{if } S(t) > K. \end{cases}$$

(a) What is the optimal exercise strategy?

(b) What is the time 0 price of this option? You may use the fact that for $\mu \in \mathbb{R}$, $m > 0$, $X(t) = \mu t + B(t)$, $\tau_m = \inf\{t \geq 0 : X(t) = m\}$, and for all $\lambda > 0$

$$E(e^{-\lambda \tau_m} 1_{\{\tau_m < \infty\}}) = e^{m\mu - m\sqrt{\mu^2 + 2\lambda}}.$$

Solution. Let $r \geq 0$ be the interest rate and the stock price follow a GBM with some drift parameter (irrelevant here) and volatility $\sigma > 0$.

(a) Exercise at the first time $t \geq 0$ when $S(t) \leq K$. Since the paths are continuous, exercise at $T_K := \inf\{t \geq 0 : S(t) = K\}$.

(b) If $S(0) \leq K$ then the option can be exercised immediately and its price is 1. Hence, we assume from now on that $S(0) > K$. The time 0 price of this option is given by the expression $\tilde{E}(e^{-rT_K} \mathbb{1}_{\{T_K < \infty\}})$ (the risk-neutral expectation of the present value of the payoff). Under the risk-neutral measure the stock price process is a GBM with drift parameter $r - \sigma^2/2$ and volatility parameter σ. Converting the problem from GBM to BM (i.e. taking logarithms) we get

$$S(0)e^{(r-\sigma^2/2)t + \sigma B(t)} = K \quad \Leftrightarrow \quad \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) t + B(t) = \frac{1}{\sigma} \ln \frac{K}{S(0)} < 0.$$

Since the result about BM is given for $m > 0$, we shall switch the sign and use the fact that $(-B(t))_{t \geq 0}$ is a BM. We have

$$\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right) t + (-B(t)) = \frac{1}{\sigma} \ln \frac{S(0)}{K} > 0.$$

Now T_K has the same distribution as τ_m for the process $\mu t + (-B(t))$, $t \geq 0$, where

$$m = \frac{1}{\sigma} \ln \frac{S(0)}{K}, \quad \mu = \frac{\sigma}{2} - \frac{r}{\sigma}.$$

Therefore, applying the given result with $\lambda = r$ we get for all $r > 0$ that

$$\tilde{E}(e^{-rT_K} \mathbb{1}_{\{T_K < \infty\}}) = e^{m\mu - m\sqrt{\mu^2 + 2r}}.$$

Substituting the values for μ and m and simplifying we conclude that for all $r > 0$

$$\tilde{E}(e^{-rT_K} \mathbb{1}_{\{T_K < \infty\}}) = \left(\frac{K}{S(0)}\right)^{\frac{2r}{\sigma^2}}.$$

Case $r = 0$ is best handled separately because random times T_K depend on r through the drift μ. So instead of passing to the limit as $r \to 0$ in the above expression[1] we first take the limit as $\lambda \to 0$ in the stated result (for fixed $m > 0$ and μ)[2] and see that for $\mu > 0$, $P(\tau_m < \infty) = 1$. Applying this result directly (when $r = 0$ we have $\mu = \sigma/2 > 0$) we conclude that the price is equal to $\tilde{E}(\mathbb{1}_{\{T_K<\infty\}}) = \tilde{P}(T_K < \infty) = 1$.

Problem 3. Let $(B(t))_{t\geq 0}$ be a standard Brownian motion (BM), $(\mathcal{F}(t))_{t\geq 0}$ be a filtration for this BM, and $(\Delta(t))_{t\geq 0}$ be an adapted process such that for all $t \geq 0$

$$\int_0^t E(\Delta^2(s))ds < \infty.$$

(a) Use basic properties of an Itô integral to show that for all $s, t \geq 0$,

$$E\left(\int_0^t \Delta(u)\,dB(u) \cdot \int_0^s \Delta(v)\,dB(v)\right) = \int_0^{s\wedge t} E(\Delta^2(r))\,dr.$$

State clearly which properties you used.

(b) Let $Y(t) = \int_0^t B^2(u)\,dB(u)$. Compute the covariance and correlation between $Y(s)$ and $Y(t)$ for all $s, t \geq 0$.

Solution. (a) Let

$$M(t) = \int_0^t \Delta(s)\,dB(s), \quad t \geq 0.$$

Then by the properties of Itô integral $(M(t))_{t\geq 0}$ is a martingale relative to $(\mathcal{F}(t))_{t\geq 0}$. Assume without loss of generality that $0 \leq s \leq t$. Then

$$E[M(t)M(s)] = E[(M(t) - M(s))M(s)] + E\left[(M(s))^2\right].$$

The orthogonality (in L^2) of martingale increments gives that the first term is 0. In more detail, by the properties of conditional expectation and the martingale property,

$$\begin{aligned} E[(M(t) - M(s))M(s)] &= E[E[(M(t) - M(s))M(s) \mid \mathcal{F}(s)]] \\ &= E[M(s)E[(M(t) - M(s)) \mid \mathcal{F}(s)]] = 0. \end{aligned}$$

We conclude that if $0 \leq s \leq t$ then

$$E[M(t)M(s)] = E\left[(M(s))^2\right].$$

Exchanging the roles of s and t we get (a).

(b) This is just an application of (a) with $\Delta(t) = B^2(t)$. The square integrability condition is satisfied because $E(\Delta^2(s)) = E(B^4(s)) = 3s^2$, which is clearly integrable from 0 to t for all $t \geq 0$. Therefore,

$$\mathrm{Cov}(Y(t), Y(s)) = E(Y(t)Y(s)) = \int_0^{s\wedge t} 3r^2\,dr = (s \wedge t)^3 = s^3 \wedge t^3.$$

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[1] We would have to argue that $rT_K(r,\omega) \to 0$ as $r \to 0$ a.s.. This is true but can be avoided.

[2] This is easily justified by the bounded convergence theorem as the function $e^{-\lambda\tau_m}\mathbb{1}_{\{\tau_m<\infty\}} \geq 0$ is bounded above by 1 and converges to 0 as $\lambda \to 0$ a.s..

Finally the correlation is

$$\frac{\operatorname{Cov}(Y(t),Y(s))}{\sqrt{\operatorname{Cov}(Y(s),Y(s))}\cdot\sqrt{\operatorname{Cov}(Y(t),Y(t))}} = \frac{s^3\wedge t^3}{s^{3/2}\,t^{3/2}} = \left(\frac{s\wedge t}{s\vee t}\right)^{3/2}.$$

Problem 4. Let $S(0)=x$ and for some fixed $a,\mu\ge 0$ and $\sigma>0$

$$dS(t) = (\mu - aS(t))dt + \sigma\,dB(t).$$

(a) Find the expectation and variance of $S(t)$.

(b) Find a closed form solution of the above equation. Then find the distribution of $S(t)$ directly from the closed form solution.

Solution. (a) Write the equation in the integral form and take expectations:

$$S(t) = S(0) + \int_0^t \mu - aS(u))du + \int_0^t \sigma\,dB(u);$$

$$E(S(t)) = S(0) + \int_0^t \mu - aE(S(u))du.$$

Let $m(t)=E(S(t))$. Then $m(0)=S(0)$ and $m'(t)=\mu - am(t)$, and

$$m(t) = xe^{-at} + \frac{\mu}{a}\,(1-e^{-at}).$$

Remark. There are several ways to solve a linear non-homogeneous equation. I shall give only one way ("variation of a constant"), but if you are used to a different approach, use the one you know.

Step 1. Find the general solution of the homogeneous equation $m_h'(t) = -am_h(t)$. Get $m_h(t) = Ce^{-at}$, where C is an arbitrary constant.

Step 2. Try to find a solution of the original equation in the form $m(t)=C(t)e^{-at}$, $m(0)=C(0)=x$. Substitution gives

$$m'(t) = C'(t)e^{-at} - aC(t)e^{-at} = \mu - aC(t)e^{-at} \;\Rightarrow\; C'(t)=\mu e^{at},\; C(0)=x.$$

Therefore, $C(t) = x + \frac{\mu}{a}\,(e^{at}-1)$ and we get the above formula for $m(t)$.

A standard approach for finding the variance is to find the equation for $dS^2(t)$ using Itô's formula and proceed in the same way as for the expectation to compute $E(S^2(t))$. Then we have to subtract the square of the expectation to get the variance. By the definition, the variance is the expectation of $(S(t)-m(t))^2$, which we shall try to compute directly.

$$d(S(t)-m(t))^2 = 2(S(t)-m(t))d(S(t)-m(t)) + d[S-m,S-m](t),$$

where $d[S-m,S-m](t) = d[S,S](t) = \sigma^2 dt$ and

$$\begin{aligned} d(S(t)-m(t)) &= (\mu - aS(t))dt + \sigma\,dB(t) - m'(t)dt \\ &= (\mu - aS(t))dt + \sigma\,dB(t) - (\mu - am(t))dt \\ &= -a(S(t)-m(t))\,dt + \sigma\,dB(t). \end{aligned}$$

Therefore,

$$d(S(t)-m(t))^2 = -2a(S(t)-m(t))^2\,dt + 2\sigma(S(t)-m(t))\,dB(t) + \sigma^2 dt.$$

Setting $v(t) := E(S(t)-m(t))^2$ we get that $v(0)=0$ and

$$v(t) = -2a\int_0^t v(u)\,du + \sigma^2 t.$$

This equation is of the same type as the one for $m(t)$. Applying the last remark we obtain

$$v(t) = \frac{\sigma^2}{2a}(1-e^{-2at}).$$

(b) We see that the equation has the term $-aS(t)dt$. In other words, $S(t)$ gets "discounted". This tells us that the undiscounted process might look simpler and we consider the process $e^{at}S(t)$ instead (this approach comes from ODEs, we are finding an "integrating factor"). We have

$$d(e^{at}S(t)) = e^{at}((\mu - aS(t))dt + \sigma\,dB(t)) + ae^{at}S(t)\,dt = e^{at}(\mu\,dt + \sigma\,dB(t)). \tag{1}$$

Then

$$e^{at}S(t) = x + \mu\int_0^t e^{au}\,du + \sigma\int_0^t e^{au}\,dB(u) = x + \frac{\mu}{a}\,(e^{at}-1) + \sigma\int_0^t e^{au}\,dB(u).$$

and

$$S(t) = xe^{-at} + \frac{\mu}{a}\,(1-e^{-at}) + \sigma e^{-at}\int_0^t e^{au}\,dB(u).$$

The integrand in the last line is a deterministic process. Therefore, the distribution of the integral and, thus, of the whole process is normal. The variance of the integral term is equal to $\sigma^2 e^{-2at}\int_0^t e^{2au}\,du$, and the answer is:

$$S(t) \sim \text{Norm}\left(xe^{-at} + \frac{\mu}{a}\,(1-e^{-at}),\ \frac{\sigma^2}{2a}\,(1-e^{-2at})\right).$$

Problem 5. Let $B(t) = (B_1(t),\ B_2(t))^T$ be a standard two-dimensional Brownian motion and $R(t) = \sqrt{B_1^2(t)+B_2^2(t)}$. Determine which of the following processes are standard Brownian motions (dimension 1 or 2). Clearly state and check all the required conditions.

(a) $dX(t) = R^{-1}(t)(B_1(t)dB_1(t) + B_2(t)dB_2(t))$, $X(0)=0$.

(b) $W(t) = (W_1(t), W_2(t))^T$, where

$$W(t) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\beta & \sin\beta \end{pmatrix} B(t),$$

and $\alpha,\ \beta \in \mathbb{R}$ are fixed numbers.

Solution. (a) Let $(\mathcal{F}(t))_{t\ge 0}$ be the natural filtration of our 2-dimensional standard BM. We shall use Lévy's characterization of BM ($d = 1$). First of all, by the properties of Itô integral[3], the process

$$X(t) := \int_0^t \frac{B_1(s)}{R(s)}\, dB_1(s) + \int_0^t \frac{B_2(s)}{R(s)}\, dB_2(s), \quad t \ge 0,$$

starts from 0, has continuous paths[4] and is a martingale with respect to $(\mathcal{F}(t))_{t\ge 0}$. Next, its quadratic variation is

$$\begin{aligned} d[X]_t &= \frac{B_1^2(t)}{R^2(t)}\, d[B_1]_t + \frac{2B_1(t)B_2(t)}{R^2(t)} d[B_1, B_2]_t + \frac{B_2^2(t)}{R^2(t)}\, d[B_2]_t \\ &= \frac{B_1^2(t) + B_2^2(t)}{R^2(t)}\, dt = dt. \end{aligned}$$

We used the fact that for independent Brownian motions B_1 and B_2 the cross-variation process is identically 0. Thus, the conditions of Lévy's characterization are satisfied, and W is a standard BM.

(b) $W(t)$ has two-dimensional state space. Here we see from the definition that $W(0) = (0, 0)^T$. The path continuity is inherited from Brownian motion, since the process is just a linear transformation of $(B(t))_{t\ge 0}$. We need to check whether the covariance matrix is equal to tI_2. We can write

$$dW(t) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\beta & \sin\beta \end{pmatrix} dB(t) =: A\, dB(t)$$

Then

$$\begin{aligned} d[W]_t &= A\, d[B, B](t) A^T = A I_2 A^T dt = AA^T dt \\ &= \begin{pmatrix} \sin^2\alpha + \cos^2\alpha & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\cos\beta\sin\alpha + \sin\beta\cos\alpha & (-\cos\beta)^2 + \sin^2\beta \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & \sin(\beta - \alpha) \\ \sin(\beta - \alpha) & 1 \end{pmatrix} dt \end{aligned}$$

Therefore, the answer is

$$\begin{cases} \text{no}, & \text{if } \beta - \alpha \ne \pi k,\ k \in \mathbb{Z} \\ \text{yes}, & \text{if } \beta - \alpha = \pi k,\ k \in \mathbb{Z}. \end{cases}$$

Problem 6. Let $\mu \in \mathbb{R},\ r, a \ge 0$ and $S(t)$ satisfy under $\mathbb{P}$

$$dS(t) = (\mu - aS(t))dt + \sigma\, dB(t), \quad 0 \le t \le T.$$

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[3]The integrands $B_i(t)/\sqrt{Y(t)}$ are adapted and bounded in absolute value by 1 for all $t \ge 0$ (as they are simply cos and sin of the angle between the "tip" of the 2-dimensional BM vector and the positive direction of the first coordinate axis.

[4]Lévy's theorem as stated in Theorem 4.3 of lecture 4, i.e. requiring $[M]_t \equiv t$, holds without the additional assumption of continuity of $M(t)$ in t. This is due to the fact that continuity of the quadratic variation process implies continuity of paths. Yet in our very first version of Lévy's theorem, Theorem 3.4 from lecture 3, where we ask $M^2(t) - t,\ t \ge 0$, to be a martingale, continuity of paths is an essential condition. Example: let $M(t) := N(t) - t$ be a compensated Poisson process with intensity 1. Then $M(0) = 0$, it is a martingale, and $M^2(t) - t$ is also a martingale. But, clearly, $M(t)$ is not a Brownian motion.

(a) Find the probability measure $\tilde{\mathbb{P}}$ such that the process $e^{-(r-a)t}S(t)$, $0 \le t \le T$, is a martingale under $\tilde{\mathbb{P}}$. Give explicitly the Radon-Nikodym derivative $d\tilde{\mathbb{P}}/d\mathbb{P}$.

(b) Find the equation for $S(t)$ under $\tilde{\mathbb{P}}$.

Solution. Our goal is to eliminate the drift for the process $e^{-(r-a)t}S(t)$.

$$\begin{aligned}
d(e^{-(r-a)t}S(t)) &= e^{-(r-a)}(-(r-a)S(t)\,dt + dS(t)) \\
&= e^{-(r-a)}(-(r-a)S(t)\,dt + (\mu - aS(t))dt + \sigma\,dB(t)) \\
&= e^{-(r-a)}((\mu - rS(t))dt + \sigma\,dB(t)) = e^{-(r-a)}\sigma\left(\frac{\mu - rS(t)}{\sigma}dt + dB(t)\right).
\end{aligned}$$

Now it is clear that we can set

$$\theta(t) = \frac{\mu - rS(t)}{\sigma}, \quad \tilde{B}(t) = B(t) + \int_0^t \theta(u)\,du.$$

Define $\tilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by its Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta(u)\,dB(u) - \frac{1}{2}\int_0^T \theta^2(u)\,du\right) =: Z(T),$$

i.e.

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}, \quad \forall A \in \mathcal{F}(T).$$

Then by Girsanov's theorem (assuming that all conditions check out) under $\tilde{\mathbb{P}}$ the process $\tilde{B}(t)$ is a standard Brownian motion, and by construction $e^{-(r-a)t}S(t)$, $0 \le t \le T$, is a martingale under $\tilde{\mathbb{P}}$.

(b) Under $\tilde{\mathbb{P}}$ we have $dB(t) = d\tilde{B}(t) - \theta(t)\,dt$. Substituting this in the equation for $dS(t)$ we get

$$\begin{aligned}
dS(t) = (\mu - aS(t))dt + \sigma\,(d\tilde{B}(t) - \theta(t)\,dt) &= (\mu - aS(t))dt + \sigma d\tilde{B}(t) - (\mu - rS(t))\,dt \\
&= (r-a)S(t)\,dt + \sigma\,d\tilde{B}(t).
\end{aligned}$$

This equation is of the same type as in problem 2 (take $\mu = 0$ and replace a with $a - r$).

Problem 7. Let $(B(t))_{t\ge0}$ be a standard Brownian motion, $(\mathcal{F}(t))_{t\ge0}$ be its natural filtration, and $T > 0$ be a fixed time. Define $M(t) := E(B^3(T)\,|\,\mathcal{F}(t))$, $0 \le t \le T$. Then we know (by the tower property of conditional expectations) that $(M(t))_{0\le t\le T}$ is a martingale. This problem will guide you through finding an explicit representation for this martingale, i. e. finding a stochastic process $\Gamma(t)$ such that

$$M(t) = \int_0^t \Gamma(u)\,dB(u), \quad 0 \le t \le T.$$

(a) Use Jensen's inequality for conditional expectations to show that $M(t)$ is square integrable for each $t \in [0, T]$.

(b) Compute $dB^3(t)$ and write $B^3(T)$ as a sum of a stochastic and a regular integral.

(c) Integrate your regular integral by parts. Write the result as a single stochastic integral.

(d) Use parts (b) and (c) together with one of the basic properties of Itô integral to write $E(B^3(T)\,|\,\mathcal{F}(t))$ as a single stochastic integral. State explicitly your answer, i.e. $\Gamma(t) = \ldots$.

Solution. (a) We use the conditional Jensen inequality $\phi(E(X|\mathcal{F})) \leq E(\phi(X)|\mathcal{F})$ a.s. for the convex function $\phi(x) = x^2$ and then take the expectation:

$$M^2(T) = (E(B^3(T)\,|\,\mathcal{F}(t)))^2 \leq E(B^6(T)\,|\,\mathcal{F}(t)) \quad \text{a.s.} \;\Rightarrow$$
$$E(M^2(t)) \leq E(E(B^6(T)\,|\,\mathcal{F}(t))) = E(B^6(T))) = 15T^3 < \infty.$$

(b) We obtain

$$dB^3(t) = 3B^2(t)\,dB(t) + 3B(t)\,dt;\; B^3(T) = 3\int_0^T B^2(u)\,dB(u) + 3\int_0^T B(u)\,du.$$

(c) Integrating the last integral by parts we get

$$\int_0^T B(u)\,du = B(T)T - \int_0^T u\,dB(u) = \int_0^T T\,dB(u) - \int_0^T u\,dB(u) = \int_0^T (T-u)\,dB(u).$$

(d) Therefore,

$$B^3(T) = 3\int_0^T (B^2(u) + (T-u))dB(u).$$

Using the martingale property of the Ito integral, namely, the fact that the process

$$Y(t) := \int_0^t (B^2(u) + (T-u))dB(u), \quad t \in [0,T],$$

is an $\mathcal{F}(t)$-martingale we conclude that $E(Y(T) \mid \mathcal{F}(t)) = Y(t)$, $0 \leq t \leq T$. Recalling that $Y(T) = B^3(T)$, we get that for $0 \leq t \leq T$

$$M(t) = E(B^3(T)\,|\,\mathcal{F}(t)) = E(Y(T) \mid \mathcal{F}(t)) = 3\int_0^t (B^2(u) + (T-u))dB(u),$$

and $\Gamma(t) = 3(B^2(t) + (T-t))$.