Group 5 (Wenxin Shi, Mingxiang Jia, Leyao Zhang, Chenyu Zhao) HW1

(1)

Let $d = \sqrt{B_1(t)^2 + B_2(t)^2}$, let $y = d^2$, we have $\frac{Y}{t} = z_1^2 + z_2^2 + z_3^2 \sim \chi_2^2$, where z_1, z_2 are standard normal distribution.

$$P(D < x) = P(Y < x^2) = P\left(\frac{Y}{t} < \frac{x^2}{t}\right) = F_{\chi_2^2}(\frac{x^2}{t})$$

$$f_{d(x)} = f_{\chi_2^2}\left(\frac{x^2}{t}\right) \cdot \frac{2x}{t} = \frac{x}{t} e^{-\frac{1}{2} \cdot \frac{x^2}{t}} \mathbf{1}_{x>0}$$

(2). Solution:

- Because $B_1(t)$ and $B_2(t)$ are continuous, $X(t)$ is a linear combination of them. Thus, $X(t)$ is also continuous.
- $X(0) = \rho B_1(0) + \sqrt{1-\rho^2} B_2(0) = 0$
- for each $m \in N$ and $t_0 = 0 < t_1 < \cdots < t_m$

$$X(t_1) - X(t_0) = \rho\big(B_1(t_1) - B_1(t_0)\big) + \sqrt{1-\rho^2}(B_2(t_1) - B_2(t_0))$$

$$X(t_2) - X(t_1) = \rho\big(B_1(t_2) - B_1(t_1)\big) + \sqrt{1-\rho^2}(B_2(t_2) - B_2(t_1))$$

……

$$X(t_m) - X(t_{m-1}) = \rho\big(B_1(t_m) - B_1(t_{m-1})\big) + \sqrt{1-\rho^2}(B_2(t_m) - B_2(t_{m-1}))$$

Because $B_1(t_1) - B_1(t_0), B_1(t_2) - B_1(t_1), \ldots, B_1(t_m) - B_1(t_{m-1})$ are independent random variables;$(B_2(t_1) - (B_2(t_0),\ B_2(t_2) - B_2(t_1),\ B_2(t_m) - B_2(t_{m-1})$ are independent random variables; $B_1(t)$ and $B_2(t)$ are independent.

Thus, $X(t_1) - X(t_0), X(t_2) - X(t_1), X(t_m) - X(t_{m-1})$ are independent random variables.

- For all s>0, and t >= 0, $B_1(t)$ and $B_2(t)$ are independent, their covariance = 0

$$E[X(t+s) - X(t)] = \rho E[B_1(t+s) - B_1(t)] + \sqrt{1-\rho^2} E[B_2(t+s) - B_2(t)] = 0$$

$$var[X(t+s) - X(t)] = \rho^2 var[B_1(t+s) - B_1(t)] + (1-\rho^2) var[B_1(t+s) - B_1(t)] = s$$

Because $[B_1(t+s) - B_1(t)] \sim N\ (0, s)\ and\ [B_2(t+s) - B_2(t)] \sim N\ (0, s),$
the increment $X(t+s) - X(t)$ also has a normal distribution with mean 0 and variance s

So we can conclude that X(t) is a Brownian motion.

- $corr[X(t), B_1(t)] = corr\big[\rho B_1(t) + \sqrt{1-\rho^2} B_2(t), B_1(t)\big]$

$$= \frac{E\big[(\rho B_1(t) + \sqrt{1-\rho^2} B_2(t)) * B_1(t)\big]}{t}$$

$$= \frac{\rho E[B_1(t) * B_1(t)] + \sqrt{1-\rho^2}\ E(B_2(t) * B_1(t))}{t}$$

$$= \frac{\rho t + 0}{t}$$

$$= \rho$$

(3).

(a) X(t) = -B(t)

- The negative multiplication maintains the continuity and independence of increments
- X(0) = -B(0) = 0
- $E[X(t+s)-X(t)] = E[-(B(t+s)-B(t))] = 0$
  var$[X(t+s)-X(t)] = var[-(B(t+s)-B(t))] = var[B(t+s)-B(t)] = s$
  Because $B(t+s)-B(t)$ has normal distribution, X(t) still maintains normal distribution with mean 0 and variance s
  Thus, $(-B(t))_{t\geq 0}$ is a Brownian motion.

(b) X(t) = $(cB(t/c^2))_{t\geq 0}$ where c > 0 is a constant

- Continuity and independence of increments still maintains.
- X(0) = $cB(0)$ = 0
- Because $B\left(\frac{t+s}{c^2}\right)-B\left(\frac{t}{c^2}\right)$ has normal distribution,
  $\mathrm{X(t+s)-x(t)}$ still maintains normal distribution
  $E[X(t+s)-X(t)] = E\left[c(B\left(\frac{t+s}{c^2}\right)-B\left(\frac{t}{c^2}\right))\right] = 0$
  $$\mathrm{var}[X(t+s)-X(t)] = var\left[c\left(B\left(\frac{t+s}{c^2}\right)-B\left(\frac{t}{c^2}\right)\right)\right]$$
  $$= c^2 * var\left[B\left(\frac{t+s}{c^2}\right)-B\left(\frac{t}{c^2}\right)\right]$$
  $$= c^2 * \frac{s}{c^2}$$
  = s
  Thus, $(cB(t/c^2))_{t\geq 0}$ is a Brownian motion.

(c) X(t) = $\left(\sqrt{t}B(1)\right)_{t\geq 0}$

- $$\mathrm{var}[X(t+s)-X(t)] = \mathrm{var}\left[\sqrt{t+s}B(1)-\sqrt{t}B(1)\right]$$
  $$= (t+s)var[B(1)] + t*var[B(1)] - 2\sqrt{t+s} * \sqrt{t}\ cov(B(1),B(1))$$
  $$= 2t+s-2\sqrt{t+s}*\sqrt{t}$$

The variance of increments is still related to t, so it's not a Brownian motion.

(d) X(t) = $\left(B(2t)-B(t)\right)_{t\geq 0}$

- for all s > 0, t >= 0
  $$\mathrm{var}[X(t+s)-X(t)] = \mathrm{var}[B(2t+2s)-B(t+s)-B(2t)+B(t)]$$
  $$= \begin{cases} \mathrm{var}[B(2t+2s)-B(t+s)] + \mathrm{var}[B(2t)-B(t)] & s\geq t \\ \mathrm{var}[B(2t+2s)-B(2t)] + \mathrm{var}[B(s+t)-B(t)] & s<t \end{cases}$$
  $$= \begin{cases} \mathrm{s+2t} & s\geq t \\ \mathrm{3s} & s<t \end{cases}$$

Thus, the variance of increments is still related to t for s $\geq t$. It's not a Brownian motion.

(e) X(t) = $\left(B(s) - B(s-t)\right)_{0\leq t\leq s}$, where s is fixed

- Because X(t) is a linear combination of Brownian motions, it maintains the independence of increments and is almost surely continuous. The increments are also normally distributed.
- For all m > 0, $t + m \leq s$

$$E[X(t+\mathrm{m}) - X(t)] = E[B(s) - B(s-t-m) - B(s) + B(s-t)]$$
$$= E[B(s-t) - B(s-t-m)]$$
$$= 0$$
$$\mathrm{var}[X(t+s) - X(t)] = var[B(s-t) - B(s-t-m)]$$
$$= \mathrm{m}$$

Thus, it's a Brownian motion.

(4).

$$\mathrm{P}(B^*(t) \geq a, B(t) \leq \mathrm{x}) = \mathrm{P(B(t) \leq x) - P(B^*(t) < a, B(t) \leq x) = P(B(t) \leq x) - P(\tau_a > t)}$$

We apply reflection principle,

$$\mathrm{P(\tau_t > t) = 1 - P(\tau_a < t) = 1 - 2P(B(t) > a)}$$
$$\mathrm{P(B^*(t) \geq a, B(t) \leq x) = P(B(t) \leq x) - 1 + 2P(B(t) > a) = 2N\left(\frac{-a}{\sqrt{t}}\right) - N(\frac{-x}{\sqrt{t}})}$$

Where $\mathrm{N}(\cdot)$ is cumulative function of standard normal distribution.

(5).

Since for square-integrable mean zero random variables X and Y, their inner product is defined to be E(XY), then following the general procedure of Gram-Schmidt orthogonalization, we have

$$\mathrm{Y_1} = X_1$$
$$\mathrm{Y_2} = X_2 - \frac{E(X_2, Y_1)}{E(Y_1^2)} Y_1$$
$$\mathrm{Y_3} = X_3 - \frac{E(X3, Y_1)}{E(Y_1^2)} Y_1 - \frac{E(X_3, Y_2)}{E(Y_2^2)} Y_2$$
$$\cdots\cdots$$
$$\mathrm{Y_n} = X_n - \frac{E(X_n, Y_1)}{E(Y_1^2)} Y_1 - \frac{E(X_n, Y_2)}{E(Y_2^2)} Y_2 - \cdots - \frac{E(X_n, Y_{n-1})}{E(Y_{n-1}^2)} Y_{n-1}$$

Then $\mathrm{E(Y_i Y_j) = 0, \forall i \neq j}$, and the collection of all random variables from set $\{\mathrm{Y_1, Y_2, \cdots, Y_n}\}$ forms an orthogonal basis of the span of $\mathrm{X_1}$, $\mathrm{X_2}, \cdots, X_n$:

$$\mathrm{X_1} = Y_1$$
$$\mathrm{X_2} = \frac{E(X_2, Y_1)}{E(Y_1^2)} Y_1 + Y_2$$

$$X_3 = \frac{E(X_3,Y_1)}{E(Y_1^2)}Y_1 + \frac{E(X_3,Y_2)}{E(Y_2^2)}Y_2 + Y_3$$

$$\cdots\cdots$$

$$X_n = \frac{E(X_n,Y_1)}{E(Y_1^2)}Y_1 + \frac{E(X_n,Y_2)}{E(Y_2^2)}Y_2 + \cdots \frac{E(X_n,Y_{n-1})}{E(Y_{n-1}^2)}Y_{n-1} + Y_n$$

For simplicity, we write $X = AY$, where $X = (X_1, X_2, \cdots, X_n)^T$, $Y = (Y_1, Y_2, \cdots, Y_n)^T$, and $Y_1, Y_2, \cdots, Y\_n$ are independent normal random variables, $Y_i \sim N(0, \sigma_i^2)$.

(a)

First, let's assume $E(X) = 0$, then according to the discussion above, we have

$$X = AY$$

Where $Y = (Y_1, Y_2, \cdots, Y_n)^T$, and $Y_1, Y_2, \cdots, Y_n$ are independent normal random variables, $Y_i \sim N(0, \sigma_i^2)$.

Write Y as $Y = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)Z$, where $\operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a diagonal matrix and Z is a standard normal vector, $Z \sim N(0, I)$, then we have

$$X = A \cdot \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)Z$$

Let $B = A \cdot \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, then apply Singular Value Decomposition to B, we have

$$B = U \cdot D \cdot V$$

where U, V are orthogonal matrices and $D = \operatorname{diag}(\widetilde{\sigma_1}, \widetilde{\sigma_2}, \cdots, \widetilde{\sigma_n})$ is a diagonal matrix, then we have

$$X = BZ = U \cdot D \cdot VZ$$

Since V is orthogonal, $Z \sim N(0, I)$, we have $\operatorname{Var}(VZ) = V \cdot \operatorname{Var}(Z) \cdot V^T = V \cdot V^T = I$, which means VZ is also a standard normal vector.

Thus, let $\tilde{Y} = DV \cdot Z$, then it's easy to see that $\tilde{Y} \sim N(0, D^2)$, where $D^2 = diag(\widetilde{\sigma_1}^2, \widetilde{\sigma_2}^2, \cdots, \widetilde{\sigma_2}^2)$.

In conclusion, we have

$$X = U \cdot \tilde{Y}$$

Where U is an orthogonal matrix, $\tilde{Y} = (\widetilde{Y_1}, \widetilde{Y_2}, \cdots, \widetilde{Y_n})$, and $\widetilde{Y_i} \sim N(0, \tilde{\sigma}_i^2)$ are independent normal random variables.

As for $E(X) = \mu = (\mu_1, \mu_2, \cdots, \mu_n)^T \neq 0$, where $\mu_i = E(X_i), \forall i = 1, 2, \cdots, n$. Denote $\tilde{X} = X - \mu$, then using the conclusion proven above, we know that there exists an orthogonal matrix A and independent normal random variables $Y_i \sim N(0, \sigma_i^2), i = 1, 2, \cdots, n$, such that

$$\tilde{X} = \tilde{A}Y$$

so $X = \mu + X - \mu = \mu + \tilde{X} = \mu + \tilde{A}Y$. Thus, we have proven Theorem 1.6 from Lecture 1.

(b)

From the process of Gram-Schmidt orthogonalization, we know that $X_k = a_1Y_1 + a_2Y_2 + \cdots + a_kY_k$, where $a_i = \frac{E(X_k, Y_i)}{E(Y_i^2)}, \forall i = 1, 2, \cdots, k$, thus

$$E(X_k | X_1, X_2, \cdots, X_{k-1}) = \sum_{i=1}^{k} a_i E(Y_i | X_1, X_2, \cdots, X_{k-1})$$

Also, $\forall i = 1, 2, \cdots, k-1, Y_i$ is a linear combination of $X_1, X_2, \cdots, X_{k-1}$, denote it as $Y_i =$

$\sum_{j=1}^{k-1} b_{ij}X_j$, then

$$\mathrm{E}(\mathrm{Y_i}|\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}) = \sum_{j=1}^{k-1} b_{ij}X_j\,, \forall \mathrm{i} = 1,2\cdots,\mathrm{k}-1$$

Next, we prove that $\mathrm{Y_k}$ is independent with $\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}$.

Since $\mathrm{Y_k},\mathrm{X_i},\forall 1 \le i \le k-1$ is a linear combination of $\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_n}$, we can tell that they are jointly normal thus to prove that they are independent , we just need to prove that

$$\mathrm{Cov}\big((\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}),\mathrm{Y_k}\big) = 0$$

For all $1 \le \mathrm{i} \le \mathrm{k}-1$,

$$\mathrm{Cov}(\mathrm{X_i},\mathrm{Y_k}) = E(X_iY_k) = E\left[\left(\sum_{j=1}^{i} a_jY_j\right)\cdot Y_k\right] = \sum_{j=1}^{i} a_jE\big(Y_jY_k\big) = 0$$

Thus, $\mathrm{E}(\mathrm{Y_k}|\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}) = \mathrm{E}(\mathrm{Y_k}) = 0$.

In conclusion, we have proven that

$$\mathrm{E}(\mathrm{X_k}|\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}) = \sum_{i=1}^{k-1} a_i \sum_{j=1}^{k-1} b_{ij}X_j = \sum_{j=1}^{k-1}(\sum_{i=1}^{k-1} a_ib_{ij})X_j$$

Is a linear function of $\mathrm{X_1},\mathrm{X_2},\cdots,\mathrm{X_{k-1}}$.

(c)

Denote $\mathbf{X_1} = (X_1,X_2.\cdots,X_k),\boldsymbol{X_2} = (X_{k+1},X_{k+2},\cdots,X_n)$, then what we need to do is to prove $(\mathbf{X\_2}|\mathbf{X_1} = \mathbf{x_1} = (\mathrm{x_1},\mathrm{x_2},\cdots,\mathrm{x_k})^{\mathrm{T}})$ is Gaussian and that the dependence of the parameters on $\mathrm{x_1},\mathrm{x_2},\cdots,\mathrm{x_k}$ is linear.

Denote the mean vector of $(\mathbf{X_1},\mathbf{X_2})$ as $(\boldsymbol{\mu_1},\boldsymbol{\mu_2})$, and the covariance matrix as $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, then we claim that

$$(\mathbf{X_2}|\mathbf{X_1} = \boldsymbol{x_1}) \sim \mathrm{N}(\Sigma_{21}\Sigma_{11}^{-1}\mathbf{x_1},\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

To prove it, denote $\mathrm{Z} = \mathbf{X_2} - \Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{X_1}$, we want to show that Z is independent with $\mathbf{X_1}$. Obviously, $\mathrm{X_1},\mathrm{Z}$ are jointly normal, then we only need to show that $\mathrm{Cov}(\mathrm{X_1},\mathrm{Z}) = 0$.

$$\mathrm{Cov}(\mathbf{X_1},\mathrm{Z}) = \mathrm{Cov}(\mathrm{X_1},\mathrm{X_2}-\Sigma_{21}\Sigma_{11}^{-1}\mathbf{X_1}) = Cov(X_1\ \ X_2) - Cov(X_1,X_1)\cdot(\Sigma_{21}\Sigma_{11}^{-1})^T = \Sigma_{12}-\Sigma_{12} = 0$$

Also we have $\mathrm{E}(\mathrm{Z}) = 0, \mathrm{Var}(\mathrm{Z}) = \mathrm{Var}(\mathbf{X_2}) - (\Sigma_{21}\Sigma_{11}^{-1})\mathrm{Var}(\mathbf{X_1})(\Sigma_{21}\Sigma_{11}^{-1})^{\mathrm{T}} = \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, then, according to the property of conditional expectation, we know that

$$\mathrm{Z}|\mathbf{X_1} \sim Z = N(0,\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

Where $\sim$ denote identically distributed.

Write Z as $\mathrm{Z} = \mathrm{AW}$, where $\mathrm{AA^T} = \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ and $\mathrm{W} \sim \mathrm{N}(0,\mathrm{I_{(n-k)\times(n-k)}})$, so $\mathbf{X_2} = Z + \Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{X_1}$,

$$\mathbf{X_2}|\boldsymbol{X_1} = (Z+\Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{X_1})|(\boldsymbol{X_1} = \boldsymbol{x_1}) \sim N(\Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{x_1},\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

Hence, we have proven that the conditional distribution of

$$(\mathrm{X_{k+1}},\mathrm{X_{k+2}},\cdots,\mathrm{X_n})|((\mathrm{X_1},\mathrm{X_2}\cdots,\mathrm{X_k}) = (x_1,x_2,\cdots,x_k))$$

Is Gaussian and that the dependence of the parameters on $\mathrm{x_1},\mathrm{x_2},\cdots,\mathrm{x_k}$ is linear.

(6).

The conditional distribution is Gaussian by the previous problem

we assume

$$X_2 - \mu_2 = \Lambda(X_1 - \mu_1) + W$$

where Λ is a non-random matrix and W is a mean zero Gaussian vector independent from X_1

$$(X_2 - \mu_2)(X_1 - \mu_1)^t = \Lambda(X_1 - \mu_1)(X_1 - \mu_1)^t + W(X_1 - \mu_1)^t$$

$$E((X_2 - \mu_2)(X_1 - \mu_1)^t) = E(\Lambda(X_1 - \mu_1)(X_1 - \mu_1)^t) + E(W(X_1 - \mu_1)^t)$$

We have X_1 and W are independent and C_{11} is non-degenerate, so

$$C_{21} = \Lambda C_{11}$$

$$\Lambda = C_{21}C_{11}^{-1}$$

$$(X_2 - \mu_2)(X_2 - \mu_2)^t = \Lambda(X_1 - \mu_1)(X_2 - \mu_2)^t + W(X_2 - \mu_2)^t$$

Then multiply both parts by

$$(X_2 - \mu_2)^t = (\Lambda(X_1 - \mu_1) + W)^t$$

$$E((X_2 - \mu_2)(X_2 - \mu_2)^t) = E(\Lambda(X_1 - \mu_1)(X_2 - \mu_2)^t) + E(W(\Lambda(X_1 - \mu_1) + W)^t)$$

$$C_{22} = \Lambda C_{12} + E(WW^t)$$

$$Cov(W) = C_{22} - \Lambda C_{12}$$

$$Cov(W) = C_{22} - C_{21}C_{11}^{-1}C_{12}$$

We have

$$W \sim N(0, C_{22} - C_{21}C_{11}^{-1}C_{12})$$

$$(X_2|X_1 = x) = C_{21}C_{11}^{-1}(x - \mu_1) + W$$

$$E(X_2|X_1 = x) = E(\Lambda(X_1 - \mu_1) + W + \mu_2|X_1 = x)$$

$$E(X_2|X_1 = x) = \Lambda(x - \mu_1) + \mu_2$$

$$E(X_2|X_1 = x) = C_{21}C_{11}^{-1}(x - \mu_1) + \mu_2$$

Then

$$(X_2|X_1 = x) \sim N(C_{21}C_{11}^{-1}(x - \mu_1) + \mu_2, C_{22} - C_{21}C_{11}^{-1}C_{12})$$