Basic uses of Itô's formula

Notation: $(B(t))_{t\geq 0}$ is a standard Brownian motion, $(\mathcal{F}(t))_{t\geq 0}$ is the filtration generated by $(B(t))_{t\geq 0}$.

A. (Itô's formula for Brownian motion.) Let $f \in C^{1,2}([0,\infty)\times\mathbb{R})$. Then with probability 1 for all $t > 0$

$$f(t, B(t)) - f(0, B(0)) = \int_0^t f_x(u, B(u))\, dB(u) + \int_0^t f_t(u, B(u))\, du + \frac{1}{2}\int_0^t f_{xx}(u, B(u))\, du.$$

To proceed to the next step it is convenient to think about du in the last integral as $d[B,B](u)$ (since $[B,B](t) \equiv t$ a.s.). Then the next formula will not look surprising.

B. (Itô's formula for Itô processes.) Let

$$X(t) = X(0) + \int_0^t \Delta(u)\, dB(u) + \int_0^t \Theta(u)\, du$$

be an Itô process and $f \in C^{1,2}([0,\infty)\times\mathbb{R})$. Then with probability 1 for all $t > 0$

$$f(t, X(t)) - f(0, X(0)) = \int_0^t f_x(u, X(u))\, dX(u) + \int_0^t f_t(u, X(u))\, dt + \frac{1}{2}\int_0^t f_{xx}(u, X(u))\, d[X,X](u).$$

I prefer not to substitute $dX(u) = \Delta(u)\, dB(u) + \Theta(u)\, du$ and $d[X,X](u) = \Delta^2(u)\, du$ in the right hand side of the formula, since this substitution hides the natural structure of the formula.

C. (Itô's integral for deterministic integrands.) Let $(\Delta(t))_{t\geq 0}$ be a non-random square integrable function on $[0,t]$. Then

$$I(t) = \int_0^t \Delta(u)\, dB(u) \ \sim\ N\left(0, \int_0^t \Delta^2(u)\, du\right).$$

Exercises:

(1) Apply Itô's formula to the following processes:

(a) $B^2(t)$;

(b) $tB(t)$;

(c) $(B(t)+t)\exp(-B(t)-t/2)$;

(d) $t^2B(t)-2\int_0^t uB(u)\,du$;

(e) $\log S(t)$, where $dS(t)=\nu S(t)\,dt+\sigma S(t)\,dB(t)$;

(f) $\exp\left(\int_0^t \Delta(u)\,dB(u)-\frac{1}{2}\int_0^t \Delta^2(u)\,du\right)$.

(2) Use Itô's formula to compute

$$\int_0^t B(u)\,dB(u).$$

Solution. Our guess from the regular calculus is that the answer should contain $B^2(t)$. Applying Itô's formula, we get

$$dB^2(t)=2B(t)\,dB(t)+dt.$$

This expression simply means (we usually say "integrating from 0 to t we get...", even though the above differential form is just a short-hand for the integral)

$$B^2(t)-B^2(0)=2\int_0^t B(u)\,dB(u)+t,$$

from which we find that

$$\int_0^t B(u)\,dB(u)=\frac{1}{2}\left(B^2(t)-t\right).$$

(3) Use Itô's formula to compute the moment generating function of $B(t)$.

Solution. Apply Itô's formula to $f(B(t))$ where $f(x)=e^{\lambda x}$. Then

$$df(B(t))=\lambda f(B(t))\,dB(t)+\frac{1}{2}\,\lambda^2 f(B(t))\,dt.$$

Writing this in the integral form and taking the expectations we get

$$Ef(B(t))=Ef(B(0))+\frac{1}{2}\,\lambda^2\int_0^t Ef(B(u))\,du.$$

Denoting $Ee^{\lambda B(t)}=Ef(B(t))$ by $m(t)$, we see that $m(t)$ satisfies

$$m'(t)=\frac{1}{2}\,\lambda^2 m(t),\quad m(0)=1.$$

This gives the answer

$$m(t)=Ee^{\lambda B(t)}=e^{\lambda^2 t/2}.$$

(4) Compute the distribution of the signed area under the graph of Brownian motion on the interval $[0, t]$,

$$\int_0^t B(u)\, du.$$

Solution. From part (b) of Exercise 1 we get $d(tB(t)) = B(t)\, dt + t\, dB(t)$. Writing this in the integral form and rearranging the terms we obtain the following "integration by parts" formula and the solution to our problem:

$$\begin{aligned}\int_0^t B(u)\, du &= tB(t) - \int_0^t u\, dB(u)\\ &= t\int_0^t dB(u) - \int_0^t u\, dB(u) = \int_0^t (t-u)\, dB(u).\end{aligned}$$

By Tool C, the last integral has normal distribution with mean 0 and variance $\int_0^t (t-u)^2\, du = \frac{1}{3}\, t^3$.

(5) (From Black-Karasinski to Vasicek model.) Let α, β, σ be positive constants. A (special case of) Black-Karasinski interest rate model states that the interest rate process satisfies

$$dR(t) = \left(\alpha + \frac{1}{2}\,\sigma^2 - \beta \log R(t)\right) R(t)\, dt + \sigma R(t)\, dB(t).$$

Set $r(t) = \log R(t)$ and find the equation on $r(t)$.

Solution. By Itô's formula

$$\begin{aligned}d\log R(t) &= \frac{1}{R(t)}\, dR(t) - \frac{1}{2R^2(t)} d[R, R](t)\\ &= \left(\alpha + \frac{1}{2}\,\sigma^2 - \beta \log R(t)\right) dt + \sigma\, dB(t) - \frac{1}{2}\,\sigma^2\, dt\\ &= (\alpha - \beta \log R(t))\, dt + \sigma\, dB(t).\end{aligned}$$

Therefore $dr(t) = (\alpha - \beta r(t))\, dt + \sigma\, dB(t)$. The equation on $r(t)$ is known as Vasicek model, which admits a closed form solution (see Exercise 11). Once we know $r(t)$, we set $R(t) = e^{r(t)}$ and obtain a solution to the Black-Karasinski equation.

Other useful tools:

D. (Integration by parts formula, regular case.) The "integration by parts" formula we obtained in Exercise 4 of part 1 for a specific case can be generalized. Here is a version, which is not hard to prove (see p. 46 of B. Øksendal, Stochastic Differential Equations, Sixth Edition). Let

$(F(t))_{t\geq 0}$ be a stochastic process with a.s. continuously differentiable trajectories on $[0,t]$ and $(X(t))_{t\geq 0}$ be an Itô process. Then

$$\int_0^t F(u)\,dX(u) = F(u)X(u)\Big|_0^t - \int_0^t X(u)\,dF(u).$$

Derive this useful fact from Itô's formula.

E. Given a "nice" $f(t,x)$ satisfying appropriate integrability conditions, how can we determine whether the process $(f(t,B(t)))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale?

Solution. Apply Itô's formula to $f(t,B(t))$:

$$df(t,B(t)) = (f_t(t,B(t)) + \frac{1}{2}\,f_{xx}(t,B(t)))\,dt + f_x(t,B(t))\,dB(t).$$

This allows us to say that whenever function f satisfies the partial differential equation (PDE) $f_t(t,x) + \frac{1}{2}\,f_{xx}(t,x) = 0$ for all $(t,x) \in (0,T)\times\mathbb{R}$, then the process $(f(t,B(t)))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale.

The necessity of this condition is harder to prove. It can be treated as a consequence of the Martingale Representation Theorem (Shreve II, Section 5.3.1) which we shall discuss later. See also B. Øksendal, Stochastic Differential Equations, Sixth Edition, Exercise 4.12 on p. 59 for a direct proof in a slightly more general setting.

Exercises:

(6) Use the method of Exercise 3 to compute the variance of the process $S(t)$, which solves the equation $dS(t) = \sigma S(t)\,dB(t)$, $S(0) = A$. Hint: apply Itô's formula to $S^2(t)$ and then take the expected value. In addition, solve this problem directly by first verifying the fact that $S(t) = Ae^{\sigma B(t) - \sigma^2 t/2}$ solves the equation and then computing the variance directly.

(7) Find the mean and variance of the process $\int_0^t S(u)\,du$, where $dS(t) = \sigma S(t)\,dB(t)$, $S(0) = A$. Give a solution based on integration by parts (similar to the solution of Exercise 4). Use the result of Exercise 6.

(8) As an application of Tool E, determine which of the processes in Exercise 1 are martingales. When it is possible to give an alternative argument based on definition and/or basic properties of martingales or processes in question, provide such an argument as well. For example, the process $t + B(t)$ is not a martingale, since its expectation is not constant.

(9) (Review of properties of conditional expectation.) Using the definition of a martingale (without Itô's formula or any of its consequences) show that the process in Exercise 1(d) is a martingale.

(10) (The Ornstein-Uhlenbeck process.) Let $(X(t))_{t\geq 0}$ satisfy

$$dX(t) = -\beta X(t)\,dt + \sigma dB(t), \quad X(0) = x,$$

where $\beta \in \mathbb{R}$, $\sigma > 0$ are constants. For $\beta > 0$ this is a special case of Vasicek interest rate model (see the next exercise). Apply Itô's formula to $e^{\beta t}X(t)$ and show that $X(t)$ admits a closed form solution

$$X(t) = e^{-\beta t}x + \sigma e^{-\beta t}\int_0^t e^{\beta u}\,dB(u).$$

Use this expression to find the mean and variance of $X(t)$. Then compute the mean and variance not using the solution but applying the same approach as in Exercise 6.

(11) (Vasicek model.) Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\,dB(t).$$

Find a closed form solution of this equation. Compute the mean and variance of $r(t)$.

(12) (Cox-Ingersoll-Ross (CIR) model.) Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dB(t).$$

(a) Compute the mean and variance of $r(t)$.

(b) Assume that $4\alpha = \sigma^2$. Let $X(t) = \sqrt{r(t)}$. Derive the equation for $X(t)$.

(c) Using part (b) determine the distribution of $r(t)$. Compute its moment generating function.