MTH 9831. Solutions to Quiz 11.

Notation: $(B(t))_{t\geq 0}$ is a standard Brownian motion, $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ, $M(t) = N(t) - \lambda t$, $t \geq 0$, is a compensated Poisson process, $\{Q(t)\}_{t\geq 0}$ is a compound Poisson process with jumps distributed as Y_1, and $M_Q(t) = Q(t) - \beta\lambda t$ ($\beta = \mathbb{E}Y_1$), $t \geq 0$, be a compensated compound Poisson process. All processes are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration $(\mathcal{F}(t))_{t\geq 0} \subseteq \mathcal{F}$.

(1) (2+4 points) In addition to the conditions given above assume that the MGF of Y_1, $M_{Y_1}(v)$, is finite for all $v \in \mathbb{R}$.

 (i) Show that for every $v \in \mathbb{R}$ the process $e^{vQ(t)-\lambda t(M_{Y_1}(v)-1)}$, $t \geq 0$, is an $\mathcal{F}(t)$-martingale.

 (ii) Show that $B(t)$ and $Q(t)$ are independent random variables for each $t \geq 0$.

Solution. (i) The easiest check is by the definition. Recall that a compound Poisson process has stationary independent increments. Hence, for $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}(e^{vQ(t)-\lambda t(M_{Y_1}(v)-1)} \mid \mathcal{F}(s)) &= e^{vQ(s)-\lambda t(M_{Y_1}(v)-1)}\mathbb{E}(e^{v(Q(t)-Q(s))} \mid \mathcal{F}(s)) \\
&= e^{vQ(s)-\lambda t(M_{Y_1}(v)-1)}\mathbb{E}(e^{vQ(t-s)}) \\
&= e^{vQ(s)-\lambda t(M_{Y_1}(v)-1)}e^{\lambda(t-s)(M_{Y_1}(v)-1)} \\
&= e^{vQ(s)-\lambda s(M_{Y_1}(v)-1)}.
\end{aligned}$$

Alternatively, one can use Itô's formula but the computation is a bit longer.

(ii) We shall show that that the joint MGF of $(B(t), Q(t))$ splits into a product of the MGF of $B(t)$ and the MGF of $Q(t)$, i.e. that for all $u, v \in \mathbb{R}$

$$\mathbb{E}(e^{uB(t)+vQ(t)}) = e^{tu^2/2}e^{\lambda t(M_{Y_1}(v)-1)}. \tag{1}$$

For this we shall argue that the process

$$e^{X(t)}, \quad \text{where} \quad X(t) = uB(t) + vQ(t) + \mu t \text{ and } \mu = -u^2/2 - \lambda(M_{Y_1}(v) - 1),$$

is an $\mathcal{F}(t)$-martingale (u, v are fixed). Then equating the expectations

$$\mathbb{E}(e^{X(t)}) = \mathbb{E}(e^{X(0)}) = 1$$

we shall get (1).

I'll give first a "bird's view" argument without details and then a detailed argument similar to the one on pp. 487-488 of the textbook.

We can write

$$e^{X(t)} = e^{uB(t)-tu^2/2}e^{vQ(t)-\lambda t(M_{Y_1}(v)-1)} =: M_1(t)M_2(t),$$

where both M_1 and M_2 are martingales. By Itô's product formula, we see that

$$de^{X(t)} = d(M_1(t)M_2(t)) = M_1(t-)dM_2(t) + M_2(t-)dM_1(t) + d[M_1, M_2](t).$$

Note that M_1 is a continuous martingale and the martingale M_2 is a product of a pure jump process $J(t) = e^{vQ(t)}$ and of a regular process $R(t) = e^{-\lambda t(M_{Y_1}(v)-1)}$. Formally, we could write

$$\begin{aligned}
d[M_1, M_2](t) &= dM_1(t)dM_2(t) = dM_1(t)(R(t)dJ(t) + J(t)dR(t) + \underbrace{d[J, R](t)}_{=0}) \\
&= R(t)d[M_1, J](t) + J(t)d[M_1, R](t) = 0.
\end{aligned}$$

Therefore, $[M_1, M_2] \equiv 0$ a.s., and $e^{X(t)}$ is a martingale.

Here is a detailed basic argument. We have

$$X(t) = uB(t) + vQ(t) + \mu t = (uB(t) + \mu t) + vQ(t) = X^c(t) + Q(t).$$

By Itô's formula,

$$\begin{aligned}
e^{X(t)} - 1 &= \int_0^t e^{X(s)} dX^c(s) + \frac{1}{2}\int_0^t e^{X(s)} d[X^c, X^c](s) + \sum_{0<s\le t} (e^{X(s)} - e^{X(s-)}) \\
&= u\int_0^t e^{X(s)} dB(s) + \mu \int_0^t e^{X(s)} ds + \frac{u^2}{2}\int_0^t e^{X(s)} ds + \sum_{0<s\le t} e^{X(s-)}(e^{v\Delta Q(s)} - 1) \\
&= u\int_0^t e^{X(s)} dB(s) + \left(\mu + \frac{u^2}{2}\right)\int_0^t e^{X(s)} ds + \sum_{0<s\le t} e^{X(s-)}(e^{vY_{N(s)}} - 1)\Delta N(s) \\
&= u\int_0^t e^{X(s)} dB(s) - \lambda(M_{Y_1}(v) - 1)\int_0^t e^{X(s-)} ds + \int_0^t e^{X(s-)}\, dH(s),
\end{aligned}$$

where

$$H(s) = \sum_{i=1}^{N(s)} (e^{vY_i} - 1)$$

is a compound Poisson process with jumps distributed as $e^{vY_1} - 1$. Therefore, the process $M_H(t) = H(t) - \lambda t(M_{Y_1}(v) - 1)$, $t \ge 0$, is an $\mathcal{F}(t)$-martingale. We conclude that

$$e^{X(t)} = 1 + u\int_0^t e^{X(s)} dB(s) + \int_0^t e^{X(s-)}\, dM_H(t)$$

is a martingale.

Remark. Note that this time we wrote $e^{X(t)}$ as a sum of two martingales while in the first argument we represented $e^{X(t)}$ as a product of two martingales with zero cross variation.

(2) (5 points) Suppose that under a risk-neutral measure the stock price can be represented as follows:

$$S(t) = S^*(t)e^{Q(t)}, \quad 0 \le t \le T,$$

where $S^*(t) = S(0)e^{\sigma B(t) + \mu t}$, $B(t)$ is a standard Brownian motion, $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process with intensity λ, random variables Y_i, $i \ge 1$, are normal with mean μ_0 and variance σ_0^2, and $\mu = r - \sigma^2/2 - \lambda(\tilde{E}e^{Y_1} - 1)$, where r is the annual nominal interest rate. Find the time 0 cost of a European call option with strike price K and expiration T.

Solution. It is assumed that all processes are adapted to the same filtration so that, in particular, $B(T)$ and $Q(T)$ are independent. Note that $M_{Y_1}(1) = \tilde{E}(e^{Y_1}) = e^{\mu_0 + \sigma_0^2/2}$. Let us rewrite $S(T)$ conditioned on the event $\{N(T) = n\}$ in a convenient form:

$$\begin{aligned}
\ln S(T) - \ln S(0) &= \sigma B(T) + \mu T + \sum_{i=1}^n Y_i = \sigma\sqrt{T} Z_0 + \sum_{i=1}^n (\mu_0 + \sigma_0 Z_i) + \mu T \\
&= (\mu T + \mu_0 n) + \sqrt{\sigma^2 T + \sigma_0^2 n}\, Z.
\end{aligned}$$

Here $Z_0, Z_1, \dots, Z_n$ are independent standard normals and Z is a standard normal. Hence, conditional on $\{N(T) = n\}$, the random variable $\ln S(T) - \ln S(0)$ is normal with mean $\mu T + \mu_0 n$ and variance $\sigma^2 T + \sigma_0^2 n$. Let

$$\sigma^2(n) := \sigma^2 + \frac{n}{T}\sigma_0^2, \quad r(n) := \mu + \mu_0\frac{n}{T} + \frac{\sigma^2(n)}{2} = r - \lambda(M_{Y_1}(1) - 1) + \frac{n}{T}\left(\mu_0 + \frac{\sigma_0^2}{2}\right).$$

With this notation, $\ln S(T) - \ln S(0)$ is normal with mean $\left(r(n) - \frac{\sigma^2(n)}{2}\right) T$ and variance $\sigma^2(n)T$. Using a standard Black-Scholes formula we get that

$$e^{-r(n)T} \tilde{E}\left[(S(T) - K)_+ \middle| N(T) = n\right] = C(S(0), T, K, \sigma(n), r(n)),$$

where $C(x, t, K, \sigma, r)$ is the time zero price of a call option with strike K and expiration t, if $S(0) = x$, interest rate is r, and stock volatility is σ.

We conclude that the price we are looking for is equal to

$$\begin{aligned}
e^{-rT} \tilde{E}[(S(T) - K)_+] &= e^{-rT} \sum_{n=0}^{\infty} \tilde{E}\left[(S(T) - K)_+ | N(T) = n\right] \tilde{P}(N(T) = n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} e^{(r(n)-r)T} C(S(0), T, K, \sigma(n), r(n)) \\
&= e^{-\lambda T M_{Y_1}(1)} \sum_{n=0}^{\infty} \frac{(\lambda T M_{Y_1}(1))^n}{n!} C(S(0), T, K, \sigma(n), r(n)).
\end{aligned}$$

Remark. Using part (i) of problem 1 we see that $e^{-rt}S(t)$, $t \geq 0$, is a martingale, so modeling the stock price under the risk-neutral measure as it is done in this problem is not unreasonable.