MTH 9831. Solutions to Quiz 8.

(1) (4 points) Let $S(t)$, $t \geq 0$, be a geometric Brownian motion $dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t)$, where $W(t)$, $t \geq 0$, is a standard Brownian motion with respect to $\mathbb{P}$, $\sigma > 0$, and let $Y(t) = \max_{0 \leq u \leq t} S(u)$. Show that the process $(S(t), Y(t))$, $t \geq 0$, is Markov.

Solution. Let T be fixed, $t \in [0, T]$, and $f(x, y)$ be a non-negative Borel-measurable function. Filtration $\mathcal{F}(t)$ is the filtration associated with $W(t)$. We need to show that there exists a function $g(x, y)$ such that

$$\mathbb{E}\big[f\big(S(T), Y(T)\big)\big|\mathcal{F}(t)\big] = g\big(S(t), Y(t)\big).$$

Notice that

$$S(T) = S(t)\,\frac{S(T)}{S(t)} = S(t)\exp\left(\sigma(W(T) - W(t)) + (r - \sigma^2/2)(T - t)\right)$$
$$= S(t)\exp\left(\sigma\left(\hat{W}(T) - \hat{W}(t)\right)\right),$$

where $\hat{W}(u) = W(u) + \alpha u$, $\alpha = \sigma^{-1}(r - \sigma^2/2)$, $u \geq 0$, is a drifted Brownian motion under $\mathbb{P}$. Moreover,

$$\begin{aligned}
Y(T) = Y(t) + Y(T) - Y(t) &= Y(t) + \left(\max_{t \leq u \leq T} S(u) - Y(t)\right)_+ \\
&= Y(t) + S(t)\left(\max_{t \leq u \leq T}\frac{S(u)}{S(t)} - \frac{Y(t)}{S(t)}\right)_+ \\
&= Y(t) + S(t)\left(\exp\left(\sigma\max_{t \leq u \leq T}\big(\hat{W}(u) - \hat{W}(t)\big)\right) - \frac{Y(t)}{S(t)}\right)_+
\end{aligned}$$

Since $\hat{W}(u) - \hat{W}(t)$ is independent of $\mathcal{F}(t)$ for all $u \in [t, T]$ and has the same distribution as $\hat{W}(u - t)$, $u \geq t$, by the independence lemma

$$\mathbb{E}\big[f\big(S(T), Y(T)\big)\big|\mathcal{F}(t)\big] = g\big(S(t), Y(t)\big),$$

where

$$g(x, y) = \mathbb{E}\left(f\left(xe^{\sigma\hat{W}(T-t)}, y + x\left(e^{\sigma\hat{M}(T-t)} - \frac{y}{x}\right)_+\right)\right),$$

and $\hat{M}(u) = \max_{0 \leq s \leq u}\hat{W}(s)$, $u \geq 0$. This finishes the proof of the Markov property.

Just to elaborate a bit further,[1] recall that the joint density of $(\hat{M}(u), \hat{W}(u))$ is given by

$$f_{\hat{M}(u),\hat{W}(u)}(m, w) = \frac{2(2m - w)}{u\sqrt{2\pi u}}\,e^{\alpha w - \frac{1}{2}\alpha^2 u - \frac{1}{2u}(2m-w)^2}\,1_{\{w \leq m, m \geq 0\}}.$$

Thus, we can express $g(x, y)$ as the following integral:

$$g(x, y) = \int_0^\infty \int_{-\infty}^m f\left(xe^{\sigma w}, y + x\left(e^{\sigma m} - \frac{y}{x}\right)_+\right)$$
$$\times\,\frac{2(2m - w)}{(T - t)\sqrt{2\pi(T - t)}}\,e^{\alpha w - \frac{1}{2}\alpha^2(T-t) - \frac{1}{2(T-t)}(2m-w)^2}\,dw\,dm.$$

Clearly, function g depends on $T - t$, r, and σ but these dependencies are not reflected in the notation.

Remark. Observe that (up to the discounting factor) we obtained a formula for the time t value of any derivative security with expiration T whose payoff is a function of $S(T)$ and $Y(T)$.

[1] this is not a part of the problem.

(2) (2 points) Let $S(t)$, $Y(t)$, $t \geq 0$, be as above and Π be a partition of $[0,t]$ with $t_0 = 0 < t_1 < \cdots < t_m = t$. Show that $\lim\limits_{\|\Pi\| \to 0} \mathbb{E}\left[\left(\sum\limits_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))(S(t_i) - S(t_{i-1}))\right)^2\right] = 0$. You may assume that as $\|\Pi\| \to 0$

$$\mathbb{E}\left(\sum_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))^2\right)^2 \to 0 \quad \text{and} \quad \mathbb{E}\left(\sum_{i=1}^{m}(S(t_i) - S(t_{i-1}))^2\right)^2 \to \sigma^2 \int_0^t \mathbb{E}(S^2(u))\, du. \tag{1}$$

Solution. Let $t_0 = 0 < t_1 < \cdots < t_m = t$ be a partition of $[0,t]$. Then by the Cauchy-Schwarz inequality we get

$$\Big|\sum_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))(S(t_i) - S(t_{i-1}))\Big| \leq \left(\sum_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))^2\right)^{1/2}\left(\sum_{i=1}^{m}(S(t_i) - S(t_{i-1}))^2\right)^{1/2}.$$

Squaring both sides, taking expectations, and again using the Cauchy-Schwarz inequality (now for expectations) we see that

$$\mathbb{E}\left(\sum_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))(S(t_i) - S(t_{i-1}))\right)^2 \leq \sqrt{\mathbb{E}\left(\sum_{i=1}^{m}(Y(t_i) - Y(t_{i-1}))^2\right)^2}\sqrt{\mathbb{E}\left(\sum_{i=1}^{m}(S(t_i) - S(t_{i-1}))^2\right)^2}.$$

By (1) the right hand side converges to 0 as $\|\Pi\| \to 0$.

(3) (4 points) Assume Black-Scholes model and consider a zero-strike Asian call option, i.e. the option with the payoff

$$V(T) = \frac{1}{T}\int_0^T S(u)\, du.$$

Suppose that at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)\, du = y \geq 0$. Use the fact that $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\tilde{\mathbb{E}}\left(\frac{1}{T}\int_0^T S(u)\, du \,\Big|\, \mathcal{F}(t)\right).$$

Call your answer $v(t,x,y)$. Then find explicitly $\Delta(t)$ for all $0 \leq t \leq T$.

Solution. Let $Y(t) = \int_0^t S(u)\, du$. By Fubini's theorem for conditional expectations and the fact that $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$ we have

$$\begin{aligned}
\tilde{\mathbb{E}}\left(\int_0^T S(u)\, du \,\Big|\, \mathcal{F}(t)\right) &= \tilde{\mathbb{E}}\left(\int_0^t S(u)\, du + \int_t^T S(u)\, du \,\Big|\, \mathcal{F}(t)\right) \\
&= Y(t) + \int_t^T \tilde{\mathbb{E}}\left(S(u) \,|\, \mathcal{F}(t)\right)\, du = Y(t) + \int_t^T e^{ru}\tilde{\mathbb{E}}\left(e^{-ru}S(u) \,|\, \mathcal{F}(t)\right)\, du \\
&= Y(t) + \int_t^T e^{ru}e^{-rt}S(t)\, du = Y(t) + S(t)\int_t^T e^{r(u-t)}\, du \\
&= Y(t) + S(t)\,\frac{1}{r}\left(e^{r(T-t)} - 1\right).
\end{aligned}$$

Multiplying by $e^{-r(T-t)}$ and dividing by T we get

$$v(t, S(t), Y(t)) = \frac{1}{T}\, e^{-r(T-t)}Y(t) + \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) S(t).$$

Therefore,

$$v(t,x,y) = \frac{1}{T}\, e^{-r(T-t)}y + \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) x, \quad v_x(t,x,y) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right),$$

and $\Delta(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)$, $0 \leq t \leq T$, a non-random smooth function.