

MTH 9831 Assignment 8 (8/11/2015 - 11/14/2015).

Read Lecture 8. Additional references for this material are:

1. S. Shreve, Stochastic Calculus for Finance II, Sections 6.2 - 6.6.
2. A. Etheridge, A Course in Financial Calculus, Section 4.8.

- (1) Suppose that for $0 \leq t \leq u \leq T$

$$dX(u) = b(u, X(u)) du + \sigma(u, X(u)) dB(u), \quad X(t) = x.$$

Let $f(x)$ and $h(x)$ be given deterministic functions. Find the PDE and the terminal condition satisfied by

$$g(t, x) = E^{t,x}[h(X(T))] + \int_t^T E^{t,x}[f(X(u))] du.$$

Hint: the game here is to find a relevant martingale, after that the PDE is obtained by applying Ito's formula and setting the drift term to 0. If I give you the martingale, the game is over. Instead, I shall tell you how to see it, since this point of view is applicable to many situations that involve finding a PDE. Start by looking at the following sequence of equalities, think why they are true (all but one are just rewriting of the same thing), and find a martingale:

$$\begin{aligned} g(t, x) &= E^{t,x}[h(X(T))] + \int_t^T E^{t,x}[f(X(u))] du; \\ g(t, x) &= E\left[h(X(T)) + \int_t^T f(X(u)) du \mid X(t) = x\right]; \\ g(t, X(t)) &= E\left[h(X(T)) + \int_t^T f(X(u)) du \mid X(t)\right]; \\ g(t, X(t)) &= E\left[h(X(T)) + \int_t^T f(X(u)) du \mid \mathcal{F}(t)\right]; \\ g(t, X(t)) &= E\left[h(X(T)) + \int_0^T f(X(u)) du \mid \mathcal{F}(t)\right] - \int_0^t f(X(u)) du; \\ g(t, X(t)) + \int_0^t f(X(u)) du &= E\left[g(T, X(T)) + \int_0^T f(X(u)) du \mid \mathcal{F}(t)\right]. \end{aligned}$$

Which transition from line to line is non-trivial? Justify this transition and find the required PDE.

- (2) (From a PDE to an SDE) Find a diffusion process (i.e. an SDE) whose generator \mathcal{A} has the following form:

(a) ($d = 1$) $\mathcal{A} = (2 - x) \frac{\partial}{\partial x} + 2x^2 \frac{\partial^2}{\partial x^2}$. Can you solve the obtained SDE?

(b) ($d = 2$)

$$\mathcal{A} = \frac{1+x^2}{2} \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \frac{\partial^2}{\partial y^2} + 2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}.$$

Give at least two different answers.

(3) (Feynman-Kac+Girsanov) Consider the following terminal value problem:

$$g_t + \frac{1}{2} g_{xx} + \theta(x) g_x = 0, \quad (x, t) \in \mathbb{R} \times [0, T), \quad g(T, x) = h(x).$$

Show that the solution of this problem can be written as

$$g(t, x) = E^{t, x} \left(e^{\int_t^T \theta(B(s)) dB(s) - \frac{1}{2} \int_t^T \theta^2(B(s)) ds} h(B(T)) \right).$$

Hint: use Feynman-Kac and Girsanov theorems. You may assume that h and θ are such that the problem has a unique solution and both theorems are applicable.

(4) (Bond price for CIR model) Read Example 6.5.2 and do Exercise 6.4.

(5) (Heston stochastic volatility model) Exercise 6.7, parts (ii)-(v).

(6) (Derivation of Kolmogorov forward equation) Exercise 6.9.

(7) (Implying the volatility surface) Exercise 6.10.