MTH 9831. LECTURE 10

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ABSTRACT. This lecture continues the discussion of Asian options and then turns to American options.
1. Asian options (continued: reduction of dimension through the change of numeraire)
2. American options. Part 1: perpetual American put.

1. ASIAN OPTIONS

Recall the BSM framework:

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{B}(t), \quad 0 \le t \le T.$$
$$dD(t) = -rD(t)dt, \quad D(0) = 1, \quad \text{i.e. } D(t) = e^{-rt}, \quad 0 \le t \le T.$$

Here $\widetilde{B}$ is a standard BM with respect to the risk-neutral measure $\widetilde{\mathbb{P}}$. Previously we derived the following equation:

$$d(D(t)S(t)) = \sigma D(t)S(t)d\widetilde{B}(t), \quad 0 \le t \le T. \tag{1}$$

The time t value of an option with the payoff $V(T)$ is

$$V(t) = \widetilde{\mathbb{E}}(e^{-r(T-t)}V(T) \mid \mathcal{F}(t)), \quad 0 \le t \le T,$$

and

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}(e^{-rT}V(T) \mid \mathcal{F}(t)), \quad 0 \le t \le T,$$

is a $\widetilde{\mathbb{P}}$-martingale. We assume that $r > 0$[1] and $V(T) = (\frac{1}{c}\int_{T-c}^{T} S(t)dt - K)_+$ where c is a fixed constant, $0 < c \le T$. The pricing will be done in 2 steps.

Step 1: Create a portfolio process $X(t)$ such that

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(t)dt - K. \tag{2}$$

Step 2: Use 1 share of underlying as a unit, reduce the dimension, and set up a simpler PDE and boundary conditions than those in Lecture 9.

Step 1. Let $X(t) = \gamma(t)S(t) + (X(t) - \gamma(t)S(t))$, where $\gamma(t)$ is non-random and differentiable. Our goal is to find $\gamma(t)$, $0 \le t \le T$, such that $X(T)$ matches (2).

First of all, recall that under the self-financing condition

$$dX(t) = \gamma(t)dS(t) + r(X(t) - \gamma(t)S(t))dt.$$

Discounting, we get the equation

$$d(D(t)X(t)) = -rD(t)X(t)dt + D(t)dX(t) = D(t)\gamma(t)dS(t) - rD(t)\gamma(t)S(t)dt.$$

Expressing[2] $D(t)\gamma(t)dS(t)$ as

$$d(D(t)\gamma(t)S(t)) - S(t)d(D(t)\gamma(t)) = d(D(t)\gamma(t)S(t)) + rD(t)S(t)\gamma(t)dt - D(t)S(t)d\gamma(t)$$

and canceling $\pm rD(t)\gamma(t)S(t)dt$ terms we arrive at

$$d(D(t)X(t)) = d(D(t)\gamma(t)S(t)) - D(t)S(t)d\gamma(t). \tag{3}$$

[1] See Exercise 7.8 in the textbook for the case $r = 0$.

[2] $D(t)\gamma(t)$ is a regular process, so the usual product rule applies.

Observe for the future use that, (3) and (1) imply

$$d(X(t)D(t)) = \gamma(t)\sigma D(t)S(t)d\widetilde{B}(t). \tag{4}$$

Next we integrate (3) from $T-c$ to T and put $D(t)=e^{-rt}$:

$$e^{-rT}X(T) - e^{-r(T-t)}X(T-c) = e^{-rT}\gamma(T)S(T) - e^{-r(T-c)}\gamma(T-c)S(T-c) - \int_{T-c}^{T} e^{-rt}S(t)\gamma'(t)dt;$$

$$X(T) = e^{rc}X(T-c) + \gamma(T)S(T) - e^{rc}\gamma(T-c)S(T-c) - \frac{1}{c}\int_{T-c}^{T} S(t)e^{-r(t-T)}c\gamma'(t)dt.$$

To get a match with the expression in (2) we should set

$$-c\gamma'(t)e^{-r(T-t)} = 1, \quad \text{for all } \ t\in[T-c,T]; \tag{5}$$

$$e^{rc}X(T-c) + \gamma(T)S(T) - e^{rc}\gamma(T-c)S(T-c) = -K. \tag{6}$$

Solving (5) we get

$$\gamma(t) = \gamma(T-c) - \frac{1}{rc}(e^{-r(T-t)} - e^{-rc}),\ t\in[T-c,T]. \tag{7}$$

At time $T-c$, we know $X(T-c)$ and $S(T-c)$, but we can not deterministically control $S(T)$. Thus, the only way (6) can hold is when $\gamma(T)=0$ and (6) becomes

$$e^{rc}X(T-c) - e^{rc}\gamma(T-c)S(T-c) = -K. \tag{8}$$

The condition $\gamma(T)=0$ and (7) lead to

$$\begin{aligned} 0 = \gamma(T) &= \gamma(T-c) - \frac{1}{rc}(1-e^{-rc}),\\ \gamma(T-c) &= \frac{1}{rc}(1-e^{-rc}), \end{aligned}$$

and from (8),

$$X(T-c) = \frac{1}{rc}(1-e^{-rc})S(T-c) - Ke^{-rc}.$$

The last equation corresponds to the following position at time $T-c$:

- long $\frac{1}{rc}(1-e^{-rc})$ shares of stock;
- short Ke^{-rT} shares of MMA (the cost of 1 share of MMA at time $T-c$ is $e^{r(T-c)}$).

What should we do at time 0 to ensure that at time $T-c$? we shall have this position? If at time 0 we purchase $\frac{1}{rc}(1-e^{-rc})$ shares by borrowing $e^{-rT}K$ from a bank and just simply hold this portfolio up to time $T-c$, then we shall get what we want. Such portfolio at time 0 costs

$$X(0) = \frac{1}{rc}(1-e^{-rc})S(0) - e^{-rT}K.$$

Our hedging strategy is as follows: at time 0 start with $X(0)$ in cash and keep over time the portfolio that has

$$\gamma(t) = \begin{cases} \frac{1}{rc}(1-e^{-rc}), & \text{if } 0\le t\le T-c;\\ \frac{1}{rc}(1-e^{-r(T-t)}), & \text{if } T-c\le t\le T. \end{cases} \tag{1.1}$$

shares of stock borrowing/depositing money from/to MMA as needed. Then at time T, we have (2).

Step 2. Now back to pricing. The payoff is $V(T)=(X(T))_+$ where $X(T)$ is given by (2). Under $\widetilde{\mathbb{P}}$, the price is

$$V(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}((X(T))_+|\mathcal{F}(t)).$$

Our tool is "change of numeraire". Define $Y(t) = \frac{X(t)}{S(t)}$. This means that we use one unit of stock as numeraire. Recall (1) and (4):

$$d(D(t)X(t)) = \gamma(t)\sigma D(t)S(t)d\widetilde{B}(t) = \frac{\gamma(t)\sigma}{Y(t)} D(t)X(t)d\widetilde{B}(t);$$
$$d(D(t)S(t)) = \sigma D(t)S(t)d\widetilde{B}(t).$$

By Theorem 3.3 of Lecture 7, we have that

$$dY(t) = Y(t)\left(\frac{\gamma(t)\sigma}{Y(t)} - \sigma\right) d\widetilde{B}^S(t) = \sigma(\gamma(t) - Y(t))d\widetilde{B}^S(t),$$

where $\widetilde{B}^S(t) = \widetilde{B}(t) - \sigma t$ is a BM under $\widetilde{\mathbb{P}}^S$, and

$$\widetilde{\mathbb{P}}^S(A) = \frac{1}{S(0)}\int_A D(T)S(T)d\widetilde{\mathbb{P}}, \quad A \in \mathcal{F}(T).$$

Recall that Radon-Nikodým derivative process

$$Z(t) = \frac{S(t)D(t)}{S(0)}, \quad 0 \le t \le T.$$

Since $Y(t)$, $0 \le t \le T$, satisfies $dY(t) = \sigma(\gamma(t) - Y(t))d\widetilde{B}^S(t)$, it is a Markov process and a martingale under $\widetilde{\mathbb{P}}^S$. We have

$$\begin{aligned}
V(t) &= e^{rt}\widetilde{\mathbb{E}}\left(e^{-rT}(X(T))_+ \mid \mathcal{F}(t)\right)\\
&= e^{rt}\widetilde{\mathbb{E}}\left(\underbrace{e^{-rT}S(T)}_{=S(0)Z(T)}\left(\frac{e^{-rT}X(T)}{e^{-rT}S(T)}\right)_+ \Big| \mathcal{F}(t)\right)\\
&= e^{rt}S(0)\widetilde{\mathbb{E}}(Z(T)(Y(T))_+|\mathcal{F}(t))\\
&\overset{\text{Lec. 5, Lem. 5.4}}{=} e^{rt}S(0)Z(t)\widetilde{\mathbb{E}}^S((Y(T))_+|\mathcal{F}(t))\\
&= S(t)\widetilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(t)).
\end{aligned}$$

Since Y is a Markov process, there is $g(t, y)$ such that

$$g(t, Y(t)) = \widetilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(t)), \quad 0 \le t \le T.$$

In particular,

$$g(T, Y(T)) = \widetilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(T)) = (Y(T))_+,$$

and

$$g(T, y) = y_+, \quad y \in \mathbb{R}. \tag{9}$$

(Recall that $Y(T) = \frac{X(T)}{S(T)}$, where $X(T)$ can be positive or negative.) As usual, by the tower property, $g(t, Y(t))$ is a martingale under $\widetilde{\mathbb{P}}$. Compute

$$dg(t, Y(t)) = \left(g_t(t, Y(t)) + \frac{1}{2}\sigma^2(\gamma(t) - Y(t))^2 g_{yy}(t, Y(t))\right)dt + \sigma(\gamma(t) - Y(t))g_y(t, Y(t))d\widetilde{B}^S(t).$$

We conclude that $g(t, y)$ has to solve

$$g_t(t, y) + \frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t, y) = 0, \quad 0 \le t < T, \quad y \in \mathbb{R}. \tag{10}$$

We have so far one boundary condition (9), we need conditions at infinity to ensure uniqueness. If $y \to -\infty$, then it means that $Y(t)$ is very negative, so the chances that $(Y(T))_+ > 0$ are going to 0. Thus, in the limit $\widetilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(t)) = 0$, and

$$\lim_{y\to-\infty} g(t, y) = 0, \quad 0 \le t \le T. \tag{11}$$

If $y \to +\infty$, then the chances that $(Y(T))_+ > 0$ are going to 1, and the limit of $\widetilde{\mathbb{E}}^S((Y(T))_+|\mathcal{F}(t))$ as $y \to \infty$ is approximately the same as that of $\widetilde{\mathbb{E}}^S(Y(T)|\mathcal{F}(t)) = Y(t)$. This gives

$$\lim_{y\to\infty}(g(t,y) - y) = 0, \quad 0 \le t \le T. \tag{12}$$

We have shown the following result.

Theorem 1.1 (Večeř). *For $0 \le t \le T$, the price $V(t)$ at time t of the Asian call option paying $\left(\frac{1}{c}\int_{T-c}^T S(t)dt - K\right)_+$ at time T is given by*

$$V(t) = S(t)g\left(t, \frac{X(t)}{S(t)}\right),$$

where $g(t,y)$ satisfies (10) *with boundary conditions* (9), (11), (12)*, and $X(t)$ is a portfolio value process with $\gamma(t)$ given by* (1.1).

Remark 1.2. Exactly the same steps can be used to price a *discretely sampled Asian call* with payoff

$$V(T) = \left(\frac{1}{m}\sum_{i=1}^m S(t_i) - K\right)_+, \quad 0 < t_1 < ... < t_m = T.$$

See pp. 329–330 of the textbook.

2. Perpetual American put

We shall need the following notions and facts:

(a) *Stopping time; first passage time.*
Fact: the first passage time of a continuous process is stopping time, i.e.

$$\forall m \in \mathbb{R}, \quad \tau_m := \inf\{t \ge 0 : \ X(t) = m\}$$

is a stopping time with respect to the natural filtration of X. Proof is a required reading: see Shreve II, p. 342

(b) *Optional stopping theorem*, Shreve II, p. 342.

(c) *Laplace transform of the first passage time for the drifted Brownian Motion* (see HW2 or Shreve II, Theorem 8.3.2). More precisely, let $m > 0$, $\mu \in \mathbb{R}$, $X(t) = B(t) + \mu t$, and $\tau_m = \inf\{t \ge 0 : X(t) = m\}$. Then $\forall \lambda > 0$,

$$\mathbb{E}e^{-\lambda\tau_m} = e^{m\mu - m\sqrt{\mu^2+2\lambda}} \tag{13}$$

Assume the BSM model: under $\widetilde{\mathbb{P}}$ (risk-neutral)

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{B}(t), \quad r, \sigma > 0 \text{ (constants)}.$$

Consider a perpetual American put with strike K. Denote by $v_*(x)$ its price when $S(0) = x$. How to determine $v_*(x)$?

Notice that the decision to exercise the put at time t can only depend on the information up to time t. This tells us that the exercise time should be a stopping time. When is it optimal to exercise? When the risk-neutral expectation of the present value of the payoff is maximal, i.e. when τ (the exercise time) is such that $\widetilde{\mathbb{E}}(e^{-\tau r}(K - S(\tau)))$ is maximized.[3] Maximized over which variable(s)? Over all possible stopping times. Thus, we define

$$v_*(x) := \max_{\tau\in\mathcal{T}} \widetilde{\mathbb{E}}\left(e^{-\tau r}(K - S(\tau))\right),$$

where $\mathcal{T}$ is the set of all stopping times relative to the natural filtration generated by $(\widetilde{B}(t))_{t\le 0}$. This formula does not seem to help much, since we do not know how to maximize over all stopping times. How does one even start describing all possible stopping times?

[3]We can drop the subscript $+$, because the option will not be exercises unless $K - S(\tau) > 0$.

This is where the "perpetuity" simplifies the problem. Since the option never expires, the decision to exercise should not depend on time to expiration (it is always ∞), it can only depend on the path. By the Markov property, if we are given that $S(t) = x$, the future and the past of the process S relative to time t are independent. Then it seems reasonable to seek the optimal exercise time as the first time when the price falls to some level $L_* < K$. Then our problem is reduced to finding this optimal level L_* and setting the optimal exercise time τ_* to be the first time when S hits level L_*.

Let $\tau_L = \min\{t \le 0 : S(t) = L\}$, $0 < L < K$, and if $S(0) \ge L$

$$v_L(S(0)) := \widetilde{\mathbb{E}}(e^{-\tau_L r}(K - S(\tau_L))).$$

But $S(\tau_L) = L$, and for $0 < L < K$ and $S(0) \ge L$,

$$v_L(S(0)) = (K - L)\,\widetilde{\mathbb{E}}e^{-\tau_L r}.$$

If $S(0) < L$, then at time 0, the stock is already below L, and the option should be exercised immediately. Thus

$$v_L(S(0)) = \begin{cases} K - S(0), & \text{if } S(0) < L; \\ (K-L)\,\widetilde{\mathbb{E}}e^{-\tau_L r}, & \text{if } S(0) \ge L. \end{cases}$$

When $S(0) = L$, $\tau_L = 0$, the function $v_L(S(0))$ is continuous at $S(0) = L$, and

$$v_L(S(0)) = \begin{cases} K - S(0), & \text{if } S(0) \le L; \\ (K-L)\,\widetilde{\mathbb{E}}e^{-\tau_L r}, & \text{if } S(0) \ge L. \end{cases}$$

Lemma 2.1.

$$v_L(x) = \begin{cases} K - x, & \textit{if } x \in [0, L]; \\ (K-L)\left(\dfrac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & \textit{if } x \ge L. \end{cases}$$

Proof. We have already discussed the case $S(0) = x \in [0, L]$. Assume now that $x > L$, i.e. $S(0) > L$. Then τ_L is the first time when

$$S(t) = xe^{\sigma\widetilde{B}(t) + (r - \frac{1}{2}\sigma^2)t}$$

reaches level L. Hence, $S(t) = L$ iff

$$\sigma\widetilde{B}(t) + \left(r - \frac{1}{2}\sigma^2\right)t = \log\frac{L}{x};$$

$$\underbrace{\widetilde{B}(t) + \frac{r - \frac{1}{2}\sigma^2}{\sigma}t}_{\text{drifted BM}} = \underbrace{\frac{1}{\sigma}\log\frac{L}{x}}_{\text{level } m,\text{ but } m<0 \text{ as } x>L}.$$

This does not fit into the assumptions of (13). The change of sign will help.

$$\underbrace{-\widetilde{B}(t)}_{\text{still BM}} + \underbrace{\left(\frac{1}{2}\sigma - \frac{r}{\sigma}\right)}_{\mu\in\mathbb{R}}t = \underbrace{\frac{1}{\sigma}\log\frac{x}{L}}_{=m>0}.$$

Using (13) with $\lambda = r$, $\mu = \frac{1}{2}\sigma - \frac{r}{\sigma}$, and $m = \frac{1}{\sigma}\log\frac{x}{L}$, we get

$$\widetilde{\mathbb{E}}e^{-\tau_L r} = e^{m\mu - m\sqrt{\mu^2+2r}};$$

$$\mu^2 + 2r = \frac{\sigma^2}{4} - r + \frac{r^2}{\sigma^2} + 2r = \left(\frac{\sigma}{2} + \frac{r}{\sigma}\right)^2;$$

$$m\mu - m\sqrt{\mu^2+2r} = \left(\frac{1}{\sigma}\log\frac{x}{L}\right)\left(\frac{1}{2}\sigma - \frac{r}{\sigma} - \frac{\sigma}{2} - \frac{r}{\sigma}\right) = -\frac{2r}{\sigma^2}\log\frac{x}{L};$$

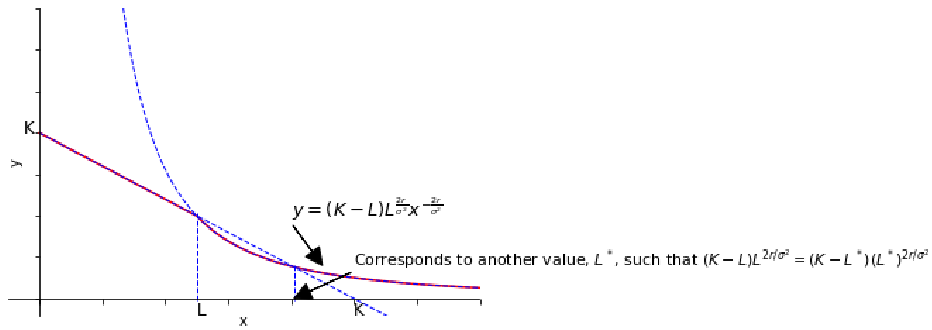
$$\widetilde{\mathbb{E}}e^{-\tau_L r} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

$\square$

Now we have to find out which level L^* is optimal, i.e. for which L the function $v_L(x)$ is the largest. It could be that for different x, we get different L^*, but, as we shall see, this does not happen.

$$v_L(x) = \begin{cases} K - x, & \text{if } x \in [0, L]; \\ (K-L)L^{\frac{2r}{\sigma^2}} x^{-\frac{2r}{\sigma^2}}, & \text{if } x \geq L. \end{cases}$$

Set $g(L) = (K-L)L^{2r/\sigma^2}$ and maximize it over $L \in [0, K]$.



$$\begin{aligned} g'(L) &= \frac{2r}{\sigma^2} K L^{\frac{2r}{\sigma^2}-1} - \left(\frac{2r}{\sigma^2}+1\right) L^{\frac{2r}{\sigma^2}} \\ &= L^{\frac{2r}{\sigma^2}-1}\left(\frac{2r}{\sigma^2}K - \frac{2r+\sigma^2}{\sigma^2}L\right) = 0; \\ L_* &= \frac{2rK}{2r+\sigma^2} \in (0, K) \quad (r, \sigma > 0). \end{aligned}$$

Since $g(0) = g(K) = 0$, $g(L) \geq 0$, and this is the only critical point, it is the point where $g(L)$ attains the absolute maximum on $[0, K]$. Putting this L_* in the equation for v_L, we get

$$v_{L_*}(x) = \begin{cases} K - x, & \text{if } x \in [0, L_*]; \\ (K - L_*)\left(\dfrac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & \text{if } x \geq L_*. \end{cases}$$

$$v'_{L_*}(x) = \begin{cases} -1, & \text{if } x \in [0, L_*]; \\ -(K - L_*)\dfrac{2r}{\sigma^2}\frac{1}{L_*}\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}-1}, & \text{if } x \geq L_*. \end{cases}$$

$$v'_{L_*}(L_{*+}) = -\frac{(K-L_*)}{L_*}\frac{2r}{\sigma^2} = -\frac{2rK}{L_*\sigma^2} + \frac{2r}{\sigma^2} = -1.$$

Conclusion: $v_{L_*}(x)$ is a continuously differentiable function on $[0, \infty)$, and $v'_{L_*}(L_*) = -1$.

It is obvious that $v''_{L_*}(x)$ can not be continuous and will have a jump at $x = L_*$.

$$v''_{L_*}(x) = \begin{cases} 0 & x \in [0, L_*] \\ (K - L_*)\frac{2r}{\sigma^2}(\frac{2r}{\sigma^2}+1)\frac{1}{L_*^2}(\frac{x}{L_*})^{-\frac{2r}{\sigma^2}-2} & x \geq L_* \end{cases}$$

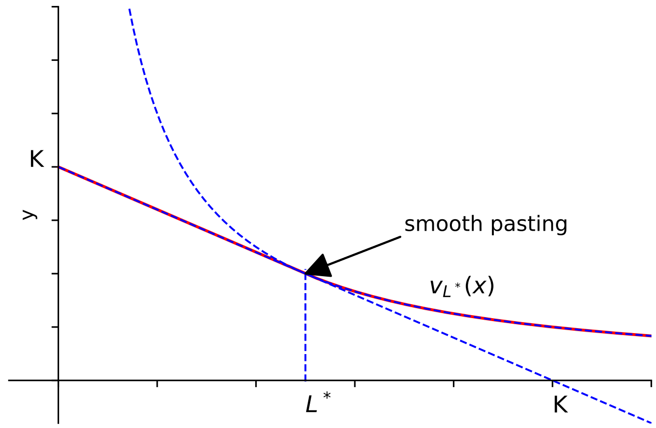
$$v''_{L_*}(L_{*+}) > 0$$

For $x > L_*$, it can be verified by substitution that

$$rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = (K - L_*)\left(r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}} = 0. \tag{14}$$

For $x \in [0, L_*]$,

$$rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = r(K-x) + rx = rK. \tag{15}$$



This implies that $v_{L_*}(x)$ satisfies "linear complementarity conditions"

$$v(x) \geq (K - x)_+; \tag{16}$$

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0, \quad \forall x \geq 0; \tag{17}$$

$$\text{and for each } x \geq 0, \text{ equality holds in either (16) or (17).} \tag{18}$$

At the point $x = L_*$ we can use either $v''_{L_*}(L_{*+})$ or $v''_{L_*}(L_{*-})$ in (17), the inequality still holds.

It turns out that $v_{L_*}(x)$ is the only bounded continuously differentiable function on $[0, \infty)$ ($C^1([0, \infty))$ that satisfies (16)–(18). (This is Exercise 8.3 in Shreve II.)

To reiterate: analytically, the perpetual put price is the only bounded $C^1([0, \infty)$ function which satisfies (16)–(18).