MTH 9831. LECTURE 8

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ABSTRACT. Connections with PDEs
1. SDEs: definition, Markov property of solutions, associated differential operators.
2. Feynman-Kac formula.
3. Kolmogorov's backward and forward equations.

1. SDEs: definition, Markov property of solutions, associated differential operators

Definition 1.1. *Let $b(t,x)$ be a deterministic d-dimensional vector-valued function on $[0,\infty)\times\mathbb{R}^d$ and $\sigma(t,x)$ be a deterministic $d\times r$ matrix for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$. Denote by $B(t)$ the r-dimensional standard BM. A stochastic differential equation with drift $b(t,x)$ and volatility $\sigma(t,x)$ is an equation of the form*

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dB(t). \tag{$\mathcal{E}$}$$

A strong solution to $(\mathcal{E})$ with the initial condition $x\in\mathbb{R}^d$ is a stochastic process $(X(t))_{t\ge 0}$ with continuous sample paths such that

(i) $X(0)=x$ *a.s.;*
(ii) $(X(t))_{t\ge0}$ *is adapted to the filtration $(\mathcal{F}(t))_{t\ge0}$ generated by $(B(t))_{t\ge0}$;*
(iii) $\int_0^t |b_i(s,X(s))| + \sigma_{ij}^2(s,X(s))ds < \infty$ *a.s. for $1\le i\le d$, $1\le j\le r$, $0\le t<\infty$;*
(iv) *for $t\in[0,\infty)$,*

$$X(t) = X(0) + \int_0^t b(u,X(u))du + \int_0^t \sigma(u,X(u))dB(u) \quad a.s..$$

Remark 1.2. Often in applications we want to start the process at time t from a given point $x\in\mathbb{R}^d$ and find $X=(X(u))_{u\ge t}$ such that $X(t)=x$ and for $u\ge t$

$$X(u) = X(t) + \int_t^u b(s,X(s))ds + \int_t^u \sigma(s,X(s))dB(s) \quad \text{a.s..}$$

Some conditions on $b(t,x)$ and $\sigma(t,x)$ are needed to ensure the existence and uniqueness of solutions to $(\mathcal{E})$.

Example 1.3. Find a solution to

$$dX(t) = X^3(t)dt - X^2(t)dB(t), \quad X(0)=1.$$

Let us try to find $X(t)$ in the form $X(t)=f(t,B(t))$ for some deterministic function $f(t,x)$. Then

$$\begin{aligned} dX(t) = df(t,B(t)) &= f_t dt + f_x dB(t) + \frac{1}{2}f_{xx}dt \\ &= (f_t + \frac{1}{2}f_{xx})dt + f_x dB(t). \end{aligned}$$

We conclude that f has to satisfy: $f_x = -f^2$ and $f_t + \frac12 f_{xx} = f^3$. Solving the first equation we get that

$$f(t,x) = \frac{1}{x+C(t)}$$

for some $C(t)$. Substituting f in the second equation gives

$$-\frac{-C'(t)}{(x+C(t))^2}+\frac{1}{2}\frac{2}{(x+C(t))^3}=\frac{1}{(x+C(t))^3}$$

We conclude that $C'(t)=0$, and $C(t)\equiv C$. Therefore, $f(t,x)=(x+C)^{-1}$. Using the initial condition $X(0)=0$ we find

$$X(0)=f(0,0)=\frac{1}{C}=1 \quad\Rightarrow\quad C=1,$$

and $X(t)=f(t,B(t))=(B(t)+1)^{-1}$. The problem is that with probability 1, $B(t)$ hits -1, i.e. $P(T_{-1}<\infty)=1$. Thus, the solution exists only up to the "explosion time" T_{-1}.

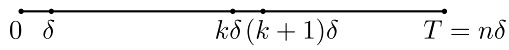
To prevent explosion and guarantee the existence and uniqueness of $(\mathcal{E})$ are it is sufficient that there is a constant K such that for all $t\in[0,\infty)$, $x,y\in\mathbb{R}$

$$\begin{aligned}&|b(t,x)|,\|\sigma(t,x)\|\le K(1+|x|) \quad \text{and}\\ &|b(t,x)-b(t,y)|+\|\sigma(t,x)-\sigma(t,y)\|\le K|x-y|,\end{aligned}$$

where for any $d\times r$ matrix A we set $\|A\|=\sqrt{\sum_{i=1}^d\sum_{j=1}^r a_{ij}^2}$. In fact, for $d=1$ the conditions can be significantly weakened. (For example, CIR can be included.)

Markov Property. Intuition for $d=1$. How can one try to simulate $X(t)$, $0\le t\le T$? Suppose that $X(0)=x$ is given. Consider a small increment of time δ. Then according to $(\mathcal{E})$,

$$\begin{aligned}X(\delta)&\approx x+b(0,x)\delta+\sigma(0,x)\underbrace{B(\delta)}_{\sim N(0,\delta)}\\ &=x+b(0,x)\delta+\sigma(0,x)\sqrt{\delta}Z_1,\quad Z_1\sim N(0,1).\end{aligned}$$



Continuing for $k\in\{0,1,2,\ldots,n-1\}$ we get

$$\begin{aligned}X((k+1)\delta)&=X(k\delta)+b(k\delta,X(k\delta))\delta+\sigma(k\delta,X(k\delta))(B((k+1)\delta)-B(k\delta))\\ &=X(k\delta)+b(k\delta,X(k\delta))\delta+\sigma(k\delta,X(k\delta))\sqrt{\delta}Z_{k+1};\end{aligned}$$

and $Z_1,Z_2\ldots,Z_n$ are i.i.d. $N(0,1)$. Then $X(n\delta)=X(T)$ gives us a simulated value at time T. If we could justify passing to the limit (in some sense) as $\delta\to 0$, then the limiting process would give us a solution to $(\mathcal{E})$. Moreover, our procedure suggests that the limiting process would be memoryless ($\equiv$ Markov process), since given $X(k\delta)$ we can simulate $X((k+1)\delta),...,X(n\delta)$ without knowing $X(0),X(\delta),...,X((k-1)\delta)$. More rigorously, the following theorem holds.

Theorem 1.4. *Let $0\le t\le T$ and h be a Borel measurable function such that the expectations below are well-defined. Set*

$$g(t,x)=\mathbb{E}^{t,x}h(X(T)).$$[1]

Then for $t\in[0,T]$,

$$\mathbb{E}(h(X(T))|\mathcal{F}(t))=g(t,X(t)),$$

i.e. solutions to SDEs are Markov process.

[1] Take $X(t)=x$, solve $(\mathcal{E})$ for $u\in[t,T]$, get $X(T)$, and compute the expectation of $h(X(T))$, repeat for all $t\in[0,T]$, $x\in\mathbb{R}$. Get a deterministic function $g(t,x)$.

The proof is omitted.

Interpretation of the drift and diffusion coefficients. Here is a heuristic approach (for $d = 1$).

$$\mathbb{E}(X(\delta + t) - X(t) \mid X(t) = x) \approx$$
$$\mathbb{E}(b(t, X(t))\delta + \sigma(t, X(t))\sqrt{\delta}Z \mid X(t) = x) = b(t, x)\delta;$$
$$\Rightarrow \lim_{\delta \to 0} \frac{\mathbb{E}(X(t + \delta) - X(t)|X(t) = x)}{\delta} = b(t, x).$$
$$\mathbb{E}((X(t + \delta) - X(t))^2 \mid X(t) = x) \approx$$
$$\mathbb{E}((b(t, X(t))\delta + \sigma(t, X(t))\sqrt{\sigma}Z)^2 \mid X(t) = x) = b^2(t, x)\delta^2 + \delta\sigma^2(t, x);$$
$$\Rightarrow \lim_{\delta \to 0} \frac{\mathbb{E}((X(t + \delta) - X(t))^2 \mid X(t) = x)}{\delta} = \sigma^2(t, x) =: a(t, x)$$

From the above, it is natural that $b(t, x)$ is called the drift coefficient and $a(t, x)$ is called the diffusion coefficient.

Let's take the next step. Take an arbitrary smooth function $u(t, x)$. Then looking at the first few terms in the Taylor expansion we get

$$\mathbb{E}(u(t + \delta, X(t + \delta)) - u(t, X(t)) \mid X(t) = x) \approx$$
$$\mathbb{E}(u_t(t, X(t))\delta + u_x(t, X(t))(X(t + \delta) - X(t)) + \frac{1}{2}u_{xx}(t, X(t))(X(t + \delta) - X(t))^2|X(t) = x)$$
$$= \delta(u_t(t, x) + b(t, x)u_x(t, x) + \frac{1}{2}u_{xx}(t, x)a(t, x)) \quad \Rightarrow$$
$$\lim_{\delta \to 0} \frac{1}{\delta}\, \mathbb{E}(u(t + \delta, X(t + \delta)) - u(t, X(t))|X(t) = x) = u_t(t, x) + b(t, x)u_x(t, x) + \frac{1}{2}a(t, x)u_{xx}(t, x)$$
$$=: u_t(t, x) + (\mathcal{A}_t u)(t, x),$$

where

(1)
$$\mathcal{A}_t := b(t, x)\frac{\partial}{\partial x} + \frac{1}{2}a(t, x)\frac{\partial^2}{\partial x^2}, \quad (t \geq 0)$$

Similarly, for $d \geq 1$, we set

(2)
$$\mathcal{A}_t := \sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad (t \geq 0),$$

where

$$b^T(t, x) = \lim_{\delta \to 0} \delta^{-1}\mathbb{E}((X(t + \delta) - X(t))^T|X(t) = x);$$
$$a(t, x) = \lim_{\delta \to 0} \delta^{-1}\mathbb{E}((X(t + \delta) - X(t))(X(t + \delta) - X(t))^T|X(t) = x)$$
$$= (\sigma\sigma^T)(t, x);$$

Remark 1.5. When b and σ do not depend on t, i.e. $b = b(x)$, $\sigma = \sigma(x)$, X is called a (time homogeneous, or Itô) diffusion process and $\mathcal{A}$ (now it does not depend on t) is called the generator of X.

2. Feynman-Kac formula.

This formula gives a stochastic representation of a solution to a terminal problem for a linear PDE.

Theorem 2.1 (Feynman-Kac formula)**.** *Let $g(t,x)$ be the solution of the following terminal value problem*

$$g_t + b(t,x)g_x + \frac{1}{2}\sigma^2(t,x)g_{xx} = r(t,x)g, \quad (t,x) \in [0,T) \times \mathbb{R}$$
$$g(T,x) = h(x).$$

Let $(X(u))_{t\le u\le T}$ solve $(\mathcal{E})$ with $X(t)=x$. Then,

$$g(t,x) = \mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))ds}h(X(T))\right).$$

(We assume that some mild integrability conditions are satisfied so that all integrals make sense.)

The idea of the proof. For $0 \le t \le u \le T$, we compute (t is fixed and u is changing)

$$\begin{aligned}
&d\left(e^{-\int_t^u r(s,X(s))ds}g(u,X(u))\right)\\
&= -r(u,X(u))e^{-\int_t^u r(s,X(s))ds}g(u,X(u))du + e^{-\int_t^u r(s,X(s))ds}dg(u,X(u))\\
&= e^{-\int_t^u r(s,X(s))ds}[(-r(u,X(u))g(u,X(u)) + g_t(u,X(u))\\
&+ b(u,X(u))g_x(u,X(u)) + \frac{1}{2}\sigma^2(u,X(u))g_{xx}(u,X(u)))du + g_x(u,X(u))\sigma(u,X(u))dB(u)].
\end{aligned}$$

Since g solves the PDE, the drift term is 0. Integrating from $u=t$ to $u=T$ we get

$$e^{-\int_t^T r(s,X(s))ds}g(T,X(T)) - g(t,X(t)) = \int_t^T e^{-\int_t^u r(s,X(s))ds}g_x(u,X(u))\sigma(u,X(u))dB(u).$$

Taking conditional expectation (given $X(t)=x$), we get

$$\mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))ds}h(X(T))\right) = g(t,x).$$ □

Theorem 2.2 (Multidimensional Feynman-Kac formula)**.** *Let $a(t,x) = (\sigma\sigma^T)(t,x)$ and assume that $g(t,x)$ satisfies on $[0,T)\times\mathbb{R}^d$ the equation*

$$\partial_t g + \sum_{j=1}^d \frac{\partial}{\partial x_j}g + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{ij}(t,x)\frac{\partial^2 g}{\partial x^2} = r(t,x)g$$

with the terminal condition $g(T,x)=h(x)$, $x\in\mathbb{R}^d$. Then (under integrability conditions)

$$g(t,x) = \mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))ds}h(X(T))\right),$$

where $(X(u))_{t\le u\le T}$ solves $(\mathcal{E})$ with $X(t)=x$; i.e. $\forall j = 1,2,...,d$,

$$dX_j(u) = b_j(u,X(u))du + \sum_{k=1}^d \sigma_{jk}(u,X(u))dB_k(u);$$
$$X_j(t) = x, \quad j=1,2,...,d; \quad t\le u\le T.$$

Example 2.3 (Heston's stochastic volatility model; $d=2$)**.** Suppose that under $\widetilde{\mathbb{P}}$ the stock price follows

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)d\widetilde{B_1}(t),$$

where $V(t)$ is a stochastic process and

$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}d\widetilde{B_2}(t),$$

$a,b,\sigma > 0$ and $d[B_1,B_2]_t = \rho dt$, $-1<\rho<1$.

- Find the corresponding generator $\mathcal{A}$.

- What does the FK formula say about the solution of

$$\frac{\partial g}{\partial t} + \mathcal{A}g = rg; \quad g(T, x, v) = h(x, v)?$$

It is easy to find the drift $b(t, x, v) = \begin{bmatrix} rx \\ a - bv \end{bmatrix}$.

There are 2 ways to find $\| a_{ij}(t, x, v) \|$.

I. Find $\sigma(t, x, v)$ and compute $\sigma\sigma^T(t, x, v)$.
For this, we have to "decorrelate" our BM:

$$\begin{aligned} d\widetilde{B_1}(t) &= d\widetilde{W_1}(t), \\ d\widetilde{B_2}(t) &= \rho d\widetilde{W_1}(t) + \sqrt{1-\rho^2} d\widetilde{W_2}(t), \end{aligned}$$

where $(\widetilde{W_1}(t), \widetilde{W_2}(t))$ is a standard 2-dimensional BM.

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)d\widetilde{W_1}(t); \\ dV(t) &= (a - bV(t))dt + \sigma\sqrt{V(t)}(\rho d\widetilde{W_1}(t) + \sqrt{1-\rho^2}d\widetilde{W_2}(t)); \\ \sigma(t, x, v) &= \begin{bmatrix} \sqrt{v}x & 0 \\ \rho\sigma\sqrt{v} & \sqrt{1-\rho^2}\sigma\sqrt{v} \end{bmatrix}; \\ \sigma\sigma^T(t, x, v) &= \begin{bmatrix} vx^2 & \sigma xv\rho \\ \sigma xv\rho & \sigma^2 v \end{bmatrix}. \end{aligned}$$

II. Compute directly $\| a_{ij}(t, x, v) \|$.

$$\begin{aligned} d[S, S](t) &= V(t)S^2(t)dt \Rightarrow a_{11}(t, x, v) = vx^2; \\ d[S, V](t) &= \sigma V(t)S(t)\rho dt \Rightarrow a_{12}(t, x, v) = a_{21}(t, x, v) = \sigma\rho vx; \\ d[V, V](t) &= \sigma^2 V(t)dt \Rightarrow a_{22}(t, x, v) = \sigma^2 v. \end{aligned}$$

Why does this work? Recall our heuristics:

$$\begin{aligned} &\mathbb{E}((S(t+\delta) - S(t))^2 | S(t) = x, V(t) = v); \\ &\approx \mathbb{E}((rS(t)\delta + \sqrt{V(t)}S(t)\sqrt{\delta}Z)^2 | S(t) = x, V(t) = v); \\ &\approx vx^2\delta + o(\delta) \quad \Rightarrow a_{11}(t, x, v) = vx^2; \\ &\mathbb{E}((S(t+\delta) - S(t))(V(t+\delta) - V(t)) | S(t) = x, V(t) = v); \\ &\approx \mathbb{E}((rS(t)\delta + \sqrt{V(t)}S(t)\sqrt{\delta}Z_1)((a - bV(t)\delta + \sigma\sqrt{V(t)}\sqrt{\delta}Z_2)\delta) | S(t) = x, V(t) = v); \\ &= \sigma vx\delta \underbrace{\mathbb{E}(Z_1 Z_2)}_{\rho} + o(\delta) \quad \Rightarrow a_{12}(t, x, v) = \sigma vx\rho. \end{aligned}$$

$a_{22}(t, x, v)$ is determined similarly.

Thus,

$$\mathcal{A} := rx\frac{\partial}{\partial x} + (a - bv)\frac{\partial}{\partial v} + \frac{1}{2}vx^2\frac{\partial^2}{\partial x^2} + \sigma\rho vx\frac{\partial^2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2}.$$

- By the FK formula, $g(t, x, v) = \mathbb{E}^{t,x,v}h(S(T), V(T))$ represents the solution to

$$\frac{\partial g}{\partial t} + \mathcal{A}g = 0,$$

with the terminal condition

$$g(T, x, v) = h(x, v).$$

3. Kolmogorov's backward and forward equations.

Main point: We have a SDE $(\mathcal{E})$ and its solution X such that $X(t) = x$. For a given T, $X(T)$ is a random variable, whose distribution depends on parameters of $(\mathcal{E})$ and also on t, x.

- $X(T)$ has a density denoted by $p(t, x; T, y)$. Here t, x, T are fixed parameters, and to get the density of $X(T)$, you have to consider it as a function of y.
- Variables (x, t) are called "backward variables" as they corresponds to the starting point of the process $X : X(t) = x$.
- Variables (T, y) are called "forward variables" as y is the "location" of the process at the future time T.

Question: how to find $p(t, x; T, y)$?

- $(X(u))_{t \le u \le T}$ is a Markov process, and $p(t, x; T, y)$ is its transition probability density, that is
$$P(X(T) \in B | X(t) = x) = \int_B p(t, x; T, y) dy, \quad \forall B \in \mathcal{B}.$$

Remark 3.1. When the coefficients b and σ do not depend on t, that is when $b = b(x)$ and $\sigma = \sigma(x)$, then P is a function of $(T - t)$, x, y (as for BM).

- All in all, $p(t, x; T, y)$ has 2 pairs of variables: (t, x)-backward and (T, y)-forward.

- Kolmogorov's <u>backward</u> equation is the PDE to which $p(t, x; T, y)$ is a solution (with some special <u>terminal</u> data) when T, y are treated as fixed parameters and t, x as variables.
- Kolmogorov's <u>forward</u> equation is the PDE to which $p(t, x; T, y)$ is a solution (with some special initial data) when x, t are treated as fixed parameters and T, y as variables. In particular, the density of $X(T)$ (when $X(t) = x$ and (t, x) are fixed) satisfies the forward equation in variables T, y.

We shall start with the backward heat equation.

Example 3.2 (Backward heat equation)**.**

$$\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0; \quad g(x, T) = h(x) \tag{3}$$

$(d = 1)$ $b \equiv 0$; $a \equiv 1 \Rightarrow$ the corresponding SDE is
$$dX(u) = dB(u); \quad X(t) = x.$$

Thus
$$X(u) = x + B(u) - B(t), \quad t \le u \le T.$$

Generator $\mathcal{A} = \frac{\partial^2}{\partial x^2}$, $g(t, x) = \mathbb{E}^{t,x} h(X(T))$ is the solution to (3).
$$g(t, x) = \mathbb{E}^{t,x} h(x + B(T) - B(t)).$$

Since $B(T) - B(t) \sim N(0, T - t)$, we can compute $B(T) - B(t) \overset{d}{=} \sqrt{T - t} Z$.

$$g(t, x) = \int_{-\infty}^{+\infty} h(x + \sqrt{T - t} z) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz. \tag{4}$$

Look at it now from the perspective of $p(t, x; T, y)$,

$$g(t, x) = \mathbb{E}^{t,x} h(X(T)) = \int_{-\infty}^{+\infty} h(y) p(t, x; T, y) dy. \tag{5}$$

- If $h(x) = 1_B(x)$ for some set B, then
$$\begin{aligned} g(t, x) &= \int_B p(t, x; T, y) dy = P(X(T) \in B | X(t) = x) \\ &= \mathbb{E}^{t,x}(1_B(X(T)))(\text{FK formula}). \end{aligned}$$

- Comparing (4) and (5) (setting $y = x + \sqrt{T-t}z$ in (4)), we get that
$$p(t,x;T,y) = \frac{1}{\sqrt{(2\pi)(T-t)}} e^{-\frac{(x-y)^2}{2(T-t)}},$$
the transition probability density of BM. Informally, $p(t,x;T,y)dy$ is "the probability that the BM that started at x at time t will be in $(y, y+dy)$ at time T".

It is easy to check by differentiation that

- for a fixed $(T,y) \in [0,\infty) \times \mathbb{R}$ the function $p(t,x;T,y)$ satisfies the backward heat equation in variables (t,x) for all $t < T$ and $x \neq y$;
- for a fixed $(t,x) \in [0,\infty) \times \mathbb{R}$ the function $p(t,x;T,y)$ satisfies the forward heat equation in variables (T,y) for all $T > t$ and $y \neq x$.

It is natural to ask whether there are initial conditions which correspond to $p(t,x;T,y)$ in each of these cases. This is the content of the next two statements. In them we denoted by $p(t,x;T,y)$ the transition probability density of a general diffusion process.

Theorem 3.3 ($d = 1$)**.** *Let X solve $(\mathcal{E})$ on $[t,T]$ and $X(t) = x$. Let*
$$p(t,x;T,y) = \lim_{|B|\to 0} \frac{1}{|B|} P(X(T) \in B | X(t) = x).$$
($|B|$ is the Lebesgue measure of set $B \in \mathcal{B}$). Then for $x \in \mathbb{R}^d$, $0 \leq t < T$,

(6) $$\frac{\partial P}{\partial t} + \mathcal{A}_t P = 0 \quad \textit{(in variables t,x) with the terminal condition } p(T,x;T,y) = \delta_y(x)$$

The latter means that $\forall f \in \mathcal{C}_B(\mathbb{R})$,
$$\lim_{t\uparrow T} \int f(x)p(t,x;T,y)dx = f(y), \text{ or}$$
equivalently, that for every $B \in \mathcal{B}$,
$$\lim_{t\uparrow T} \int_B p(t,x;T,y)dx = 1_B(y).$$
((6) is Kolmogorov's backward equation, $\mathcal{A}_t$ is given by (2))

Theorem 3.4 (Kolmogorov's forward equation)**.** *Define*
$$\mathcal{A}_T^* f(T,y) = -\frac{\partial}{\partial y}(b(T,y)f(T,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)f(T,y)) \quad (\textit{adjoint operator}).$$
Then $p(t,x;T,y)$ satisfies for $T > t$, $x \in \mathbb{R}^d$,
$$\frac{\partial P}{\partial T} = \mathcal{A}_T^* P, \textit{ with the initial condition } p(t,x;t,y) = \delta_x(y) \textit{ (in variables T,y,)}$$
meaning that $\forall f \in \mathcal{C}_B(\mathbb{R})$,
$$\lim_{T\downarrow t} \int_{\mathbb{R}} f(y)p(t,x;T,y)dy = f(x), \textit{ or}$$
equivalently, that $\forall B \in \mathcal{B}$,
$$\lim_{T\downarrow t} \int_B p(t,x;T,y)dy = 1_B(x).$$

An example to have in mind: $X(t)$ is a standard Brownian motion (d-dimensional),
$$\mathcal{A}_t = \frac{1}{2}\Delta_x = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2};$$
$$p(t,x;T,y) = \frac{1}{(2\pi)^{d/2}(T-t)^{1/2}} \exp\left(-\frac{1}{2(T-t)}\sum_{i=1}^d (y_i - x_i)^2\right).$$

Then the backward equation:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\Delta_x P = 0, \quad 0 \le t < T,\, x \in \mathbb{R}^d$$

$$\lim_{t\uparrow T} \int_{\mathbb{R}^d} f(x) p(t, x; T, y) dx \underset{\text{set } z_i = \frac{x_i - y_i}{\sqrt{T-t}}}{=} \lim_{t\uparrow T} \int_{\mathbb{R}^d} f(y + \sqrt{2(T-t)}z) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|z\|^2}{2}} dz = f(y),$$

for all continuous bounded functions f.

Computing the adjoint operator, we get that in this case $\mathcal{A}_T^* = \frac{1}{2}\Delta_y = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial y_i^2}$ and the forward equation is

$$\frac{\partial P}{\partial T} = \frac{1}{2}\Delta_y P; \quad t < T,\, x \in \mathbb{R}^d$$

and

$$\lim_{T\downarrow t} \int_{\mathbb{R}^d} f(y) p(t, x; T, y) dy \underset{\text{set } z_i = \frac{x_i - y_i}{\sqrt{T-t}}}{=} \lim_{T\downarrow t} \int_{\mathbb{R}^d} f(x + \sqrt{T-t}z) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|z\|^2}{2}} dz = f(x)$$

for all $f \in \mathcal{C}_B(\mathbb{R}^d)$.