MTH 9831. Solutions to Quiz 1.

(1) State one of the equivalent definitions of Brownian motion.

Please see Lecture 1.

Note: we require **continuity of sample paths** in the definition (even though it can be shown[1] that every process which has all other properties of, say, our original definition will have a version with continuous paths, and, thus, this version can be taken to be our process).

(2) Let $(B(t))_{t\geq 0}$ be a standard Brownian motion. Given that $B(1) = x$ and $B(1/2) = y$, find the distribution of $B(3/4)$.

Solution. When dealing with Gaussian vectors it is often much simpler and faster to use linear algebra than density calculations.

We need to find the conditional distribution of $B(3/4)$ given that $B(1/2) = y$ and $B(1) = x$. The key observation is that this distribution is the same as the one for $y + \widetilde{B}(1/4)$ given that $\widetilde{B}(1/2) = x - y$, where $\widetilde{B}(t)$, $t \geq 0$, is a standard Brownian motion.

Then we try to find a constant c such that

$$\widetilde{B}(1/4) = c\,\widetilde{B}(1/2) + W, \text{ where } W \text{ is independent from } \widetilde{B}(1/2):$$
$$\mathrm{Cov}(\widetilde{B}(1/2), W) = \mathrm{Cov}(\widetilde{B}(1/2), \widetilde{B}(1/4) - c\widetilde{B}(1/2)) = 1/4 - c\,\mathrm{Var}(\widetilde{B}(1/2)) = 0 \;\Rightarrow\; c = 1/2.$$

Computing variances in the first line we get

$$1/4 = c^2/2 + \mathrm{Var}(W) \;\Rightarrow\; \mathrm{Var}(W) = 1/8 \text{ and conclude that}$$
$$\widetilde{B}(1/4) = 1/2\,\widetilde{B}(1/2) + W, \text{ where } W \sim N(0, 1/8).$$

Since W is independent[2] from $\widetilde{B}(1/2)$, conditioning on $\widetilde{B}(1/2) = x - y$ does not change the distribution of W. Thus, we get that, conditionally on $\widetilde{B}(1/2) = x - y$,

$$\widetilde{B}(1/4) = (x - y)/2 + W.$$

From this we see that the distribution of $y + \widetilde{B}(1/4)$ is normal with mean $(x + y)/2$ and variance $1/8$.

Remark. Let's take a closer look at the key observation. The best way to do this, in fact, is to draw a picture. The following manipulations are based on the definition of Brownian motion.

$$\begin{aligned}
B(3/4)\,|\,B(1/2) &= y,\; B(1) = x \\
B(1/2) + (B(3/4) - B(1/2))\,|\,B(1/2) &= y,\; B(1) - B(1/2) = x - y \\
y + (B(3/4) - B(1/2))\,|\,B(1) - B(1/2) &= x - y \quad (B(3/4) - B(1/2) \text{ is independent from } B(1/2)) \\
y + \widetilde{B}(1/4)\,|\,\widetilde{B}(1/2) &= x - y, \quad \text{where}
\end{aligned}$$

$\widetilde{B}(t) := B(t + 1/2) - B(1/2)$, $t \geq 0$, is a standard Brownian motion.

[1] see, for example, M. Marcus, J. Rosen, Markov processes, Gaussian processes, and Local Times, 2006, Lemma 6.4.6.

[2] Since the joint distribution of W and $\widetilde{B}(1/2)$ is normal, independence follows from the fact that W and $\widetilde{B}(1/2)$ are uncorrelated.