MTH 9831. LECTURE 12

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Abstract. This lecture continues our short excursion into stochastic calculus for jump processes.

1. Another useful version of Itô-Doeblin formula ($d = 1$), Itô's isometry for jump process.
2. Itô-Doeblin formula for $d = 2$. "Product rule" for jump processes.
3. Change of measure for
   (a) Poisson process;
   (b) Compound Poisson process when the jump distribution is supported on a finite set;
   (c) Compound Poisson process when the jump distribution has a density;
   (d) Compound Poisson process and Brownian motion.
4. Pricing a European call in a jump model.

1. Another useful version of Itô-Doeblin formula ($d = 1$).

Let X be a jump process, that is

$$X(t) = \underbrace{X(0) + I(t) + R(t)}_{X^c(t)} + J(t), \tag{JP}$$

where

$$I(t) = \int_0^t \Gamma(s)dB(s); \quad R(t) = \int_0^t \theta(s)ds.$$

Recall that:

$$\Delta X(t) = X(t) - X(t-) = J(t) - J(t-) = \Delta J(t);$$
$$[X, X](t) = [X^c, X^c](t) + [J, J](t) = \int_0^t \Gamma^2(s)ds + \sum_{0<s\le t} (\Delta J(s))^2.$$

Itô formula states that for a function $f \in C^2(\mathbb{R})$,[1]

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s-))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s-))d[X^c, X^c](s) + \sum_{0<s\le t} (f(X(s)) - f(X(s-))). \tag{1}$$

Let us add and subtract $\int_0^t f'(X(s-))dJ(s) = \sum_{0<s\le t} f'(X(s-))\Delta X(s)$. Then (1) can be rewritten as

$$\begin{aligned} f(X(t)) = f(X(0)) &+ \int_0^t f'(X(s-))dX(s) + \frac{1}{2}\int_0^t f''(s)d[X^c, X^c](s) \\ &+ \sum_{0<s\le t} (f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s)). \end{aligned} \tag{2}$$

This version is sometimes more convenient to use than the original one. Here is an example.

Example 1.1. Let $M(t) = N(t) - \lambda t$. We would like to find $\mathrm{Var}(\int_0^t M(s-)\, dM(s))$. Since the process $M(t-)$ is left continuous, we know that $\int_0^t M(s-)\, dM(s)$ is a martingale. Thus, $\mathbb{E}\int_0^t M(s-)dM(s) \equiv 0$

[1] Note that as long as the integrator is $X^c(s)$ or $[X^c, X^c](s)$ we are free to use s or $s-$ as we please.

and

$$\operatorname{Var}\left(\int_0^t M(s-)dM(s)\right)=\mathbb{E}\left[\left(\int_0^t M(s-)dM(s)\right)^2\right]. \tag{3}$$

In your most recent homework you computed that

$$\int_0^t M(s-)dM(s)=\frac{1}{2}\left(M^2(t)-N(t)\right) \tag{4}$$

Therefore, we can find the variance (3) using equation (4). It is not difficult[2] but we shall take a different route which will lead us to a more general result. We shall compute (3) without using (4). Set

$$X(t)=\int_0^t M(s-)dM(s)=\int_0^t M(s-)d(-\lambda s)+\sum_{0<s\le t}M(s-)\Delta N(s)=X^c(t)+J(t).$$

Note that $[X^c,X^c]\equiv 0$ and

$$\sum_{0<s\le t}(\Delta X(s))^2=\sum_{0<s\le t}M^2(s-)(\Delta N(s))^2=\sum_{0<s\le t}M^2(s-)\Delta N(s)=\int_0^t M^2(s-)\,dN(s).$$

Now we can apply (2) with $f(x)=x^2$.

$$\begin{aligned}X^2(t)&=\int_0^t 2X(t-)dX(t)+\sum_{0<s\le t}\left((X(s-)+\Delta X(s))^2-X^2(s-)-2X(s-)\Delta X(s)\right)\\&=2\int_0^t X(t-)dX(t)+\sum_{0<s\le t}(\Delta X(s))^2=2\underbrace{\int_0^t X(t-)dX(t)}_{\text{martingale}}+\int_0^t M^2(s-)dN(s).\end{aligned}$$

Taking the expectation, we get that

$$\begin{aligned}\mathbb{E}(X^2(t))&=\mathbb{E}\left(\int_0^t M^2(s-)dN(s)\right)=\mathbb{E}\Big(\underbrace{\int_0^t M^2(s-)dM(s)}_{\text{martingale}}+\lambda\int_0^t M^2(s-)ds\Big)\\&=\lambda\int_0^t\mathbb{E}\left(M^2(s)\right)ds=\lambda^2\int_0^t sds=\frac{1}{2}\lambda^2t^2;\end{aligned}$$

Thus, we have shown that

$$X^2(t)=2\int_0^t X(t-)dX(t)+\int_0^t M^2(s-)dN(s),\quad\text{and} \tag{5}$$

$$\mathbb{E}(X^2(t))=\frac{1}{2}\lambda^2t^2. \tag{6}$$

There is a general fact, a "jump version" of Itô isometry.

Proposition 1.2 (Itô's isometry)**.** *Let $X(t)$, $t\ge 0$, be a jump process of the form* (JP) *(see p. 1). Assume that $X(t)$, $t\ge 0$, is a martingale. Then*

$$\mathbb{E}(X^2(t))-X^2(0)=\mathbb{E}([X,X](t))=\mathbb{E}([X^c,X^c](t))+\mathbb{E}\sum_{0<s\le t}(\Delta J(s))^2.$$

[2]I recommend to do this as an exercise.

The proof is left as an exercise.[3] The same method which we used when we derived (6) can be used to show that for a left-continuous adapted square-integrable process Φ

$$\mathbb{E}\left[\left(\int_0^t \Phi(s)dM(s)\right)^2\right] = \lambda \int_0^t \mathbb{E}\Phi^2(s)ds.$$

More generally, if $M_Q(t) := Q(t) - \beta\lambda t$, where Q is a compound Poisson process, $\beta = \mathbb{E}[Y_1]$, then for a left-continuous adapted square-integrable process Φ

$$\mathbb{E}\left[\left(\int_0^t \Phi(s)dM_Q(s)\right)^2\right] = \lambda\beta \int_0^t \mathbb{E}[\Phi^2(s)]ds.$$

Proofs are again left as exercises. Now we turn to the case of more than 1 dimension.

2. Itô-Doeblin formula for $d = 2$. "Product rule" for jump processes

Theorem 2.1. *Let X_1, X_2 be jump processes and $f(t, x_1, x_2)$ be a $C^{1,2}([0,T] \times \mathbb{R}^2)$ function. Then*

$$\begin{aligned}
f(t, X_1(t), X_2(t)) &= f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))ds + \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) \\
&+ \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) + \frac{1}{2}\int_0^t f_{x_1x_1}(s, X_1(s), X_2(s))d[X_1^c, X_1^c](s) \\
&+ \frac{1}{2}\int_0^t f_{x_1x_2}(s, X_1(s), X_2(s))d[X_1^c, X_2^c](s) + \frac{1}{2}\int_0^t f_{x_2x_2}(s, X_1(s), X_2(s))d[X_2^c, X_2^c](s) \\
&+ \sum_{0<s\le t} (f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))).
\end{aligned}$$

The proof is omitted.

Corollary 2.2 (Itô's product formula for jump processes)**.** *Let X_1, X_2 be jump processes. Then*

$$\begin{aligned}
(*) \qquad X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^s X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) + [X_1^c, X_2^c](t) \\
&\qquad + \sum_{0<s\le t} (X_1(s)X_2(s) - X_1(s-)X_2(s-)) \\
&= X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1, X_2](t).
\end{aligned}$$

To see (*) note that

$$\begin{aligned}
\sum_{0<s\le t} (X_1(s)X_2(s) - X_1(s-)X_2(s-)) &= \sum_{0<s\le t} ((X_1(s-) + \Delta J_1(s))(X_2(s-) + \Delta J_2(s)) - X_1(s-)X_2(s-)) \\
&= \sum_{0<s\le t} (X_2(s-)\Delta J_1(s) + X_1(s-)\Delta J_2(s) + \Delta J_1(s)\Delta J_2(s)) \\
&= \int_0^t X_2(s-)dJ_1(s) + \int_0^t X_1(s-)dJ_2(s) + [J_1, J_2](t).
\end{aligned}$$

Example 2.3. By the product formula,

$$tN(t) = \int_0^t N(s)ds + \int_0^t sdN(s) \quad \text{(The cross variation term is 0.)}$$

[3] Apply Itô's formula to $X^2(t)$ and take expectations of both sides.

3. Change of measure

We shall always work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that $(\mathcal{F}(t))_{t\ge 0} \subset \mathcal{F}$ is a filtration for the process or processes we consider.[4]

3.1. **(a) Poisson process.** Let $(N(t))_{t\ge 0}$ be a Poisson process with intensity λ under $\mathbb{P}$. Given $\widetilde{\lambda} > 0$ and $T > 0$ we shall find a measure $\widetilde{\mathbb{P}}$ under which $N(t)$, $0 \le t \le T$, will be a Poisson process with intensity $\widetilde{\lambda}$. Let

$$Z(t) = e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N(t)}.$$

Then Z is a Geometric Poisson process with parameter $\sigma = (\widetilde{\lambda} - \lambda)/\lambda$. According to Example 6.3 of Lecture 11 it satisfies $dZ(t) = \frac{\widetilde{\lambda}-\lambda}{\lambda} Z(t-)dM(t)$ and, in particular, is a martingale.[5]

Define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P} \quad \text{for all } A \in \mathcal{F}(T).$$

Theorem 3.1. *Under $\widetilde{\mathbb{P}}$, the process $N(t)$, $0 \le t \le T$, is Poisson process with intensity $\widetilde{\lambda}$.*

Here is an idea as to why the result holds.

$$\begin{aligned}\widetilde{\mathbb{E}}\left(e^{uN(t)}\right) \underset{\text{Lem. 5.4 in Lec. 5}}{=} \mathbb{E}\left(e^{uN(t)}Z(t)\right) &= \mathbb{E}\left[e^{uN(t)}e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N(t)}\right]\\ &= e^{(\lambda-\widetilde{\lambda})t}\mathbb{E}e^{(u+\ln(\widetilde{\lambda}/\lambda))N(t)} = e^{(\lambda-\widetilde{\lambda})t}e^{\lambda t(e^{u+\ln(\widetilde{\lambda}/\lambda)}-1)}\\ &= e^{(\lambda-\widetilde{\lambda})t}e^{\lambda t(\frac{\widetilde{\lambda}}{\lambda}e^u-1)} = e^{\widetilde{\lambda}t(e^u-1)} \quad \Rightarrow \quad N(t) \sim \text{Poisson}(\widetilde{\lambda}t)\end{aligned}$$

Remark 3.2. We checked that the distribution of $N(t)$ under $\widetilde{\mathbb{P}}$ is Poisson$(\widetilde{\lambda}t)$. But how can one figure out what should be $\dfrac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$ if I have $N(t) \sim$ Poisson(λt) under $\mathbb{P}$ and I want $N(t) \sim$ Poisson$(\widetilde{\lambda}t)$ under $\widetilde{\mathbb{P}}$? We did this in the summer course in general for finite state spaces (see pp. 5-6 of summer Lecture 3). Countably infinite state spaces are treated in the same way.

Let $\Omega = \{0, 1, \dots\}$, $\mathcal{F} =$ all subsets of Ω, $\mathbb{P}(i) = p_i$, $p_i > 0$, $\sum_{i\in\Omega} p_i = 1$. Then

$$\mathbb{P}(A) = \sum_{i\in A} p_i; \quad \mathbb{E}X(\omega) = \sum_{i\in\Omega} X(i)p_i$$

We want $\widetilde{\mathbb{P}}(i) = \widetilde{p}_i$, $\widetilde{p}_i > 0$, $\sum_{i\in\Omega}\widetilde{p}_i = 1$. Then we simply write

$$\begin{aligned}\widetilde{\mathbb{P}}(A) &= \sum_{i\in A}\widetilde{p}_i = \sum_{i\in A}\frac{\widetilde{p}_i}{p_i}p_i = \sum_{i\in A} Z(i)p_i\\ \widetilde{\mathbb{E}}X(i) &= \sum_{i\in\Omega} X(i)\widetilde{p}_i = \sum_{i\in\Omega} X(i)\frac{\widetilde{p}_i}{p_i}p_i\\ &= \mathbb{E}(X(i)Z(i)), \quad Z(i) = \frac{\widetilde{p}_i}{p_i}, \quad \forall i \in \Omega.\end{aligned}$$

In our case $N(t)$ takes values in $\{0, 1, \dots\}$. So if we are only interested in $N(t)$ for a fixed t we can take $p_i = (\lambda t)^i e^{-\lambda t}/i!$ and then

$$Z(i) = \frac{\widetilde{p}_i}{p_i} = \frac{\frac{(\widetilde{\lambda}t)^i}{i!}e^{-\widetilde{\lambda}t}}{\frac{(\lambda t)^i}{i!}e^{-\lambda t}} = \left(\frac{\widetilde{\lambda}}{\lambda}\right)^i e^{(\lambda-\widetilde{\lambda})t}.$$

[4]See Definition 4.1 of Lecture 11.

[5]The martingale property of Z is easiest to check directly using the definition rather than stochastic calculus.

Note that the claim of Theorem 3.1 is much more profound than this, since the change of measure is done not for a single fixed t, but for a whole process, so that elementary outcomes are paths of our Poisson process. But the above tells you how to guess and quickly recover the answer.

3.2. **(b) Compound Poisson process when the jump distribution is supported on a finite set.** Let $Q(t) = \sum_{i=1}^{N(t)} Y_i$ and $\mathbb{P}(Y_1 = y_m) = p_m$, $m = 1, 2, \dots, M$, where $p_m \in (0,1)$ and $\sum_{m=1}^M p_m = 1$. Recall (see Lecture 11) that we can write $N(t) = \sum_{m=1}^M N_m(t)$ and $Q(t) = \sum_{m=1}^M y_m N_m(t)$, where $N_1, \dots, N_M$ are independent Poisson process with intensities $p_1\lambda, \dots, p_M\lambda$ respectively. Set $\lambda_m := \lambda p_m$.

Remark 3.3. Which parameters can we hope to change? We can change λ to $\widetilde{\lambda}$ and p_m to $\widetilde{p}_m$, $m = 1, \dots, M$. But we can not change y_m's, since then $\widetilde{\mathbb{P}}$ will never be absolutely continuous w.r.t. $\mathbb{P}$. Say, if $\mathbb{P}(Y_1 = 3) > 0$ and $\widetilde{\mathbb{P}} \sim \mathbb{P}$, then $\widetilde{\mathbb{P}}(Y_1 = 3) > 0$ (can not be 0). If $\mathbb{P}(Y_1 = 2) = 0$ and $\widetilde{\mathbb{P}} \sim \mathbb{P}$ then $\widetilde{\mathbb{P}}(Y_1 = 2) = 0$ (can not be > 0). Thus both measures have to "charge" the same collection of y_i's.

Let $\widetilde{\lambda}$ and $\widetilde{p}_1, \dots, \widetilde{p}_M$ be given ($\widetilde{p}_m > 0$ and $\sum_{m=1}^M \widetilde{p}_m = 1$). Set $\widetilde{\lambda}_m = \widetilde{\lambda}\widetilde{p}_m$ and define

$$Z_m(t) := e^{(\widetilde{\lambda}_m - \lambda_m)t} \left(\frac{\widetilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}, \quad Z(t) := \prod_{m=1}^M Z_m(t).$$

Fix $T > 0$, set

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T) d\mathbb{P}, \quad \forall A \in \mathcal{F}(T).$$

Theorem 3.4. *Under $\widetilde{\mathbb{P}}$, the process $Q(t)$, $0 \le t \le T$, is a compound Poisson process with intensity $\widetilde{\lambda}$ and jump distribution*

$$\widetilde{\mathbb{P}}(Y_1 = y_m) = \widetilde{p}_m.$$

The proof is omitted. Intuitively, we change measure for each Poisson process $N_m(t)$, $m = 1, 2, \dots, M$, and use there independence.

3.3. **(c) Compound Poisson process when the jump distribution has a density.** Assume now that Y_1 has density $f(y)$. Let us guess how $Z(t)$ should look like in this case. In the previous subsection we had

$$\begin{aligned} Z(t) = \prod_{m=1}^M Z_m(t) &= e^{\sum_{m=1}^M (\lambda_m - \widetilde{\lambda}_m t)} \prod_{m=1}^M \left(\frac{\widetilde{\lambda}\widetilde{p}_m}{\lambda p_m}\right)^{N_m(t)} \\ &= e^{(\widetilde{\lambda} - \lambda)t} \prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)}, \end{aligned}$$

where $\widetilde{p}(Y_i) := \widetilde{p}_m$ and $p(Y_i) := p_m$ when the i-th jump is of size y_m. Here we think of $p(\cdot)$ and $\widetilde{p}(\cdot)$ as functions from $\{y_1, y_2, \dots, y_M\}$ to $(0,1)$ and $p(Y_i)$ and $\widetilde{p}(Y_i)$ as random variables. Our guess then is that if Y_1 has density $f(y)$ then if we want Q to become a compound Poisson process with intensity $\widetilde{\lambda}$ and jump density $\widetilde{f}$ the we should set

$$Z(t) = e^{(\widetilde{\lambda} - \lambda)t} \prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)}.$$

We again fix $T > 0$ and define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T) d\mathbb{P}, \quad A \in \mathcal{F}(T).$$

Theorem 3.5. *Under $\widetilde{\mathbb{P}}$, the process $Q(t)$, $0 \le t \le T$, is a compound Poisson process with intensity $\widetilde{\lambda}$ and jump distribution*

$$\widetilde{\mathbb{P}}(Y_1 \le y) = \int_{-\infty}^y \widetilde{f}(x) dx.$$

3.4. **(d) Compound Poisson process and Brownian motion.** Suppose that on the same $(\Omega, \mathcal{F}, \mathbb{P})$, we have Brownian motion $B(t)$, $t \geq 0$, and a compound Poisson process $Q(t)$, $t \geq 0$. We assume that they are adapted to the same filtration $\mathcal{F}(t)$, $t \geq 0$. By Exercise 11.6 (which will be a part of the last homework) we know that B and Q are independent. We would like to change the measure in such a way that under $\widetilde{\mathbb{P}}$

$$\widetilde{B}(t) := B(t) + \int_0^t \theta(s)ds, \tag{7}$$

where $\theta(s)$ is an adapted process, is a standard Brownian motion and Q is a compound Poisson process with intensity $\widetilde{\lambda}$ and modified jump distribution. Moreover, we would like $\widetilde{B}$ and Q be independent under $\widetilde{\mathbb{P}}$. Define

$$\begin{aligned} Z_1(t) &= e^{-\int_0^t \theta(u)dB(u) - \frac{1}{2}\int_0^t \theta^2(u)du}, \\ Z_2(t) &= \begin{cases} e^{(\lambda-\widetilde{\lambda})t}\prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)} & \text{if jumps have density } f \text{ and we want it to be } \widetilde{f}, \\ e^{(\lambda-\widetilde{\lambda})t}\prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)} & \text{if } Y_1 \text{ has only finitely many jumps sizes,} \end{cases} \\ Z(t) &= Z_1(t)Z_2(t). \end{aligned}$$

Fix $T > 0$ and let

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P}, \quad \forall A \in \mathcal{F}(T).$$

Theorem 3.6. *Under $\widetilde{\mathbb{P}}$, the process (7), $0 \leq t \leq T$, is a standard BM and $Q(t)$, $0 \leq t \leq T$, is a compound Poisson process with intensity $\widetilde{\lambda}$ and jump distribution given by density $\widetilde{f}$ or by the probability mass function $\widetilde{p}_m$, $1 \leq m \leq M$. Moreover, $\widetilde{B}$ and Q are independent under $\widetilde{\mathbb{P}}$.*

4. Pricing a European call in a jump model

We shall assume that the dynamics of the stock price under the market measure $\mathbb{P}$ is given by the following equation:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t) + S(t-)d(Q(t) - \beta\lambda t). \tag{8}$$

The expected rate of return of this stock is α. We assume that possible jumps of Q are $-1 < y_1 < y_2 < \cdots < y_M$, none of which is 0, and that

$$P(Y_1 = y_m) = p_m, \quad m = 1, 2, \ldots, M.$$

Theorem 4.1. *The solution to (8) is*

$$S(t) = \underbrace{S(0)e^{\sigma B(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t}}_{X(t)} \underbrace{\prod_{i=1}^{N(t)} (Y_i + 1)}_{J(t)}.$$

Proof. We shall apply Itô's product formula to $S(t) = X(t)J(t)$. We have

$$\begin{aligned} &dX(t) = (\alpha - \beta\lambda)X(t)dt + \sigma(t)X(t)dB(t); \\ &\Delta J(t) = J(t) - J(t-) = J(t-)\left(\frac{J(t)}{J(t-)} - 1\right) = J(t-)\underbrace{Y_{N(t)}\Delta N(t)}_{\Delta Q(t)}, \text{ i.e. } dJ(t) = J(t-)dQ(t); \\ &S(t) = X(t)J(t) = S(0) + \int_0^t X(s-)dJ(s) + \int_0^t J(s)dX(s) + [X, J](t). \end{aligned}$$

Since J is a pure jump process and X is continuous, $[X,J](t) \equiv 0$. Substituting $dJ(s)$ and $dX(s)$ we obtain

$$S(t) = S(0) + \int_0^t X(s-)J(s-)dQ(s) + (\alpha - \beta\lambda)\int_0^t J(s)X(s)ds + \sigma \int_0^t J(s)X(s)dB(s),$$

or in the differential form,

$$dS(t) = S(t-)dQ(t) + (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dB(t), \tag{9}$$

which is exactly (8). □

We want to construct $\widetilde{\mathbb{P}}$ such that under $\widetilde{\mathbb{P}}$

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{B}(t) + S(t-)d(Q(s) - \widetilde{\beta}\,\widetilde{\lambda}s),$$

where $d\widetilde{B}(t) = dB(t) + \theta dt$. Rewriting the above equation we get

$$dS(t) = S(t-)dQ(t) + (r + \theta\sigma - \widetilde{\beta}\,\widetilde{\lambda})S(t)dt + \sigma S(t)dB(t).$$

Comparing with (9), we see that

$$r + \theta\sigma - \widetilde{\beta}\,\widetilde{\lambda} = \alpha - \beta\lambda \quad \Leftrightarrow \quad \theta\sigma + \beta\lambda - \widetilde{\beta}\,\widetilde{\lambda} = \alpha - r \quad \Leftrightarrow$$
$$\theta\sigma + \sum_{m=1}^M y_m p_m \lambda + \sum_{m=1}^M y_m \widetilde{p}_m \widetilde{\lambda} = \alpha - r;$$

Recalling that $\lambda_m = p_m\lambda$ and $\widetilde{\lambda}_m = \widetilde{p}_m\widetilde{\lambda}$ we get

$$\theta\sigma + \sum_{m=1}^M y_m(\lambda_m - \widetilde{\lambda}_m) = \alpha - r. \tag{10}$$

We have one equation and $M+1$ unknowns θ, $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_M$. Therefore, the model is incomplete. These extra parameters can be used to calibrate the model to market prices.

Pick any one solution of (10). Under $\widetilde{\mathbb{P}}$,

$$S(t) = S(0)e^{\sigma\widetilde{B}(t) + (r - \widetilde{\beta}\widetilde{\lambda} - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N(t)} (Y_i + 1),$$

where N has intensity $\widetilde{\lambda}$ under $\widetilde{\mathbb{P}}$ and

$$\widetilde{\mathbb{P}}(Y_1 = y_m) = \widetilde{p}_m = \widetilde{\lambda}_m / \widetilde{\lambda}.$$

The call price at time t is now $e^{-r(T-t)}\widetilde{\mathbb{E}}((S(T) - K)_+ | \mathcal{F}(t))$. There is an explicit expression (see Theorem 11.7.5 of the textbook). But I shall neither state nor derive it.

Theorem 4.2. *Let $c(t,x)$ be the price of a call at time t when $S(t) = x$. Then $c(t,x)$ satisfies*

$$-rc(t,x) + c_t(t,x) + (r - \widetilde{\beta}\widetilde{\lambda})xc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) + \widetilde{\lambda}\sum_{m=1}^M [\widetilde{p}_m c(t,(y_m+1)x) - c(t,x)] = 0,$$

$0 \le t < T$, $x > 0$, *and the terminal condition*

$$c(T,x) = (x-K)_+, \quad x \ge 0.$$

Proof is left as an exercise. The process $e^{-rt}c(t,S(t))$ should be a martingale. Apply Itô-Doeblin formula and set the "dt" term to zero. The details are given in Theorem 11.7.7 of the textbook.