Probability and Stochastic Processes for Finance I (MTH 9831).

Final Examination.

Instructions: Please **print** your name below. Show all work and write legibly. Full credit corresponds to 100 points. **Good luck!**

Student's name:

Grade

Problem	Out of	Score	Comments
1	15		
2	15		
3	20		10+5+5
4	20		
5	15		
6	25		
7	30		
Total	140		

Problem 1. Find the conditional distribution of $B(s)$ given that $B(t) = x$.

Problem 2. Let $\tau_m = \inf\{t \geq 0 : \ B(t) = m\}$. Show that for every $m \neq 0$

$$E(\tau_m^\alpha) < \infty \quad \iff \quad \alpha < 1/2.$$

Problem 3. Let $B(s)$, $s \geq 0$, be a standard Brownian motion.

(a) Solve the following SDE for $t \geq 0$:

$$dY(s) = (2 - Y(s))dt + 2dB(s), \quad Y(0) = x.$$

(b) Fix $t \in [0, 1]$ and let $X(u)$, $u \geq t$, be the solution of

$$dX(u) = (2 - X(u))\, du + 2dB(u), \quad X(t) = x.$$

Find the distribution, expectation, and variance of $X(u)$, $u \geq t$.

(c) Compute the second moment of $X(1)$.

Problem 4. Let $g(t,x)$ be the solution to the following terminal value problem on $[0,1]\times\mathbb{R}$:

$$g_t + (2-x)g_x + 2g_{xx} = x, \quad g(1,x) = x^2.$$

(a) Write the stochastic representation of $g(t,x)$. Hint: apply Itô-Doeblin formula to $g(u, X(u))$ for an appropriate process X, use the PDE, integrate, and take the expectation to arrive at the required stochastic representation.

(b) Compute the expectation in the previous part and find $g(t,x)$.

Problem 5. Let $(B(t))_{t\geq 0}$ be a standard Brownian motion and $B^*(t) = \max_{0\leq u\leq t} B(u)$. Show that the two-dimensional process $(B(t), B^*(t))_{t\geq 0}$ is a Markov process with respect to the natural filtration of $(B(t))_{t\geq 0}$. Hint: start by writing for $0 \leq s \leq t$

$$B^*(t) = B^*(s) + (\max_{s\leq u\leq t} B(u) - B^*(s))_+$$

and use the definition of a Markov process, properties of Brownian motion, and the independence lemma.

Problem 6. Let $B(t)$, $t \geq 0$, be a standard Brownian motion, $Q(t)$, $t \geq 0$, be a compound Poisson process with respect to the same filtration. Assume that the intensity of the Poisson process is λ and the jump distribution is normal with mean η and variance v. Set

$$S(t) = S(0)e^{\mu t + \sigma B(t) + Q(t)},$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are constants.

(a) Find $\mathrm{Cov}(S(t), \ln S(t))$.

(b) Determine under which condition on μ, σ, η, v the process $S(t)$, $t \geq 0$, is a martingale. Justify your answer. Hint: it is simpler to use the definition of a martingale than Itô-Doeblin formula.

Problem 7. Let $B(t)$, $t \geq 0$, be a standard Brownian motion, $Q(t)$, $t \geq 0$, be a compound Poisson process (relative to the same filtration). Assume that the jump intensity is λ and that the jump distribution is discrete with possible values $y_1, y_2, \ldots, y_m$ taken with probabilities $p_1, p_2, \ldots, p_m$ respectively. Define $X(t) = \mu t + \sigma B(t) + Q(t)$, $\mu \in \mathbb{R}$, $\sigma > 0$.

(a) Let the stock price $S(t)$, $t \geq 0$, under $\mathbb{P}$ satisfy the following equation

$$S(t) = S(0) + \int_0^t S(u)\, dX^c(u) + \sum_{0<u\leq t} S(u-)\Delta X(u).$$

Solve this equation and find an explicit formula for $S(t)$, $t \geq 0$.

(b) Describe all equivalent measures $\tilde{\mathbb{P}}$ under which $e^{-rt}S(t)$ is a $\tilde{\mathbb{P}}$-martingale (give conditions on parameters and the Radon-Nikodym derivative with respect to $\mathbb{P}$).

(c) Let $c(t, x)$ be the price of a call option at time t when $S(t) = x$. Which equation should $c(t, x)$ satisfy in order for $e^{-rt}c(t, S(t))$ to be a martingale under $\tilde{\mathbb{P}}$ (pick and fix one from the previous part)? Derive the equation.