(5)

Note that

$$Z(t) = Z(0)e^{(\lambda-\tilde{\lambda})t}(\frac{\tilde{\lambda}}{\lambda})^{N(t)} = Z(0)e^{(\lambda-\tilde{\lambda})t+N(t)\ln\frac{\tilde{\lambda}}{\lambda}}$$

Let

$$X(t) \triangleq (\lambda-\tilde{\lambda})t + N(t)\ln\frac{\tilde{\lambda}}{\lambda}$$

then using Ito's formula, we have

$$\begin{aligned}
dZ(t) &= Z(0)e^{X(t-)}dX^c(t) + \frac{1}{2}Z(0)e^{X(t)}d[X^c, X^c]_t + Z(0)e^{X(t)} - Z(0)e^{X(t-)} \\
&= Z(0)e^{X(t-)}(\lambda-\tilde{\lambda})dt + 0 + Z(0)e^{X(t-)}(e^{X(t)-X(t-)} - 1) \\
&= -Z(0)e^{X(t-)}\frac{\tilde{\lambda}-\lambda}{\lambda}d(\lambda t) + Z(0)e^{X(t-)}(e^{\ln\frac{\tilde{\lambda}}{\lambda}} - 1)dN(t) \\
&= Z(0)e^{X(t-)}\frac{\tilde{\lambda}-\lambda}{\lambda}d(N(t)-\lambda t) \\
&= Z(0)e^{X(t-)}\frac{\tilde{\lambda}-\lambda}{\lambda}dM(t)
\end{aligned}$$

Since $\{M(t)\}$ is a martingale, $Z(t-)$ is left-continuous and adapted, and

$$\mathbb{E}\int_0^t \Gamma^2(s)\Phi^2(s)ds = \mathbb{E}\int_0^t 0 \times (Z(0)e^{X(s-)}\frac{\tilde{\lambda}-\lambda}{\lambda})^2 ds = 0 < \infty$$

, according to Theorem 4.5, we conclude that

$$Z(t) = Z(0) + \int_0^t Z(0)e^{X(s-)}\frac{\tilde{\lambda}-\lambda}{\lambda}dM(s)$$

is a martingale.

(6)

Applying Ito's formula to $\log S(t)$, we have

$$d\log S(t) = \frac{1}{S(t-)}dS^c(t) - \frac{1}{2}\cdot\frac{1}{S^2(t-)}d[S^c, S^c]_t + \log S(t) - \log S(t-)$$

since

$$dS(t) = \sigma S(t-)dM(t) = -\lambda\sigma S(t-)dt + \sigma S(t-)dN(t)$$

we have

$$dS^c = -\lambda\sigma S(t-)dt$$

Thus,

$$d\log S(t) = -\lambda\sigma dt + 0 + \log(1 + \frac{S(t)-S(t-)}{S(t-)})$$

Note that

$$S(t) - S(t-) = \sigma S(t-)\Delta N(t) \Rightarrow \frac{S(t) - S(t-)}{S(t-)} = \sigma \Delta N(t)$$

and since $\Delta N(t)$ can either be 1 or 0, we conclude that

$$d \log S(t) = -\lambda\sigma dt + \log(1+\sigma)\Delta N(t)$$

Integrating, we have

$$\log S(t) - \log S(0) = -\lambda\sigma t + \log(1+\sigma)N(t)$$

and finally, exponentiating, we see that

$$S(t) = S(0)e^{-\lambda\sigma t}(1+\sigma)^{N(t)}$$