MTH 9831 Assignment 9 (11/15/2018 - 11/21/2018).

(1) (Markov property for geometric Brownian motion and its maximum to date.) Exercise 7.3.

(2) (Properties of the running maximum of a GBM.) Let $S(t)$, $t \geq 0$, be a geometric Brownian motion

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dB(t),$$

where $B(t)$, $t \geq 0$ is a standard Brownian motion with respect to $\mathbb{P}$, and let

$$Y(t) = \max_{0 \leq u \leq t} S(u).$$

(i) Read the proof of the fact that for each $T > 0$

$$\lim_{\|\Pi\| \to 0} \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))^2 = 0 \quad \text{a.s.}$$

on p. 309 of the textbook. Here $0 = t_0 < t_1 < \cdots < t_m = T$ is a partition of $[0, T]$ and $\|\Pi\| = \max_{1 \leq j \leq m}(t_j - t_{j-1})$. Conclusion: the quadratic variation process $[Y]_T$, $T \geq 0$, when defined as an a.s. limit, exists and is equal to 0. Convince yourself that this proof works for any stochastic process with continuous monotone paths. There is nothing to submit for this part (but see the footnote).

(ii) Show that

$$\sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))^2 \leq (Y(T) - Y(0))^2$$

and use this, part (i), and an appropriate convergence theorem to prove that the left hand side converges to 0 in L^2, i.e. that

$$\lim_{\|\Pi\| \to 0} \mathbb{E}\left[\left(\sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))^2\right)^2\right] = 0.$$

Conclusion: $[Y]_T$, defined as an L^2 limit (see Definition 1.1 and 1.4 in Lecture 3), exists and is equal to 0 for every $T > 0$.[1]

(iii) Show that for each $T > 0$ the cross-variation $[S, Y]_T$ exists in a.s. and L^2 sense (both!) and is equal to 0.

(iv) Read p. 310 (for general education): you will learn the connection between points of increase of BM and Cantor set as well as what is a singularly continuous function ($Y(t)$ is an example of such a function). This part of the homework is deemed not "examinable".

(3) (Zero strike Asian call) Exercise 7.7.

[1] The very last conclusion is automatic from the previous part as the value of the limit is unique as long as it exists, no matter in which sense it is considered (a.s., in probability, or in L^2).

(4) Read Section 8.3.1 of the textbook. This part reviews the Laplace transform of the first passage time for a BM with drift. Realize that you have already seen this before and that your midterm problem 2 was essentially to prove Lemma 8.3.4. There is nothing to submit for this part of the homework.[2] We shall use this material in the next class.

[2]Parts of this homework for which you have nothing to submit can be easily made into quiz questions.