Lim, sup, and max.

Below a_n, b_n $n \in \mathbb{N}$, are sequences of real numbers. Recall one of the definitions of $\limsup$:

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} (\sup_{k\geq n} a_k).$$

It always exists (can be ∞), since the sequence $s_n := \sup_{k\geq n} a_k$ is monotone non-increasing.

Exercise 1. (4 points) Suppose that

$$\limsup_{n\to\infty} a_n = L \quad \text{and} \quad \liminf_{n\to\infty} a_n = -L \quad (\text{i.e. } \limsup_{n\to\infty}(-a_n) = L).$$

Show that $L \geq 0$ and $\limsup_{n\to\infty} |a_n| = \limsup_{n\to\infty}(a_n \vee (-a_n)) = L$. Redo Exercise 3 from Lecture 2.

Exercise 2. (3 points) Is it always the case that

$$\limsup_{n\to\infty}(a_n \vee b_n) = (\limsup_{n\to\infty} a_n) \vee (\limsup_{n\to\infty} b_n)?$$

Give a proof or a counterexample.

Exercise 3. (3 points) Is it always the case that

$$\limsup_{n\to\infty} \sup_{m\in\mathbb{N}} a(m,n) = \sup_{m\in\mathbb{N}} \limsup_{n\to\infty} a(m,n)?$$

Give a proof or a counterexample.