MTH 9831 Assignment 1 (9/06 - 9/12).

Read Lecture 1. Some additional references for this material are:

1. J. Jacod, Ph. Protter, Probability Essentials, Chapter 16.
2. S. Shreve, Stochastic Calculus for Finance II, Chapter 3.
3. A. Etheridge, A Course in Financial Calculus, Chapter 3.

Rules of the game: When solving these problems you are allowed to use (with a proper reference) any statement from lecture notes, even if it was given without a proof. Citing a convenient theorem from some other source and deriving your solution from it is not allowed.

Solve:

(1) Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$, be a two dimensional Brownian motion (assume that the coordinates are independent). Find the distribution of the distance from $B(t)$ to the origin.

(2) Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$, be a two dimensional Brownian motion (assume that the coordinates are independent), and $\rho \in [-1, 1]$. Is the process $X(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)$ a Brownian motion? What is the correlation between $X(t)$ and $B_1(t)$?

(3) Let $(B(t))_{t\geq 0}$ be a Brownian motion. Determine, which of the following processes are Brownian motions. Justify your answers.

(a) $(-B(t))_{t\geq 0}$;
(b) $(cB(t/c^2))_{t\geq 0}$, where $c > 0$ is a constant;
(c) $(\sqrt{t}B(1))_{t\geq 0}$;
(d) $(B(2t) - B(t))_{t\geq 0}$;
(e) $(B(s) - B(s-t))_{0\leq t\leq s}$, where s is fixed.

(4) Let $(B(t))_{t\geq 0}$ be a Brownian motion. Define $B^*(t) := \max_{0\leq s\leq t} B(s)$. For $0 \leq a \leq x$ calculate $P(B^*(t) \geq a,\, B(t) \leq x)$.

The next two problems are review problems for Gaussian random vectors. For your convenience I first recall some linear algebra. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then *inner product* (dot product) of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i,$$

the *norm* (length) of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$, and $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$. Given (non-zero) vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n} \in \mathbb{R}^n$ we can perform the Gram-Schmidt orthogonalization by setting

$$\begin{aligned}
\mathbf{y_1} &= \mathbf{x_1} \\
\mathbf{y_2} &= \mathbf{x_2} - \frac{(\mathbf{x_2} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} \\
\mathbf{y_3} &= \mathbf{x_3} - \frac{(\mathbf{x_3} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} - \frac{(\mathbf{x_3} \cdot \mathbf{y_2})}{(\mathbf{y_2} \cdot \mathbf{y_2})}\mathbf{y_2} \\
&\cdots\cdots \\
\mathbf{y_n} &= \mathbf{x_n} - \frac{(\mathbf{x_n} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} - \frac{(\mathbf{x_n} \cdot \mathbf{y_2})}{(\mathbf{y_2} \cdot \mathbf{y_2})}\mathbf{y_2} - \cdots - \frac{(\mathbf{x_n} \cdot \mathbf{y_{n-1}})}{(\mathbf{y_{n-1}} \cdot \mathbf{y_{n-1}})}\mathbf{y_{n-1}}.
\end{aligned}$$

Then $\mathbf{y_i} \cdot \mathbf{y_j} = 0$ for all $\mathbf{i} \neq \mathbf{j}$ and the collection of all non-zero vectors[1] from the set $\{\mathbf{y_1}, \mathbf{y_2}, \dots, \mathbf{y_n}\}$ forms an orthogonal basis of the span of $\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}$:

$$
\begin{aligned}
\mathbf{x_1} &= \mathbf{y_1} \\
\mathbf{x_2} &= \frac{(\mathbf{x_2} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} + \mathbf{y_2} \\
\mathbf{x_3} &= \frac{(\mathbf{x_3} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} + \frac{(\mathbf{x_3} \cdot \mathbf{y_2})}{(\mathbf{y_2} \cdot \mathbf{y_2})}\mathbf{y_2} + \mathbf{y_3} \\
&\dots\dots \\
\mathbf{x_n} &= \frac{(\mathbf{x_n} \cdot \mathbf{y_1})}{(\mathbf{y_1} \cdot \mathbf{y_1})}\mathbf{y_1} + \frac{(\mathbf{x_n} \cdot \mathbf{y_2})}{(\mathbf{y_2} \cdot \mathbf{y_2})}\mathbf{y_2} + \dots + \frac{(\mathbf{x_n} \cdot \mathbf{y_{n-1}})}{(\mathbf{y_{n-1}} \cdot \mathbf{y_{n-1}})}\mathbf{y_{n-1}} + \mathbf{y_n}.
\end{aligned}
$$

Now let's go back to probability. For any square-integrable mean zero random variables X and Y define their inner product to be $\mathbb{E}(XY)$ and the L^2-norm $\|X\|_2 := \sqrt{\mathbb{E}(X^2)} = \sigma(X)$. Note that if (X, Y) is a mean zero Gaussian vector then orthogonality of X and Y (i.e. $\mathbb{E}(XY) = 0$) is equivalent to independence of X and Y.

(5) Let $(X_1, X_2, \dots, X_n)$ be a Gaussian vector with mean zero. Write down the Gram-Schmidt procedure for this vector.

(a) Use the result to give an alternative proof of Theorem 1.6 from Lecture 1.

(b) Show that for every $1 < k \leq n$, $\mathbb{E}(X_k \mid X_1, X_2, \dots, X_{k-1})$ is a linear function of $X_1, X_2, \dots, X_{k-1}$.

(c) Show that for every $1 < k < n$, the conditional distribution of

$$((X_{k+1}, X_{k+2}, \dots, X_n)|(X_1, X_2, \dots, X_k) = (x_1, x_2, \dots, x_k))$$

is Gaussian and that the dependence of the parameters on $x_1, x_2, \dots, x_k$ is linear.

(6) Let $(\mathbf{X_1}, \mathbf{X_2})$ be a Gaussian vector, $\dim \mathbf{X_1} = k$ and $\dim \mathbf{X_2} = m$. The mean vector of $(\mathbf{X_1}, \mathbf{X_2})$ is $(\mu_\mathbf{1}, \mu_\mathbf{2})$ and the covariance matrix is

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where C_{11} the covariance matrix of $\mathbf{X_1}$, C_{22} is the covariance matrix of $\mathbf{X_2}$ and $C_{12} = C_{21}^t = \|\mathrm{Cov}(X_1^i, X_2^j)\|_{1 \leq i \leq k, 1 \leq j \leq m}$. Assume that C_{11} is non-degenerate and find the distribution of $(\mathbf{X_2}|\mathbf{X_1} = \mathbf{x})$.

Hint: the conditional distribution is Gaussian by the previous problem. One possible way to find the parameters is to use the idea presented in class, i.e. write

$$\mathbf{X_2} - \mu_\mathbf{2} = \Lambda(\mathbf{X_1} - \mu_\mathbf{1}) + \mathbf{W},$$

where Λ is a non-random matrix and $\mathbf{W}$ is a mean zero Gaussian vector independent from $\mathbf{X_1}$[2]. First multiply both sides by $(\mathbf{X_1} - \mu_\mathbf{1})^t$, compute the expectation, and find Λ. Then multiply both parts by $(\mathbf{X_2} - \mu_\mathbf{2})^t = (\Lambda(\mathbf{X_1} - \mu_\mathbf{1}) + \mathbf{W})^t$ and compute the expectation to find the covariance matrix of $\mathbf{W}$. Finally perform the conditioning and write down the parameters.

[1] Some vectors $\mathbf{y_i}$ can be zero vectors as we did not assume that the original vectors were linearly independent. We agree to replace such terms in the Gram-Schmidt process simply by $\mathbf{0}$.

[2] This is a version of Gram-Schmidt idea of orthogonalization: the vector $\Lambda(\mathbf{X_1} - \mu_\mathbf{1})$ is the orthogonal (in the L^2 sense) projection of $(\mathbf{X_2} - \mu_\mathbf{2})$ on $(\mathbf{X_1} - \mu_\mathbf{1})$. Subtracting it from $(\mathbf{X_2} - \mu_\mathbf{2})$ gives a vector $\mathbf{W}$ which is orthogonal to $(\mathbf{X_1} - \mu_\mathbf{1})$.