MTH 9831 Assignment 2 (09/13 - 09/26).

Read Lecture 2. Some additional references for this material are:

1. S. Shreve, Stochastic Calculus for Finance II, Section 3.7.
2. A. Etheridge, A Course in Financial Calculus, Chapter 3.

Solve:

(1) Exercise 2 from Lecture 2.

(2) The proof of Lemma 2.2 contains the proof of the following statement.
Lemma (Borel-Cantelli, part 1). Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_n \in \mathcal{F}$, $n \geq 1$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then

$$P\left(\bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n\right) = 0.$$

The event $\cap_{N=1}^{\infty} \cup_{n \geq N} A_n$ is usually denoted by $\limsup A_n$ or $\{A_n \text{ i.o.}\}$, where i.o. stands for "infinitely often", and consists of those and only those ω which belong to infinitely many sets A_n of the sequence.[1] Give a detailed proof of the above lemma and use it to show that if $(X_n)_{n \geq 1}$ is a sequence of identically distributed integrable random variables then

$$P\left(\lim_{n \to \infty} \frac{X_n}{n} = 0\right) = 1.$$

(3) Exercise 3 from Lecture 2.

(4) Use martingales in Theorem 3.2(a),(b) to compute[2] (a) $P(\tau_a < \tau_b)$; (b) $E(\tau_a \wedge \tau_b)$, where $a < 0 < b$, $\tau_x := \inf\{t \geq 0 \,|\, B(t) = x\}$, and $(B(t))_{t \geq 0}$ is a standard Brownian motion.

(5) Exercise 3.7 from the textbook.

(6) Use the stochastic representation (4.2) of Lecture 2 to find the solution of the backward heat equation with the terminal function[3] (a) $f(x) = x^2$; (b) $f(x) = e^x$; (c) $f(x) = e^{-x^2}$.

(7) Exercise 4 from Lecture 2.

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[1] i.e. for which there is a subsequence n_k, $k \geq 1$, $n_k \to \infty$ such that $\omega \in \cap_{k=1}^{\infty} A_{n_k}$.

[2] You may assume that the optional stopping theorem A (Theorem 2.4 from refresher lecture 5) and the optional sampling theorem (Theorem 2.5 from refresher lecture 5 or Theorem 8.2.4 of the textbook) hold in the continuous time setting.

[3] In our derivations we assumed that f, f', f'' were continuous and bounded. The boundedness condition does not hold for f in parts (a) and (b). But you should still use the representation and then check that the obtained function is indeed a solution. In all three examples the solution of the terminal problem is unique.