MTH 9831. LECTURE 6

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Abstract. We start the lecture with applications of Girsanov's theorem.
1. Application of Girsanov's theorem: finding the joint distribution of the Brownian motion with a constant drift and its running maximum.
2. Application of Girsanov's theorem: risk-neutral measure for a simple market (stock + MMA).
3. Martingale representation theorem. Application to a single stock model: pricing and hedging.
4. Girsanov's theorem and martingale representation theorem for $d > 1$.
5. Multidimensional market model.

1. Application: finding the joint distribution of the Brownian motion with a constant drift and its running maximum.

Let $\widetilde{B}(t) = \alpha t + B(t)$, where $(B(t))_{t\geq 0}$ is a standard Brownian motion and $\alpha > 0$ is a constant. Denote be $\widetilde{B}^*(T)$ the maximum of $\widetilde{B}$ up to time T, that is $\widetilde{B}^*(T) = \max_{0\leq t\leq T} \widetilde{B}(t)$. We are going to compute the joint density $f(x,a)$ of $(\widetilde{B}(T), \widetilde{B}^*(T))$.

Note that since $\widetilde{B}^*(T) \geq 0$ and $\widetilde{B}^*(T) \geq \widetilde{B}(T)$, the density $f(x,a)$ is equal to zero on the set $\{a < 0\} \cup \{a < x\}$. Thus, we need to compute $f(x,a)$ on the set $\{a \geq 0\} \cap \{a \geq x\}$. The idea is to perform a change of measure so that the process $\widetilde{B}$ becomes a standard Brownian motion for which we already know the joint distribution of its end point at time T and its maximum of time T.[1] We write

$$\begin{aligned} P(\widetilde{B}(T) \leq x, \widetilde{B}^*(T) \leq a) &= E(\mathbb{1}_{\{\widetilde{B}(T)\leq x, \widetilde{B}^*(T)\leq a\}}) \\ &= E(\mathbb{1}_{\{\widetilde{B}(T)\leq x, \widetilde{B}^*(T)\leq a\}}(Z(T))^{-1}Z(T)) = \widetilde{E}(\mathbb{1}_{\{\widetilde{B}(T)\leq x, \widetilde{B}^*(T)\leq a\}}(Z(T))^{-1}), \end{aligned} \tag{1.1}$$

where $Z(T) = e^{-\alpha B(T) - \alpha^2 T/2}$ and $\widetilde{E}$ is the expectation with respect to the new probability measure

$$\widetilde{P}(A) = \int_A Z(T)\, dP.$$

This can be done according to Theorem 5.1(2) of lecture 5 since $E(Z(T)) = 1$ and $P(Z(T) > 0) = 1$. Note that this choice of measure corresponds to the case $\Theta(t) \equiv \alpha$ in Girsanov's theorem. Then the Radon-Nikodym derivative which appears there is exactly

$$Z(T) = \exp\left(-\int_0^T \Theta(t)\, dB(t) - \frac{1}{2}\int_0^T \Theta^2(t)\, dt\right) = e^{-\alpha B(T) - \alpha^2 T/2}.$$

Therefore, under $\widetilde{P}$ the process $\widetilde{B}$ is a standard Brownian motion and

$$\begin{aligned} \widetilde{P}(\widetilde{B}(T) \leq x, \widetilde{B}^*(T) \leq a) &= N\left(\frac{2a - x}{\sqrt{T}}\right) - N\left(-\frac{x}{\sqrt{T}}\right); \\ \widetilde{f}(x,a) &= \frac{\partial^2}{\partial x \partial a}\left(N\left(\frac{2a-x}{\sqrt{T}}\right) - N\left(-\frac{x}{\sqrt{T}}\right)\right) = \frac{\partial}{\partial x}\left(N'\left(\frac{2a-x}{\sqrt{T}}\right)\frac{2}{\sqrt{T}}\right) \\ &= \frac{2}{\sqrt{T}}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{2\pi}}e^{-(2a-x)^2/(2T)}\right) = \frac{2(2a-x)}{T\sqrt{2\pi T}}\, e^{-(2a-x)^2/(2T)}, \quad x \leq a,\ a \geq 0. \end{aligned}$$

[1] See Theorem 1.8 and Exercise 2 from lecture 2

Since we know the joint density and

$$(Z(T))^{-1} = e^{\alpha B(T)+\alpha^2 T/2} = e^{\alpha(\widetilde{B}(t)-\alpha T)+\alpha^2 T/2} = e^{\alpha \widetilde{B}(T)-\alpha^2 T/2},$$

(1.1) can be written as

$$\int_{-\infty}^{x}\int_{-\infty}^{a} e^{\alpha y-\alpha^2 T/2}\widetilde{f}(y,m)\,dy\,dm.$$

Differentiating in x and then a we find that

$$f(x,a) = e^{\alpha x-\alpha^2 T/2}\,\frac{2(2a-x)}{T\sqrt{2\pi T}}\,e^{-(2a-x)^2/(2T)}, \quad x \le a,\ a \ge 0.$$

It is now a just calculus exercise to compute the distribution function and the density of $\widetilde{B}^*(T)$.

Exercise 1. *Compute the distribution function and the density of* $\widetilde{B}^*(T)$.

These distributions and, more generally, the change of measure play key roles in pricing of exotic options (see Chapter 7 of the textbook).

2. Application of Girsanov's theorem: risk-neutral measure for a simple market.

Let $(B(t))_{t\ge 0}$ be a standard Brownian motion on $\Omega, \mathcal{F}, P)$, $(\mathcal{F}(t))_{t\ge 0}$ be the filtration for this Brownian motion, $\mathcal{F}(t) \subset \mathcal{F}$ for all $t \ge 0$. We shall consider a market with

- an MMA account with an adapted interest rate process $(R(t))_{t\ge 0}$ so that the discounting process $(D(t))_{t\ge 0}$ satisfies

$$dD(t) = -R(t)D(t)\,dt, \quad t \ge 0, \tag{2.1}$$

  i.e. $D(t) = e^{-\int_0^t R(u)\,du}$;
- a single stock whose price under P satisfies the equation

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t), \tag{2.2}$$

  where $\alpha(t),\ \sigma(t),\ t \ge 0$, are adapted processes, $\sigma(t) \ne 0$ a.s.. We know that the solution of (2.2) is given by

$$S(t) = S(0)\exp\left(\int_0^t \sigma(u)\,dB(u) + \int_0^t \alpha(u) - \sigma^2(u)/2\,du\right)$$

  which a *generalized geometric Brownian motion (GBM)*

We call the above model *a simple market model.*

Definition 2.1. *A risk-neutral measure* $\widetilde{P}$ *(also called an equivalent martingale measure) is a measure on* $(\Omega, \mathcal{F}, P)$ *that is*

- *equivalent to* P *and*
- *under which the discounted stock price process* $(D(t)S(t))_{t\ge 0}$ *is an* $\mathcal{F}(t)$*-martingale.*

To find $\widetilde{P}$ we calculate $d(D(t)S(t))$ and see if we can get rid of the "dt" term. Since $D(t)$ is a regular process, $d[S,D]_t = 0$ and

$$d(D(t)S(t)) = D(t)dS(t) + S(t)dD(t) = \sigma(t)D(t)S(t)\left(dB(t) + \frac{\alpha(t)-R(t)}{\sigma(t)}dt\right).$$

Let $\Theta(t) = (\sigma(t))^{-1}(\alpha(t) - R(t))$ be *the market price of risk.* It is *the excess instantaneous rate of return of the stock (over MMA) per unit of volatility.* Set

$$\widetilde{P}(A) = \int_A Z\,dP, \quad \text{where} \quad Z = Z(T) = \exp\left(-\int_0^T \Theta(t)\,dB(t) - \frac{1}{2}\int_0^T \Theta^2(t)\,dt\right).$$

Then under $\widetilde{P}$ the process $\widetilde{B}(t) = B(t) + \int_0^t \Theta(s)\, ds$, $0 \le t \le T$, is a standard Brownian motion, and the process

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\widetilde{B}(t), \quad 0 \le t \le T, \tag{2.3}$$

is an $\mathcal{F}(t)$-martingale under $\widetilde{P}$. Since $P(Z > 0) = 1$, $\widetilde{P}$ is equivalent to P. Moreover, from (2.3) we also get that

$$dS(t) = R(t)S(t)\, dt + \sigma(t)S(t)\, d\widetilde{B}(t).$$

This shows that the instantaneous rate of return of the stock under $\widetilde{P}$ is the same as for the MMA,

$$M(t) := (D(t))^{-1} = \exp\left(\int_0^t R(u)\, du\right); \quad dM(t) = R(t)M(t)\, dt.$$

This should explain the name "risk-neutral" measure for $\widetilde{P}$.

Let $X(t)$ be the value of a portfolio which consists of $\Delta(t)$ shares of stock and $\Gamma(t)$ shares of MMA. We shall show that under the self-financing condition (see problem 4.10 of the textbook included in HW 5) not only the discounted stock price but the discounted portfolio value process $(D(t)X(t))_{0\le t\le T}$ is an $\mathcal{F}(t)$-martingale under $\widetilde{P}$ constructed above. We have $X(t) = \Delta(t)S(t) + \Gamma(t)M(t)$ and

$$dX(t) = \Delta(t)\, dS(t) + S(t)\, d\Delta(t) + d[S,\Delta]_t + \Gamma(t)\, dM(t) + M(t)\, d\Gamma(t) + d[\Gamma, M]_t.$$

The self-financing condition states that

$$S(t)\, d\Delta(t) + d[S,\Delta]_t + M(t)\, d\Gamma(t) + d[\Gamma, M]_t = 0, \ 0 \le t \le T.$$

Therefore,

$$\begin{aligned} dX(t) = \Delta(t)\, dS(t) + \Gamma(t)\, dM(t) &= \Delta(t)\, dS(t) + (X(t) - \Delta(t)S(t))\frac{dM(t)}{M(t)} \\ &= \Delta(t)\, dS(t) + R(t)(X(t) - \Delta(t)S(t))\, dt \\ = R(t)X(t)\, dt + \Delta(t)S(t)\sigma(t)\, &\underbrace{(\Theta(t)\, dt + dB(t))}_{=d\widetilde{B}(t)} = R(t)X(t)\, dt + \Delta(t)S(t)\sigma(t)d\widetilde{B}(t). \end{aligned}$$

Finally, since D is a regular process, using the above equation with find that

$$d(D(t)X(t)) = D(t)\, dX(t) + X(t) \underbrace{dD(t)}_{=-R(r)D(t)\, dt} = \Delta(t)D(t)\sigma(t)S(t)\, d\widetilde{B}(t) \overset{(2.3)}{=} \Delta(t)d(D(t)S(t))$$

From the last two equalities in the above formula we conclude that

- $(D(t)X(t))_{0\le t\le T}$ is an $\mathcal{F}(t)$-martingale under $\widetilde{P}$;
- changes to $D(t)X(t)$ are entirely due to the changes in $D(t)S(t)$.

3. Martingale representation theorem (MRT)

Bottom line: we know that stochastic integrals with respect to Brownian motion are martingales. The MRT states that (under conditions) the converse is also true.

Theorem 3.1 (Martingale representation theorem)**.** *Let $(B(t))_{t\ge0}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}(t))_{t\ge0}$ be the filtration* ***generated by the Brownian motion****. Let $(M(t))_{0\le t\le T}$ be a square-integrable $\mathcal{F}(t)$-martingale. Then there is an adapted process $(\Gamma(t))_{0\le t\le t}$ such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dB(u), \quad 0 \le t \le T. \tag{3.1}$$

The proof is omitted.

Remark 3.2. *There is no a priori imposed condition that M has continuous paths. This is one of the conclusions of this theorem, since* (3.1) *implies that M has continuous paths. Unlike the discrete case (see HW 6, exercise 1), there is no explicit formula for the process Γ.*

Return to a simple market model. While Girsanov's theorem guarantees the existence of the risk-neutral measure for this model, MRT ensures that every contingent claim can be hedged (even though MRT does not say how). We have to assume though that $(\mathcal{F}(t))_{t\geq 0}$ is the natural filtration of the driving Brownian motion. Let $V(T)$ be $\mathcal{F}(T)$-measurable. $V(T)$ represents the payoff at time T of some (possibly path dependent) derivative security.

Question: what initial capital $X(0)$ and portfolio process $(\Delta(t))_{0\leq t\leq T}$ ensure a perfect hedge for a short position in the derivative security, i.e. ensure that $X(T)=V(T)$ a.s.?

We need a portfolio process $(\Delta(t))_{0\leq t\leq T}$ such that $X(T)=V(T)$ a.s., or equivalently, $D(T)X(T)=D(T)V(T)$ a.s.. Define

$$V(t)=\frac{1}{D(t)}\,\widetilde{E}\left(D(T)V(T)\,|\,\mathcal{F}(t)\right),\quad 0\leq t\leq T. \tag{3.2}$$

Then $(D(t)V(t))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale under $\widetilde{P}$. By the MRT, there is an adapted process $(\widetilde{\Gamma}(t))_{0\leq t\leq T}$ such that

$$D(t)V(t)=\underbrace{D(0)}_{=1}V(0)+\int_0^t\widetilde{\Gamma}(u)\,\widetilde{B}(u),\quad 0\leq t\leq T.$$

In order to have $V(t)=X(t),\ 0\leq t\leq T$, we should choose $V(0)=X(0)$ and, since

$$D(t)X(t)=X(0)+\int_0^t D(u)\Delta(u)\sigma(u)S(u)\,d\widetilde{B}(u),$$

we should also choose $\Delta(t)$ so that

$$D(t)\Delta(t)\sigma(t)S(t)=\widetilde{\Gamma}(t),\quad 0\leq t\leq T.$$

Since $D(t)$ and $S(t)$ are positive and $\sigma(t)\neq 0$ a.s. by assumption

$$\Delta(t)=\frac{\widetilde{\Gamma}(t)}{D(t)\sigma(t)S(t)},\quad 0\leq t\leq T.$$

Thus, this choice of $X(0)$ and $(\Delta(t))_{0\leq t\leq T}$ provides a perfect hedge for this derivative security. Since the claim $V(T)$ was arbitrary, we conclude that every contingent claim can be hedged, i.e. the model is complete.

Let us turn now to pricing. Once the portfolio process $(\Delta(t))_{0\leq t\leq T}$ is shown to exist, we have that

$$D(t)X(t)=X(0)+\int_0^t D(u)\Delta(u)S(u)\sigma(u)\,d\widetilde{B}(u)$$

is a martingale, and, thus,

$$D(t)X(t)=\widetilde{E}(D(T)X(T)\,|\,\mathcal{F}(t))=\widetilde{E}(D(T)V(T)\,|\,\mathcal{F}(t))=D(t)V(t).$$

$X(t)$ is the capital needed at time t to hedge a short position in the derivative security with payoff $V(T)$. Thus, we can refer to $X(t)$ (and $V(t)$) as the price of the derivative security at time t. In short, (3.2) becomes our risk-neutral pricing formula for a single stock model:

$$\begin{aligned}V(t)&=\frac{1}{D(t)}\widetilde{E}(D(T)V(T)\,|\,\mathcal{F}(t))=\widetilde{E}\left(\frac{D(T)}{D(t)}\,V(T)\,|\,\mathcal{F}\right)\\&=\widetilde{E}\left(e^{-\int_t^T R(u)\,du}V(T)\,|\,\mathcal{F}(t)\right),\quad 0\leq t\leq T.\end{aligned}$$

Example 3.3 (Black-Scholes-Merton (BSM) model)**.** We assume that $R(t) \equiv r \geq 0$, $\sigma(t) \equiv \sigma > 0$, and $\alpha(t)$ a stochastic process adapted to the filtration generated by the driving Brownian motion. We shall compute the price of a European derivative security with payoff $V(T) = f(S(T))$, where f is a non-negative Borel function. We already know that under the risk-neutral measure $\widetilde{P}$

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{B}(t),$$

and $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\widetilde{B}(t)}$. We need to compute

$$V(t) = \widetilde{E}\left(e^{-r(T-t)}f(S(T))\,|\,\mathcal{F}(t)\right) = e^{-r(T-t)}\,\widetilde{E}\left(f(S(T))\,|\,\mathcal{F}(t)\right), \quad 0 \leq t \leq T.$$

Writing $S(T) = S(t)e^{(r-\sigma^2/2)(T-t)+\sigma(\widetilde{B}(T)-\widetilde{B}(t))}$ and using the independence lemma ($\widetilde{B}(T) - \widetilde{B}(t) =: \sqrt{T-t}\,Z$ is independent of $\mathcal{F}(t)$ and $S(t)$ is $\mathcal{F}(t)$-measurable) we get that

$$\widetilde{E}\left(f\left(S(t)e^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}\,Z}\right)\,|\,\mathcal{F}(t)\right) = g(t, S(t)),$$

where

$$g(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f\left(xe^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}\,z}\right)e^{-z^2/2}\,dz.$$

We have accomplished two things: (1) we have found a closed form formula for the price

$$V(t) = e^{-r(T-t)}g(t, S(t)), \quad 0 \leq t \leq T,$$

and (2) we have shown that a GBM with a constant drift and volatility parameters is a Markov process.

Taking $f(x) = (x-K)_+$ and computing the integral will give a standard BSM formula. The details are left as an exercise.[2]

4. Girsanov's theorem and martingale representation theorem for $d > 1$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B = (B(t))_{t\geq 0}$, $B(t) = (B_1(t), B_2(t), \ldots, B_d(t))^T$ be a standard d-dimensional Brownian motion on it. Measure P will be called "market", "real world", or "actual" probability measure, i.e. the one that can be observed from the market data. Denote by $(\mathcal{F}(t))_{t\geq 0}$ a filtration associated with Brownian motion B (not necessarily the one generated by B). Time horizon, or expiration date, will be denoted by T. We shall restrict everything to times $0 \leq t \leq T$ and also assume that $\mathcal{F}(T) = \mathcal{F}$.

Theorem 4.1 (Girsanov, $d \geq 1$)**.** *Let $\Theta(t) = (\Theta_1(t), \Theta_2(t), \ldots, \Theta_d(t))$, $0 \leq t \leq T$, be an $\mathcal{F}(t)$-adapted process. Define*

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dB(u) - \frac{1}{2}\int_0^t \|\Theta(t)\|^2\,dt\right), \quad \widetilde{B}(t) = B(t) + \int_0^t \Theta(u)\,du,$$

where $\|\Theta(t)\|^2 = \sum_{i=1}^d \Theta_i^2(t)$. Assume that

$$E\int_0^T \|\Theta(t)\|^2 Z^2(t)\,dt < \infty. \tag{4.1}$$

Set $Z = Z(T)$. Then $E(Z) = 1$, and under the probability measure

$$\widetilde{P}(A) = \int_A Z(\omega)\,dP(\omega) \tag{4.2}$$

the process $\widetilde{B}(t)$, $0 \leq t \leq T$, is a standard d-dimensional Brownian motion.

The proof is similar to the one-dimensional version and uses a d-dimensional version of Lévy's characterization.

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[2]Or see p. 219-220 of the textbook, where you have to use $y = -z$ as a variable of integration.

Theorem 4.2 (MRT, $d \geq 1$). *Assume that $(\mathcal{F})_{0\leq t\leq T}$ is the filtration generated by the standard d-dimensional Brownian motion $(B(t))_{0\leq t\leq T}$, $B(t) = (B_1(t), B_2(t), \ldots, B_d(t))^T$. Let $(M(t))_{0\leq t\leq T}$ be a (one-dimensional) $\mathcal{F}(t)$-martingale with respect to P. Then there is an adapted d-dimensional process $(\Gamma(t))_{0\leq t\leq T}$, $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t), \ldots, \Gamma_d(t))$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dB(u), \quad 0 \leq t \leq T.$$

If, in addition, we assume the notation and conditions of Theorem 4.1 and if $(\widetilde{M}(t))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale with respect to $\widetilde{P}$, then there is an adapted d-dimensional process $(\widetilde{\Gamma(t)})_{0\leq t\leq T}$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{B}(u), \quad 0 \leq t \leq T.$$

5. Multidimensional market model

We shall consider a model which consists of m stocks (driven by d independent Brownian motions with the associated filtration $(\mathcal{F}(t))_{0\leq t\leq T}$) and a money market account (MMA).

- MMA: interest rate $(R(t))_{0\leq t\leq T}$, a non-negative adapted process. As before we introduce the discounting factor $D(t) = e^{-\int_0^t R(u)\,du}$, $0 \leq t \leq T$, which satisfies $D'(t) = -R(t)D(t)$, or $dD(t) = -R(t)D(t)dt$, and is a regular process.
- m stocks with the price vector $S(t) = (S_1(t), S_2(t), \ldots, S_m(t))^T$, $0 \leq t \leq T$. We assume that for some adapted vector process $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_m(t))^T$, $0 \leq t \leq T$, and an $m \times d$ matrix process $\sigma(t) = \|\sigma_{ij}(t)\|_{i=1,2,\ldots,m;j=1,2,\ldots,d}$ such that

  $$\sigma_i^2(t) := \sum_{j=1}^d \sigma_{ij}^2(t) > 0 \ \text{ for all } 0 \leq t \leq T \text{ (a.s.) and } i = 1, 2, \ldots, m,$$

  the price vector satisfies

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + S_i(t)\sum_{j=1}^d \sigma_{ij}(t)\,dB_j(t), \quad m = 1, 2, \ldots, m. \tag{5.1}$$

Just as in an example with two stocks we shall first show that each of $(S_i(t))_{0\leq t\leq T}$, $i = 1, 2, \ldots, m$, is a generalized geometric Brownian motion. To see this, define $W_i(0) = 0$,

$$dW_i(t) = \frac{1}{\sigma_i(t)}\sum_{j=1}^d \sigma_{ij}(t)\,dB_j(t), \quad i = 1, 2, \ldots, m. \tag{5.2}$$

Then $(W_i(t))_{0\leq t\leq T}$ is a continuous martingale, and (using $d[B_i, B_j]_t = 0$, $k \neq j$, and $d[B_j]_t = dt$)

$$d[W_i]_t = \frac{1}{\sigma_i^2(t)}\sum_{j=1}^d \sigma_{ij}^2(t)\,dt = dt.$$

By Lévy's theorem we conclude that $(W_i(t)_{0\leq t\leq T}$ is a standard Brownian motion. Rewriting the equation for $dS_i(t)$ in terms of the newly defined Brownian motion we get

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + S_i(t)\sigma_i(t)dW_i(t), \quad i = 1, 2, \ldots, m, \tag{5.3}$$

and conclude that each S_i is a generalized geometric Brownian motion.

Next we discuss how different stock prices, S_k and S_ℓ, are correlated. They are correlated through their driving Brownian motions W_k and W_ℓ,

$$d[W_k, W_\ell]_t \overset{(5.2)}{=} \frac{1}{\sigma_k(t)\sigma_\ell(t)}\sum_{j=1}^d\sum_{n=1}^d \sigma_{kj}(t)\sigma_{\ell n}(t)d[B_j, B_n]_t = \frac{1}{\sigma_k(t)\sigma_\ell(t)}\sum_{j=1}^d \sigma_{kj}(t)\sigma_{\ell j}(t)dt =: \rho_{k\ell}(t)\,dt.$$

The process $(\rho_{k\ell}(t))_{0\leq t\leq T}$ is called *instantaneous correlation* of W_k and W_ℓ.

Exercise 2. Use Itô's product rule to show that $\mathrm{Cov}(W_k(t), W_\ell(t)) = E\int_0^t \rho_{k\ell}(u)\,du$.

Computing the cross variation between S_k and S_ℓ we get

$$d[S_k, S_\ell]_t \stackrel{(5.3)}{=} \rho_{k\ell}(t)S_k(t)S_\ell(t)\sigma_k(t)\sigma_\ell(t)\,dt.$$

In the case when $\rho_{k\ell}(t)$, $\sigma_k(t)$ and $\sigma_\ell(t)$, $0 \le t \le T$, are non-random we can explicitly compute the covariance and correlation of $S_k(t)$ and $S_\ell(t)$ just as we did in Section 3 of lecture 5.

Finally, using Itô-Doeblin formula and (5.3) we can show that

$$d(D(t)S_i(t)) = D(t)S_i(t)((\alpha_i(t) - R(t))\,dt + \sigma_i(t)\,dW_i(t)), \quad i = 1, 2, \dots, m.$$

Exercise 3. Derive the above formula.

Next time we shall address the question of existence and uniqueness of a risk-neutral measure for this model.