(1)

Let , let we have , where are standard normal distribution.

(2). Solution:

* Because are continuous, is a linear combination of them. Thus, is also continuous.
* for each and

……

Because are independent random variables;, are independent random variables; and are independent.

Thus, are independent random variables.

* For all s>0, and t >= 0, and are independent, their covariance = 0

Because ],

the increment also has a normal distribution with mean 0 and variance s

So we can conclude that X(t) is a Brownian motion.

=

=

=

=

(3).

(a) X(t) = -B(t)

* The negative multiplication maintains the continuity and independence of increments
* X(0) = -B(0) = 0

Because has normal distribution, X(t) still maintains normal distribution with mean 0 and variance s

Thus, is a Brownian motion.

(b) X(t) = where c > 0 is a constant

* Continuity and independence of increments still maintains.
* X(0) = = 0

= s

Thus, is a Brownian motion.

(c) X(t) =

The variance of increments is still related to t, so it’s not a Brownian motion.

(d) X(t) =

* for all s > 0, t >= 0

Thus, the variance of increments is still related to t for s . It’s not a Brownian motion.

(e) X(t) = , where s is fixed

* Because X(t) is a linear combination of Brownian motions, it maintains the independence of increments and is almost surely continuous. The increments are also normally distributed.
* For all m > 0,

=

= 0

=

= m

Thus, it’s a Brownian motion.

(4).

We apply reflection principle,

Where is cumulative function of standard normal distribution.

(5).

Since for square-integrable mean zero random variables and , their inner product is defined to be , then following the general procedure of Gram-Schmidt orthogonalization, we have

Then , and the collection of all random variables from set forms an orthogonal basis of the span of :

For simplicity, we write , where , , and are independent normal random variables, .

First, let’s assume , then according to the discussion above, we have

Where , and are independent normal random variables, .

Write Y as , where is a diagonal matrix and Z is a standard normal vector, , then we have

Let , then apply Singular Value Decomposition to B, we have

where are orthogonal matrices and is a diagonal matrix, then we have

Since V is orthogonal, , we have , which means is also a standard normal vector.

Thus, let  , then it’s easy to see that , where .

In conclusion, we have

Where U is an orthogonal matrix, , and are independent normal random variables.

As for , where . Denote  , then using the conclusion proven above, we know that there exists an orthogonal matrix A and independent normal random variables , such that

so . Thus, we have proven Theorem 1.6 from Lecture 1.

(b)

From the process of Gram-Schmidt orthogonalization, we know that , where , thus

Also, is a linear combination of , denote it as , then

Next, we prove that is independent with .

Since is a linear combination of , we can tell that they are jointly normal thus to prove that they are independent , we just need to prove that

For all ,

Thus, .

In conclusion, we have proven that

Is a linear function of .

(c)

Denote then what we need to do is to prove is Gaussian and that the dependence of the parameters on is linear.

Denote the mean vector of as , and the covariance matrix as , then we claim that

To prove it, denote , we want to show that is independent with Obviously, are jointly normal, then we only need to show that .

Also we have , then, according to the property of conditional expectation, we know that

Where denote identically distributed.

Write Z as , where and , so ,

Hence, we have proven that the conditional distribution of

Is Gaussian and that the dependence of the parameters on is linear.

(6).

The conditional distribution is Gaussian by the previous problem

we assume

where is a non-random matrix and is a mean zero Gaussian vector independent from

We have and are independent and is non-degenerate, so

Then multiply both parts by

We have

Then