Optimization Techniques in Finance

Homework assignment #3

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Problems

1. A function $f(x)$, $x \in \mathbb{R}^n$, is called *log-concave* (respectively, *log-convex*), if it is positive for all x, and $\log f(x)$ is a concave (respectively, convex) function.

   (i) Show that the densities of the following distributions are log-concave (over their domains):

   - normal
   - multinormal
   - exponential
   - logistic

   (ii) Consider the Kullback-Leibler distance between two discrete distributions $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$,

   $$\mathrm{KL}(p\|q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}\,,$$

   and assume that $q = q(\theta)$ is a log-concave function of the parameters $\theta \in \mathbb{R}^n$. Show that the function

   $$f(\theta) = \mathrm{KL}(p\|q(\theta))$$

is a convex function. As a consequence, the minimization problem $\min_\theta \mathrm{KL}(p\|q(\theta))$ is a convex optimization problem.

2. Prove (analytically) that $x^* = (1, 0.5, -1)$ is the optimal solution to the following optimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\, x^{\mathsf{T}} A x + a^{\mathsf{T}} x + b, \qquad \text{subject to } -1 \le x_i \le 1,\ i = 1, 2, 3,$$

where

$$A = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix},$$
$$a = \begin{pmatrix} -22 \\ 14.5 \\ 13 \end{pmatrix},$$
$$b = 1.$$

Determine the values of the Lagrange multipliers corresponding to all the (inequality) constraints.
Hint: Carefully analyze the implications of the KKT complementary slackness condition.

This assignment is due on December 3