Optimization Techniques in Finance

Homework assignment #4

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Problems

1. Consider the following *analytic centering problem* with equality constraints:

$$\min -\sum_{i=1}^{n} \log x_i, \qquad \text{subject to } Ax = b. \tag{1}$$

   (i) Show that in the simplest case of a single constraint $\sum_{i=1}^{n} x_i = 1$, the solution to this problem is $x_i^* = 1/n$, $\lambda^* = n$).

   (ii) Show that the dual Lagrange function is

$$q(\lambda) = -b^{\mathrm{T}}\lambda + \sum_{i=1}^{n} \log(A^{\mathrm{T}}\lambda)_i + n. \tag{2}$$

   Formulate the dual problem.

   (iii) Show that the solution to the first order conditions is

$$x_i = \frac{1}{(A^{\mathrm{T}}\lambda)_i}\,.$$

   Therefore, to solve the equality constraint analytic centering problem, we solve the unconstraint dual problem for λ^* and then obtain x^* from these formulas.

(iv) Implement Newton's method in Python in order to solve the optimization problem (1). Assume $\alpha = 0.1$ and $\beta = 0.5$ in the backtracking line search algorithm, and make sure that you take the structure of the problem into account. Carry out the convergence analysis of the algorithm for the problem with $p = 100$ and $n = 500$ (choose your own $p \times n$ constraint matrix!) by following the trajectories of Netwon iterates (until they reach the exit criterion) starting at five different initial points.

(v) Carry out a similar analysis for solving the corresponding dual problem.

2. Consider a KKT matrix

$$\begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix}, \tag{3}$$

where $H \in \mathrm{Mat}_n(\mathbb{R})$ is positive semidefinite, $A \in \mathrm{Mat}_{pn}(\mathbb{R})$ is of maximum rank $\mathrm{rank}(A) = p < n$.

(a) Show that each of the following statements is equivalent to nonsingularity of the KKT matrix:

(i) $\mathcal{N}(H) \cap \mathcal{N}(A) = \{0\}$, where $\mathcal{N}(P)$ denotes the null-space of the matrix P.

(ii) $Ax = 0,\ x \neq 0$ imply that $x^{\mathrm{T}} H x > 0$.

(iii) $F^{\mathrm{T}} H F > 0$, where $F \in \mathrm{Mat}_{n,n-p}(\mathbb{R})$ is a matrix for which $\mathcal{R}(F) = \mathcal{N}(A)$. Here $\mathcal{R}(F)$ denotes the range of F.

(iv) $H + A^{\mathrm{T}} Q A > 0$, for some $Q > 0$.

(b) Show that if the KKT matrix is nonsingular, then it has exactly n positive and p negative eigenvalues.

This assignment is due on December 17