Chenyu Zhao homework3

3.1 $$\frac{d\mu}{d\sigma} = - \frac{\int_R z\, u'(\mu+\sigma z)\, g_1(z^2)\, dz}{\int_R u'(\mu+\sigma z)\, g_1(z^2)\, dz}$$

$$0 = \int_R u'(\mu+\sigma z)\left(z + \frac{\partial\mu}{\partial\sigma}\right) g_1(z^2)\, dz$$

take derivative w.r.t. σ

$$0 = \int_R u'' \left(z+\frac{\partial\mu}{\partial\sigma}\right)^2 g_1(z^2)\, dz + \int_R u' \frac{d^2\mu}{d\sigma^2} g_1(z^2)\, dz$$

$$\therefore \frac{d^2\mu}{d\sigma^2} = - \frac{\int_R u'' \left(z+\frac{\partial\mu}{\partial\sigma}\right)^2 g_1(z^2)\, dz}{\int_R u'\, g(z^2)\, dz}$$

$\because u''<0 \quad u'>0$

$\therefore \frac{d^2\mu}{d\sigma^2} > 0$

3.2 $\mathbb{E}\,u(x) = \int_{\mathbb{R}} \log(x)\, f(x)\, dx \qquad z = \log x \qquad dx = e^{z}\, dz$

$$= \int_{\mathbb{R}} z \frac{1}{\sqrt{2\pi}\, s} e^{-\frac{(z-m)^2}{2s^2}}\, dz$$

$$= m$$

$$= \log \mu - \frac{1}{2} \log\left(\frac{\sigma^2}{\mu^2} + 1\right)$$

$\frac{d}{d\lambda}$:

$$0 = \frac{\mu'}{\mu} - \frac{1}{2} \frac{1}{\frac{\sigma^2}{\mu^2} + 1} \left[\frac{2\sigma}{\mu^2} \sigma' - \frac{2\sigma^2}{\mu^3} \mu' \right]$$

$$= \frac{\mu'}{\mu} + \frac{1}{\sigma^2 + \mu^2} \frac{\sigma^2}{\mu} \mu' - \frac{\sigma}{\sigma^2 + \mu^2} \sigma'$$

$$\therefore \frac{\mu'}{\sigma'} = \frac{\sigma \mu}{2\sigma^2 + \mu^2} \qquad \because \mu > 0 \quad \sigma > 0$$

$\therefore \frac{d\mu}{d\sigma} > 0$ along the curve

$$\frac{d^2\mu}{d\sigma^2} = \frac{d}{d\sigma}\left(\frac{d\mu}{d\sigma}\right) = \frac{\mu(2\sigma^2 + \mu^2) - \sigma\mu \cdot 4\sigma}{(2\sigma^2 + \mu^2)^2} = \frac{\mu^3 - 2\sigma^2 \mu}{(2\sigma^2 + \mu^2)^2}$$

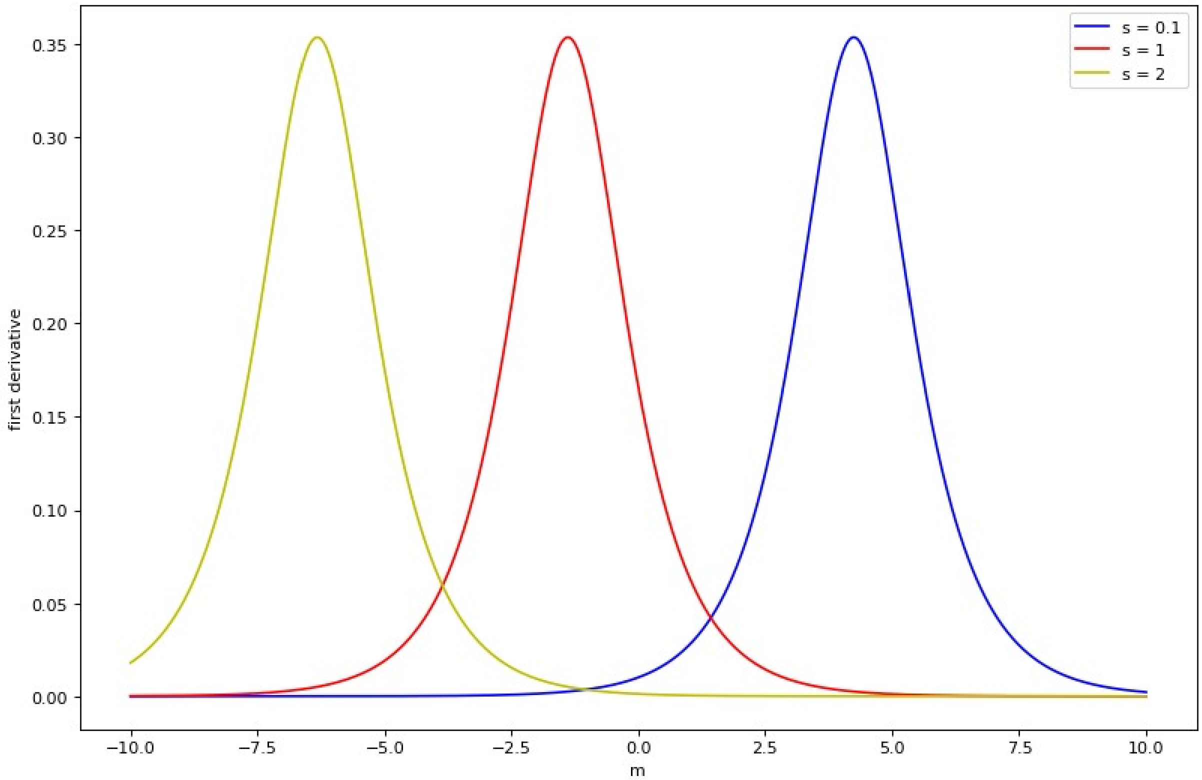
$$\because \sigma^2 = \mu^2 (e^{s^2} - 1)$$

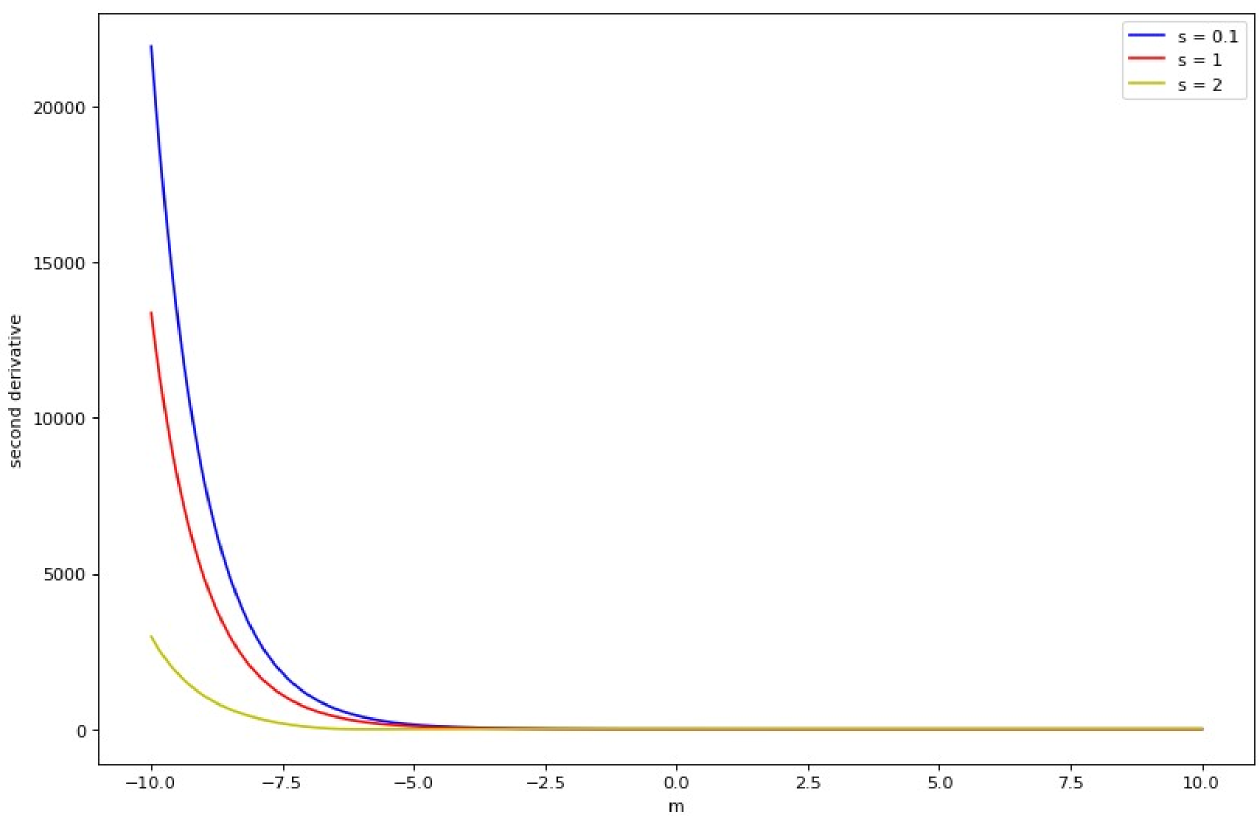
$$\therefore 1 > 2(e^{s^2} - 1) \iff s^2 < \log \frac{3}{2} \qquad \frac{d^2\mu}{d\sigma^2} > 0$$

$$s^2 = \log \frac{3}{2} \qquad \frac{d^2\mu}{d\sigma^2} = 0$$

$$s^2 > \log \frac{3}{2} \qquad \frac{d^2\mu}{d\sigma^2} < 0$$

Plot the curve using m as x-axis and draw different lines corresponding to different s. The first plot is first derivative and the second plot is second derivative.


s = 0.1
s = 1
s = 2
first derivative
m
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


s = 0.1
s = 1
s = 2
second derivative
m
0
5000
10000
15000
20000
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0