Interest Rate and Credit Models

Homework Assignment #3

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Problems

1. Show that the value of a receiver swaption, expressed in the units of the annuity function, is given by

$$\mathsf{E}^{\mathsf{Q}_{T_0,T}}\big[(K - S(T_0,T))^+\big], \tag{1}$$

where $\mathsf{Q}_{T_0,T}$ denotes the swap (martingale) measure.

Hint: Follow the line of reasoning analogous to the case of a caplet, discussed in Lecture Notes #5.

2. Consider an at the money option.

   (i) Show that the implied normal volatility σ_{n} can be expressed in terms of the implied lognormal volatility σ_{ln} as

$$\sigma_{\mathrm{n}}(T, F_0, F_0, \sigma_{\mathrm{ln}}) = \sqrt{\frac{2\pi}{T}}\, F_0 \operatorname{erf}\Big(\frac{\sqrt{T}}{2\sqrt{2}}\,\sigma_{\mathrm{ln}}\Big), \tag{2}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{3}$$

is the *error function*.

(ii) Show that, for $\varepsilon = \sigma_{\mathrm{ln}}^2 T$ small, the following approximation holds:

$$\sigma_{\mathrm{n}} = F_0\, \sigma_{\mathrm{ln}} \Big(1 - \frac{1}{24}\, \sigma_{\mathrm{ln}}^2 T + \frac{1}{640}\, (\sigma_{\mathrm{ln}}^2 T)^2 + \dots \Big). \tag{4}$$

You can view this calculation as the simplest example of the type of analysis that is used in the derivation of the SABR asymptotic formula.

3. The asymptotic formulas for the implied normal and lognormal volatilities $\sigma_{\mathrm{n}}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$ and $\sigma_{\mathrm{ln}}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$, respectively, in the SABR model, as given in Lecture Notes #5, contain singularities of the type $\frac{0}{0}$ for the at the money strikes. Use l'Hopital's rule (or any other method of your choice) in order to calculate the at the money implied volatilities $\sigma_{\mathrm{n}}(T, F_0, F_0, \sigma_0, \alpha, \beta, \rho)$ and $\sigma_{\mathrm{ln}}(T, F_0, F_0, \sigma_0, \alpha, \beta, \rho)$. You will find the result very useful when implementing the SABR model in computer code.

4. The purpose of the first four problems is to implement the standard SABR model.

   Note: In all option models below assume the annuity functions to be 1.

   (i) Implement the normal and lognormal option models (in case you have not done it yet...).

   (ii) Implement the SABR implied normal and lognormal volatility functions. This function should take as arguments the model parameters (α, β, ρ), the relevant market parameters (σ, F) and the contract terms (option maturity and strike). Remember to use the result of the calculation in Problem 3 for strikes near the money.

   (iii) Using the enclosed data sheet, calibrate the parameters of the SABR model. Fix the beta parameter to be $\beta = 0.5$. You can use an NLS optimization routine of your choice (or simply the Excel Solver utility). I suggest starting the searches with the following initial guess: $\sigma_0 = 0.1$, $\alpha = 0.3$, $\rho = 0.0$.

5. The purpose of this problem is to assess the accuracy of the asymptotic SABR formula for implied volatility. Specifically, we will run a large number of Monte Carlo simulations of the exact SABR model, and compare the MC prices of European options to the values obtained by applying the asymptotic SABR formula.

   Note: The annuity functions are assumed to be 1.

(i) Use the Python function `numpy.random.normal` to generate uncorrelated Gaussian random numbers. For the purposes of this assignment, you will need to generate a sequence of two dimensional Gaussian random numbers (x, y), $(x, y \sim N(0, 1))$ whose correlation coefficient is $\mathsf{Corr}(x, y) = \rho$. Explain how to do it, and verify your algorithm.

(ii) Write code for generating MC paths for the SABR model. I suggest the following approach. Instead of using the naive Euler scheme for both equations, integrate first the equation for σ, $\sigma(t) = \sigma_0\, e^{\alpha Z(t) - \alpha^2 t/2}$. Then discretize:

$$
\begin{aligned}
F_{i+1} &= \max\big(F_i + \sigma_i F_i^{\beta}\sqrt{\delta t}\, x_i,\ 0\big),\\
\sigma_{i+1} &= \sigma_i e^{\alpha\sqrt{\delta t}\, y_i - \alpha^2\,\delta t/2},
\end{aligned}
$$

where δt is the time step, and (x_i, y_i), $i = 1, \ldots, numSteps$, are the Gaussian random vectors described in (i). Keep in mind that the "long jump" method does not work for SABR, and you have to sample at intermediate time steps in order to assure accuracy of the procedure. In fact, in order to assure that the correlation coefficient between dW and dZ is respected by the Monte Carlo simulation, δt ought to be fairly small. Note the presence of $\max(\,\cdot\,, 0)$ in the algorithm above, which implements the Dirichlet boundary condition.

(iii) Use (at least) 5000 MC paths to value each of the options in Table 1. Note: A call or put option refers here to the call on the rate or put on the rate, respectively (and not on the market). Compare the results to those obtained from Black's formula (lognormal or normal), with the appropriate SABR implied volatility used for the volatility argument.

This assignment is due on March 25

	T	F_0	K	σ_0	α	β	ρ
call	0.25	0.005	0.01	0.0072	1.1	0.2	0.8
call	0.25	0.05	0.06	0.0224	0.8	0.5	-0.2
call	0.5	0.01	0.02	0.0400	0.8	0.5	0.7
call	0.5	0.05	0.06	0.0182	0.7	0.2	-0.3
call	1	0.015	0.02	0.0232	0.6	0.2	0.4
call	1	0.04	0.05	0.0500	0.5	0.5	-0.3
call	2	0.02	0.03	0.0707	0.5	0.5	0.3
call	2	0.06	0.07	0.0176	0.5	0.2	-0.1
call	5	0.05	0.06	0.0146	0.4	0.2	0.2
call	5	0.05	0.07	0.0447	0.3	0.5	-0.5
put	0.25	0.005	0.002	0.0354	0.9	0.5	0.8
put	0.25	0.04	0.02	0.0095	0.9	0.2	-0.0
put	0.5	0.01	0.005	0.0400	0.7	0.5	0.4
put	0.5	0.03	0.02	0.0577	0.6	0.5	-0.2
put	1	0.03	0.01	0.0202	0.6	0.2	0.5
put	1	0.05	0.04	0.0182	0.6	0.2	-0.5
put	2	0.04	0.025	0.0500	0.5	0.5	0.2
put	2	0.05	0.035	0.0447	0.5	0.5	-0.1
put	5	0.05	0.035	0.0146	0.4	0.2	0.1
put	5	0.06	0.04	0.0176	0.4	0.2	0.0

Table 1