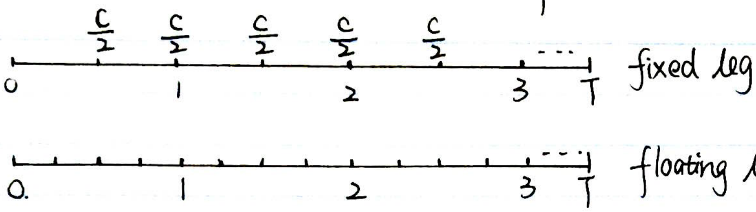
Luxi Tang
Bo Feng
Yifan Cui

MTH9878 HW01

1. (1).



Assume that the coupon rate on fixed leg is c.

Value of the fixed leg is $V_{fixed} = \sum_{i=1}^{2T} \left(\frac{c}{2}\right) e^{-\frac{i}{2}\cdot f_0} = \frac{c}{2}\cdot \frac{e^{-\frac{f_0}{2}}(1-e^{-f_0T})}{1-e^{-f_0/2}}$

Since the LIBOR rate is L_0, the payment at each payment date is $e^{L_0/4}-1$.

Value of the floating leg is $V_{floating} = \sum_{i=1}^{4T}\left(e^{\frac{L_0}{4}}-1\right)e^{-f_0\cdot\frac{i}{4}} = \left(e^{\frac{L_0}{4}}-1\right)\frac{e^{-\frac{f_0}{4}}(1-e^{-f_0T})}{1-e^{-f_0/4}}$

(2). To determine the par coupon of the swap, we should let $V_{fixed} = V_{floating}$.

$$\frac{c^*}{2}\cdot\frac{e^{-f_0/2}\cancel{(1-e^{-f_0T})}}{1-e^{-f_0/2}} = \left(e^{\frac{L_0}{4}}-1\right)\cdot\frac{e^{-\frac{f_0}{4}}\cancel{(1-e^{-f_0T})}}{1-e^{-f_0/4}}.$$

$$c^* = \frac{2e^{f_0/4}\left(e^{\frac{L_0}{4}}-1\right)\left(1-e^{-f_0/2}\right)}{1-e^{-f_0/4}}$$

$$= \frac{2\left(e^{\frac{L_0}{4}}-1\right)\left(e^{f_0/4}-e^{-f_0/4}\right)}{1-e^{-f_0/4}}$$

$$= \frac{2\left(e^{\frac{L_0}{4}}-1\right)\left(e^{\frac{f_0}{2}}-1\right)}{\left(e^{f_0/4}-1\right)}$$

$$= 2\left(e^{\frac{L_0}{4}}-1\right)\left(e^{\frac{f_0}{4}}+1\right).$$

2. Prove: $\boxed{\frac{d}{dt}B_k^{(d)}(t) = \frac{d}{t_{k+d}-t_k}B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t).}$ (30).

This can be proved by induction:

(1). $d=0$ we have $B_k^{(0)}(t) = \begin{cases} 1, & \text{if } t_k \le t < t_{k+1} \\ 0, & \text{otherwise.} \end{cases}$

The derivative is equal to 0 everywhere.

The equation holds while $n=0$.

(2). Suppose that the formula is valid for $d=0,1,2,\cdots,n$

Now we try to prove it holds while $d=n+1$.

Since $B_k^{(n+1)}(t) = \frac{t-t_k}{t_{k+n+1}-t_k}B_k^{(n)}(t) + \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}B_{k+1}^{(n)}(t).$

We have

$$\frac{d}{dt}B_k^{(n+1)}(t) = \frac{\partial}{\partial t}\left(\frac{t-t_k}{t_{k+n+1}-t_k}\right)\cdot B_k^{(n)}(t) + \frac{t-t_k}{t_{k+n+1}-t_k}\cdot\left(\frac{\partial}{\partial t}B_k^{(n)}(t)\right)$$

$$+\frac{\partial}{\partial t}\left(\frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\right)\cdot B_{k+1}^{(n)}(t) + \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\cdot\left(\frac{\partial}{\partial t}B_{k+1}^{(n)}(t)\right).$$

$$= \boxed{\frac{1}{t_{k+n+1}-t_k}\cdot B_k^{(n)}(t)} + \frac{t-t_k}{t_{k+n+1}-t_k}\left[\frac{n}{t_{k+n}-t_k}B_k^{(n-1)}(t) - \frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t)\right]$$

$$-\boxed{\frac{1}{t_{k+n+2}-t_{k+1}}B_{k+1}^{(n)}(t)} + \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\left[\frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t) - \frac{n}{t_{k+n+2}-t_{k+2}}B_{k+2}^{(n-1)}(t)\right] \quad (1)$$

The desirable form is

$$\frac{d}{dt}B_k^{(n+1)}(t) = \frac{n+1}{t_{k+n+1}-t_k}B_k^n(t) - \frac{n+1}{t_{k+n+2}-t_{k+1}}B_{k+1}^n(t).$$

$$= \frac{n}{t_{k+n+1}-t_k}B_k^n(t) + \frac{1}{t_{k+n+1}-t_k}B_k^n(t)$$

$$-\frac{n}{t_{k+n+2}-t_{k+1}}B_{k+1}^{(n)}(t) - \frac{1}{t_{k+n+2}-t_{k+1}}B_{k+1}^{(n)}(t).$$

$$= \frac{n}{t_{k+n+1}-t_k}\left[\frac{t-t_k}{t_{k+n}-t_k}B_k^{(n-1)}(t) + \frac{t_{k+n+1}-t}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t)\right] + \boxed{\frac{1}{t_{k+n+1}-t_k}B_k^{(n)}(t)}$$

$$- \frac{n}{t_{k+n+2}-t_{k+1}}\left[\frac{t-t_{k+1}}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t) + \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+2}}B_{k+2}^{(n-1)}(t)\right] - \boxed{\frac{1}{t_{k+n+2}-t_{k+1}}B_{k+1}^{(n)}(t)}. \quad \text{— (2)}$$

We now go back to (1).

1st and 3rd term in (1) is the same as the 2nd and 4th term in (2).

The rest of (1) is:

$$\frac{t-t_k}{t_{k+n+1}-t_k}\cdot\frac{n}{t_{k+n}-t_k}B_k^{(n-1)}(t) - \frac{t-t_k}{t_{k+n+1}-t_k}\cdot\frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t)$$

$$+ \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\cdot\frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t) - \frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\cdot\frac{n}{t_{k+n+2}-t_{k+2}}B_{k+2}^{(n-1)}(t). \quad \text{—(3)}$$

1st and 4th term in (3) is the same as in (2).

The rest of (1) is

$$\frac{t_{k+n+2}-t}{t_{k+n+2}-t_{k+1}}\cdot\frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t) - \frac{t-t_k}{t_{k+n+1}-t_k}\cdot\frac{n}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t). \quad \text{—— (4)}$$

The rest of (2) is

$$\frac{n}{t_{k+n+1}-t_k}\cdot\frac{t_{k+n+1}-t}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t) - \frac{n}{t_{k+n+2}-t_{k+1}}\cdot\frac{t-t_{k+1}}{t_{k+n+1}-t_{k+1}}B_{k+1}^{(n-1)}(t). \text{—— (5).}$$

Exclude the common factor $nB_{k+1}^{(n-1)}(t)$.

(4) becomes

$$\frac{t_{k+n+2}-t}{(t_{k+n+2}-t_{k+1})(t_{k+n+1}-t_{k+1})} - \frac{t-t_k}{(t_{k+n+1}-t_k)(t_{k+n+1}-t_{k+1})} \quad \text{—— (6)}$$

(5) becomes

$$\frac{t_{k+n+1}-t}{(t_{k+n+1}-t_k)(t_{k+n+1}-t_{k+1})} - \frac{t-t_{k+1}}{(t_{k+n+2}-t_{k+1})(t_{k+n+1}-t_{k+1})} \quad \text{—— (7).}$$

$$(6) = \frac{(t_{k+n+2}-t)(t_{k+n+1}-t_k)-(t-t_k)(t_{k+n+2}-t_{k+1})}{(t_{k+n+2}-t_{k+1})(t_{k+n+1}-t_{k+1})(t_{k+n+1}-t_k)} = \cancel{\frac{t_{k+n+2}t_{k+n+1}-t\,t_{k+n+1}-t_{k+n+}}{(*)}}$$

$$= \frac{t_{k+n+2} \cdot t_{k+n+1} - t \cdot t_{k+n+1} - \cancel{t_{k+n+2} \cdot t_k} + t \cdot t_k - \left(t \cdot t_{k+n+2} - \cancel{t_k t_{k+n+2}} - t \cdot t_{k+1} + t_k t_{k+1}\right)}{(\cdot)}$$

$$= \frac{t_{k+n+2} \cdot t_{k+n+1} - t \cdot t_{k+n+1} + t \cdot t_k - t \cdot t_{k+n+2} + t \cdot t_{k+1} - t_k t_{k+1}}{(\cdot)} \quad \text{---} (8)$$

$$(7) = \frac{(t_{k+n+1} - t)(t_{k+n+2} - t_{k+1}) - (t - t_{k+1})(t_{k+n+1} - t_k)}{(t_{k+n+2} - t_{k+1})(t_{k+n+1} - t_{k+1})(t_{k+n+1} - t_k)}.$$

$$= \frac{t_{k+n+1} t_{k+n+2} - t \cdot t_{k+n+2} - \cancel{t_{k+n+1} t_{k+1}} + t \cdot t_{k+1} - \left(t \cdot t_{k+n+1} - \cancel{t_{k+1} t_{k+n+1}} - t \cdot t_k + t_{k+1} t_k\right)}{(\cdot)}.$$

$$= \frac{t_{k+n+2} \cdot t_{k+n+1} - t \, t_{k+n+1} + t \cdot t_k - t \cdot t_{k+n+2} + t \cdot t_{k+1} - t_k t_{k+1}}{(\cdot)}. \quad \text{---} (9).$$

We ~~d~~ can see that ⑧ = ⑨.

So the formula is proved.

2. Prove $$\boxed{\int_{-\infty}^{t} B_k^{(d)}(\tau)\,d\tau = \sum_{i=k}^{\infty} \frac{t_{k+d+1}-t_k}{d+1} B_i^{(d+1)}(t)} \quad (31)$$

Take derivative of t on both side:

LHS: $$\frac{\partial}{\partial t}\int_{-\infty}^{t} B_k^{(d)}(\tau)\,d\tau = B_k^{(d)}(t). \quad \text{------} ①$$

RHS: $$\frac{\partial}{\partial t}\sum_{i=k}^{\infty} \frac{t_{k+d+1}-t_k}{d+1} B_i^{(d+1)}(t). \quad \text{------} (*)$$

Since $$\frac{d}{dt}B_k^{(d)}(t) = \frac{d}{t_{k+d}-t_k} B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1}-t_{k+1}} B_{k+1}^{(d-1)}(t).$$

We have $$\frac{\partial B_i^{(d+1)}}{\partial t}(t) = \frac{d+1}{t_{i+d+1}-t_i} B_i^d(t) - \frac{d+1}{t_{i+d+2}-t_{i+1}} B_{i+1}^d(t).$$

Therefore:

$$\begin{aligned}
(*) &= \frac{t_{k+d+1}-t_k}{d+1}\left(\frac{d+1}{t_{k+d+1}-t_k} B_k^d(t) - \frac{d+1}{t_{k+d+2}-t_{k+1}} B_{k+1}^d(t)\right) \\
&+ \frac{t_{k+d+1}-t_k}{d+1}\left(\frac{d+1}{t_{k+d+2}-t_{k+1}} B_{k+1}^d(t) - \frac{d+1}{t_{k+d+3}-t_{k+2}} B_{k+2}^d(t)\right) \\
&+ \frac{t_{k+d+1}-t_k}{d+1}\left(\frac{d+1}{t_{k+d+3}-t_{k+2}} B_{k+2}^d(t) - \frac{d+1}{t_{k+d+4}-t_{k+3}} B_{k+3}^d(t)\right) \\
&+ \cdots \\
&= (t_{k+d+1}-t_k)\Bigg[\frac{B_k^d(t)}{t_{k+d+1}-t_k} - \cancel{\frac{B_{k+1}^d(t)}{t_{k+d+2}-t_{k+1}}} + \cancel{\frac{B_{k+1}^d(t)}{t_{k+d+2}-t_{k+1}}} - \cancel{\frac{B_{k+2}^d(t)}{t_{k+d+3}-t_{k+2}}} \\
&\qquad + \cancel{\frac{B_{k+2}^d(t)}{t_{k+d+3}-t_{k+2}}} - \frac{B_{k+3}^d(t)}{t_{k+d+4}-t_{k+3}} + \cdots\Bigg] \\
&= (t_{k+d+1}-t_k)\cdot\frac{B_k^d(t)}{t_{k+d+1}-t_k} = B_k^d(t). \quad \text{---} ②
\end{aligned}$$

Now ① = ②. So the formula is proved.

3. By Ito's product rule: $d(XY) = XdY + YdX + dXdY$.

$$d\left(\frac{X}{Y}\right) = Xd\left(\frac{1}{Y}\right) + \frac{1}{Y}dX + d\left(\frac{1}{Y}\right)dX$$

$$= -\frac{X}{Y^2}dY + \frac{1}{Y}dX - \frac{1}{Y^2}dXdY$$

In the original frictionless market, $V = \sum_{i=1}^{N} w_i(t) \cdot S_i(t)$.

$$dV = \sum_{i=1}^{N} w_i(t) \cdot dS_i(t).$$

We'd like to show that given the numeraire $N(t)$.

$$dU^N = \sum_{i=1}^{N} w_i dS_i^N(t).$$

$$dV^N = d\left(\frac{V}{N}\right) = -\frac{V}{N^2}dN + \frac{1}{N}dV - \frac{1}{N^2}dVdN$$

$$= -\frac{\sum w_i S_i}{N^2}dN + \frac{1}{N}\sum_{i=1}^{N} w_i(t)dS_i(t) - \frac{1}{N^2}dN \cdot \sum_{i=1}^{N} w_i(t)dS_i(t).$$

$$= \sum_{i=1}^{N}\left(w_i\left[-\frac{S_i}{N^2}dN + \frac{1}{N}dS_i - \frac{dN\,dS_i}{N^2}\right]\right)$$

$$= \sum_{i=1}^{N} w_i d\left(\frac{S_i}{N}\right)$$

$$= \sum_{i=1}^{N} w_i dS_i^N.$$

$\therefore$ the ~~pro~~ portfolio expressed in terms of the relative price is frictionless