HW02

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Q1. Shifted lognormal model: $dF(t) = [\sigma_1 F(t) + \sigma_0]\, dW(t)$.

$$d[\log(\sigma_1 F(t)+\sigma_0)] = \frac{\sigma_1}{\sigma_1 F(t)+\sigma_0} dF(t) - \frac{\sigma_1^2}{2(\sigma_1 F(t)+\sigma_0)^2} d[F]_t.$$

$$= \frac{\sigma_1}{\sigma_1 F(t)+\sigma_0}[\sigma_1 F(t)+\sigma_0]\, dW_t - \frac{\sigma_1^2}{2(\sigma_1 F(t)+\sigma_0)^2}[\sigma_1 F(t)+\sigma_0]^2 dt.$$

$$= \sigma_1 dW_t - \frac{\sigma_1^2}{2} dt.$$

$$\Rightarrow \log(\sigma_1 F_T + \sigma_0) - \log(\sigma_1 F_0 + \sigma_0) = \sigma_1 W_T - \frac{\sigma_1^2}{2} T.$$

$$\Rightarrow \quad F_T = \frac{(\sigma_1 F_0 + \sigma_0) e^{\sigma_1 W_T - \frac{\sigma_1^2}{2}T} - \sigma_0}{\sigma_1}$$

(28).* Payoff of call at $T = \begin{cases} F_T - K, & \text{while } F_T \geq K \\ 0, & \text{otherwise.} \end{cases}$

$$F_T \geq K \Rightarrow \frac{(\sigma_1 F_0 + \sigma_0) e^{\sigma_1 W_T - \frac{\sigma_1^2}{2}T} - \sigma_0}{\sigma_1} \geq K$$

$$\Rightarrow e^{\sigma_1 W_T - \frac{\sigma_1^2}{2}T} \geq \frac{\sigma_1 K + \sigma_0}{\sigma_1 F_0 + \sigma_0}$$

$$\Rightarrow W_T \geq \left(\log \frac{\sigma_1 K + \sigma_0}{\sigma_1 F_0 + \sigma_0} + \frac{\sigma_1^2}{2} T\right) / \sigma_1$$

We can denote W_T as $\sqrt{T} Z$, where $Z \sim N(0,1)$. then we have:

(29)* $$Z \geq \frac{1}{\sigma_1 \sqrt{T}} \log \frac{\sigma_1 K + \sigma_0}{\sigma_1 F_0 + \sigma_0} + \frac{\sigma_1 \sqrt{T}}{2} = -d_-, \quad \left(\text{let } d_- = \frac{\log \frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} - \frac{\sigma_1^2 T}{2}}{\sigma_1 \sqrt{T}},\ d_+ = d_- + \sigma_1 \sqrt{T}\right)$$

(27)* Price of call: (Risk-neutral pricing) $$\frac{P^{call}}{N(0)} = \int_{-d_-}^{+\infty} (F_T - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-d_-}^{+\infty} \left[\frac{(\sigma_1 F_0 + \sigma_0) e^{\sigma_1 \sqrt{T} x - \frac{\sigma_1^2}{2}T} - \sigma_0}{\sigma_1} - K\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-d_-}^{+\infty} \left[\left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma_1\sqrt{T})^2}{2}} - \left(K + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right] dx$$

$$= \int_{-d_+}^{+\infty} \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du - \int_{-d_-}^{+\infty} \left(K + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

(28) $$= \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot N(d_+) - \left(K + \frac{\sigma_0}{\sigma_1}\right) \cdot N(d_-)$$

(30). (*) In a similar way.

$$\text{Payoff of put at } T = \begin{cases} K - F_T, & \text{while } F_T \le K \\ 0, & \text{otherwise.} \end{cases}$$

$F_T \le K \Rightarrow \quad Z \le -d_-$.

$$\text{Price of put : } \frac{P^{put}}{N(0)} = \int_{-\infty}^{-d_-} (K - F_T) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{-d_-} \left[K - \frac{(\sigma_1 F_0 + \sigma_0) e^{\sigma_1 \sqrt{T} x - \frac{\sigma_1^2 T}{2}} - \sigma_0}{\sigma_1} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{-d_-} \left(K + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{-\infty}^{-d_-} \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma_1\sqrt{T})^2}{2}} dx$$

$$= \left(K + \frac{\sigma_0}{\sigma_1}\right) \cdot N(-d_-) - \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot \int_{-\infty}^{-d_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

$$= \left(K + \frac{\sigma_0}{\sigma_1}\right) N(-d_-) - \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) \cdot N(-d_+).$$

Q2. (a) For normal volatility σ_n. according to lec 3 (24), we have

$$P_n^{call}(T, K, F_0, \sigma_n) = \sigma\sqrt{T}\,\left(d_+ N(d_+) + N'(d_+)\right) \cdot N(0)$$

$$P_n^{put}(T, K, F_0, \sigma_n) = \sigma\sqrt{T}\,\left(d_- N(d_-) + N'(d_-)\right) \cdot N(0)$$

$$d_\pm = \pm \frac{F_0 - K}{\sigma\sqrt{T}}$$

For lognormal volatility. σ_{ln}, according to Black-Scholes formula, we have.

$$P_{ln}^{call}(T, K, F_0, \sigma_{ln}) = N(0) \cdot \left[F_0 N(d_1) - K \cdot N(d_2)\right]$$

$$P_{ln}^{put}(T, K, F_0, \sigma_{ln}) = N(0) \cdot \left[K \cdot N(-d_2) - F_0 \cdot N(-d_1)\right]$$

$$d_{1,2} = \frac{\ln\frac{F_0}{K} \pm \frac{1}{2}\sigma_{ln}^2 T}{\sigma_{ln}\sqrt{T}}.$$

For ATM Option: ($F_0 = K$),

$$P_n^{call} = \sigma_n\sqrt{T}\left(0 \cdot N(0) + N'(0)\right) \cdot N(0) = N(0) \cdot \sigma_n\sqrt{T} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}}\right) = N(0) \cdot \frac{\sigma_n\sqrt{T}}{\sqrt{2\pi}}$$

$$P_{ln}^{call} = N(0) \cdot F_0 \left(N(d_1) - N(d_2)\right), \qquad d_{1,2} = \pm \frac{1}{2}\sigma_{ln}\sqrt{T}.$$

$$= N(0) F_0 \left(N\left(\tfrac{1}{2}\sigma_{ln}\sqrt{T}\right) - N\left(-\tfrac{1}{2}\sigma_{ln}\sqrt{T}\right)\right)$$

$$= N(0) F_0 \cdot \int_{-\frac{1}{2}\sigma_{ln}\sqrt{T}}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

let $P_n^{call} = P_{ln}^{call}$, we have $\frac{\sigma_n\sqrt{T}}{\sqrt{2\pi}} = F_0 \cdot \int_{-\frac{1}{2}\sigma_{ln}\sqrt{T}}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$\Rightarrow \sigma_n = \sqrt{\frac{2\pi}{T}} F_0 \cdot \int_{-\frac{1}{2}\sigma_{ln}\sqrt{T}}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\xRightarrow{\text{let } u = \frac{\sqrt{2}}{2}x} \sigma_n = \sqrt{\frac{2\pi}{T}} F_0 \cdot \int_{-\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-u^2} \cdot \sqrt{2}\, du$$

$$= \sqrt{\frac{2\pi}{T}} F_0 \left(\frac{2}{\sqrt{\pi}} \int_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}} e^{-u^2} du\right) = \sqrt{\frac{2\pi}{T}} F_0 \cdot \mathrm{erf}\left(\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}\right).$$

(b). The taylor expansion of e^{-u^2} is $e^{-u^2} = 1 - u^2 + \frac{(-u^2)^2}{2!} + \frac{(-u^2)^3}{3!} + \cdots\cdots$

$$= 1 - u^2 + \frac{u^4}{2} - \frac{u^6}{6} + \frac{u^8}{24} - \cdots$$

According to the conclusion in (a), we know that,

$$\sigma_n = \sqrt{\frac{2\pi}{T}} F_0 \cdot \frac{2}{\sqrt{\pi}} \int_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{1n}} \left(1 - u^2 + \frac{u^4}{2} - \frac{u^6}{6} + \frac{u^8}{24} - \cdots\right) du$$

$$= \frac{2\sqrt{2}}{\sqrt{T}} F_0 \cdot \left(u - \frac{u^3}{3} + \frac{u^5}{10} - \frac{u^7}{42} + \frac{u^9}{216} - \cdots\right)\Big|_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{1n}}$$

$$= \frac{2\sqrt{2}}{\sqrt{T}} F_0 \cdot \left[u\left(1 - \frac{u^2}{3} + \frac{u^4}{10} - \frac{u^6}{42} + \frac{u^8}{216} - \cdots\right)\right]\Big|_0^{\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{1n}}$$

$$= F_0 \cdot \sigma_{1n} \left(1 - \frac{1}{3} \cdot \frac{T}{8}\sigma_{1n}^2 + \frac{1}{10} \cdot \frac{T^2}{64}\sigma_{1n}^4 - \frac{1}{42} \cdot \frac{T^3}{512}\sigma_{1n}^6 + \cdots\right)$$

$$= F_0 \sigma_{1n} \left(1 - \frac{1}{24}\sigma_{1n}^2 T + \frac{1}{640}(\sigma_{1n}^2 T)^2 - \cdots\right).$$

HW 2 – Problem 3

Recall that the formula (13) from the Lecture Notes 4 states that

$$\begin{aligned}
\sigma_n &= \alpha \frac{F_0 - K}{D(\zeta)} \left(1 + O(\varepsilon)\right), \quad \text{where} \\
\zeta &= \frac{\alpha \left(F_0^{1-\beta} - K^{1-\beta}\right)}{\sigma_0 (1-\beta)} \quad \text{and} \quad D(\zeta) = \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1-\rho}.
\end{aligned}$$

We will now find the asymptotic of $\frac{F_0 - K}{D(\zeta)}$ when $F_0 - K \to 0$. Let us denote $F_0 = K + h$. According to Taylor's theorem we have

$$\begin{aligned}
D(\zeta) &= \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1-\rho} = \log \frac{1 + \frac{1}{2}\left(-2\rho\zeta + \zeta^2\right) + o(\zeta) + \zeta - \rho}{1-\rho} \\
&= \log\left(1 + \zeta + o(\zeta)\right) = \zeta + o(\zeta) \\
&= \frac{\alpha}{\sigma_0(1-\beta)} \left((K+h)^{1-\beta} - K^{1-\beta}\right) \\
&= \frac{\alpha K^{1-\beta}}{\sigma_0(1-\beta)} \left(\left(1 + \frac{h}{K}\right)^{1-\beta} - 1\right) \\
&= \frac{\alpha K^{1-\beta}}{\sigma_0(1-\beta)} \left(1 + (1-\beta)\frac{h}{K} + o(h) - 1\right) \\
&= \frac{\alpha h}{\sigma_0 K^{\beta}} + o(h).
\end{aligned}$$

Therefore

$$\sigma_n = \sigma_0 K^{\beta} \left(1 + O(\varepsilon)\right) + o(h).$$

Formula (16) from the Lecture Notes 4 implies

$$\sigma_{\ln} = \alpha \frac{\log F_0 - \log K}{D(\zeta)} \left(1 + O(\varepsilon)\right).$$

We have derived that $D(\zeta) = \frac{\alpha h}{\sigma_0 K^{\beta}} + o(h)$. We now obtain

$$\begin{aligned}
\sigma_{\ln} &= \alpha \frac{\log(K+h) - \log K}{D(\zeta)} \left(1 + O(\varepsilon)\right) = \alpha \frac{\log\left(1 + \frac{h}{K}\right)}{\frac{\alpha h}{\sigma_0 K^{\beta}} + o(h)} \left(1 + O(\varepsilon)\right) \\
&= \alpha \frac{\frac{h}{K} + o(h)}{\frac{\alpha h}{\sigma_0 K^{\beta}} + o(h)} \left(1 + O(\varepsilon)\right) \\
&= \sigma_0 K^{\beta-1} \left(1 + O(\varepsilon)\right) + o(h).
\end{aligned}$$