MATH 9878 ASSIGNMENT 5

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1. Problem 1

Show that, under the spot measure, the LMM dynamics has the form given by formula (6) of Lecture Notes 7. Note: you should follow the logic of the calculation that we discussed in class (that was done for the forward measure), with the appropriate numeraire replacing the zero coupon bonds.

Answer:

From the lecture note, we know that the dynamics of $L_j(t)$ is

$$dL_j(t) = \Delta_j(t) + C_j(t)dW_j(t)$$

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, and the drift and instantaneous volatility in this way should be:

$$\Delta_j(t) = \Delta_j(t, L(t))$$
$$C_j(t) = C_j(t, L(t))$$

Assume that the numeraire for measure Q_j is the price $P(t, T_{j+1})$ of the zero coupon bond expiriting at T_{j+1} .

Then, let F_i denotes the OIS forward spanning the accrual period $[T_i, T_{i+1})$, and $\gamma : [0, T_N] \to \mathbb{Z}$, define

$$\gamma(t) = m + 1, \text{ if } t \in [T_m, T_{m+1})$$

,we then have

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \prod_{\gamma(t) \le i \le j} \frac{1}{1 + \delta_j F_j(t)}$$

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Note that the numeraire for the spot measure Q_0 is

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1 \le i \le \gamma(t)} P(T_{i-1}, T_i)}$$

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Since the drift of $L_j(t)$ under Q_j is zero,

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[1] Formula 1, Lecture 7.

[2] Formula 2, Lecture 7.

[3] Formula 5, Lecture 7.

$$E^P[exp(\frac{1}{2}\int_0^t \theta(s)^T\theta(s)ds)] < \infty$$
$$dD(t) = \theta(t)^T D(t)dW(t)$$

[4]

, when we switch from Q_j measure to spot measure, $\Delta_j(t)$ becomes

$$\begin{aligned}
\Delta_j(t) &= \frac{d}{dt}[L_j, log\frac{B(t)}{P(t,T_{j+1})}](t) \\
&= \frac{d}{dt}[L_j, log(\frac{\frac{P(t,T_{\gamma(t)})}{\prod_{1\le i\le\gamma(t)} P(T_{i-1},T_i)}}{P(t,T_{\gamma(t)})\prod_{\gamma(t)\le i\le j}\frac{1}{1+\delta_i F_i(t)}}](t) \\
&= \frac{d}{dt}[L_j, log(\frac{1}{\prod_{1\le i\le\gamma(t)} P(T_{i-1},T_i)}\Pi_{\gamma(t)\le i\le j}(1+\delta_i F_i(t))](t) \\
&= \frac{d}{dt}[L_j, log(\frac{1}{\Pi_{1\le i\le\gamma(t)}\frac{1}{1+\delta_i F_i(t)}})\Pi_{\gamma(t)\le i\le j}(1+\delta_i F_i(t))](t) \\
&= \frac{d}{dt}[L_j, log\prod_{0\le i\le\gamma(t)-1}(1+\delta_i F_i(t))\prod_{\gamma(t)\le i\le j}(1+\delta_i F_i(t))](t) \\
&= \frac{d}{dt}[L_j, log\prod_{0\le i\le j}(1+\delta_i F_i)](t)
\end{aligned}$$

[5]

Since $T_i < t$, for $0 \le i \le \gamma(t)-1$, $\prod_{0\le i\le\gamma(t)-1}(1+\delta_i F_i(T_i))$ is constant .

Then,

$$\begin{aligned}
\Delta_j(t) &= \sum_{\gamma(t)\le i\le j} dL_j(t)dlog(1+\delta_i F_i(t)) \\
&= \sum_{\gamma(t)\le i\le j} dL_j(t)\frac{\delta_i dF_i(t)}{1+\delta_i F_i(t)} \\
&= C_j(t)\sum_{\gamma(t)\le i\le j}\frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}
\end{aligned}$$

[6]

Since the change of measure does not affect the diffusion part,

$$dL_j(t) = C_j(t)[\sum_{\gamma(t)\le i\le j}\frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}dt + dW_j(t)]$$

under the spot measure.

[4] Formula 26 & 27, Lecture 2.
[5] Page 9, Lecture 7.
[6] Page 9, Lecture 7

2. **Problem 2**

Prove the identity stated in formula (32) of Lecture Notes 8.

Answer:

$$\begin{aligned}
I_{(a,b)} + I_{(b,a)} &= \int_t^{t+\Delta}\int_t^s (dZ_a(u)dZ_b(s) + dZ_b(u)dZ_a(s)) \\
&= \int_t^{t+\Delta}(\int_t^s dZ_a(u))dZ_b(s) + \int_t^{t+\Delta}(\int_t^s dZ_b(u))dZ_a(s) \\
&= \int_t^{t+\Delta}(Z_a(s) - Z_a(t))dZ_b(s) + \int_t^{t+\Delta}(Z_b(s) - Z_b(t))dZ_a(s) \\
&= \int_t^{t+\Delta} Z_a(s)dZ_b(s) - \int_t^{t+\Delta} Z_a(t)dZ_b(s) + \int_t^{t+\Delta} Z_b(s)dZ_a(s) - \int_t^{t+\Delta} Z_b(t)dZ_a(s) \\
&= \int_t^{t+\Delta} Z_a(s)dZ_b(s) + \int_t^{t+\delta} Z_b(s)dZ_a(s) - Z_a(t)\int_t^{t+\Delta} dZ_b(s) - Z_b(t)\int_t^{t+\Delta} dZ_a(s) \\
&= \int_t^{t+\Delta}[Z_a(s)dZ_b(s) + Z_b(s)dZ_a(s)] - Z_a(t)\Delta Z_b(t) - Z_b(t)\Delta Z_a(t) \\
&= \int_t^{t+\Delta}[Z_a(s)dZ_b(s) + Z_b(s)dZ_a(s)] - 2\Delta Z_a(t)Z_b(t)
\end{aligned}$$

[7]

Apply Ito's Lemma to $Z_a(u)Z_b(s)$, we should have

$$\begin{aligned}
d(Z_a(s)Z_b(s)) &= Z_a(s)dZ_b(s) + Z_b(s)dZ_a(s) \\
\Longrightarrow Z_a(s)Z_b(s)|_t^{t+\Delta} &= \int_t^{t+\Delta}(Z_a(s)dZ_b(s) + Z_b(s)dZ_a(s))
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I_{(a,b)} + I_{(b,a)} &= \int_t^{t+\Delta}[Z_a(s)dZ_b(s) + Z_b(s)dZ_a(s)] - 2\Delta Z_a(t)Z_b(t) \\
&= Z_a(s)Z_b(s)|_t^{t+\Delta} - 2\Delta Z_a(t)Z_b(t) \\
&= (Z_a(t) + Z_a(t)\Delta)(Z_b(t) + \Delta Z_b(t)) - Z_a(t)Z_b(t) - 2\Delta Z_a(t)Z_b(t) \\
&= \Delta Z_a \Delta Z_b
\end{aligned}$$

[8]

3. Problem 3

Show that the dynamics of LMM satisfies the integrability condition (30) (or (31)) of Lecture Notes 8, and it thus can be numerically implemented by means of Milstein's scheme.

Answer:

Rewrite the dynamics of the model in terms of the independent Brownian motions:

$$dL_j(t) = \Delta_j(t)dt + \sum_{1\le a\le d} B_{ja}(t)dZ_a(t)$$

[7]Formula 29, Lecture 8

[8]Formula 32, Lecture 8

[9]

where $B_{ia}(t) = U_{ia}C_i(t)$[10] and $Z_a(t)$ are independent Brownian Motions.

$$\mathcal{L}^a B_{ib} = \sum_{1\le k\le n} B_{ka}\frac{\partial B_{ib}}{\partial X_k} = \sum_{1\le k\le n} U_{ka}C_k(t)\frac{\partial U_{ib}C_i(t)}{\partial \mathcal{L}_k}$$

[11]

Since $C_i(t, \mathcal{L}_i(t))$ is a function of t and $L_i(t)$, $\frac{\partial C_i(t)}{\partial L_k} = \begin{cases} 0, & k \neq i \\ \frac{\partial C_i(t)}{\partial \mathcal{L}_i}, & k = i \end{cases}$, U_{ka} is just a coefficient.

Thus, we then should be able to get

$$\begin{aligned}
\mathcal{L}^a B_{ib} &= \sum_{1\le k\le n} U_{ka}C_k(t)\frac{\partial U_{ib}C_i(t)}{\partial L_k} \\
&= U_{ia}C_i(t)\cdot U_{ib}\frac{\partial C_i(t)}{\partial \mathcal{L}_i} \\
&= U_{ia}U_{ib}C_i(t)\frac{\partial C_i(t)}{\partial \mathcal{L}_i} \\
\mathcal{L}^b B_{ia} &= \sum_{1\le k\le n} U_{kb}C_k(t)\frac{\partial U_{ia}C_i(t)}{\partial L_k} \\
&= U_{ib}C_i(t)\cdot U_{ia}\frac{\partial C_i(t)}{\partial \mathcal{L}_i} \\
&= U_{ib}U_{ia}C_i(t)\frac{\partial C_i(t)}{\partial \mathcal{L}_i} \\
\Longrightarrow \mathcal{L}^a B_{ib} &= \mathcal{L}^b B_{ia}
\end{aligned}$$

In this way we proved that the integrability condition is satisfied.

[9]Formula 12, Lecture 7
[10]Formula 13, Lecture 7
[11]Formula 25, Lecture 7