Market impact with multi-timescale liquidity

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Outline

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   - Deficiencies of LLOB model
2. Market impact profile under finite memory
   - Price trajectories with finite cancel and deposit rates
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   - LLOB model with fast and slow agents
   - From linear to square-root impact
4. The multi-frequency framework
   - Resolution of the "diffusivity puzzle"
   - Meta-order impact

Introduction to LLOB model

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Introduction to LLOB model

The latent volume densities of limit orders in the order book $\phi_{\mathrm{b}}(x,t)$ (bid side) and $\phi_{\mathrm{a}}(x,t)$ (ask side) follows

$$\begin{aligned}\partial_t\phi_{\mathrm{b}} &= D\partial_{xx}\phi_{\mathrm{b}} - \nu\phi_{\mathrm{b}} + \lambda\Theta(x_t - x) - R_{\mathrm{ab}}(x)\\ \partial_t\phi_{\mathrm{a}} &= D\partial_{xx}\phi_{\mathrm{a}} - \nu\phi_{\mathrm{a}} + \lambda\Theta(x - x_t) - R_{\mathrm{ab}}(x)\end{aligned} \tag{1}$$

The price x_t is defined as the solution of

$$\begin{aligned}\phi(x_t,t) &= \phi_{\mathrm{b}}(x,t) - \phi_{\mathrm{a}}(x,t) = 0\\ \partial_t\phi &= D\partial_{xx}\phi - \nu\phi + \lambda\mathrm{sign}(x_t - x)\end{aligned} \tag{2}$$

Introduction to LLOB model

The stationary order book is given by

$$\phi^{\mathrm{st}}(\xi) = -\frac{\lambda}{\nu}\mathrm{sign}(\xi)\left[1 - \exp\left(-\frac{|\xi|}{\xi_c}\right)\right] \tag{3}$$

where $\xi_c = \sqrt{D\nu^{-1}}$ and $\xi = x - x_t$.

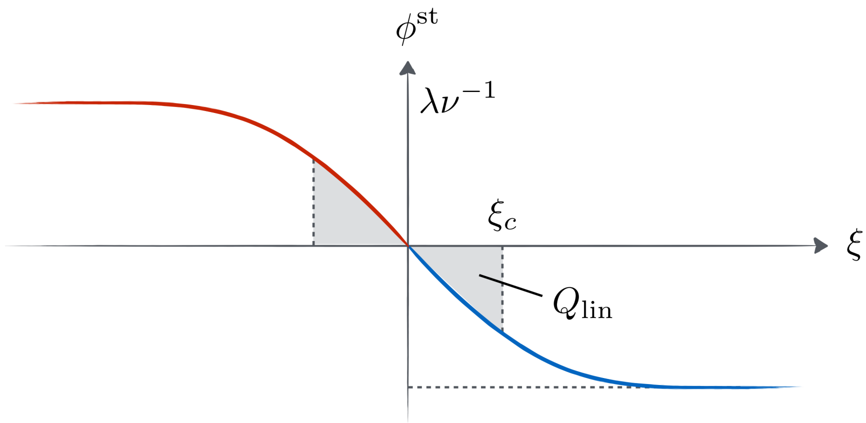

Figure 1: Stationary order book under LLOB model.

Deficiencies of LLOB model

- The original LLOB model focused on the **infinite memory** limit, namely $\nu, \lambda \to 0$ while $\mathcal{L} \sim \lambda\nu^{-1/2}$, such that the latent order book becomes exactly linear; while in reality we are facing finite memory.

- Square-root law is only recovered where the **execution rate** m_0 of the meta-order is larger than the normal execution rate J of the market itself; whereas most meta-order impact data is in the opposite limit $m_0 \lesssim 0.1J$.

- The theoretical inverse square-root impact decay is too fast and leads to significant short time **mean-reversion effects**, not observed in real prices.

Price and impact profile under finite memory

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Price and impact profile under finite memory

We introduce a buy meta-order as the source of market impact with intensity rate m_t, then equation (2) becomes

$$\partial_t \phi = D\partial_{xx}\phi - \nu\phi + \lambda \text{sign}(x_t - x) + m_t \delta(x - x_t) \mathbb{1}_{[0,T]} \tag{4}$$

Focusing on constant participation rates $m_t = m_0$, we consider

- Small **participation rate** $m_0 \ll J$ *v.s.* large participation rate $m_0 \gg J$.
- Fast **execution speed** $\nu T \ll 1$ *v.s.* slow execution speed $\nu T \gg 1$.
- Small **meta-order volumes** $Q = m_0 T \ll Q_{\text{lin}}$ *v.s.* large meta-order volumes $Q = m_0 T \gg Q_{\text{lin}}$.

Price trajectories with finite cancel and deposit rates

For fast execution and small meta-order volumes, we have the price trajectory as

$$x_t = \alpha \left[z_t^0 + \sqrt{\nu} z_t^1 + \mathcal{O}(\nu) \right] \tag{5}$$

For slow execution and/or large meta-order volumes,

$$x_t = \frac{m_0 \nu}{\lambda} t \tag{6}$$

Price trajectories with finite cancel and deposit rates

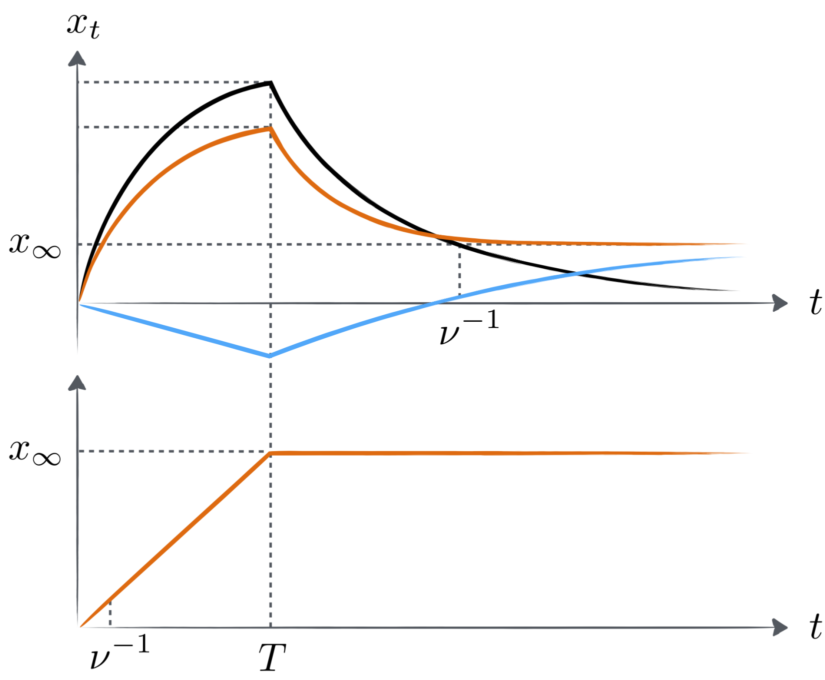

Figure 2: Top graph: Price trajectory during and after a buy meta-order execution for $\nu T \ll 1$. Bottom graph: Price trajectory for $\nu T \gg 1$.

Linear permanent impact with finite memory

We find that the permanent impact I_∞ follows

$$I_\infty = \frac{1}{2}\xi_c \frac{Q}{Q_{\text{lin}}} \tag{7}$$

The result is dictated by non-arbitrage arguments and compatible with the classical Kyle model.

The double-frequency framework

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The double-frequency framework

Consider there are two sorts of agents co-exists in the market:

- **Slow agents** with vanishing cancellation and deposition rates: $\nu_{\mathrm{s}} T \to 0$, while keeping $\mathcal{L}_{\mathrm{s}} := \lambda_{\mathrm{s}} / \sqrt{\nu_{\mathrm{s}} D}$ finite.
- **Fast agents** with large cancellation and deposition rates: $\nu_{\mathrm{f}} T \gg 1$, such that $\mathcal{L}_{\mathrm{f}} := \lambda_{\mathrm{f}} / \sqrt{\nu_{\mathrm{f}} D} \gg \mathcal{L}_{\mathrm{s}}$.

LLOB model with fast and slow agents

With the conditions below,

$$\begin{aligned} m_{\mathrm{s}t} + m_{\mathrm{f}t} &= m_0 \\ x_{\mathrm{s}t} = x_{\mathrm{f}t} &= x_t \end{aligned} \tag{8}$$

the total order book volume is given by

$$\phi^{\mathrm{st}}(x) = \phi_{\mathrm{s}}^{\mathrm{st}}(x) + \phi_{\mathrm{f}}^{\mathrm{st}}(x) \tag{9}$$

where

$$\begin{aligned} \phi_{\mathrm{s}}^{\mathrm{st}} &\approx -\mathcal{L}_{\mathrm{s}} x \\ \phi_{\mathrm{f}}^{\mathrm{st}} &\approx -\frac{\lambda_{\mathrm{f}}}{\nu_{\mathrm{f}}} \mathrm{sign}(x) \end{aligned} \tag{10}$$

LLOB model with fast and slow agents

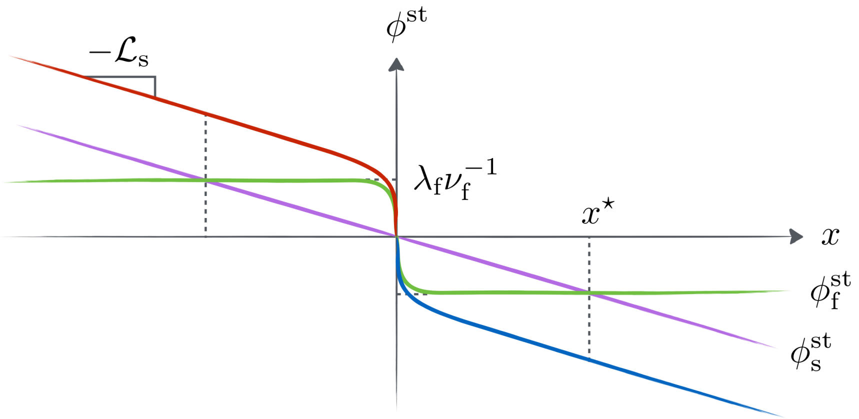

Figure 3: Stationary double-frequency order book.

From linear to square-root impact

Consider the meta-order intensity is in between of slow traders and fast traders. That is $J_{\mathrm{s}} \ll m_0 \ll J_{\mathrm{f}}$.

Equation (4) and equation (8) yield that

$$
\begin{aligned}
m_{\mathrm{f}t} &= \frac{m_0}{\sqrt{1+\frac{t}{t^\star}}}, \quad t^\star := \frac{1}{2\nu_{\mathrm{f}}}\frac{J_{\mathrm{f}}^2}{J_{\mathrm{s}} m_0} \\
m_{\mathrm{s}t} &= m_0 - m_{\mathrm{f}t}
\end{aligned}
\tag{11}
$$

From linear to square-root impact

The resulting price trajectory reads

$$x_t = \frac{\lambda_{\mathrm{f}}}{\mathcal{L}_{\mathrm{s}}\nu_{\mathrm{f}}}\left(\sqrt{1+\frac{t}{t^\star}}-1\right) \tag{12}$$

The most of the incoming meta-order is executed against the fast agents for $t < t^\star$ but the slow agents then take over for $t > t^\star$.

The market impart crosses over from a linear regime when $t \ll t^\star$ to a square root regime for $t \gg t^\star$.

From linear to square-root impact

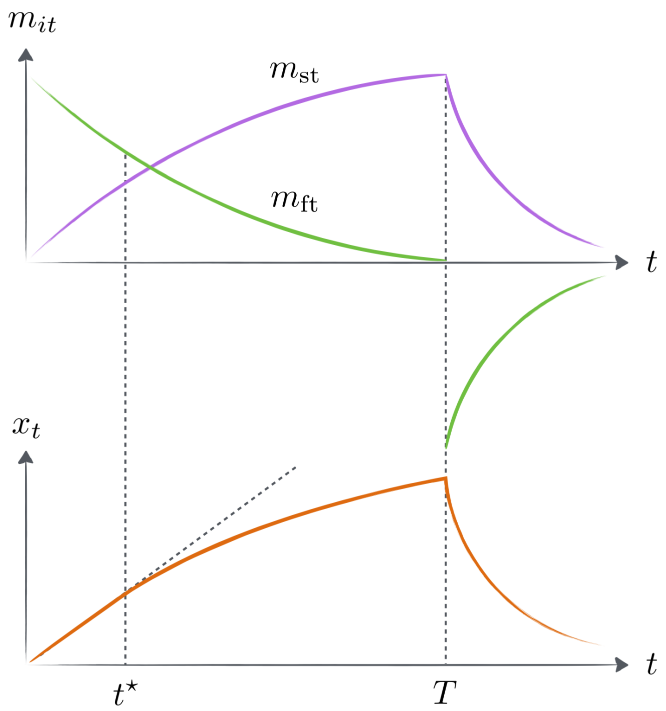

Figure 4: Execution rates m_{it} (top) and price trajectory (bottom) within the double-frequency order book model.

The multi-frequency framework

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The multi-frequency framework

The double-frequency framework can be extended to the more realistic case of a continuous range of cancellation and deposition rates:

$$\partial_t \phi_\nu = D\partial_{xx}\phi_\nu - \nu\phi_\nu + \lambda_\nu \text{sign}(x_{\nu t} - x) + m_{\nu t}\delta(x - x_{\nu t}) \tag{13}$$

with the following conditions,

$$\begin{aligned} &\int_0^\infty \rho(\nu) m_{\nu t} \mathrm{d}\nu = m_t \\ &x_{\nu t} = x_t, \quad \forall \nu \end{aligned} \tag{14}$$

where $\rho(\nu)$ denotes the distribution of cancellation rates ν, and m_t denotes an arbitrary order flow .

Resolution of the "diffusivity puzzle"

With $\langle x_t^2 \rangle \propto t^{1-\gamma}$, the latent liquidity in the original LLOB case is too persistent and prevents the price from diffusing.

With the power-law distribution $\rho(\nu)$ as

$$\rho(\nu) = Z\nu^{\alpha-1}e^{-\nu t_c} \tag{15}$$

the price diffusion under multi-frequency framework is given by

$$\langle x_t^2 \rangle \propto t^{1+2\alpha-\gamma} \tag{16}$$

Meta-order impact

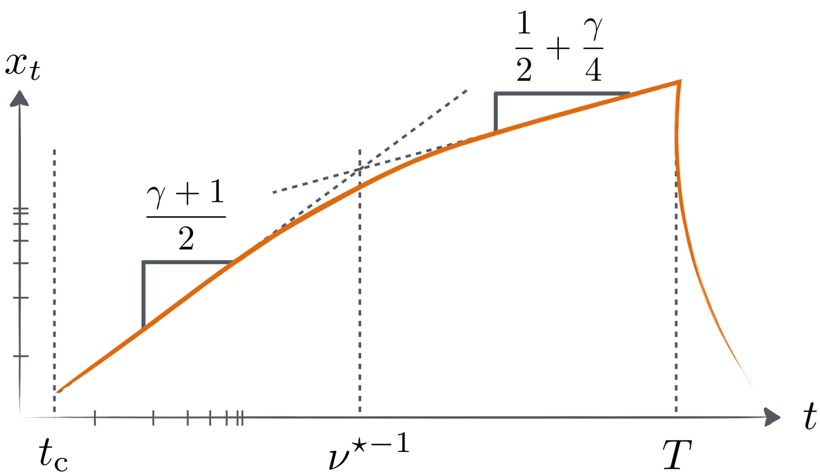

Figure 5: Price trajectory during a constant rate meta-order execution within the multi-frequency order book model. For $\gamma = 1/2$, the impact crosses over from a $t^{3/4}$ to a $t^{5/8}$ regime.

References

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Market impact with multi-timescale liquidity

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A fully consistent, minimal model for non-linear market impact