MTH 9879 Market Microstructure Models, Spring 2019

Lecture 5: Price formation under asymmetric information: The Kyle model

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Outline of Lecture 5

- Bayesian learning
- Asymmetric information
- Equilibrium price under naïve expectations
- Asymmetric information under perfect competition: The Grossman-Stiglitz (1980) model
- Rational expectations with strategic traders under imperfect competition: The Kyle model
- Linear regression and rational expectations

Price formation

- According to classical economics, supply and demand are matched at the efficient price (though the mechanism is unclear). Information is symmetric.

- In reality, some traders have more or better information than others. That is, information is asymmetric.
    - Even in a purely mechanical model such as ZI, there are trading signals. Faster and better traders have an advantage.
    - This informational advantage requires investment. For example, gaining a latency advantage requires colocation.

- How does this information get impounded into prices?

CARA utility again

- CARA stands for *Constant Absolute Risk Aversion*.

- The utility function is
$$U(W) = -e^{-\alpha W}$$

- If W is normally distributed,
$$\mathbb{E}\left[U(W)\right] = \mathbb{E}\left[-e^{-\alpha W}\right] = -\exp\{-\alpha\,(\mathbb{E}[W] - \alpha/2\,\mathrm{Var}[W])\}$$
  so maximizing CARA utility is equivalent to maximizing
$$\mathcal{L}(W) = \mathbb{E}[W] - \alpha/2\,\mathrm{Var}[W]$$
  which is just mean-variance optimization.

- $\mathbb{E}[W] - \alpha/2\,\mathrm{Var}[W]$ is called the *certainty equivalent* of W.

Trader demand under CARA utility

Let p be the current asset price and $\tilde{S}$ be the terminal asset price with $\tilde{S} \sim N(\mu, \sigma^2)$.

The trader is assumed to maximize utility of terminal wealth $W = y\,(\tilde{S} - p)$:
$$0 = \partial_y\,\left\{\mathbb{E}[y\,(\tilde{S} - p)] - \frac{1}{2}\,\alpha\,\mathrm{Var}[y\,\tilde{S}]\right\}$$
$$= \partial_y\,\left\{y\,(\mu - p) - \frac{1}{2}\,\alpha\,y^2\,\sigma^2\right\}$$
$$= \mu - p - \alpha\,y\,\sigma^2$$

Then the optimal trader demand y is given by
$$y = \frac{\mathbb{E}[\tilde{S}] - p}{\alpha\,\sigma^2} = \frac{\mu - p}{\alpha\,\sigma^2}.$$

Bayes' Rule for continuous random variables

- Suppose the trader has a prior belief on the parameter y described by the distribution $g(y)$.

- This trader then makes an observation x (such as a trade price).
  - Denote the conditional density of x by $f(x|y)$.

- Then because
$$g(y|x)\,f(x) = f(x|y)\,g(y),$$
  we have

Bayes' Rule
(1)

$$g(y|x) = \frac{f(x|y)\,g(y)}{f(x)} = \frac{f(x|y)\,g(y)}{\int f(x|y)\,g(y)\,dy}$$

Bayes' Rule for normal random variables

Suppose

$$g(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y}\,\exp\left\{-\frac{(y-\bar{y})^2}{2\,\sigma_y^2}\right\}$$

and

$$f(x|y) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\,\exp\left\{-\frac{(x-y)^2}{2\,\sigma_x^2}\right\}.$$

Substituting into (1) gives

$$g(y|x) = \sqrt{\frac{\tau_y + \tau_x}{2\pi}}\,\exp\left\{-\frac{(y-\beta x-(1-\beta)\,\bar{y})^2}{2\,(1-\beta)\,\sigma_y^2}\right\}$$

where $\tau_i = 1/\sigma_i^2$ is *precision* and

$$\beta = \frac{\tau_x}{\tau_y + \tau_x}.$$

The filter (a simple Kalman filter)

So, if $y \sim N(\bar{y}, \sigma_y^2)$ and $x|y \sim N(y, \sigma_x^2)$,

(2)

$$y|x \sim N(\beta x + (1-\beta)\,\bar{y}, (1-\beta)\,\sigma_y^2)$$

- If both prior and observation distributions are normal, the posterior distribution will also be normal.
    - The mean of the posterior distribution is a linear combination of the prior mean and the observation, with weights proportional to their precisions.
    - The variance of the posterior is lower than both the variance of the prior and the variance of the measurement.
- As we make more observations, the true value of the variable of interest is progressively revealed.

A model with asymmetric information

- Consider a model with a single risky asset (a stock).

- There are N_i informed traders, N_u uninformed traders, and N_z noise traders. They have demand functions y_i, y_u and y_z respectively with

$$y_z \sim N(0, \sigma_z^2).$$

- Agents hold a prior distribution on the expected future value $\tilde{S}$ of the asset:

$$\tilde{S} \sim N(\mu, \sigma^2).$$

- Agents do not take into account market impact when trading.

- At time $t = 0$, informed traders receive a signal x on $\tilde{S}$ which is drawn from the distribution

$$x|\tilde{S} = \tilde{S} + \sigma_x W \sim N(\tilde{S}, \sigma_x^2),$$

where $W \sim N(0, 1)$, independent of $\tilde{S}$.

Computation of trader demands for the asset

- As above, an application of Bayes' Rule then shows that

$$\tilde{S}|x \sim N(\beta x + (1-\beta)\mu, (1-\beta)\sigma^2)$$

  with

$$\beta = \frac{\tau_x}{\tau_S + \tau_x}$$

- At $t = 1$, trading takes place.

- Optimal trader demand assuming CARA utility (see above) is given by

$$y_u = \frac{\mathbb{E}[\tilde{S}] - p}{\alpha\,\mathrm{Var}[\tilde{S}]}; \quad y_i = \frac{\mathbb{E}[\tilde{S}|x] - p}{\alpha\,\mathrm{Var}[\tilde{S}|x]}$$

- At $t = 2$, the value of the asset is revealed and becomes public information.

Equilibrium price under naïve expectations

The market clearing condition is just

$$N_i\,y_i + N_u\,y_u + N_z\,y_z = 0$$

so

$$N_i\,\frac{\mathbb{E}[\tilde{S}|x] - p}{\alpha\,\mathrm{Var}[\tilde{S}|x]} + N_u\,\frac{\mathbb{E}[\tilde{S}] - p}{\alpha\,\mathrm{Var}[\tilde{S}]} + N_z\,y_z = 0$$

Solving for p gives an expression of the form

$$p = \theta\,\mathbb{E}[\tilde{S}|x] + (1-\theta)\,\mu + \lambda\,y_z,$$

where

$$\theta = \frac{\frac{N_i}{\mathrm{Var}[\tilde{S}|x]}}{\frac{N_i}{\mathrm{Var}[\tilde{S}|x]} + \frac{N_u}{\sigma^2}} \in [0,1] \quad \text{and} \quad \lambda = \frac{\alpha N_z}{\frac{N_i}{\mathrm{Var}[\tilde{S}|x]} + \frac{N_u}{\sigma^2}}.$$

Substituting $\mathbb{E}[\tilde{S}|x] = \beta x + (1-\beta)\mu = \mu + \beta(x-\mu)$ gives

(3)

$$p = \mu + \beta\,\theta\,(x - \mu) + \lambda\,y_z$$

The last term in (3) is just noise and the product $\beta\,\theta \le 1$ so the equilibrium price p is a noisy version of the signal x.

- If there are only informed traders, $p = \mathbb{E}[\tilde{S}|x]$ and $y_i = 0$. Thus, no trading.
- If there are only uninformed traders, $p = \mathbb{E}[\tilde{S}] = \mu$ and $y_u = 0$. Thus, no trading either.
- If there are only noise traders, the price is undetermined.

In this version of the model, uninformed traders don't update their prior using information contained in the market price. That's why this equilibrium is termed *naïve*.

- Let's now see what happens under rational expectations.

Grossman and Stiglitz (1980)

- The setup is as before: Both $\mathcal{I}$ and $\mathcal{U}$ have CARA utility, $\tilde{S} \sim N(\mu, \sigma^2)$ is unknown, and $\mathcal{I}$ receives a signal $x \sim N(\tilde{S}, \sigma_x^2)$.

- In contrast to the naïve equilibrium case however, $\mathcal{U}$ updates his prior after observing the market price. Quoting Grossman and Stiglitz (but with our notation),

> If over time uninformed traders observe many realizations of $\{\tilde{S}, p\}$, then they learn the joint distribution of $\{\tilde{S}, p\}$. After all learning about the joint distribution of $\{\tilde{S}, p\}$ ceases, all traders will make allocations and form expectations such that this joint distribution persists over time.

Fully revealing rational expectations equilibrium (REE)

- From equation (3), with no noise trading ($y_z = 0$),

$$p = \mu + \beta\theta(x - \mu)$$

so $\mathcal{U}$ can obtain the value of x by regression.
- The informed trader $\mathcal{I}$'s signal x can be perfectly inferred by $\mathcal{U}$ from the price p.
  - Thus the equilibrium is *fully revealing*.
  - $y_i = y_u = 0$ and there is no trading.

- The equilibrium price is given by

(4)

$$p = \mathbb{E}[\tilde{S}|x].$$

- Prices are strong-form efficient because they fully incorporate public and private information.

The need for noise traders

- The realization that under the Efficient Markets hypothesis, uninformed traders can learn the signal at zero cost from the market price was the motivation for Grossman and Stiglitz (1980) to introduce a random noise trader demand y_z.
  - If all information is fully impounded in the price, informed traders conclude they should pay nothing for information. But then there are no informed traders and there is no equilibrium.
  - Having no one informed is also not an equilibrium, because then each trader, taking the price as given, feels that there are profits to be made from becoming informed.

- With the random demand y_z, the market clearing condition becomes

(5)

$$N_u y_u + N_i y_i + N_z y_z = 0$$

and x can no longer be inferred perfectly from p.

Another quote from Grossman and Stiglitz

> The only way informed traders can earn a return on their activity of information gathering is if they can use their information to take positions in the market which are "better" than the positions of uninformed traders. "Efficient Markets" theorists have claimed that "at any time prices fully reflect all available information" (see Eugene Fama, p. 383). If this were so then informed traders could not earn a return on their information.

Imperfect vs perfect competition

Grossman and Stiglitz assume perfect competition. Under this assumption, traders are schizophrenic:

- They don't consider the effect of their own trades on the market price.

Kyle addressed this problem by introducing the notion of *imperfect competition*. With this assumption

- Traders do factor in the effect of their trading on the market price.
- Prices cannot reveal more than half the information available to insiders (at least in the one-step model).

To quote Kyle:

> ...the competitive rational expectations concept requires each trader to behave "schizophrenically" by taking the equilibrium price as given when making his trading decision, even though he influences that price when he trades.

Kyle (1985)

- There is a single informed trader who trades strategically.
  - When computing his optimal course of action, this trader takes into account the rational reactions of other traders.

- This informed trader sets his trade sizes taking cost of execution into account.

- In the multiperiod version, the trader can return to the market, spreading out his trades over time.
  - This feature of the model is particularly relevant now with small ticks and fragmented markets leading to only very small size typically available on each side of the market.
  - A trader with large size to trade must split the trade into smaller pieces.

Kyle model setup

- One asset economy.

- One informed trader, a group of risk-neutral market makers, and a group of uninformed (liquidity) traders.

- At time $t = 0$, the informed trader (insider) discovers what the final price $\tilde{S}$ will be.

- All traders know the distribution of $\tilde{S}$:

$$\tilde{S} \sim N(\mu, \sigma^2).$$

- At time $t = 1$, uninformed (liquidity) traders submit a random net demand $y_u \sim N(0, \sigma_u^2)$, independent of $\tilde{S}$.

- Competition drives market makers to offer the price function

$$p(y) = \mathbb{E}[\tilde{S}|y]$$

  so that expected profits are

$$\mathbb{E}[(p(y) - \tilde{S}) \, y|y] = 0.$$

  - In average, market makers have zero profit.
  - The market makers know that there are informed traders.
  - However, they see only the aggregate order flow $y = y_i + y_u$. They cannot distinguish orders between informed and uninformed.

- If there were no informed traders, the price function would be

$$p(y) = \mathbb{E}[\tilde{S}|y_u] = \mathbb{E}[\tilde{S}] = \mu$$

  which is independent of size so the market would be infinitely deep.

A market mechanism

- In the above, we just assume that the market mechanism will drive market maker profits to zero (*perfect competition*).

- On page 137 of [Foucault, Pagano and Röell][2], there is a nice explanation of a possible mechanism by which perfect competition might be achieved:
  - The entire order flow $y = y_i + y_u$ is displayed to the market makers.
  - Each market maker responds with his price for executing the order in full. The entire flow will be routed to the market maker with the best price.
  - Market makers undercut each other until the zero-profit price $p(y) = \mathbb{E}[\tilde{S}|y]$ is reached.

- Note that the price achieved depends in general on the market mechanism.

Characterization of equilibrium

Traders are risk-neutral. In equilibrium,

- Market makers ($\mathcal{M}$) maximize expected profits (in fact, market makers "set price" because under perfect competition they have zero expected profit) conditioned on the observed order flow.
  - The order flow has an informed component so $\mathcal{M}$ can extract information about the price.

- The informed trader ($\mathcal{I}$) sets y_i so as to maximize his expected profit given his rational expectation of the impact of his order on the market price.
  - $\mathcal{I}$ knows that his order will end up giving information to $\mathcal{M}$.

The informed trader's strategy

- $\mathcal{I}$ chooses his demand function y_i so as to maximize his expected profit
$$\mathbb{E}[y_i\,(\tilde{S} - p(y))|\tilde{S}].$$
His optimal strategy thus depends on $p(y)$.

- $\mathcal{I}$ conjectures that the pricing function set by $\mathcal{M}$ is of the form
$$p(y) = \mu + \lambda\, y,$$
i.e., half spread is proportional to the total demand.

- $\mathcal{I}$'s expected profit is then

$$\mathbb{E}[y_i\,(\tilde{S} - p(y))|\tilde{S}] = \mathbb{E}[y_i\,(\tilde{S} - \mu - \lambda(y_i + y_u))|\tilde{S}]$$
$$= y_i\,(\tilde{S} - \mu - \lambda\, y_i)$$

since the uninformed demand y_u is independent of $\tilde{S}$ and has mean 0.

- Maximizing with respect to y_i then gives $\mathcal{I}$'s optimal demand as
$$y_i = \frac{\tilde{S} - \mu}{2\,\lambda}.$$

The Kyle lambda

- Recall that
$$p(y) = \mu + \lambda\, y.$$

- The *Kyle lambda* λ is a measure of market liquidity;
  - The lower λ is, the more liquid the market.
  - Equivalently, λ is a market impact coefficient in a linear model of market impact. The higher λ is, the greater the market impact of a trade.
  - The Kyle lambda may be viewed as the definition of a market statistic. Empirically, λ is the slope in a linear regression of p against y.

Quote from Kyle (1985)

> Roughly speaking, Black defines a liquid market as one which is almost infinitely tight, which is not infinitely deep, and which is resilient enough so that prices eventually tend to their underlying value.
>
> Our continuous auction equilibrium has exactly the characteristics described by Black. Furthermore, these aspects of market liquidity acquire a new prominence in our model because the insider, who does not trade as a perfect competitor, must make rational conjectures about tightness, depth, and resiliency in choosing his optimal quantity to trade.

The market markers' pricing strategy

- $\mathcal{M}$ conjectures that $\mathcal{I}$'s demand function is of the form
$$y_i = a + b\,\tilde{S}.$$

- Knowing the optimization process followed by $\mathcal{I}$, $\mathcal{M}$ can solve for a and b:
$$y_i = \frac{\tilde{S} - \mu}{2\,\lambda} = a + b\,\tilde{S} \quad \forall \tilde{S}$$

  So

  (6)

$$a = -\frac{\mu}{2\,\lambda} \text{ and } b = \frac{1}{2\,\lambda}.$$

  In particular, $\mathcal{I}$ trades less as λ increases.

- $\mathcal{M}$ sets the pricing function as
$$p(y) = \mu + \lambda\,y = \mathbb{E}[\tilde{S}|y]$$

  so we need to compute $\mathbb{E}[\tilde{S}|y]$.

Normal regression

Let x and y be bivariate normally distributed. Then, conditioned on x, $y|x$ is also normally distributed. Moreover, we have

(7)

$$\mathbb{E}[y|x] = \mathbb{E}[y] + \beta\,(x - \mathbb{E}[x]),$$
$$\mathrm{Var}[y|x] = \mathrm{Var}[y] - \beta^2\,\mathrm{Var}[x],$$

where $\beta = \dfrac{\mathrm{Cov}_{[y,x]}}{\mathrm{Var}_{[x]}}$.

Note

The posterior variance $\mathrm{Var}[y|x]$ is always smaller than the prior variance $\mathrm{Var}[y]$ unless y and x are uncorrelated.

Regression of $\tilde{S}$ against y

- Recall that $\tilde{S} \sim N(\mu, \sigma^2)$ and $y_u \sim N(0, \sigma_u^2)$.
- Also, $y_i = a + b\,\tilde{S} = \frac{\tilde{S} - \mu}{\lambda}$ must be normal.
- $\tilde{S}$ and y are jointly normal.
- To compute $\mathbb{E}[\tilde{S}|y]$, use (7):

(8)

$$\mathbb{E}[\tilde{S}|y] = \mathbb{E}[\tilde{S}] + \frac{\mathrm{Cov}[\tilde{S}, y]}{\mathrm{Var}[y]}(y - \mathbb{E}[y])$$

- The (unconditional) expectation of y is given by
$$\mathbb{E}[y] = \mathbb{E}[y_i] + \mathbb{E}[y_u] = 0.$$
- The variance and covariance are given by
$$\mathrm{Var}[y] = \mathrm{Var}[y_u] + \mathrm{Var}[y_i] = \sigma_u^2 + \frac{\sigma^2}{4\lambda^2} \text{ and } \mathrm{Cov}[\tilde{S}, y] = \frac{\sigma^2}{2\lambda}.$$

Computation of equilibrium

Substituting back into (8) gives

(9)

$$\mathbb{E}[\tilde{S}|y] = \mu + \frac{\frac{\sigma^2}{2\lambda}}{\sigma_u^2 + \frac{\sigma^2}{4\lambda^2}}\, y$$

Also

(10)

$$p(y) = \mu + \lambda\, y = \mathbb{E}[\tilde{S}|y]$$

Equating (9) and (10) and solving for all y gives

$$\lambda = \frac{\frac{\sigma^2}{2\lambda}}{\sigma_u^2 + \frac{\sigma^2}{4\lambda^2}}.$$

Solve for λ we obtain

$$\lambda = \frac{\sigma}{2\sigma_u}$$

Also, recall that $\mathcal{M}$'s conjecture (6) about $\mathcal{I}$'s demand gives

$$b = \frac{1}{2\,\lambda} = \frac{\sigma_u}{\sigma}.$$

Final result and intuition

Optimal choice of λ

$$b = \frac{\sigma_u}{\sigma} \text{ and } \lambda = \frac{1}{2}\frac{\sigma}{\sigma_u}$$

Thus, the market maker's pricing function reads

$$p(y) = \mu + \frac{\sigma}{2\sigma_u}y$$

and the informed trader's demand is given by

$$y_i = \frac{\sigma_u}{\sigma}(\tilde{S} - \mu).$$

Intuition is as follows:

- The greater the demand of uninformed traders (as proxied by σ_u), the smaller the proportion of informed trading on average and the less the adverse selection faced by $\mathcal{M}$.

Informed trader's profit

The informed trader's expected profit is

$$\mathbb{E}[y_i\,(\tilde{S} - p)|\tilde{S}] = \frac{(\tilde{S} - \mu)^2}{4\,\lambda} = \frac{(\tilde{S} - \mu)^2}{2}\frac{\sigma_u}{\sigma}$$

Profit is increasing in σ_u:

- The more liquidity trading there is, the more $\mathcal{I}$ can hide his intentions.
  - Note that σ_u is the typical magnitude of y_u ($\mathbb{E}[|y_u|] = \sqrt{\frac{2}{\pi}}\,\sigma_u$).

- The more liquid the stock, the easier it is to trade!

The unconditional expectation of the informed trader's profit is

$$\mathbb{E}[y_i\,(\tilde{S} - p)] = \frac{\mathbb{E}\left[(\tilde{S} - \mu)^2\right]}{2}\frac{\sigma_u}{\sigma} = \frac{1}{2}\sigma\,\sigma_u.$$

The market makers' posterior distribution

- $\mathcal{M}$'s posterior distribution is $\tilde{S}|y$.

- If for example there was big demand from liquidity traders, the mean $p = \mathbb{E}[\tilde{S}|y]$ may be further away from the true value $\tilde{S}$ than the mean μ of the prior.

- The variance of the posterior distribution is of course much smaller than the variance of the prior.

- We now compute the mean and variance of the posterior distribution.

Net order flow y as a signal for $\tilde{S}$

The order flow y is a signal for $\tilde{S}$. We apply the following lemma to derive the posterior distribution $\tilde{S}|y$.

Lemma
Let X and Y be jointly normal with means μ_x, μ_y and variances σ_x^2 and σ_y^2, respectively. Let ρ be the correlation coefficient between X and Y. Then

$$X|Y \sim N(\mu, \sigma^2),$$

where

$$\mu = \mu_x + \frac{\rho\sigma_x}{\sigma_y}(Y - \mu_y), \quad \sigma^2 = \left(1 - \rho^2\right)\sigma_x^2.$$

Therefore, since

$\mathbb{E}\left[\tilde{S}\right] = \mu$, $\mathbb{E}\left[y\right] = 0$, $\mathrm{Var}[\tilde{S}] = \sigma^2$, $\mathrm{Var}[y] = 2\sigma_u^2$, and $\mathrm{Cov}[\tilde{S}, y] = \sigma\sigma_u$, we have

$$\mathbb{E}\left[\tilde{S}|y\right] = \mu + \frac{1}{2}\frac{\sigma}{\sigma_u}y = \mu + \lambda y$$

and

$$\mathrm{Var}[\tilde{S}|y] = (1 - \rho^2)\mathrm{Var}[\tilde{S}] = \frac{\sigma^2}{2}.$$

Proportion of information revealed

Thus, the posterior distribution

$$\tilde{S}|y \sim N\left(\mu + \lambda y, \frac{\sigma^2}{2}\right),$$

where recall that $\lambda = \frac{\sigma}{2\sigma_u}$.

- No matter what the trade outcome is, the posterior variance is exactly half of the prior variance.
- Since σ proxies the information that $\mathcal{I}$ holds (inside information), we can say following Kyle that $\mathcal{I}$'s optimal strategy will result in half of his information being revealed by the market price.
- Note that the mean of the posterior distribution is exactly the $\mathcal{M}$'s pricing rule.

Extension to noisy signal

- In this extension of the model, $\mathcal{I}$ receives imperfect (as opposed to perfect) information at $t = 0$.

- The signal $x = \tilde{S} + \epsilon$ where $\epsilon \sim N(0, \sigma_\epsilon^2)$ is independent of everything else.

- In this case, we can show (see HW5 problem 1) that

  (11)
  $$\lambda = \frac{\sigma}{2\,\sigma_u}\,\sqrt{\phi}$$

  with

  $$\phi = \frac{\sigma^2}{\sigma^2 + \sigma_{\tilde{\epsilon}}^2}$$

- This clearly has the right limit as $\sigma_{\epsilon} \to 0$.

Extension to multiple informed traders

- The solution for k traders can be shown to be:
  $$\lambda_k = \frac{1}{k+1}\,\frac{\sigma\,\sqrt{k}}{\sigma_u};\ b_k = \frac{\sigma_u}{\sigma\,\sqrt{k}}$$

- The greater the number k of informed traders, the lower the value of information each holds and so the smaller the b_k (recall that $y_i = a + b\,\tilde{S}$).

- Competition between informed traders drives down the optimal λ, decreasing market impact and increasing liquidity.
    - This contrasts with the piggybacking case explored in HW5 Problem 3.

Imperfect competition

- In the above, we assumed perfect competition between market makers so that $p = \mathbb{E}\left[\tilde{S}|y\right]$.

- On page 144 of [Foucault, Pagano and Röell][2], an alternative market mechanism is explored with imperfect competition between K dealers.
    - The kth market maker $\mathcal{M}_k$ must taken into account the strategic behavior of the other market markers when setting a price schedule.
    - The price $p^\star$ thus depends in particular on the number K of competing market makers.

The cost of imperfect competition

- With this market mechanism and with K market makers, each $\mathcal{M}_k$ posts less quantity than that in the case of perfect competition, by a factor
  $$\frac{K-2}{K-1}.$$

- Relative to the case of perfect competition, market impact with K market makers is given by
  $$\lambda_K = \lambda\,\frac{K-1}{K-2} = \lambda + \frac{\lambda}{K-2}$$

  so relative to the perfect competition case, the Kyle lambda increases by the factor $(K-1)/(K-2)$.

- As $K \to \infty$, we retrieve the perfect competition case.

- We also see that the cost of trading depends on market structure.
  - Different auction mechanisms lead to different market maker rents.

Extension to multiple trading periods

Suppose there are N auctions at intervals $\Delta t = 1/N$ and indexed by $t \in \{1, \ldots, N\}$.

The equilibrium has the following properties:

- $\mathcal{I}$'s demand is $\Delta y_{i_t} = b_t \left(\tilde{S} - p_{t-1} \right) \Delta t$ where b_t is trading intensity at time t.

- The price change at the tth auction is given by

$$\Delta p_t = \lambda_t \left(\Delta y_{i_t} + \Delta y_{u_t} \right)$$

- Market efficiency gives

$$p_t = \mathbb{E}[\tilde{S} | y_t]$$

where y_t is cumulative order flow up to time t.

Properties of the solution

- $\mathcal{I}$ slices and dices his order flow, just like real traders splitting orders.

- $\mathcal{I}$'s optimal trade sizes change over time with the random uninformed demand y_u.
  - $\mathcal{I}$ hides behind the uninformed order flow in such a way that $\mathcal{M}$ cannot predict on the basis of net order flow what $\mathcal{I}$ will do next.

- There is no autocorrelation in the total order flow y. If there were, it would induce autocorrelation in prices and that is ruled out by market efficiency.
  - Note that this is unlike sequential trader models like Glosten and Milgrom where there is autocorrelation of order flow.

- In the limit $N \to \infty$, the volatility of the price process tends to a constant.

More properties of the solution

- As time progresses, information is progressively revealed.
- The Kyle model shows how information can be incorporated into prices.
- In the Kyle model, the process of price formation is explicit.
- In the perfect competition version, market makers get zero profits, informed traders make profits and liquidity traders face transaction costs.

Price manipulation

- One might wonder whether traders could manipulate the market price.
  - A potential strategy would be to submit a large buy order, driving the price up then selling back slowly at newer higher prices.

- [Huberman and Stanzl][4] showed manipulation is not possible with the linear market impact function in the Kyle model and in fact that *only* linear models of this type have this property.

- We will revisit price manipulation later in this lecture series.

Correlation of order flow

- In real markets, order flow is in fact highly autocorrelated.

- However, to a very accurate approximation, price changes are unpredictable.

- In subsequent lectures we will try to understand how this can be so.

The econophysics critique of Kyle

Amongst other things, [Bouchaud, Farmer, Lillo][1] complain that:

> The rational investors, in contrast, are assumed to correspond to skilled professionals. Their trading is perfect, in the sense that they know everything. Examples of what they must know includes the strategies of all the noise traders, and the fraction of capital traded by noise traders as opposed to rational investors. In such models prices can deviate from fundamental values due to the action of the noise traders and the desire of the rational agents to exploit them as much as possible, but the rational agents always keep them from deviating too much.

The following simulations suggest that perhaps all an agent needs to do to be considered as acting rationally, is to perform linear regression.

Kyle model simulation with Bayesian learning

Simulation pseudo code

- $\mathcal{M}$ assumes some prior λ with some (low) precision

- $\mathcal{I}$ determines his optimal demand y_i.

- The true value $\tilde{S}$ is revealed.

- $\mathcal{M}$ regresses $\tilde{S}$ against y to get a new observation of λ and computes its precision.

- This new observation is combined with the prior to generate a new posterior estimate of λ.

$$\lambda_2 = \frac{\tau_0\,\lambda_0 + \tau_1\,\lambda_1}{\tau_0 + \tau_1}.$$

```
In [1]: sigu <- .5
        sig <- .5
        mu <- 1
        nTrials <- 1

        #Initialize vectors
        numRuns <- 2000
        sTildeVec <- numeric(numRuns)
        yVec <- numeric(numRuns)
        lamVec <- numeric(numRuns)
```

- Suppose the market maker $\mathcal{M}$'s prior is such that he sets the Kyle lambda to 1/3.

```
In [2]: lam0 <- 1/3
        #lam0 <- .1
        sd0 <- 1 # Std error of prior
        tau0 <-.01 # Precision of prior

        lamK <- lam0
        lam2 <- 0 # initialize lambda_2
```

```
In [3]: for(count in 1:numRuns){

            # Generate a true value
            sTilde <- rnorm(nTrials, mean=mu, sd=sig)

            # Generate uninformed demand
            yu <- rnorm(nTrials, mean=0, sd=sigu)

            # Generate informed demand
            yi <- (sTilde - mu)/(2*lamK)

            # Get updated value of lamK from regression
            y <- yi + yu
            sTildeVec[count] <- sTilde
            yVec[count] <- y
            lamVec[count] <- lamK
            if(count > 2) {
                tmp <- coef(summary(lm(sTildeVec[1:count] ~ yVec[1:count])))
                lam1 <- tmp[2, 1]
                stderr <- tmp[2, 2]
                tau1 <- 1/stderr^2
                lam2 <- (lam0*tau0 + lam1*tau1)/(tau0 + tau1)
                if (lam2 > .001) {lamK <- lam2}
            }
            lamVec[count] <- lamK
        }
```
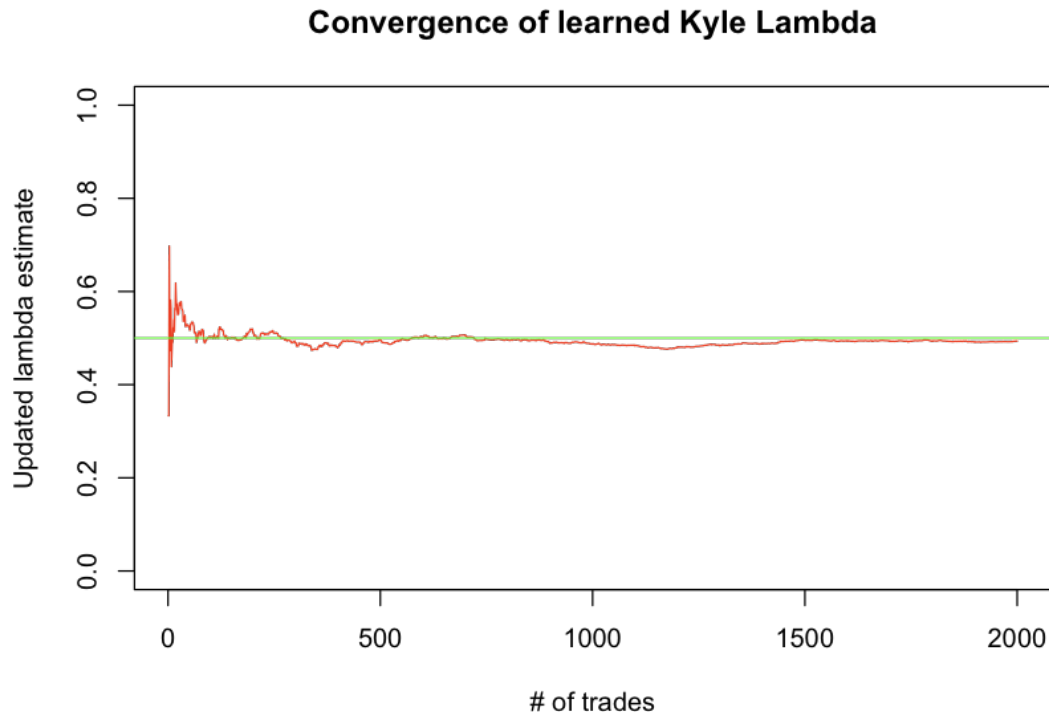
Convergence plot

```
In [4]: library(repr)
        options(repr.plot.height=5)
```

```
In [5]: plot(lamVec, type="l", ylim=c(0,1), col="red", ylab="Updated lambda estimate",
             xlab="# of trades", main="Convergence of learned Kyle Lambda")
        abline(h=.5, col="green")
```

Convergence of learned Kyle Lambda

Convergence to equilibrium

- The question then arises as to whether this always works: If agents just perform linear regressions, does the system converge?

- The answer according to [Marcet and Sargent][5] is that there is a rational expectations equilibrium under certain conditions.

- *Rational expectations* should then be understood in this sense: that agents updating their beliefs by regressing the results of their actions (reactions) against price changes may (sometimes) cause equilibrium to emerge.
  - In that case, the system behaves as if the agents are rational; capable of computing the rational reaction of other agents to their rational actions.

- In this light, the above critique of the econophysicists appears naïve.

Inventory models vs asymmetric information models

- Studies of market makers inventory suggest that the effect of inventory on price is negligible.

- This has led market microstructure theorists to reject inventory models in favor of asymmetric information models.

- However, this seems to be an overly restrictive view of inventory models.
  - Some portion of the activity of most market participants could be classed as market-making. For example, the simple choice to send a limit order rather than a market order.
  - Ultimately, there is almost always a market maker who will accept new inventory in return for spread.
- Taking this view, cumulative order flow is inventory and everyone is a market maker.

Models and algorithmic trading

- With the advent of algorithmic trading, models take on a new importance - as inspiration for algorithms rather than as guides to market design.
  - Managing inventory is key!
- Thus we see a new literature starting with Avellaneda and Stoikov (2008) that focuses on inventory.
- Adverse selection (asymmetric information) and inventory are both very important.

Final remarks

- In economics, the rôle of prices is not just to allocate resources efficiently but also to transmit information about the values of assets.
- The Kyle model exhibits a mechanism through which information may be impounded into market prices.
  - Note however that the market price can depart very substantially from fair value if there is large uninformed demand.
  - If fair value is itself evolving dynamically, the market price may never correspond to fair value!

References

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