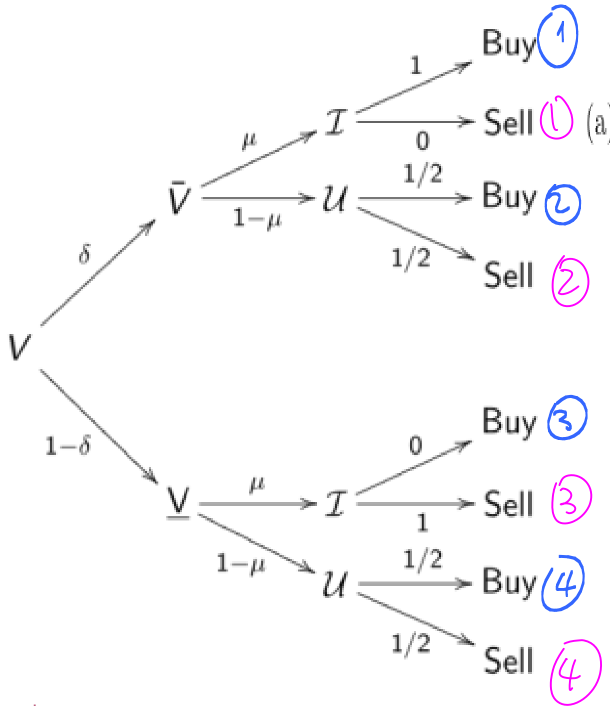


(a) Determine the unconditional probabilities of buy and sell orders respectively in terms of the model parameters.

$$\mathbb{P}[\text{Buy}) = \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4)$$

$$= \mu \cdot \delta \cdot 1 + \delta(1-\mu)\cdot\frac{1}{2} + 0 + (1-\delta)(1-\mu)\cdot\frac{1}{2}$$

$$= \mu\delta + \frac{1}{2}(1-\mu)$$

$$\mathbb{P}[\text{sell}] = \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4)$$

$$= 0 + \delta(1-\mu)\frac{1}{2} + (1-\delta)\mu + (1-\delta)(1-\mu)\frac{1}{2}$$

$$= \mu(1-\delta) + \frac{1}{2}(1-\mu)$$

(b) Under perfect competition, $\mathcal{M}$ sets his ask A and bid B as

$$A = \mathbb{E}[V|\text{Buy}], \quad B = \mathbb{E}[V|\text{Sell}].$$

Determine A and B in terms of model parameters.

$$A = \mathbb{E}[V|\text{Buy}] = \bar{V}\,\mathbb{P}(V=\bar{V}\,|\,\text{Buy}) + \underline{V}\,\mathbb{P}(V=\underline{V}\,|\,\text{Buy})$$

$$= \bar{V} \cdot \frac{\overset{\alpha}{\mathbb{P}(V=\bar{V}, \text{Buy})}}{\mathbb{P}(\text{Buy})} + \underline{V} \, \frac{\overset{\beta}{\mathbb{P}(V=\underline{V}, \text{Buy})}}{\mathbb{P}(\text{Buy})}$$

$$\alpha = \mathbb{P}(①) + \mathbb{P}(②) = \mu \cdot \delta \cdot 1 + \delta(1-\mu) \cdot \frac{1}{2}$$

$$\beta = \mathbb{P}(③) + \mathbb{P}(④) = 0 + (1-\delta)(1-\mu) \cdot \frac{1}{2}$$

$$= \bar{V} \cdot \frac{\mu\delta + \frac{1}{2}\delta(1-\mu)}{\mu\delta + \frac{1}{2}(1-\mu)} + \underline{V} \, \frac{(1-\delta)(1-\mu)\frac{1}{2}}{\mu\delta + \frac{1}{2}(1-\mu)}$$

$$= \frac{\bar{V}\delta(1+\mu) + \underline{V}(1-\delta)(1-\mu)}{2\mu\delta + (1-\mu)}$$

$$B = \frac{\bar{V}\delta(1-\mu) + \underline{V}(1-\delta)(1+\mu)}{2\mu(1-\delta) + 1 - \mu}$$

(c) Are the bid B and ask A prices set symmetrically around the efficient price $\mathbb{E}[V]$?

$$\mathbb{E}(V) = \delta\bar{V} + (1-\delta)\underline{V}$$

No, it's not, but you need to do some algebra.

(d) Determine the spread $s = A - B$ in terms of the model parameters. Comment on the case where $\delta = \frac{1}{2}$.

$$S = A - B$$

$$= \frac{\bar{V}\delta(1+\mu) + \underline{V}(1-\delta)(1-\mu)}{2\mu\delta + 1-\mu} - \frac{\bar{V}\delta(1-\mu) + \underline{V}(1-\delta)(1+\mu)}{2\mu(1-\delta) + 1-\mu}$$

when $\delta = 1/2$

$$A = \frac{1}{2}\left[(1+\mu)\bar{V} + \underline{V}(1-\mu)\right], \quad B = \frac{1}{2}\left[\bar{V}(1-\mu) + \underline{V}(1+\mu)\right]$$

$$S = \mu(\bar{V} - \underline{V})$$

(e) What happens in the limit as $\mu \to 1$, i.e., the limiting case where no uninformed traders exist? ~~Let $\delta = 1/2$~~

as $\mu \to 1 \Rightarrow A \to \bar{V}, \quad B \to \underline{V}$

$$S \to \bar{V} - \underline{V}$$

$$\mathbb{E}[S] \approx M + \frac{2s}{\pi} \arctan I, \quad {/\!/ \mu} \qquad s = A - B, \ \varepsilon \qquad S \sim N\left(M + \frac{2s}{\pi} \arctan I, \ \sigma^2\right)$$

Assume that S is normally distributed with variance σ^2, that the dealer has exponential utility $u(x) = -e^{-\alpha x}$ with $\alpha > 0$, and that his current inventory is q.

(a) As in the Stoll model, show that the indifference size n_B the dealer should quote at the bid B is given by

$$n_B = \frac{s}{\alpha \sigma^2}\left(1 + \frac{4}{\pi} \arctan I\right) - 2q.$$

no trade expected utility = expected utility with trade

$$\mathbb{E}(q\, S) - \frac{\alpha}{2} V(q\, S) = \mathbb{E}\left[(q+n)S - n B\right] - \frac{\alpha}{2} V\left((q+n) S\right)$$

$$\Rightarrow q \cdot \mu - \frac{\alpha}{2} q^2 \sigma^2 = (q+n)\mu - nB - \frac{\alpha}{2}(q+n)^2 \sigma^2$$

$$\Rightarrow \cdots \Rightarrow n = n_B = \frac{s}{\alpha \sigma^2}\left(1 + \frac{4}{\pi} \arctan I\right) - 2q.$$

(b) Derive a similar formula for the indifference offered quantity n_A.

$$\mathbb{E}(q\, S) - \frac{\alpha}{2} V(q\, S) = \mathbb{E}\left((q-n)S + nA\right) - \frac{\alpha}{2} V\left((q-n)S\right)$$

$$\Rightarrow q\mu - \frac{\alpha}{2} q^2 \sigma^2 = (q-n)\mu + nA - \frac{\alpha}{2}(q-n)^2\sigma^2$$

$$\Rightarrow \cdots\cdots \Rightarrow n_A = \frac{s}{\alpha\sigma^2}\left(-1 + \frac{4}{\pi}\arctan I\right) + 2q$$

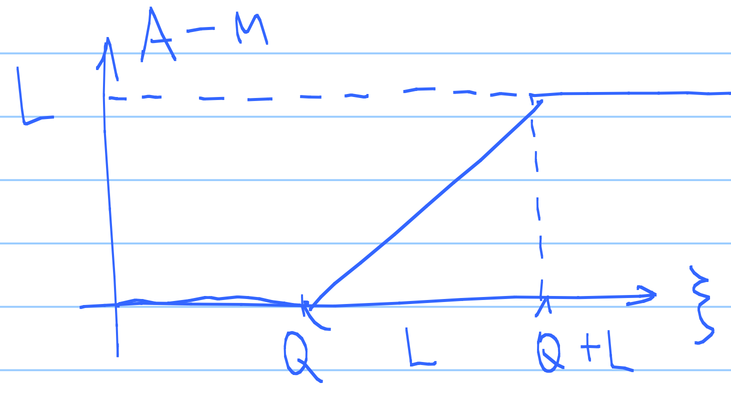
up to 85% confi.

(c) Give your intuition for the sensitivities of n_B and n_A to each of the various inputs.

SAY something if you see something.

(3) In the Cont-Kukanov setting, the trader is facing the problem of submitting in total X orders to a single venue. He has to decide between a market buy of M orders and a limit buy of L orders. Assuming it is not optimal to submit either above or blow his target orders, i.e., $X = L + M$. Hence, $M = X - L$. There are currently Q orders in the limit order queue. Assume the make rebate is r, take fee is f, and half-spread is h. Let ξ be the (random) outflow of limit orders (from market sell order fills and cancelations). X will be considered as a fixed constant hereafter.

(a) Determine the executed quantity $\mathcal{A}(L, \xi)$ as a function of L, and ξ.



$$A(L,\xi) = (\xi - Q)^+ - (\xi - Q - L)^+ + M$$

spread/2 take fee

(b) Explain why the cost relative to mid-quote is given by $\mathcal{C} = (h+f)M - (h+r)(\mathcal{A}(L,\xi) - M)$.

rebate spread/2

(c) The penalty of under fills relative to target quantity X is defined as $\mathcal{P} = \lambda(X - \mathcal{A}(L,\xi))$, where λ is a positive constant. The trader's optimal decision between market and limit orders is determined by minimizing the expected total cost $\mathbb{E}[\mathcal{C}+\mathcal{P}]$. Show that the optimal decision (L^*, M^*) is given by

$$L^* = F_\xi^{-1}\left(\frac{2h+f+r}{\lambda+h+r}\right) - Q, \qquad M^* = X - L^*,$$

where $F_\xi(\cdot)$ is the cumulative distribution function of the random variable ξ.

$$\frac{d}{dL}\mathbb{E}(C+P) = (h+f)M - (h+r)\left(A(L,\xi) - M\right)$$

$$+ \lambda\left(X - \underbrace{A(L,\xi)}_{(\xi-Q)^+ + (\xi-Q-L)^+ - M}\right)$$

$$0 = (h+f)(X-L) - (h+r)\mathbb{E}\left[(\xi-Q)^+ - (\xi-Q-L)^+\right]$$

$$+ \lambda\mathbb{E}\left[L - (\xi-Q)^+ + (\xi-Q-L)^+\right]$$

$$\Rightarrow 0 = -(h+f) + (h+r)\frac{d}{dL}\mathbb{E}(\xi-Q-L)^+$$

$$+\lambda + \lambda\frac{d}{dL}\mathbb{E}(\xi-Q-L)^+$$

$$\Rightarrow \quad h + f - \lambda = (h + r + \lambda) \frac{d}{dL} \mathbb{E}(\xi - Q - L)^+$$

$$\Rightarrow \quad h + f - \lambda = (h + r + \lambda)\left(F_\xi(Q+L) - 1\right)$$

$$\Rightarrow \quad 2h + r + f = (h + r + \lambda) F_\xi(Q+L)$$

$$\Rightarrow F_\xi^{-1}\left(\frac{2h + r + f}{h + r + \lambda}\right) - Q = L$$

$$F_\xi(x) = \mathbb{P}[\xi \le x]$$

$$S_\xi(x) = \mathbb{P}[\xi > x] = 1 - F_\xi(x)$$

survival function

POE (probability of excedance)

(d) As X increases, the optimal limit orders L^* increases or decreases? How about M^*?

No, L doesn't depend on X

(5) Almgren-Chriss model assumes that the stock price S_t evolves as

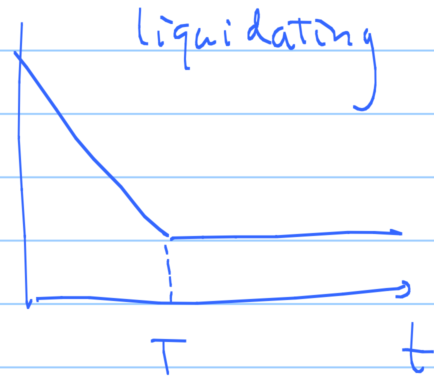
$$dS_t = \gamma dx_t + \sigma\, dZ_t \longrightarrow S_t - S_0 = \gamma(x_t - x_0) + \sigma Z_t$$

and the price $\tilde{S}_t$ at which we transact is given by

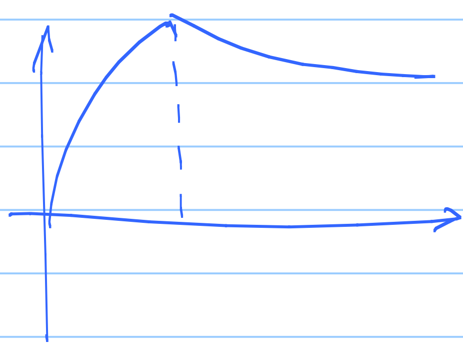
$$\tilde{S}_t = S_t - \eta\, v_t,$$

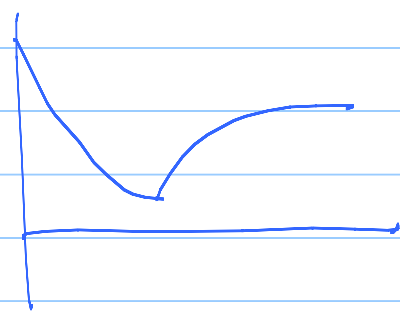
where $v_t := -\dot{x}_t$ is the rate of trading with $x_0 = X$ and $x_T = 0$. γ, σ, and η are positive constants.

(a) Plot the price trajectory in the Almgren-Chriss model during a VWAP execution. Is the expected price trajectory consistent with the empirical observation? Why or why not?



No,





(b) Almgren-Chriss determined the optimal strategy by minimizing the following expected risk adjusted cost

$$\mathbb{E}\left[\int_0^T (\tilde{S}_t - S_0)dx_t + \lambda\sigma^2 \int_0^T x_t^2 dt\right].$$

Assume x_t is of finite variation, apply the technique of integration by parts to show that the problem of minimizing expected cost reduces to the variational problem

$$\min_x \left\{\eta \int_0^T \dot{x}_t^2 dt + \lambda\sigma^2 \int_0^T x_t^2 dt\right\}$$

subject to $x_0 = X$ and $x_T = 0$.

$$S_t - S_0 = \gamma(x_t - x_0) + \sigma Z_t$$

$$\Rightarrow \mathbb{E}\int_0^T (\tilde{S}_t - S_0)\,dx_t = \int_0^T \left(\gamma(x_t - x_0) + \sigma Z_t - \eta v_t\right)dx_t$$

$$= \frac{\gamma}{2}(x_T - x_0)^2 + \sigma\int_0^T Z_t\,dx_t + \eta\int_0^T v_t^2\,dt$$

$$= \frac{\gamma}{2}X^2 - \sigma\,\mathbb{E}\int_0^T \cancel{x_t\,dZ_t}^{\,0} + \sigma\,\cancel{(Z_t x_t)}^{\,0}\Big|_0^T + \eta\int_0^T v_t^2\,dt$$

$$\therefore\ \mathbb{E}\left[\int_0^T (\tilde{S}_t - S_0)\,dx_t\right] = \frac{\gamma}{2}X^2 + \eta\int_0^T \mathbb{E}(v_t^2)\,dt$$

$\Rightarrow$ The expected risk adjusted cost is

$$\frac{\gamma}{2}\cancel{X^2} + \eta\int_0^T \dot{x}_t^2\,dt + \lambda\sigma^2\int_0^T x_t^2\,dt$$

(c) Recall that, associated with the variation problem

$$\min_x \left\{ \int_0^T L(t, x_t, \dot{x}_t) dt \right\},$$

the Euler-Lagrange equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Determine the Almgren-Chriss strategy by solving the Euler-Lagrange equation associated with the variational problem obtained in (b), taking into account the boundary conditions $x_0 = X$ and $x_T = 0$.

In our case, $L = \eta \dot{x}_t^2 + \lambda \sigma^2 x_t^2$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(2\eta \dot{x}_t) - 2\lambda\sigma^2 x_t = 0$$

$$\Rightarrow \quad \ddot{x}_t = \frac{\lambda\sigma^2}{\eta} x_t \qquad \frac{\lambda\sigma^2}{\eta} = K^2$$

$$\Rightarrow \quad x_t = X \frac{\sinh(K(T-t))}{\sinh(KT)}.$$

(6) In the Gatheral-Schied model, the stock price process is assumed

$$\frac{dS_t}{S_t} = \sigma dW_t,$$

$$dx_t = -v_t dt.$$

The optimal strategy is determined by the minimizer of the problem:

$$\min_{v \in \mathcal{G}[0,T]} \mathbb{E}\left[\int_0^T v_t^2 dt + \lambda \int_0^T S_t x_t dt\right].$$

where $\mathcal{G}$ is the set of admissible strategies. Define the value function C as

$$C(t,x,s) = \min_{v \in \mathcal{G}[t,T]} \mathbb{E}\left[\int_t^T v_\tau^2 d\tau + \lambda \int_t^T S_\tau x_\tau d\tau \middle| X_t = x, S_t = s\right].$$

(a) What is the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function $C(t,x,s)$?

$$C_t + \min_v \left\{ \frac{\sigma^2 s^2}{2} C_{ss} - v\, C_x + v^2 + \lambda s x \right\} = 0$$

$$\Rightarrow C_t + \frac{\sigma^2 s^2}{2} C_{ss} + \lambda s x + \underbrace{\min_v \left(v^2 - v\, C_x \right)}_{v = \frac{C_x}{2}} = 0$$

$$\Rightarrow C_t + \frac{\sigma^2 s^2}{2} C_{ss} + \lambda s x - \frac{C_x^2}{4} = 0$$

(b) By applying the first order criterion, determine the optimal trading rate v_t^* at any time $t \in [0,T]$ in terms of the value function $C(t,x,s)$. $v^* = \frac{C_x}{2}$

(c) Imposing the ansatz $C(t,x,s) = \alpha(t)s^2 + \beta(t)sx + \gamma(t)x^2$, derive a system of ODEs satisfied by α, β, and γ from the HJB equation obtained in (a).

$$C_t + \frac{\sigma^2 s^2}{2} C_{ss} + \lambda s x - \frac{C_x^2}{4} = 0$$

$$\alpha' s^2 + \beta' s x + \gamma' x^2$$

$$+ \frac{\sigma^2 s^2}{2} \cdot 2\alpha + \lambda s x - \frac{1}{4} \cdot (\beta s + 2\gamma x)^2 = 0$$

$$s^2 : \quad \alpha' + \sigma^2 \alpha - \frac{\beta^2}{4} = 0$$

$$sx : \quad \beta' + \lambda - \gamma \beta = 0$$

$$x^2 : \quad \gamma' - \gamma^2 = 0$$

(d) Verify that

$$\alpha(t) = \frac{\lambda^2}{8\sigma^6}\left[1 - e^{\sigma^2(T-t)} + \sigma^2(T-t) + \frac{1}{2}\sigma^4(T-t)^2\right],$$

$$\beta(t) = \frac{\lambda}{2}(T-t),$$

$$\gamma(t) = \frac{1}{T-t}.$$

is the solution to the system of ODEs obtained in (c) with the terminal condition

$$\lim_{t \uparrow T} C(t,x,s) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

(e) Determine the optimal trading strategy v_t^* explicitly in terms of t, s_t, and x_t (and of course λ and T). Does the optimal trading strategy depend on stock price S_t during execution?

$C = \alpha s^2 + \beta s x + \gamma x^2$

$v = \frac{C_x}{2} = \frac{\beta}{2} s + \gamma x$

Yes, it is.

(4) Consider the price process

$$S_t = S_0 + \int_0^t f(\dot{x}_s)\, G(t-s)\, ds + \sigma Z_t, \quad 0 \le t \le T.$$

where $\dot{x}_s$ is the rate of trading in dollars at time $s < t$, $f(\dot{x}_s)$ represents the impact of trading at time s and $G(t-s)$ is a decay factor. The cost of trading $C[x]$ corresponding to the trading strategy x_t is given by

$$C[x] = \int_0^T (S_t - S_0) dx_t.$$

(a) With the choices

$\mathrm{sgn}(v)\sqrt{|v|}$

$$f(v) = \sqrt{v}, \quad G(\tau) = \frac{1}{1+\tau},$$

derive an expression for the expected cost of accumulating shares at the rate v_1 and liquidating them at the rate v_2, assuming $x_0 = x_T = 0$.

$$\int_0^T (S_t - S_0)\,dX_t = \int_0^T \left[\int_0^t \frac{\sqrt{V_s}}{1+t-s}\,ds + \sigma Z_t\right] dX_t$$

$\operatorname{sgn}(V_s)\sqrt{|V_s|}$

$$= \int_0^T\int_0^t \frac{\sqrt{V_s}}{1+t-s}\,V_t\,ds\,dt + \sigma\int_0^T Z_t\,dX_t$$

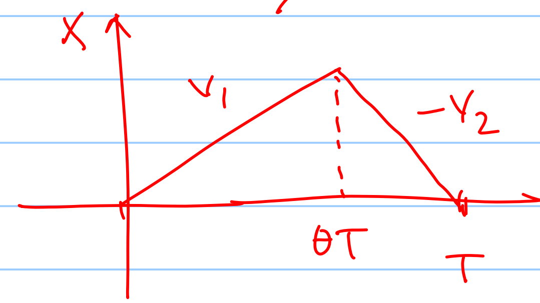
$$+ \sigma Z_t X_t\Big|_0^T - \sigma\int_0^T X_t\,dZ_t$$

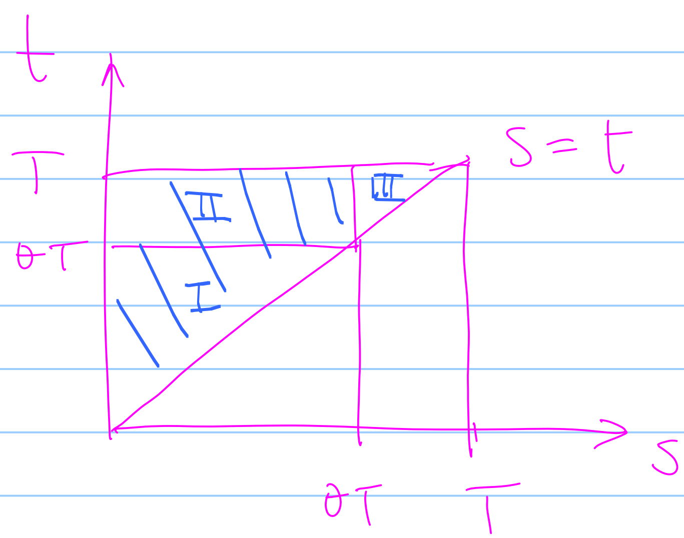
$\operatorname{sgn}(V_s)\sqrt{|V_s|}$

$$\therefore\ \mathbb{E}(C) = \mathbb{E}\int_0^T\int_0^t \frac{\sqrt{V_s}}{1+t-s}\,V_t\,ds\,dt + \sigma\,\mathbb{E}(\cancel{X_T}^{\,0} Z_T)$$

$X_T = 0$

$$V_t = \begin{cases} V_1 & \text{if } 0 \le t \le \theta T \\ -V_2 & \theta T \le t \le T \end{cases}$$





$$\int_0^T\int_0^t (\cdots)\,ds\,dt$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

$$= \int_0^{\theta T}\int_0^t \cdots\,ds\,dt$$

$$+ \int_{\theta T}^T\int_0^{\theta T} \cdots\,ds\,dt + \int_{\theta T}^T\int_{\theta T}^t \cdots\,ds\,dt$$

$$= \underbrace{\int_0^{\theta T}\int_0^t \frac{\sqrt{V_1}\cdot V_1}{1+t-s}\,ds\,dt}_{\mathrm{I}} + \underbrace{\int_{\theta T}^{T}\int_0^{\theta T} \frac{\sqrt{V_1}\cdot(-V_2)}{1+t-s}\,ds\,dt}_{\mathrm{II}}$$

$$+ \underbrace{\int_{\theta T}^{T}\int_{\theta T}^{t} \frac{-\sqrt{V_2}\,(-V_2)}{1+t-s}\,ds\,dt}_{\mathrm{III}}$$

(b) What is a price manipulation? Googling is NOT allowed.

(c) With $T = 1$, does the choice of f and G in (a) admit price manipulation? Prove or disprove your answer. Apply the Theorem.