MTH 9879 Market Microstructure Models, Spring 2019

Lecture 6: Variance and covariance estimation

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Outline of Lecture 6

- Uses of volatility estimates
- Market microstructure bias
- A survey of estimation and forecasting algorithms
- Experimental results

Motivation

- We often need to estimate in-sample volatility. Typically we need a good volatility estimate to reduce errors in estimating parameters of:
  - Market impact models
  - Limit order fill models
  - Market making: Avellaneda and Stoikov (Lecture 3) and other such algorithms need volatility forecasts.

- Volatility forecasting models
  - Without good in-sample volatility estimates, how can we even assess the quality of volatility forecasts?
  - Also, it has been shown that the performance of volatility forecasting models may be improved if they use better estimates of realized variance.

Uses of volatility forecasts

- Option valuation
- Risk estimation
- Order fill probability
- For the first two of these, we need to estimate the width of the distribution of relatively long-timescale returns.

- For the order fill probability, we need a volatility to estimate the first passage time density. Given that the underlying stochastic process is not Brownian motion, there is no a priori reason why the volatility numbers required for these quite different computations should be the same.
- We will focus on measuring the width of the distribution of returns.

Choice of sampling scheme

It is important to sample in transaction time rather than in business time or calendar time.

- In transaction time, empirically observed returns (price change) from trade data are consistent with MA(1) (the Roll model).
  - Autocorrelation coefficients are insignificant after one lag.
- In calendar time, with varying intensity, empirically observed returns are ARIMA.

TAQ data cleaning

Data cleaning is critical. In Lecture 4, we already gave the Barndorff-Nielsen cleaning recipe implemented in the R-package `highfrequency`:

Integrated variance or quadratic variation

Given a set of tick data, how can we measure the, say daily, variance?
A possibility is to estimate the *integrated variance*, also known as the *quadratic variation* in the theory of semimartingale. We shall use both terms interchangeably hereafter.

Recall that the *quadratic variation* $\langle X \rangle_t$ of the continuous stochastic process X_t is defined by

$$\langle X \rangle_T := \lim_{\|\Pi_n\| \to 0} \sum_{t_i \in \Pi_n} |\Delta X_{t_i}|^2$$

provided the limit exist (in probability).

Thus, the goal is to estimate the quadratic variation of the efficient log price from the transacted log price, i.e., tick data. However, the subtlety is that efficient price is not directly observable and is contaminated by market microstructure noises.

Note

- If the process X has jumps, the quadratic variation $\langle X \rangle$ becomes

  $$\langle X \rangle_t = \langle X^c \rangle_t + \sum_{0 < s \leq t} |\Delta X_s|^2,$$

  where X^c denotes the continuous part of X and $\Delta X_s := X_s - X_{s^-}$ is the jump size at time s. In this case, the integrated variance usually refers to $\langle X^c \rangle$, i.e., the quadratic variation of the continuous part.
- We shall always assume X is a continuous process, thus no jumps, in the sequel.

Realized Variance

How to estimate the integrated variance from a given set of tick data? The naïve answer would be to compute the statistic

$$\sum_{i=1}^{n} \left(Y_{t_i} - Y_{t_{i-1}} \right)^2 = \sum_{i=1}^{n} \left(\Delta Y_{t_i} \right)^2,$$

where $Y_t = \log S_t$ and S_t are successive prices in the dataset.

- This estimator is called the *Realized Variance (RV)* estimator

Notations

- Realized variance and realized covariance
  Given a partition $\Pi = \{0 = t_1 < \cdots < t_n = T\}$ of the interval $[0, T]$, the realized variance $[X]_T^\Pi$ of the process X_t sampled at Π is defined by
  $$[X]_T^\Pi = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2.$$
  Similiarly, the realized covariance between X_t and Y_t sampled at Π is given by
  $$[X, Y]_T^\Pi = \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$
- Quadratic variation (integrated variance) and covariation
  The quadratic variation of X is defined by the limit
  $$\langle X \rangle_t = \lim_{\|\Pi_n\| \to 0} [X]_T^{\Pi_n}$$
  provided the limit exists. Π_n denotes a sequence of partitions of the interval $[0, T]$ such that $\|\Pi_n\| \to 0$ as $n \to \infty$, where $\|\Pi_n\|$ denotes the mesh of the partition Π_n.

  Likewise, the covariation between and X and Y is defined by the limit
  $$\langle X, Y \rangle_t = \lim_{\|\Pi_n\| \to 0} [X, Y]_T^{\Pi_n}.$$

Assumption

The log price X_t follows the Ito process
$$dX_t = \mu_t dt + \sigma_t dW_t,$$
where W_t is a Brownian motion. Under the assumption, $\langle X \rangle_t = \int_0^t \sigma_\tau^2 d\tau$.

An exact relation

- Let $X_t := \log S_t$ be the log-price and $r_t := \Delta X_t = X_t - X_{t-1}$ the log-return.
- The following relation is exact:
  $$(X_T - X_0)^2 = \sum_{i=1}^{T} r_i^2 + 2 \sum_{k=1}^{T-1} \gamma(k),$$
  where $\gamma(k) = \sum_{i=1}^{T-k} r_i \, r_{i+k}$ is the kth realized autocovariance.
- So if returns r_t are serially uncorrelated then an unbiased and efficient estimate of the daily (realized) variance can be obtained as the sum of squared intra-day returns (RV).

Note

Should X_t be a martingale, ΔX_i and ΔX_j will be uncorrelated for $i \neq j$.

- However, intra-day returns sampled at the highest frequency will generally exhibit serial correlation thereby invalidating RV as a reliable variance estimator.
- The estimators we discuss below are all motivated by this reasoning and aim to provide improved measures of the realized variance.

Microstructure noise

In the limit of very high sampling frequency, RV picks up mainly market microstructure noise. To see this, suppose that the observed price Y_t is given by

$$Y_t = X_t + \epsilon_t,$$

where X_t is the value of the underlying (log-)price process of interest and ϵ_t is a random market microstructure-related noise term, assumed independent of X_t. Suppose we sample the price series $n+1$ times (so that there are n price changes) at $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ in the time interval $[0, T]$.

Note that, conditioned on $\mathcal{F}_T^X$, the conditional expectation of the realized variance of transacted (log) price satisties

$$\begin{aligned}
\mathbb{E}\left[[Y]_T^{\Pi} | \mathcal{F}_T^X\right] &:= \sum_{i=1}^{n} \mathbb{E}\left[(\Delta Y_{t_i})^2 | \mathcal{F}_T^X\right] \\
&= \sum_{i=1}^{n} (\Delta X_{t_i})^2 + 2\sum_{i=1}^{n} \Delta X_{t_i} \mathbb{E}\left[\Delta \epsilon_{t_i} | \mathcal{F}_T^X\right] + \sum_{i=1}^{n} \mathbb{E}\left[(\Delta \epsilon_{t_i})^2 | \mathcal{F}_T^X\right] \\
&= [X]_T + 2n \operatorname{var}[\epsilon] \\
&\approx \langle X \rangle_T + 2n \operatorname{var}[\epsilon].
\end{aligned}$$

Note

The difference between $[X]_T$ and $\langle X \rangle_T$ is referred to as the *discretization error*, which is usually controled by the integrated quarticity $\int_0^T \sigma_t^4 dt$.

Asymptotic result

A more detailed, but more technical, asymptotic analysis shown in [Zhang, Mykland and Aït-Sahalia][9] yields that as $n \to \infty$

$$[Y]_T^{\Pi} \overset{\mathcal{L}}{\approx} \langle X \rangle_T + 2\, n \operatorname{var}[\epsilon] + \sqrt{4n\mathbb{E}\left[\epsilon^4\right] + \frac{2T}{n} \int_0^T \sigma_t^4 dt}\; Z,$$

where $Z \sim N(0, 1)$.

Note

- The naive RV estimator $[Y]_T^{\Pi}$ is biased by the variance of market microstructure noise ϵ. The biasedness increases as the sampling frequency increases.
- We see that as $n \to \infty$, the naive RV estimator $[Y]_T^{\Pi}$ picks up mainly the microstructure noise.

BAC trades

In [89]:
```
download.file(url="https://mfe.baruch.cuny.edu/wp-content/uploads/2018/02/tqDataB
AC_20170919.zip", destfile="tq.zip")
unzip(zipfile="tq.zip")
download.file(url="https://mfe.baruch.cuny.edu/wp-content/uploads/2015/03/RvEstim
ators.R.zip", destfile="RvEstimators.R.zip")
unzip(zipfile="RvEstimators.R.zip")
```

In [2]:
```
load("tqDataBAC_20170919.rData")
Sys.setenv(TZ='EST')
tqBAC <- tqdata
```

In [4]:
```
library(highfrequency)
source("RvEstimators.R") # Code mostly due to Roel Oomen
```

In [5]:
```
# rescale the plot
options(repr.plot.height=5, repr.plot.width=10)
```

BAC data from 19-Sep-2017.

In [6]:
```
head(tqBAC[5000:5100, ], 20)
```

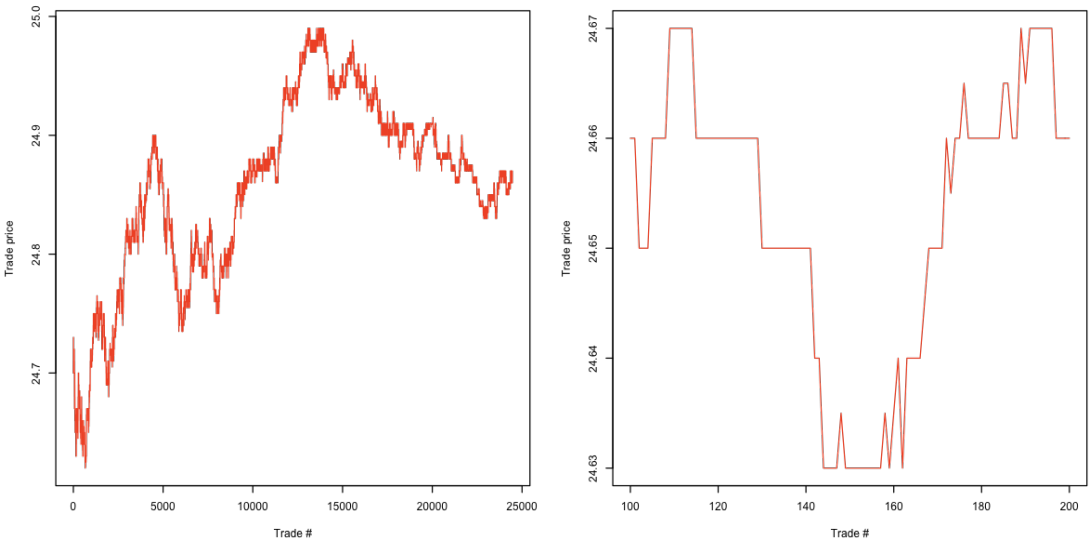
```
                     SYMBOL EX  PRICE   SIZE    COND BID     BIDSIZ OFR
2017-09-19 10:19:52 "BAC"  "K" "24.86" "100"   " "  "24.85" "1428" "24.86"
2017-09-19 10:19:54 "BAC"  "V" "24.86" "100"   " "  "24.85" "421"  "24.86"
2017-09-19 10:19:56 "BAC"  "B" "24.86" "400"   " "  "24.85" "5748" "24.86"
2017-09-19 10:19:56 "BAC"  "Z" "24.86" "100"   " "  "24.85" "394"  "24.86"
2017-09-19 10:19:56 "BAC"  "J" "24.86" "900"   " "  "24.85" "3953" "24.86"
2017-09-19 10:19:56 "BAC"  "Z" "24.86" "100"   " "  "24.85" "3953" "24.86"
2017-09-19 10:19:56 "BAC"  "Z" "24.86" "200"   " "  "24.85" "864"  "24.86"
2017-09-19 10:19:56 "BAC"  "Z" "24.86" "100"   " "  "24.85" "864"  "24.86"
2017-09-19 10:19:57 "BAC"  "V" "24.86" "100"   " "  "24.85" "419"  "24.86"
2017-09-19 10:19:58 "BAC"  "N" "24.86" "100"   " "  "24.85" "399"  "24.86"
2017-09-19 10:19:58 "BAC"  "Z" "24.86" "100"   " "  "24.85" "399"  "24.86"
2017-09-19 10:19:59 "BAC"  "Z" "24.86" "300"   " "  "24.85" "396"  "24.86"
2017-09-19 10:19:59 "BAC"  "Z" "24.86" "100"   " "  "24.85" "396"  "24.86"
2017-09-19 10:20:00 "BAC"  "Y" "24.85" "2440"  " "  "24.85" "5059" "24.86"
2017-09-19 10:20:00 "BAC"  "Y" "24.85" "100"   " "  "24.85" "1142" "24.86"
2017-09-19 10:20:00 "BAC"  "J" "24.85" "2600"  " "  "24.85" "1914" "24.86"
2017-09-19 10:20:00 "BAC"  "T" "24.85" "23529" " "  "24.85" "922"  "24.86"
2017-09-19 10:20:00 "BAC"  "Y" "24.85" "13117" " "  "24.84" "1686" "24.85"
2017-09-19 10:20:00 "BAC"  "Y" "24.85" "1876"  " "  "24.84" "4041" "24.85"
2017-09-19 10:20:00 "BAC"  "N" "24.85" "4133"  " "  "24.84" "2473" "24.85"
                     OFRSIZ
2017-09-19 10:19:52 "231"
2017-09-19 10:19:54 "89"
2017-09-19 10:19:56 "2280"
2017-09-19 10:19:56 "145"
2017-09-19 10:19:56 "969"
2017-09-19 10:19:56 "969"
2017-09-19 10:19:56 "144"
2017-09-19 10:19:56 "144"
2017-09-19 10:19:57 "80"
2017-09-19 10:19:58 "83"
2017-09-19 10:19:58 "83"
2017-09-19 10:19:59 "90"
2017-09-19 10:19:59 "90"
2017-09-19 10:20:00 "7381"
2017-09-19 10:20:00 "2065"
2017-09-19 10:20:00 "4353"
2017-09-19 10:20:00 "3457"
2017-09-19 10:20:00 "3801"
2017-09-19 10:20:00 "7177"
2017-09-19 10:20:00 "4315"
```

In [7]:
```
px <- tqBAC$PRICE
p <- as.numeric(px)

# Plot time series of trades
par(mar=c(5, 4, 1, 2) + 0.1)
par(mfrow=c(1,2), cex=0.6)
plot(p,col="red", type="l", ylab="Trade price", xlab="Trade #")

# Plot trades from 100 to 200
plot(100:200, p[100:200], col="red", type="l", ylab="Trade price", xlab="Trade #"
)

# restore plot setting
par(mfrow=c(1,1))
```



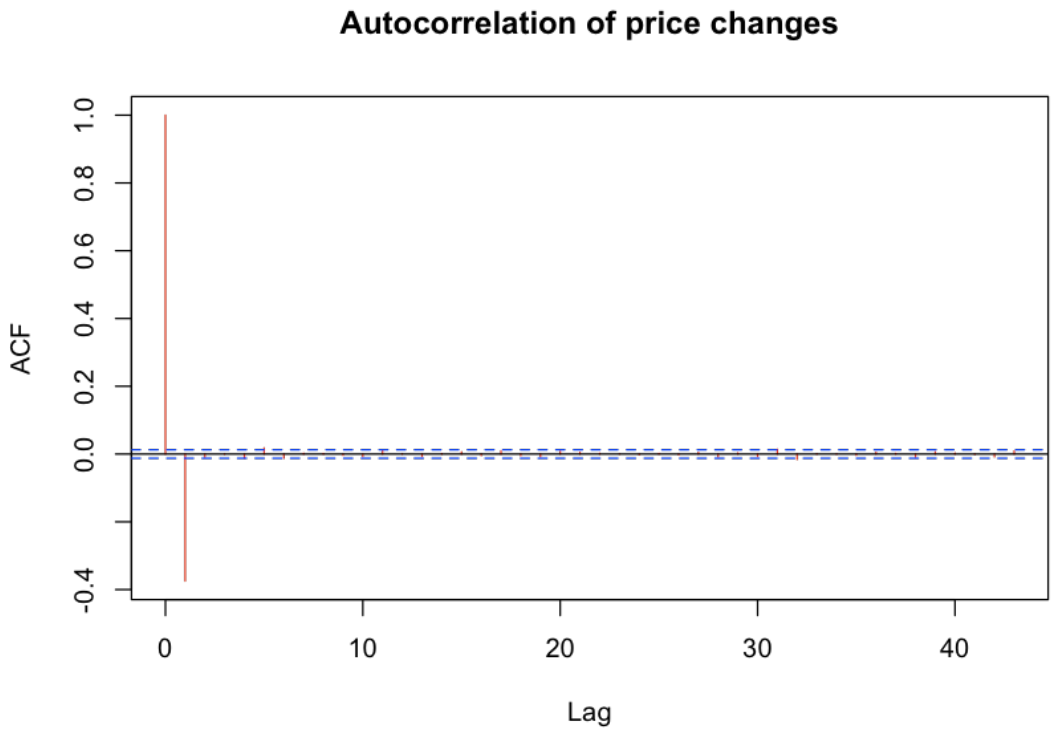
Autocorrelation of BAC trade price changes

In [8]:
```
options(repr.plot.width=7)
par(mar=c(5, 4, 4, 2) + 0.1, cex=0.8)
dp <- diff(log(p))
ac <- acf(dp, plot=FALSE)
ac[1:5]
```

```
Autocorrelations of series ‘dp’, by lag

     1      2      3      4      5
-0.375 -0.009 -0.001 -0.012  0.018
```

In [9]:
```
# autocorrelation of price changes
plot(ac, main="Autocorrelation of price changes", col="red")
```



BAC signature plot

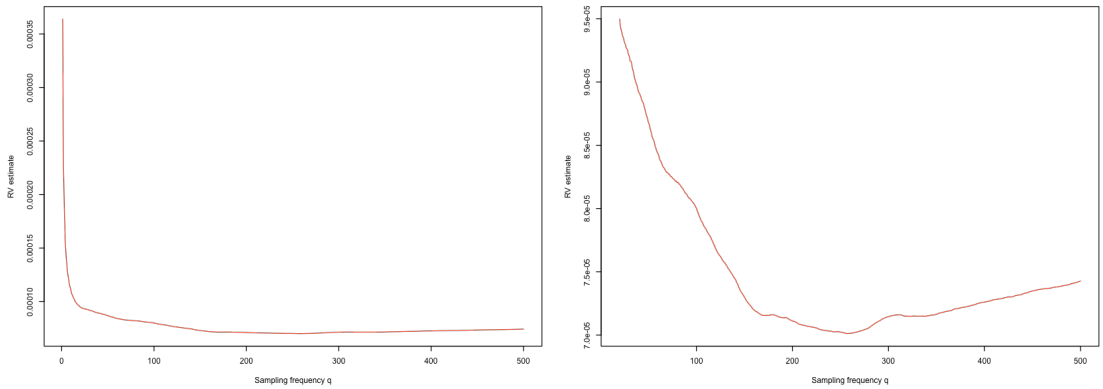
BAC data from 19-Sep-2017.

A *signature plot* is a plot of RV as a function of the sampling frequency q.

In [10]:
```
options(repr.plot.width=14)
nn <- 1000
rv <- sapply(1:nn, function(q){rCov(diff(log(p), lag=q))/q})
# The code above is in fact the boosted subsample RV estimator

# Signature plot
m <- 500
par(mfrow=c(1,2), cex=.6, mar=c(5,4,1,2)+.1)
plot(1:m, rv[1:m], col="red", type="l", ylab="RV estimate", xlab="Sampling freque
ncy q")

# zoom into 20:m
plot(20:m, rv[20:m], col="red", type="l", ylab="RV estimate", xlab="Sampling freq
uency q")
par(mfrow=c(1,1))
```



Oomen's noise-to-signal ratio

Following Oomen, we define the noise-to-signal ratio:

$$\xi = \frac{\text{var}[\epsilon]}{\sigma^2}.$$

ξ may be efficiently estimated using

$$\text{var}[\epsilon] \sim -\frac{1}{n-1}\sum_{i=1}^{n-1} \Delta Y_{t_{i+1}} \Delta Y_{t_i}$$

which is just the square of the half-spread in the Roll model.

Note

By using the asymptotic result, we can also estimate $\text{var}[\epsilon]$ by the consistent estimator $\frac{1}{2n}[Y]_T^{\Pi}$.

The Roll model

Recall that in the Roll model, with

$$p_t = m_t + \epsilon_t = m_t + c\, e_t,$$

where m_t is a martingale and e_t is iid with mean 0 and variance 1, the effective half-spread is given by

$$c = \sqrt{-\gamma_1}$$

and

$$\sigma^2 = \gamma_0 + 2\,\gamma_1$$

Some features of the data

What was the volume?

In [11]:
```
sum(as.numeric(tqBAC$SIZE))
```

29531519

How many trades?

In [12]:
```
(n.trades <- dim(tqdata)[1])
```

24460

How many trades per 5 minutes in average?

In [13]:
```
n.trades*5/390
```

313.589743589744

Roll estimate of volatility

In [14]:
```
res <- acf(dp, type="covariance", plot=F)
gam0 <- res$acf[1]
gam1 <- res$acf[2]
```

In [15]:
```
sig2 <- gam0 + 2*gam1
```

In [16]:
```
sqrt(sig2*n.trades)
```

0.0095520984724023

In [17]:
```
sqrt(sig2*n.trades*252*1.4) # 1.4 factor for overnight moves
```

0.179416788701333

An annualized volatility of around 18%.

The conventional solution

- The conventional solution is to sample at most every five minutes or so.
  - In our BAC dataset, there are 24,460 trades, roughly 314 trades per 5 minutes.
  - We get an annualized volatility of around 18%. Compare this with around 23% both 1-month historical and implied respectively on 19-Sep-2017.
  - Sampling only every 5 minutes corresponds to throwing out 99.7% of the points!

- To quote [Zhang, Mykland and Aït-Sahalia][9], "It is difficult to accept that throwing away data, especially in such quantities, can be an optimal solution."
- From a more practical perspective, if we believe that volatility is time-varying, it makes sense to try and measure it from recent data over the last few minutes rather than from a whole day of trading.

Asymptotics for subsamples

Let $\Pi' = \{0 \leq t'_1 < \cdots < t'_{n'} \leq T\} \subset \Pi$ be a subsample of Π, $0 < n' \leq n$. The same aymptotics applies to subsamples as well. As $n' \to \infty$, we have

$$[Y]_T^{\Pi'} \overset{\mathcal{L}}{\approx} \langle X \rangle_T + 2\,n'\,\mathrm{var}[\epsilon] + \sqrt{4n'\mathbb{E}\left[\epsilon^4\right] + \frac{2T}{n'}\int_0^T \sigma_t^4 dt}\; Z,$$

where $Z \sim N(0,1)$.

- One should not sample too frequent but not too infrequent (it increases the discretization error) either.
- Thus, a natural question to ask is: what is the optimal sampling frequency? Or equivalently, what is the optimal subsample size n'?

Optimal sampling frequency

- The optimal sampling frequency is such that MSE is minimized.
  - As sampling frequency increases, the variance of the estimate of realized variance decreases but its bias typically increases.
  - This optimal sampling frequency may be computed theoretically for various estimators of realized variance under idealized assumptions.

Review: Mean-squared error of an estimator

- Mean-squared error (MSE) of an estimator θ of some quantity θ is defined by:

$$\mathrm{MSE}[\hat{\theta}] = \mathbb{E}[(\hat{\theta} - \theta)^2]$$

- Bias is given by $\mathbb{E}[\hat{\theta} - \theta]$.
- Variance of the estimator is given by

$$\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- Then \begin{eqnarray} \mbox{MSE}&=&\mathbb{E} [(\hat \theta- \mathbb{E} [\hat \theta]+\mathbb{E} [\hat \theta]-\theta)^2]\ &=&\mathbb{E} [(\hat \theta- \mathbb{E} [\hat \theta])^2]+(\mathbb{E} [\hat \theta-\theta])^2\ &=&\mbox{Variance} + \mbox{Bias}^2 \end{eqnarray}

Review: Consistency and efficiency

- An estimator $\hat{\theta}$ is *(asymptotically) consistent* if it converges in probability to the true value θ.
- Estimator A is more *efficient* than estimator B if $\mathbb{E}[(A-\theta)^2] < \mathbb{E}[(B-\theta)^2]$, i.e., A has smaller MSE.
- An *efficient* estimator is the most efficient one.

- An article in econometrics describing a new estimator typically proves that this new estimator is asymptotically consistent and efficient in the limit that the number of samples $n \to \infty$.
  - Assuming some specific process of course.

Optimal subsample size

It is shown in [Zhang, Mykland and Aït-Sahalia][9] that the optimal subsample size n^* is given by

$$n^* = \sqrt[3]{\frac{T}{4\text{var}^2[\epsilon]} \int_0^T \sigma_t^4 dt}.$$

Subsampling

Let $\Pi^{(k)} = \{0 \le t_0^{(k)} < \cdots < t_{n_k}^{(k)} \le T\}$, for $1 \le k \le K$, be a collection of nonoverlapping subsampling times in Π. That is,

$$\bigcup_{k=1}^{K} \Pi^{(k)} = \Pi \quad \text{and} \quad \Pi^{(k)} \cap \Pi^{(\ell)} = \emptyset \text{ for } k \ne \ell.$$

A typical example that we shall be using in the following is by sampling every K ticks from the kth tick on. That is,

$$\begin{aligned}
\Pi^{(1)} &= \{t_1 < t_{1+K} < t_{1+2K} < \cdots < t_{1+n_1 K} \le T\},\\
\Pi^{(2)} &= \{t_2 < t_{2+K} < t_{2+2K} < \cdots < t_{2+n_2 K} \le T\},\\
&\vdots\\
\Pi^{(K)} &= \{t_0 < t_K < t_{2K} < \cdots < t_{n_K K} \le T\}.
\end{aligned}$$

We denote by $[Y]_T^{\Pi^{(k)}}$ the RV estimate of Y using the subsamples that are sampled from the sampling times in $\Pi^{(k)}$, for $1 \le k \le K$.

By the same token, we have the following asymptotics for each subsample $k \in \{1, \cdots, K\}$

$$[Y]_T^{\Pi^{(k)}} \overset{\mathcal{L}}{\approx} \langle X \rangle_T + 2\, n_k\, \text{var}[\epsilon] + \sqrt{4 n_k \mathbb{E}\left[\epsilon^4\right] + \frac{2T}{n_k} \int_0^T \sigma_t^4 dt}\; Z_k$$

where $Z_k \sim N(0,1)$.

Boosting RV estimator

We can boost the RV estimator by averaging over the "weak learners" $[Y]_T^{\Pi^{(k)}}$

$$\begin{aligned}[Y]_T^{avg} &= \frac{1}{K}\sum_{k=1}^{K}[Y]_T^{\Pi^{(k)}} \\ &\approx \langle X\rangle_T + 2\,\bar{n}_K\,\mathrm{var}[\epsilon] + \sqrt{4\frac{\bar{n}_K}{K}\mathbb{E}\left[\epsilon^4\right] + \frac{4T}{3\bar{n}_K}\int_0^T \sigma_t^4 dt}\;Z\end{aligned}$$

where $\bar{n}_K := \frac{1}{K}\sum_k n_k$ is the average number of ticks in each subsample, roughly equal to $\frac{n}{K}$.

Note

- Boosting reduces biasedness and variance by a factor of K, but is unable to completely remove the biasedness.
- The optimal average subsample size $\bar{n}^*$ is given by

$$\bar{n}^* = \sqrt[3]{\frac{T}{6\mathrm{var}^2[\epsilon]}\int_0^T \sigma_t^4 dt}.$$

  Thus, the whole sample set is splitted into roughly $K^* \approx \frac{n}{\bar{n}^*}$ sets of subsamples.

The ZMA estimator

Recall that

$$\begin{aligned}[Y]_T^{\Pi} &\approx \langle X\rangle_T + n\,\mathrm{var}[\epsilon], \\ [Y]_T^{avg} &\approx \langle X\rangle_T + \bar{n}_K\mathrm{var}[\epsilon].\end{aligned}$$

We can eliminate bias by forming

$$\frac{1}{\bar{n}_K}[Y]_T^{avg} - \frac{1}{n}[Y]_T^{\Pi} \approx \left(\frac{1}{\bar{n}_K} - \frac{1}{n}\right)\langle X\rangle_T.$$

Thus we obtain the [Zhang, Mykland and Aït-Sahalia][9] (ZMA) bias-corrected estimator of $\langle X\rangle_T$:

$$[Y]_T^{ZMA} := \frac{1}{n - \bar{n}_K}\left\{n\,[Y]_T^{avg} - \bar{n}_K\,[Y]_T^{\Pi}\right\}.$$

Moreover, we have the asymptotic behavior for $[Y]_T^{ZMA}$ as

$$[Y]_T^{ZMA} \approx \langle X\rangle_T + \frac{1}{\sqrt[6]{n}}\sqrt{\frac{8}{c^2}\mathrm{var}^2[\epsilon] + c\frac{4T}{3}\int_0^T \sigma_t^4 dt}\;Z$$

where $Z \sim N(0,1)$. The optimal constant c^* is given by

$$c^* = \left(\frac{T}{12\,\mathrm{var}^2[\epsilon]}\int_0^T \sigma_t^4 dt\right)^{-\frac{1}{3}}.$$

Note

In the original paper [Zhang, Mykland and Aït-Sahalia][9], the authors suggested the estimator as $[Y]_T^{avg} - \frac{\bar{n}_K}{n}[Y]_T^{\Pi}$, whereas the estimator $[Y]_T^{ZMA}$ obtained above is referred to as the *small-sample adjustment* in the paper.

The Zhou estimator

Define

$$[Y]_T^{\Pi,Z} := \sum_{i=1}^{n} (\Delta Y_{t_i})^2 + \sum_{i=2}^{n} \Delta Y_{t_i} \Delta Y_{t_{i-1}} + \sum_{i=1}^{n-1} \Delta Y_{t_i} \Delta Y_{t_{i+1}}$$
$$= \sum_{i=1}^{n} (Y_{t_i} - Y_{t_{i-1}})(Y_{t_{i+1}} - Y_{t_{i-2}}).$$

Thus, under the assumption $Y = X + \epsilon$ of serially uncorrelated noise independent of returns X, we obtain $\mathbb{E}\left[[Y]^{\Pi,Z}\right] = \mathbb{E}\left[[X]_T\right]$.

By further assume $X_t = \sigma W_{\tau(t)}$ (a time-changed Brownian motion) for some Brownian motion W and deterministic increasing function $\tau(\cdot)$, since

$$\mathbb{E}\left[(\Delta X_{t_i})^2\right] = \sigma^2 \mathbb{E}\left[\left\{W_{\tau(t_i)} - W_{\tau(t_{i-1})}\right\}^2\right] = \sigma^2\{\tau(t_i) - \tau(t_{i-1})\},$$

we have

$$\mathbb{E}\left[[Y]^{\Pi,Z}\right] = \sigma^2\{\tau(T) - \tau(0)\} = \langle X \rangle_T.$$

In other words, in this case $[Y]^{\Pi,Z}$ is an unbiased estimator of $\langle X \rangle_T$.

Boosting Zhou estimator

As suggested by [Zhou][9] himself, we may boost the Zhou estimator from subsamples of the data obtaining

$$[Y]^{avg,Z} := \frac{1}{K} \sum_{k=1}^{K} [Y]^{\Pi^{(k)},Z}$$

Notation

- Given an observed log price series $\{p_i\}_{i=0}^{T}$, let

  (1)
  $$\gamma_{h,q}(k) = \sum_{i=1}^{m} (p_{iq+h} - p_{(i-1)q+h})(p_{(i+k)q+h} - p_{(i-1+k)q+h}),$$

  where $m = \lfloor (T - h + 1)/q \rfloor - k$.
  - Thus q is the sub-sampling frequency, h is the index of a given subsample and k is a time-offset. $\gamma_{h,q}(k)$ is the kth realized autocovariance of the subsampled series.

Zhou and ZMA estimators in updated notation

With our updated notation:

- The Zhou estimator becomes

  (2)
  $$ZHOU = \frac{1}{q} \sum_{h=0}^{q-1} (\gamma_{h,q}(0) + 2\gamma_{h,q}(1))$$

- The ZMA (or two-scale RV) estimator becomes:

  (3)

$$TSRV = \left(1 - \overline{T}/T\right)^{-1} \left(\frac{1}{q}\sum_{h=0}^{q-1}\gamma_{h,q}(0) - \frac{\overline{T}}{T}\gamma_{0,1}(0)\right),$$

  where $\overline{T} = (T - q + 1)/q$

Multiscale RV

Zhang's multiscale RV estimator is given by

(4)

$$MSRV = \sum_{j=1}^{q}\frac{a_j}{j}\sum_{h=0}^{j-1}\gamma_{h,j}(0),$$

where

$$a_j^* = (1 - 1/q^2)^{-1}\left(\frac{j}{q^2}h(j/q) - \frac{j}{2q^3}h'(j/q)\right)$$

and

$$h(x) = 12\,(x - 1/2).$$

Realized Kernel estimator of Barndorff-Nielsen, Hansen, Lunde and Shephard

(5)

$$KRV = \gamma_{0,1}(0) + 2\sum_{s=1}^{k}\kappa\left(\frac{s-1}{k}\right)\gamma_{0,1}(s).$$

Choices for the kernel $\kappa(x)$ include:

- Modified Tukey-Hanning kernel TH_2: $\kappa\,(x) = \sin^2\left\{\frac{\pi}{2}(1-x)^2\right\}$.
- Cubic kernel: $\kappa\,(x) = 1 - 3x^2 + 2x^3$.

Both of these are so-called *flat-topped* kernels where the coefficient of first order autocovariance is 1. KRV estimators are designed to maximize efficiency. However, their derivation assumes independent noise.

Large's alternation estimator

(6)

$$ALT = \frac{C}{R}\,\gamma_{0,1}(0),$$

where R (C) are the number of reversals (continuations) in the sample.

- Note that $C = \sum_{i=2}^{T}(I_i^p I_{i-1}^p + I_{i-1}^n I_i^n)$ and $R = \sum_{i=2}^{T}(I_i^p I_{i-1}^n + I_{i-1}^p I_i^n)$ where $I_i^p = \mathbf{1}_{\{r_i>0\}}$ and $I_i^n = \mathbf{1}_{\{r_i<0\}}$.
- If there are zero returns in the sample then these are first removed, *i.e.*, the estimator is implemented using tick data.

Large's estimator

- Note once again that

$$\gamma_{0,1}(0) = \sum_t \Delta p_t^2$$

  which is just RV. So

$$ALT = \frac{C}{R} RV.$$

- If the efficient price is constant, then $C = 0$ and $ALT = 0$.
- If there is no microstructure noise, $C = R$ for a random walk and $ALT = RV$.
- This estimator only works for large tick markets where the price moves by only one tick whenever it moves.

Maximum Likelihood estimator of Aït-Sahalia, Mykland and Zhang

(7)

$$MLRV = M \widehat{\delta^2} (1 + \widehat{\eta})^2,$$

where $(\widehat{\eta}, \widehat{\delta^2})$ are the maximum likelihood estimates of an MA(1) model for observed returns, *i.e.* $r_i = \varepsilon_i + \eta\varepsilon_{i-1}$ where the ε_i's are serially uncorrelated with mean zero and variance δ^2.

Practical implementation

- Ideally, we should be able to update a RV estimator in real time as ticks come in.
  - An *online algorithm* in computer science terminology.
- From this perspective, estimators where we can just add the most recent observation and drop the oldest observation are preferred.

Convergence of TSRV, MSRV and kernel estimators

- The TSRV (two-timescale RV) measure yields a consistent estimator of IV that converges at rate $M^{-1/6}$.
- The rate of the TSRV estimator can be improved to $M^{-1/4}$ – the fastest attainable in this setting – by using multiple time scales as in the MSRV.
- The realized kernels provide an equally efficient alternative to the subsampling estimators with rates of convergence of $M^{-1/6}$ or $M^{-1/4}$ depending on the choice of kernel (the TH_2 and cubic kernels converge at the fastest rate).

- Finally, both the ALT and MLRV estimators are also consistent and converge at rate $M^{-1/4}$, albeit under more restrictive (semi-) parametric assumptions.

- An important feature of the non-parametric RV measures TSRV, MSRV, and KRV is that they allow for stochastic volatility, leverage and can be made robust to dependent noise.
  - ZHOU is biased with dependent noise
  - ALT rules out leverage effects and requires uncorrelated noise
  - Although MLRV can be modified to take account of dependent noise it does not allow for stochastic volatility.

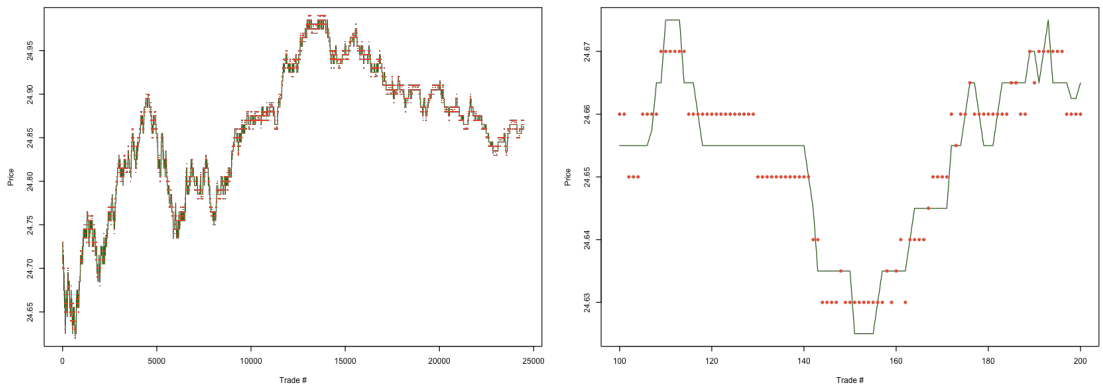
Using the mid-quote

- What if quote data is available?
- According to practitioners, using mid-quotes eliminates bid-ask bounce.
  - If we use mid-quotes, do we get the same volatility estimate?
  - When should we sample the mid-quotes? Every quote change? Every second?
- According to Bouchaud, Gefen, Potters and Wyart(2004) and also Bandi and Russell (2006), it's best to sample the mid-quote just before each trade.
- Let's see how the autocorrelation plot and noise plot look with mid-quotes.

BAC trades and quotes

In [18]:
```
bid <- as.numeric(tqBAC$BID)
ask <- as.numeric(tqBAC$OFR)
mid <- (bid + ask)/2
```

In [19]:
```
par(mfrow=c(1,2),mar=c(5,4,1,2)+.1,cex=.6)
plot(mid,col="dark green",type="l",ylab="Price",xlab="Trade #")
points(p,col="red",type="p",pch=".")
plot(100:200,mid[100:200],col="dark green",type="l",ylab="Price",xlab="Trade #")
points(100:200,p[100:200],col="red",type="p",pch=20)
par(mfrow=c(1,1))
```

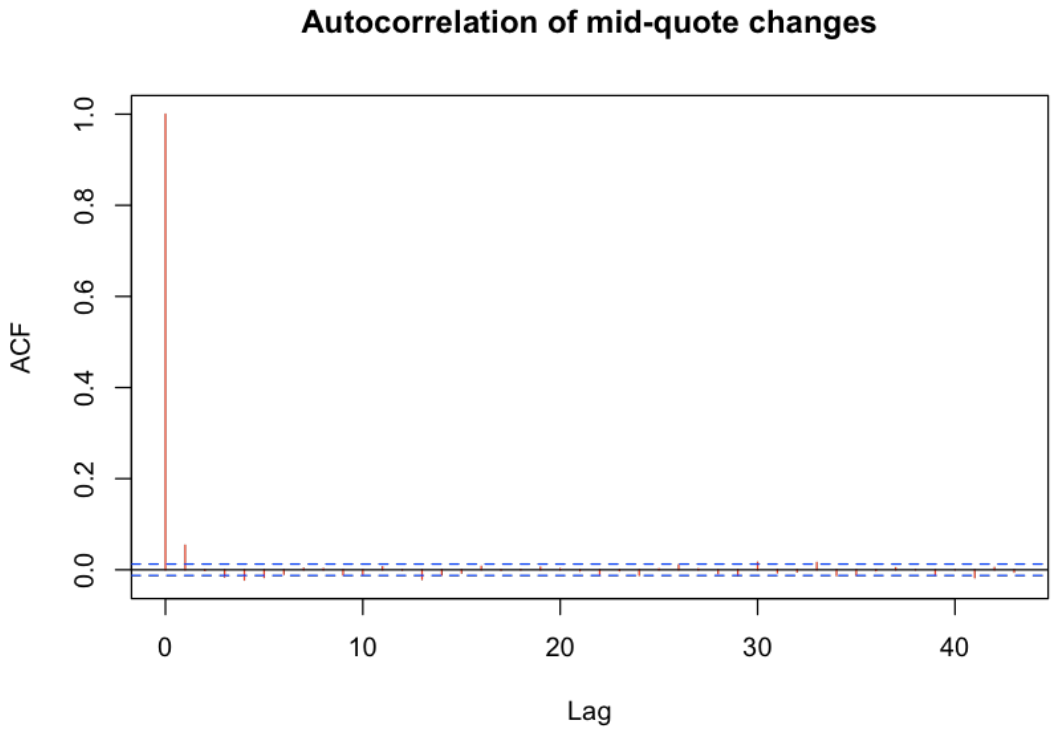


BAC data from 19-Sep-2017. Mid-quotes in green, trade prices in red.

Autocorrelation of BAC mid-quote changes

BAC data from 19-Sep-2017.

In [20]:
```
# Plot autocorrelation function of mid-quote changes
options(repr.plot.width=7)
dm <- diff(log(mid))
acm <- acf(dm, plot=F)
plot(acm, main="Autocorrelation of mid-quote changes", col="red")
```



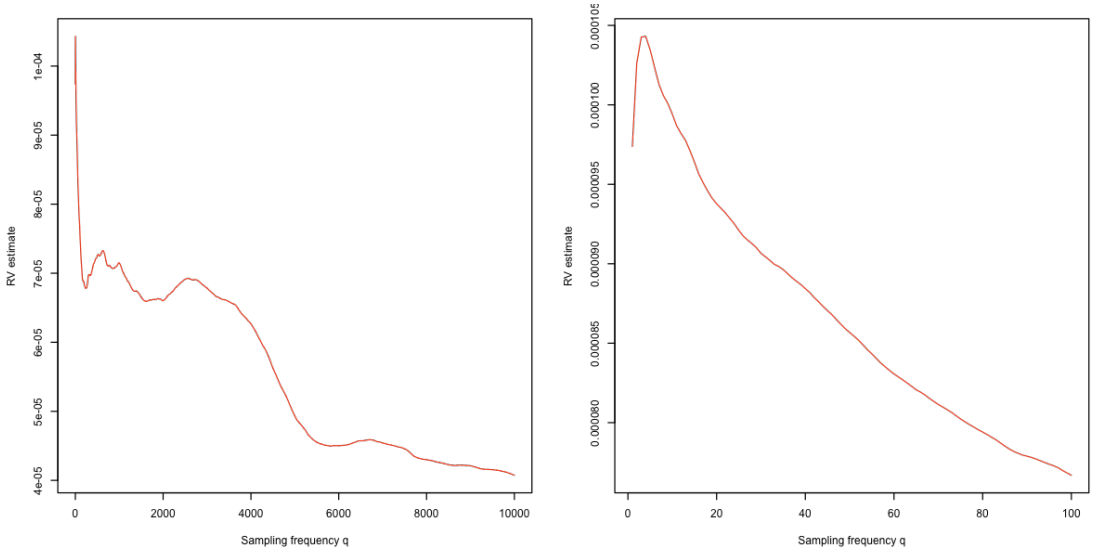
- Much less autocorrelation in the mid-quote data.

BAC signature plot using mid-prices

Using BAC mid-price data, we again plot RV as function of sampling frequency q.

In [21]:
```
nn <- 10000
rvm <- sapply(1:nn, function(q){rCov(diff(log(mid), lag=q))/q})
```

In [22]:
```
options(repr.plot.width=10)
par(mfrow=c(1,2), cex=.6,mar=c(5,4,1,2)+.1)
plot(1:nn, rvm,col="red", type="l", ylab="RV estimate", xlab="Sampling frequency
q")
plot(1:100, rvm[1:100], col="red", type="l", ylab="RV estimate",
     xlab="Sampling frequency q")
par(mfrow=c(1,1))
```



ZI simulation

- Given a set of tick data, what was the realized variance?
- If we don't know what the true value was, how can we test the performance of different estimators?
  - One way is to simulate from a model which could be stochastic volatility or jump-diffusion for example.
  - A noise process is typically added to the efficient price process.
- [Gatheral and Oomen][5] generated simulations from the Smith Farmer zero-intelligence (ZI) model. This has the benefit that microstructure noise is in some sense much more realistically modeled.
- We tested many different estimators and came up with a ranking.

Gatheral Oomen results for mid-quotes

Table 3 Performance of alternative realized variance measures with ZI quote-price data

		$M = 1{,}000$					$M = 5{,}000$					$M = 10{,}000$				
		mean	stdev	MSE	loss	q^*	mean	stdev	MSE	loss	q^*	mean	stdev	MSE	loss	q^*
1.	Realized variance															
	(a) highest ($q = 1$)	1.112	0.138	−3.452	0.479	1.00	1.113	0.062	−4.097	1.360	1.00	1.113	0.043	−4.224	1.899	1.00
	(b) ad hoc (5 mins)	1.008	0.198	−3.238	0.694	12.00	1.001	0.169	−3.555	1.901	64.00	0.999	0.165	−3.609	2.513	128.0
	(c) q^*_{RV}	1.049	0.146	−3.736	0.196	1.93	1.035	0.070	−5.087	0.370	3.61	1.027	0.051	−5.689	0.434	4.63
2.	Bias-corrected RV of Zhou [28]															
	(a) highest ($q = 1$)	1.013	0.143	−3.876	0.056	1.00	1.015	0.064	−5.447	**0.009**	1.00	1.015	0.045	−6.098	**0.024**	1.00
	(b) ad hoc (5 mins)	0.995	0.278	−2.561	1.371	12.00	0.998	0.266	−2.650	2.806	64.00	0.996	0.263	−2.674	3.449	128.0
	(c) q^*_{Zhou}	1.013	0.143	−3.876	0.056	1.00	1.015	0.064	−5.447	**0.009**	1.00	1.015	0.045	−6.098	**0.024**	1.00
3.	Two-scale RV of Zhang, Mykland, and Aït-Sahalia [27]															
	(a) ad hoc ($q = 5$)	0.996	0.153	−3.752	0.179	5.00	1.002	0.068	−5.383	0.073	5.00	1.003	0.048	−6.079	0.044	5.00
	(b) q^*_{ZMA}	1.011	0.143	−3.880	0.052	2.00	1.015	0.064	−5.450	**0.006**	2.00	1.015	0.045	−6.101	**0.022**	2.00
4.	Multi-scale RV of Zhang [26]															
	(a) ad hoc ($q = 5$)	0.995	0.154	−3.741	0.191	5.00	1.002	0.068	−5.375	0.081	5.00	1.002	0.048	−6.071	0.051	5.00
	(b) q^*_Z	1.011	0.143	−3.880	0.052	2.00	1.015	0.064	−5.450	**0.006**	2.00	1.015	0.045	−6.101	**0.022**	2.00

Note This table reports the mean ("mean"), standard deviation ("stdev"), logarithmic MSE ("MSE"), the difference in log MSE relative to the best estimator ("loss"), and the average (sub)sampling frequency or bandwidth ("q^*") for each realized variance measure and across sample size M. Loss levels in boldface are insignificantly different from zero at a 1% bootstrapped confidence level

Conclusion of the ZI simulation: Which estimators are best?

- Sample prices at the highest available frequency and then measure realized variance using one of:
  - The Two-Scale ZMA estimator with a subsampling frequency of 5,
  - The Multi-Scale RV of Zhang with 5 subsamples,
  - The Realized Kernel of BNS with a bandwidth of 5.
- The performance of these estimators is largely equivalent and their implementation equally straightforward, so which particular one to use would be a matter of taste.
- Relative to the widely used sum of sparsely sampled returns following the "5-minute" rule prescribed in earlier literature, the efficiency gain achieved with these is likely to be substantial.

Conclusion of the ZI simulation: What series to sample?

- In terms of data sampling, use mid-quotes.
- When sampled immediately prior to a trade, we ensure the same number of observations as for the trade data but with a heavily reduced level of microstructure noise.
- The micro price is also preferred over the trade data but, despite some seemingly appealing features, does not seem to improve over mid-quote data. At least in the ZI simulation, it appears that micro-price had higher order autocorrelations.

Conclusion of the ZI simulation: What bandwidth to use?

- The rule for choosing q should be to let q grow with the level of noise so that with little noise we compute something that is close to RV and with high levels of noise we effectively reduce the sampling frequency so as to mitigate its impact.
- The optimal bandwidth q^* can be computed in closed-form for realized kernels, TSRV and MSRV. It is of the form:

$$q^* = A\sqrt{\xi}.$$

  for some estimator-dependent constant A where ξ is the Oomen noise-to-signal ratio.
- Alternatively, $q = 5$ seems to work well nearly all of the time.

Trades vs mid-quotes

- [Barndorff-Nielsen et al.][1] point out that one way to assess the performance of an estimator on real data is to see if it gives the same result on trades and quotes.
  - They find that results depend on the data-cleaning protocol followed.
  - It's better to take data from each exchange separately and average the results rather than supposing that all data comes from the same source.
- This might be less true now post Reg. NMS (Regulation National Market System).
- The realized kernel estimators do fine, giving similar results for trades and quotes.
- There are some days with lengthy strong trends which are not compatible with standard models of microstructure noise.

Estimates of realized variance of BAC data

Estimates from trade data

In [23]:
```
RV1 <- RVplain(log(p), 1)*sqrt(n.trades)
RV314 <- RVplain(log(p), 314)*sqrt(n.trades)  # Roughly 5 minutes
TS5 <- TSRV(log(p), 5)*sqrt(n.trades)
MS5 <- MSRV(log(p), 5)*sqrt(n.trades)
KTH5 <- KRVTH(log(p), 5)*sqrt(n.trades)
KC5 <- KRVC(log(p), 5)*sqrt(n.trades)

data.frame(RV1, RV314, TS5, MS5, KTH5, KC5)
```

RV1	RV314	TS5	MS5	KTH5	KC5
0.0569063	0.009852493	0.01304519	0.01305223	0.01344506	0.01305842

Estimates from mid-quote data

In [24]:
```
RV1 <- RVplain(log(mid),1)*sqrt(n.trades)
RV314 <- RVplain(log(mid),314)*sqrt(n.trades)
TS5 <- TSRV(log(mid),5)*sqrt(n.trades)
MS5 <- MSRV(log(mid),5)*sqrt(n.trades)
KTH5 <- KRVTH(log(mid),5)*sqrt(n.trades)
KC5 <- KRVC(log(mid),5)*sqrt(n.trades)

data.frame(RV1, RV314, TS5, MS5, KTH5, KC5)
```

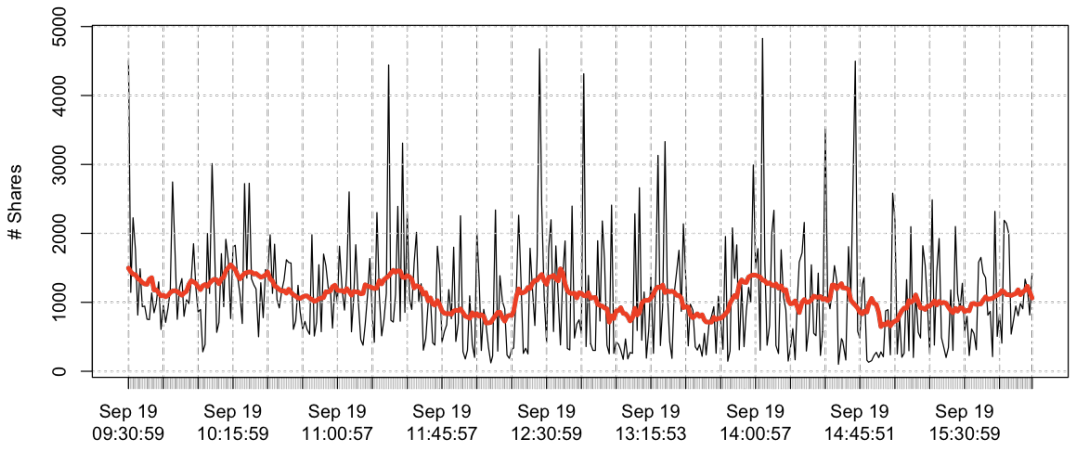
RV1	RV314	TS5	MS5	KTH5	KC5
0.01522992	0.01039115	0.01642153	0.01645283	0.01665178	0.0162197

BAC volume profile

Using BAC size data, we plot the volume profile.

In [25]:
```
ep <- endpoints(tqBAC,'minutes', k=1)
ne <- length(ep)
ep <- ep[-ne]
sz <- function(x){mean(as.numeric(x$SIZE))}
szVec <- period.apply(tqBAC, INDEX=ep, FUN= sz)
ns <- length(szVec)
```

In [105]:
```
plot(szVec, ylab = "# Shares", main=NA)
ks <- ksmooth(1:ns,szVec, bandwidth=20)$y
sz2 <- as.xts(cbind(szVec,ks))
names(sz2) <- c("SIZE", "kSIZE")
lines(sz2$kSIZE,col="red",lwd=4)
```



- The red line is kernel-smoothed volume

BAC volume and realized variance profiles

Using the same data, we compute realized variance from mid-quotes with the Tukey Hanning kernel ($q = 5$) (KRV).

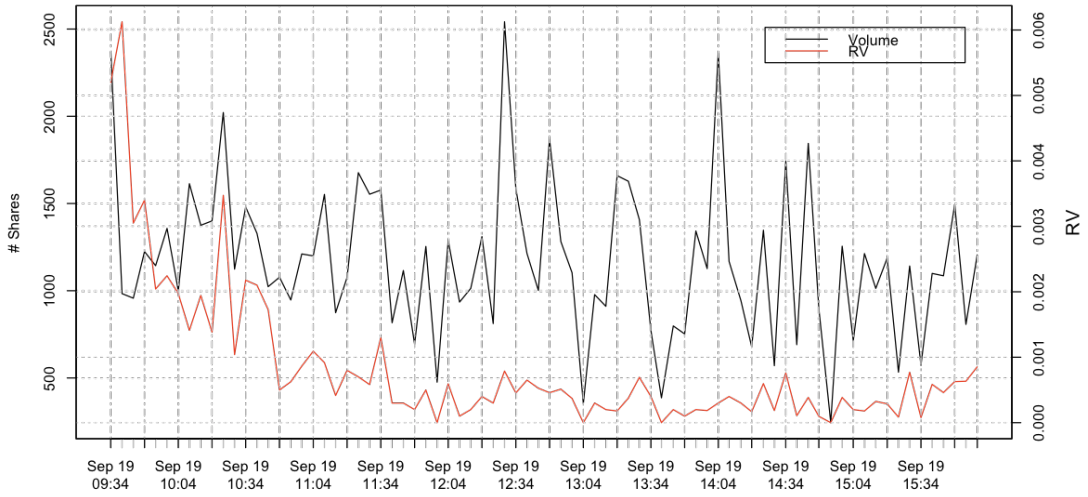
In [58]:
```
rv <- function(x){
    mid <- (as.numeric(x$BID) + as.numeric(x$OFR))/2

    KRVTH(mid, 5)
}
sz <- function(x){mean(as.numeric(x$SIZE))}

ep <- endpoints(tqBAC, 'minutes', k=5)
ep <- ep[-length(ep)]
szVec <- period.apply(tqBAC, INDEX=ep, FUN=sz)
rvVec <- period.apply(tqBAC, INDEX=ep, FUN=rv)
```

Now, we show the volume and RV plots together.

In [59]:
```
par(mar = c(5,5,2,5),cex=0.8)
plot(szVec, ylab="# Shares",  main=NA)
par(new = T)
plot(rvVec, col="red", axes=F, xlab=NA, ylab=NA, main=NA)
axis(side=4)
mtext(side=4, line=3, "RV")
legend("topright", inset=0.05, legend=c("Volume    ","RV"), col=c("black","red"),
       lwd=c(1,1))
```



- The red line is kernel-smoothed volume; the black line is KRV.

Estimating covariance

- The natural estimator of covariance is just

$$\frac{1}{T}\sum_{i=1}^{T}\Delta p_i^{(1)}\,\Delta p_i^{(2)}$$

  where $p^{(1)}$ and $p^{(2)}$ are prices of two assets.

- However the prices $p^{(i)}$ are usually asynchronous.

- Consequently, this estimator decreases as sampling frequency increases.
  - In fact, [Hayashi and Yoshida][6] show under pretty innocuous assumptions that it tends to zero as sampling frequency increases.
- This is called the *Epps Effect*.

The Hayashi-Yoshida estimator

$$HY_t = \sum_{i \le t} \sum_{j \le t} (p_i^{(1)} - p_{i-1}^{(1)})\,(p_j^{(2)} - p_{j-1}^{(2)})\,v_{ij}$$

where

$$v_{ij} = \mathbf{1}_{[t_{i-1},t_i] \cap [t_{j-1},t_j] \neq \emptyset}$$

In words, the sum is over all overlapping intervals.

How does Hayashi-Yoshida work?

Suppose

$$dp_t^{(1)} = \sigma_1\, dZ_1$$
$$dp_t^{(2)} = \sigma_2\, dZ_2$$

with $\mathbb{E}[dZ_1\, dZ_2] = \rho\, dt$. Then

$$\begin{aligned}\mathbb{E}\left[(p_i^{(1)} - p_{i-1}^{(1)})\,(p_j^{(2)} - p_{j-1}^{(2)})\right] &= \mathbb{E}\left[\int_{t_{i-1}}^{t_i} \sigma_1\, dZ_1 \int_{t_{j-1}}^{t_j} \sigma_2\, dZ_2\right] \\ &= \int_{[t_{i-1},t_i] \cap [t_{j-1},t_j]} \mathbb{E}\left[\rho\, \sigma_1\, \sigma_2\right]\, dt.\end{aligned}$$

Summing over all partitions of the interval $[0, T]$ gives the result.

Volatility forecasting

- It is by now well established that volatility forecasts are substantially improved by using accurate estimates of realized variance.
- We now briefly review two of the best performing estimators:
  - The HAR-RV (Heterogeneous Autoregressive Realized Variance) estimator of [Corsi][2].
  - The Rough Volatility estimator of [Gatheral, Jaisson and Rosenbaum][4].

The Corsi HAR-RV forecast

- The package `highfrequency` implements a regression to fit the parameters HAR-RV.
- This model can be regarded as a poor man's version of a long memory model such as ARFIMA.
  - True long-memory models such as ARFIMA are notoriously hard to fit.
- HAR-RV can also be considered an intelligent alternative to GARCH.
- The model boils down to the regression

  $$RV_{t,t+h} = \beta_0 + \beta_D\, RV_t + \beta_W\, RV_{t-5,t} + \beta_M\, RV_{t-22,t} + \epsilon_{t,t+h}.$$

  In words, the RV forecast for h days from now is a linear combination of the current realized variance and (aggregate) RV estimates for the last week and the last month.

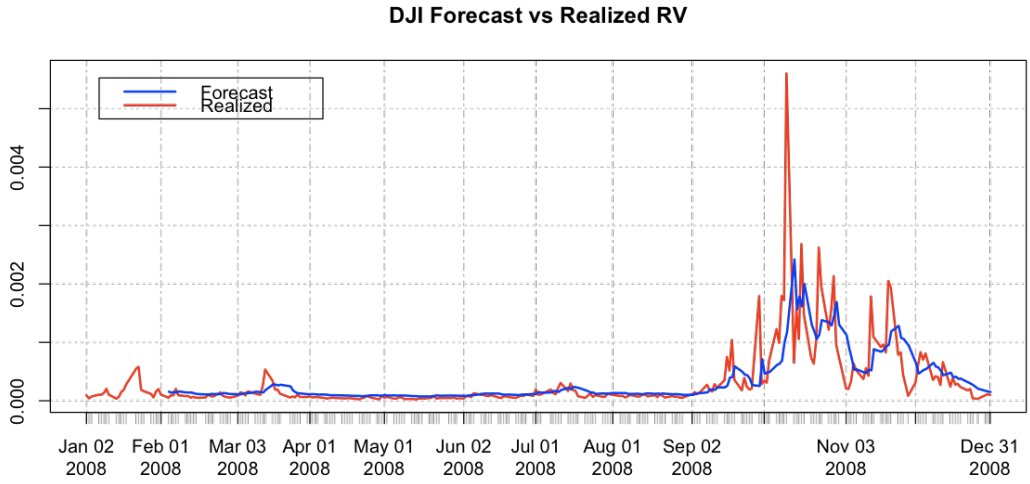
Example

In [107]:
```
# Forecasting daily Realized volatility for DJI 2008 using the basic harModel: HA
R-RV

data(realized_library); # Get sample daily Realized Volatility data
DJI_RV <- realized_library$Dow.Jones.Industrials.Realized.Variance; #Select DJI
DJI_RV <- DJI_RV[!is.na(DJI_RV)] #Remove NA's
DJI_RV <- DJI_RV['2008']

x <- harModel(data=DJI_RV, periods=c(1,5,22), RVest=c("rCov"), type="HARRV", h=1,
              transform=NULL)
```

In [108]:
```
plot.xts(DJI_RV,type="n", main="DJI Forecast vs Realized RV")
lines(DJI_RV, col="red", lwd=2)
lines(as.xts(x$fitted.values), col="blue", lwd=2)
legend("topleft", inset=0.05, legend=c("Forecast", "Realized"), col=c("blue","red
"),
       lwd=c(2, 2))
```



In [62]:
```
print(summary(x))
```

```
Call:
"RV1 = beta0  +  beta1 * RV1 +  beta2 * RV5 +  beta3 * RV22"

Residuals:
       Min         1Q     Median         3Q        Max
-0.0017683 -0.0000626 -0.0000427 -0.0000087  0.0044331

Coefficients:
       Estimate Std. Error t value Pr(>|t|)
beta0 4.432e-05  3.695e-05   1.200   0.2315
beta1 1.586e-01  8.089e-02   1.960   0.0512 .
beta2 6.213e-01  1.362e-01   4.560 8.36e-06 ***
beta3 8.721e-02  1.217e-01   0.716   0.4745
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.0004344 on 227 degrees of freedom
Multiple R-squared:  0.4679,    Adjusted R-squared:  0.4608
F-statistic: 66.53 on 3 and 227 DF,  p-value: < 2.2e-16
```

The Rough Volatility forecast

Variance forecast formula

(3)

$$\mathbb{E}\left[v_{t+\Delta} \middle| \mathcal{F}_t\right] = \exp\left\{\mathbb{E}\left[\log(v_{t+\Delta}) \middle| \mathcal{F}_t\right] + 2\,c\,\nu^2\Delta^{2H}\right\}$$

where

$$\mathbb{E}\left[\log v_{t+\Delta} \middle| \mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds.$$

HAR and rough volatility forecasts

The HAR forecast looks like

$$\widehat{\log RV} = \sum_{j=1}^{3}\beta_j\int_{-t-\Delta_j}^{t}\log v_s\,ds = \int_{-\infty}^{t}\kappa(t-s)\,\log v_s\,ds$$

with

$$\kappa(\tau) = \beta_0 + \sum_{j=1}^{3}\beta_j\,\mathbb{1}_{\tau\le\Delta_j}.$$

The rough volatility forecast looks (up to a factor) like

$$\widehat{\log RV} = \int_{-\infty}^{t} \kappa(t-s)\ \log v_s\ ds$$

with

$$\kappa(\tau) = \frac{1}{(t-s+\Delta)(t-s)^{H+1/2}}.$$

The HAR kernel looks like a piecewise constant approximation to the rough volatility kernel.

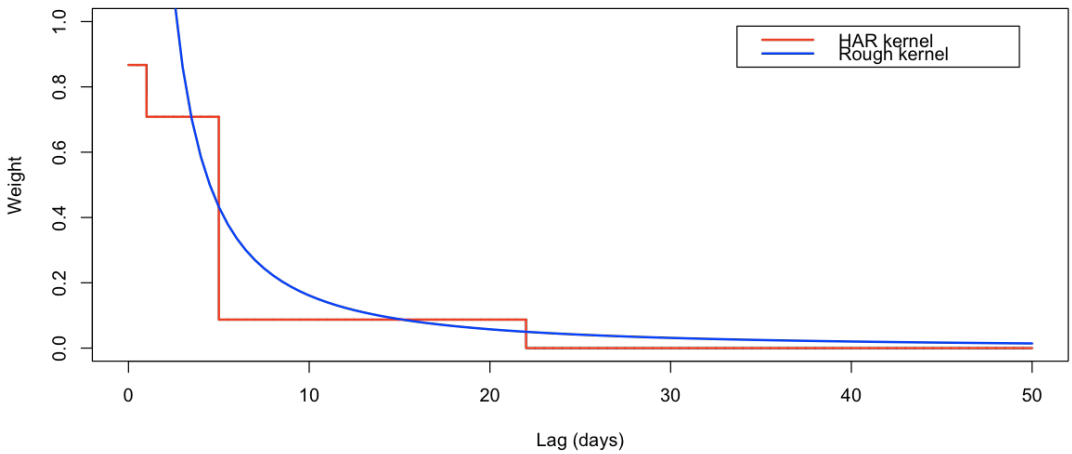
In [109]:
```
kappa.HAR.raw <- function(tau){
    vec <- (tau <= c(1,5,22))
    x$coefficients[2:4] %*% vec
}

kappa.HAR <- Vectorize(kappa.HAR.raw)

kappa.Rough <- function(tau){
    H <- 0.05
    cos(H*pi)/(H*pi)/((tau+1)*tau^(.5+H))}
```

Graphical comparison of kernels

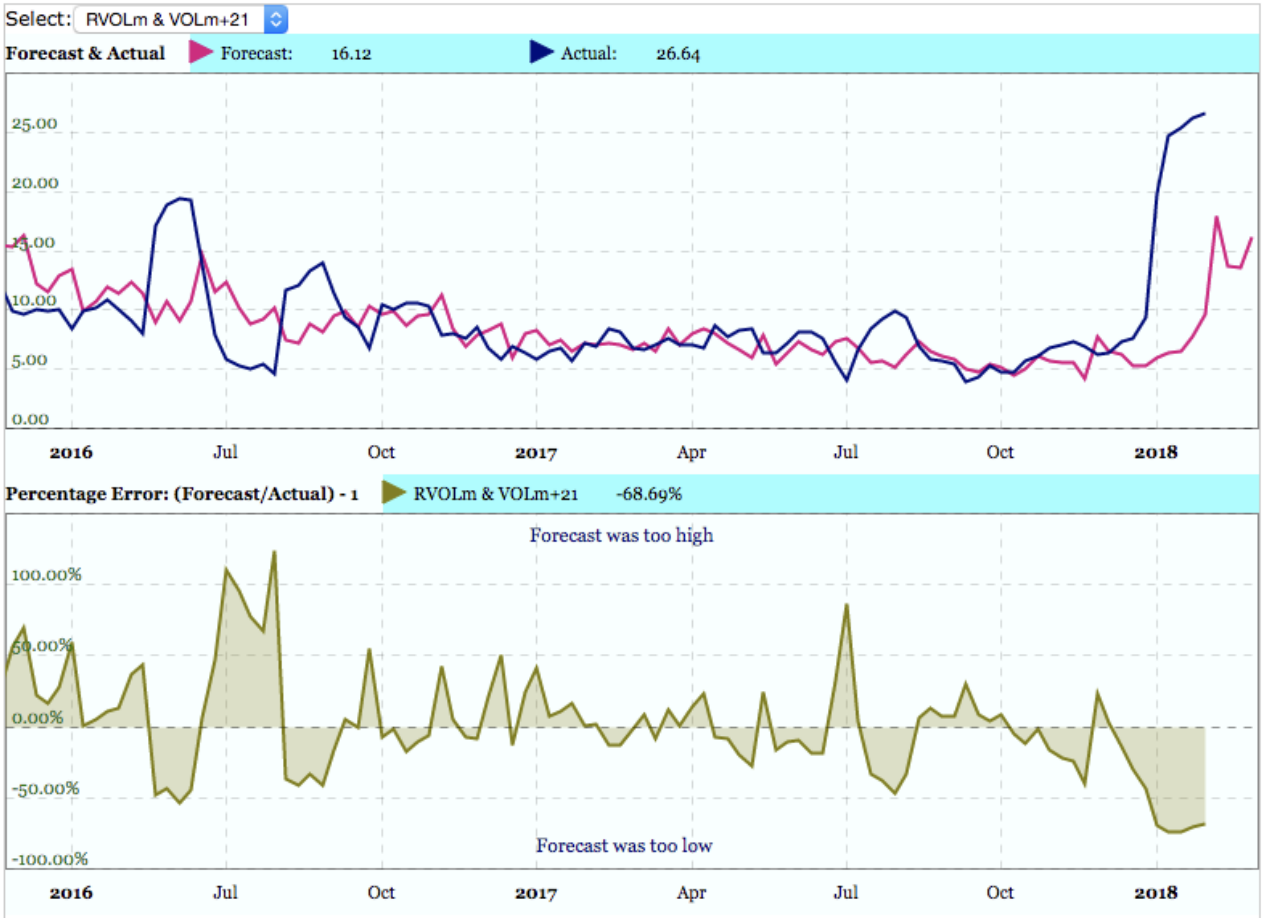
In [110]:
```
curve(kappa.HAR,from=0,to=50,col="red",lwd=2,n=10000,xlab="Lag (days)",ylab="Weig
ht",ylim=c(0,1))
curve(kappa.Rough, from=0,to=50,col="blue",lwd=2,add=T)
legend("topright",inset=0.05,legend=c("HAR kernel","Rough kernel"),col=c("red","b
lue"),lwd=c(2,2))
```

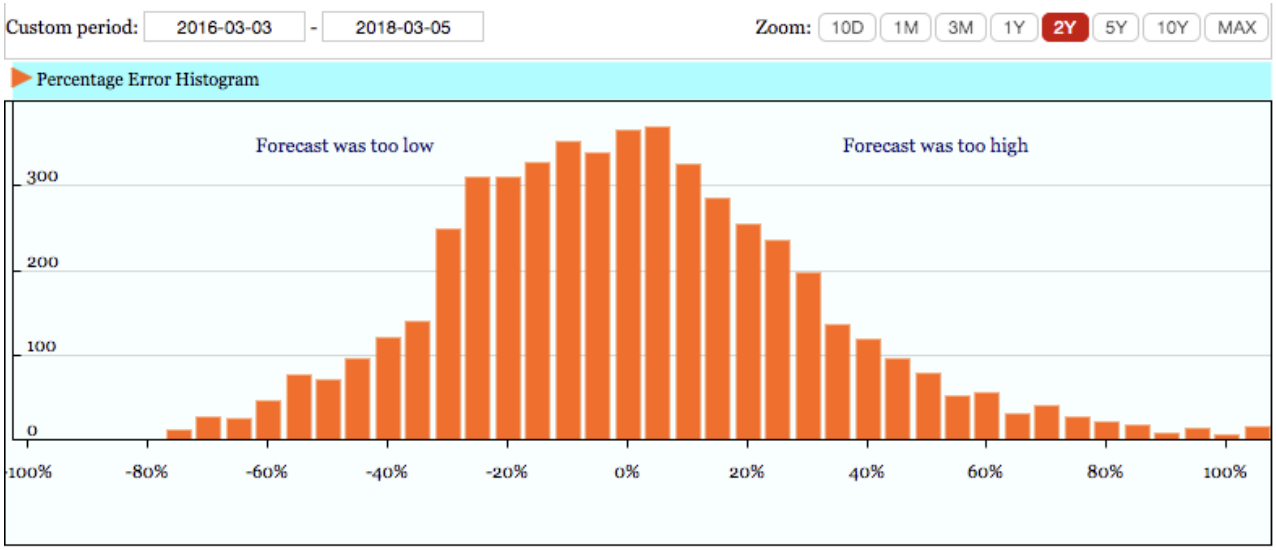


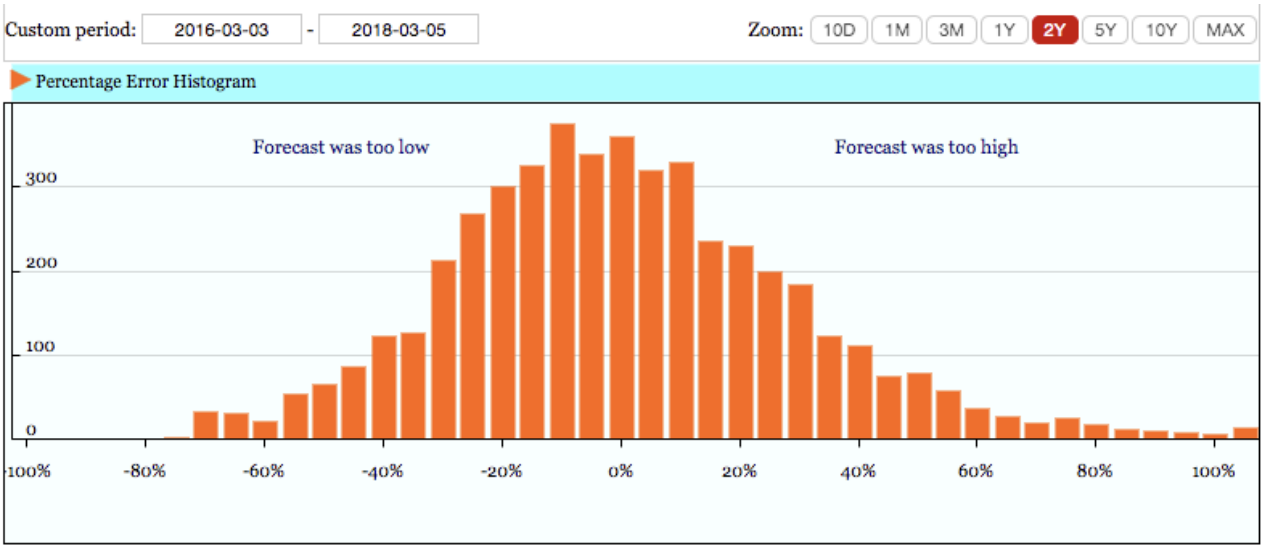
VolX

- The commercial company VolX (http://volx.us (http://volx.us)) has developed a number of RealVol Instruments and RealVol Indices based on realized volatility as defined by the RealVol Formulas.
  - Their business model is to license these indices to exchanges and information providers.
- They publish daily forecasts of RV using HARK (which is HAR-RV with Kalman filtering, and RVOL, an implementation of the Rough Volatility forecast.
- You can compare forecast versus actual volatility for the two estimators here: http://www.volx.us/volatilitycharts.shtml?2&SPY&PRED (http://www.volx.us/volatilitycharts.shtml?2&SPY&PRED).

VolX screenshots







RV data from the Oxford-Man Institute

- In principle, it is straightforward to compute RV every day for any given underlying using one of the better RV estimators we presented above.
- In practice, this involves a lot of data work, especially cleaning.

- The R-package `highfrequency` has some historical RV data sourced from the Oxford-Man Institute of Quantitative Finance Realized Library.
  - More such data is publicly available at http://realized.oxford-man.ox.ac.uk (http://realized.oxford-man.ox.ac.uk).
    - In particular, various historical RV estimates for SPX are available.
    - Data is updated daily.

Conditional and unconditional variances

- The HAR and rough volatility forecasts are both impressive.
  - Much superior to alternatives such as GARCH.
- However, HAR is a regression and rough volatility is a proper model.
- One practical consequence is that we can put error bars on our volatility forecasts.

So how good is the forecast?

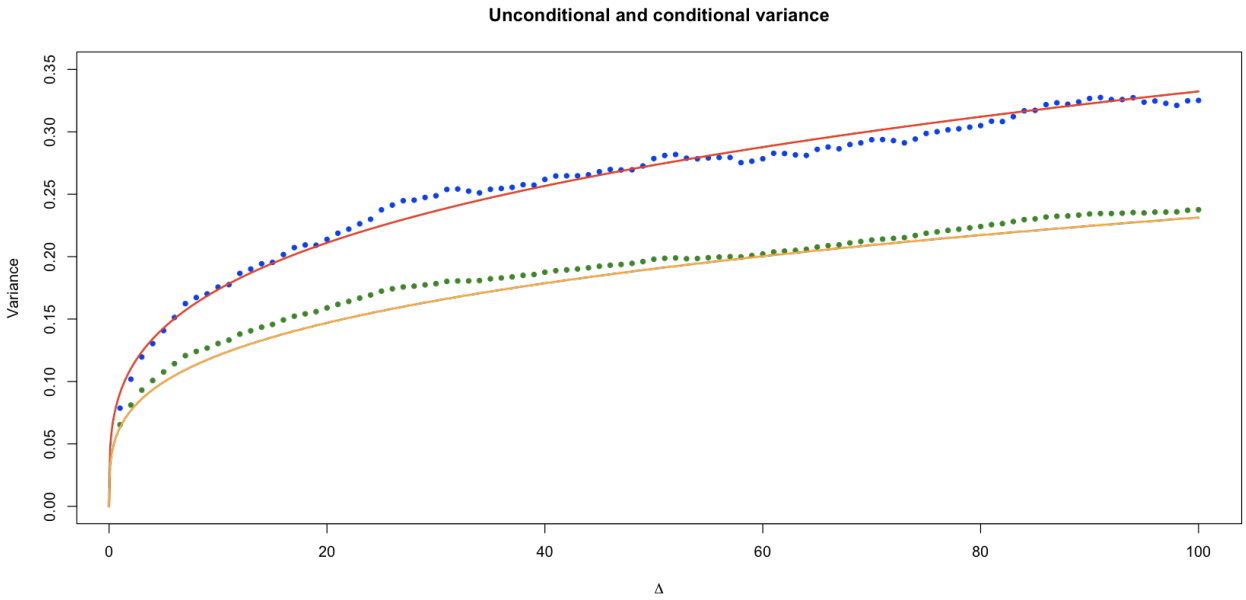
Specifically, by how much is the variance of the future variance reduced by taking into account the whole history of the fBm?

- In practice of course, we only consider some finite history, 200 points say.
- We know this again from [Nuzman and Poor]^{[10]} who showed that the ratio of the conditional to the unconditional variance of the $\log v_t$ is

$$\tilde{c} = \frac{\Gamma(3/2 - H)}{\Gamma(H + 1/2)\,\Gamma(2 - 2H)}.$$

- We can compute this ratio empirically and compare with the model prediction.

Unconditional and conditional variance vs lag Δ



Actual unconditional variance in blue, rough volatility prediction in red; Actual conditional variance in green, rough volatility prediction in orange.

Amazing agreement between data and model

- We observe that the ratio of conditional to unconditional variance is more or less *exactly* as predicted by the model!

Summary

- There has been a huge expansion in the literature on realized variance and covariance estimation since around 2003 with many very interesting papers.
- As a result, we now have very efficient estimators for realized variance that take into account all of the available information.
  - The newer volatility estimators are all very much more efficient that RV sampled every 5 minutes.
  - Moreover, kernel-based estimators are easily updated in real time by adding the most recent tick and dropping the oldest tick.
- The article by [McAleer and Medeiros][7] is a nice review of the literature up to 2008 or so.
- The rough volatility forecast seems to be the simplest and best!

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