MTH 9879 Market Microstructure Models, Spring 2019

Lecture 9: Optimal trading strategies: Almgren-Chriss

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Outline of Lecture 9

- The Euler-Lagrange equation
- The Hamilton-Jacobi-Bellman equation
- The Almgren-Chriss optimal liquidation strategy
- Dependence on dynamical assumptions: ABM vs GBM
- Almgren (2005)
- Optimal portfolio liquidation with a dark pool

Overview of execution algorithm design

Typically, an execution algorithm has three layers:

- The macrotrader
  - This highest level layer decides how to slice the order: when the algorithm should trade, in what size and for roughly how long.
- The microtrader
  - Given a slice of the order to trade (a child order), this level decides whether to place market or limit orders and at what price level(s).
- The smart order router
  - Given a limit or market order, which venue should this order be sent to?

In this lecture, we are concerned with the highest level of the algorithm: How to slice the order.

Statement of the problem

- Given a model for the evolution of the stock price, we would like to find an optimal strategy for trading stock, the strategy that minimizes some cost function over all permissible strategies.
  - We will specialize to the case of stock liquidation where the initial position $x_0 = X$ and the final position $x_T = 0$.
- A *static* strategy is one determined in advance of trading.

- A *dynamic* strategy is one that depends on the state of the market during execution of the order, *i.e.* on the stock price.
  - Delta-hedging is an example of a dynamic strategy. VWAP is an example of a static strategy.

- It will turn out, surprisingly, that in many models, a statically optimal strategy is also dynamically optimal.

The Euler-Lagrange equation

Suppose that the strategy x_t minimizes the cost functional

$$C[x, \dot{x}] = \int_0^T L(t, x_t, \dot{x}_t)\, dt$$

with boundary conditions $x_0 = 0, x_T = X$ (assume we are acquiring X shares).

Let φ be a perturbation with $\varphi(0) = \varphi(T) = 0$. The first order criterion for optimal strategy (if there exists one) is

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} C[x + \epsilon\, \varphi, \dot{x} + \epsilon\, \dot{\varphi}] = 0.$$

Computing this derivative explicitly (suppressing explicit dependence on *t*),

$$\begin{aligned}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} C &= \int_0^T \left\{ \frac{\partial L}{\partial x}\, \varphi + \frac{\partial L}{\partial \dot{x}}\, \dot{\varphi} \right\} dt \\ &= \int_0^T \left\{ \frac{\partial L}{\partial x}\, \varphi - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}}\, \varphi \right\} dt + \left.\frac{\partial L}{\partial \dot{x}}\, \varphi \right|_{t=0}^{T} \quad \text{(applying integration by parts)} \\ &= \int_0^T \left\{ \frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} \right\} \varphi\, dt \quad (\text{since } \varphi(0) = \varphi(T) = 0)\end{aligned}$$

Since, this must hold for any perturbation φ with $\varphi(0) = \varphi(T) = 0$, we deduce

The Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Stochastic control problem

A stochastic control problem is a control problem which aims to minimize certain expected costs among all admissible controls.
Specifically, consider

$$\max_{v \in \mathcal{G}[0,T]} \mathbb{E}\left[g(X_T^{(v)}) + \int_0^T h(s, X_s^{(v)}, v_s) ds \right]$$

where

- g is referred to as the *terminal cost* and h is the *running cost*

- the state variable $X_t^{(v)}$ is driven by the controlled SDE

  $$
  dX^{(v)}_t = \mu(t,X^{(v)}_t, v_t)\, dt + \sigma(t,X^{(v)}_t, v_t)\, dW_t.
  $$

- $\mathcal{G}[0,T]$ is the collection of admissible controls in the time interval $[0,T]$

Bellman’s principle of optimality (Dynamic Programming Principle)

“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

(See Bellman, 1957, Chap. III.3.)

Value function

- For a given admissible control $v \in \mathcal{G}[t,T]$, define the expected objective functional $J^{(v)}$ as

  $$
  J^{(v)}(t,x) = \E\left[\left. g(X^{(v)}_T) + \int_0^T h(s,X^{(v)}_s,v_s) ds \right| X_t = x \right].
  $$

- The value function $J(t,x)$ for a stochastic control problem is defined as

  $$
  J(t,x) = \max_{v\in\mathcal G[t,T]} J^{(v)}(t,x).
  $$

- The value function J at (t,x) is the optimal value of the control problem conditioned on the process starting at (t,x) and applying the optimal control thereafter.

Bellman's Dynamic Progamming Principle again

Bellman's DPP can be recast in terms of the value function as follows. For any $0 < \epsilon < T - t$,

$$J(t,x) = \max_{v\in\mathcal{G}[t,t+\epsilon]} \mathbb{E}\left[\int_t^{t+\epsilon} h(s,X_s^{(v)},v_s)ds + J(t+\epsilon, X_{t+\epsilon}^{(v)}) \middle| \mathcal{F}_t\right].$$

The Hamilton-Jacobi-Bellman (HJB) equation

The value function J satisfies the terminal value problem

$$\partial_t J(t,x) + \max_{v \in \mathcal{G}[t]} \{\mathcal{L}^{(v)} J(t,x) + h(t,x,v)\} = 0, \text{ for } t < T$$

with terminal condition

$$J(T,x) = g(x),$$

where $\mathcal{L}^{(v)} := \frac{\sigma^2}{2}\partial_x^2 + \mu \partial_x$ is the associated infinitesimal generator of the controlled process

$$dX_t^{(v)} = \mu(t, X_t^{(v)}, v_t)\, dt + \sigma(t, X_t^{(v)}, v_t)\, dW_t.$$

- The HJB equation is effectively an infinitesimal version of Bellman's principle.
- The optimal policy (control) is given implicitly in terms of the value function J.

Deterministic and stochastic optimal control

- In *deterministic* optimal control, the evolution of the state vector is deterministic.
- In *stochastic* optimal control, the evolution of the state vector is stochastic.

Almgren and Chriss

- The seminal paper of [Almgren and Chriss][2] treats the execution of a meta order as a tradeoff between risk and execution cost.
- According to their formulation:
  - The faster an order is executed, the higher the execution cost
  - The faster an order is executed, the lower the risk (which is increasing in position size).
- Note that this is inconsistent with the empirical success of the square-root formula in describing the cost of meta orders.

Almgren and Chriss

For simplicity, we consider liquidation of an existing position X. Denote the position at time t by x_t with $x_0 = X$ and $x_T = 0$.

[Almgren and Chriss][2] model market impact and slippage as follows. The stock price S_t evolves as

$$dS_t = \gamma\, dx_t + \sigma\, dZ_t$$

and the price $\tilde{S}_t$ at which transactions occur is given by

$$\tilde{S}_t = S_t - \eta\, v_t$$

where $v_t := -\dot{x}_t$ is the rate of trading.

Price path in the Almgren and Chriss model

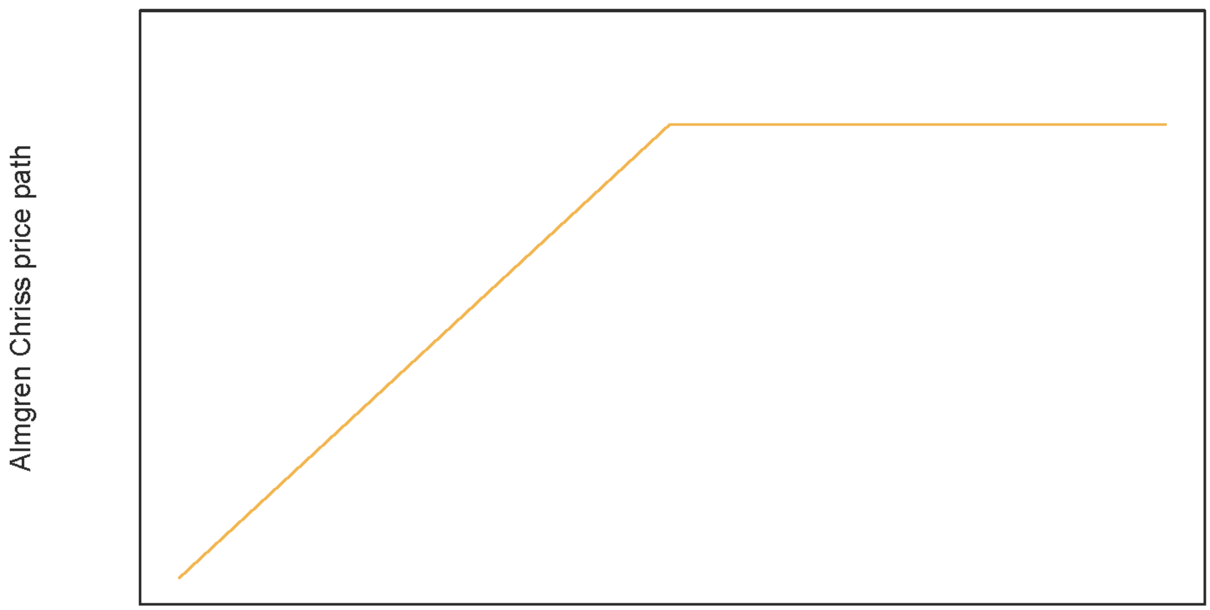


Figure 1: The Almgren and Chriss price path is plotted in orange.

Consistency with empirical observation

- This price path is inconsistent with empirical observation:
  - The average price path during execution is linear.
  - There is no price reversion after completion of the order.

P&L and cost associated with a trading strategy

Let x_t be a trading strategy. The corresponding P&L (up to time t), denoted by $\Pi_t(x)$, is identified as

$$\Pi_t(x) := x_t(S_t - S_0) + \int_0^t (S_0 - \tilde{S}_\tau)\, dx_\tau.$$

- The first term represents unrealized P&L on transactions (purchases or sales) yet to be executed.
- The second term corresponds to realized P&L on transactions executed up to time t.

Implementation shortfall

- The P&L $\Pi_T(x)$ associated with a trading strategy x terminating at time T is on average negative.
- The quantity $C_T(x) := -\Pi_T(x)$ is often referred to as *implementation shortfall*.
  - Our objective will be to find the strategy x that minimizes the cost $C_T(x)$.

P&L in the Almgren-Chriss model

Note that, at the end of execution period T, the P&L reads

$$\Pi_T(x) = x_T\,(S_T - S_0) + \int_0^T (S_0 - \tilde{S}_u)\,dx_u,$$

should there be x_T shares yet to be transacted.

Hence, in the Almgren-Chriss model

$$\begin{aligned}\Pi_T(x) &= x_T(S_T - S_0) + \int_0^T (S_0 - \tilde{S}_u)dx_u \\ &= \int_0^T [-\gamma(x_u - X) - \sigma W_u + \eta v_u]dx_u \quad \text{(note that } x_T = 0) \\ &= -\frac{\gamma}{2}X^2 + \sigma\int_0^T x_u dW_u - \eta\int_0^T v_u^2 du\end{aligned}$$

using integration by parts and that $v_u = -\dot{x}_u$.

The expected cost corresponding to the trading strategy x is then given by

$$\mathbb{E}\left[C_T(x)\right] = \frac{\gamma}{2}X^2 + \eta\int_0^T \mathbb{E}\left[v_u^2\right]du.$$

- The first term corresponds to permanent market impact and the second to temporary market impact.

An observation from Predoiu, Shaikhet and Shreve

Suppose the cost associated with a strategy depends on the stock price only through the term

$$\int_0^T S_t\,dx_t.$$

with S_t a martingale. Then assuming that the trading strategy x is of bounded variation, integration by parts gives

$$\mathbb{E}\left[\int_0^T S_t\,dx_t\right] = \mathbb{E}\left[S_T\,x_T - S_0\,x_0 - \int_0^T x_t\,dS_t\right] = -S_0\,X$$

which is independent of the trading strategy and we may proceed as if $S_t = 0$.

Quote from [Predoiu, Shaikhet, and Shreve][10]

> ...there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time.

Corollary

- This observation enables us to easily determine whether or not a statically optimal strategy will be dynamically optimal.
  - In particular, if the price process is of the form
$$S_t = S_0 + \text{ impact of prior trading} + \text{noise},$$
and if there is no risk term, a statically optimal strategy will be dynamically optimal.
  - If there is a risk term independent of the current stock price, a statically optimal strategy will again be dynamically optimal.

Expected cost of a VWAP in the Almgren and Chriss model

For a VWAP, $v_t = X/T$ where X is the total trade size and T is the duration of the order.
$$\begin{aligned}\mathbb{E}[C_T(x)] &= \mathbb{E}\left[\frac{\gamma}{2}X^2 - \sigma\int_0^T x_u dW_u + \eta\int_0^T \left(\frac{X}{T}\right)^2 du\right] \\ &= \frac{\gamma}{2}X^2 + \eta\frac{X^2}{T} \\ &= \left(\frac{\gamma}{2} + \frac{\eta}{T}\right)X^2.\end{aligned}$$

The cost per share of executing an order using VWAP is therefore
$$\hat{C} = \left(\frac{\gamma}{2} + \frac{\eta}{T}\right)X$$

which is linear in the trade size X.

The optimal strategy of a risk neutral trader

For a risk neutral trader whose objective is to minimize his total cost, the optimal control problem reads
$$\begin{aligned}&\min_v \mathbb{E}\left[C_T(x)\right] \\ &= \min_v \left\{\frac{\gamma}{2}X^2 + \eta\int_0^T \mathbb{E}\left[v_u^2\right] du\right\} \\ &= \frac{\gamma}{2}X^2 + \eta \min_v \int_0^T \mathbb{E}\left[v_u^2\right] du,\end{aligned}$$

where the state variable x_t is driven by $dx_t = -v_t dt$ with the constraints $x_0 = X$ and $x_T = 0$.

- Since S_t is not involved in the last expression, we can apply the observation of Predoiu, Shaikhet and Shreve to see that the optimal strategy v_t must be deterministic.

Variational problem

$$\min_v \int_0^T v_t^2\, dt$$

with $v_t = -\dot{x}_t$ and $x_0 = X, x_T = 0$.

The Euler-Lagrange equation is then

$$\partial_t v_t = -\partial_{t,t} x_t = 0$$

with boundary conditions $x_0 = X$ and $x_T = 0$ and the solution is obviously

$$v_t = \frac{X}{T};\ x_t = X\left(1 - \frac{t}{T}\right)$$

Adding a risk term

[Almgren and Chriss][2] add a risk term that penalizes quadratic variation, which is approximately the variance of the trading cost.

$$Var\left[\int_0^T x_t\, dS_t\right] \approx \sigma^2 \int_0^T x_t^2\, dt$$

In fact, by Itô's isometry, it is an equality if x_t is deterministic.

Ignoring the strategy independent permanent impact term $\frac{\gamma}{2} X^2$, the risk-adjusted cost to be minimized is then given by

$$\eta \int_0^T \dot{x}_t^2\, dt + \lambda\, \sigma^2 \int_0^T x_t^2\, dt$$

for some price of risk λ.

- Note the analogies to physics and portfolio theory.
  - The first term looks like kinetic energy and the second term like potential energy.
  - The expression looks like the objective in mean-variance portfolio optimization.

The Euler-Lagrange equation becomes

$$\ddot{x} - \kappa^2\, x = 0$$

with

$$\kappa^2 = \frac{\lambda\, \sigma^2}{\eta}$$

- The solution is a linear combination of terms of the form $e^{\pm \kappa t}$ that satisfies the boundary conditions $x_0 = X$, $x_T = 0$.

The solution is then

Almgren-Chriss optimal strategy
(2)

$$x(t) = X\, \frac{\sinh \kappa (T - t)}{\sinh \kappa T}$$

Once again, this statically optimal solution is dynamically optimal.

Efficient frontier in the Almgren Chriss model

- Impact cost is proportional to

$$C_I = \int_0^T \dot{x}_t^{\,2}\, dt.$$

- Risk cost is proportional to

$$C_R = \kappa^2 \int_0^T x_t^{\,2}\, dt.$$

VWAP

$$\begin{aligned} C_I &= \int_0^T \left(\frac{X}{T}\right)^2 dt = \frac{X^2}{T} \\ C_R &= \kappa^2 \int_0^T \left(\frac{X\,t}{T}\right)^2 dt = \kappa^2\, X^2\, \frac{T}{3}. \end{aligned}$$

Optimal

$$\begin{aligned} C_I &= \int_0^T \left(\frac{\frac{\partial}{\partial t}\sinh\kappa(T-t)}{\sinh\kappa T}\right)^2 dt = \frac{\kappa}{4}\,\frac{\sinh(2\kappa T)+2\kappa T}{\sinh^2\kappa T} \\ C_R &= \kappa\, X^2 \int_0^T \left(\frac{\sinh\kappa(T-t)}{\sinh\kappa T}\right)^2 dt = \frac{\kappa}{4}\,\frac{\sinh(2\kappa T)-2\kappa T}{\sinh^2\kappa T}. \end{aligned}$$

In [1]:
```r
# Figure 2: Almgren-Chriss Efficient frontier plot
costOpt <- function(kappa,T){1/4*kappa*(1/sinh(kappa*T))^2*(2*kappa*T + sinh(2*kappa*T))}
riskOpt <- function(kappa,T){1/4*kappa*(1/sinh(kappa*T))^2*(-2*kappa*T + sinh(2*kappa*T))}

costVWAP <- function(kappa,T){1/T}
riskVWAP <- function(kappa,T){kappa^2*T/3}

tt <- (20:1000)/200
rOpt <- riskOpt(1,tt)
cOpt <- costOpt(1,tt)
rVWAP <- riskVWAP(1,tt)
cVWAP <- costVWAP(1,tt)
```

In [2]:
```r
library(repr)
options(repr.plot.height=5)
```

In [3]:
```
plot(rOpt,cOpt,type="l",col="red",xlab="Risk",ylab="Cost",xaxt="n",yaxt="n")
points(rVWAP,cVWAP,type="l",col="blue",lty=2)
```

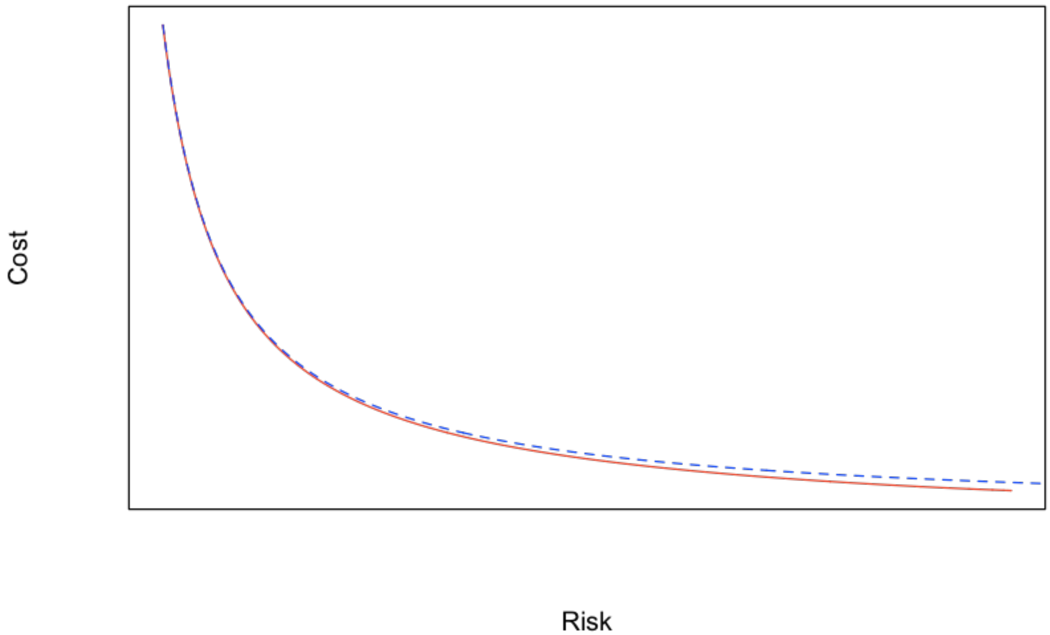


Figure 2: The optimal strategy (2) is in red and VWAP in blue.

Deterministic control problem

Alternatively, since there is no dependence on S_t, we can recast the problem of minimizing expected risk-adjusted cost as the following deterministic control problem:

$$\min_v \left\{ \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \, \sigma^2 \int_0^T x_t^2 \, dt \right\}$$

with state variable x_t driven by $dx_t = -v_t dt$ and $x_0 = X$.

The value function J is defined by

$$J(t, x) = \min_{v \in \mathcal{G}[t,T]} \left\{ \eta \int_t^T \dot{x}_t^2 \, dt + \lambda \, \sigma^2 \int_t^T x_t^2 \, dt \right\}$$

with $x_t = x$.

The HJB equation

The HJB equation reads

(3)
$$\frac{\partial J}{\partial t} + \lambda\, \sigma^2\, x^2 + \min_{v \in \mathcal{G}} \left\{ -v\, J_x + \eta\, v^2 \right\} = 0.$$

The optimal choice of v is
$$v^\star = \frac{J_x}{2\eta}.$$

Substituting back into (3) gives

(4)
$$\frac{\partial J}{\partial t} + \lambda \sigma^2 x^2 - \frac{J_x^2}{4\eta} = 0$$

which we solve by imposing the ansatz $J(t,x) = \eta\, a(t)\, x^2$.

Remark

To take into account the terminal condition that $x_T = 0$, we set the terminal condition for the HJB equation as
$$\lim_{t \uparrow T} J(t,x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

Solution to HJB equation

By substituting the ansatz $J(t,x) = \eta\, a(t)\, x^2$ into the HJB equation (4), we obtain
$$\dot{a}\, x^2 + \kappa^2 x^2 - a\, x^2 = 0.$$

Setting the coefficient of x^2 equal to zero yields the ODE for a:
$$\dot{a} + \kappa^2 - a^2 = 0.$$

Taking into account the terminal condition, we obtain the unique solution for a as
$$a(t) = \frac{\kappa}{\tanh \kappa\, (T - t)}.$$

Note

$a(t) \to \infty$ as $t \to T^-$.

Optimal strategy

The optimal trading rate v^* is obtained via the value function J as

$$v_t^* = \frac{J_x}{2\eta} = a(t)\, x_t^\star = \frac{\kappa\, x_t^\star}{\tanh \kappa\,(T-t)}.$$

Finally, recall that $v_t = -\dot{x}_t$, by solving the ODE

$$-\dot{x}_t = \frac{\kappa\, x_t}{\tanh \kappa\,(T-t)}.$$

with initial conditional $x_0 = X$, we obtain the Almgren-Chriss solution

$$x_t^\star = X\, \frac{\sinh \kappa\,(T-t)}{\sinh \kappa\, T}.$$

What happens if we change the risk term?

Suppose we penalize average VaR instead of variance. This choice of risk term has the particular benefit of being linear in the position size. The expected risk-adjusted cost is then given by

$$C = \eta \int_0^T \dot{x}_t^2\, dt + \lambda\, \sigma \int_0^T x_t\, dt$$

for some price of risk λ.

The Euler-Lagrange equation becomes

$$\ddot{x} - A = 0$$

with

$$A = \frac{\lambda\, \sigma}{2\, \eta}$$

The solution is a quadratic of the form $A\, t^2/2 + B\, t + C$ that satisfies the boundary conditions $x_0 = X, x_T = 0$. The solution is then

(5)

$$x(t) = \left(X - \frac{A\, T}{2}\, t\right)\, \left(1 - \frac{t}{T}\right)$$

In contrast to the previous case where the cost function is monotonic decreasing in the trading rate and the optimal choice of liquidation time is ∞, in this case, we can compute an optimal liquidation time.

When T is optimal, we have

$$\frac{\partial C}{\partial T} \propto \dot{x}_T^2 + A\, x_T = 0$$

from which we deduce that $\dot{x}_T = 0$.

Substituting into (5) and solving for the optimal time $T^\star$ gives

$$T^\star = \sqrt{\frac{2\, X}{A}}$$

With this optimal choice $T = T^\star$, the optimal strategy becomes

$$x(t) = X \left(1 - \frac{t}{T}\right)^2$$
$$u(t) = -\dot{x}(t) = 2\,X\,\left(1 - \frac{t}{T}\right)$$

Again, the static strategy is dynamically optimal, independent of the stock price.

ABM vs GBM

- One of the reasons that the statically optimal strategy is dynamically optimal is that the stock price process is assumed to be arithmetic Brownian motion (ABM).
- If for example geometric Brownian motion (GBM) is assumed, the optimal strategy depends on the stock price.
- How dependent is the optimal strategy on dynamical assumptions for the underlying stock price process?

Forsyth et al.

- [Forsyth et al.][5]</sup> solve the HJB equation numerically under geometric Brownian motion with variance as the risk term so that the (random) cost is given by

$$\mathcal{C} = \eta \int_0^T \dot{x}_t^2\,dt + \lambda\,\sigma^2 \int_0^T S_t^2\,x_t^2\,dt$$

- The efficient frontier is found to be virtually identical to the frontier computed in the arithmetic Brownian motion case.

- The problem of finding the optimal strategy is ill-posed; many strategies lead to almost the same value of the cost function.
- It is optimal to trade faster when the stock price is high so as to reduce variance. The optimal strategy is aggressive-in-the-money when selling stock and passive-in-the-money when buying stock.

Gatheral and Schied

[Gatheral and Schied][8] take time-averaged VaR as the risk term so that

(6)

$$C(0, X, S_0) = \inf_{v \in \mathcal{G}} \mathbb{E}\Big[\int_0^T v_t^2\,dt + \lambda \int_0^T S_t\,x_t\,dt\Big],$$

where the state variables are driven by

$$dS_t = \sigma\,S_t dW_t,$$
$$dx_t = -v_t dt,$$

and $\mathcal{G}$ is the set of admissible strategies.

The value function $C(t, x, s)$ should then satisfy the following Hamilton-Jacobi-Bellman PDE:

(7)

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{s,s} + \lambda s x + \inf_{v \in \mathbb{R}}(v^2 - v C_x) = 0.$$

with terminal condition

(8)

$$\lim_{t \uparrow T} C(t, x, s) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

The first order criterion

By substitute the first order criterion $v = \frac{C_x}{2}$ into the HJB equation we have

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + \lambda s x - \frac{C_x^2}{4} = 0.$$

Using the ansatz $C(t, s, x) = a(t) s^2 + b(t) s x + c(t) x^2$ we have

$$\dot{a} s^2 + \dot{b} s x + \dot{c} x^2 + \sigma^2 s^2 a + \lambda s x - \frac{(b s + 2 c x)^2}{4} = 0.$$

Compare the coefficients and obtain the following system of ODEs

$$\begin{aligned} s^2 &: \dot{a} + \sigma^2 a - \frac{b^2}{4} = 0, \\ s x &: \dot{b} + \lambda - b c = 0, \\ x^2 &: \dot{c} - c^2 = 0. \end{aligned}$$

Solution to the HJB equation

Taking into account the boundary condition (8), we obtain the solution to the system of ODEs as

$$\begin{aligned} a(t) &= \frac{\lambda^2}{8\sigma^6}\left[1 - e^{\sigma^2(T-t)} + \sigma^2(T-t) + \frac{1}{2}\sigma^4(T-t)^2\right], \\ b(t) &= \frac{\lambda}{2}(T-t), \\ c(t) &= \frac{1}{T-t}. \end{aligned}$$

Hence, the optimal trading rate v^* is given by the value function as

$$v_t^\star = \frac{C_x}{2} = \frac{\lambda}{4}(T-t)\, s_t + \frac{x_t^\star}{T-t}.$$

The optimal strategy under GBM

Theorem

The unique optimal trade execution strategy attaining the infimum in (6) is
(9)

$$x^\star_t = \left(\frac{T-t}{T}\right)\left[X - \frac{\lambda\,T}{4}\int_0^t S_u\,du\right]$$

Moreover, the value of the minimization problem in (6) is given by
(10) \begin{eqnarray} *C&=&\mathbb{E} \bigg[\,\int_0^T\bigg{(\dot x*^_t)^2+\lambda \,x^\star_t\,S_t\bigg}\,dt\,\bigg] \nonumber\ &=&\frac{X^2}T+\frac12\lambda \,T\,X\,S_0 \ &&+ \frac{\lambda^2}{8\,\sigma^6}\,S_0^2\,\Big(1 - e^{\sigma^2\, T} + \sigma^2 \,T + \frac12\,\sigma^4\,T^2\Big).\end{eqnarray*}

The optimal strategy under ABM

If we assume ABM, $S_t = S_0\,(1+\sigma W_t)$, instead of GBM, the risk term becomes

(11)

$$\hat{\lambda}\,S_0\int_0^T x_t\,dt.$$

As we already showed, the optimal strategy under ABM is just the static version of the dynamic strategy (9) obtained by replacing S_t with its expectation $\mathbb{E}[S_t] = S_0$, a strategy qualitatively similar to the Almgren-Chriss optimal strategy.

Comparing optimal strategies under ABM and GBM

As before, define the characteristic timescale

$$T^\star = \sqrt{\frac{4\,X}{\lambda\,S_0}}$$

and choose the liquidation time T to be $T^\star$.

With $T = T^\star$, the optimal trading rate under ABM becomes

(12)

$$v^A(t) = \frac{x_t}{T-t} + \frac{X}{T^2}\,(T-t) = \frac{2\,X}{T}\left(1-\frac{t}{T}\right)$$

and the optimal trading rate under GBM becomes

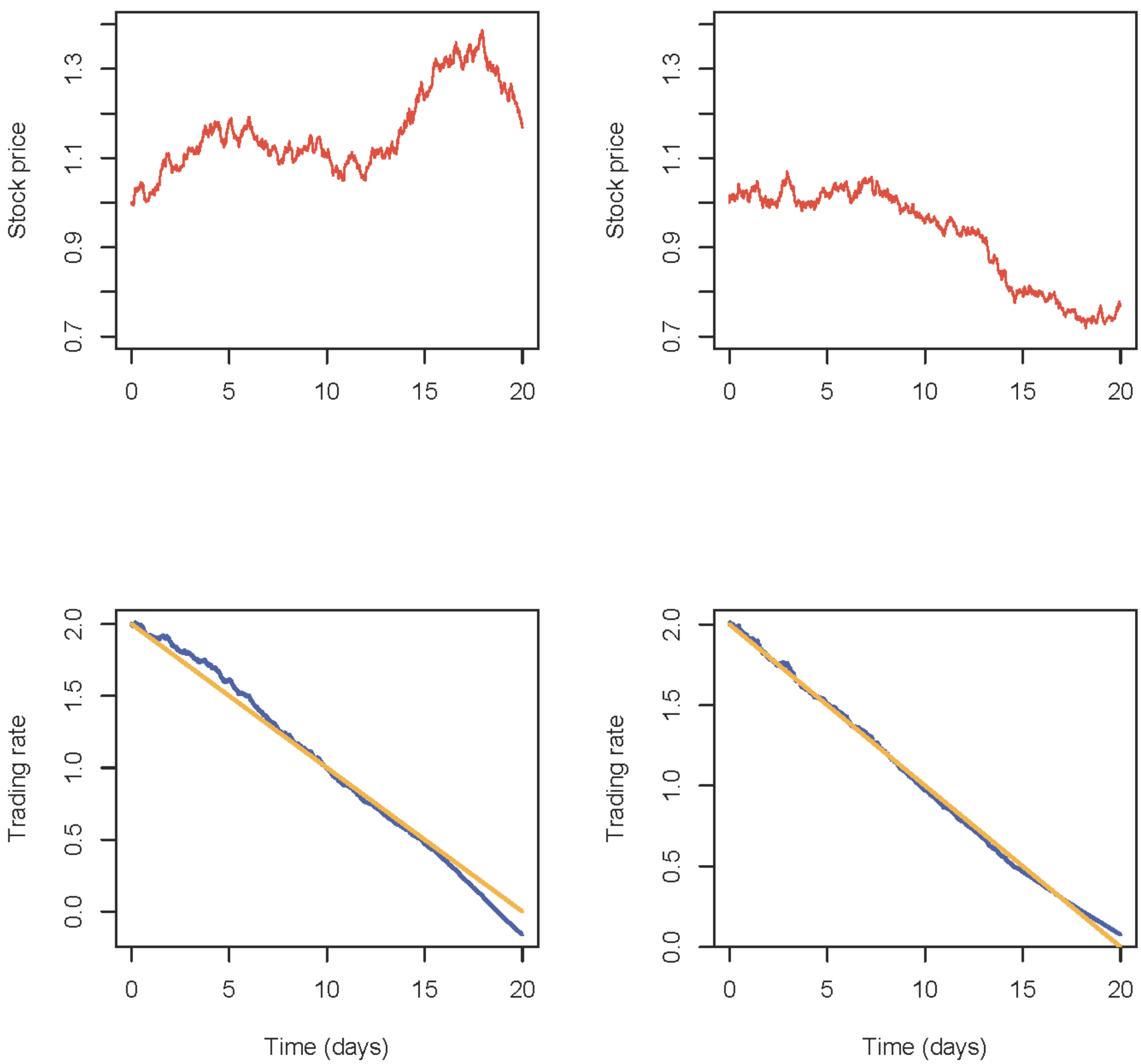
(13)

$$v^G(t) = \frac{x_t}{T-t} + \frac{X}{T^2}\,\frac{S_t}{S_0}\,(T-t).$$

Comparing optimal strategies under ABM and GBM

In the following slide:

- The upper plots show rising and falling stock price scenarios respectively; the trading period is 20 days and daily volatility is 4%.
- The lower plots show the corresponding optimal trading rates from (12) and (13); the optimal trading rate under ABM is in orange and the optimal trading rate under GBM is in blue.
- Even with such extreme parameters and correspondingly extreme changes in stock price, the differences in optimal trading rates are minimal.



Remarks

- For reasonable values of $\sigma^2\, T \ll 1$, there is almost no difference in expected costs and risks between the optimal strategies under ABM and GBM assumptions.
- Intuitively, although the optimal strategy is stock price-dependent under GBM assumptions but not under ABM assumptions, when $\sigma^2\, T \ll 1$, the difference in optimal frontiers is tiny because the stock-price S_t cannot diffuse very far away from S_0 in the short time available.
- Equivalently, as in the plots, there can only be a small difference in optimal trading rates under the two assumptions.

The Almgren 2005 model

In the (2005) model of [Almgren][1], the stock price S_t evolves as

$$dS_t = \gamma\, dx_t + \sigma\, dZ_t$$

and the price $\tilde{S}_t$ at which we transact is given by

$$\tilde{S}_t = S_t - \eta\, v_t^\delta$$

where $v_t := -\dot{x}_t$ is the rate of trading.

The expected cost of trading is then given by

$$\begin{aligned} C &= \mathbb{E}\left[\int_0^T \tilde{S}_t\, v_t\, dt\right] \\ &= \int_0^T (\gamma\, x_t + \eta\, v_t^\delta)\, v_t\, dt \\ &= \frac{\gamma}{2}\,(x_T^2 - x_0^2) + \eta\, \int_0^T v_t^{1+\delta}\, dt \end{aligned}$$

where wlog, we have set $S_0 = 0$.

We see that the first term corresponding to permanent impact is independent of the trading strategy, as it should be. The second term is convex in the trading rate so the minimum cost strategy is again VWAP.

Applications of the Almgren-Chriss framework

- Although the Almgren and Chriss price process is not particularly realistic, it leads to a tractable framework for solving a number of interesting practical problems.

- Applications include:
  - Portfolio liquidation (*e.g.* [Schöneborn][12])
  - Optimal liquidation with a dark pool (*e.g.* [Kratz and Schöneborn][9])
  - Optimal delta-hedging of options under transactions costs (*e.g.* [Li and Almgren][10])
  - Optimal liquidation in the presence of a trading signal (*e.g.* [Almgren(2012)][3])

- Various other practical problems such as targeting VWAP are solved under Almgren-Chriss assumptions using the HJB equation in the book by [Cartea, Jaimungal and Penalva][5].

- As an example, following [Kratz and Schöneborn][8], we will now solve the optimal liquidation problem in the presence of a dark pool under Almgren-Chriss assumptions.

Optimal liquidation with a dark pool

- We suppose we have one primary lit venue and one dark pool.
- Trading in the lit venue incurs temporary market impact according to the usual Almgren-Chriss assumptions.
- Trading in the dark pool is costless.
- Dark pool executions do not affect the price in the primary venue.
- Trade executions arrive in the dark pool as a Poisson process with intensity θ.
- When a trade occurs in the dark pool, your entire posted quantity y_t is executed.

State variables and equations

Denote the current position by x_t, the (negative) trading rate in the lit pool by $v_t = -\dot{x}_t$, and the quantity currently posted in the dark pool by y_t.

The state variable is thus x_t and the controls are v_t and y_t. The state equation is

$$dx_t = -v_t\,dt - y_t\,dN_t$$

where N_t is a Poisson process with intensity θ.

Note

The infinitesimal generator $\mathcal{L}$ associated the process x_t is given by

$$\mathcal{L}u = -v_t \partial_x u + \theta(u(x-y) - u(x)).$$

Cost of liquidation

As before, the expected cost of liquidation (with risk penalty), under the admissible strategy $(v, y) \in \mathcal{G}[t, T]$, is given (as of time t) by

$$C^{(v,y)}(t,x) = \eta\,\mathbb{E}_t\left[\int_t^T (v_u^2 + \kappa^2\,x_u^2)\,du\right]$$

with

$$\kappa^2 = \frac{\lambda\,\sigma^2}{\eta}.$$

The HJB equation

The value function $\mathcal{C}$

$$\mathcal{C}(t,x) := \min_{v,y\in\mathcal{G}[t,T]} C^{(v,y)}(t,x)$$

satisfies the following HJB equation

Kratz-Schöneborn HJB equation

(14)

$$\frac{\partial\mathcal{C}}{\partial t} + \eta\,\kappa^2\,x^2 + \min_{v,y\in\mathcal{G}}\left\{-v\,\mathcal{C}_x + \eta\,v^2 + \theta\,\left[\mathcal{C}(t,x-y) - \mathcal{C}(t,x)\right]\right\} = 0$$

with boundary condition $C(t, 0) = 0$ for all $0 \le t < T$ and terminal condition

$$\lim_{t\uparrow T} C(t,x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

The first order conditions

The value of y that minimizes $[C(t, x-y) - C(t,x)]$ is obviously $y^\star = x$.

- This is completely intuitive; if there is no cost of execution in the dark pool and such executions don't affect the price in the primary venue, continuously post the full amount in the dark pool.

Differentiating wrt v, we get the first order condition on v

$$v^\star = \frac{1}{2\,\eta}\,C_x.$$

Substituting back into (14), we get

$$\frac{\partial C}{\partial t} + \eta\,\kappa^2\,x_t^2 - \frac{1}{4\,\eta}\,(C_x)^2 - \theta\,C = 0$$

As before, we solve the HJB equation by imposing the ansatz

$$C(t,x) = \eta\,a(t)\,x^2.$$

Solution to the HJB equation

Substituting the ansatz into the HJB equation yields the ODE for a

$$\dot{a} + \kappa^2 - a^2 - \theta a = 0.$$

Taking into account the terminal condition as in (8), the solution of a is given by

$$a(t) = \frac{\tilde{\theta}}{2}\coth\left(\frac{\tilde{\theta}}{2}\,(T-t)\right) - \frac{\theta}{2},$$

where $\tilde{\theta}^2 = \theta^2 + 4\,\kappa^2$.

Optimal strategy

The optimal trading rate $v_t^\star$ is determined via the value function C as

$$v^\star = \frac{1}{2\,\eta}\,C_x = a(t)\,x_t = \left\{\frac{\tilde{\theta}}{2}\coth\left(\frac{\tilde{\theta}}{2}\,(T-t)\right) - \frac{\theta}{2}\right\}x_t^\star.$$

Hence, by solving

$$-\dot{x}_t = \left\{\frac{\tilde{\theta}}{2}\coth\left(\frac{\tilde{\theta}}{2}\,(T-t)\right) - \frac{\theta}{2}\right\}x_t^\star$$

with the initial condition $x_0 = X$, the optimal trading strategy $x_t^\star$ is given by

(15)

$$x^\star(t) = X\,e^{\frac{\theta\,t}{2}}\,\frac{\sinh\left(\frac{1}{2}(T-t)\sqrt{\theta^2+4\kappa^2}\right)}{\sinh\left(\frac{1}{2}T\sqrt{\theta^2+4\kappa^2}\right)}$$

which obviously gives the usual Almgren-Chriss strategy in the limit $\theta \to 0$.

Remarks on the solution

Recall that

$$v^\star(t) = \left\{ \frac{\tilde{\theta}}{2} \coth\left(\frac{\tilde{\theta}}{2} \, (T - t) \right) - \frac{\theta}{2} \right\} \, x^\star(t).$$

where $\tilde{\theta}^2 = \theta^2 + 4\,\kappa^2$.

- The more likely an execution in the dark pool, the slower the optimal rate of trading in the lit venue.
- “every explicit solution (of the HJB equation) is a triumph over nature” – Bernt Øksendal.

Optimal single stock strategy from [Kratz and Schöneborn][7]

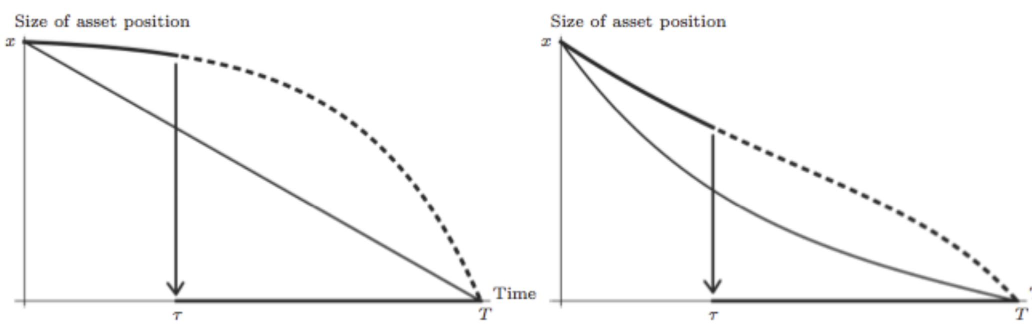


Figure 1: Optimal trading trajectories for risk-neutral (left picture) respectively risk-averse traders (right picture). In both cases, the thick solid lines denote the optimal trading trajectory with dark pools in the scenario where the dark pool order is executed at time τ. The dashed lines denote the scenario where the dark pool order is not executed during the entire trading horizon. The thin solid lines denote optimal liquidation without dark pools. $T = 1$, $\Lambda = 1$, $\theta = 4$ and (for the right picture) $\alpha = 6$, $\Sigma = 1$.

Optimal portfolio liquidation strategy

- In the portfolio case, it is no longer optimal in general to place the entire remaining quantity in the dark pool
    - because obtaining a dark pool execution may unbalance the portfolio and incur a greater risk cost.
- If the starting portfolio is balanced, the optimal quantity to place in the dark pool tends to be small
    - The optimal strategy is to trade out of the position almost linearly.

- If the starting portfolio is unbalanced, the optimal quantity to place in the dark pool tends to be a large proportion of the remaining quantity.
- It may be optimal to short a stock that one is trying to liquidate if it is a good hedge for other less liquid positions (think of SPY).

Optimal portfolio strategy from [Kratz and Schöneborn][7]

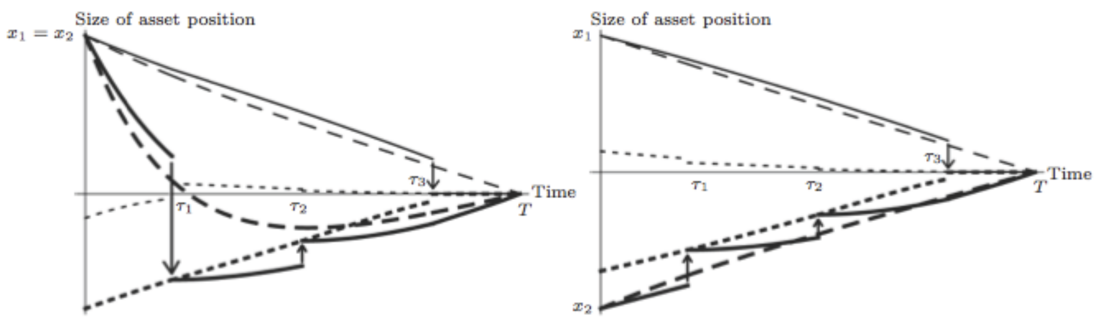


Figure 2: Evolution of a portfolio consisting of two highly correlated stocks over time. The left figure illustrates the poorly diversified portfolio, the right figure the well diversified portfolio. In both pictures thin lines are used for the less liquid first stock and thick lines for the more liquid second stock. Dashed lines correspond to trading without the dark pool and solid lines correspond to a realization of the liquidation process using the dark pool, where dark pool orders for the second stock are executed at times τ_1 and τ_2 and for the first stock only at time τ_3. Dotted lines correspond to the position which the investor aims to reach by her dark pool order for the respective stock.

Summary

- The Almgren-Chriss price process is in practice the most widely-used.
- It forms the basis for many of the algorithms and most of the thinking in algorithmic execution.
  - despite the fact that it is unrealistic: market impact decays instantaneously and it is completely incompatible with the square-root law.
- Because of the analytical tractability of the Almgren-Chriss framework, there are closed-form or quasi-closed-form solutions for many problems of practical interest.

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