MTH 9879 Market Microstructure Models, Spring 2019

Lecture 2: Order book and order flow: The market or limit order decision

Jim Gatheral
Department of Mathematics

Outline of Lecture 2

- Biais, Hillion, and Spatt (1995)
- Order flow from order book: The market order or limit order decision
  - The Parlour (1998) model
  - The Foucault Kadan Kandel (2005) model
  - The Roşu (2009) model
  - Cont and Kukanov (2013)
- Order book from order flow
  - Bouchaud, Mézard and Potters order book shape approximation
  - Mike and Farmer empirical law of order arrivals

Price signal in the ZI simulation

- Recall from Lecture 1 that even in the ZI model, the shape of the order book allows prediction of price movements.
  - Traders really would need to have zero intelligence not to condition on book shape!

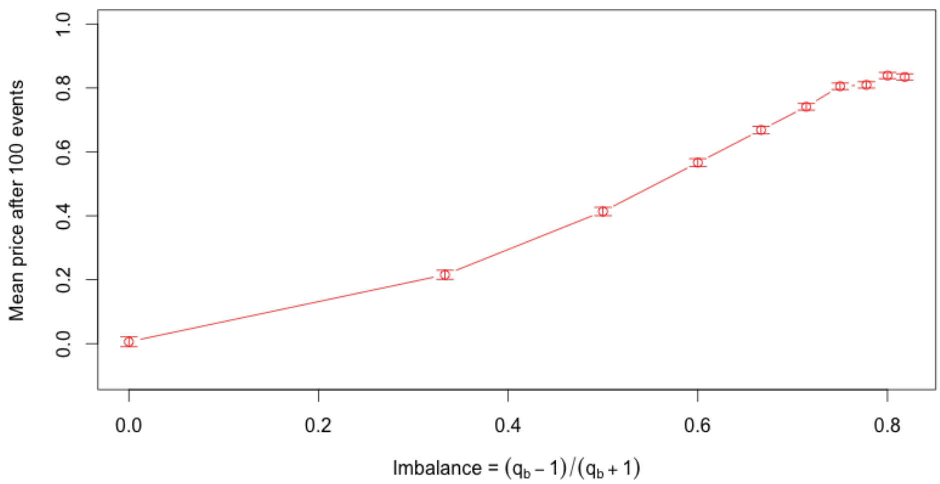


Figure 1: With one share at best offer, future price change vs book imbalance.

Biais, Hillion and Spatt

- In the zero-intelligence models, limit order, market order and cancelation processes are all independent.
  - In particular, these processes do not depend on the state of the order book.
- [Biais, Hillion, and Spatt][1] performed an empirical analysis of the interaction between the order book and order flow processes on the Paris Bourse.

Biais, Hillion and Spatt results

Table IV

Frequency of Orders and Trades, Given the Last Order or Trade

For the 19 trading days in the period between October 29 and November 26, 1991, for the stocks included in the CAC 40 index at that time, Table IV reports the empirical percent frequency on each of the 15 events, conditional upon the type of the previous event. Each row corresponds to a given event at time *t*-1. Each column corresponds to a given event at time *t*. Each row can be thought of as a probability vector and adds up to 100 percent. The empirical frequencies are obtained after pooling all stocks. To facilitate interpretation, the three largest numbers in each column are in bold face type.

t-1	Application	Large Buy	Market Buy	Small Buy	New Bid Within	New Bid At	New Bid Below	Cancel Bid	Large Sell	Market Sell	Small Sell	New Ask Within	New Ask At	New Ask Above	Cancel Ask
Application	0.00	**3.56**	1.98	13.25	9.45	5.42	6.44	5.93	4.15	2.49	21.53	9.10	4.87	6.19	5.64
Large buy	**2.94**	**7.29**	**3.41**	15.34	**13.05**	5.01	5.90	5.21	1.80	1.11	14.39	7.26	4.65	6.83	**5.81**
Market buy	1.94	3.20	2.47	**17.11**	2.14	**6.88**	**11.33**	**7.92**	3.35	**6.16**	23.49	2.78	2.45	4.84	3.94
Small buy	2.43	**3.71**	**2.89**	**20.49**	9.67	5.27	6.33	4.02	2.35	1.75	19.77	5.88	4.54	6.20	4.72
New bid within	2.33	3.19	2.14	13.12	**13.62**	6.87	**9.21**	5.22	3.01	1.98	18.58	8.28	2.92	5.42	4.11
New bid at	1.92	2.23	2.40	14.01	**16.52**	**6.88**	7.36	4.93	1.61	0.90	22.64	6.05	3.84	5.30	3.40
New bid below	1.90	1.90	1.60	11.50	7.50	**7.99**	**18.24**	4.45	3.23	2.41	19.99	6.92	3.37	5.58	3.42
Cancel bid	**2.67**	2.27	1.71	10.56	12.45	5.21	8.05	**9.70**	3.80	2.26	19.54	8.70	3.57	5.79	3.72
Large sell	**2.49**	0.92	0.93	6.46	8.82	4.54	6.53	**5.97**	**9.84**	**3.33**	**23.91**	12.57	3.33	4.99	5.37
Market sell	1.27	2.89	**6.33**	**16.16**	2.28	2.92	4.28	3.60	**4.82**	**3.04**	**29.35**	2.21	4.98	**9.50**	**6.37**
Small sell	2.16	1.59	1.60	10.27	6.92	4.78	5.80	4.37	**4.82**	2.96	**33.87**	7.88	4.00	5.29	3.71
New ask within	2.15	2.01	1.87	9.86	9.43	4.03	6.04	4.06	4.44	2.60	22.67	**12.61**	**5.53**	**7.76**	4.94
New ask at	1.94	1.41	0.97	12.79	6.41	4.81	5.65	3.83	3.00	2.48	23.10	**14.74**	**6.65**	7.30	4.92
New ask above	2.08	2.24	2.15	10.81	7.24	4.12	5.86	3.36	2.81	2.10	21.35	5.85	**7.07**	**19.36**	3.60
Cancel ask	2.48	2.03	1.85	9.95	8.73	4.39	6.10	4.51	4.06	2.13	19.19	**13.84**	3.95	7.57	**9.24**
Unconditional	2.2	2.4	2.1	12.5	9.0	5.3	7.4	4.7	3.8	2.4	24.0	8.3	4.3	7.0	4.8

Limit Order Book and Order Flow in Paris Bourse 1673

Biais, Hillion and Spatt: Observations

- We note the *diagonal* effect:
    - Events with the same sign are more frequent that events with different signs.
    - The probability that a given type of order or trade occurs is larger after this event has just occurred than it would be unconditionally.
- This is inconsistent with the zero-intelligence (ZI) picture.
    - In the ZI picture, the relative probabilities of events are independent of previous events.
    - In real markets, order flow depends on the shape of the order book.
- In a ZI market, traders submit orders and cancelations without looking either at the current state of the book or analyzing order flow.
- In real markets, traders look at the screen before trading!

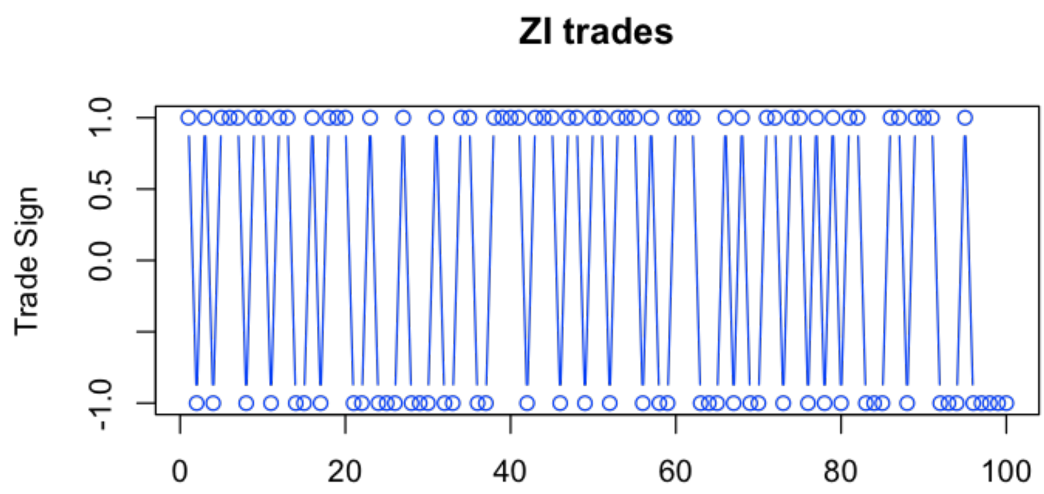
Time series of trade signs: MSFT vs ZI

In [1]:
```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2015/01/ziSetup.zip", destfile="ziSetup.zip")
unzip(zipfile="ziSetup.zip")
source("ziSetup.R")
```

In [2]:
```
# rescale plot
# library(repr)  # load required package if necessary
options(repr.plot.width=6, repr.plot.height=3.5)
```

In [3]:
```
# Simulate and generate time series of returns
alpha <-1; mu<-10; delta <- 0.2; # Asymptotic book depth is 5
numEvents <- 10000
initializeBook5()
for(count in 1:numEvents){generateEvent()}

tradeLog <- eventLog[(eventLog$Type=="MB")|(eventLog$Type=="MS"), ]
tradeSigns <- ifelse(tradeLog$Type=="MB", +1, -1)
someTrades <- 1:100
plot(tradeSigns[someTrades], type="b", col="blue", xlab=NA, ylab="Trade Sign",
     main="ZI trades")
```



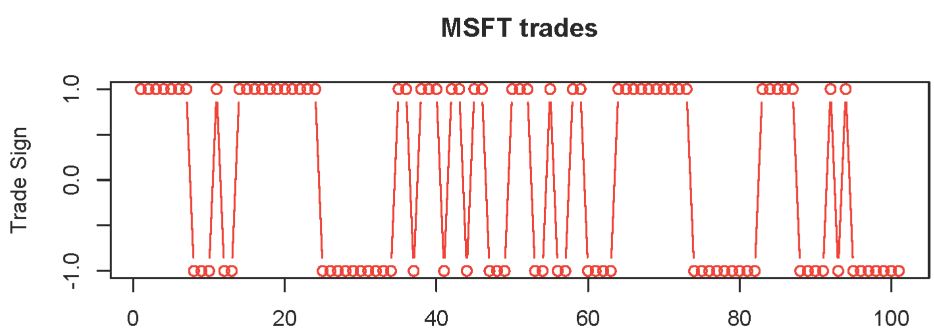


Figure 2: Sequences of trade signs from real data and from ZI simulated data

Observations

- We don’t see any particular pattern in the ZI market.
- In the real MSFT market though, we see that MB tends to follow MB and MS tends to follow MS.
  - It seems that real traders condition their trades on something.
  - In real markets, traders look at the screen before trading!

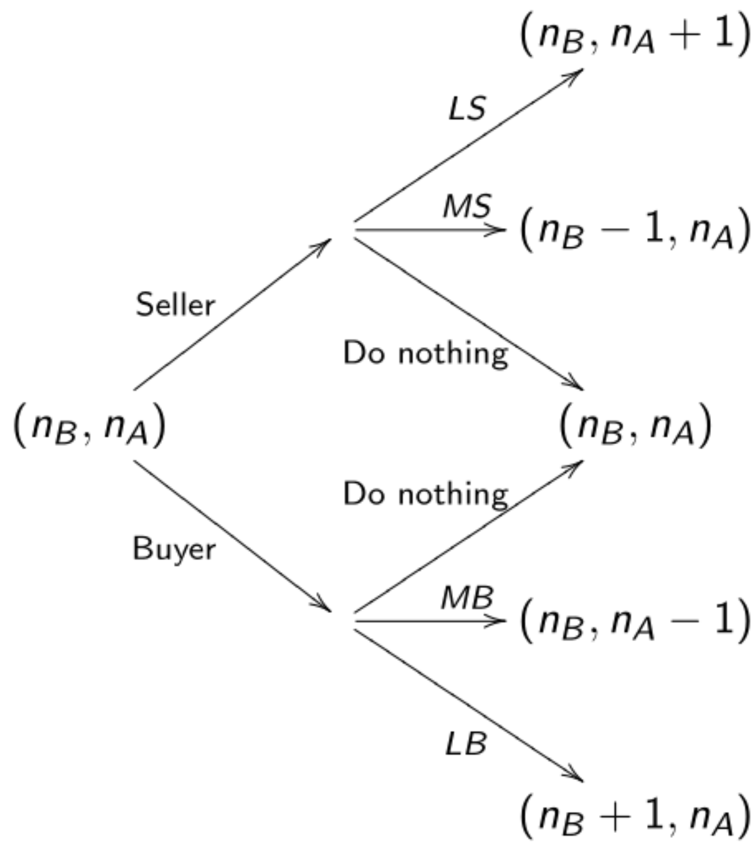
The limit or market order decision

- We start by describing one of the cleverest models of strategic trading due to Christine Parlour.
  - Traders correctly compute the probabilities of future events conditional on the current state of the order book and their own actions.
  - Traders send a market order or a limit order, whichever is better.
  - Philosophically the *polar opposite* of the zero-intelligence assumption.

The Parlour (1998) model

- Trading occurs on day 1, starting at time $t = 1$ and ending at $t = T$.
- Everyone knows that on day 2, the asset will be worth V.
- Infinite order-book depth at the bid B and the ask A so these don’t change over time.
  - $B = V - s/2;\ A = V + s/2;$
- At each time t during the trading day, one (new and different) trader arrives who is a buyer or seller with probability $1/2$.
  - This trader may submit a market or a limit order or do nothing.
- The trader at time t has utility
$$U_t = C_1 + \beta_t\, C_2 \ \text{ with } \beta_t \in \left[\underline{\beta}, \bar{\beta}\right]$$
where β_t is a time-preference parameter and the C_i denote consumption on day $i = 1, 2$.
  - For example, if trader t is a seller, the lower β_t is, the more likely he is to submit a market order.

A generic timestep in the Parlour model



At time T

- There is no point in sending a limit order because the probability of a fill is zero.
- For a seller, the increase in utility (benefit) associated with a market sell would be

$$\Delta U_T = B - \beta_T\, V.$$

  Thus a seller will enter a market order only if $B - \beta_T\, V > 0$ or equivalently if $\beta_T < B/V$.

- We can then compute the unconditional probability as of time $T-1$ of a market sell (MS) in period T:

$$\mathbb{P}^S_T = \frac{1}{2}\,\mathbb{P}\left(\beta_T < \frac{B}{V}\right).$$

- Similarly,

$$\mathbb{P}^B_T = \frac{1}{2}\,\mathbb{P}\left(\beta_T > \frac{A}{V}\right).$$

At time $T - 1$

- There is now some potential benefit to submitting a limit order.
- The upside associated with a limit order depends on the probability of a fill (respectively MB or MS) in period T.
  - These fill probabilities depend in particular on the distribution of β_T.
- There is no point in sending a limit order if there is existing quantity in the order book.
  - The limit order at the front of the queue has priority.

At generic time t

- The upside associated with a limit order depends on the probability of a fill (respectively MB or MS) in subsequent periods.
  - These fill probabilities depend in particular on the distribution of β_t.
- The decision to send a limit order depends on position in the queue.
  - So the market order/ limit order decision depends on the number of shares n_t on the same side of the book.
- The market order/ limit order decision depends also on the length of the queue on the opposite side of the order book.
  - Consider the case of a seller, the longer the queue on the bid side, the more likely it is that a market buy order will be submitted in subsequent periods and the more likely that a limit sell order will be filled.

Time t payoffs

Table 1: Time t payoffs

Strategy	**Payoff**
Market sell (MS)	$B - \beta_t\, V$
Limit sell (LS)	$(A - \beta_t\, V)\, \mathbb{P}^B_{t+i}$
Limit buy (LB)	$(\beta_t\, V - B)\, \mathbb{P}^S_{t+i}$
Market buy (MB)	$\beta_t\, V - A$

- From the payoffs in [Table 1](#tab:payoffs), we can compute the probabilities of a market buy and market sell and by recursion, compute the same for period $t - 1$.

A numerical example

Following [Parlour][7], we assume the following parameters:

$$V = 5.5;\ B = 5;\ A = 6;\ T = 3 \text{ and } \beta_t \sim U(0, 2).$$

At time $t = T = 3$,
$$ {\mathbb{P} }(MS|S)=\frac{1}{2}\,\int_0^2\,d\beta\,{\mathbf{1} }_{\{\beta**Thus**

$$\mathbb{P}^S_3 = \frac{5}{22}.$$

By symmetry,

$$\mathbb{P}^B_3 = \frac{5}{22}.$$

Time $t = 2$: No existing quantity in the order book

Suppose the order book is empty at time $t = 2 = T - 1$.

- **The utility of a market sell order is $B - \beta_t\, V$.**
- **The utility of a limit sell order is $(A - \beta_t\, V)\, \mathbb{P}_3^B$.**

Thus, submit a MS only if $B - \beta_t\, V \geq (A - \beta_t\, V)\, \mathbb{P}_3^B$ or equivalently

$$\beta_2 \leq \frac{B - A\, \mathbb{P}_3^B}{V\,(1 - \mathbb{P}_3^B)} = \frac{B - A\, \mathbb{P}_3^B}{V\,(1 - \mathbb{P}_3^B)} = \frac{160}{187}.$$

Continuing in this way, we compute that a seller will choose as follows:

Action	Condition	Numerically
Marketsell(MS)	if $B - \beta_t\, V \geq (A - \beta_t\, V)^+\, \mathbb{P}_3^B$	$0 \leq \beta_t < \frac{160}{187}$
Limitsell(LS)	if $(A - \beta_t\, V)\, \mathbb{P}_3^B > (B - \beta_t\, V)^+$	$\frac{160}{187} \leq \beta_t < \frac{12}{11}$
Do nothing	otherwise	$\frac{12}{11} \leq \beta_t \leq 2$

Time $t = 2$: No existing quantity in the order book

Thus, conditional on an empty book and the trader being a seller,

$$\mathbb{P}_{MS} = \mathbb{P}\left(\beta_2 < \frac{160}{187}\right) = \frac{80}{187}$$
$$\mathbb{P}_0 = \mathbb{P}\left(\beta_2 > \frac{12}{11}\right) = \frac{5}{11}$$
$$\mathbb{P}_{LS} = 1 - \mathbb{P}_{MS} - \mathbb{P}_0 = 1 - \frac{80}{187} - \frac{5}{11} = \frac{2}{17}.$$

What if the book is not empty?

- **Assume $n_B > 0$ and $n_A > 0$. It is not optimal to leave a limit order.**
  - **This case reduces to the case $t = 3 = T$.**
- **Assume $n_B > 0$ and $n_A = 0$.**
  - **It is not optimal for buyer to leave a limit order.**
  - **Seller behaves as in the case of empty book. For example, seller will submit a limit order if $160/187 < \beta < 12/11$.**
- **We need to expand our state space to include the state of the book.**
  - **The market/limit order decision depends on the state of the order book.**
  - **We need richer notation. For example $\mathbb{P}_{\text{event}}[n_B, n_A](t)$.**

Model predictions

- **An increase in book depth n_B on the bid-side decreases the probability of a limit buy order.**
- **A decrease in book depth n_A on the ask-side decreases the probability of a market sell order in subsequent periods and so decreases the probability of a limit buy order.**
- **Order signs will be autocorrelated - a *herding* effect.**
  - **A market sell order increases the probability of a subsequent market sell and decreases the probability of a subsequent market buy.**

Comments

- **In the Parlour model, it is optimal to herd even though the bid and ask prices do not move.**
  - **Conditional on MB, the probability of MB increases if agents are rational.**
- **Even in the ZI market, we can predict the expected price change from the current shape of the order book.**
  - **We are far from efficient markets at this micro timescale.**
  - **It is optimal to herd not only for the strategic reasons illuminated by the Parlour model but also because order book shape and order flow are informative for future price movements.**
- **It follows in particular that both market and limit orders have market impact.**
  - **In fact, even hidden orders will have market impact due to pinging!**

Zero intelligence vs efficient markets

- **Traders certainly do condition their actions on the current state of the market as well as the history of order flow.**
- **However, we cannot believe the caricature that rational traders are able to compute the rational reaction of other agents to their own rational actions.**
  - **We will see later when we study the Kyle model that this caricature may nevertheless describe an equilibrium resulting from agents performing linear regression.**
- **Reality lies somewhere between these two extremes.**

Foucault, Kadan and Kandel (2005)

In this model:

- **Liquidity traders choose between market orders and limit orders.**
  - **This choice depends on their degree of impatience.**
- **Different traders value immediacy differently.**
  - **Type $\mathcal{P}$ traders incur a waiting cost at rate δ_P and type $\mathcal{I}$ traders at rate δ_I with $\delta_I > \delta_P > 0$.**
  - **Traders arrive as a Poisson process at rate λ.**
  - **The proportion of patient traders is θ_P. $\theta_I = 1 - \theta_P$.**
- **Order book depth is infinite at B and A, where $A > B$. The best quotes b and a satisfy $b \geq B$ and $a \leq A$, so $s := a - b < K := A - B$. Prices are in discrete ticks Δ.**

Trading rules

- **A trader arrives only once, submits an order and leaves the market. Orders cannot be modified or canceled.**
- **Limit orders must be price improving; they must narrow the spread.**
- **Buyers and sellers alternate. The first trade is a buy with probability $\frac{1}{2}$.**

Computation of equilibrium

Wlog, take tick size $\Delta = 1$.

- **A buyer can either take the offer at a or submit a limit buy order at $a - j$ with $j \in \{1, \cdots, s-1\}$.**
  - **Higher j's are less aggressive.**
  - **$j = 0$ corresponds to market order.**
- **The trader needs to compute**
$$\max_{j \in \{0,\ldots,s-1\}} \pi_i(j) := j - \delta_i \, T(j)$$
where $T(j)$ is the expected time to execution and $i \in \{\mathcal{I}, \mathcal{P}\}$. $T(0) = 0$ since market order is executed immediately.
- **For trader $i \in \{\mathcal{I}, \mathcal{P}\}$, an order placement strategy o_i is a mapping that, for a given spread $s \in \{1, \cdots, K\}$, assigns a limit order at $j \in \{0, \cdots, s-1\}$, i.e., $j = o_i(s)$.**
- **Optimal strategies assign a j to each possible spread s.**
- **An equilibrium is a pair of order placement strategies $o^*_{\mathcal{I}}$, $o^*_{\mathcal{P}}$ and a expected waiting time T^* such that**
  - **$o^*_{\mathcal{I}}$ and $o^*_{\mathcal{P}}$ maximizes their respective expected utility in expected waiting time T^***
  - **the expected waiting time T^* is computed assuming that the traders follow their optimal strategies.**

Equilibrium solution

- **Start at the end node: If $s = 1$, the trader, whether $\mathcal{P}$ or $\mathcal{I}$, needs to submit a market order.**
- **If $s = 2$ compare submission of a limit order with $j = 1$ or a market order.**
  - **If a limit order is submitted, it will shrink the spread to 1 and will be executed next period. So $T(1) = \frac{1}{\lambda}$.**
  - **The payoff from a market order is $0 - \delta_i \, T(0) = 0$.**
  - **The payoff from a limit order is $1 - \frac{\delta_i}{\lambda}$ which is greater for $\mathcal{P}$ than for $\mathcal{I}$.**
- **Continuing by induction, we can compute optimal actions for both $\mathcal{P}$ and $\mathcal{I}$ traders for all possible spreads $s \leq K = A - B$.**
  - **In particular, there is some minimum spread level s^* such that both $\mathcal{P}$ and $\mathcal{I}$ traders will in general submit only market orders if $s \leq s^*$.**
  - **Also, there is a maximum spread level beyond which both $\mathcal{I}$ and $\mathcal{P}$ traders will submit limit orders.**

Final result

- **The final expression for the expected time to execution is**

  (1)
$$T(s_m) = \frac{1}{\lambda}\left\{1 + 2\sum_{k=1}^{m-1}\left(\frac{\theta_P}{1-\theta_P}\right)^k\right\}$$
where s_m is the mth spread value in the table of equilibrium spreads given below.
- **The equilibrium spreads are equal to:**

  (2)
$$s_1 = s^*; \; s_m = s_1 + \sum_{k=2}^{m} \psi_k \text{ for } m = 2, \ldots, q$$
with
$$\psi_k = \left\lceil 2\,\frac{\delta_P}{\lambda\,\Delta}\left(\frac{\theta_P}{1-\theta_P}\right)^{k-1}\right\rceil \text{ and } s_q = K.$$

Comparative statics

From (1) and (2), we see that:

- **As the proportion θ_P of patient traders increases:**
  - **The expected time to execution increases.**
  - **The equilibrium spreads increase.**
- **As the arrival rate λ increases, spreads decrease.**
- **As the cost of waiting decreases, spreads increase.**
  - **Scaling λ and the δ_i by the same factor leaves all results unchanged.**

- **Note that we can define a measure of resiliency $R = \theta_P^{q-1}$ which is the probability that the spread will revert from its maximum level $K = A - B$ to s^* before the next transaction.**

FKK (2005) model summary

- **The Foucault, Kadan and Kandell model gives us an equilibrium distribution of spreads.**
- **The optimal strategy for both $\mathcal{I}$ and $\mathcal{P}$ traders depends (only) on the spread.**
- **If the spread is very wide, both $\mathcal{I}$ and $\mathcal{P}$ will submit limit orders.**
- **If the spread is sufficiently narrow, both $\mathcal{I}$ and $\mathcal{P}$ will submit market orders.**

The Roşu (2009) model

In the [Roşu][8] model:

- **Four types of traders: Patient buyers, patient sellers, impatient buyers, impatient sellers arrive randomly as Poisson processes at rates respectively $\lambda_{PB},\ \lambda_{PS},\ \lambda_{IB},\ \lambda_{IS}$.**
- **As in Foucault, Kadan and Kandel, traders lose utility proportionally to expected waiting time at rate r called the *patience coefficient*.**
  - **r' for impatient traders is assumed to be so much greater than r that impatient traders always submit market orders.**
- **Order book depth is infinite at B and $A > B$.**
- **Time is continuous and prices are continuous.**
- **Traders may submit market orders or limit orders.**
  - **Limit orders may be canceled at any time.**
  - **All orders are executed instantaneously.**

Equilibrium: One side of the order book

A special case of this model can be solved in closed form:

- **Assume $\lambda_{PB} = \lambda_{IS} = 0$ so there are only patient sellers and impatient buyers.**
  - **The best bid is stuck at the minimum level B.**
- **Put $\lambda_{PS} = \lambda_1$ and $\lambda_{IB} = \lambda_2$ and denote the seller's patience coefficient by r.**
  - **If there are m (sell) orders in the book at levels a_i, $i \in \{1, .., m\}$, the utility of the ith seller is given by**
  $$f_i = a_i - r\,\tau_i^\star = f_m$$
  where $\tau_i^\star$ is the expected time to execution for the ith order.
  - **The last equality holds because in equilibrium, every trader must have the same utility; allowing costless cancelations makes this argument work.**

The main idea

- **Suppose there is no quantity in the book.**
- **It is optimal for the first patient seller (trader 1) to submit a sell order at A (the highest possible price).**
- **Now suppose a second patient seller (trader 2) comes to the market.**
- **Traders 1 and 2 engage in a price war, undercutting each other until their utilities are equal:**
  - **The optimal solution is for trader 1 to post at $a_1 = A$ and trader 2 to post at $a_2**
  - **Trader 1 waits longer for an execution but gets a higher price.**

Definitions

$$\lambda = \lambda_1 + \lambda_2 = \textbf{activity}$$
$$c = \frac{\lambda_1}{\lambda_2} = \textbf{competition}$$
$$\epsilon = \frac{r}{\lambda} = \textbf{granularity}$$

Maximum order book size

- **The number of orders in the book must be finite and $\leq M$ for some $M > 0$.**
- **Otherwise, the expected execution time at the top of the book would be $+\infty$ and the associated utility $-\infty$.**
  - **An agent encountering a full book with M orders would always rather hit the bid at B.**

Theorem
The maximum number M of sellers in the book is given by

$$M = \frac{\log\left(\frac{A-B}{\epsilon}\frac{(c-1)^2}{c+1}\right)}{\log c} + s,\ \text{with } s \in \left(-1, \frac{\log 2}{\log c}\right)$$

Minimum spread

When the book is full ($m = M$), the utility of placing an order at a_M must equal the utility of hitting the bid at B. Thus

$$a_M - r\,\tau_M^\star = B$$

(B being the utility of a market sell order at B). Since

$$\tau_M^\star = \frac{1}{\lambda_2},$$

even though the tick size is zero, we can see that:

Corollary

There is a minimum spread

$$s_{\min} = a_M - B = \frac{r}{\lambda_2}$$

A recursive expression for f_m

- From a book with m sellers, either
  - A patient seller arrives at random time $T_1 \sim \exp(\lambda_1)$
    - $\hookrightarrow$ a book with $m+1$ sellers.
    - $\mathbb{P}(\text{PS}|\text{New event}) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.
  - A impatient buyer arrives at random time $T_2 \sim \exp(\lambda_2)$.
    - $\hookrightarrow$ a book with $m-1$ sellers.
    - $\mathbb{P}(\text{IB}|\text{New event}) = \frac{\lambda_2}{\lambda_1+\lambda_2}$.
- Thus, the expected utility of the mth patient seller must be given by
$$f_m = \frac{\lambda_1}{\lambda_1+\lambda_2} f_{m+1} + \frac{\lambda_2}{\lambda_1+\lambda_2} f_{m-1} - r\,\frac{1}{\lambda_1+\lambda_2}$$
- Note that if an IB arrives, the seller with the lowest offer gets a_m and the other sellers get f_{m-1}. Thus
$$a_m = f_{m-1}$$
- All of the f_m may then be computed by induction.

The general case

- Roşu is able to compute equilibrium numerically in the general case where there are also impatient sellers and patient buyers.
- He can also extend to the case where market orders are for more than one share.
- The one-sided case is helpful for gaining intuition; the best bid in the two-sided case may be thought of as the fixed bid price B in the one-sided case.

Summary of Roşu model results

- Hump-shaped limit order book.
  - Intuition: Patient sellers correctly anticipate the arrival of large market orders.
- Assuming small market orders are more likely than large market orders, temporary impact is larger than permanent impact.
  - Intuition (for MB): The large market buy increases the spread. The spread reverts due to competition between patient sellers. The higher $c = \lambda_1/\lambda_2$, the greater the reversion.
- Bid and ask prices comove.
  - Intuition: In this model, an MS order causes the best bid price b to drop. The best ask a also falls but by less unless the offered side of the book is full.
- Higher activity and competition cause smaller spreads.
  - Intuition: The higher the activity λ is, the shorter the time that a patient seller has to wait. Obviously greater competition between patient sellers reduces spreads.
- In some cases, traders may submit fleeting limit orders.
  - Intuition: When the order book is full (which is when the spread is at a minimum), a buyer (for example) may place a LB at the best bid. The most aggressive seller will immediately cancel his limit order and hit this new bid.
  - The model prediction is that fleeting limit orders will appear only when the spread is at a minimum and will be placed inside the spread.

Cont and Kukanov: Fill probability for large tick stocks

- **As in the Parlour model, whether or not to send a limit order must depend in particular on the length of the queue and your position in it.**
  - **This particularly significant for large tick stocks where the queue is long.**
- **So far, we have surveyed only stylized financial economics models.**
- **[Cont and Kukanov][4] recently developed an elegant engineering approach to this problem that is practical to implement.**

Cont and Kukanov: optimal limit order placement

- **Assume a buy order**
- **Venues $k = 1, \ldots, K$ are characterized by**
  - **Bid queue lengths Q_k**
  - **Make rebates r_k, take fees f_k, half-spread $h = s/2$.**
- **Simplify by assuming that any market order is sent to the venue with the smallest take fee f.**
- **Order placement strategy is $X = (M, L_1, \ldots, L_K)$ where the components represent volumes of the single market order and the kth limit orders respectively.**
- **The tradeoff is between cost and execution probability.**
  - **On a given exchange, the lower the price of the limit order, the lower the probability of execution**
  - **On multiple exchanges, the higher the rebate, the longer the queue length, the lower the probability of execution.**

Objective function

- **Denote duration of child order by T.**
- **ξ_k is the (random) outflow of limit orders (from market order fills and cancelations) at venue k in the interval $[0, T]$.**
- **Executed quantity is given by**
$$\mathcal{A}(X, \xi) = M + \sum_k \left\{ (\xi_k - Q_k)^+ - (\xi_k - Q_k - L_k)^+ \right\}$$
- **Cost relative to mid-quote is given by**
$$\mathcal{C}(X, \xi) = (h + f)\, M - \sum_k (h + r_k)\, \left\{ (\xi_k - Q_k)^+ - (\xi_k - Q_k - L_k)^+ \right\}$$
- **Penalize under- or over-fills relative to target quantity $\tilde{X}$ with penalty**
$$\mathcal{P}(X, \xi) = \lambda_u (\tilde{X} - \mathcal{A})^+ + \lambda_o\, (\mathcal{A} - \tilde{X})^+.$$
  - **λ_u and λ_o price adverse selection.**

Optimal order placement strategy

- **The optimal order placement strategy X^* is given by**
$$X^* = \arg\min_X \mathbb{E}\, [\mathcal{C}(X, \xi) + \mathcal{P}(X, \xi)]$$
- **Assuming it is suboptimal to exceed the target $\tilde{X}$ and that it is better to execute with limit orders than market orders, Cont and Kukanov show that there is an optimal strategy.**

Solution for single venue

- **If λ_u is in a suitable range,**
$$L^* = F^{-1}\left(\frac{2h+f+r}{\lambda_u+h+r}\right) - Q, \qquad M^* = \tilde{X} - L^*.$$
where $F(\cdot)$ denotes the distribution of ξ (which is of course increasing in ξ).
- **Comparative statics. L^* is:**
  - **decreasing in Q,**
  - **increasing in h,**
  - **increasing in f,**
  - **decreasing in λ_u,**
  - **increasing in r.**
- **As the target size $\tilde{X}$ increases, L^* is fixed and M^* increases.**
- **$F(\cdot)$ may be estimated in real time using recent outflow data (and potentially other signals).**
  - **$F(\cdot)$ need not be parametric.**

Single venue proof: Suboptimality of $L + M < \tilde{X}$

- **If $L + M \leq \tilde{X}$, conditional on a given outflow realization ξ, the cost of execution (with under fill penalty) is:**
$$\mathcal{C} + \mathcal{P} = (h+f)\,M - (h+r)\;(\mathcal{A} - M) + \lambda_u\,(\tilde{X} - \mathcal{A})^+$$
where the (random) number of shares executed is given by
$$\mathcal{A} = M + (\xi - Q)^+ - (\xi - Q - L)^+.$$
- **In a scenario where $\xi > Q + L$, $\mathcal{A} = M + L$ and so**
$$\mathcal{C} + \mathcal{P} = (h+f)\,M - (h+r)\,L + \lambda_u\,(\tilde{X} - M - L)^+$$
- **Increasing L then reduces the cost by increasing the rebate and decreasing the penalty. Optimality thus requires $M + L \geq \tilde{X}$.**

Single venue proof: Suboptimality of $L + M > \tilde{X}$

- **Then, conditional on a given outflow realization ξ, the cost of execution (with penalty) is:**

(3)
$$\mathcal{C} + \mathcal{P} = (h+f)\,M - (h+r)\;(\mathcal{A} - M) + \lambda_u\,(\tilde{X} - \mathcal{A})^+ + \lambda_o\,(\mathcal{A} - \tilde{X})^+.$$

- **We only care about cases where $\mathcal{A} > \tilde{X}$ when from (3) above**

$$\mathcal{C} + \mathcal{P} = (2\,h + f + r)\,M + (\lambda_o - (h+r))\;\mathcal{A} - \lambda_o\,\tilde{X}$$

which is increasing in $\mathcal{A}$ if $\lambda_o > (h+r)$. The optimal strategy is therefore to reduce L until there is no over-fill possibility.

Single venue proof: Optimal choice of M

- Since it is optimal to put $L + M = \tilde{X}$, we can write the value function in terms of M only.
- From (3), the value function is given by

$$V(M) = \mathbb{E}\left[(h+f)\,M - (h+r)\,\left(\mathcal{A} - M\right) + \lambda_u\,(\tilde{X} - \mathcal{A})^+\right]$$

  where $\mathcal{A} = M + (\xi - Q)^+ - (\xi - Q - \tilde{X} + M)^+$.

- Then

$$V'(M) = \mathbb{E}\left[(2\,h + f + r) - (h+r)\,\partial_M \mathcal{A} - \lambda_u\,\mathbf{1}_{\{\mathcal{A} < \tilde{X}\}}\,\partial_M \mathcal{A}\right]$$

- Note that since $\mathcal{A} < \tilde{X}$ if and only if $\xi < Q + \tilde{X} - M$, we have

$$\mathbf{1}_{\{\mathcal{A} < \tilde{X}\}} = \mathbf{1}_{\{\xi < Q + \tilde{X} - M\}}.$$

  Also

$$\partial_M \mathcal{A} = 1 - \mathbf{1}_{\{\xi \geq Q + \tilde{X} - M\}} = \mathbf{1}_{\{\xi < Q + \tilde{X} - M\}}$$

- Then

$$V'(M) = \mathbb{E}\left[(2\,h + f + r) - (h + r + \lambda_u)\,\partial_M \mathcal{A}\right]$$

- Setting $V'(M) = 0$, the optimal choice of M satisfies

$$F\left(Q + \tilde{X} - M^\star\right) = F\left(Q + L^\star\right) = \frac{2\,h + f - r}{h + r + \lambda_u}$$

  where $F(Y) = \mathbb{P}\left[\xi \leq Y\right]$ is the cumulative distribution function of ξ.

Solution for multiple venues

- When the allocation is optimal,

  \begin{eqnarray} \mathbb{P}\left({\mathcal{A} }(X^,\xi)<\tilde X\right)&=& \frac{h+f+\lambda_o}{\lambda_u+ \lambda_o}\ \mathbb{P}\left(\left.{\mathcal{A} }(X^,\xi)<\tilde X \right| \xi_j > Q_j + L^_j\right)&=& \frac{\lambda_o-(h+r_j)}{\lambda_u+\lambda_o}\end{eqnarray*}
  - Note that the bigger the rebate r_j on a given exchange, the lower the conditional shortfall probability.
  - This can be used to define λ_u and λ_o in terms of maximal under-fill probabilities.

Cont and Kukanov practical implications

- **One interesting insight from the solution is that execution cost is lower with multiple venues relative to a single venue if the outflows ξ_k are sufficiently uncorrelated.**
  - **This amounts to a condition for optimality of order-flow fragmentation.**
- **In equilibrium, under competition and with smart order routing, the queue sizes should adjust to reflect rebate and fee structures. Indeed this has been shown empirically.**
  - **Short term deviations from equilibrium however (which always exist in practice) permit smart traders to lower execution costs.**

Fragmentation may or may not be good for society, but it is certainly good for individual traders who know how to optimize order placement.

Bouchaud, Mézard and Potters

[Bouchaud, Mézard and Potters][3] present a simple model that explains the empirically-observed shape of the typical order book. They assume that

- **Limit orders arrive at some distance u to the best quote at some rate $\lambda(u)$.**
- **Limit orders are canceled at the constant rate $\delta(u)$ (constant in fact in the BMP version).**
- **The stock price diffuses, eating up all of the quantity at the best quote as it moves.**

Heuristic derivation of the master equation

- **Consider the order density $\rho(u,t)$ on the offered side of the book where $u = L - S_t$ is the current distance to the mid-price S_t.**
- **For fixed u, this density will change because:**
  - **The stock price S_t moves.**
  - **Limit orders arrive at rate $\lambda(u)$.**
  - **Cancelations occur at the proportional rate $\delta(u)$.**
- **Itô's Lemma gives**

$$d\rho(u,t) = \partial_S \rho(u,t)\, dS + \frac{1}{2}\, \partial_{S,S}\rho(u,t)\, dS^2 + \lambda(u)\, dt - \delta(u)\, \rho(u,t)\, dt.$$

- **Assuming arithmetic Brownian motion so that $dS^2 = \sigma^2\, dt$, we obtain the master equation for the evolution of the expected order book density $\hat{\rho}(u,t) = \mathbb{E}[\rho(u,t)]$.**

  (4)

$$\frac{\partial}{\partial t}\hat{\rho}(u,t) = \frac{1}{2}\,\sigma^2\,\frac{\partial^2}{\partial u^2}\hat{\rho}(u,t) - \delta(u)\,\hat{\rho}(u,t) + \lambda(u)$$

  with the boundary condition $\hat{\rho}(0,t) = 0$.

Steady-state solution of the master equation

For simplicity, set $\sigma = 1$ (we can reinstate later using dimensional analysis)

Denoting the steady state density by $\bar\rho$ and setting the time derivative in (4) to zero, we obtain

(5)

$$\frac{1}{2}\,\bar\rho''(u) - \delta(u)\,\bar\rho(u) + \lambda(u) = 0$$

We can show that the general solution of (5) is of the form

(6)

$$\rho(\Delta) = f(\Delta)\,\int_0^\Delta\,g(u)\,\lambda(u)\,du + g(\Delta)\,\int_\Delta^\infty\,f(u)\,\lambda(u)\,du.$$

Substitution into (5) implies that: \begin{eqnarray} *\frac{f''(u)}{f(u)}=\frac{g''(u)}{g(u)}&=&2\,\delta(u)\end{eqnarray}* with boundary conditions

$$g(0) = 0;\,f(\infty) = 0.$$

The BMP steady-state density

Example (Constant cancelation rate)
If $\delta(u) = \delta$, a constant, one solution is given by

$$f(u) = \sqrt{\frac{2}{\delta}}\,e^{-\sqrt{2\,\delta}\,u} \text{ and } g(u) = \sinh(\sqrt{2\,\delta}\,u)$$

which gives us the BMP solution

(7) \begin{eqnarray} *\rho_{BMP}(\Delta)&=&\sqrt{\frac 2 \delta}\,\left{e^{-\sqrt{2\,\delta}\,\Delta}\,\int0^\Delta \,\sinh(\sqrt{2\,\delta}\,u)\,\lambda(u)\,du + \sinh(\sqrt{2\,\delta}\,\Delta)\,\int\Delta^\infty\,e^{-\sqrt{2\,\delta}\,u}\, \lambda(u)\,du\right}* \end{eqnarray}

Order book density with constant cancelation rate

Define the the characteristic distance

$$\tilde\Delta = \sqrt{\frac{\sigma^2}{2\,\delta}}.$$

Then, reinstating σ, we obtain \begin{eqnarray} *\rho_{BMP}(\Delta) &=& \frac{2\,\tilde \Delta}{\sigma^2}\,\left{e^{-\Delta/{\tilde \Delta}}\,\int0^\Delta\,\sinh(u/{\tilde \Delta})\,\lambda(u)\,du + \sinh(\Delta/{\tilde \Delta})\,\int\Delta^ \infty\,e^{-u/{\tilde \Delta}}\,\lambda(u)\,du\right}\ &=& \frac 1 \delta\,\left{e^{-\Delta/{\tilde \Delta}} \,\int0^{\Delta/{\tilde\Delta}}\,\sinh(u)\,\lambda(\tilde\Delta u)\,du + \sinh(\Delta/{\tilde \Delta})\,\int{\Delta/{\tilde \Delta}}^\infty\,e^{-u}\,\lambda(\tilde\Delta u)\,du\right}.\end{eqnarray}*

Asymptotics of SFGK

Specializing even further to the zero-intelligence SFGK model where $\lambda(u) = \lambda$, a constant, we obtain

(8) \begin{eqnarray} \rho_{BMP}(\Delta) &=& \frac{\lambda}{\delta}\,\left{e^{-\Delta/{\tilde \Delta}} \,\int0^{\Delta/{\tilde\Delta}}\,\sinh(u)\,du+ \sinh(\Delta/{\tilde \Delta})\,\int{\Delta/{\tilde\Delta}}^\infty\,e^{-u} \,du\right}\nonumber\ &=& \frac{\lambda}{\delta}\,\left{1 - e^{-\Delta/{\tilde \Delta}} \right} \end{eqnarray}

which is consistent with the asymptotic limit λ/δ we derived earlier for the zero-intelligence (ZI) model using a physical argument.

Plot of approximate book density

Following [Bouchaud, Mézard and Potters][3], assuming limit orders arrive as a power-law with tail exponent μ so that

$$\lambda(u) = \frac{\lambda}{u^{1+\mu}},$$

we obtain the shape of the order book should depend only on μ and the rescaled distance to best quote $\hat{\Delta} := \Delta/\tilde{\Delta}$:

(9)

$$\rho(\hat{\Delta}) \propto e^{-\hat{\Delta}} \int_0^{\hat{\Delta}} du\, \frac{\sinh(u)}{u^{1+\mu}} + \sinh(\hat{\Delta}) \int_{\hat{\Delta}}^{\infty} du\, \frac{e^{-u}}{u^{1+\mu}}$$

This shape matches average book shapes generated by zero-intelligence simulations very well.

Plot of approximate book density

In [4]:
```
# Density computation
rho <- function(delhat,mu){
f1 <- function(u){sinh(u)/u^(1+mu)};
tmp1 <-  if(delhat < 0.0000001) 0 else exp(-delhat)*integrate(f1, lower = 0, uppe
r = delhat)$value;
f2 <- function(u){exp(-u)/u^(1+mu)};
tmp2 <- if(delhat < 0.0000001) 0 else sinh(delhat)*integrate(f2, lower = delhat,
upper = Inf)$value;
return(tmp1+tmp2)
}

rhov <- function(u,rho){sapply(u,function(u){rho(u,rho)})}

# Book shape plot
curve(rhov(x,.5),from=.0,to=10,col="red",xlab=expression(hat(Delta)),ylab=express
ion(rho(hat(Delta))),n=1000)
```

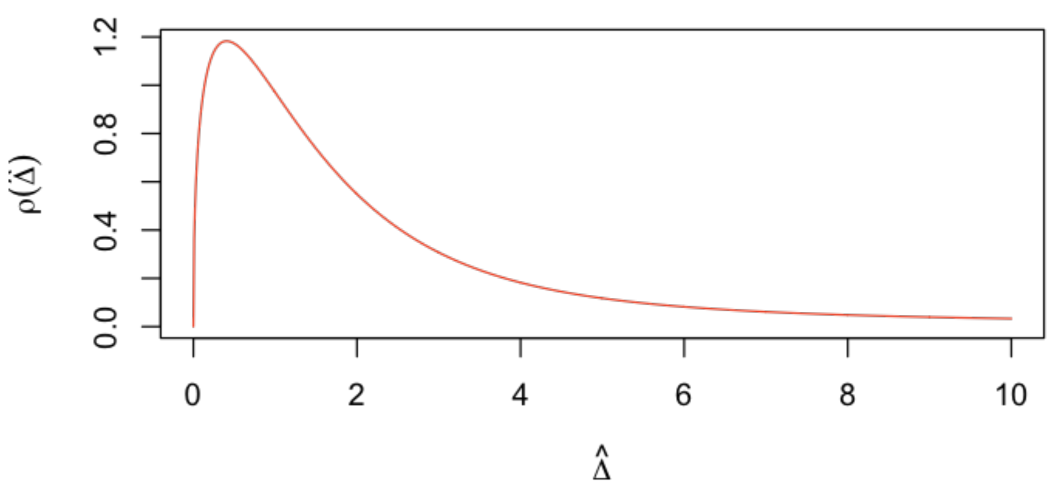


Figure 3: Approximate book density with $\mu = 1/2$

Approximate density vs zero intelligence simulation

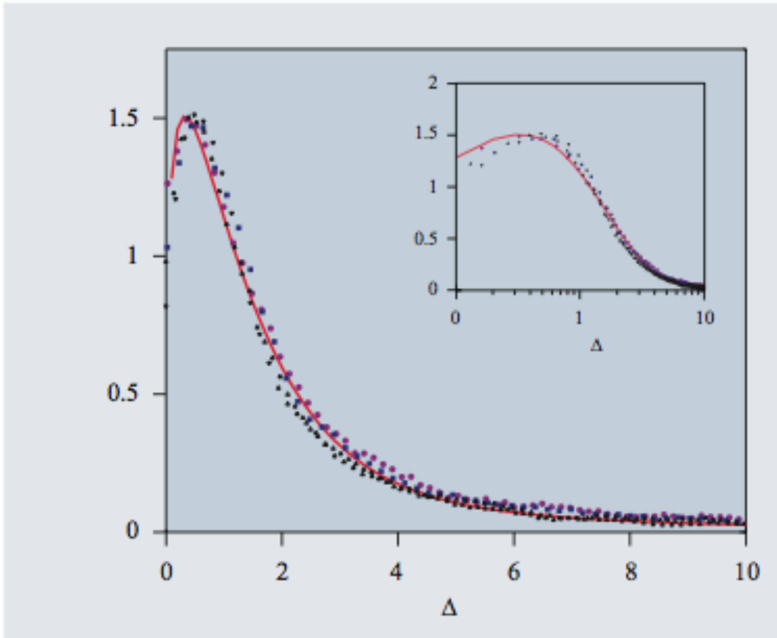


Figure 4. The average order book of the numerical model with various choices of parameters ($\mu = 0.6$, $p_m \in \{1/4, 1/6\}$ and $\Gamma \in \{10^{-3}, 5 \times 10^{-4}\}$ is compared to the approximate analytical prediction (full curve), equation (7). After rescaling the axes, the various results roughly scale on the same curve, which is well reproduced by our simple analytic argument. The inset shows the same data in a log–linear representation.

Approximate density vs empirically observed order books

QUANTITATIVE FINANCE Statistical properties of stock order books: empirical results and models

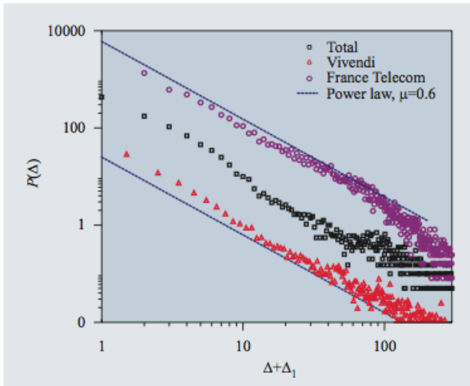


Figure 1. Number of incoming orders arriving at a distance Δ from the current best price, as a function of $\Delta_1 + \Delta$ (in ticks), in log–log coordinates. We chose $\Delta_1 = 1$ for FT, 0.5 for Vivendi and 0 for Total. The data have been shifted vertically for clarity. We also found $P(\Delta = -1) \approx P(\Delta = 0) \approx P(\Delta = 1)$ (not shown). The symbols correspond to buy orders, but sell orders show an identical distribution. The straight lines correspond to $\mu = 0.6$. Note that the power-law crosses over to a faster decay beyond $\Delta \approx 100$.

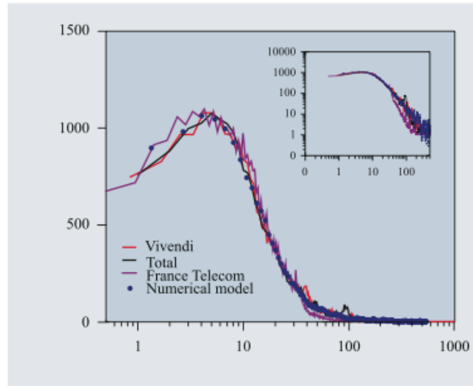


Figure 2. Average volume of the queue in the order book for the three stocks, as a function of the distance Δ from the current bid (or ask) in a log–linear scale. Both axes have been rescaled in order to collapse the curves corresponding to the three stocks. The thick dots correspond to the numerical model explained below, with $\Gamma = 10^{-3}$ and $p_m = 0.25$. Inset: same data in log–log coordinates.

Power-law arrival of orders

- **The Bouchaud et al. and Farmer et al. groups both published papers showing that the density (intensity) of arrival of limit orders into the order book is a power-law function of distance Δ to the (same side) best quote.**
- **[Mike and Farmer][6] found and described a further beautiful regularity:**
  - **The distribution of distance Δ to the same-side best quote looks like Student-t.**
  - **Recall the Student-t distribution:**

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\,\pi}\,\Gamma(\frac{\nu}{2})}\,\frac{1}{\left(1+\frac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}} \sim \frac{1}{x^{\nu+1}} \text{ as } |x| \to \infty$$

  so Student-t with ν degrees of freedom has *tail exponent* ν.
- **This means in particular that market orders are less likely when the spread is wide than when the spread is narrow.**

```
In [ ]:
```

Power-law order placement

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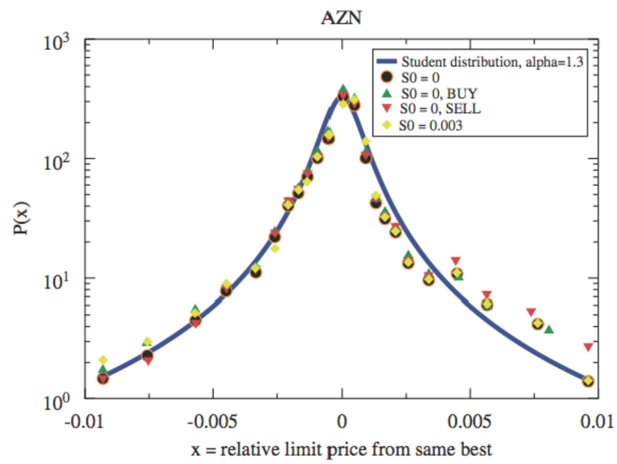

Fig. 2. Reconstruction of the probability density function $P^*(x)$ describing limit order prices as a function of x, the limit price relative to the same best price. The reconstruction is done both for buy orders (green upward pointing triangles) and sell orders (red downward pointing triangles), and for two different spread conditions. $s > s_0 = 0$ allows all 410, 000 points that survive the data filterings described in the text and that satisfy the condition $x \leqslant s - T/p$; there are 211, 000 buy orders and 199, 000 sell orders. There are only 26, 000 points that satisfy $s > s_0 = 0.003$. The fitted blue curve is a Student distribution with 1.3 degrees of freedom.

Empirically observed tail exponents

- **Bouchaud, Mézard and Potters found $\mu \approx 0.6$ on the Paris Stock Exchange (data from Feb-2001).**
- **Zovko and Farmer (2002) found $\mu \approx 1.5$ on the London Stock Exchange (data from Aug-1998 to Apr-2000).**
- **Mike and Farmer found $\mu \approx 1.3$ on the London Stock Exchange (data from May-2000 to Dec-2002).**

Large tick and small tick stocks

If the minimum tick size (and so the minimum spread) is more than the volatility per trade, we say that the stock is a *large tick* stock. The spread for such stocks is typically one tick and there is huge size at the best quote (C for example). When volatility per trade is larger than the tick size, we say that the stock is a *small tick* stock. The typical spread for such a stock is more than one tick and there is only small size at the best quote.

- **We expect the Bouchaud Mézard and Potters results to hold for small tick stocks.**
- **For large tick stocks, there should be an accumulation of volume at the best quote.**
- **The minimum spread cannot be smaller than the minimum tick size!**

Summary

- **Parlour (1998) shows that a rational market order/ limit order decision should depend on the state of the order book**
- **Foucault, Kadan and Kandel (2005) model the order book as a market for immediacy, relating the spread to the ratio between patient and impatient traders**
- **Roşu (2009) removes many over-stylized features of FKK (2005) by allowing instantaneous cancelation of orders**
- **Cont and Kukanov (2013) show how to incorporate the fee structures and current queue lengths in different venues top optimize the market/limit order mix.**
- **Bouchaud, Mézard and Potters show that the average order book shape, consistent with ZI simulation and empirical observation, may be derived using a simple price diffusion approximation**
- **Mike and Farmer find a simple empirical relationship between the arrival rates of limit and market orders**

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