SRM VALLIAMMAI ENGINEERING COLLEGE

1918402 - PROBABILITY & QUEUEING THEORY

UNIT I - RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

PART - A

1) The number of hardware failures of a computer system in a week of operations has the following pdf. Find the mean of the number of failures in a week.

No/: of failures	0	1	2	3	4	5	6
Probability	0.18	0.28	0.25	0.18	0.06	0.04	0.01

SOLUTION:

$$\text{Mean} = E[X] = \sum x P(x)$$

$$= 0 + 1(0.28) + 2(0.25) + 3(0.18) + 4(0.06) + 5(0.04) + 6(0.01)$$

$$= 0.28 + 0.5 + 0.54 + 0.24 + 0.2 + 0.06$$

$$\boxed{E[X] = 1.82}$$

2. The number of hardware failures of a computer system in a week of operations has the following pdf. Calculate the value of K.

No/: of failures	0	1	2	3	4	5	6
Probability	k	2k	2k	k	3k	k	4k

SOLUTION:

We know that $\sum P(x) = 1$

$$K + 2K + 2K + K + 3K + K + 4K = 1 \Rightarrow 14K = 1$$

$$\Rightarrow \boxed{K = \frac{1}{14}}$$

3. Check whether the function given by $f(x) = \frac{x+2}{25}$ for $x = 1, 2, 3, 4, 5$. can serve as the probability distribution of a discrete random variable.

SOLUTION:

x	1	2	3	4	5
$P(x)$	$3/25$	$4/25$	$5/25$	$6/25$	$7/25$

Now $\sum P(x) = \frac{3}{25} + \frac{4}{25} + \frac{5}{25} + \frac{6}{25} + \frac{7}{25} = \frac{25}{25} = 1.$

$\therefore$ the given $f(x)$ is a Probability distribution.

4) If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution of X.

SOLUTION:

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$ (say)

$\therefore P(X=1) = \frac{k}{2}$, $P(X=2) = \frac{k}{3}$, $P(X=3) = k$, $P(X=4) = k/5$

$\therefore \sum P(x) = 1 \Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$$\Rightarrow \frac{15k + 10k + 30k + 6}{2 \times 3 \times 5} = 1$$

$$\Rightarrow \frac{61k}{30} = 1 \Rightarrow \boxed{k = \frac{30}{61}}$$

5) The R.V X has the following probability distribution.

x:	-2	-1	0	1
$P(x)$:	0.4	k	0.2	0.3

Find k and mean value of X.

SOLUTION:

$$\Sigma P(x) = 1$$

$$\Rightarrow 0.4 + k + 0.2 + 0.3 = 1$$

$$\Rightarrow k + 0.9 = 1$$

$$\Rightarrow k = 1 - 0.9$$

$$\Rightarrow \boxed{k = 0.1}$$

6. If $f(x) = k(x + x^2)$ in $1 < x < 5$ is a pdf of a continuous random variables. Find the value of k.

SOLUTION:

We know that $\int_{-\infty}^{\infty} f(x)\,dx$

$$\int_1^5 k(x + x^2) = 1$$

$$k\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^5 = 1$$

$$k\left[\frac{25}{2} + \frac{125}{3} - \left(\frac{1}{2} + \frac{1}{3}\right)\right] = 1$$

$$\Rightarrow k\left[\frac{75 + 250}{6} - \frac{5}{6}\right] = 1$$

$$\Rightarrow k\,\frac{320}{6} = 1$$

$$\Rightarrow \boxed{k = \frac{6}{320}} \quad \boxed{k = \frac{3}{160}}$$

7) The pdf of a continuous random variable X is $f(x) = k(1 + x)$, $2 < x < 5$. Find k.

SOLUTION:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{2}^{5} k(1+x)\,dx = 1$$

$$\Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$\Rightarrow k\left[5 + \frac{25}{2} - (2+2)\right] = 1$$

$$\Rightarrow k\left[5 - 4 + \frac{25}{2}\right] = 1$$

$$\Rightarrow k\left[1 + \frac{25}{2}\right] = 1 \Rightarrow k\left[\frac{27}{2}\right] = 1$$

$$\Rightarrow \boxed{k = \frac{2}{27}}$$

8. For a continuous distribution $f(x) = k(x - x^2)$, $0 \le x \le 1$. where k is a constant. Find k.

SOLUTION:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow k\int_{0}^{1} (x - x^2)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$\Rightarrow k\left[\frac{3-2}{6}\right] = 1 \Rightarrow k\,\frac{1}{6} = 1$$

$$\Rightarrow \boxed{k = 6}$$

9) If $f(x) = kx^2, 0 < x < 3$ is to be a density function, find the value of k.

SOLUTION:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^3 kx^2\,dx = 1$$

$$\Rightarrow k\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow k\,\frac{27}{3} = 1 \Rightarrow k9 = 1$$

$$\Rightarrow \boxed{k = \frac{1}{9}}$$

10) A test engineer discovered that the cumulative distribution function of the life time of an experiment (in years) is given by $F(x) = 1 - e^{-x/5}$, $x \geq 0$. What is the expected lifetime of an equipment?

SOLUTION:

Gn: $F(x) = 1 - e^{-x/5}$

$$\therefore f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(1 - e^{-x/5}\right)$$

$$= -e^{-x/5}\left(\frac{-1}{5}\right)$$

$$\boxed{f(x) = \frac{1}{5}e^{-x/5}}$$

$\therefore E[X] =$ Expected life time $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_0^{\infty} x\,\frac{1}{5}e^{-x/5}\,dx$$

$$\cancel{u=x} = \frac{1}{5}\int_0^{\infty} xe^{-x/5}\,dx$$

$u = x \qquad v = e^{-x/5}$

$u' = 1 \qquad v_1 = \frac{e^{-x/5}}{-1/5} = -5e^{-x/5}$

$u'' = 0 \qquad v_2 = \frac{e^{-x/5}}{1/25} = 25e^{-x/5}$

$$\therefore E[x] = \left[x\left(-5e^{-x/5}\right) - (1)\,25e^{-x/5} \right]_0^{\infty}$$

$$= 0 - 25e^{-\infty} - \left(0 - 25e^{0}\right)$$

$$\boxed{E[x] = 25}$$

Bernoulli's formula

$\int uv = uv_1 - u'v_2 + u''v_3 - \cdots$

11) The mean of Binomial distribution is 20 and Standard deviation is 4. Find the parameters of the distribution.

SOLUTION:

Mean of Binomial distribution $= np = 20$ — (1)

Variance $= npq$

$S.D = \sqrt{npq} = 4$

$\Rightarrow npq = 16$ — (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{\cancel{np}}{\cancel{np}q} = \frac{20}{16} \Rightarrow \frac{1}{q} = \frac{5}{4} \Rightarrow \boxed{q = \frac{4}{5}}$$

$$\therefore p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$\boxed{p = \frac{1}{5}}$ and $np = 20$

$\Rightarrow n\left(\frac{1}{5}\right) = 20$

$\Rightarrow n = 20 \times 5$

$\Rightarrow \boxed{n = 100}$

$\therefore$ The parameters of the distributions are

$n = 100$, $p = \frac{1}{5}$, $q = \frac{4}{5}$

12) In 256 sets of 8 tosses of a coin, in how many sets one may expect heads and tails in equal number.

SOLUTION:

$$n = 8 \quad , p = \frac{1}{2} \quad , q = \frac{1}{2}$$

$$P(X=x) = nC_x \, p^x q^{n-x}$$

$$P(X=x) = 8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x}$$

$$= 8C_x \frac{\cancel{\left(\frac{1}{2}\right)^x} \left(\frac{1}{2}\right)^8}{\cancel{\left(\frac{1}{2}\right)^x}}$$

$$P(X=x) = 8C_x \frac{1}{256}$$

$$\Rightarrow P(X=4) = 8C_4 \frac{1}{256} = \frac{\cancel{8}^{2} \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \frac{1}{256}$$

$$= \cancel{1680 \times \frac{1}{256}} \; 70 \times \frac{1}{256}$$

There are 256 sets, ∴ The required number

$$= \cancel{1680}^{70} \times \frac{1}{\cancel{256}} \times \cancel{256}$$

$$= \cancel{1680}. \, 70$$

13) If 3% of the electric bulbs manufactured by a company are defective. Find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective.

SOLUTION:

Gn: $p = 3\% = \frac{3}{100}$, $n = 100$ $\Rightarrow \lambda = np = 100 \times \frac{3}{100}$

~~$p = 0.03$~~ , ~~$q = 1 - 0.03 = 0.97$~~

$\boxed{\lambda = 3}$

~~$P(X=x) = nC_x \, p^x q^{n-x}$~~

$$\therefore E[x] = \left[x(-5e^{-x/5}) - (1)\,25e^{-x/5} \right]_0^{\infty}$$

$$= 0 - 25e^{-\infty} - (0 - 25e^{0})$$

$$\boxed{E[x] = 25}$$

Bernoulli's formula

$\int uv = uv_1 - u'v_2 + u''v_3 - \cdots$

11) The mean of Binomial distribution is 20 and Standard deviation is 4. Find the parameters of the distribution.

SOLUTION:

Mean of Binomial distribution $= np = 20$ — (1)

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$$\frac{(1)}{(2)} \Rightarrow \frac{\cancel{np}}{\cancel{np}q} = \frac{20}{16} \Rightarrow \frac{1}{q} = \frac{5}{4} \Rightarrow \boxed{q = \frac{4}{5}}$$

$$\therefore p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$\boxed{p = \frac{1}{5}}$ and $np = 20$

$\Rightarrow n\left(\frac{1}{5}\right) = 20$

$\Rightarrow n = 20 \times 5$

$\Rightarrow \boxed{n = 100}$

$\therefore$ The parameters of the distributions are

$n = 100$, $p = \frac{1}{5}$, $q = \frac{4}{5}$

12) In 256 sets of 8 tosses of a coin, in how many sets one may expect heads and tails in equal number.

SOLUTION:

$$n = 8, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}$$

$$P(X=x) = nC_x \, p^x q^{n-x}$$

$$P(X=x) = 8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x}$$

$$= 8C_x \frac{\cancel{\left(\frac{1}{2}\right)^x} \left(\frac{1}{2}\right)^8}{\cancel{\left(\frac{1}{2}\right)^x}}$$

$$P(X=x) = 8C_x \frac{1}{256}$$

$$\Rightarrow P(X=4) = 8C_4 \frac{1}{256} = \frac{\cancel{8}^{2} \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \frac{1}{256}$$

$$= \cancel{1680 \times \frac{1}{256}} \; 70 \times \frac{1}{256}$$

There are 256 sets, $\therefore$ The required number

$$= \cancel{1680}^{70} \times \frac{1}{\cancel{256}} \times \cancel{256}$$

$$= \cancel{1680}.\ 70$$

13) If 3% of the electric bulbs manufactured by a company are defective. Find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective.

SOLUTION:

Gn: $p = 3\% = \frac{3}{100}$, $n = 100$ $\Rightarrow \lambda = np = 100 \times \frac{3}{100}$

$\cancel{p = 0.03}$, $\cancel{q = 1 - 0.03 = 0.97}$ $\boxed{\lambda = 3}$

$\cancel{P(X=x)} = \cancel{nC_x} \, \cancel{p^x} \, q^{n-x}$

Poisson distribution $P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

$$P(X=x) = \frac{e^{-3} 3^{x}}{x!}$$

P(exactly 5 bulbs are defective) = $P(X=5)$

$$= \frac{e^{-3} 3^{5}}{5!}$$

$$= \frac{0.04979 \times 243}{120}$$

$$\boxed{P(X=5) = 0.1008}$$

14) Suppose that, on an average in every three pages of a book there is one typographical error. If the number of typographical errors on a single page of the book is a Poisson variable. What is the probability if at least one error on a specific page of the book?

SOLOTION :

$$\lambda = 0.5$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!} = \frac{e^{-0.5} (0.5)^{x}}{x!}$$

$\binom{\text{P(atleast}}{\text{one error)}}$ $= P(X \geq 1) = 1 - P(X \leq 0)$

$$= 1 - P(X=0)$$

$$= 1 - \frac{e^{-0.5} (0.5)^{0}}{0!} = 1 - e^{-0.5}$$

$$= 1 - 0.6065 = 0.3935$$

15) The no/-of monthly breakdown of a computer is a R.V having Poisson distribution with mean 1.8. Find the probability that this computer will function for a month with only one breakdown?

SOLUTION:

Gn: $\lambda = 1.8$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-1.8}(1.8)^{x}}{x!}$$

$$P(\text{only one breakdown}) = P(X=1)$$

$$= \frac{e^{-1.8}(1.8)}{1!} = e^{-1.8}(1.8)$$

$$P(X=1) =$$

16) If X is a Poisson variate such that $2P(X=0)+P(X=2) = 2P(X=1)$.

Find $E[X]$.

SOLUTION:

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Gn: $2P(X=0)+P(X=2) = 2P(X=1)$

$$2\frac{e^{-\lambda}\lambda^{0}}{0!} + \frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{2e^{-\lambda}\lambda}{1!}$$

$$\cancel{e^{-\lambda}}\left[2+\frac{\lambda^2}{2}\right] = 2\cancel{e^{-\lambda}}\lambda$$

$$\Rightarrow \frac{\lambda^2+4}{2} = 2\lambda \Rightarrow \lambda^2+4 = 4\lambda$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda-2)^2 = 0 \Rightarrow \boxed{\lambda = 2}$$

$\therefore E[X] = \text{Mean} = \lambda = 2 \Rightarrow \boxed{E[X]=2}$

$$4$$
$$-2 \mid -2$$

17. The probability that a candidate can pass in an examination is 0.6. What is the probability that he will pass in third trial.

SOLUTION:

Gn: $p = 0.6 \Rightarrow q = 0.4$

$$P(\text{he will pass in } 3^{rd} \text{ trial}) = P(X=3)$$

$$= pq^r = pq^3$$

$$= (0.6)(0.4)^3$$

$$\boxed{P(X=3) = 0.0384}$$

18. If X is a geometric variate, taking the values $1, 2, 3 \ldots$ Find $P(X \text{ is odd})$.

SOLUTION: In Geometric distribution

$$P(X=x) = q^{x-1}p$$

$$\therefore P(X = \text{odd}) = P(X=1) + P(X=3) + P(X=5) + \cdots$$

$$= p + q^2 p + q^4 p + \cdots$$

$$= p\left[1 + q^2 + (q^2)^2 + \cdots\right]$$

$$= p(1-q^2)^{-1} = \frac{p}{1-q^2} = \frac{p}{(1-q)(1+q)}$$

$$= \frac{\cancel{p}}{\cancel{p}(1+q)}$$

$$= \frac{1}{1+q}$$

$$\therefore \boxed{P(X \text{ is odd}) = \frac{1}{1+q}.}$$

19. If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8. What is the probability that he will finally pass the test on the fourth trial.

SOLUTION:

$$p = 0.8 \quad , \quad q = 0.2$$

$$P(X = x) = q^{x-1} p$$

$$P(X = 4) = q^{4-1} p = (0.2)^3 (0.8)$$

$$= (0.008)(0.8)$$

$$\boxed{P(X=4) = 0.0064}$$

20) If X has uniform distribution in $(-3, 3)$, find $P(|X-2| < 2)$.

SOLUTION:

X has uniform distribution, $\therefore f(x) = \frac{1}{b-a}$ if $a \le x \le b$.

$$\therefore \text{Here } f(x) = \frac{1}{3-(-3)} = \frac{1}{6} \, , -3 < x < 3$$

$$\therefore P(|X-2| < 2) \Rightarrow P(-2 < X - 2 < 2)$$

$$= P(-2+2 < X < 2+2)$$

$$= P(0 < X < 4)$$

$$= \int_0^4 f(x)\, dx = \int_0^3 \frac{1}{6}\, dx = \frac{1}{6}\left[x\right]_0^3$$

$$= \frac{1}{6} \times 3 = \frac{1}{2}.$$

21) If the MGF of a continuous RV is $\frac{1}{t}(e^{5t} - e^{4t})$ what is the distribution of x? what are the mean and variance of x?

SOLUTION:

$$MGF = \frac{1}{t}(e^{5t} - e^{4t})$$

$$E[x] = \left\{\frac{d}{dt}[M_x(t)]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left(\frac{e^{5t} - e^{4t}}{t}\right)\right\}_{t=0}$$

$$= \left\{\frac{t(5e^{5t} - 4e^{4t}) - (e^{5t} - e^{4t})}{t^2}\right\}_{t=0}$$

$$M_x(t) = \frac{e^{5t} - e^{4t}}{t}, \; t \neq 0.$$

and $M(0) = 1$

This is the MGF of uniform distribution.

Now, $$M_x(t) = \frac{e^{bx} - e^{ax}}{t(b-a)}, \; t \neq 0$$

Here $b = 5$, $a = 4$

Now, Mean $= \frac{a+b}{2} = \frac{4+5}{2} = \frac{9}{2} = 4.5$

Variance $= \frac{(a-b)^2}{12} = \frac{(4-5)^2}{12} = \frac{1}{12}$.

$\therefore$ Mean $= 4.5$

Variance $= \frac{1}{12}$.

22) Suppose that the life of industrial lamp (in thousand of hours) is exponentially distributed with mean life of 3000 hours. Evaluate the probability that the lamp will last between 2000 and 3000 hours.

SOLUTION:

Here $\lambda = \frac{1}{3}$ $\left(\because \frac{1}{3000} \times 1000 = \frac{1}{3} \right)$

Exponential Distribution,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$\therefore$ Here $f(x) = \frac{1}{3} e^{-1/3 x}$

Now $P(2000 \leq X \leq 3000) = P(2 \leq x \leq 3)$

$$= \int_2^3 f(x)\, dx$$

$$= \int_2^3 \frac{1}{3} e^{-1/3 x}\, dx$$

$$= \frac{1}{3} \left[\frac{e^{-1/3 x}}{-1/3} \right]_2^3$$

$$= - \left[e^{-1} - e^{-2/3} \right]$$

$$= e^{-2/3} - e^{-1}$$

$$= 0.5134 - 0.3679$$

$$= 0.1455$$

23) A continuous RV X has the density function $Ce^{-x/5}$, $x > 0$, Find c. Create $E[X]$ and $Var(X)$.

SOLUTION:

Gn: $f(x) = c e^{-x/5}$, $x > 0$

Now $\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \int_0^{\infty} c e^{-x/5}\, dx = 1$

$$c\left[\frac{e^{-x/5}}{-1/5}\right]_0^{\infty}=1$$

$$\Rightarrow -5c\left[e^{-\infty}-e^{0}\right]=1$$

$$\Rightarrow -5c\,[0-1]=1 \Rightarrow 5c=1 \Rightarrow \boxed{c=\frac{1}{5}}$$

$$E[x]=\int_{-\infty}^{\infty} x\, f(x)\,dx = \int_0^{\infty} c\,x\,e^{-x/5}\,dx$$

$$=\frac{1}{5}\int_0^{\infty} x\,e^{-x/5}\,dx$$

Put $u=x$ $\quad v=e^{-x/5}$

$u'=1$ $\quad v_1=\dfrac{e^{-x/5}}{-1/5}=-5e^{-x/5}$

$u''=0$ $\quad v_2=\dfrac{e^{-x/5}}{(-1/5)^2}=25e^{-x/5}$

$$\therefore E[x]=\frac{1}{5}\left[x(-5e^{-x/5})-(1)\,25e^{-x/5}\right]_0^{\infty}$$

$$E[x]=\frac{1}{5}\left[+25\right]=5$$

$$E[x^2]=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_0^{\infty} x^2\,c\,e^{-x/5}\,dx$$

$$=\frac{1}{5}\int_0^{\infty} x^2 e^{-x/5}\,dx$$

$u=x^2$ $\quad v=e^{-x/5}$

$u'=2x$ $\quad v_1=\dfrac{e^{-x/5}}{-1/5}=-5e^{-x/5}$

$u''=2$ $\quad v_2=\dfrac{e^{-x/5}}{(-1/5)^2}=25e^{-x/5}$

$u'''=0$ $\quad v_3=\dfrac{e^{-x/5}}{(-1/5)^3}=-125e^{-x/5}$

$\ldots \cdot / X > 2)$

$$\therefore E[x^2] = \frac{1}{5}\left[-5x^2 e^{-x/5} - (2x)25e^{-x/5} + (2)(-125)e^{-x/5}\right]_0^\infty$$

$$= \frac{1}{5}\left[0 - (-250\, e^0)\right]$$

$$E[x^2] = \frac{1}{5}\, 250 = 50$$

$$\therefore \text{variance} = E[x^2] - (E[x])^2$$

$$= 50 - (5)^2 = 50 - 25$$

$$\boxed{\text{Variance} = 25}$$

24) If x is a normal random variable with mean 3 and variance 9. Find the probability that x lies between 2 and 5

SOLUTION:

Here given $\mu = 3 = \text{mean}$

Variance $= 9$

$S.D = \sqrt{var} = \sqrt{9} = 3 = \sigma$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 3}{3}$$

Now If $X = 2 \Rightarrow Z = \frac{2-3}{3} = \frac{-1}{3}$

If $X = 5 \Rightarrow Z = \frac{5-3}{3} = \frac{2}{3}$

$$\therefore P(2 \le X \le 5) = P\left(\frac{-1}{3} \le Z \le \frac{2}{3}\right)$$

$$= P\left(0 \le Z \le \frac{1}{3}\right) + P\left(0 \le Z \le \frac{2}{3}\right)$$

$$= P(0 \le Z \le 0.33) + P(0 \le Z \le 0.67)$$

$$= 0.1293 + 0.2486$$

$$= 0.3779$$

25. A normal distribution has mean $\mu = 20$ and Standard deviation $\sigma = 10$. Evaluate $P(15 \le x \le 40)$

SOLUTION:

Gn: Mean $= \mu = 20$

$S.D = \sigma = 10$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-20}{10}$$

$\therefore$ If $X = 15$, $Z = \frac{15-20}{10} = \frac{-5}{10} = -0.5$

If $X = 40$, $Z = \frac{40-20}{10} = \frac{20}{10} = 2$.

$\therefore P(15 \le X \le 40) = P(-0.5 \le Z \le 2)$

$= P(0 \le Z \le 0.5) + P(0 \le Z \le 2)$

$= 0.1915 + 0.4772$

$= 0.6687$.

PART-B

1)a) A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k (ii) $P(1.5 < x < 4.5 / x > 2)$

SOLUTION:

We know that $\Sigma P(x) = 1$

$k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

$(10k-1)(k+1) = 0 \Rightarrow 10k = 1$ & $k = -1$

$\Rightarrow \boxed{k = \frac{1}{10}}$ & $\boxed{k = -1}$ (not possible)

-10

$-\frac{1}{10} \quad \frac{+10}{10}$

(ii) $P(1.5 < X < 4.5 / X > 2)$

$$P(X > 2) = 1 - P(X \leq 2)$$

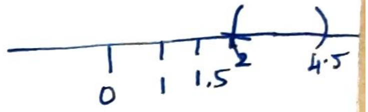
$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - [0 + k + 2k] = 1 - 3k$$

$$= 1 - \frac{3}{10} = \frac{7}{10}$$

$$P(1.5 < X < 4.5 / X > 2) = \frac{P((1.5 < X < 4.5) \cap X > 2)}{P(X > 2)}$$

$$= \frac{P(2 < X < 4.5)}{P(X > 2)}$$



$$\therefore P(2 < X < 4.5) = P(X=3) + P(X=4)$$

$$= 2k + 3k = 5k = 5/10$$

$$\therefore P(1.5 < X < 4.5 / X > 2) = \frac{5/10}{7/10} = \frac{5}{7}.$$

1) b) Find the MGF of Binomial distribution and hence find its mean and variance.

SOLUTION:

Binomial Distribution $P(X = x) = nC_x \, p^x q^{n-x}$

<u>MGF:</u> $M_X(t) = E[e^{tx}]$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x \, (pe^t)^x \, q^{n-x}$$

$$= nc_0 (pe^t)^0 q^n + nc_1 (pe^t)^1 q^{n-1} + \dots + nc_n (pe^t)^n q^0$$

$$= q^n + nc_1 (pe^t)^1 q^{n-1} + \dots + (pe^t)^n$$

$$M_x(t) = (q + pe^t)^n$$

<u>mean:</u>

$$E[x] = \left\{ \frac{d}{dt} M_x(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} (q + pe^t)^n \right\}_{t=0}$$

$$= \left\{ n (q + pe^t)^{n-1} pe^t \right\}_{t=0}$$

$$= np (q + pe^0)^{n-1} e^0$$

$$= np (q + p)^{n-1}$$

$$\boxed{E[x] = np} \qquad [\because p + q = 1]$$

$$E[x^2] = \left\{ \frac{d^2}{dt^2} M_x(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(npe^t (q + pe^t)^{n-1} \right) \right\}_{t=0}$$

$$= np \left\{ e^t (n-1) (q + pe^t)^{n-2} pe^t + e^t (q + pe^t)^{n-1} \right\}_{t=0}$$

$$= np \left[(n-1) p + 1 \right]$$

$$= np^2 (n-1) + np$$

$$E[x^2] = n^2 p^2 - np^2 + np$$

$\therefore$ variance $= E[x^2] - (E[x])^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$

$= np(1-p)$

$$\boxed{\text{Variance} = npq}$$

2)a) The probability mass function of a discrete R.V X is given in the following table:

X	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	2k	0.3	K

(i) Find the value of K
(ii) $P(X<1)$
(iii) $P(-1<X\leq 2)$

SOLUTION:

(i) We know that $\sum P(x) = 1$

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$0.6 + 4k = 1$

$4K = 1 - 0.6 \Rightarrow 4k = +0.4$

$\Rightarrow k = \frac{0.4}{4}$

$$\boxed{k = 0.1}$$

(ii) $P(X<1) = P(x=-2) + P(X=-1)$

$= 0.1 + k$

$= 0.1 + 0.1 = 0.2$

(iii) $P(-1<X\leq 2) = P(x=0) + P(x=1) + P(x=2)$

$= 0.2 + 2K + 0.3$

$= 0.2 + 2(0.1) + 0.3 = 0.2 + 0.2 + 0.3$

$= 0.7$

2)b) Obtain the MGF of Poisson distribution and hence find its mean and variance:

SOLUTION:

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad : x = 0, 1, 2 \ldots$$

MGF:

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[\frac{(\lambda e^{t})^{0}}{0!} + \frac{(\lambda e^{t})^{1}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^{t}}$$

$$M_X(t) = e^{\lambda(e^{t}-1)}$$

Mean:

$$E[X] = \left\{\frac{d}{dt} M_X(t)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt} e^{\lambda(e^{t}-1)}\right\}_{t=0}$$

$$= \left\{e^{\lambda(e^{t}-1)} \lambda e^{t}\right\}_{t=0}$$

$$= \lambda e^{0} e^{\lambda(e^{0}-1)} = \lambda$$

$$\boxed{E[X] = \lambda}$$

$$E[X^2] = \left\{\frac{d^2}{d^2} M_X(t)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left(\lambda e^t e^{\lambda(e^t-1)}\right)\right\}_{t=0}$$

$$= \lambda\left[e^t e^{\lambda(e^t-1)} \lambda e^t + e^t e^{\lambda(e^t-1)}\right]_{t=0}$$

$$= \lambda\left[\lambda e^0 + e^0\right] = \lambda(\lambda+1)$$

$$E[X^2] = \lambda^2 + \lambda$$

$\therefore$ Variance $= E[X^2] - (E[X])^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Variance} = \lambda}$$

3)a) The probability mass function of a discrete R.V X is given in the following table.

X	0	1	2	3	4	5	6	7	8
$P(X)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find (i) the value of a (ii) $P(X<3)$ (iii) Mean of X (iv) Variance of X.

SOLUTION:

We know that $a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$$81a = 1$$

$$\Rightarrow \boxed{a = \frac{1}{81}}$$

(ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$$= a+3a+5a = 9a = \frac{9}{81}$$

(iii) Mean of $X = E[X] = \sum x\, P(x)$

$$= 0+3a+10a+21a+36a+55a+78a+105a+136a$$

$$= 444a$$

$$= 444 \frac{1}{81}$$

$$= \frac{148}{27}$$

(iv) $E[x^2] = \sum x^2\, P(x)$

$$= 0+1(3a)+4(5a)+9(7a)+16(9a)+25(11a)$$
$$+36(13a)+49(15a)+64(17a)$$

$$= 3a+20a+63a+144a+275a+468a+735a$$
$$+1088a$$

$$= 2796a = 2796\times\frac{1}{81}$$

$$= \frac{932}{27}$$

$\therefore$ Variance $= E[x^2] - (E[x])^2$

$$= \frac{932}{27} - \left(\frac{148}{27}\right)^2$$

$$= \frac{932\times 27 - 148\times 148}{27\times 27}$$

$$= \frac{25164 - 21904}{729} = \frac{3260}{729}$$

$\therefore$ Variance $= \frac{3260}{729}$

3) b) Deduce the MGF of a geometric distribution and hence find the mean and variance.

SOLUTION:

Geometric distribution $P(X=x) = pq^x$; $x = 0, 1, 2, \ldots$

MGF:

$$M_X(t) = E[e^{tX}]$$

$$= \sum e^{tx} P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} pq^x$$

$$= \sum_{x=0}^{\infty} p(qe^t)^x$$

$$= p\sum_{x=0}^{\infty} (qe^t)^x$$

$$= p\left[1 + (qe^t) + (qe^t)^2 + \cdots\right]$$

$$= p(1-qe^t)^{-1}$$

$$M_X(t) = \frac{p}{1-qe^t}$$

Mean:

$$E[X] = \left\{\frac{d}{dt} M_X(t)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left(\frac{p}{1-qe^t}\right)\right\}_{t=0}$$

$$= \left\{-p(1-qe^t)^{-2}(-qe^t)\right\}_{t=0}$$

$$= pq(1-q)^{-2}$$

$$= \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$\boxed{\text{mean} = \frac{q}{p}}$$

Variance :

$$E[x^2] = \left\{\frac{d^2}{dt^2} M_x(t)\right\}_{t=0}$$

$$= \left(\frac{d}{dt}\left\{\frac{pqe^t}{(1-qe^t)^2}\right\}\right)_{t=0}$$

$$= pq\left\{\frac{d}{dt}\left(\frac{e^t}{(1-qe^t)^2}\right)\right\}_{t=0}$$

$$= pq\left\{\frac{(1-qe^t)^2 e^t - e^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right\}_{t=0}$$

$$= pq\left[\frac{(1-q)^2 + 2q(1-q)}{(1-q)^4}\right]$$

$$= pq\,\frac{[p^2 + 2qp]}{p^4}$$

$$= \frac{q}{p^3}(p(p+2q)$$

$$= \frac{q}{p^2}(p+2q)$$

$$\therefore \text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{q}{p^2}(p+2q) - \frac{q^2}{p^2}$$

$$= \frac{1}{p^2}[pq + 2q^2 - q^2]$$

$$= \frac{1}{p^2}[pq + q^2] = \frac{q}{p^2}(p+q)$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

4) a) If the discrete random variable X has the probability function given by the table.

x :	1	2	3	4
$P(x)$:	$K/3$	$K/6$	$K/3$	$K/6$

Find the value of K and cumulative distribution of X.

SOLUTION:

We know that $\sum P(x) = 1$

$$\frac{K}{3} + \frac{K}{6} + \frac{K}{3} + \frac{K}{6} = 1$$

$$\frac{2K + K + 2K + K}{6} = 1$$

$$\frac{6K}{6} = 1 \Rightarrow \boxed{K = 1}$$

Cumulative Distribution:

x	$F(x)\ P(X \leq x)$
1	$K/3 = 1/3$
2	$\frac{K}{3} + \frac{K}{6} = \frac{3K}{6} = \frac{K}{2} = \frac{1}{2}$
3	$\frac{K}{3} + \frac{K}{6} + \frac{K}{3}$ $\frac{5K}{6} = \frac{5}{6}$
4	$\frac{5K}{6} + \frac{K}{6} = \frac{6K}{6} = 1.$

4)b) Derive the MGF of Uniform distribution and hence deduce the mean and variance.

SOLUTION:

Uniform Distribution is $f(x) = \begin{cases} \frac{1}{b-a} & , a<x<b \\ 0 & , \text{otherwise} \end{cases}$

MGF:

$$M_X(t) = E[e^{tx}]$$

$$= \int e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx = \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$M_X(t) = \frac{1}{(b-a)t} \left[e^{bt} - e^{at}\right]$$

Mean:

$$E[x] = \int_a^b x\, f(x)\, dx$$

$$= \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)} \left[b^2 - a^2\right] = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$\boxed{E[x] = \frac{a+b}{2}}$$

$$E[x^2] = \int_a^b x^2 f(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}(b^3-a^3) = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3(a+b)^2}{12}$$

$$= \frac{4a^2+4ab+4b^2-3(a^2+b^2+2ab)}{12}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$\boxed{\text{Variance} = \frac{(a-b)^2}{12}}$$

5) a) The probability mass function of a R.V is given by $P(X=r) = kr^3$; $r = 1, 2, 3, 4$. Find (i) the value of k.

(ii) $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right)$.

SOLUTION:

$$P\left(\tfrac{1}{2} < x < 5/2 \,/\, x > 1\right) = \frac{P\left(\tfrac{1}{2} < x < 5/2\right)}{P(x>1)}$$

$$P(x>1) = P(x=2) + P(x=3) + P(x=4)$$

$$= k\,2^3 + k.3^3 + k.4^3$$

$$= k(8+27+64)$$

$$= k.99$$

$$\boxed{P(x>1) = \frac{99}{100}}$$

$$\sum_{k=1}^{4} P(x) = 1$$

$$k + 2^3k + 3^3k + 4^3k = 1$$

$$k(1+8+27+64) = 1$$

$$100\,k = 1 \Rightarrow \boxed{k = \frac{1}{100}}$$

$$P\left(\left(\tfrac{1}{2} < x < 5/2\right) \cap \cancel{E}(x>1)\right)$$

$$= \frac{P(1 < x < 5/2)}{P(x>1)}$$

Now $P(1 < x < 5/2) = P(x=2)$

$$= k.2^3 = \frac{8}{100}$$

$$\therefore\ P\left(\tfrac{1}{2} < x < 5/2 \,/\, x>1\right) = \frac{8/100}{99/100} = \frac{8}{99}$$

5)b) Deduce the MGF of Exponential distribution and hence find its mean and variance.

SOLUTION:

Exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int e^{tx} f(x)\,dx$$

$$= \int_0^\infty \lambda e^{tx} e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty = \frac{-\lambda}{\lambda-t} \left[e^{-\infty} - e^0 \right]$$

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t}}$$

Mean:

$$E[X] = \left\{ \frac{d}{dt} M_X(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(\frac{\lambda}{\lambda - t} \right) \right\}_{t=0}$$

$$= \left\{ \lambda (-1) (\lambda - t)^{-2} (-1) \right\}_{t=0}$$

$$= \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[X] = \frac{1}{\lambda}$$

Variance:

$$E[X^2] = \left\{ \frac{d^2}{dt^2} M_X(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(\lambda (\lambda - t)^{-2} \right) \right\}_{t=0}$$

$$= \left\{ \lambda (-2) (\lambda - t)^{-3} (-1) \right\}_{t=0}$$

$$= 2 \lambda \lambda^{-3} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$\therefore$ Variance $= E[x^2] - (E[X])^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2}.$

6) a) The probability distribution of an infinite discrete ~~mat~~ distribution is given by $P[X=j] = \frac{1}{2^j}$ $(j = 1, 2, 3 \ldots)$.

Find (i) Mean of X (2) P(x is even) (iii) P(x is odd) (iv) P(x is divisible by 3)

SOLUTION:

$$P[x=j] = \frac{1}{2^j} \quad (j = 1, 2, 3 \ldots)$$

(i) Mean of X $= E[X] = \sum_{j=1}^{\infty} x \cdot P(x)$

$$= \cancel{x}\, \frac{1}{2} + \cancel{2^2} \cdot \frac{1}{2^2} + \cancel{3^3} \cdot \frac{1}{2^3} + \cdots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{1}{2} \cdot \frac{1}{2^{-2}} = \frac{1}{2} \; 4 = 2$$

$E[x] = 2$

(ii) P(x is even) $= P(x=2) + P(x=4) + P(x=6) + \cdots$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2}\right]^{-1}$$

$$= \frac{1}{4}\left[1-\frac{1}{4}\right]^{-1} = \frac{1}{4}\left[\frac{3}{4}\right]^{-1}$$

$$= \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$

(iii) $P(x \text{ is odd}) = P(x=1) + P(x=3) + P(x=5) + \ldots$

$$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{2}\right]^{-1} = \frac{1}{2}\left(\frac{1}{2}\right)^{-1}$$

$$= \frac{1}{2}\, 2 = 1$$

(iv) $P(x \text{ is divisible by } 3) = P(x=3) + P(x=6) + P(x=9) + \ldots$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots$$

$$= \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots\right]$$

$$= \frac{1}{8}\left[1 + \left(\frac{1}{2^3}\right) + \left(\frac{1}{2^3}\right)^2 + \cdots\right]$$

$$= \frac{1}{8}\left[1-\frac{1}{2^3}\right]^{-1} = \frac{1}{8}\left[1-\frac{1}{8}\right]^{-1}$$

$$= \frac{1}{8}\left[\frac{7}{8}\right]^{-1} = \frac{1}{\cancel{8}}\cdot\frac{\cancel{8}}{7}$$

$$= \frac{1}{7}$$

6) b) State and prove the memory less property of exponential distribution.

SOLUTION:

Statement:

If X is exponentially distributed, then

$P(X>s+t / X>s) = P(X>t)$ for any $s,t>0$.

Proof:

$$P(X>k) = \int_k^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty = -1\left[e^{-\infty} - e^{-k\lambda}\right]$$

$$P(X>k) = e^{-k\lambda}$$

$$\therefore P(X>s+t) = e^{-k(s+t)}$$

$$P(X>s) = e^{-ks}$$

$$P(X>t) = e^{-kt}$$

$$\therefore P(X>s+t / X>s) = \frac{P(X>s+t \cap X>s)}{P(X>s)}$$

$$= \frac{P(X>s+t)}{P(X>s)}$$

$$= \frac{e^{-k(s+t)}}{e^{-ks}} = \frac{\cancel{e^{-ks}}\, e^{-kt}}{\cancel{e^{-ks}}}$$

$$= e^{-kt}.$$

$$= P(X>t)$$

$\therefore$ Memoryless Property proved.

4 L 2² J

7)a) Find the mean and variance of the following probability distribution.

X :	1	2	3	4	5	6	7	8
P(X) :	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

SOLUTION:

$$\text{Mean} = E[X] = \Sigma x\, P(x)$$

$$= 1(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$

$$= 0.08 + 0.24 + 0.57 + 0.96 + 0.8 + 0.6 + 0.49 + 0.32$$

$$\boxed{E[X] = 4.06}$$

$$E[X^2] = 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$

$$= 0.08 + 0.48 + 1.71 + 3.84 + 4 + 3.6 + 3.43 + 2.56$$

$$= 19.7$$

$$\therefore \text{Variance} = E[X^2] - (E[X])^2$$

$$= 19.7 - (4.06)^2$$

$$= 19.7 - 16.5$$

$$\therefore \boxed{\text{Variance} = 3.2}$$

7)b) Assume that 50% of all Engineering students are good in mathematics. Determine the probabilities that among 18 engineering students (i) exactly 10 (ii) atleast 15 are good in mathematics.

SOLUTION:

Gn: $p = 50\% = \frac{5\cancel{0}}{10\cancel{0}} = \frac{1}{2} = 0.5$

$q = \frac{1}{2}$, $n = 18$

$P(X=x) = nC_x \, p^x \, q^{n-x}$

$= 18C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{18-x}$

(i) P(exactly 10) $= P(X=10)$

$= 18C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{18-10}$

$= \frac{18 \times 17 \times \cancel{16} \times \cancel{15} \times \cancel{14} \times 13 \times \cancel{12} \times 11}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6} \times \cancel{7} \times \cancel{8}} \left(\frac{1}{2}\right)^{18}$

$= 43,758 \frac{1}{262144} = \frac{43758}{2^{18}}$

(ii) P(atleast 15 are good in maths) $= P(X \geq 15)$

$= P(X=15) + P(X=16) + P(X=17) + P(X=18)$

$= 18C_{15} \left(\frac{1}{2}\right)^{15} \left(\frac{1}{2}\right)^{18-15} + 18C_{16} \left(\frac{1}{2}\right)^{16} \left(\frac{1}{2}\right)^{18-16}$

$+ 18C_{17} \left(\frac{1}{2}\right)^{17} \left(\frac{1}{2}\right)^{18-17} + 18C_{18} \left(\frac{1}{2}\right)^{18} \left(\frac{1}{2}\right)^{18-18}$

$= \frac{\cancel{18} \times 17 \times 16}{1 \times \cancel{2} \times \cancel{3}} \cdot \frac{1}{2^{18}} + \frac{\cancel{18} \times 17}{1 \times \cancel{2}} \frac{1}{2^{18}} + 18 \left(\frac{1}{2^{18}}\right) + \left(\frac{1}{2}\right)^{18}$

$= \frac{1}{2^{18}} \left[816 + 153 + 18 + 1\right] = \frac{988}{2^{18}}$

8) Obtain the MGF of a normal distribution and hence find its mean and variance.

SOLUTION: Normal Distribution is

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

MGF :

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$\Rightarrow x = \mu + \sigma z$

$\Rightarrow dx = \sigma\,dz \Rightarrow \frac{dx}{\sigma} = dz$

$$\therefore M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu} e^{t\sigma z} e^{\frac{-1}{2}z^2} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z-\sigma t)^2 - \sigma^2 t^2\right]} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}\sigma^2 t^2} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

Let $z - \sigma t = u$

$\Rightarrow dz = du$

$$\therefore M_x(t) = \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \, 2 \int_{0}^{\infty} e^{-u^2/2} \, du$$

$$= \frac{\cancel{\sqrt{2\pi}} \; 2}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{0}^{\infty} e^{-u^2/2} \, du$$

Let $\frac{u^2}{2} = t^2 \Rightarrow u^2 = 2t^2$

$\Rightarrow u = \sqrt{2}\, t$

$\Rightarrow du = \sqrt{2}\, dt$

$$\therefore M_x(t) = \frac{2}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{0}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{2}{\sqrt{\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{0}^{\infty} e^{-t^2} dt$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{\cancel{\sqrt{\pi}}}{\cancel{2}} \qquad \left[\because \int_{0}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \right]$$

$$\therefore \boxed{M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

Mean:

$$E[X] = \left\{ \frac{d}{dt} M_x(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(e^{\mu t + \frac{t^2 \sigma^2}{2}} \right) \right\}_{t=0}$$

$$= \left\{ e^{\mu t + \frac{t^2\sigma^2}{2}} \left(\mu + \frac{2t\sigma^2}{2} \right) \right\}_{t=0}$$

$$= \mu e^0 = \mu$$

$$\boxed{E[x] = \mu}$$

$$E[x^2] = \left\{ \frac{d^2}{dt^2} M_x(t) \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2) \right) \right\}_{t=0}$$

$$= \left\{ e^{\mu t + \frac{t^2\sigma^2}{2}} (\sigma^2) + e^{\mu t + \frac{t^2\sigma^2}{2}} \left(\mu + \frac{2t\sigma^2}{2} \right)^2 \right\}_{t=0}$$

$$= e^0 \sigma^2 + e^0 (\mu) = \mu^2 + \sigma^2$$

$\therefore$ Variance $= E[x^2] - (E[x])^2$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$\boxed{\text{Variance} = \sigma^2}$$

9)a) If a random variable x has pdf $f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & \text{Otherwise} \end{cases}$

Find (a) $P(x<1)$ (b) $P(|x|>1)$ (c) $P(2x+3) > 5)$

SOLUTION:

$$f(x) = \begin{cases} \frac{1}{4}, & 2 < x < 2 \\ 0, & \text{Otherwise} \end{cases}$$

$|x| < 2 \Rightarrow$

$-2 < x < 2$.

(i) $P(X<1) = \int_{-2}^{1} f(x)\,dx = \int_{-2}^{1} \frac{1}{4}\,dx$

$= \frac{1}{4}[x]_{-2}^{1} = \frac{1}{4}[1+2] = \frac{3}{4}$

(ii) $P(|X|>1) = 1 - P[|X|<1]$

$= 1 - P(-1<X<1)$

$= 1 - \int_{-1}^{1} f(x)\,dx$

$= 1 - \int_{-1}^{1} \frac{1}{4}\,dx = 1 - \frac{1}{4}[x]_{-1}^{1}$

$= 1 - \frac{1}{4}(1+1) = 1 - \frac{2}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

(iii) $P(2X+3>5) = P(2X > 5-3)$

$= P(2X>2)$

$= P(X>1)$

$= \int_{1}^{2} f(x)\,dx = \int_{1}^{2} \frac{1}{4}\,dx$

$= \frac{1}{4}[x]_{1}^{2} = \frac{1}{4}(2-1) = \frac{1}{4}$

9)b) Out of 2000 families with 4 children each. Find how many family would you expect to have (i) atleast 1 boy (ii) 2 boys.

SOLUTION:

Gn: $n=4$, $x \to$ no/: of boys

$\therefore p = \frac{1}{2}$, $q = \frac{1}{2}$

Binomial Distribution is

$P(X=x) = nC_x\, p^x\, q^{n-x}$

$\therefore P(X=x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

(i) P(atleast 1 boy) $= P(X \geq 1)$

$= 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{2^4} = 1 - \frac{1}{16}$

$= \frac{15}{16}$

$\therefore$ Out of 2000 families $= \frac{15}{16} \times 2000 = 1875$

(ii) P(2 boys) $= P(X=2)$

$= 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$

$= \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}} \frac{1}{2^4} = \frac{6}{16} = \frac{3}{8}$

$\therefore$ Out of 2000 families $= \frac{3}{8} \times 2000 = 750$

10)a) Find the MGF of the random variable X having the probability density function $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x>0 \\ 0, & \text{otherwise.} \end{cases}$

Also find the mean and variance.

SOLUTION:

MGF:

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{x}{4} e^{-x/2} \, dx$$

$$= \frac{1}{4} \int_0^\infty x \, e^{-\frac{1}{2}(1-2t)x} \, dx = \frac{1}{4} \int_0^\infty x \, e^{-(\frac{1}{2}-t)x} \, dx$$

$u = x \qquad v = e^{-(\frac{1}{2}-t)x}$

$u' = 1 \qquad v_1 = \dfrac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)}$

$u'' = 0 \qquad v_2 = \dfrac{e^{-(\frac{1}{2}-t)x}}{(\frac{1}{2}-t)^2}$

$$\therefore M_X(t) = \frac{1}{4}\left[x \frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)} - (1) \frac{e^{-(\frac{1}{2}-t)}}{(\frac{1}{2}-t)^2} \right]_0^\infty$$

$$= \frac{1}{4}\left[0 - \left(0 - \frac{e^0}{(\frac{1}{2}-t)^2}\right)\right]$$

$$M_X(t) = \frac{1}{4\left(\frac{1-2t}{2}\right)^2} = \frac{4}{4(1-2t)^2} = \frac{1}{(1-2t)^2}$$

$$\text{Mean} = E[X] = \left\{\frac{d}{dt}(M_X(t))\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left(\frac{1}{(1-2t)^2}\right)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}(1-2t)^{-2}\right\}_{t=0}$$

$$= \left\{(-2)(1-2t)^{-3}(-2)\right\}_{t=0}$$

$$= 4$$

$$E[X^2] = \left\{\frac{d^2}{dt^2} M_X(t)\right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(4(1-2t)^{-3} \right) \right\}_{t=0}$$

$$= \left[4 (1-2t)^{-4} \times (-3)(-2) \right]_{t=0}$$

$$= 4 \times 6 = 24.$$

$$\therefore \text{ Variance} = E[X^2] - (E[X])^2$$

$$= 24 - 16 = 8$$

10)b) 4 Coins were tossed simultaneously. What is the probability of getting (i) 2 heads (ii) atleast 2 heads (iii) atmost 2 heads.

SOLUTION:

$$n=4, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}$$

Binomial Distribution $P(X=x) = nC_x \, p^x q^{n-x}$

$$P(X=x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

(i) $P(\text{2 heads}) = P(X=2)$

$$= 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= 6 \frac{1}{2^4} = \frac{6}{16} = \frac{3}{8}$$

(ii) $P(\text{atleast 2 heads}) = P(X \geq 2)$.

$$= 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} + 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1} \right]$$

$$= 1 - \left[\frac{1}{2^4} + 4 \frac{1}{2^4} \right] = 1 - \left[\frac{1}{16} + \frac{4}{16} \right]$$

$$= 1 - \frac{5}{16} = \frac{11}{16}$$

11)a) A random variable X has cdf $F(x) = \begin{cases} 0 & , \text{ if } x < -1 \\ a(1+x) & , \text{ if } -1 < x < 1 \\ 1 & , \text{ if } x \geq 1 \end{cases}$

Find the value of a. Also $P(X > 1/4)$ and $P(-0.5 \leq X \leq 0)$

SOLUTION:

$$f(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx} a(1+x)$$

$$= a[1] = a$$

$$\therefore f(x) = \begin{cases} 0 & \text{if } x < -1 \\ a & \text{if } -1 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

$\therefore$ We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{-1}^{1} a\, dx = 1$$

$$a[x]_{-1}^{1} = 1 \Rightarrow a[1-(-1)] = 1$$

$$\Rightarrow 2a = 1 \Rightarrow \boxed{a = 1/2}$$

$$\therefore P(X > 1/4) = \int_{1/4}^{1} f(x)\, dx = \int_{1/4}^{1} a\, dx = \frac{1}{2}[x]_{1/4}^{1} = \frac{1}{2}\left[1 - \frac{1}{4}\right]$$

$$= \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

$$P(-0.5 \leq X \leq 0) = \int_{-0.5}^{0} f(x)\, dx = \int_{-0.5}^{0} \frac{1}{2}\, dx = \frac{1}{2}[x]_{-0.5}^{0}$$

$$= \frac{1}{2}[0 + 0.5] = \frac{0.5}{2} = 0.25$$

11) b) The atoms of a radioactive element are randomly disintegrating. If every gram of this element, on average, emits 3.9 alpha particles per second, then what is the probability that during the next second the number of alpha particles emitted from 1 gram is (i) at most 6 (ii) atleast 2 and (iii) atleast 3 and atmost 5.

SOLUTION:

Gn: $\lambda = 3.9$

Poisson distribution is $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!} = \dfrac{e^{-3.9}(3.9)^{x}}{x!}$

(i) P(at most 6) $= P(X \le 6)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= e^{-3.9}\left[1 + 3.9 + \dfrac{(3.9)^2}{2} + \dfrac{(3.9)^3}{6} + \dfrac{(3.9)^4}{24} + \dfrac{(3.9)^5}{120} + \dfrac{(3.9)^6}{720}\right]$

$= e^{-3.9}\left[1 + 3.9 + \dfrac{15.21}{2} + \dfrac{59.319}{6} + \dfrac{231.344}{24} + \dfrac{902.242}{120} + \dfrac{3518.744}{720}\right]$

$= e^{-3.9}\left[1 + 3.9 + 7.605 + 9.887 + 9.639 + 7.519 + 4.887\right]$

$= e^{-3.9}\left[44.437\right] = (0.0202)(44.437)$

$= 0.8995$

(ii) $P(\text{atleast } 2) = P(x \leq 2)$

$$= P(x=0) + P(x=1) + P(x=2)$$

$$= e^{-3.9} + e^{-3.9}(3.9) + e^{-3.9}\frac{(3.9)^2}{2}$$

$$= (0.0202)(1 + 3.9 + 7.605)$$

$$= (0.0202)(12.505)$$

$$= 0.2526$$

(iii) $P(\text{atleast } 3 \text{ and atmost } 5) = P(3 \leq x \leq 5)$

$$= P(x=3) + P(x=4) + P(x=5)$$

$$= e^{-3.9}\left[9.887 + 9.639 + 7.519\right]$$

$$= (0.0202)(27.045) = 0.5463$$

12) If $f(x) = \begin{cases} ax & , 0 \leq x \leq 1 \\ a & , 1 \leq x \leq 2 \\ 3a - ax & , 2 \leq x \leq 3 \\ 0 & , \text{elsewhere.} \end{cases}$ is the pdf of X.

Calculate (i) The value of a

(ii) The cumulative distribution function of X.

(iii) If X_1, X_2 and X_3 are 3 independent observations of X. Find the probability that exactly one of these ~~is~~ 3 is greater than 1.5?

SOLUTION:

We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{0}^{3} f(x)\, dx = 1$$

$$= \frac{1}{2}\frac{1}{2} + \frac{1}{2}(2-1) + 3\frac{1}{2}x - \frac{1}{2}\frac{x^2}{2} - \left(6\frac{1}{2} - \frac{1}{\cancel{4}}\cancel{4}\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3}{2}x - \frac{x^2}{4} - 3 + 1$$

$$= \frac{1+2}{4} + \frac{6x - x^2}{4} - 2 = \frac{3}{4} + \frac{6x-x^2}{4} - 2$$

$$= \frac{3+6x-x^2-8}{4} = \frac{-x^2+6x-5}{4}$$

(iv) If $x > 3$, then $F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx$

$$+ \int_3^x f(x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - (3-1)$$

$$= \frac{1}{4} + 5 - \frac{9}{4} - 2 = \cancel{5} - 1 + 2 = 1.$$

$\therefore$ CDF is

$$F(x) = \begin{cases} x^2/4, & 0 \le x \le 1 \\ \frac{1}{4}(2x-1), & 1 \le x \le 2 \\ \frac{-x^2+6x-5}{4}, & 2 \le x \le 3 \\ 1, & x > 3. \end{cases}$$

(iii) $P(X > 1.5) = \int_{1.5}^3 f(x)\,dx = \int_{1.5}^2 a\,dx + \int_2^3 (3a - ax)\,dx$

$$= \frac{1}{2}[x]_{1.5}^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^3$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a-ax)\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$a\frac{1}{2} + a(2-1) + 9a - \frac{9a}{2} - (6a-2a) = 1$$

$$\frac{a}{2} + a + 9a - \frac{9a}{2} - 4a = 1$$

$$-4a + 6a = 1 \Rightarrow 2a = 1 \Rightarrow \boxed{a = \frac{1}{2}}$$

Cumulative Distribution of X: $F(x) = P(X \le x)$

(i) If $x < 0$ then $F(x) = 0$

(ii) If $0 \le x \le 1$, then $F(x) = \int_0^x f(x)\,dx$

$$= \int_0^x ax\,dx = a\left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{1}{2}\,\frac{x^2}{2} = \frac{x^2}{4}$$

(iii) If $1 \le x \le 2$, then $F(x) = \int_0^1 ax\,dx + \int_1^x a\,dx$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^x$$

$$= \frac{1}{4} + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$$

$$= \frac{1}{4}(2x-1)$$

(iii) If $2 \le x \le 3$, then

$$F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a-ax)\,dx$$

$$= a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^x$$

$$= \frac{1}{2}(2-1.5) + \frac{9}{2} - \frac{9}{4} - (3-1)$$

$$= 0.25 + \frac{18-9}{4} - 2$$

$$= 0.25 + \frac{9}{4} - 2$$

$$= 0.25 + 2.25 - 2 = 0.5$$

Choosing an X and observing its value can be considered as a trial and $X > 1.5$ can be considered as follows.

$$P = P(X > 1.5) = \frac{1}{2}$$

$$\therefore p = \frac{1}{2}, \quad q = \frac{1}{2}, \quad n = 3 \text{ (3 observations)}$$

By Bernoulli's theorem, P(exactly one value > 1.5)

$$= P(1 \text{ success}) = 3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{3-1}$$

$$= 3 \cdot \frac{1}{2} \cdot \frac{1}{2^2} = \frac{3}{8}.$$

13)a) The probability distribution function of a R.V X is given by $f(x) = \frac{4x(9-x^2)}{81}$, $0 \le x \le 3$. Find the mean and variance.

SOLUTION:

Mean: $E[X] = \int x \, f(x)\, dx$

$$= \int_0^3 x \, \frac{4x(9-x^2)}{81}\, dx$$

$$= \frac{4}{81} \int_0^3 (9x^2 - x^4)\, dx$$

$$= \frac{4}{81}\left[\frac{\cancel{9}^{3}x^3}{\cancel{3}} - \frac{x^5}{5}\right]_0^3$$

$$= \frac{4}{81}\left[81 - \frac{243}{5}\right] = \frac{4}{81}\left[\frac{405-243}{5}\right] = \frac{\cancel{162}^{2} \times 4}{\cancel{81} \times 5}$$

$$\boxed{\text{mean} = 8/5}$$

$$E[X^2] = \int x^2 f(x)\,dx$$

$$= \int_0^3 x^2 \frac{4x(9-x^2)}{81}\,dx$$

$$= \frac{4}{81}\int_0^3 (9x^3 - x^5)\,dx$$

$$= \frac{4}{81}\left[\frac{9x^4}{4} - \frac{x^6}{6}\right]_0^3 = \frac{4}{81}\left[\frac{729}{4} - \frac{729}{6}\right]$$

$$= \frac{4}{\cancel{81}} \times \cancel{729}^{9}\left(\frac{1}{4} - \frac{1}{6}\right) = \cancel{36}^{3}\left(\frac{6-4}{\cancel{24}_{2}}\right)$$

$$= \frac{3}{2}(2) = 3$$

$$\therefore \text{Variance} = E[X^2] - (E[X])^2$$

$$= 3 - \frac{64}{25} = \frac{75-64}{25} = \frac{11}{25}$$

$$\therefore \text{Variance} = \frac{11}{25}.$$

13) b) The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (1) without breakdown (2) with only one breakdown and (3) with atleast one breakdown.

SOLUTION:

Here $\lambda = 1.8$

Poisson distribution : $P(X=x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!} = \dfrac{e^{-1.8} (1.8)^{x}}{x!}$

(i) P(without breakdown) $= P(X=0)$

$$= \frac{e^{-1.8} (1.8)^{0}}{0!} = e^{-1.8} = 0.1653$$

(ii) P(with only one breakdown) $= P(X=1) = \dfrac{e^{-1.8} (1.8)^{1}}{1!}$

$$= (1.8)\, e^{-1.8}$$

$$= 0.2975.$$

(iii) P(atleast one breakdown) $= P(X \geq 1)$

$$= 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - 0.1653 = 0.8347$$

14) a) Messages arrive at a switch board in a Poisson manner at an average rare of 6 per hour. Find the probability that exactly 2 messages arrive within one hour, no messages arrive within one hour and atleast 3 messages arrive within one hour.

SOLUTION:

$$\lambda = 6, \quad P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \frac{e^{-6} 6^{x}}{x!}$$

$$P(\text{exactly 2 messages}) = P(X=2)$$

$$= \frac{e^{-6} 6^{2}}{2!} = \frac{e^{-6} \times 36}{2}$$

$$= 18 e^{-6} = 0.0446$$

$$P(\text{no messages arrive}) = P(X=0) = \frac{e^{-6} 6^{0}}{0!}$$

$$= e^{-6} = 0.0025$$

$$P(\text{atleast 3 messages arrive}) = P(X \geq 3)$$

$$= 1 - P(X < 3)$$

$$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

$$= 1 - [e^{-6} + e^{-6} 6 + 18 e^{-6}]$$

$$= 1 - e^{-6}(1 + 6 + 18)$$

$$= 1 - e^{-6}(25)$$

$$= 1 - 0.0625 = 0.9378$$

14)b) Suppose that the life of an industrial lamp in 1000 of hours is exponentially distributed with mean life of 3000 hours. Find the probability that (i) the lamp last more than the mean life (ii) the lamp last between 2000 and 3000 hours (iii) the lamp last another 1000 hours given that it has already lasted for 250 hours.

$$= \frac{1}{3000}\left[\frac{e^{-x/3000}}{-1/3000}\right]_{750}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-\frac{750}{3000}}\right]$$

$$= e^{-0.25} = 0.7788$$

15)a) The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

(a) What is the probability that the repair time exceeds 2 hours?

(b) what is the conditional probability that a repair time exceeds at least 10 hours that its distribution exceeds 9 hours?.

SOLUTION:

$$\lambda = 1/2 \quad , \quad P(X=x) = \lambda e^{-\lambda x}$$

$$= \frac{1}{2} e^{-1/2 x}$$

(a) P(the repair time exceeds 2 hrs) $= P(X>2)$

$$= \int_{2}^{\infty} \frac{1}{2} e^{-1/2 x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-1/2 x}}{-1/2}\right]_{2}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1}\right]$$

$$= e^{-1}$$

$$= 0.3679$$

SOLUTION:

$\frac{1}{\lambda} = 3000$, Exponential Distribution

$\Rightarrow \lambda = \frac{1}{3000}$ $\quad f(x) = \lambda e^{-\lambda x} = \frac{1}{3000} e^{\frac{-x}{3000}}$

(i) P(lamp last more than mean) $= P(x > 3000)$

$$= \int_{3000}^{\infty} \frac{1}{3000} e^{\frac{-x}{3000}} dx$$

$$= \frac{1}{\cancel{3000}} \left[\frac{e^{-x/3000}}{-\cancel{1/3000}} \right]_{3000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1} \right]$$

$$= e^{-1} = 0.3679$$

(ii) P(lamp last between 2000 and 3000 hrs) $= P(2000 \leq x \leq 3000)$

$$= \int_{2000}^{3000} \frac{1}{3000} e^{-x/3000} dx$$

$$= \frac{1}{\cancel{3000}} \left[\frac{e^{-x/3000}}{-\cancel{1/3000}} \right]_{2000}^{3000}$$

$$= -\left[e^{-1} - e^{-2/3} \right]$$

$$= e^{-2/3} - e^{-1}$$

$$= 0.5134 - 0.3679$$

$$= 0.1455$$

(iii) P (lamp last another 1000 given that already last for 250 hrs)

$= P(x > 1000 / x > 250) = P(x > 750)$

[By Memoryless Property]

$$= \int_{750}^{\infty} \frac{1}{3000} e^{-x/3000} dx$$

(b) $P(X>10/X>9) = P(X>1)$ [By Memoryless Property]

$$= \int_1^\infty \frac{1}{2} e^{-1/2 x}\, dx$$

$$= \cancel{\frac{1}{2}} \left[\frac{e^{-1/2 x}}{-\cancel{1/2}} \right]_1^\infty$$

$$= -\left[e^{-\infty} - e^{-1/2} \right]$$

$$= +e^{-1/2} = e^{-0.5} = 0.6065$$

15) b) Let X be a Uniformly distributed Random variable over $[-5, 5]$. Evaluate (i) $P(X \leq 2)$ (ii) $P(|X| > 2)$.

(iii) Cumulative distribution function of X (iv) Var(X)

SOLUTION:

$$f(x) = \frac{1}{5-(-5)} = \frac{1}{10}, \quad -5 \leq x \leq 5$$

(i) $P(X \leq 2) = \int_{-5}^{2} \frac{1}{10}\, dx = \frac{1}{10}[x]_{-5}^{2}$

$$= \frac{1}{10}[2+5] = \frac{7}{10} = 0.7$$

(ii) $P(|X| > 2) = 1 - P(|X| \leq 2)$

$$= 1 - \int_{-2}^{2} f(x)\, dx = 1 - \int_{-2}^{2} \frac{1}{10}\, dx$$

$$= 1 - \frac{1}{10}[x]_{-2}^{2} = 1 - \frac{1}{10}[2+2]$$

$$= 1 - \frac{4}{10} = \frac{6}{10} = 0.6$$

(iii) Cumulative Distribution of X.

$$CDF = P(X \leq x)$$

$$= \int_{-5}^{x} \frac{1}{10}\,dx = \frac{1}{10}\left[x\right]_{-5}^{x}$$

$$F(x) = \frac{1}{10}(x+5) \quad , \quad -5 \le x \le 5$$

(iv) $$E[X] = \int_{-5}^{5} x \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[\frac{x^2}{2}\right]_{-5}^{5} = \frac{1}{20}\left[25-25\right] = 0$$

$$E[X^2] = \int_{-5}^{5} x^2 \frac{1}{10}\,dx = \frac{1}{10}\left[\frac{x^3}{3}\right]_{-5}^{5}$$

$$= \frac{1}{30}\left[125-(-125)\right] = \frac{1}{3\cancel{0}}(25\cancel{0})$$

$$= \frac{25}{3}$$

$\therefore$ Variance of $X = E[X^2] - (E[X])^2$

$$= \frac{25}{3}.$$

16) a) Buses arrive at a specified stop at 15 minutes intervals starting at 7 am that is 7, 7.15, 7.30, 7.45 and so on. If a passenger arrive at the stop at a random time that is uniformly distributed between 7 and 7.30 am, evaluate the probability that he waits (i) Less than 5 minutes for a bus

(ii) Atleast 12 minutes for a bus

SOLUTION:

Let X denotes the time that the time that a passenger arrive between 7 and 7.30 a.m

$\therefore \quad f(x) = \dfrac{1}{30-0} = \dfrac{1}{30}$

(a) P(waits less than 5 minutes) = $P(10 \leq X \leq 15) +$
[7.10 to 7.15 (or) 7.25 – 7.30] $\quad P(25 \leq X \leq 30)$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[x\right]_{10}^{15} + \frac{1}{30}\left[x\right]_{25}^{30}$$

$$= \frac{1}{30}(15-10) + \frac{1}{30}(30-25)$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \frac{1}{3}$$

(b) P(atleast 12 minutes) = $P(0 \leq x \leq 3) + P(15 \leq x \leq 18)$
(7 – 7.03 or 7.15 – 7.18)

$$= \int_{0}^{3} \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[x\right]_{0}^{3} + \frac{1}{30}\left[x\right]_{15}^{18}$$

$$= \frac{3}{30} + \frac{1}{30}(18-15) = \frac{3}{30} + \frac{3}{30}$$

$$= \frac{6}{30} = \frac{1}{5}$$

16)b) The marks obtained by a number of students for a certain subject is assumed to be normally distributed with mean 65 and standard deviation 5. If 3 students are taken at random from this set. Find the probability that exactly 2 of them will have marks over 70?

SOLUTION:

Gn: $\mu = 65, \quad \sigma = 5$

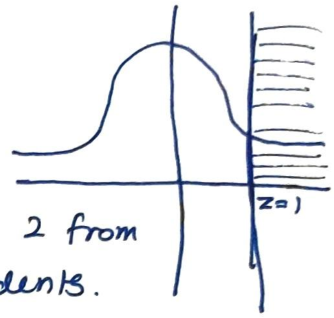
$Z = \frac{X-\mu}{\sigma}$ (Standard Normal Variate)

$P(x > 70) = P(Z > 1)$ If $X = 70$, $Z = \frac{70-65}{5} = \frac{5}{5} = 1$

$Z = 1$

$= 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413$



$P(x > 70) = 0.1587$, $\therefore p = 0.1587$

$q = 0.3413$

Apply Binomial Distribution, we select 2 from 3 students.

$\therefore n = 3$, $P(X = x) = nC_x\, p^x q^{n-x}$

$= 3C_2\,(0.1587)^2\,(0.3413)^{3-2}$

$= 3\,(0.1587)^2\,(0.3413)$

P(exactly 2 of them will have marks over 70) $= 0.0258$

17) In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Find the number of bulbs likely to burn for (i) more than 2150 hours (ii) less than 1950 hours and (iii) more than 1920 hours and less than 2160 hours.

SOLUTION:

$\mu = 2040$, $\sigma = 60$

Standard Normal variate $Z = \frac{X-\mu}{\sigma}$

$= \frac{X - 2040}{60}$

(i) More than 2150 hrs $\Rightarrow X = 2150$

$$Z = \frac{2150 - 2040}{60} = \frac{110}{60} = 1.83$$

$\therefore$ P(more than 2150 hrs) $= P(X > 2150)$

$$= P(Z > 1.83)$$

$$= 0.5 - P(1.83)$$

$$= 0.5 - 0.4664$$

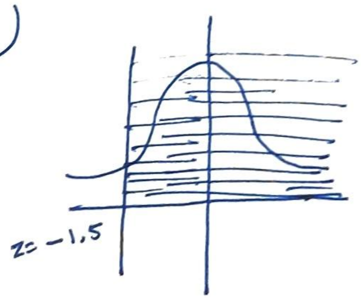
$$= 0.0336$$

(ii) If $X = 1950$, $Z = \frac{1950 - 2040}{60}$

$$= \frac{-90}{60} = \frac{-3}{2} = -1.5$$

$\therefore$ P(less than 1950 hrs) $= P(X < -1.5)$

$$= 0.5 + P(0 \le Z \le 1.5)$$



$$= 0.5 + 0.4332$$

$$= 0.9332$$

(iii) more than 1920 hrs and less than 2160 hrs.

If $X = 1920$, $Z = \frac{1920 - 2040}{60} = \frac{-120}{60} = -2$

If $X = 2160$, $Z = \frac{2160 - 2040}{60} = \frac{120}{6} = 2$

$\therefore P(1920 \le X \le 2160) = P(-2 \le Z \le 2)$

$$= P(0 \le Z \le 2) + P(0 \le Z \le 2)$$

$$= \cancel{0.444}\ 0.4772 \times 2$$

$$= 0.9544$$

18) a) The mileage which car owners get with a certain kind of radial tire is a R.V having an exponential distribution with mean 40000 km. Find the probabilities that

one of these will last (a) atleast 20,000 km and (b) at most 30,000 km.

SOLUTION:

Gn: $\frac{1}{\lambda} = 40000 \Rightarrow \lambda = \frac{1}{40000}$

Exponential Distribution

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{40000} e^{\frac{-x}{40000}}$$

(a) $P(\text{atleast } 20000) = P(X \geq 20000)$

$$= \int_{20000}^{\infty} \frac{1}{40000} e^{\frac{-x}{40000}} dx$$

$$= \frac{\cancel{1}}{\cancel{40000}} \left[\frac{e^{-x/40000}}{-\cancel{1}/\cancel{40000}} \right]_{20000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1/2} \right]$$

$$= e^{-1/2} = 0.6065$$

(b) $P(\text{at most } 30000) = P(X \leq 30000)$

$$= \int_{0}^{30000} \frac{1}{40000} e^{\frac{-x}{40000}} dx$$

$$= \frac{\cancel{1}}{\cancel{40000}} \left[\frac{e^{-x/40000}}{-\cancel{1}/\cancel{40000}} \right]_{0}^{30000}$$

$$= -\left[e^{-3/4} - e^{0} \right] = 1 - e^{-3/4}$$

$$= 1 - 0.4724$$

$$= 0.5276$$

18) b) The annual rainfall in inches in a certain region has a normal distribution with a mean of 40 and variance of 16. What is the probability that the rainfall in a given year is between 30 and 48 inches?

SOLUTION:

Gn: mean $\mu = 40$, $\sigma^2 = 16 \Rightarrow \sigma = 4$.

$$Z = \frac{X-\mu}{\sigma} = \frac{X-40}{4}$$

If $X = 30$, $Z = \frac{30-40}{4} = \frac{-10}{4} = -2.5$

If $X = 48$, $Z = \frac{48-40}{4} = \frac{8}{4} = 2$

$\therefore P(30 \le X \le 48) = P(-2.5 \le Z \le 2)$

$= P(0 \le Z \le 2.5) + P(0 \le Z \le 2)$

$= 0.4938 + 0.4772$

$= 0.9710$

PART-C

1) Out of 2000 families with 4 children each, Create how many families would you expect to have (i) atleast 1 boy (ii) 2 boys and 2 girls (iii) at most 2 girls (iv) Children of both genders.

SOLUTION:

$P \rightarrow$ probabity boy baby $= \frac{1}{2}$

$q \rightarrow \frac{1}{2}$

$n = 4$, Binomial Distribution is

$$P(x = x) = nC_x \, p^x \, q^{n-x}$$

$\therefore P(X=x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

(i) P(atleast 1 boy) $= P(X \geq 1)$

$= 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{2^4} = 1 - \frac{1}{16} = \frac{15}{16}$

Out of 2000 families $= \frac{15}{16} \times 2000 = 1875$

(ii) P(2 boys and 2 girls) $= P(X=2)$

$= 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$

$= \frac{\cancel{4} \times 3}{1 \times 2} \; \frac{1}{4} \; \frac{1}{\cancel{4}}$

$= \frac{3}{8}$

Out of 2000 families $= \frac{3}{8} \times 2000 = 750$

(iii) P(at most 2 girls) $= P(X \leq 2)$

$= P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{16} + 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 + \frac{3}{8}$

$= \frac{1}{16} + 4 \; \frac{1}{16} + \frac{3}{8} = \frac{1+4+6}{16} = \frac{11}{16}$

Out of 2000 families $= \frac{11}{16} \times 2000 = 1375$

(IV) P(children of both genders) = P(no girls)

$= P(X=0) = \frac{15}{16}$

Out of 2000 families $= \frac{15}{16} \times 2000 = 1875$

2) In a certain factory manufacturing razor blades, there is a small chance of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) No defective (ii) One defective (iii) Two defective blades respectively in a consignment of 10000 packet.

SOLUTION:

Here $n = 10$, $p = \frac{1}{500}$ $\quad \therefore np = \lambda$

$$\Rightarrow \lambda = 1\cancel{0} \times \frac{1}{50\cancel{0}} = \frac{1}{50} = 0.02$$

Poisson Distribution is $P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!}$

$$= \frac{e^{-0.02} (0.02)^x}{x!}$$

(i) P(no defective) $= P(X=0)$

$$= \frac{e^{-0.02} (0.02)^0}{0!} = e^{-0.02} = 0.9802$$

Consignment of 10000 packet $= 0.9802 \times 10000 = 9802$

(ii) P(one defective) $= P(X=1)$

$$= \frac{e^{-0.02} (0.02)^1}{1!} = (0.02)(e^{-0.02}) = 0.0196$$

Consignment of 10000 packet $= 0.0196 \times 10000 = 196$

(iii) P(two defective) $= P(X=2) = \frac{e^{-0.02} \times (0.02)^2}{2}$

$$= 0.0002$$

$\therefore$ Consignment of 10000 packet $= 2$

3) Buses arrive at a specified stop at 15 minutes interval starting at 6 AM ie, they arrive at 6 AM, 6.15 AM, 6.30 AM and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 6 and 6.30 AM. Evaluate the probability that he waits (i) Less than 5 minutes for a bus (ii) more than 10 minutes for a bus.

SOLUTION:

Here $f(x) = \frac{1}{30-0} = \frac{1}{30}$, $0 \leq x \leq 30$

(i) Less than 5 minutes (6.10 – 6.15) (or) (6.25 – 6.30)

$$\therefore P(\text{less than 5 minutes}) = \int_{10}^{15} f(x)\,dx + \int_{25}^{30} f(x)\,dx$$

$$= \frac{1}{30}\int_{10}^{15} dx + \frac{1}{30}\int_{25}^{30} dx$$

$$= \frac{1}{30}\left[x\right]_{10}^{15} + \frac{1}{30}\left[x\right]_{25}^{30}$$

$$= \frac{1}{30}(5) + \frac{1}{30}(5) = \frac{10}{30} = \frac{1}{3}$$

(ii) more than 10 minutes. (6 – 6.05) (or) (6.20 – 6.25)

$$\therefore P(\text{more than 10 mins}) = \int_{0}^{5} \frac{1}{30}\,dx + \int_{20}^{25} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[x\right]_{0}^{5} + \frac{1}{30}\left[x\right]_{20}^{25}$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \frac{1}{3}.$$

4) The daily consumption of milk in excess of 20000 litres in a ~~the~~ town is approximately exponentially distributed with parameter $1/3000$. The town has a daily stock of 35,000 L. What is the probability that of 2 days selected at random, the stock is insufficient for both days?

SOLUTION:

Let $X \rightarrow$ Excess amount of milk consumed in a day

If Y denotes the daily consumption of milk, then

$X = Y - 20000$ follows an Exponential Distribution

$$\therefore \text{Mean} = \frac{1}{\lambda} = 3000 \Rightarrow \lambda = \frac{1}{3000}$$

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{3000} e^{-\frac{x}{3000}}$$

$\therefore$ The probability that the stock is insufficient for one day is $= P(Y > 35000)$

$$= P(X + 20000 > 35000)$$

$$= P(X > 15000)$$

$$= \int_{15000}^{\infty} \frac{1}{3000} e^{-\frac{x}{3000}} dx$$

$$= \frac{\cancel{1}}{3000} \left[\frac{e^{-x/3000}}{-\cancel{1}/3000} \right]_{15000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-5} \right] = e^{-5}$$

Hence the probability that of 2 days selected a random, the stock is insufficient for both days $= e^{-5} \cdot e^{-5} = e^{-10}$

5) In an Engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students get distinction. Find the percentage of students who have got first class and second class. Assume normal distribution of marks.

SOLUTION:

Failed 10% for less than 45%

Value = 45%

Z score for 10% = −1.281

$$\therefore -1.281 = \frac{45-\mu}{\sigma} \Rightarrow -1.281\sigma = 45-\mu \quad -(1)$$

5% got distinction ⇒ z score for 100 − 5 = 95% = 1.645

Value = 75%

$$1.645 = \frac{75-\mu}{\sigma} = 1.645\sigma = 75-\mu \quad -(2)$$

$$(1)-(2) \Rightarrow \quad -2.926\sigma = -30$$

$$\Rightarrow \sigma = \frac{30}{2.926} = 10.253$$

$$\boxed{\sigma = 10.253}$$

Sub in (1),

$$-1.281(10.253) = 45-\mu$$

$$-13.134 = 45-\mu$$

$$\Rightarrow -\mu = -58.134$$

$$\Rightarrow \boxed{\mu = 58.134}$$

First class between 60% and 75%

$$Z \text{ value for } 60\% = \frac{60 - 58.134}{10.253} = 0.182 = 57.2\%$$

$$Z \text{ value for } 75\% = \frac{75 - 58.134}{10.253} = 1.645 = 95\%$$

$\therefore$ Percentage of students who have got first class

$= 95 - 57.2 = 37.8\%$

Second Class between 45% and 60%

$$Z \text{ value for } 45\% = \frac{45 - 58.134}{10.253} = -1.281 = 10\%$$

$$Z \text{ value for } 60\% = \frac{60 - 58.134}{10.253} = 0.182 = 57.2\%$$

$\therefore$ Percentage of students who have got second class $= 57.2\% - 10\% = 47.2\%$