UNIT II – TWO DIMENSIONAL RANDOM VARIABLES

PART-A:

1. The joint probability distribution of X and Y is given by $P(x, y) = \frac{x+y}{21}$; $x = 1, 2, 3$; $y = 1, 2$. Find the marginal probability distributions of X.

SOLUTION:

Y \ X	1	2	3	
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
	5/21	7/21	9/21	1

$f(x, y) = \frac{x+y}{21}$

Marginal Distributions of x

$P(x = 1) = \frac{5}{21}$

$P(x = 2) = \frac{7}{21}$

$P(x = 3) = \frac{9}{21}$

2) The joint probability function (x, y) is given by $P(x, y) = k(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find the value of k.

SOLUTION:

Y \ X	0	1	2	
1	3k	5k	7k	15k
2	6k	8k	10k	24k
3	9k	11k	13k	33k
	18k	24k	30k	1

$72k = 1$

$\Rightarrow \boxed{k = \frac{1}{72}}$

3) Find the probability distribution of $X+Y$ from the bivariate distribution of (X,Y) given below.

X \ Y	1	2
1	0.4	0.2
2	0.3	0.1

SOLUTION:

The probability distribution of $X+Y$.

$X+Y$	$P(X+Y)$
2	0.4
3	$0.2+0.3 = 0.5$
4	0.1

4) Let X and Y have the joint pdf

Y \ X	0	1	2
0	0.1	0.4	0.1
1	0.2	0.2	0

Find $P(X+Y>1)$

SOLUTION:

$X+Y$	$P(X+Y)$
0	0.1
1	0.6
2	0.3
3	0

$\therefore P(X+Y>1) = P(X+Y=2) + P(X+Y=3)$

$= 0.3$

5. Find the marginal distributions of X and Y from the bivariate distribution of (X, Y) given below.

X \ Y	1	2
1	0.1	0.2
2	0.3	0.4

SOLUTION:

X \ Y	1	2	
1	0.1	0.2	0.3
2	0.3	0.4	0.7
	0.4	0.6	1

Marginal Distribution of X

$P(X=1) = 0.3$

$P(X=2) = 0.7$

Marginal Distribution of Y

$P(Y=1) = 0.4$

$P(Y=2) = 0.6$

6. The joint probability distribution function of the random variable (X, Y) is given by $f(x,y) = k(x^3y - xy^3)$, $0 \le x \le 2$, $0 \le y \le 1$.

Derive the value of k.

SOLUTION:

We know that $\int_0^1 \int_0^2 f(x,y)\,dx\,dy = 1$

$$k\int_0^1 \int_0^2 (x^3y - xy^3)\,dx\,dy = 1$$

$$\int_0^2 k\left[\frac{x^4}{4}y - \frac{x^2}{2}y^3\right]_0^2 dy = 1$$

$$k\int_0^2 \left(\frac{y}{4} - \frac{y^3}{2}\right) dy = 1$$

$$k\left[\frac{y^2}{8} - \frac{y^4}{8}\right]_0^2 = 1$$

$$k\left[\frac{4}{8} - \frac{16}{8}\right] = 1$$

$$k\left[\right.$$

$$\int_0^1 k\left(\frac{x^4}{4}y - \frac{x^2}{2}y^3\right)_0^2 dy =$$

$$k\int_0^1 (4y - 2y^3)\, dy = 1$$

$$k\left[4\frac{y^2}{2} - \frac{2\cdot y^4}{4}\right]_0^1 = 1$$

$$k\left[2 - \frac{1}{2}\right] = 1$$

$$k\left[\frac{4-1}{2}\right] = 1$$

$$\Rightarrow \frac{3k}{2} = 1 \Rightarrow \boxed{k = \frac{2}{3}}$$

7) If the joint probability density function of a random variable X and Y, is given by $f(x,y) = \begin{cases} \frac{x^3y^3}{16}, & 0<x<2,\ 0<y<2 \\ 0, & \text{otherwise} \end{cases}$

Obtain the marginal density function of X.

SOLUTION:

marginal density function of X

$$f(x) = \int_0^2 f(x,y)\, dy$$

$$= \int_0^2 \frac{x^3y^3}{16}\, dy$$

$$= \frac{1}{16}\left[x^3\frac{y^4}{4}\right]_0^2 = \frac{x^3}{4} \Rightarrow \boxed{f(x) = \frac{x^3}{4}}$$

8) The joint probability density of a two dimensional random variable X and Y is given by $f(x,y) = \begin{cases} kxe^{-y}, & 0 \le x < 2,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$

Evaluate k.

SOLUTION:

$$\int_0^2 \int_0^2 kxe^{-y}\, dx\, dy = 1$$

$$k \int_0^\infty e^{-y} \left[\frac{x^2}{2} \right]_0^2 dy = 1$$

$$\Rightarrow k \int_0^\infty e^{-y} (2)\, dy = 1$$

$$\Rightarrow 2k \left[\frac{e^{-y}}{-1} \right]_0^\infty = 1$$

$$= -2k \left[e^{-\infty} - e^{0} \right] = 1$$

$$\Rightarrow 2k = 1 \Rightarrow \boxed{k = \frac{1}{2}}$$

9. The joint probability density function of a random variable (X,Y) is $f(x,y) = ke^{-(2x+3y)}$, $x \geq 0$, $y \geq 0$. Point out the value of k.

SOLUTION:

$$\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = 1$$

$$k \int_0^\infty \int_0^\infty e^{-2x} e^{-3y}\, dx\, dy = 1$$

$$k \int_0^\infty e^{-2x}\,dx \int_0^\infty e^{-3y}\,dy = 1$$

$$k \left[\frac{e^{-2x}}{-2}\right]_0^\infty \left[\frac{e^{-3y}}{-3}\right]_0^\infty = 1$$

$$\frac{k}{6}\left[e^{-\infty} - e^{0}\right]\left[e^{-\infty} - e^{0}\right] = 1$$

$$\frac{k}{6}(-1)(-1) = 1 \Rightarrow \frac{k}{6} = 1 \Rightarrow \boxed{k = 6}$$

10. If the joint pdf of (X, Y) is $f(x,y) = \begin{cases} \frac{1}{4}, & 0 < x, y < 2 \\ 0, & \text{otherwise} \end{cases}$.

Find $P(X+Y \leq 1)$

SOLUTION:

$$P(X+Y \leq 1) = P(Y \leq 1 - X)$$

$$= \iint f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{4}\,dy\,dx$$

$$= \frac{1}{4}\int_0^1 \left[y\right]_0^{1-x} dx$$

$$= \frac{1}{4}\int_0^1 (1-x)\,dx$$

$$= \frac{1}{4}\left[x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{4}\left(\frac{2-1}{2}\right)$$

$$= \frac{1}{8}$$

Sub in (1) $\Rightarrow 3x + 2\left(\frac{11}{3}\right) = 19$

$$3x + \frac{22}{3} = 19$$

$$3x = 19 - \frac{22}{3}$$

$$= \frac{57-22}{3}$$

$$3x = \frac{35}{3}$$

$$\Rightarrow \boxed{x = \frac{35}{9}}$$

24. The equation of two regression lines are $3x + 2y = 19$ and $3y + 9x = 46$. Derive the correlation coefficient ~~of~~ between X and Y.

SOLUTION:

$$3x + 2y = 19$$

$$3x = -2y + 19$$

$$\Rightarrow x = \frac{-2}{3}y + \frac{19}{3}$$

$$\therefore b_{xy} = \frac{-2}{3}$$

$$3y + 9x = 46$$

$$\Rightarrow 3y = -9x + 46$$

$$\Rightarrow y = \frac{-9}{3}x + \frac{46}{3}$$

$$\therefore b_{yx} = \frac{-9}{3}$$

$$\therefore r^2 = b_{xy}\, b_{yx}$$

$$= \left(\frac{-2}{\cancel{3}}\right)\left(\frac{-\cancel{9}}{\cancel{3}}\right) = 2$$

$$r^2 = 2$$

$$r = \pm\sqrt{2}$$

$$3x + 2y = 19$$

$$2y = -3x + 19$$

$$y = \frac{-3}{2}x + \frac{19}{2}$$

$$\therefore b_{yx} = \frac{-3}{2}$$

$$3y + 9x = 46$$

$$9x = -3y + 46$$

$$x = \frac{-3}{9}y + \frac{46}{9}$$

$$\therefore b_{xy} = \frac{-3}{9}$$

$$\therefore r^2 = b_{xy}\, b_{yx}$$

$$= \frac{-\cancel{3}}{9} \cdot \frac{-\cancel{3}}{2} = \frac{1}{2}$$

$$r = \pm\sqrt{\frac{1}{2}}$$

$$r = \pm\frac{1}{\sqrt{2}} \Rightarrow r = \frac{1}{\sqrt{2}}$$

25. If $x = R\cos\phi$ and $Y = R\sin\phi$, how are the joint probability density function (X, Y) and (R, ϕ) are related?

SOLUTION:

$$J = \frac{\partial(x, y)}{\partial(R, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$x = R\cos\phi$

$\frac{\partial x}{\partial R} = \cos\phi$

$\frac{\partial x}{\partial \phi} = -R\sin\phi$

$$= \begin{vmatrix} \cos\phi & -R\sin\phi \\ \sin\phi & R\cos\phi \end{vmatrix}$$

$Y = R\sin\phi$

$\frac{\partial Y}{\partial R} = \sin\phi$

$\frac{\partial Y}{\partial \phi} = R\cos\phi$

$$= R\cos^2\phi + R\sin^2\phi$$

$$J = R.$$

$$\therefore f(R, \phi) = |J|\, f(x, Y)$$

$$\therefore f(R, \phi) = R\, f(x, Y)$$

PART-B

1. From the following table for bivariate distribution of (X, Y) Find (i) $P(X \le 1)$ (ii) $P(Y \le 3)$ (iii) $P(X \le 1, Y \le 3)$ (iv) $P(X \le 1 / Y \le 3)$ (v) $P(Y \le 3 / X \le 1)$ (vi) $P(X + Y \le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

SOLUTION:

X \ Y	1	2	3	4	5	6	P(X=x)
0	0	0	1/32	2/32	2/32	3/32	2/8
1	1/16	1/16	1/8	1/8	1/8	1/8	5/8
2	1/32	1/32	1/64	1/64	0	2/64	1/8
P(Y=y)	3/32	3/32	11/64	13/64	6/32	16/64	1

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

$= 2/8 + 5/8 = 7/8$

(ii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$= 3/32 + 3/32 + 11/64$

$= \frac{6}{32} + \frac{11}{64} = \frac{12+11}{64} = \frac{23}{64}$

(iii) $P(X \leq 1, Y \leq 3) = \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$= \frac{1+4+4}{32}$

$= \frac{9}{32}.$

(iv) $P(X \leq 1 / Y \leq 3) = \frac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}$

$= \frac{9/32}{23/64} = \frac{9}{32} \times \frac{64}{23} = \frac{18}{23}$

(v) $P(Y \le 3 / X \le 1) = \dfrac{P(X \le 1, Y \le 3)}{P(X \le 1)}$

$$= \frac{9/32}{7/8} = \frac{9}{\cancel{32}_{4}} \times \frac{\cancel{8}}{7}$$

$$= \frac{9}{28}$$

(vi) $P(X + Y \le 4) = P(X+Y = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= 0 + \left(0 + \frac{1}{16}\right) + \left(\frac{1}{32} + \frac{1}{16}\right) + \left(\frac{2}{32} + \frac{1}{32} + \frac{1}{8}\right)$$

$$= \frac{1}{16} + \frac{1+2}{32} + \frac{3+4}{32}$$

$$= \frac{2+3+7}{32} = \frac{12}{32} = \frac{6}{16} = \frac{3}{8}$$

2)a) The two dimensional random variable (X, Y) has the joint probability mass function $f(x, y) = \dfrac{x + 2y}{27}$, $x = 0, 1, 2$, $y = 0, 1, 2$.

Find the marginal distributions of X and Y. Also find the conditional distribution of Y given $X = 1$. Also find the conditional distribution of X given $Y = 1$.

SOLUTION:

Y \ X	0	1	2	P(Y=y)
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
P(X=x)	6/27	9/27	12/27	1

Marginal Distribution of X:

$P(X=0) = 6/27$

$P(X=1) = 9/27$

$P(X=2) = 12/27$

Marginal Distribution of Y:

$P(Y=0) = 3/27$

$P(Y=1) = 9/27$

$P(Y=2) = 15/27$

Conditional distribution of Y given X=1

$$P(Y=0/X=1) = \frac{P(X=1, Y=0)}{P(X=1)} = \frac{1/27}{6/27} = 1/6$$

$$P(Y=1/X=1) = \frac{P(X=1, Y=1)}{P(X=1)} = \frac{3/27}{6/27} = 3/6$$

$$P(Y=2/X=1) = \frac{P(X=1, Y=2)}{P(X=1)} = \frac{5/27}{6/27} = 5/6$$

Conditional Distribution of X given Y=1:

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{2/27}{9/27} = \frac{2}{9}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{3/27}{9/27} = \frac{3}{9}$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{4/27}{9/27} = 4/9.$$

2)b) The joint pdf of a bivariate R.V (X, Y) is given by

$$f(x,y) = \begin{cases} kxy & , 0<x<1, \ 0<y<1 \\ 0 & , \text{Otherwise} \end{cases}$$

Find k

(2) Find $P(X+Y<1)$ (3) Are X and Y independent R.V's

SOLUTION:

We know that,

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 1$$

$$\int_0^1 \int_0^1 k\, xy\, dx\, dy = 1 \Rightarrow k \int_0^1 y \left[\frac{x^2}{2}\right]_0^1 dy = 1$$

$$\Rightarrow \frac{k}{2} \int_0^1 y\, dy = 1$$

$$\Rightarrow \frac{k}{2} \left[\frac{y^2}{2}\right]_0^1 = 1$$

$$\Rightarrow \frac{k}{4} = 1 \Rightarrow \boxed{k=4}$$

$$P(X+Y<1) = P(X<1-Y)$$

$$= \int_0^1 \int_0^{1-y} f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} 4xy\, dx\, dy$$

$$= 4 \int_0^1 y \left[\frac{x^2}{2}\right]_0^{1-y} dy$$

$$= 2 \int_0^1 y (1-y)^2\, dy$$

$$= 2 \int_0^1 y (1+y^2-2y)\, dy$$

$$= 2 \int_0^1 (y^3 - 2y^2 + y)\, dy$$

$$= 2\left[\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right]$$

$$= 2\left[\frac{3-8+6}{12}\right] = \frac{2}{12} = \frac{1}{6}$$

<u>Marginal density of X:</u>

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 4xy\,dy = \cancel{4}^{2}x\left[\frac{y^2}{\cancel{2}}\right]_0^1$$

$$= 2x$$

<u>Marginal density of y:</u>

$$f(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 4xy\,dx = 4y\left[\frac{x^2}{2}\right]_0^1$$

$$= 2y$$

Now, $f(x)\,f(y) = 2x \cdot 2y = 4xy = f(x,y)$

$\therefore f(x,y) = f(x)\,f(y)$.

$\therefore$ X and Y are independent.

3) (a) The joint pdf of (X,Y) is given by $P(x,y) = k(2x+3y)$;

$x = 0,1,2,3$

$y = 1,2,3$.

Find all the marginal probability distribution. Also find the probability distribution of $X+Y$.

SOLUTION:

Y \ X	0	1	2	3	P(Y=y)
1	3k	5k	7k	9k	24k
2	6k	8k	10k	12k	36k
3	9k	11k	13k	15k	48k
P(X=x)	18k	24k	30k	36k	108k

$108k = 1$

$\Rightarrow \boxed{k = \frac{1}{108}}$

Y \ X	0	1	2	3	P(Y=y)
1	3/108	5/108	7/108	9/108	24/108
2	6/108	8/108	10/108	15/108	36/108
3	9/108	11/108	13/108	36/108	48/108
P(X=x)	18/108	24/108	30/108	36/108	1

<u>Marginal distribution of X:</u>

$P(X=0) = 18/108$

$P(X=1) = 24/108$

$P(X=2) = 30/108$

$P(X=3) = 36/108$

<u>Marginal Distribution of Y:</u>

$P(Y=1) = 24/108$

$P(Y=2) = 36/108$

$P(Y=3) = 48/108$

$x+y$	$P(x+y)$
1	$3/108$
2	$6/108 + 5/108 = 11/108$
3	$9/108 + 8/108 + 7/108 = 24/108$
4	$9/108 + 10/108 + 11/108 = 30/108$
5	$15/108 + 13/108 = 28/108$
6	$36/108$

3b) The joint pdf of the R.V (X, Y) is given by $f(x,y) = kxy\, e^{-(x^2+y^2)}$, $x>0, y>0$. Find the value of k. Also prove that X and Y are independent.

SOLUTION:

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

$$\int_0^{\infty}\int_0^{\infty} kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$k \int_0^{\infty} x e^{-x^2} dx \int_0^{\infty} y e^{-y^2} dy = 1$$

Put $x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{dt}{2}$

$$k \int_0^{\infty} e^{-t} \frac{dt}{2} \int_0^{\infty} e^{-t} \frac{dt}{2} = 1$$

$$\frac{k}{4} \left[\frac{e^{-t}}{-1}\right]_0^{\infty} \left[\frac{e^{-t}}{-1}\right]_0^{\infty} = 1$$

$$\frac{k}{4} \left[e^{-\infty} - e^{0}\right] \left[e^{-\infty} - e^{0}\right] = 1$$

$$\frac{k}{4}(-1)(-1) = 1 \Rightarrow \frac{k}{4} = 1 \Rightarrow \boxed{k = 4}$$

<u>Marginal density of x:</u>

$$f(x) = \int_0^\infty f(x,y)\,dy$$

$$= \int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4\int_0^\infty x\,y\,e^{-x^2}e^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^\infty e^{-t}\frac{dt}{2}$$

$$= 2xe^{-x^2}\left[\frac{e^{-t}}{-1}\right]_0^\infty = -2xe^{-x^2}\left[e^{-\infty} - e^{0}\right]$$

$$= -2xe^{-x^2}(-1)$$

$$f(x) = 2xe^{-x^2}$$

<u>Marginal density of y:</u>

$$f(y) = \int_0^\infty f(x,y)\,dx$$

$$= \int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dx$$

$$= 4ye^{-y^2}\int_0^\infty x\,e^{-x^2}\,dx$$

$$= 4ye^{-y^2}\int_0^\infty e^{-t}\frac{dt}{2}$$

$$= 2ye^{-y^2}\left[\frac{e^{-t}}{-1}\right]_0^\infty = -2ye^{-y^2}\left[e^{-\infty} - e^{0}\right]$$

$$f(y) = 2ye^{-y^2}$$

Now, $f(x)\,f(y) = 2x\,e^{-x^2}\;2y\,e^{-y^2}$

$$= 4xy\,e^{-(x^2+y^2)}$$

$$= f(x,y)$$

$\therefore\; f(x,y) = f(x)\,f(y)$

$\therefore$ X and Y are independent.

5. Find the marginal distribution of X and Y and also $P(X\le 1, Y\le 1)$, $P(X\le 1)$, $P(Y\le 1)$. Check whether X and Y are independent.

The joint probability mass function X and Y is

X \ Y	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

SOLUTION:

X \ Y	0	1	2	$P(X=x)$
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.50
$P(Y=y)$	0.24	0.38	0.38	1

$P(X \le 1) = P(X=0) + P(X=1)$

$= 0.16 + 0.34 = 0.50$

$P(Y \le 1) = P(Y=0) + P(Y=1)$

$= 0.24 + 0.38 = 0.62$

$P(X \le 1, Y \le 1) = 0.10 + 0.04 + 0.08 + 0.20$

$= 0.42$

$P(X=0)\,P(Y=0) = 0.16 \times 0.24 = 0.0384 \neq 0.10 = P(X=0, Y=0)$

$\therefore$ X and Y are not independent.

6. A machine is used for a particular job in the forenoon and for a different job in the afternoon. The joint probability distribution of (X, Y), where X and Y represent the number of times the machine breakdown, in the forenoon and in the afternoon respectively is given in the following table. Examine if X and Y are independent R.V's.

X \ Y	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

SOLUTION:

X \ Y	0	1	2	P(X=x)
0	0.1	0.04	0.06	0.20
1	0.2	0.08	0.12	0.40
2	0.2	0.08	0.12	0.40
P(Y=y)	0.5	0.20	0.30	1

$P(X=0)\,P(Y=0) = (0.20)(0.5) = 0.1 = P(X=0, Y=0)$

$P(X=0)\,P(Y=1) = (0.20)(0.20) = 0.04 = P(X=0, Y=1)$

$P(X=0)\,P(Y=2) = (0.20)(0.30) = 0.06 = P(X=0, Y=2)$

$P(X=1)\,P(Y=0) = (0.40)(0.5) = 0.2 = P(X=1, Y=0)$

$P(X=1)\,P(Y=1) = (0.40)(0.20) = 0.8 = P(X=1, Y=1)$

$P(X=1)\,P(Y=2) = (0.40)(0.30) = 0.12 = P(X=1, Y=2)$

$P(X=2)\,P(Y=0) = (0.40)(0.5) = 0.2 = P(X=2, Y=0)$

$P(X=2)\,P(Y=1) = (0.40)(0.20) = 0.08 = P(X=2, Y=1)$

$P(X=2)\,P(Y=2) = (0.40)(0.30) = 0.12 = P(X=2, Y=2)$

$\therefore$ X and Y are independent.

7. If the joint pdf of a two-dimensional random variable (X,Y) is given by $f(x,y) = x^2 + \frac{xy}{3}$, $0<x<1$, $0<y<2$. Find (i) $P(X > 1/2)$ (ii) $P(Y < 1/2, X < 1/2)$ (iii) $P(Y < 1/2 / X < 1/2)$

SOLUTION:

The marginal density function of X

$$f(x) = \int_0^2 f(x,y)\,dy$$

$$= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \left[x^2 y + \frac{xy^2}{6} \right]_0^2$$

$$f(x) = 2x^2 + \frac{2x}{3}$$

<u>Marginal density of Y:</u>

$$f(y) = \int_0^1 f(x,y)\, dx$$

$$= \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2 y}{6} \right]_0^1$$

$$= \frac{1}{3} + \frac{y}{6}$$

(i) $P(X > 1/2) = \int_{1/2}^1 f(x)\, dx = \int_{1/2}^1 \left(2x^2 + \frac{2x}{3}\right) dx$

$$= \left[2\frac{x^3}{3} + \frac{2x^2}{6} \right]_{1/2}^1 = \frac{2}{3} + \frac{1}{3} - \left(\frac{1}{12} + \frac{1}{12}\right)$$

$$= 1 - \frac{2}{12} = 1 - \frac{1}{6} = \frac{5}{6}$$

(ii) $P(Y < 1/2, X < 1/2) = \int_0^{1/2} \int_0^{1/2} f(x,y)\, dx\, dy$

$$= \int_0^{1/2} \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^{1/2} \left[\frac{x^2 y^2}{2} + \frac{x^3}{24} \right]_0^{1/2} dy$$

$$= \int_0^{1/2} \left[\frac{y^2}{8} + \frac{1}{8(24)}\right] dy$$

$$= \left[\frac{y^3}{24} + \frac{y}{192}\right]_0^{1/2} = \frac{1}{8(24)} + \frac{1}{2(192)}$$

$$= \frac{1}{192} + \frac{1}{2(192)} = \frac{2+1}{2(192)}$$

$$= \frac{3}{2(192)} = \frac{1}{2(64)} = \frac{1}{128}$$

(iii) $P(Y < 1/2 \,/\, X < 1/2) = \dfrac{P(X < 1/2,\ Y < 1/2)}{P(X < 1/2)}$

$$= \frac{1/128}{1/6} = 1/128 \times 6 = \frac{3}{64}$$

$\left[\because P(X<1/2) = 1 - P(X > 1/2) = 1 - \frac{5}{6} = \frac{1}{6}\right]$

8) The joint pdf of a two dimensional random variable (X, Y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$.

Compute (i) $P(X > 1 / Y < 1/2)$ (ii) $P(Y < 1/2 \,/\, X > 1)$

(iii) $P(X + Y \le 1)$.

SOLUTION:

<u>Marginal density of X:</u>

$$f(x) = \int_0^1 f(x, y)\, dy$$

$$= \int_0^1 \left(xy^2 + \frac{x^2}{8}\right) dy$$

$$= \left[\frac{xy^3}{3} + \frac{x^2}{8} y\right]_0^1 = \frac{x}{3} + \frac{x^2}{8}$$

Marginal density of y:

$$f(y) = \int_0^2 f(x,y)\,dx$$

$$= \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx$$

$$= \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2$$

$$= 2y^2 + \frac{1}{3}$$

$$P(X>1) = \int_1^2 f(x)\,dx = \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx$$

$$= \left[\frac{x^2}{6} + \frac{x^3}{24}\right]_1^2 = \frac{4}{6} + \frac{8}{24} - \left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= \frac{2}{3} + \frac{1}{3} - \left(\frac{4+1}{24}\right)$$

$$= 1 - \frac{5}{24} = \frac{19}{24}$$

$$P(Y<1/2) = \int_0^{1/2} f(y)\,dy = \int_0^{1/2} \left(2y^2 + \frac{1}{3}\right) dy$$

$$= \left[\frac{2y^3}{3} + \frac{y}{3}\right]_0^{1/2} = 2\,\frac{1}{8\times 3} + \frac{1}{6}$$

$$= \frac{1}{12} + \frac{1}{6} = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}.$$

(ii) $$P(X>1, Y<1/2) = \int_0^{1/2}\int_1^2 f(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$

$$= \int_0^{1/2} \left[\frac{x^2y^2}{2} + \frac{x^3}{24} \right]_1^2 dy$$

$$= \int_0^{1/2} \left[2y^2 + \frac{1}{3} - \left(\frac{y^2}{2} + \frac{1}{24} \right) \right] dy$$

$$= \int_0^{1/2} \left(\frac{3y^2}{2} + \frac{7}{24} \right) dy$$

$$= \left[\frac{3y^3}{6} + \frac{7y}{24} \right]_0^{1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{8} + \frac{7}{48}$$

$$= \frac{1}{16} + \frac{7}{48} = \frac{3+7}{48} = \frac{10}{48} = \frac{5}{24}$$

(ii) $P(Y < 1/2 / X > 1) = \dfrac{P(X > 1, Y < 1/2)}{P(X > 1)}$

$$= \frac{5/24}{19/24}$$

$$= 5/19$$

(iii) $P(X+Y \leq 1) = P(X \leq 1-Y)$

$$= \int_0^1 \int_0^{1-y} f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24} \right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24} \right] dy$$

$$= \int_0^1 \left[\frac{(1+y^2-2y)y^2}{2} + \frac{(1-3y+3y^2-y^3)}{24} \right] dy$$

$$= \int_0^1 \left[\frac{y^4-2y^3+y^2}{2} + \frac{1}{24}(-y^3+3y^2-3y+1) \right] dy$$

$$= \left[\frac{1}{2}\left(\frac{y^5}{5} - \frac{2y^4}{4} + \frac{y^3}{3} \right) + \frac{1}{24}\left(\frac{-y^4}{4} + \frac{3y^3}{3} - \frac{3y^2}{2} + y \right) \right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right] + \frac{1}{24}\left[\frac{-1}{4} + 1 - \frac{3}{2} + 1 \right]$$

$$= \frac{1}{2}\left[\frac{6-15+10}{30} \right] + \frac{1}{24}\left[\frac{-1+8-6}{4} \right]$$

$$= \frac{1}{2} \cdot \frac{1}{30} + \frac{1}{24} \cdot \frac{1}{4} = \frac{1}{60} + \frac{1}{96}$$

$$= \frac{8+5}{480} = \frac{13}{480}$$

9) The joint pdf of X and Y is given by

$$f(x,y) = \begin{cases} kx(x-y) & , 0 < x < 2, -x < y < x \\ 0 & , \text{Otherwise.} \end{cases}$$

Find (i) k (ii) $f_X(x)$ and (iii) $f_Y(y)$.

SOLUTION:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$\int_0^2 \int_{-x}^{x} kx(x-y)\, dy\, dx = 1$$

$$k \int_0^2 \int_{-x}^{x} (x^2 - xy)\, dy\, dx = 1$$

$$k \int_0^2 \left[x^2 y - \frac{xy^2}{2} \right]_{-x}^{x} dx = 1$$

$$k\int_0^2 \left[x^3 - \frac{x^3}{2} - \left(-x^3 - \frac{x^3}{2}\right)\right] dx = 1$$

$$k\int_0^2 \left(\frac{x^3}{2} + \frac{3x^3}{2}\right) dx = 1$$

$$\frac{k}{2}\int_0^2 4x^3\, dx = 1$$

$$\frac{3k}{2}\left[\frac{4x^4}{4}\right]_0^2 = 1 \Rightarrow \frac{3k}{2}(16) = 1$$

$$\Rightarrow 24k = 1$$

$$\Rightarrow \boxed{k = \frac{1}{24}}$$

<u>Marginal density of x:</u>

$$f_x(x) = \int_{-x}^{x} f(x,y)\, dy$$

$$= \int_{-x}^{x} k(x^2 - xy)\, dy$$

$$= \frac{1}{24}\left[x^2 y - \frac{xy^2}{2}\right]_{-x}^{x}$$

$$= \frac{1}{24}\left[x^3 - \frac{x^3}{2} - \left(-x^3 - \frac{x^3}{2}\right)\right]$$

$$= \frac{1}{24}\left[\frac{x^3}{2} + \frac{3x^3}{2}\right]$$

$$= \frac{1}{\cancel{24}_{6}} \frac{\cancel{4}x^3}{2}$$

$$f_x(x) = \frac{x^3}{12}$$

<u>Marginal density of y:</u>

$$f_y(y) = \int_0^2 f(x,y)\, dx$$

$$= \int_0^2 k\,(x^2 - xy)\,dx$$

$$= \frac{1}{24}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^2$$

$$= \frac{1}{24}\left[\frac{8}{3} - 2y\right]$$

$$f_y(y) = \frac{1}{9} - \frac{y}{12}$$

10. Find the co-efficient of correlation between industrial production and export using the following table.

Production X.	14	17	23	21	25
Export (Y)	10	12	15	20	23

SOLUTION:

X	Y	$x-\bar{x}$ $= x-20$	$y-\bar{y}$ $= y-16$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
14	10	-6	-6	36	36	36
17	12	-3	-4	9	16	12
23	15	3	-1	9	1	-3
21	20	1	4	1	16	4
25	23	5	7	25	49	35
				80	118	84

$$\bar{x} = \frac{14+17+23+21+25}{5} = \frac{100}{5} = 20$$

$$\bar{y} = \frac{10+12+15+20+23}{5} = \frac{80}{5} = 16$$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{84}{118} = 0.7119$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{84}{80} = 1.05$$

$$\therefore r^2 = b_{xy}\, b_{yx} = 0.7119 \times 1.05 = 0.7475$$

$$\Rightarrow r = \pm\sqrt{0.7475} = 0.8646$$

$\therefore$ correlation coefficient $= 0.8646$.

11. Find the correlation coefficient for the following heights of fathers X, their sons Y and also find the equations of regression lines. Hence find the height of son when the height of father is 71.

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

SOLUTION:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

X	Y	$x-\bar{x}$ $=x-68$	$y-\bar{y}$ $=y-69$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
65	67	−3	−2	9	4	6
66	68	−2	−1	4	1	2
67	65	−1	−4	1	16	4
67	68	−1	−1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
				36	44	24

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{24}{44} = 0.5455$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{24}{36} = 0.6154$$

$$\therefore r^2 = b_{xy}\, b_{yx}$$

$$= (0.5455)(0.6154)$$

$$r^2 = 0.3357$$

$$\Rightarrow \boxed{r = 0.5794}$$

$\therefore$ correlation coefficient $= 0.5794$

The equation of Regression Lines:

Regression of Y on X is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y-69 = 0.6154(x-68)$$

$$y-69 = 0.6154x - 41.8472$$

$$\Rightarrow \boxed{y = 0.6154x + 27.1528}$$

Regression of X on Y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

$$x-68 = (0.5455)(y-69)$$

$$x = 0.5455y - 37.6395 + 68$$

$$\boxed{x = 0.5455y + 30.3605}$$

When $x = 71$, $y = 0.6154(71) + 27.1528$

$$y = 43.6934 + 27.1528$$

$$\boxed{y = 70.8462}$$

12. Obtain the lines of regression

X	50	55	50	60	65	65	65	60	60
Y	11	14	13	16	16	15	15	14	13

SOLUTION:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{530}{9} = 58.9$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{127}{9} = 14.1$$

X	Y	$x-\bar{x}$ $x-58.9$	$y-\bar{y}$ $y-14.1$	$(x-x)^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
50	11	-8.9	-3.1	79.21	9.61	27.59
55	14	-3.9	-0.1	15.21	0.01	0.39
50	13	-8.9	-1.1	79.21	1.21	9.79
60	16	1.1	1.9	1.21	3.61	2.09
65	16	6.1	1.9	37.21	3.61	11.59
65	15	6.1	0.9	37.21	0.81	5.49
65	15	6.1	0.9	37.21	0.81	5.49
60	14	1.1	-0.1	1.21	0.01	-0.11
60	13	1.1	-1.1	1.21	1.21	1.21
				288.89	20.89	63.53

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{63.53}{20.89} = 3.0411$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{63.53}{288.89} = 0.2199$$

$\therefore$ Regression of Y on X is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y-14.1 = 0.2199(x-58.9)$$

$$y-14.1 = 0.2199x - 12.8861$$

$\Rightarrow y = 0.2199x - 12.8861 + 14.1$

$\Rightarrow \boxed{y = 0.2199x + 1.2139}$

Regression of X on Y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 58.9 = (3.0411)(y - 14.1)$$

$$x - 58.9 = 3.0411y - 42.8795$$

$$\boxed{x = 3.0411y + 16.0205}$$

13. If $f(x,y) = \frac{6-x-y}{8}$; $0 \le x \le 2$, $2 \le y \le 4$ for a bivariate random variable (X,Y). Evaluate the correlation coefficient ρ.

SOLUTION:

Marginal density of X:

$$f(x) = \int_2^4 \frac{6-x-y}{8}\, dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{8}\left[24 - 4x - 8 - (12 - 2x - 2)\right]$$

$$= \frac{1}{8}\left[6 - 2x\right] = \frac{3-x}{4}$$

Marginal density of Y:

$$f(y) = \int_0^2 \frac{6-x-y}{8}\, dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[12 - 2 - 2y\right]$$

$$= \frac{1}{8}\left[10 - 2y\right] = \frac{5-y}{4}$$

$$E[X] = \int_0^2 x\, f(x)\, dx$$

$$= \int_0^2 x\left(\frac{3-x}{4}\right) dx$$

$$= \frac{1}{4}\int_0^2 (3x - x^2)\, dx$$

$$= \frac{1}{4}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}\left[6 - \frac{8}{3}\right] = \frac{1}{4}\left[\frac{18-8}{3}\right] = \frac{1}{4}\,\frac{10}{3} = \frac{5}{6}$$

$$E[Y] = \int_2^4 y\, f(y)\, dy$$

$$= \int_2^4 y\left(\frac{5-y}{4}\right) dy$$

$$= \frac{1}{4}\int_2^4 (5y - y^2)\, dy$$

$$= \frac{1}{4}\left[\frac{5y^2}{2} - \frac{y^3}{3}\right]_2^4 = \frac{1}{4}\left[40 - \frac{64}{3}\right] - \left(10 - \frac{8}{3}\right)\Big]$$

$$= \cancel{\frac{1}{4}\left[\frac{120-64}{3}\right] = \frac{1}{4}\,\frac{56}{3} = \frac{14}{3}}$$

$$= \frac{1}{4}\left[30 - \frac{56}{3}\right] = \frac{1}{4}\left[\frac{90-56}{3}\right] = \frac{1}{4}\left[\frac{34}{3}\right]$$

$$E[Y] = \frac{17}{6}$$

$$= \frac{1}{8}\int_2^4 \left[\frac{\cancel{6}^{3} x^2 y}{\cancel{2}} - \frac{x^3 y}{3} - \frac{x^2 y^2}{2} \right]_0^2 dy$$

$$= \frac{1}{8}\int_2^4 \left[12y - \frac{8y}{3} - 2y^2 \right] dy$$

$$= \frac{1}{8}\left[\frac{\cancel{12}^{6} y^2}{\cancel{2}} - \frac{\cancel{8}^{4} y^2}{\cancel{6}_{3}} - \frac{2y^3}{3} \right]_2^4$$

$$= \frac{1}{8}\left[96 - \frac{64}{3} - \frac{128}{3} - \left(24 - \frac{16}{3} - \frac{16}{3}\right) \right]$$

$$= \frac{1}{8}\left[72 - \frac{64}{3} - \frac{128}{3} + \frac{32}{3} \right] = \frac{1}{8}\left[72 - \frac{160}{3} \right]$$

$$= \frac{1}{8}\left[\frac{216 - 160}{3} \right] = \frac{1}{8} \; \frac{56}{3} = \frac{7}{3}$$

$$\text{Cov}(X, Y) = E[XY] - E[X]\,E[Y]$$

$$= \frac{7}{3} - \left(\frac{5}{6}\right)\left(\frac{17}{6}\right)$$

$$= \frac{7}{3} - \frac{85}{36}$$

$$= \frac{84 - 85}{36} = \frac{-1}{36}$$

$$\text{Var } X = E[X^2] - (E[X])^2$$

$$= 1 - \frac{25}{36} = \frac{36 - 25}{36} = \frac{11}{36}$$

$$\sigma_X = \sqrt{\text{Var } X} = \sqrt{\frac{11}{36}}$$

$$\text{Var } Y = E[Y^2] - (E[Y])^2$$

$$= \frac{25}{3} - \left(\frac{17}{6}\right)^2 = \frac{25}{3} - \frac{289}{36}$$

$$= \frac{300 - 289}{36} = \frac{11}{36}$$

$$\therefore \sigma_Y = \sqrt{\text{Var } Y} = \sqrt{\frac{11}{36}}$$

$\therefore$ correlation coefficient $r = \dfrac{\text{cov}(X,Y)}{\sigma_X \, \sigma_Y}$

$$= \frac{-1/36}{\sqrt{\frac{11}{36}}\sqrt{\frac{11}{36}}}$$

$$= \frac{-1/36}{11/36}$$

$$\boxed{r = \frac{-1}{11}}$$

14. Two random variables X and Y have the joint density function $f(x,y) = x+y$; $0 \le x \le 1$, $0 \le y \le 1$. Evaluate the correlation coefficient between X and Y.

SOLUTION:

<u>Marginal density of X:</u>

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy$$

$$= \left[xy + \frac{y^2}{2} \right]_0^1$$

$$f(x) = x + \frac{1}{2}$$

<u>Marginal density of Y:</u>

$$f(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 (x+y)\,dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1 = \frac{1}{2} + y$$

$$f(y) = y + \frac{1}{2}$$

$$E[X] = \int_0^1 x\, f(x)\, dx$$

$$= \int_0^1 x\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

$$E[Y] = \int_0^1 y\, f(y)\, dy$$

$$= \int_0^1 y\left(y + \frac{1}{2}\right) dy$$

$$= \int_0^1 \left(y^2 + \frac{y}{2}\right) dy$$

$$= \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4}$$

$$= \frac{4+3}{12} = \frac{7}{12}$$

$$E[X^2] = \int_0^1 x^2\, f(x)\, dx$$

$$= \int_0^1 x^2\left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12}$$

$$E[Y^2] = \int_0^1 y^2 f(y)\, dy$$

$$= \int_0^1 y^2 \left(y + \frac{1}{2}\right) dy$$

$$= \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^4}{4} + \frac{y^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6}$$

$$= \frac{6+4}{24} = \frac{10}{24} = \frac{5}{12}$$

$$E[XY] = \int_0^1 \int_0^1 xy\, f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (x^2 y + x y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{3} + \frac{y^2}{2}\right] dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\text{Var } X = E[X^2] - (E[X])^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144}$$

$$\text{Var } X = \frac{60-49}{144} = \frac{11}{144}$$

$$\therefore \sigma_X = \sqrt{\frac{11}{144}}$$

$$\text{Var } Y = E[Y^2] - (E[Y])^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144}$$

$$\text{Var } Y = \frac{60-49}{144} = \frac{11}{144}$$

$$\therefore \sigma_Y = \sqrt{\text{Var } Y} = \sqrt{\frac{11}{144}}$$

$$\text{Cov}(X,Y) = E[XY] - E[X]\,E[Y]$$

$$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12}$$

$$= \frac{1}{3} - \frac{49}{144} = \frac{48-49}{144} = \frac{-1}{144}$$

$$\therefore r = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-1/144}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-1/144}{11/144} = \frac{-1}{11}$$

15)a) 20 dice are thrown. Find the approximate probability that the sum obtained is between 65 and 75 using central limit theorem.

SOLUTION:

$$\mu = E[X] = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{1}{6}\left[1+2+3+4+5+6\right]$$

$$= \frac{1}{6}\,20 = 3.5$$

$$E[x^2] = \sum_{i=1}^{6} i^2 \left(\frac{1}{6}\right) = \frac{1}{6}\left[1^2+2^2+3^2+4^2+5^2+6^2\right]$$

$$= \frac{1}{6}\left[1+4+9+16+25+36\right] = \frac{91}{6} = 15.17$$

$$\text{Var } x = E[x^2] - (E[x])^2 = 15.17 - (3.5)^2$$

$$= 15.17 - 12.25$$

$$= 2.92$$

$$\therefore \sigma = \sqrt{\text{Var } x} = \sqrt{2.92}$$

$$\boxed{\sigma = 1.71}$$

$n = 20$

$S_n = X_1 + X_2 + \cdots + X_{20}$

If the sum of random variables follows Normal distribution then S_n follows $N(n\mu, \sigma\sqrt{n})$ by CLT.

$$\Rightarrow Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$n\mu = (20)(3.5) = 70$$

$$\sigma\sqrt{n} = (1.71)\sqrt{20} = 7.65$$

$$\therefore Z = \frac{S_n - 70}{7.65}$$

<u>To find $P(65 < S_n < 75)$</u>

When $S_n = 65$, $Z = \dfrac{65-70}{7.65} = -0.6536$

When $S_n = 75$, $Z = \dfrac{75-70}{7.65} = 0.6536$

$$\therefore P(65 < S_n < 75) = P(-0.6536 < Z < 0.6536)$$

$$= 2P(0 < Z < 0.65) = 2(0.2422)$$

$$= 0.4844$$

15) b) The two regression lines are $4x - 5y + 33 = 0$ and $20x - 9y = 107$. Find the mean of X and Y. Also find the correlation coefficient between them.

SOLUTION:

$$4x - 5y = -33 \quad \text{— (1)}$$

$$20x - 9y = 107 \quad \text{— (2)}$$

$$(1) \times 5 \Rightarrow \cancel{20x} - 25y = -165$$

$$\cancel{20x} - 9y = 107$$

$$(-) \quad (+) \quad (-)$$

$$-16y = -272$$

$$\boxed{y = 17}$$

$$4x - 5(17) = -33$$

$$4x - 85 = -33$$

$$4x = -33 + 85$$

$$4x = 52$$

$$\boxed{x = 13}$$

Mean of $x \Rightarrow \bar{x} = 13$

Mean of $y \Rightarrow \bar{y} = 17$

$$4x - 9y = -33$$

$$4x = 9y - 33$$

$$x = \frac{9}{4}y - \frac{33}{4}$$

$$b_{xy} = \frac{9}{4}$$

$$20x - 9y = 107$$

$$-9y = 107 - 20x$$

$$y = \frac{-20}{-9}x + \frac{107}{9}$$

$$b_{yx} = \frac{20}{9}.$$

$$\therefore b_{xy}\, b_{yx} = \frac{\cancel{9}}{\cancel{4}} \times \frac{\cancel{20}^{5}}{\cancel{9}}$$

$$r^2 = b_{xy}\, b_{yx} = 5$$

$$r = \sqrt{5} = 2.23 \text{ (not correct)}$$

$$4x - 9y = -33$$

$$-9y = -4x - 33$$

$$9y = 4x + 33$$

$$y = \frac{4}{9}x + \frac{33}{9} \Rightarrow b_{yx} = \frac{4}{9}$$

$$20x - 9y = 107$$

$$20x = 9y + 107$$

$$x = \frac{9}{20}y + \frac{107}{20}$$

$$\therefore b_{xy} = \frac{9}{20}$$

$$\therefore r^2 = b_{xy}\, b_{yx} = \frac{4}{\cancel{9}} \cdot \frac{\cancel{9}}{20} = \frac{1}{5} = 0.2$$

$$r = \sqrt{\frac{1}{5}} = \sqrt{0.2}$$

$$\boxed{r = 0.4472}$$

16) a) If $X_1, X_2, X_3 \ldots X_n$ are Poisson variates with mean 2, use central limit theorem to estimate $P(120 < S_n < 160)$ where $S_n = X_1 + X_2 + \cdots + X_n$ and $n = 75$.

SOLUTION:

$$\mu = E[X] = 2$$

$$\sigma^2 = Var(X) = 2$$

$$\sigma = \sqrt{2}$$

$$n = 75$$

$\therefore S_n$ follows $N(n\mu, \sigma\sqrt{n})$ by CLT,

$$\Rightarrow Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$= \frac{S_n - (75)(2)}{\sqrt{2}\sqrt{75}} = \frac{S_n - 150}{\sqrt{150}}$$

$$Z = \frac{S_n - 150}{12.25}$$

When $S_n = 120$, $Z = \frac{120 - 150}{12.25} = -2.45$

When $S_n = 160$, $Z = \frac{160 - 150}{12.25} = \frac{10}{12.25} = 0.82$

$\therefore P(120 < S_n < 160) = P(-2.45 < Z < 0.82)$

$= P(0 < Z < 2.45) + P(0 < Z < 0.82)$

$= 0.4929 + 0.2939$

$= 0.7868$

16)b) If X and Y each follow an exponential distribution with parameter 1 and are independent, find the pdf of $U = X - Y$.

SOLUTION:

The pdf of X and Y are

$f(x) = e^{-x}, \; x > 0$ and $f(y) = e^{-y}, \; y > 0$

Since X and Y are independent, the joint pdf is

$f(x, y) = f(x) f(y)$

$\therefore f(x, y) = e^{-x} e^{-y} = e^{-(x+y)}$

The density function of U is

$f(u) = \int f(u, v) \, dv$

and $f(u, v) = |J| \, f(x, y)$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Put $v = y$

$u = x - y$

$u = x - v$

$\Rightarrow u + v = x$

$\therefore x = u + v$

$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = 1$

$v = y$

$\frac{\partial y}{\partial u} = 0, \; \frac{\partial y}{\partial v} = 1$

SOLUTION:

Since X and Y are independent.

$$f(x,y) = e^{-x} e^{-y} = e^{-(x+y)} \quad ; \; x,y \geq 0$$

$$u = \frac{x}{x+y} \qquad v = x+y$$

$$u = \frac{x}{v} \Rightarrow x = uv \qquad y = \cancel{uv}\; v - uv$$
$$= v(1-u)$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = -v$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1-u$$

$$\therefore |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) + uv = v - uv + uv$$

$$|J| = v$$

$$\therefore f(u,v) = |J| \, f(x,y)$$

$$f(u,v) = v\, e^{-(x+y)} = v e^{-v}$$

The ~~density~~ function of U is

$$f(u) = \int f(u,v)\, dv$$

$$= \int_0^\infty v e^{-v}\, dv$$

$x \geq 0$ | $y \geq 0$

$\Rightarrow uv \geq 0$ | $v - uv \geq 0$

$\Rightarrow v \geq 0$ | $v \geq uv$

$\Rightarrow 1 \geq u$

$\Rightarrow u \leq 1$

$$\therefore f(u,v) = 1\, e^{-(x+y)}$$

$$f(u,v) = e^{-(u+v+v)}$$

$$= e^{-(u+2v)}$$

$$\therefore f(u,v) = e^{-(u+2v)}$$

<u>If $u<0$, then</u>

$$f(u) = \int_{-u}^{\infty} e^{-(u+2v)}\, dv$$

$$= e^{-u} \int_{-u}^{\infty} e^{-2v}\, dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty} = \frac{e^{-u}}{-2}\left[e^{-\infty} - e^{-2(-u)}\right]$$

$$= \frac{e^{-u}}{-2}\left[0 - e^{2u}\right] = \frac{e^{u}}{2}$$

<u>If $u \geq 0$, then</u>

$$f(u) = \int_{0}^{\infty} e^{-(u+2v)}\, dv = e^{-u} \int_{0}^{\infty} e^{-2v}\, dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{0}^{\infty}$$

$$= \frac{e^{-u}}{-2}\left[e^{-\infty} - e^{0}\right] = \frac{e^{-u}}{2}.$$

$$\therefore f(u) = \begin{cases} e^{u}/2 & \text{if } u<0 \\ e^{-u}/2 & \text{if } u \geq 0 \end{cases}$$

17)a) If X and Y independent random variables with pdf e^{-x}, $x \geq 0$ and e^{-y}, $y \geq 0$. Derive the density function of $U = \frac{X}{X+Y}$ and $V = X+Y$. Are they independent?

$$= \left[v\left(\frac{e^{-v}}{-1}\right) - (1)\left(\frac{e^{-v}}{1}\right) \right]_0^{\infty}$$

$$= e^{-\infty} - e^{-\infty} - (0 - e^{0})$$

$f(u) = 1.$

The density function of v is

$$f(v) = \int_0^1 f(u,v)\, du$$

$$= \int_0^1 ve^{-v}\, du = ve^{-v}\left[u\right]_0^1$$

$$f(v) = ve^{-v}$$

Now, $f(u)\, f(v) = 1 \cdot ve^{-v} = ve^{-v} = f(u,v)$

$\therefore$ U and V are independent.

17)b Two random variables X and Y have the following joint probability density function $f(x,y) = \begin{cases} x+y, & 0 \le x \le 1, \\ & 0 \le y \le 1. \\ 0, & \text{otherwise} \end{cases}$

Find the probability density function of the random variable $U = XY$.

SOLUTION:

Let $v = y$, $u = xv$

$\Rightarrow x = \frac{u}{v}$

$x = u/v$ $\quad y = v$

$$\frac{\partial x}{\partial u} = \frac{1}{v}, \quad \frac{\partial x}{\partial v} = \frac{-u}{v^2}, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial y}{\partial v} = 1$$

$$\therefore |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\therefore f(u,v) = |J| f(x,y)$$

$$\therefore f(u,v) = \frac{1}{v}(x+y) = \frac{1}{v}\left(\frac{u}{v} + v\right) = \cancel{u+v^2} \quad \frac{u}{v^2} + 1$$

The density function of U is

$$f(u) = \int_u^1 f(u,v)\, dv$$

$0 \le x \le 1$

$0 \le \frac{u}{v} \le 1$

$\boxed{0 \le u \le v}$ —(1)

$0 \le y \le 1$

$\boxed{0 \le v \le 1}$ —(2)

From (1) & (2) $\boxed{u \le v \le 1}$

$$= \int_u^1 \left(\frac{u}{v^2} + 1\right) dv$$

$$= \left[\frac{u v^{-1}}{-1} + v\right]_u^1$$

$$= \left[-u\frac{1}{v} + v\right]_u^1 = -u + 1 - (-1 + u)$$

$$= -u + 1 + 1 - u$$

$$= 2 - 2u$$

$$\therefore f(u) = 2 - 2u, \quad 0 < u < 1$$

18)a) A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using Central limit theorem to find what probability that we can assert that the mean of the sample

will not differ from μ more than 4?

SOLUTION:

$\mu = 60$

$\sigma^2 = 400 \Rightarrow \sigma = 20$

$n = 100$

$\bar{x} \rightarrow$ sample mean

$\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ by CLT

$$\Rightarrow Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 60}{20/10} = \frac{\bar{x} - 60}{2}$$

<u>To Find:</u> $P[|\bar{x} - 60| \leq 4] = P(-4 \leq \bar{x} - 60 \leq 4)$

$= P(-4 + 60 \leq \bar{x} \leq 4 + 60)$

$= P(56 \leq \bar{x} \leq 64)$

When $\bar{x} = 56$, $Z = \frac{56 - 60}{2} = \frac{-4}{2} = -2$

When $\bar{x} = 64$, $Z = \frac{64 - 60}{2} = \frac{4}{2} = 2$

$\therefore P(|\bar{x} - 60| \leq 4) = P(-2 \leq Z \leq 2) = 2P(0 \leq Z \leq 2)$

$= 2(0.4772) = 0.9544$

$\therefore P(|\bar{x} - 60| \leq 4) = 0.9544$

18)b) If X and Y follows an exponential distribution with parameter 2 and 3 respectively and are independent. Create the probability density function of $U = X + Y$.

SOLUTION:

$$f(x) = 2e^{-2x}, \quad x > 0 \quad , \quad f(y) = 3e^{-3y}, \quad y > 0$$

Since X and Y are independent

$$f(x,y) = f(x)\, f(y) = 2e^{-2x}\, 3e^{-3y}$$

$$= 6e^{-(2x+3y)}$$

Put $V = y \Rightarrow U = X + V$

$$\Rightarrow x = u - v \qquad y = v$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1$$

$$\therefore |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore f(u,v) = 6e^{-(2(u-v)+3v)}$$

$$= 6e^{-(2u-2v+3v)}$$

$$= 6e^{-(2u+v)}$$

The density function of U is

$y \geq 0$

$\Rightarrow v \geq 0$

$$f(u) = \int_0^\infty f(u,v)\, dv$$

$$= \int_0^\infty 6e^{-(2u+v)}\, dv$$