Entropy Codings

Probability, Information, Entropy

August 2018

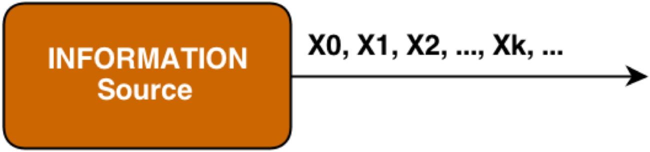
Content

- Information Model

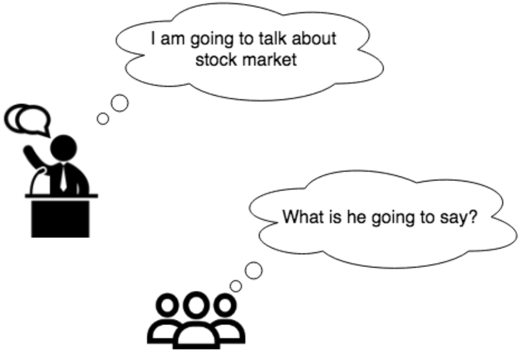
Information Source Model - Nguồn thông tin

- Some examples:
  - Tiếng gà gáy, tiếng trống canh
  - Đèn hiệu, đốt lửa, ...
  - Các file dữ liệu (e.g.: text, audio, image, video files)
  - Các "data stream"(ví dụ: bit stream)
  - A typewritter, mouse clicks of users on a webpage
  - ...
- Some common characteristics
  - Always generate *Something* as *Output* (observable, iterable, finite/infinite)
  - Roughly sequetial (Or can be seen as sequential)
- How to quantitatively characterize them?

Probabilistic Information Source Model



- Object generates "symbols"
- Randomly (Do you believe it?)



- Symbols are generated DETERMINISTICALLY. But its rules may be completely unknown => appears almost completely random to the world.
- Information source can be affected by the world so that output can be changed

Probabilistic Information Source Model

Symbols and Alphabet

- Symbols can be modeled by
  - Apply assumptions to simplify
  - Observe for a long time and figure out symbol set
- Alphabet
  - Symbol set: $X = \{x_i | i = 0..N\}$
  - And a probability distribution over symbol set $p_i = Pr(X = X_i)$
- i.i.d. Information source model
  - Symbols are identical and independent distributed random variables

Concept of independence

- A and B are independent if occurence of A doesn't affect occurence of B and vice versus
  - $P(A \cap B) = P(AB) = P(A)P(B)$
  - $P(A|B) = P(A)$ and $P(B|A) = P(B)$
- If you observe two random variables A and B, can you tell how independent they are?

Correlation

- Covariance, variance, standard deviation

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} \tag{1}$$

$$= \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sqrt{E[(X-\mu_X)^2]}\sqrt{E[(Y-\mu_Y)^2]}} \tag{2}$$

- Sample version

$$r_{X,Y} = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i-\bar{x})^2}\sqrt{\sum_{i=1}^{N}(y_i-\bar{y})^2}} \tag{3}$$

- Uncorrected sample standard deviation

$$s_N = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{4}$$

- Corrected sample standard deviation

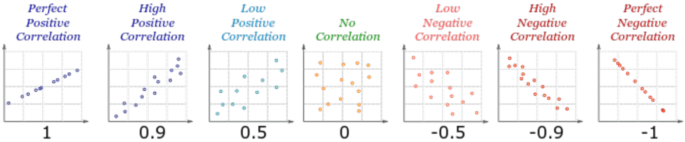
$$\dot{s}_N = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{5}$$

- Why?
- Sample correlation with corrected standard deviation

$$r_{X,Y} = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{(N-1)\dot{s}_X \dot{s}_Y} \tag{6}$$

Correlation - Application

- Just by observing 2 datasets, we can tell how two ranvars are independent or dependent,
- and how they are independent/dependent



- What is its application, then?

But, be careful

- Let X is a random variable with $\mu_X = 0$ and $Y = X^2$
  - Are they still random variables?
- Let's calculate $\rho_{X,Y}$
- The answer is $\rho_{X,Y} = 0$ (Surprise?)

- Correlation only detect linear dependency

Information content

- Symbol x with probability $Pr(X = x) = p(x)$, amount of information that symbol x carries is

$$I(x) = -log_2(p(x)) = -log(p(x)) \tag{7}$$

- High probability = highly predictable = easy to predict = little information
- $I(x)$ only depends on $p(x)$. We can "code" x with any other symbol without changing I
- *bit* is unit of $I(x)$

Consider 1 flip-flop that can store either '0' or '1' with equal probability (1 bit)

$$
\begin{aligned}
I('0') &= I('1') &(8)\\
&= -log(0.5) = -log(2^{-1}) = 1 &(9)
\end{aligned}
$$

This is true for all other similar models:

- Tung xu, xấp ngửa
- Bứt cánh hoá
- Trời mưa hay trời nắng
- ...

Information content - Some examples

- Giving 3 coins, toss them.
  - $I(\text{“}HHH'') = ?$
  - $I(\text{“}TTT'') = ?$
  - $I(\text{“}THH'') = ?$
- How about N coins?

Information measurement of a source - Entropy

Let's measure amount of information for an information source instead of a symbol

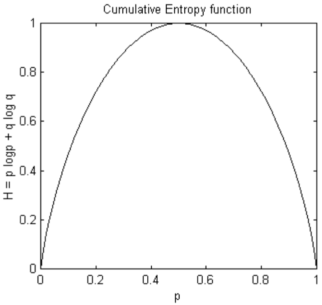
- Entropy: Average information of all symbols of a source

$$
\begin{aligned}
H(X) &= E[I(x)|\forall x \in X] && (10)\\
&= \sum_{x \in X} p(x) I(x) && (11)\\
&= -\sum_{x \in X} p(x) log(p(x)) && (12)
\end{aligned}
$$

Some examples

- Calculate entropy of 1-bit information source

$$H(\text{“}1-bit'') = -p \times log(p) - (1-p) \times log(1-p) \quad (13)$$
$$= 0.5 \times 1 + 0.5 \times 1 \quad (14)$$
$$= 1 \quad (15)$$



- Calculate entropy of N-bit information source
- Calculate entropy of binomial information source

Some examples

- Calculate entropy of 1-bit information source
- Calculate entropy of N-bit information source

$$H(``N - bit'') = \sum_{i=1}^{2^N} \frac{1}{2^N} log(2^N) \quad (16)$$

$$= N \quad (17)$$

- Calculate entropy of binomial information source

Some examples

- Calculate entropy of N-bit information source
- Calculate entropy of binomial information source

$$
\begin{aligned}
H(\text{“}N - bin''') &= \sum_{i=0}^{N} C_i^N \frac{1}{2^N} \log\left(C_i^N \frac{1}{2^N}\right) && (18) \\
&= \sum_{i=0}^{N} C_i^N \frac{1}{2^N} \left(\log(C_i^N) - \log(2^N)\right) && (19) \\
&= \sum_{i=0}^{N} C_i^N \frac{1}{2^N} \log(C_i^N) - \sum_{i=0}^{N} C_i^N \frac{N}{2^N} && (20)
\end{aligned}
$$

Binomial distribution - some cases

N	$H(N-bit)$	$H(N-bin)$
2	2	1.5
3	3	1.811
4	4	2.03

- $H(N-bin) \leq H(N-bit)$! Why?

Entropy characteristics

- Additive: If X & Y are independent

$$H(X,Y) = H(X) + H(Y) \tag{21}$$

- Given an alphabet $X = \{x_i | i = 1..N\}$

$$H(X) \leq log(N) \tag{22}$$

$$\tag{23}$$

$H(X)$ get maximized iff $p(x_i) = \frac{1}{N}, \forall i$

Data compression

- How many bit are needed to describe the outcome of an symbol (outcome of a random experiment)?

Shanon source coding theorem

Theorem:

- N i.i.d. random variables, each with entropy H(X) can ben compressed into more than $NH(X)$ bits with negligible risk of information loss, as $N \to \infty$

- Compression: compressor function that maps $x \to \dot{x} = c(x)$ which is a bit string
- Decompression: decompressor function that maps $\dot{x} \to x = d(\dot{x})$