Area Preserving Combescure Transformations
(Auxiliary computations)

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Here, we check the auxiliary computations of the proofs.

restart;

Section 1

Changing variables l_i and m_i for $i = 1, 2, 3, 4$ by:

$l_1 := \frac{\mu_1 - 1}{\lambda_1}$; $m_1 := \frac{\lambda_1 - 1}{\mu_1}$; $l_2 := \frac{\mu_2 - 1}{\lambda_2}$; $m_2 := \frac{\lambda_2 - 1}{\mu_2}$;
$l_3 := \frac{\mu_3 - 1}{\lambda_3}$; $m_3 := \frac{\lambda_3 - 1}{\mu_3}$; $l_4 := \frac{\mu_4 - 1}{\lambda_4}$; $m_4 := \frac{\lambda_4 - 1}{\mu_4}$;

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}$$

$$m_1 := \frac{\lambda_1 - 1}{\mu_1}$$

$$l_2 := \frac{\mu_2 - 1}{\lambda_2}$$

$$m_2 := \frac{\lambda_2 - 1}{\mu_2}$$

$$l_3 := \frac{\mu_3 - 1}{\lambda_3}$$

$$m_3 := \frac{\lambda_3 - 1}{\mu_3}$$

$$l_4 := \frac{\mu_4 - 1}{\lambda_4}$$

$$m_4 := \frac{\lambda_4 - 1}{\mu_4} \qquad \textbf{(1)}$$

Equating equations in Lemma 2(ii) (Lemma 2.14(ii) in Arxiv) to $1/k_i^2$ for $i = 1, 2, 3, 4$:

$$simplify\left(\frac{1 - l_1 \cdot m_1}{(1+m_1)^2}\right) = \frac{1}{k_1^2};\ simplify\left(\frac{1 - l_1 \cdot m_1}{(1+l_1)^2}\right) = \frac{1}{k_2^2};\ simplify\left(\frac{1 - l_2 \cdot m_2}{(1+m_2)^2}\right) = \frac{1}{k_2^2};$$

$$simplify\left(\frac{1 - l_2 \cdot m_2}{(1+l_2)^2}\right) = \frac{1}{k_3^2};\ simplify\left(\frac{1 - l_3 \cdot m_3}{(1+m_3)^2}\right) = \frac{1}{k_3^2};\ simplify\left(\frac{1 - l_3 \cdot m_3}{(1+l_3)^2}\right) = \frac{1}{k_4^2};$$

$$simplify\left(\frac{1 - l_4 \cdot m_4}{(1+m_4)^2}\right) = \frac{1}{k_4^2};\ simplify\left(\frac{1 - l_4 \cdot m_4}{(1+l_4)^2}\right) = \frac{1}{k_1^2};$$

$$\frac{\mu_1}{(\lambda_1 + \mu_1 - 1)\,\lambda_1} = \frac{1}{k_1^2}$$

$$\frac{\lambda_1}{(\lambda_1 + \mu_1 - 1)\,\mu_1} = \frac{1}{k_2^2}$$

$$\frac{\mu_2}{(\lambda_2 + \mu_2 - 1)\,\lambda_2} = \frac{1}{k_2^2}$$

$$\frac{\lambda_2}{(\lambda_2 + \mu_2 - 1)\,\mu_2} = \frac{1}{k_3^2}$$

$$\frac{\mu_3}{(\lambda_3 + \mu_3 - 1)\,\lambda_3} = \frac{1}{k_3^2}$$

$$\frac{\lambda_3}{(\lambda_3 + \mu_3 - 1)\,\mu_3} = \frac{1}{k_4^2}$$

$$\frac{\mu_4}{(\lambda_4 + \mu_4 - 1)\,\lambda_4} = \frac{1}{k_4^2}$$

$$\frac{\lambda_4}{(\lambda_4 + \mu_4 - 1)\,\mu_4} = \frac{1}{k_1^2} \tag{2}$$

Note that $\mu_i / \lambda_i > 0$ because of $l_i + 1 > 0,\ m_i + 1 > 0,$ where $i = 1, 2, 3, 4.$

In the above system (2), we devide equation $1-2, 3-4, 5-6, 7-8$ and taking square root, we obtain :

$$\frac{\mu_1}{\lambda_1} = \frac{k_2}{k_1};\ \frac{\mu_2}{\lambda_2} = \frac{k_3}{k_2};\ \frac{\mu_3}{\lambda_3} = \frac{k_4}{k_3};\ \frac{\mu_4}{\lambda_4} = \frac{k_1}{k_4};$$

$$\frac{\mu_1}{\lambda_1} = \frac{k_2}{k_1}$$

$$\frac{\mu_2}{\lambda_2} = \frac{k_3}{k_2}$$

$$\frac{\mu_3}{\lambda_3} = \frac{k_4}{k_3}$$

$$\frac{\mu_4}{\lambda_4} = \frac{k_1}{k_4} \qquad \mathbf{(3)}$$

Then equations of the above system (2) becomes linear :

$$\mu_1 = \frac{k_2}{k_1} \cdot \lambda_1;\ \mu_1 + \lambda_1 - 1 = k_1 \cdot k_2;\ \mu_2 = \frac{k_3}{k_2} \cdot \lambda_2;\ \mu_2 + \lambda_2 - 1 = k_2 \cdot k_3;$$

$$\mu_3 = \frac{k_4}{k_3} \cdot \lambda_3;\ \mu_3 + \lambda_3 - 1 = k_3 \cdot k_4;\ \mu_4 = \frac{k_1}{k_4} \cdot \lambda_4;\ \mu_4 + \lambda_4 - 1 = k_4 \cdot k_1;$$

$$\mu_1 = \frac{k_2\, \lambda_1}{k_1}$$

$$\lambda_1 + \mu_1 - 1 = k_1\, k_2$$

$$\mu_2 = \frac{k_3\, \lambda_2}{k_2}$$

$$\lambda_2 + \mu_2 - 1 = k_2\, k_3$$

$$\mu_3 = \frac{k_4\, \lambda_3}{k_3}$$

$$\lambda_3 + \mu_3 - 1 = k_3\, k_4$$

$$\mu_4 = \frac{k_1 \lambda_4}{k_4}$$

$$\lambda_4 + \mu_4 - 1 = k_4 k_1 \tag{4}$$

It has a solution :

$$solve\left(\left\{\mu_1 = \frac{k_2}{k_1}\cdot\lambda_1, \mu_1 + \lambda_1 - 1 = k_1\cdot k_2, \mu_2 = \frac{k_3}{k_2}\cdot\lambda_2, \mu_2 + \lambda_2 - 1 = k_2\cdot k_3, \mu_3 = \frac{k_4}{k_3}\cdot\lambda_3, \mu_3 + \lambda_3 - 1 = k_3\cdot k_4, \mu_4 = \frac{k_1}{k_4}\cdot\lambda_4, \mu_4 + \lambda_4 - 1 = k_4\cdot k_1\right\}, \{\mu_1, \lambda_1, \mu_2, \lambda_2, \mu_3, \lambda_3, \mu_4, \lambda_4\}\right)$$

$$\left\{\lambda_1 = \frac{k_1\left(k_1 k_2 + 1\right)}{k_1 + k_2}, \lambda_2 = \frac{k_2\left(k_2 k_3 + 1\right)}{k_2 + k_3}, \lambda_3 = \frac{k_3\left(k_3 k_4 + 1\right)}{k_3 + k_4}, \lambda_4 = \frac{k_4\left(k_4 k_1 + 1\right)}{k_1 + k_4}, \mu_1 = \frac{k_2\left(k_1 k_2 + 1\right)}{k_1 + k_2}, \mu_2 = \frac{k_3\left(k_2 k_3 + 1\right)}{k_2 + k_3}, \mu_3 = \frac{k_4\left(k_3 k_4 + 1\right)}{k_3 + k_4}, \mu_4 = \frac{k_1\left(k_4 k_1 + 1\right)}{k_1 + k_4}\right\} \tag{5}$$

Setting :

$$\lambda_1 := \frac{k_1\left(k_1 k_2 + 1\right)}{k_1 + k_2}; \lambda_2 := \frac{k_2\left(k_2 k_3 + 1\right)}{k_2 + k_3}; \lambda_3 := \frac{k_3\left(k_3 k_4 + 1\right)}{k_3 + k_4}; \lambda_4 := \frac{k_4\left(k_4 k_1 + 1\right)}{k_1 + k_4};$$
$$\mu_1 := \frac{k_2\left(k_1 k_2 + 1\right)}{k_1 + k_2}; \mu_2 := \frac{k_3\left(k_2 k_3 + 1\right)}{k_2 + k_3}; \mu_3 := \frac{k_4\left(k_3 k_4 + 1\right)}{k_3 + k_4}; \mu_4 := \frac{k_1\left(k_4 k_1 + 1\right)}{k_1 + k_4}$$

$$\lambda_1 := \frac{k_1\left(k_1 k_2 + 1\right)}{k_1 + k_2}$$

$$\lambda_2 := \frac{k_2\left(k_2 k_3 + 1\right)}{k_2 + k_3}$$

$$\lambda_3 := \frac{k_3\left(k_3 k_4 + 1\right)}{k_3 + k_4}$$

$$\lambda_4 := \frac{k_4\left(k_4 k_1 + 1\right)}{k_1 + k_4}$$

$$\mu_1 := \frac{k_2\left(k_1 k_2 + 1\right)}{k_1 + k_2}$$

$$\mu_2 := \frac{k_3\left(k_2 k_3 + 1\right)}{k_2 + k_3}$$

$$\mu_3 := \frac{k_4 \left(k_3 k_4 + 1\right)}{k_3 + k_4}$$

$$\mu_4 := \frac{k_1 \left(k_4 k_1 + 1\right)}{k_1 + k_4} \tag{6}$$

Then the value of l_i, m_i where i = 1, 2, 3, 4 is :

$l_1 := simplify\left(\frac{\mu_1 - 1}{\lambda_1}\right)$; $m_1 := simplify\left(\frac{\lambda_1 - 1}{\mu_1}\right)$; $l_2 := simplify\left(\frac{\mu_2 - 1}{\lambda_2}\right)$;
$m_2 := simplify\left(\frac{\lambda_2 - 1}{\mu_2}\right)$; $l_3 := simplify\left(\frac{\mu_3 - 1}{\lambda_3}\right)$; $m_3 := simplify\left(\frac{\lambda_3 - 1}{\mu_3}\right)$;
$l_4 := simplify\left(\frac{\mu_4 - 1}{\lambda_4}\right)$; $m_4 := simplify\left(\frac{\lambda_4 - 1}{\mu_4}\right)$;

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}$$

$$m_1 := \frac{\lambda_1 - 1}{\mu_1}$$

$$l_2 := \frac{\mu_2 - 1}{\lambda_2}$$

$$m_2 := \frac{\lambda_2 - 1}{\mu_2}$$

$$l_3 := \frac{\mu_3 - 1}{\lambda_3}$$

$$m_3 := \frac{\lambda_3 - 1}{\mu_3}$$

$$l_4 := \frac{\mu_4 - 1}{\lambda_4}$$

$$m_4 := \frac{\lambda_4 - 1}{\mu_4} \tag{7}$$

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restart;

$with(PolynomialTools)$:
$with(LinearAlgebra)$:

Section 2

Equations in Lemma 1 (Lemma 2.12 in Arxiv) are :

$P_1 := l_1 \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + m_1 \cdot x_2^2 - (l_1 + m_1 + 2)$

$$P_1 := l_1 x_1^2 + m_1 x_2^2 + 2 x_1 x_2 - l_1 - m_1 - 2 \quad \textbf{(8)}$$

$P_2 := l_2 \cdot x_2^2 + 2 \cdot x_2 \cdot x_3 + m_2 \cdot x_3^2 - (l_2 + m_2 + 2)$

$$P_2 := l_2 x_2^2 + m_2 x_3^2 + 2 x_2 x_3 - l_2 - m_2 - 2 \quad \textbf{(9)}$$

$P_3 := l_3 \cdot x_3^2 + 2 \cdot x_3 \cdot x_4 + m_3 \cdot x_4^2 - (l_3 + m_3 + 2)$

$$P_3 := l_3 x_3^2 + m_3 x_4^2 + 2 x_3 x_4 - l_3 - m_3 - 2 \quad \textbf{(10)}$$

$P_4 := l_4 \cdot x_4^2 + 2 \cdot x_4 \cdot x_1 + m_4 \cdot x_1^2 - (l_4 + m_4 + 2)$

$$P_4 := l_4 x_4^2 + m_4 x_1^2 + 2 x_4 x_1 - l_4 - m_4 - 2 \quad \textbf{(11)}$$

Setting the functions :

$x_1 := \dfrac{(1+t)^2 \cdot (1 - k_1) + 1 + k_1}{2 \cdot (1+t)}; x_2 := \dfrac{(1+t)^2 \cdot (1 + k_2) + 1 - k_2}{2 \cdot (1+t)};$
$x_3 := \dfrac{(1+t)^2 \cdot (1 - k_3) + 1 + k_3}{2 \cdot (1+t)}; x_4 := \dfrac{(1+t)^2 \cdot (1 + k_4) + 1 - k_4}{2 \cdot (1+t)}$

$$x_1 := \frac{(1+t)^2 (1 - k_1) + 1 + k_1}{2 + 2t}$$

$$x_2 := \frac{(1+t)^2 (1 + k_2) + 1 - k_2}{2 + 2t}$$

$$x_3 := \frac{(1+t)^2 (1 - k_3) + 1 + k_3}{2 + 2t}$$

$$x_4 := \frac{(1+t)^2 (1 + k_4) + 1 - k_4}{2 + 2t} \quad \textbf{(12)}$$

Verification of the equations in Lemma 1 (Lemma 2.12 in Arxiv) :

$simplify(P_1); simplify(P_2); simplify(P_3); simplify(P_4)$

$$0$$
$$0$$
$$0$$
$$0 \qquad \textbf{(13)}$$

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Section 3

Left side of equation (9) (equation (2.9) in Arxiv) :

$eq := collect\Big(\Big((l_2+m_2+2)\cdot\big(l_1\cdot x_1^2+2\cdot x_1\cdot x_2+m_1\cdot x_2^2-(l_1+m_1+2)\big)-(l_1+m_1+2)\cdot\Big(\big(l_2\cdot x_2^2+2\cdot x_2\cdot x_3+m_2\cdot x_3^2-(l_2+m_2+2)\big)\Big)\Big), \{x_1,x_2,x_3\}, distributed\Big)$

$$eq := (l_2+m_2+2)\,l_1\,x_1^2+(2\,l_2+2\,m_2+4)\,x_2\,x_1+\big((l_2+m_2+2)\,m_1-(l_1+m_1+2)\,l_2\big)\,x_2^2+(-2\,l_1-2\,m_1-4)\,x_3\,x_2-(l_1+m_1+2)\,m_2\,x_3^2+(l_2+m_2+2)\,(-l_1-m_1-2)-(l_1+m_1+2)\,(-l_2-m_2-2) \qquad \textbf{(14)}$$

Matrix of coefficients of the left side of equation (9) (equation (2.9) in Arxiv) :

$M := Matrix\left(\left[\left[(l_2+m_2+2)\,l_1, \frac{(2\cdot l_2+2\cdot m_2+4)}{2}, 0\right], \left[\frac{(2\cdot l_2+2\cdot m_2+4)}{2}, -l_1\,l_2+m_1\,m_2-2\,l_2+2\,m_1, \frac{(-2\,l_1-2\,m_1-4)}{2}\right], \left[0, \frac{(-2\,l_1-2\,m_1-4)}{2}, -l_1\,m_2-m_2\,(m_1+2)\right]\right]\right)$

$$M := \begin{bmatrix} (l_2+m_2+2)\,l_1 & l_2+m_2+2 & 0 \\ l_2+m_2+2 & -l_1\,l_2+m_1\,m_2-2\,l_2+2\,m_1 & -l_1-m_1-2 \\ 0 & -l_1-m_1-2 & -l_1\,m_2-m_2\,(m_1+2) \end{bmatrix} \qquad \textbf{(15)}$$

Verifying that the $\det(M)$ and left side of (10) (equation (2.10) in Arxiv) are the same :

$$simplify\Big(Determinant(M) - (l_1 + m_1 + 2)\cdot(l_2 + m_2 + 2)\cdot\big((m_2 + 1)^2\cdot(1 - m_1\cdot l_1) - (l_1 + 1)^2\cdot(1 - l_2\cdot m_2)\big)\Big)$$

$$0 \tag{16}$$

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Section 4

Equation (11) (equation (2.11) in Arxiv) :

$$Eq\_2\_11 := expand\Big(\Big(Determinant\Big(Matrix\Big(\big[[m_1, 0, l_2, 0], [2\cdot x_1, m_1, 2\cdot x_3, l_2], [l_1\cdot x_1^{\,2} - (l_1 + m_1 + 2), 2\cdot x_1, m_2\cdot x_3^{\,2} - (l_2 + m_2 + 2), 2\cdot x_3], [0, l_1\cdot x_1^{\,2} - (l_1 + m_1 + 2), 0, m_2\cdot x_3^{\,2} - (l_2 + m_2 + 2)]\big]\Big)\Big)\Big)\Big)$$

$$\begin{aligned} Eq\_2\_11 := {} & l_1^2 l_2^2 x_1^4 - 2 l_1 l_2 m_1 m_2 x_1^2 x_3^2 + m_1^2 m_2^2 x_3^4 - 2 l_1^2 l_2^2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_3^2 \\ & - 4 l_1 l_2 x_1^3 x_3 + 4 l_1 m_1 x_1^2 x_3^2 + 4 l_2 m_2 x_1^2 x_3^2 - 2 m_1^2 m_2^2 x_3^2 - 4 m_1 m_2 x_1 x_3^3 - 4 l_1 l_2^2 x_1^2 + 4 l_1 l_2 m_1 \\ & x_1^2 + 4 l_2 m_1 m_2 x_3^2 - 4 m_1^2 m_2 x_3^2 + l_1^2 l_2^2 - 2 l_1 l_2 m_1 m_2 + 4 l_1 l_2 x_1 x_3 - 4 l_1 m_1 x_3^2 - 4 l_2^2 x_1^2 \\ & + 8 l_2 m_1 x_1 x_3 - 4 l_2 m_2 x_1^2 + m_1^2 m_2^2 - 4 m_1^2 x_3^2 + 4 m_1 m_2 x_1 x_3 + 4 l_1 l_2^2 - 4 l_1 l_2 m_1 - 4 l_2 m_1 m_2 \\ & - 8 l_2 x_1^2 + 8 l_2 x_1 x_3 + 4 m_1^2 m_2 + 8 m_1 x_1 x_3 - 8 m_1 x_3^2 + 4 l_2^2 - 8 m_1 l_2 + 4 m_1^2 \end{aligned} \tag{17}$$

Expression Q in equation (13) (equation (2.13) in Arxiv) :

$$Q := l_1\cdot l_2\cdot x_1^{\,2} - m_1\cdot m_2\cdot x_3^{\,2} + m_1\cdot m_2 - l_1\cdot l_2 + 2\cdot(m_1 - l_2)$$

$$Q := l_1 l_2 x_1^2 - m_1 m_2 x_3^2 - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 \tag{18}$$

Equation (14) (equation (2.14) in Arxiv) :

$$Eq\_2\_14 := 4\cdot(l_1\cdot x_1^{\,2} - l_1 - m_1 - 2)\cdot(l_2\cdot x_1 - m_1\cdot x_3)^2 - 4\cdot x_1\cdot(l_2\cdot x_1 - m_1\cdot x_3)\cdot Q + m_1\cdot Q^2$$

$$Eq\_2\_14 := 4\,(l_1 x_1^2 - l_1 - m_1 - 2)\,(l_2 x_1 - m_1 x_3)^2 - 4 x_1\,(l_2 x_1 - m_1 x_3)\,(l_1 l_2 x_1^2 - m_1 m_2 x_3^2 \tag{19}$$

$$-l_1 l_2+m_1 m_2-2 l_2+2 m_1\big)+m_1\left(l_1 l_2 x_1^2-m_1 m_2 x_3^2-l_1 l_2+m_1 m_2-2 l_2+2 m_1\right)^2$$

Verifying that the equations (11) and (14) (equations (2.11) and (2.14) in Arxiv) are equivalent :

$simplify(m_1 \cdot Eq\_2\_11 - Eq\_2\_14)$

$$0 \tag{20}$$

:= :=
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Section 5

Equation (11) (equation (2.11) in Arxiv) :

$Eq\_2\_11 := simplify\Big(Determinant\Big(Matrix\Big(\big[[m_1, 0, l_2, 0], [2\cdot x_1, m_1, 2\cdot x_3, l_2], \big[l_1\cdot x_1{}^2-(l_1+m_1+2), 2\cdot x_1, m_2\cdot x_3{}^2-(l_2+m_2+2), 2\cdot x_3\big], \big[0, l_1\cdot x_1{}^2-(l_1+m_1+2), 0, m_2\cdot x_3{}^2-(l_2+m_2+2)\big]\big]\Big)\Big)\Big)$

$$Eq\_2\_11 := \left(l_1^2 x_1^4+\left(-2 l_1^2-4 l_1-4\right) x_1^2+\left(2+l_1\right)^2\right) l_2^2+\Big(\Big(-2 l_1\left(m_2 x_3^2-m_2-2\right) x_1^2+8 x_1 x_3+2\left(m_2 x_3^2-m_2-2\right)\left(2+l_1\right)\Big) m_1-4 x_1\Big(l_1 x_1^2 x_3+\left(-m_2 x_3^2+m_2+2\right) x_1-x_3\left(2+l_1\right)\Big)\Big) l_2+4 m_1\left(\frac{\left(x_3-1\right)\left(x_3+1\right)\left(m_2 x_3+m_2+2\right)\left(m_2 x_3-m_2-2\right) m_1}{4}+\left(l_1 x_1^2 x_3+\left(-m_2 x_3^2+m_2+2\right) x_1-x_3\left(2+l_1\right)\right) x_3\right) \tag{21}$$

Equation (11') (equation (2.11') in Arxiv) :

$Eq\_2\_11\_pr := simplify\Big(Determinant\Big(Matrix\Big(\big[[l_4, 0, m_3, 0], [2\cdot x_1, l_4, 2\cdot x_3, m_3], \big[m_4\cdot x_1{}^2-(m_4+l_4+2), 2\cdot x_1, l_3\cdot x_3{}^2-(m_3+l_3+2), 2\cdot x_3\big], \big[0, m_4\cdot x_1{}^2-(m_4+l_4+2), 0, l_3\cdot x_3{}^2-(m_3+l_3+2)\big]\big]\Big)\Big)\Big)$

$$Eq\_2\_11\_pr := \left(l_3^2 x_3^4+\left(-2 l_3^2-4 l_3-4\right) x_3^2+\left(2+l_3\right)^2\right) l_4^2+\Big(\Big(-2 m_4\left(l_3 x_3^2-l_3-2\right) x_1^2+8 x_1 x_3+2\left(m_4+2\right)\left(l_3 x_3^2-l_3-2\right)\Big) m_3-4\Big(-m_4 x_3 x_1^2+\left(l_3 x_3^2-l_3-2\right) x_1+x_3\big(m_4 \tag{22}$$

$$+2)\big)\,x_3\big)\,l_4+4\left(\frac{(x_1-1)\,(x_1+1)\,(m_4x_1+m_4+2)\,(m_4x_1-m_4-2)\,m_3}{4}+\big(-m_4x_3\right.$$
$$\left.x_1^2+\big(l_3x_3^2-l_3-2\big)\,x_1+x_3\,(m_4+2)\big)\,x_1\right)m_3$$

Comperison of coefficients of the equation $\alpha\cdot Eq\_2\_11-Eq\_2\_11\_pr$:

$Eq := collect\big(simplify(\alpha\cdot Eq\_2\_11-Eq\_2\_11\_pr),\ \{x_1,x_2,x_3\},\ distributed\big)$

$$Eq := \left(\alpha\,l_1^2\,l_2^2-m_3^2\,m_4^2\right)x_1^4+\left(-4\,\alpha\,l_1\,l_2+4\,m_3\,m_4\right)x_3\,x_1^3+\Big(-2\,l_2\,\big(\big(\big(l_1^2+2\,l_1+2\big)\,l_2-(m_2+2)\,(l_1\,m_1-2)\big)\,\alpha-2\,m_3\,\big(m_4\,(2+l_3)\,l_4+\big(-m_4^2-2\,m_4-2\big)\,m_3-2\,l_3-4\big)\Big)\,x_1^2+\big(\big(\big(-2\,l_1\,m_1\,m_2+4\,m_2\big)\,l_2+4\,l_1\,m_1\big)\,\alpha+\big(2\,l_3\,m_3\,m_4-4\,m_4\big)\,l_4-4\,l_3\,m_3\big)\,x_3^2\,x_1^2+\big(-4\,\alpha\,m_1\,m_2+4\,l_3\,l_4\big)\,x_1\,x_3^3+\big(-4\,\big(\big(-l_1-2\,m_1-2\big)\,l_2-m_1\,(m_2+2)\big)\,\alpha-4\,\big(l_3+2\,m_3+2\big)\,l_4-4\,m_3\,(m_4+2)\big)\,x_1\,x_3+\left(\alpha\,m_1^2\,m_2^2-l_3^2\,l_4^2\right)x_3^4+\Big(2\,\big(m_2\,(2+l_1)\,l_2+\big(-m_2^2-2\,m_2-2\big)\,m_1-2\,l_1-4\big)\,m_1\,\alpha+2\,\big(\big(\big(l_3^2+2\,l_3+2\big)\,l_4-(m_4+2)\,(l_3\,m_3-2)\big)\,l_4\Big)\,x_3^2+\big((2+l_1)\,l_2-m_1\,(m_2+2)\big)^2\alpha-\big((2+l_3)\,l_4-m_3\,(m_4+2)\big)^2 \tag{23}$$

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Section 6

Equation (25) (equation (2.25) in Arxiv) and $\alpha = 1$:

$l_2 := \frac{(m_3+1)\cdot(l_4+1)}{m_1+1}-1;\ l_3 := \frac{m_1\cdot m_2}{l_4};\ m_4 := \frac{l_1\cdot(m_3\cdot l_4+m_3+l_4-m_1)}{m_3\cdot(m_1+1)};\ \alpha := 1$

$$l_2 := \frac{(m_3+1)\,(l_4+1)}{m_1+1}-1$$

$$l_3 := \frac{m_1\,m_2}{l_4}$$

$$m_4 := \frac{l_1\,(l_4\,m_3+l_4-m_1+m_3)}{m_3\,(m_1+1)}$$

$$\alpha := 1 \tag{24}$$

Expression A :

$$A := (m_3 + 1)\cdot(l_1 + 1)\cdot(l_4 + 1) - (m_1 + 1)\cdot(m_1\cdot m_2 + l_1 + l_4 + m_1 - m_3 + 1)$$

$$A := (m_3 + 1)\,(l_1 + 1)\,(l_4 + 1) - (m_1 + 1)\,(m_1\,m_2 + l_1 + l_4 + m_1 - m_3 + 1) \tag{25}$$

Expression B :

$$B := m_1\cdot m_2 + m_2\cdot m_3 + l_1 + m_2 + l_4 + m_1 + 2$$

$$B := m_1\,m_2 + m_2\,m_3 + l_1 + l_4 + m_1 + m_2 + 2 \tag{26}$$

Substitution (22) in (25) ((2.22) in (2.25) in Arxiv) and comparing with required equation :

$$simplify\left(\frac{(m_1 + 1)^2}{4}\cdot\left(\left((2 + l_1)\,l_2 - m_1\,(2 + m_2)\right)^2\alpha - \left((2 + l_3)\,l_4 - m_3\,(m_4 + 2)\right)^2\right) - (m_1 + m_3 + 2)\cdot(l_4 - m_1)\cdot A\right)$$

$$0 \tag{27}$$

Substitution (25) in (23) ((2.25) in (2.23) in Arxiv) and comparing with required equation :

$$simplify\left(\frac{m_3\cdot(m_1 + 1)}{4}\cdot\left(2\,m_1\left(m_2\,(2 + l_1)\,l_2 + \left(-m_2^2 - 2\,m_2 - 2\right)m_1 - 2\,l_1 - 4\right)\alpha + 2\,l_4\left(\left(l_3^2 + 2\,l_3 + 2\right)l_4 - (m_4 + 2)\,(l_3\,m_3 - 2)\right)\right) - (l_4 - m_1)\cdot\left(A + (m_3 + 1)\cdot m_1\cdot B\right)\right)$$

$$0 \tag{28}$$

:= :=
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Section 7

Expression $A + (m_1 - m_3) \cdot B$:

$factor(A + (m_1 - m_3) \cdot B)$

$$(m_3 + 1)\ (l_1\, l_4 - m_2\, m_3) \tag{29}$$