Area Preserving Combescure Transformations
(Auxiliary computations)

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Here, we check the auxiliary computations of the proofs.

restart;

Section 1

Changing variables l_i and m_i for $i = 1, 2, 3, 4$ by:

$l_1 := \frac{\mu_1 - 1}{\lambda_1}$; $m_1 := \frac{\lambda_1 - 1}{\mu_1}$; $l_2 := \frac{\mu_2 - 1}{\lambda_2}$; $m_2 := \frac{\lambda_2 - 1}{\mu_2}$;
$l_3 := \frac{\mu_3 - 1}{\lambda_3}$; $m_3 := \frac{\lambda_3 - 1}{\mu_3}$; $l_4 := \frac{\mu_4 - 1}{\lambda_4}$; $m_4 := \frac{\lambda_4 - 1}{\mu_4}$;

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}$$

$$m_1 := \frac{\lambda_1 - 1}{\mu_1}$$

$$l_2 := \frac{\mu_2 - 1}{\lambda_2}$$

$$m_2 := \frac{\lambda_2 - 1}{\mu_2}$$

$$l_3 := \frac{\mu_3 - 1}{\lambda_3}$$

$$m_3 := \frac{\lambda_3 - 1}{\mu_3}$$

$$l_4 := \frac{\mu_4 - 1}{\lambda_4}$$

$$m_4 := \frac{\lambda_4 - 1}{\mu_4} \qquad \textbf{(1)}$$

Equating equations in Lemma 2.14(b) to $1/k_i^2$ for $i = 1, 2, 3, 4$:

$$simplify\left(\frac{1 - l_1 \cdot m_1}{(1 + m_1)^2}\right) = \frac{1}{k_1^2}; simplify\left(\frac{1 - l_1 \cdot m_1}{(1 + l_1)^2}\right) = \frac{1}{k_2^2}; simplify\left(\frac{1 - l_2 \cdot m_2}{(1 + m_2)^2}\right) = \frac{1}{k_2^2};$$
$$simplify\left(\frac{1 - l_2 \cdot m_2}{(1 + l_2)^2}\right) = \frac{1}{k_3^2}; simplify\left(\frac{1 - l_3 \cdot m_3}{(1 + m_3)^2}\right) = \frac{1}{k_3^2}; simplify\left(\frac{1 - l_3 \cdot m_3}{(1 + l_3)^2}\right) = \frac{1}{k_4^2};$$
$$simplify\left(\frac{1 - l_4 \cdot m_4}{(1 + m_4)^2}\right) = \frac{1}{k_4^2}; simplify\left(\frac{1 - l_4 \cdot m_4}{(1 + l_4)^2}\right) = \frac{1}{k_1^2};$$

$$\frac{\mu_1}{(\lambda_1 + \mu_1 - 1)\,\lambda_1} = \frac{1}{k_1^2}$$

$$\frac{\lambda_1}{(\lambda_1 + \mu_1 - 1)\,\mu_1} = \frac{1}{k_2^2}$$

$$\frac{\mu_2}{(\lambda_2 + \mu_2 - 1)\,\lambda_2} = \frac{1}{k_2^2}$$

$$\frac{\lambda_2}{(\lambda_2 + \mu_2 - 1)\,\mu_2} = \frac{1}{k_3^2}$$

$$\frac{\mu_3}{(\lambda_3 + \mu_3 - 1)\,\lambda_3} = \frac{1}{k_3^2}$$

$$\frac{\lambda_3}{(\lambda_3 + \mu_3 - 1)\,\mu_3} = \frac{1}{k_4^2}$$

$$\frac{\mu_4}{(\lambda_4 + \mu_4 - 1)\,\lambda_4} = \frac{1}{k_4^2}$$

$$\frac{\lambda_4}{(\lambda_4 + \mu_4 - 1)\,\mu_4} = \frac{1}{k_1^2} \qquad \textbf{(2)}$$

Note that $\mu_i / \lambda_i > 0$ because of $l_i + 1 > 0,\ m_i + 1 > 0,$ where $i = 1, 2, 3, 4.$

In the system (2), we devide equation $1-2, 3-4, 5-6, 7-8$ and taking square root, we obtain :

$$\frac{\mu_1}{\lambda_1}=\frac{k_2}{k_1};\ \frac{\mu_2}{\lambda_2}=\frac{k_3}{k_2};\ \frac{\mu_3}{\lambda_3}=\frac{k_4}{k_3};\ \frac{\mu_4}{\lambda_4}=\frac{k_1}{k_4};$$

$$\frac{\mu_1}{\lambda_1}=\frac{k_2}{k_1}$$

$$\frac{\mu_2}{\lambda_2}=\frac{k_3}{k_2}$$

$$\frac{\mu_3}{\lambda_3}=\frac{k_4}{k_3}$$

$$\frac{\mu_4}{\lambda_4}=\frac{k_1}{k_4} \qquad \textbf{(3)}$$

Then equations of the system (2) becomes linear :

$$\mu_1=\frac{k_2}{k_1}\cdot\lambda_1;\ \mu_1+\lambda_1-1=k_1\cdot k_2;\ \mu_2=\frac{k_3}{k_2}\cdot\lambda_2;\ \mu_2+\lambda_2-1=k_2\cdot k_3;$$

$$\mu_3=\frac{k_4}{k_3}\cdot\lambda_3;\ \mu_3+\lambda_3-1=k_3\cdot k_4;\ \mu_4=\frac{k_1}{k_4}\cdot\lambda_4;\ \mu_4+\lambda_4-1=k_4\cdot k_1;$$

$$\mu_1=\frac{k_2\,\lambda_1}{k_1}$$

$$\lambda_1+\mu_1-1=k_1\,k_2$$

$$\mu_2=\frac{k_3\,\lambda_2}{k_2}$$

$$\lambda_2+\mu_2-1=k_2\,k_3$$

$$\mu_3=\frac{k_4\,\lambda_3}{k_3}$$

$$\lambda_3+\mu_3-1=k_3\,k_4$$

$$\mu_4 = \frac{k_1 \lambda_4}{k_4}$$

$$\lambda_4 + \mu_4 - 1 = k_4 k_1 \tag{4}$$

It has a solution :

$$solve\left(\left\{\mu_1 = \frac{k_2}{k_1} \cdot \lambda_1, \mu_1 + \lambda_1 - 1 = k_1 \cdot k_2, \mu_2 = \frac{k_3}{k_2} \cdot \lambda_2, \mu_2 + \lambda_2 - 1 = k_2 \cdot k_3, \mu_3 = \frac{k_4}{k_3} \cdot \lambda_3, \mu_3 + \lambda_3 - 1 = k_3 \cdot k_4, \mu_4 = \frac{k_1}{k_4} \cdot \lambda_4, \mu_4 + \lambda_4 - 1 = k_4 \cdot k_1\right\}, \left\{\mu_1, \lambda_1, \mu_2, \lambda_2, \mu_3, \lambda_3, \mu_4, \lambda_4\right\}\right)$$

$$\left\{\lambda_1 = \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}, \lambda_2 = \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}, \lambda_3 = \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}, \lambda_4 = \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4}, \mu_1 = \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}, \mu_2 = \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}, \mu_3 = \frac{k_4 (k_3 k_4 + 1)}{k_3 + k_4}, \mu_4 = \frac{k_1 (k_4 k_1 + 1)}{k_1 + k_4}\right\} \tag{5}$$

Setting :

$$\lambda_1 := \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}; \lambda_2 := \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}; \lambda_3 := \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}; \lambda_4 := \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4};$$
$$\mu_1 := \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}; \mu_2 := \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}; \mu_3 := \frac{k_4 (k_3 k_4 + 1)}{k_3 + k_4}; \mu_4 := \frac{k_1 (k_4 k_1 + 1)}{k_1 + k_4}$$

$$\lambda_1 := \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}$$

$$\lambda_2 := \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}$$

$$\lambda_3 := \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}$$

$$\lambda_4 := \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4}$$

$$\mu_1 := \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}$$

$$\mu_2 := \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}$$

$$\mu_3 := \frac{k_4\left(k_3 k_4 + 1\right)}{k_3 + k_4}$$

$$\mu_4 := \frac{k_1\left(k_4 k_1 + 1\right)}{k_1 + k_4} \tag{6}$$

Then the value of l_i, m_i where $i = 1, 2, 3, 4$ is :

$l_1 := simplify\left(\frac{\mu_1 - 1}{\lambda_1}\right)$; $m_1 := simplify\left(\frac{\lambda_1 - 1}{\mu_1}\right)$; $l_2 := simplify\left(\frac{\mu_2 - 1}{\lambda_2}\right)$;
$m_2 := simplify\left(\frac{\lambda_2 - 1}{\mu_2}\right)$; $l_3 := simplify\left(\frac{\mu_3 - 1}{\lambda_3}\right)$; $m_3 := simplify\left(\frac{\lambda_3 - 1}{\mu_3}\right)$;
$l_4 := simplify\left(\frac{\mu_4 - 1}{\lambda_4}\right)$; $m_4 := simplify\left(\frac{\lambda_4 - 1}{\mu_4}\right)$;

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}$$

$$m_1 := \frac{\lambda_1 - 1}{\mu_1}$$

$$l_2 := \frac{\mu_2 - 1}{\lambda_2}$$

$$m_2 := \frac{\lambda_2 - 1}{\mu_2}$$

$$l_3 := \frac{\mu_3 - 1}{\lambda_3}$$

$$m_3 := \frac{\lambda_3 - 1}{\mu_3}$$

$$l_4 := \frac{\mu_4 - 1}{\lambda_4}$$

$$m_4 := \frac{\lambda_4 - 1}{\mu_4} \tag{7}$$

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restart;

$with(PolynomialTools):$
$with(LinearAlgebra):$

Section 2

Equations in Lemma 2.12 are :

$P_1 := l_1 \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + m_1 \cdot x_2^2 - (l_1 + m_1 + 2)$

$$P_1 := l_1 x_1^2 + m_1 x_2^2 + 2 x_1 x_2 - l_1 - m_1 - 2 \tag{8}$$

$P_2 := l_2 \cdot x_2^2 + 2 \cdot x_2 \cdot x_3 + m_2 \cdot x_3^2 - (l_2 + m_2 + 2)$

$$P_2 := l_2 x_2^2 + m_2 x_3^2 + 2 x_2 x_3 - l_2 - m_2 - 2 \tag{9}$$

$P_3 := l_3 \cdot x_3^2 + 2 \cdot x_3 \cdot x_4 + m_3 \cdot x_4^2 - (l_3 + m_3 + 2)$

$$P_3 := l_3 x_3^2 + m_3 x_4^2 + 2 x_3 x_4 - l_3 - m_3 - 2 \tag{10}$$

$P_4 := l_4 \cdot x_4^2 + 2 \cdot x_4 \cdot x_1 + m_4 \cdot x_1^2 - (l_4 + m_4 + 2)$

$$P_4 := l_4 x_4^2 + m_4 x_1^2 + 2 x_4 x_1 - l_4 - m_4 - 2 \tag{11}$$

Setting the functions :

$x_1 := \frac{(1+t)^2 \cdot (1-k_1) + 1 + k_1}{2 \cdot (1+t)}; x_2 := \frac{(1+t)^2 \cdot (1+k_2) + 1 - k_2}{2 \cdot (1+t)};$
$x_3 := \frac{(1+t)^2 \cdot (1-k_3) + 1 + k_3}{2 \cdot (1+t)}; x_4 := \frac{(1+t)^2 \cdot (1+k_4) + 1 - k_4}{2 \cdot (1+t)}$

$$x_1 := \frac{(1+t)^2 (1-k_1) + 1 + k_1}{2 + 2t}$$

$$x_2 := \frac{(1+t)^2 (1+k_2) + 1 - k_2}{2 + 2t}$$

$$x_3 := \frac{(1+t)^2 (1-k_3) + 1 + k_3}{2 + 2t}$$

$$x_4 := \frac{(1+t)^2 (1+k_4) + 1 - k_4}{2 + 2t} \tag{12}$$

Verification of the equations in Lemma 2.12 :

$simplify(P_1); simplify(P_2); simplify(P_3); simplify(P_4)$

$$0$$
$$0$$
$$0$$
$$0 \quad \textbf{(13)}$$

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Section 3

Left side of equation (2.9) :

$$eq := collect\Big(\big((l_2 + m_2 + 2) \cdot (l_1 \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + m_1 \cdot x_2^2 - (l_1 + m_1 + 2)) - (l_1 + m_1 + 2) \cdot ((l_2 \cdot x_2^2 + 2 \cdot x_2 \cdot x_3 + m_2 \cdot x_3^2 - (l_2 + m_2 + 2))) \big), \{x_1, x_2, x_3\}, distributed \Big)$$

$$eq := (l_2 + m_2 + 2)\, l_1\, x_1^2 + (2\, l_2 + 2\, m_2 + 4)\, x_2\, x_1 + ((l_2 + m_2 + 2)\, m_1 - (l_1 + m_1 + 2)\, l_2)\, x_2^2 + (-2\, l_1 - 2\, m_1 - 4)\, x_3\, x_2 - (l_1 + m_1 + 2)\, m_2\, x_3^2 + (l_2 + m_2 + 2)\,(-l_1 - m_1 - 2) - (l_1 + m_1 + 2)\,(-l_2 - m_2 - 2) \quad \textbf{(14)}$$

Matrix of coefficients of the left side of equation (2.9) :

$$M := Matrix\left(\left[\left[(l_2 + m_2 + 2)\, l_1, \frac{(2 \cdot l_2 + 2 \cdot m_2 + 4)}{2}, 0\right], \left[\frac{(2 \cdot l_2 + 2 \cdot m_2 + 4)}{2}, -l_1\, l_2 + m_1\, m_2 - 2\, l_2 + 2\, m_1, \frac{(-2\, l_1 - 2\, m_1 - 4)}{2}\right], \left[0, \frac{(-2\, l_1 - 2\, m_1 - 4)}{2}, -l_1\, m_2 - m_2\,(m_1 + 2)\right]\right]\right)$$

$$M := \begin{bmatrix} (l_2 + m_2 + 2)\, l_1 & l_2 + m_2 + 2 & 0 \\ l_2 + m_2 + 2 & -l_1\, l_2 + m_1\, m_2 - 2\, l_2 + 2\, m_1 & -l_1 - m_1 - 2 \\ 0 & -l_1 - m_1 - 2 & -l_1\, m_2 - m_2\,(m_1 + 2) \end{bmatrix} \quad \textbf{(15)}$$

Verifying that the det(M) and left side of (2.10) are the same :

$$simplify\Big(Determinant(M) - (l_1 + m_1 + 2)\cdot(l_2 + m_2 + 2)\cdot\big((m_2 + 1)^2\cdot(1 - m_1\cdot l_1) - (l_1 + 1)^2\cdot(1 - l_2\cdot m_2)\big)\Big)$$

$$0 \tag{16}$$

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Section 4

Equation (2.11) :

$$Eq\_2\_11 := expand\Big(\big(Determinant\big(Matrix\big(\big[[m_1, 0, l_2, 0], [2\cdot x_1, m_1, 2\cdot x_3, l_2], [l_1\cdot x_1^2 - (l_1 + m_1 + 2), 2\cdot x_1, m_2\cdot x_3^2 - (l_2 + m_2 + 2), 2\cdot x_3], [0, l_1\cdot x_1^2 - (l_1 + m_1 + 2), 0, m_2\cdot x_3^2 - (l_2 + m_2 + 2)]\big]\big)\big)\big)\Big)$$

$$Eq\_2\_11 := l_1^2 l_2^2 x_1^4 - 2 l_1 l_2 m_1 m_2 x_1^2 x_3^2 + m_1^2 m_2^2 x_3^4 - 2 l_1^2 l_2^2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_3^2 - 4 l_1 l_2 x_1^3 x_3 + 4 l_1 m_1 x_1^2 x_3^2 + 4 l_2 m_2 x_1^2 x_3^2 - 2 m_1^2 m_2^2 x_3^2 - 4 m_1 m_2 x_1 x_3^3 - 4 l_1 l_2^2 x_1^2 + 4 l_1 l_2 m_1 x_1^2 + 4 l_2 m_1 m_2 x_3^2 - 4 m_1^2 m_2 x_3^2 + l_1^2 l_2^2 - 2 l_1 l_2 m_1 m_2 + 4 l_1 l_2 x_1 x_3 - 4 l_1 m_1 x_3^2 - 4 l_2^2 x_1^2 + 8 l_2 m_1 x_1 x_3 - 4 l_2 m_2 x_1^2 + m_1^2 m_2^2 - 4 m_1^2 x_3^2 + 4 m_1 m_2 x_1 x_3 + 4 l_1 l_2^2 - 4 l_1 l_2 m_1 - 4 l_2 m_1 m_2 - 8 l_2 x_1^2 + 8 l_2 x_1 x_3 + 4 m_1^2 m_2 + 8 m_1 x_1 x_3 - 8 m_1 x_3^2 + 4 l_2^2 - 8 m_1 l_2 + 4 m_1^2 \tag{17}$$

Expression Q in equation (2.13) :

$$Q := l_1\cdot l_2\cdot x_1^2 - m_1\cdot m_2\cdot x_3^2 + m_1\cdot m_2 - l_1\cdot l_2 + 2\cdot(m_1 - l_2)$$

$$Q := l_1 l_2 x_1^2 - m_1 m_2 x_3^2 - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 \tag{18}$$

Equation (2.14) :

$$Eq\_2\_14 := 4\cdot(l_1\cdot x_1^2 - l_1 - m_1 - 2)\cdot(l_2\cdot x_1 - m_1\cdot x_3)^2 - 4\cdot x_1\cdot(l_2\cdot x_1 - m_1\cdot x_3)\cdot Q + m_1\cdot Q^2$$

$$Eq\_2\_14 := 4 (l_1 x_1^2 - l_1 - m_1 - 2) (l_2 x_1 - m_1 x_3)^2 - 4 x_1 (l_2 x_1 - m_1 x_3) (l_1 l_2 x_1^2 - m_1 m_2 x_3^2 \tag{19}$$

$$- l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1\big) + m_1 \left(l_1 l_2 x_1^2 - m_1 m_2 x_3^2 - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1\right)^2$$

Verifying that the equations (2.11) and (2.14) are equivalent :

$simplify\left(m_1 \cdot Eq\_2\_11 - Eq\_2\_14\right)$

$$0 \tag{20}$$

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Section 5

Equation (2.11) :

$Eq\_2\_11 := simplify\Big(Determinant\Big(Matrix\Big(\Big[\big[m_1, 0, l_2, 0\big], \big[2 \cdot x_1, m_1, 2 \cdot x_3, l_2\big], \big[l_1 \cdot x_1^{\,2} - \left(l_1 + m_1 + 2\right), 2 \cdot x_1, m_2 \cdot x_3^{\,2} - \left(l_2 + m_2 + 2\right), 2 \cdot x_3\big], \big[0, l_1 \cdot x_1^{\,2} - \left(l_1 + m_1 + 2\right), 0, m_2 \cdot x_3^{\,2} - \left(l_2 + m_2 + 2\right)\big]\Big]\Big)\Big)\Big)$

$$Eq\_2\_11 := \left(l_1^2 x_1^4 + \left(-2 l_1^2 - 4 l_1 - 4\right) x_1^2 + \left(2 + l_1\right)^2\right) l_2^2 + \Big(\Big(-2 l_1 \left(m_2 x_3^2 - m_2 - 2\right) x_1^2 + 8 x_1 x_3 + 2 \left(m_2 x_3^2 - m_2 - 2\right) \left(2 + l_1\right)\Big) m_1 - 4 x_1 \Big(l_1 x_1^2 x_3 + \left(-m_2 x_3^2 + m_2 + 2\right) x_1 - x_3 \left(2 + l_1\right)\Big)\Big) l_2 + 4 m_1 \left(\frac{\left(x_3 - 1\right)\left(x_3 + 1\right)\left(m_2 x_3 + m_2 + 2\right)\left(m_2 x_3 - m_2 - 2\right) m_1}{4} + \left(l_1 x_1^2 x_3 + \left(-m_2 x_3^2 + m_2 + 2\right) x_1 - x_3 \left(2 + l_1\right)\right) x_3\right) \tag{21}$$

Equation (2.11') :

$Eq\_2\_11\_pr := simplify\Big(Determinant\Big(Matrix\Big(\Big[\big[l_4, 0, m_3, 0\big], \big[2 \cdot x_1, l_4, 2 \cdot x_3, m_3\big], \big[m_4 \cdot x_1^{\,2} - \left(m_4 + l_4 + 2\right), 2 \cdot x_1, l_3 \cdot x_3^{\,2} - \left(m_3 + l_3 + 2\right), 2 \cdot x_3\big], \big[0, m_4 \cdot x_1^{\,2} - \left(m_4 + l_4 + 2\right), 0, l_3 \cdot x_3^{\,2} - \left(m_3 + l_3 + 2\right)\big]\Big]\Big)\Big)\Big)$

$$Eq\_2\_11\_pr := \left(l_3^2 x_3^4 + \left(-2 l_3^2 - 4 l_3 - 4\right) x_3^2 + \left(2 + l_3\right)^2\right) l_4^2 + \Big(\Big(-2 m_4 \left(l_3 x_3^2 - l_3 - 2\right) x_1^2 + 8 x_1 x_3 + 2 \left(m_4 + 2\right) \left(l_3 x_3^2 - l_3 - 2\right)\Big) m_3 - 4 \Big(-m_4 x_3 x_1^2 + \left(l_3 x_3^2 - l_3 - 2\right) x_1 + x_3 \big(m_4 \tag{22}$$

$$+2)\big)x_3\big)l_4+4\left(\frac{(x_1-1)(x_1+1)(m_4x_1+m_4+2)(m_4x_1-m_4-2)m_3}{4}+\big(-m_4x_3\right.$$
$$\left.x_1^2+\big(l_3x_3^2-l_3-2\big)x_1+x_3\big(m_4+2\big)\big)x_1\right)m_3$$

Comperison of coefficients of the equation $\alpha\cdot Eq\_2\_11-Eq\_2\_11\_pr$:

$Eq := collect\big(simplify(\alpha\cdot Eq\_2\_11-Eq\_2\_11\_pr),\{x_1,x_2,x_3\}, distributed\big)$

$$Eq := \big(\alpha l_1^2l_2^2-m_3^2m_4^2\big)x_1^4+\big(-4\alpha l_1l_2+4m_3m_4\big)x_3x_1^3+\Big(-2l_2\Big(\big(\big(l_1^2+2l_1+2\big)l_2-\big(m_2+2\big)\big(l_1m_1-2\big)\big)\alpha-2m_3\Big(m_4\big(2+l_3\big)l_4+\big(-m_4^2-2m_4-2\big)m_3-2l_3-4\Big)\Big)x_1^2+\Big(\Big(\big(-2l_1m_1m_2+4m_2\big)l_2+4l_1m_1\Big)\alpha+\big(2l_3m_3m_4-4m_4\big)l_4-4l_3m_3\Big)x_3^2x_1^2+\big(-4\alpha m_1m_2+4l_3l_4\big)x_1x_3^3+\Big(-4\Big(\big(-l_1-2m_1-2\big)l_2-m_1\big(m_2+2\big)\Big)\alpha-4\big(l_3+2m_3+2\big)l_4-4m_3\big(m_4+2\big)\Big)x_1x_3+\big(\alpha m_1^2m_2^2-l_3^2l_4^2\big)x_3^4+\Big(2\Big(m_2\big(2+l_1\big)l_2+\big(-m_2^2-2m_2-2\big)m_1-2l_1-4\Big)m_1\alpha+2\Big(\big(\big(l_3^2+2l_3+2\big)l_4-\big(m_4+2\big)\big(l_3m_3-2\big)\big)l_4\Big)x_3^2+\Big(\big(2+l_1\big)l_2-m_1\big(m_2+2\big)\Big)^2\alpha-\Big(\big(2+l_3\big)l_4-m_3\big(m_4+2\big)\Big)^2 \quad \mathbf{(23)}$$

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Section 6

Equation (2.25) and $\alpha=1$:

$l_2 := \dfrac{(m_3+1)\cdot(l_4+1)}{m_1+1}-1;\ l_3 := \dfrac{m_1\cdot m_2}{l_4};\ m_4 := \dfrac{l_1\cdot(m_3\cdot l_4+m_3+l_4-m_1)}{m_3\cdot(m_1+1)};\ \alpha := 1$

$$l_2 := \frac{(m_3+1)(l_4+1)}{m_1+1}-1$$

$$l_3 := \frac{m_1m_2}{l_4}$$

$$m_4 := \frac{l_1(l_4m_3+l_4-m_1+m_3)}{m_3(m_1+1)}$$

$$\alpha := 1 \quad \mathbf{(24)}$$

Expression A :

$$A := (m_3+1)\cdot(l_1+1)\cdot(l_4+1) - (m_1+1)\cdot(m_1\cdot m_2 + l_1 + l_4 + m_1 - m_3 + 1)$$

$$A := (m_3+1)\,(l_1+1)\,(l_4+1) - (m_1+1)\,(m_1\,m_2 + l_1 + l_4 + m_1 - m_3 + 1) \tag{25}$$

Expression B :

$$B := m_1\cdot m_2 + m_2\cdot m_3 + l_1 + m_2 + l_4 + m_1 + 2$$

$$B := m_1\,m_2 + m_2\,m_3 + l_1 + l_4 + m_1 + m_2 + 2 \tag{26}$$

Substitution (2.25) in (2.22) and comparing with required equation :

$$simplify\left(\frac{(m_1+1)^2}{4}\cdot\left(\left((2+l_1)\,l_2 - m_1\,(2+m_2)\right)^2\alpha - \left((2+l_3)\,l_4 - m_3\,(m_4+2)\right)^2\right) - (m_1 + m_3 + 2)\cdot(l_4 - m_1)\cdot A\right)$$

$$0 \tag{27}$$

Substitution (2.25) in (2.23) and comparing with required equation :

$$simplify\left(\frac{m_3\cdot(m_1+1)}{4}\cdot\left(2\,m_1\left(m_2\,(2+l_1)\,l_2 + \left(-m_2^2 - 2\,m_2 - 2\right)m_1 - 2\,l_1 - 4\right)\alpha + 2\,l_4\left(\left(l_3^2 + 2\,l_3 + 2\right)l_4 - (m_4+2)\,(l_3\,m_3 - 2)\right)\right) - (l_4 - m_1)\cdot\left(A + (m_3+1)\cdot m_1\cdot B\right)\right)$$

$$0 \tag{28}$$

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Section 7

Expression $A + (m_1 - m_3)\cdot B$:

$factor\left(A + \left(m_1 - m_3\right) \cdot B\right)$

$$\left(m_3 + 1\right) \left(l_1 l_4 - m_2 m_3\right) \tag{29}$$