

Area Preserving Combescure Transformations

(Auxiliary computations)

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Here, we check the auxiliary computations of the proofs.

restart;

Section 1

Changing variables l_i and m_i for $i = 1, 2, 3, 4$ by:

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}; m_1 := \frac{\lambda_1 - 1}{\mu_1}; l_2 := \frac{\mu_2 - 1}{\lambda_2}; m_2 := \frac{\lambda_2 - 1}{\mu_2};$$
$$l_3 := \frac{\mu_3 - 1}{\lambda_3}; m_3 := \frac{\lambda_3 - 1}{\mu_3}; l_4 := \frac{\mu_4 - 1}{\lambda_4}; m_4 := \frac{\lambda_4 - 1}{\mu_4};$$

$$l_1 := \frac{\mu_1 - 1}{\lambda_1}$$

$$m_1 := \frac{\lambda_1 - 1}{\mu_1}$$

$$l_2 := \frac{\mu_2 - 1}{\lambda_2}$$

$$m_2 := \frac{\lambda_2 - 1}{\mu_2}$$

$$l_3 := \frac{\mu_3 - 1}{\lambda_3}$$

$$m_3 := \frac{\lambda_3 - 1}{\mu_3}$$

$$l_4 := \frac{\mu_4 - 1}{\lambda_4}$$

$$m_4 := \frac{\lambda_4 - 1}{\mu_4}$$

(1)

Equating equations in Lemma 2 (ii) (Lemma 2.14(ii) in Arxiv) to $1/k_i^2$ for $i = 1, 2, 3, 4$:

$$\begin{aligned} \text{simplify} \left(\frac{1 - l_1 \cdot m_1}{(1 + m_1)^2} \right) &= \frac{1}{k_1^2}; \text{simplify} \left(\frac{1 - l_1 \cdot m_1}{(1 + l_1)^2} \right) = \frac{1}{k_2^2}; \text{simplify} \left(\frac{1 - l_2 \cdot m_2}{(1 + m_2)^2} \right) = \frac{1}{k_2^2}; \\ \text{simplify} \left(\frac{1 - l_2 \cdot m_2}{(1 + l_2)^2} \right) &= \frac{1}{k_3^2}; \text{simplify} \left(\frac{1 - l_3 \cdot m_3}{(1 + m_3)^2} \right) = \frac{1}{k_3^2}; \text{simplify} \left(\frac{1 - l_3 \cdot m_3}{(1 + l_3)^2} \right) = \frac{1}{k_4^2}; \\ \text{simplify} \left(\frac{1 - l_4 \cdot m_4}{(1 + m_4)^2} \right) &= \frac{1}{k_4^2}; \text{simplify} \left(\frac{1 - l_4 \cdot m_4}{(1 + l_4)^2} \right) = \frac{1}{k_1^2}; \end{aligned}$$

$$\frac{\mu_1}{(\lambda_1 + \mu_1 - 1) \lambda_1} = \frac{1}{k_1^2}$$

$$\frac{\lambda_1}{(\lambda_1 + \mu_1 - 1) \mu_1} = \frac{1}{k_2^2}$$

$$\frac{\mu_2}{(\lambda_2 + \mu_2 - 1) \lambda_2} = \frac{1}{k_2^2}$$

$$\frac{\lambda_2}{(\lambda_2 + \mu_2 - 1) \mu_2} = \frac{1}{k_3^2}$$

$$\frac{\mu_3}{(\lambda_3 + \mu_3 - 1) \lambda_3} = \frac{1}{k_3^2}$$

$$\frac{\lambda_3}{(\lambda_3 + \mu_3 - 1) \mu_3} = \frac{1}{k_4^2}$$

$$\frac{\mu_4}{(\lambda_4 + \mu_4 - 1) \lambda_4} = \frac{1}{k_4^2}$$

$$\frac{\lambda_4}{(\lambda_4 + \mu_4 - 1) \mu_4} = \frac{1}{k_1^2}$$

(2)

Note that $\mu_i/\lambda_i > 0$ because of $l_i + 1 > 0$, $m_i + 1 > 0$, where $i = 1, 2, 3, 4$.

In the above system (2), we divide equation 1 – 2, 3 – 4, 5 – 6, 7 – 8 and taking square root, we obtain :

$$\frac{\mu_1}{\lambda_1} = \frac{k_2}{k_1}; \frac{\mu_2}{\lambda_2} = \frac{k_3}{k_2}; \frac{\mu_3}{\lambda_3} = \frac{k_4}{k_3}; \frac{\mu_4}{\lambda_4} = \frac{k_1}{k_4};$$

$$\frac{\mu_1}{\lambda_1} = \frac{k_2}{k_1}$$

$$\frac{\mu_2}{\lambda_2} = \frac{k_3}{k_2}$$

$$\frac{\mu_3}{\lambda_3} = \frac{k_4}{k_3}$$

$$\frac{\mu_4}{\lambda_4} = \frac{k_1}{k_4}$$

(3)

Then equations of the above system (2) becomes linear :

$$\mu_1 = \frac{k_2}{k_1} \cdot \lambda_1; \mu_1 + \lambda_1 - 1 = k_1 \cdot k_2; \mu_2 = \frac{k_3}{k_2} \cdot \lambda_2; \mu_2 + \lambda_2 - 1 = k_2 \cdot k_3;$$

$$\mu_3 = \frac{k_4}{k_3} \cdot \lambda_3; \mu_3 + \lambda_3 - 1 = k_3 \cdot k_4; \mu_4 = \frac{k_1}{k_4} \cdot \lambda_4; \mu_4 + \lambda_4 - 1 = k_4 \cdot k_1;$$

$$\mu_1 = \frac{k_2 \lambda_1}{k_1}$$

$$\lambda_1 + \mu_1 - 1 = k_1 k_2$$

$$\mu_2 = \frac{k_3 \lambda_2}{k_2}$$

$$\lambda_2 + \mu_2 - 1 = k_2 k_3$$

$$\mu_3 = \frac{k_4 \lambda_3}{k_3}$$

$$\lambda_3 + \mu_3 - 1 = k_3 k_4$$

$$\mu_4 = \frac{k_1 \lambda_4}{k_4}$$

$$\lambda_4 + \mu_4 - 1 = k_4 k_1 \quad (4)$$

It has a solution :

$$\text{solve} \left(\left\{ \mu_1 = \frac{k_2}{k_1} \cdot \lambda_1, \mu_1 + \lambda_1 - 1 = k_1 \cdot k_2, \mu_2 = \frac{k_3}{k_2} \cdot \lambda_2, \mu_2 + \lambda_2 - 1 = k_2 \cdot k_3, \mu_3 = \frac{k_4}{k_3} \cdot \lambda_3, \mu_3 + \lambda_3 - 1 = k_3 \cdot k_4, \mu_4 = \frac{k_1}{k_4} \cdot \lambda_4, \mu_4 + \lambda_4 - 1 = k_4 \cdot k_1 \right\}, \left\{ \mu_1, \lambda_1, \mu_2, \lambda_2, \mu_3, \lambda_3, \mu_4, \lambda_4 \right\} \right)$$

$$\left\{ \lambda_1 = \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}, \lambda_2 = \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}, \lambda_3 = \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}, \lambda_4 = \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4}, \mu_1 = \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}, \mu_2 = \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}, \mu_3 = \frac{k_4 (k_3 k_4 + 1)}{k_3 + k_4}, \mu_4 = \frac{k_1 (k_4 k_1 + 1)}{k_1 + k_4} \right\} \quad (5)$$

Setting :

$$\lambda_1 := \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}; \lambda_2 := \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}; \lambda_3 := \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}; \lambda_4 := \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4};$$

$$\mu_1 := \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}; \mu_2 := \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}; \mu_3 := \frac{k_4 (k_3 k_4 + 1)}{k_3 + k_4}; \mu_4 := \frac{k_1 (k_4 k_1 + 1)}{k_1 + k_4}$$

$$\lambda_1 := \frac{k_1 (k_1 k_2 + 1)}{k_1 + k_2}$$

$$\lambda_2 := \frac{k_2 (k_2 k_3 + 1)}{k_2 + k_3}$$

$$\lambda_3 := \frac{k_3 (k_3 k_4 + 1)}{k_3 + k_4}$$

$$\lambda_4 := \frac{k_4 (k_4 k_1 + 1)}{k_1 + k_4}$$

$$\mu_1 := \frac{k_2 (k_1 k_2 + 1)}{k_1 + k_2}$$

$$\mu_2 := \frac{k_3 (k_2 k_3 + 1)}{k_2 + k_3}$$

restart;

with(PolynomialTools) :

with(LinearAlgebra) :

Section 2

Equations in Lemma 1 (Lemma 2.12 in Arxiv) are :

$$\begin{aligned} P_1 &:= l_1 \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + m_1 \cdot x_2^2 - (l_1 + m_1 + 2) \\ P_1 &:= l_1 x_1^2 + m_1 x_2^2 + 2 x_1 x_2 - l_1 - m_1 - 2 \end{aligned} \quad (8)$$

$$\begin{aligned} P_2 &:= l_2 \cdot x_2^2 + 2 \cdot x_2 \cdot x_3 + m_2 \cdot x_3^2 - (l_2 + m_2 + 2) \\ P_2 &:= l_2 x_2^2 + m_2 x_3^2 + 2 x_2 x_3 - l_2 - m_2 - 2 \end{aligned} \quad (9)$$

$$\begin{aligned} P_3 &:= l_3 \cdot x_3^2 + 2 \cdot x_3 \cdot x_4 + m_3 \cdot x_4^2 - (l_3 + m_3 + 2) \\ P_3 &:= l_3 x_3^2 + m_3 x_4^2 + 2 x_3 x_4 - l_3 - m_3 - 2 \end{aligned} \quad (10)$$

$$\begin{aligned} P_4 &:= l_4 \cdot x_4^2 + 2 \cdot x_4 \cdot x_1 + m_4 \cdot x_1^2 - (l_4 + m_4 + 2) \\ P_4 &:= l_4 x_4^2 + m_4 x_1^2 + 2 x_4 x_1 - l_4 - m_4 - 2 \end{aligned} \quad (11)$$

Setting the functions :

$$\begin{aligned} x_1 &:= \frac{(1+t)^2 \cdot (1-k_1) + 1 + k_1}{2 \cdot (1+t)}; x_2 := \frac{(1+t)^2 \cdot (1+k_2) + 1 - k_2}{2 \cdot (1+t)}; \\ x_3 &:= \frac{(1+t)^2 \cdot (1-k_3) + 1 + k_3}{2 \cdot (1+t)}; x_4 := \frac{(1+t)^2 \cdot (1+k_4) + 1 - k_4}{2 \cdot (1+t)} \\ x_1 &:= \frac{(1+t)^2 (1-k_1) + 1 + k_1}{2 + 2t} \\ x_2 &:= \frac{(1+t)^2 (1+k_2) + 1 - k_2}{2 + 2t} \\ x_3 &:= \frac{(1+t)^2 (1-k_3) + 1 + k_3}{2 + 2t} \\ x_4 &:= \frac{(1+t)^2 (1+k_4) + 1 - k_4}{2 + 2t} \end{aligned} \quad (12)$$

Verification of the equations in Lemma 1 (Lemma 2.12 in Arxiv) :

$$\begin{aligned} & \text{simplify}(P_1); \text{simplify}(P_2); \text{simplify}(P_3); \text{simplify}(P_4) \\ & 0 \\ & 0 \\ & 0 \\ & 0 \end{aligned} \tag{13}$$

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Section 3

Left side of equation (9) (equation (2.9) in Arxiv) :

$$\begin{aligned} eq &:= \text{collect}\left(\left((l_2 + m_2 + 2) \cdot (l_1 \cdot x_1^2 + 2 \cdot x_1 \cdot x_2 + m_1 \cdot x_2^2 - (l_1 + m_1 + 2)) - (l_1 + m_1 + 2) \cdot (l_2 \cdot x_2^2 \right. \right. \\ &\quad \left. \left. + 2 \cdot x_2 \cdot x_3 + m_2 \cdot x_3^2 - (l_2 + m_2 + 2))\right), \{x_1, x_2, x_3\}, \text{distributed}\right) \\ eq &:= (l_2 + m_2 + 2) l_1 x_1^2 + (2 l_2 + 2 m_2 + 4) x_2 x_1 + ((l_2 + m_2 + 2) m_1 - (l_1 + m_1 + 2) l_2) x_2^2 \tag{14} \\ &\quad + (-2 l_1 - 2 m_1 - 4) x_3 x_2 - (l_1 + m_1 + 2) m_2 x_3^2 + (l_2 + m_2 + 2) (-l_1 - m_1 - 2) - (l_1 \\ &\quad + m_1 + 2) (-l_2 - m_2 - 2) \end{aligned}$$

Matrix of coefficients of the left side of equation (9) (equation (2.9) in Arxiv) :

$$\begin{aligned} M &:= \text{Matrix}\left(\left[\left[\left(l_2 + m_2 + 2\right) l_1, \frac{(2 \cdot l_2 + 2 \cdot m_2 + 4)}{2}, 0\right], \left[\frac{(2 \cdot l_2 + 2 \cdot m_2 + 4)}{2}, -l_1 l_2 + m_1 m_2 \right. \right. \right. \\ &\quad \left. \left. -2 l_2 + 2 m_1, \frac{(-2 l_1 - 2 m_1 - 4)}{2}\right], \left[0, \frac{(-2 l_1 - 2 m_1 - 4)}{2}, -l_1 m_2 - m_2 (m_1 + 2)\right]\right]\right) \\ M &:= \begin{bmatrix} (l_2 + m_2 + 2) l_1 & l_2 + m_2 + 2 & 0 \\ l_2 + m_2 + 2 & -l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 & -l_1 - m_1 - 2 \\ 0 & -l_1 - m_1 - 2 & -l_1 m_2 - m_2 (m_1 + 2) \end{bmatrix} \end{aligned} \tag{15}$$

Verifying that the $\det(M)$ and left side of (10) (equation (2.10))

in Arxiv) are the same :

$$\begin{aligned} & \text{simplify}\left(\text{Determinant}(M) - (l_1 + m_1 + 2) \cdot (l_2 + m_2 + 2) \cdot ((m_2 + 1)^2 \cdot (1 - m_1 \cdot l_1) - (l_1 + 1)^2\right. \\ & \quad \left.\cdot (1 - l_2 \cdot m_2))\right) \end{aligned}$$

$\mathbf{(16)}$

[illegible]

Section 4

Equation (11) (equation (2.11) in Arxiv) :

$$Eq_2_11 := expand\left(\left(Determinant\left(Matrix\left(\left[\left[m_1, 0, l_2, 0\right], \left[2 \cdot x_1, m_1, 2 \cdot x_3, l_2\right], \left[l_1 \cdot x_1^2 - (l_1 + m_1 + 2), 2 \cdot x_1, m_2 \cdot x_3^2 - (l_2 + m_2 + 2), 2 \cdot x_3\right], \left[0, l_1 \cdot x_1^2 - (l_1 + m_1 + 2), 0, m_2 \cdot x_3^2 - (l_2 + m_2 + 2)\right]\right]\right)\right)\right)\right)$$

$$Eq_2_11 := l_1^2 l_2^2 x_1^4 - 2 l_1 l_2 m_1 m_2 x_1^2 x_3^2 + m_1^2 m_2^2 x_3^4 - 2 l_1^2 l_2^2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_1^2 + 2 l_1 l_2 m_1 m_2 x_3^2 \quad (17)$$

$$- 4 l_1 l_2 x_1^3 x_3 + 4 l_1 m_1 x_1^2 x_3^2 + 4 l_2 m_2 x_1^2 x_3^2 - 2 m_1^2 m_2^2 x_3^2 - 4 m_1 m_2 x_1 x_3^3 - 4 l_1 l_2^2 x_1^2 + 4 l_1 l_2 m_1 x_1^2 + 4 l_2 m_1 m_2 x_3^2 - 4 m_1^2 m_2^2 x_3^2 + l_1^2 l_2^2 - 2 l_1 l_2 m_1 m_2 + 4 l_1 l_2 x_1 x_3 - 4 l_1 m_1 x_3^2 - 4 l_2^2 x_1^2 + 8 l_2 m_1 x_1 x_3 - 4 l_2 m_2 x_1^2 + m_1^2 m_2^2 - 4 m_1^2 x_3^2 + 4 m_1 m_2 x_1 x_3 + 4 l_1 l_2^2 - 4 l_1 l_2 m_1 - 4 l_2 m_1 m_2 - 8 l_2 x_1^2 + 8 l_2 x_1 x_3 + 4 m_1^2 m_2 + 8 m_1 x_1 x_3 - 8 m_1 x_3^2 + 4 l_2^2 - 8 m_1 l_2 + 4 m_1^2$$

Expression Q in equation (13) (equation (2.13) in Arxiv) :

$$\begin{aligned} Q &:= l_1 \cdot l_2 \cdot x_1^2 - m_1 \cdot m_2 \cdot x_3^2 + m_1 \cdot m_2 - l_1 \cdot l_2 + 2 \cdot (m_1 - l_2) \\ Q &:= l_1 l_2 x_1^2 - m_1 m_2 x_3^2 - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 \end{aligned} \quad (18)$$

Equation (14) (equation (2.14) in Arxiv) :

$$\begin{aligned}
Eq_2_14 &:= 4 \cdot (l_1 \cdot x_1^2 - l_1 - m_1 - 2) \cdot (l_2 \cdot x_1 - m_1 \cdot x_3)^2 - 4 \cdot x_1 \cdot (l_2 \cdot x_1 - m_1 \cdot x_3) \cdot Q + m_1 \cdot Q^2 \\
Eq_2_14 &:= 4 \left(l_1 x_1^2 - l_1 - m_1 - 2 \right) \left(l_2 x_1 - m_1 x_3 \right)^2 - 4 x_1 \left(l_2 x_1 - m_1 x_3 \right) \left(l_1 l_2 x_1^2 - m_1 m_2 x_3^2 \right. \\
&\quad \left. - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 \right) + m_1 \left(l_1 l_2 x_1^2 - m_1 m_2 x_3^2 - l_1 l_2 + m_1 m_2 - 2 l_2 + 2 m_1 \right)^2
\end{aligned} \tag{19}$$

Verifying that the equations (11) and (14) (equations (2.11) and (2.14) in Arxiv) are equivalent:

$$\text{simplify}(m_1 \cdot \textcolor{red}{Eq_2_11} - \textcolor{blue}{Eq_2_14}) \quad \quad \quad \mathbf{0} \quad \quad \quad \mathbf{(20)}$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

Section 5

Equation (11) (equation (2.11) in Arxiv) :

$$Eq_{2_11} := simplify\left(Determinant\left(Matrix\left(\left[\left[m_1, 0, l_2, 0\right], \left[2 \cdot x_1, m_1, 2 \cdot x_3, l_2\right], \left[l_1 \cdot x_1^2 - (l_1 + m_1 + 2), 2 \cdot x_1, m_2 \cdot x_3^2 - (l_2 + m_2 + 2), 2 \cdot x_3\right], \left[0, l_1 \cdot x_1^2 - (l_1 + m_1 + 2), 0, m_2 \cdot x_3^2 - (l_2 + m_2 + 2)\right]\right]\right)\right)\right)$$

$$Eq_{2_11} := \left(l_1^2 x_1^4 + (-2 l_1^2 - 4 l_1 - 4) x_1^2 + (2 + l_1)^2\right) l_2^2 + \left((-2 l_1 (m_2 x_3^2 - m_2 - 2) x_1^2 + 8 x_1 x_3 + 2 (m_2 x_3^2 - m_2 - 2) (2 + l_1)) m_1 - 4 x_1 (l_1 x_1^2 x_3 + (-m_2 x_3^2 + m_2 + 2) x_1 - x_3 (2 + l_1))\right) l_2 + 4 m_1 \left(\frac{(x_3 - 1) (x_3 + 1) (m_2 x_3 + m_2 + 2) (m_2 x_3 - m_2 - 2) m_1}{4} + (l_1 x_1^2 x_3 + (-m_2 x_3^2 + m_2 + 2) x_1 - x_3 (2 + l_1)) x_3\right) \quad (21)$$

Equation (11') (equation (2.11') in Arxiv) :

$$Eq_2_11_pr := simplify\left(Determinant\left(Matrix\left(\left[\left[l_4, 0, m_3, 0\right], \left[2 \cdot x_1, l_4, 2 \cdot x_3, m_3\right], \left[m_4 \cdot x_1^2 - (m_4 + l_4 + 2), 2 \cdot x_1, l_3 \cdot x_3^2 - (m_3 + l_3 + 2), 2 \cdot x_3\right], \left[0, m_4 \cdot x_1^2 - (m_4 + l_4 + 2), 0, l_3 \cdot x_3^2 - (m_3 + l_3 + 2)\right]\right]\right)\right)\right)$$

$$Eq_2_11_pr := \left(l_3^2 x_3^4 + (-2 l_3^2 - 4 l_3 - 4) x_3^2 + (2 + l_3)^2\right) l_4^2 + \left((-2 m_4 (l_3 x_3^2 - l_3 - 2) x_1^2 + 8 x_1 x_3 + 2 (m_4 + 2) (l_3 x_3^2 - l_3 - 2)\right) m_3 - 4 (-m_4 x_3 x_1^2 + (l_3 x_3^2 - l_3 - 2) x_1 + x_3 (m_4 + 2)) x_3) l_4 + 4 \left(\frac{(x_1 - 1) (x_1 + 1) (m_4 x_1 + m_4 + 2) (m_4 x_1 - m_4 - 2) m_3}{4} + (-m_4 x_3\right. \quad (22)$$

$$x_1^2 + \left(l_3 x_3^2 - l_3 - 2 \right) x_1 + x_3 \left(m_4 + 2 \right) x_1 \Big) m_3$$

Comperison of coefficients of the equation $\alpha \cdot Eq_2_11 - Eq_2_11_pr$:

$$\begin{aligned} Eq &:= collect(simplify(\alpha \cdot Eq_2_11 - Eq_2_11_pr), \{x_1, x_2, x_3\}, distributed) \\ Eq &:= \left(\alpha l_1^2 l_2^2 - m_3^2 m_4^2 \right) x_1^4 + \left(-4 \alpha l_1 l_2 + 4 m_3 m_4 \right) x_3 x_1^3 + \left(-2 l_2 \left(l_1^2 + 2 l_1 + 2 \right) l_2 - \left(m_2 \right. \right. \\ &\quad \left. \left. + 2 \right) \left(l_1 m_1 - 2 \right) \right) \alpha - 2 m_3 \left(m_4 \left(2 + l_3 \right) l_4 + \left(-m_4^2 - 2 m_4 - 2 \right) m_3 - 2 l_3 - 4 \right) x_1^2 \\ &\quad + \left(\left(-2 l_1 m_1 m_2 + 4 m_2 \right) l_2 + 4 l_1 m_1 \right) \alpha + \left(2 l_3 m_3 m_4 - 4 m_4 \right) l_4 - 4 l_3 m_3 x_3^2 x_1^2 + \left(\right. \\ &\quad \left. -4 \alpha m_1 m_2 + 4 l_3 l_4 \right) x_1 x_3^3 + \left(-4 \left(-l_1 - 2 m_1 - 2 \right) l_2 - m_1 \left(m_2 + 2 \right) \right) \alpha - 4 \left(l_3 + 2 m_3 \right. \\ &\quad \left. + 2 \right) l_4 - 4 m_3 \left(m_4 + 2 \right) x_1 x_3 + \left(\alpha m_1^2 m_2^2 - l_3^2 l_4^2 \right) x_3^4 + \left(2 \left(m_2 \left(2 + l_1 \right) l_2 + \left(-m_2^2 - 2 m_2 \right. \right. \right. \\ &\quad \left. \left. - 2 \right) m_1 - 2 l_1 - 4 \right) m_1 \alpha + 2 \left(\left(l_3^2 + 2 l_3 + 2 \right) l_4 - \left(m_4 + 2 \right) \left(l_3 m_3 - 2 \right) \right) l_4 x_3^2 \\ &\quad \left. + \left(\left(2 + l_1 \right) l_2 - m_1 \left(m_2 + 2 \right) \right)^2 \alpha - \left(\left(2 + l_3 \right) l_4 - m_3 \left(m_4 + 2 \right) \right)^2 \right) \end{aligned} \quad (23)$$

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Section 6

Equation (25) (equation (2.25) in Arxiv) and $\alpha = 1$:

$$\begin{aligned} l_2 &:= \frac{(m_3 + 1) \cdot (l_4 + 1)}{m_1 + 1} - 1; l_3 := \frac{m_1 \cdot m_2}{l_4}; m_4 := \frac{l_1 \cdot (m_3 \cdot l_4 + m_3 + l_4 - m_1)}{m_3 \cdot (m_1 + 1)}; \alpha := 1 \\ l_2 &:= \frac{(m_3 + 1) (l_4 + 1)}{m_1 + 1} - 1 \\ l_3 &:= \frac{m_1 m_2}{l_4} \\ m_4 &:= \frac{l_1 (l_4 m_3 + l_4 - m_1 + m_3)}{m_3 (m_1 + 1)} \\ \alpha &:= 1 \end{aligned} \quad (24)$$

Expression A :

$$A := (m_3 + 1) \cdot (l_1 + 1) \cdot (l_4 + 1) - (m_1 + 1) \cdot (m_1 \cdot m_2 + l_1 + l_4 + m_1 - m_3 + 1)$$

$$A := (m_3 + 1) (l_1 + 1) (l_4 + 1) - (m_1 + 1) (m_1 m_2 + l_1 + l_4 + m_1 - m_3 + 1) \quad (25)$$

Expression B :

$$B := m_1 \cdot m_2 + m_2 \cdot m_3 + l_1 + m_2 + l_4 + m_1 + 2$$

$$B := m_1 m_2 + m_2 m_3 + l_1 + l_4 + m_1 + m_2 + 2 \quad (26)$$

Substitution (22) in (25) ((2.22) in (2.25) in Arxiv)
and comparing with required equation :

$$\text{simplify} \left(\frac{(m_1 + 1)^2}{4} \cdot \left(((2 + l_1) l_2 - m_1 (2 + m_2))^2 \alpha - ((2 + l_3) l_4 - m_3 (m_4 + 2))^2 \right) - (m_1 + m_3 + 2) \cdot (l_4 - m_1) \cdot A \right)$$

Substitution (25) in (23) ((2.25) in (2.23) in Arxiv)
and comparing with required equation :

$$\text{simplify}\left(\frac{m_3 \cdot (m_1 + 1)}{4} \cdot \left(2 m_1 (m_2 (2 + l_1) l_2 + (-m_2^2 - 2 m_2 - 2) m_1 - 2 l_1 - 4) \alpha + 2 l_4 \left((l_3^2 + 2 l_3 + 2) l_4 - (m_4 + 2) (l_3 m_3 - 2)\right)\right) - (l_4 - m_1) \cdot (A + (m_3 + 1) \cdot m_1 \cdot B)\right)$$

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Section 7

Expression $A + (m_1 - m_3) \cdot B$:

$$factor(A + (m_1 - m_3) \cdot B)$$

$$(m_3 + 1) (l_1 l_4 - m_2 m_3) \tag{29}$$