MA3676 - 2018 Past Paper

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Contents

1 Question 1

1.1 a

Assigning parameters: $a = 0$, $b = 2$, $c = 1$. Therefore, $\mathbb{P}[X = -1] = \frac{1}{2}$ and $\mathbb{P}[X = 0] = \frac{1}{2}$.

1.1.1 i

$$\mathbb{E}[S_n] = -1 \cdot n \cdot \frac{1}{2} + 0 \cdot n \cdot \frac{1}{2} = -\frac{1}{2}n. \tag{1}$$

1.1.2 ii

1.1.3 iii

1.1.4 iv

1.1.5 v

1.1.6 vi

1.2 b

Assigning parameters: $f = 6$, $g = 9$, $h = 9$, $u = 2$, $v = 3$.

1.2.1 i

1.2.2 ii

2 Question 2

2.1 a

Assigning parameters: $u = 0$, $v = 9$, $w = 6$. Therefore,

$$\mathbb{P}[X = 0] = \frac{1}{18}, \tag{2}$$
$$\mathbb{P}[X = 1] = \frac{5}{9}, \tag{3}$$
$$\mathbb{P}[X = 2] = \frac{7}{18}. \tag{4}$$

The probability of extinction is given by $G(\xi) = \xi$, so we must first define our generating function. Our generating function $G_{Z_n}(s)$ can be evaluated as

$$G(s) = \sum_n g_n s^n, \tag{5}$$

where $g_n = \mathbb{P}[X = n]$. Therefore, we have

$$G(s) = \frac{1}{18} + \frac{5}{9}s + \frac{7}{18}s^2. \tag{6}$$

We then use the fact $G(\xi) = \xi$ where ξ is our probability of extinction to solve:

$$G(\xi) = \xi = \frac{1}{18} + \frac{5}{9}\xi + \frac{7}{18}\xi^2 \tag{7}$$
$$0 = 7\xi^2 - 8\xi + 1 \tag{8}$$
$$= (\xi - 1)(7\xi - 1) \tag{9}$$
$$\xi = 1, \frac{1}{7}. \tag{10}$$

We take our smallest value of $\xi = \frac{1}{7}$ to represent our probability of extinction.

2.2 b

Assigning parameters: $a = 2$, $b = 9$, $c = 1$, $d = -3$. Therefore:

$$\mathbb{P}[X = +1] = p, \tag{11}$$
$$\mathbb{P}[X = -3] = q = 1 - p. \tag{12}$$

2.2.1 i

Our generating function $G_{S_n}(s)$ is obtained by evaluating $\mathbb{E}[s^{S_n}]$. This is the probability of each X occurring, multiplied by s raised to the power of the respective step taken. This is then all raised to the power n for each of the steps taken. This is numerically done through the following:

$$G_{S_n}(s) = \mathbb{E}[s^{S_n}] = (ps + qs^{-3})^n. \tag{13}$$

We use the binomial theorem

$$(a + b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}, \tag{14}$$

where a and b are ps and qs^{-3} respectively in order to get

$$G_{S_n}(s) = \sum_{m=0}^{n} \binom{n}{m} (ps)^m (qs^{-3})^{n-m} \tag{15}$$
$$= \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} s^{m-3(n-m)} \tag{16}$$
$$= \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} s^{4m-3n}. \tag{17}$$

2.2.2 ii

Firstly, we know that s can only be raised to a power of an integer, as our steps always take an integer value. Lets set introduce this varying integer to k. Therefore, we make $k = 4m - 3n$ and rearrange to $m = \frac{k+3n}{4}$. This tells us what values for m we can take, and adjusts the formula accordingly.
It's important to know here the the probability of S_n being a value k can be given by the following formula after using our substitution for k.

$$\mathbb{P}[S_n = k] = \binom{n}{\frac{k+3n}{4}} p^{\frac{k+3n}{4}} q^{\frac{n-k}{4}}. \tag{18}$$

Plugging in our value of k as 0 (as required in the question), we get the following:

$$\mathbb{P}[S_n = 0] = \binom{n}{\frac{3n}{4}} p^{\frac{3n}{4}} q^{\frac{n}{4}}, \tag{19}$$

Allowing us the find the probability of our walk being on 0 for an arbitrary value of n.

3 Question 3

Assigning parameters: $a = 2$, $b = 0$, $c = 9$, $d = 9$, $f = 6$. Therefore

$$u = \frac{1}{5}, \tag{20}$$
$$v = \frac{1}{3}, \tag{21}$$
$$w = \frac{1}{10}, \tag{22}$$
$$x = \frac{1}{11}, \tag{23}$$
$$y = \frac{1}{7}. \tag{24}$$

3.1 a

3.1.1 i

With our state space being $\{1, 2, 3, 4, 5\}$, our one-step transition matrix is

$$\mathbf{p} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & 0 & \frac{2}{5} & 0 \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{9}{11} & \frac{1}{11} & 0 & \frac{1}{11} & 0 \\ 0 & 0 & \frac{6}{7} & \frac{1}{7} & 0 \end{pmatrix}. \tag{25}$$

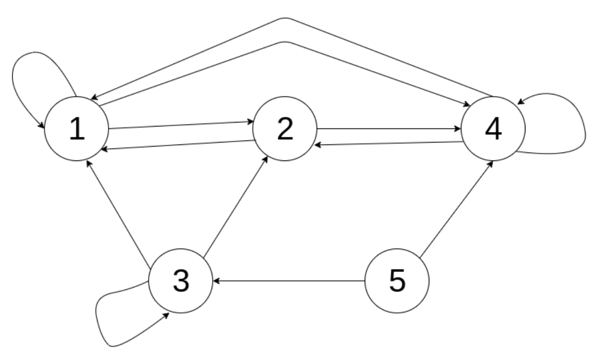
3.1.2 ii

After 8 steps (X_8), we are in state 4. We then want to find the probability of being in each state after two steps (X_10). We pass through the certainty of being in state 4 through the one-step transition matrix twice with the following:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & 0 & \frac{2}{5} & 0 \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{9}{11} & \frac{1}{11} & 0 & \frac{1}{11} & 0 \\ 0 & 0 & \frac{6}{7} & \frac{1}{7} & 0 \end{pmatrix}^2 = \begin{pmatrix} \frac{9}{11} & \frac{1}{11} & 0 & \frac{1}{11} & 0 \end{pmatrix}. \tag{26}$$

From this we can establish the fact that $\mathbb{P}[X_{10} = 3] = 0$.

3.1.3 iii



The states $\{1, 2, 4\}$ form a closed set of ergodic recurrent states, whereas 3 and 5 are transient. We can rearrange these states to give the state space $\{1, 2, 4, 3, 5\}$ and therefore the new transition matrix

$$\mathbf{p} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 \\ \frac{1}{10} & 0 & \frac{9}{10} & 0 & 0 \\ \frac{9}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{7} & \frac{6}{7} & 0 \end{pmatrix}. \tag{27}$$

We can the break this up into the form of

$$\mathbf{p} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{pmatrix} \tag{28}$$

to give us

$$\mathbf{P} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{10} & 0 & \frac{9}{10} \\ \frac{9}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}, \tag{29}$$

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{6}{7} & 0 \end{pmatrix}, \tag{30}$$

$$\mathbf{R} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}. \tag{31}$$

3.1.4 iv

We know that the probability we are in a transient state as we tend to infinity must be 0, so therefore for any j, $\mathbb{P}[X_n = i]$ as $n \to \infty$ is 0 for $i = 3, 5$.
As there is only 1 closed set of states, it doesn't matter which transient state we start at. We just need to find the stable state of $\mathbf{P}$ to find the appropriate probabilities of our 1 closed set. We establish this stable state through the following:

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_4 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{10} & 0 & \frac{9}{10} \\ \frac{9}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_4 \end{pmatrix}. \tag{32}$$

From this, we obtain the system of the questions

$$\pi_1 = \frac{2}{5}\pi_1 + \frac{1}{10}\pi_2 + \frac{9}{11}\pi_4, \tag{33}$$

$$\pi_2 = \frac{1}{5}\pi_1 + \frac{1}{11}\pi_4, \tag{34}$$

$$\pi_4 = \frac{2}{5}\pi_1 + \frac{9}{10}\pi_2 + \frac{1}{11}\pi_4. \tag{35}$$

These can then be solved in terms of π_4, along with the fact that the probabilities of these states must add up to 1 to give

$$\pi_4 \begin{pmatrix} \frac{455}{319} & 1 & \frac{120}{319} \end{pmatrix} = 1. \tag{36}$$

This is then rearranged to obtain

$$\pi_4 = \frac{319}{894}. \tag{37}$$

Now, we can sub this back into (38) to give us the stable state

$$\begin{pmatrix} \frac{455}{894} & \frac{120}{894} & \frac{319}{894} \end{pmatrix}. \tag{38}$$

With this, we now know that the probability of being at states $X_n = i$as $n \to \infty$ and $i = 1, 2, 4$ are given respectively in (38).

3.2 b

4 Question 4

Assigning parameters: $a = 0$, $b = 2$, $c = 6$, $d = 9$, $f = 9$. Therefore:

$$u = \frac{1}{2}, \tag{39}$$
$$v = \frac{1}{4}, \tag{40}$$
$$w = \frac{1}{7}, \tag{41}$$
$$x = \frac{1}{10}, \tag{42}$$
$$y = \frac{1}{9}. \tag{43}$$

4.1 a

There are a couple of things to take note from the question. One is that anybody fired or leaves can just go straight into employment level 1 as they would be instantly replaced anyway. Also, the probability given for firing or leaving from level 1 can be disregarded for now, as both current and new employees at this level are indistinguishable from one another.

4.1.1 i

With this knowledge, we obtain the following one-step transition matrix:

$$\mathbf{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} & 0 & 0 \\ \frac{1}{7} & 0 & \frac{5}{7} & \frac{1}{7} & 0 \\ \frac{1}{5} & 0 & 0 & \frac{7}{10} & \frac{1}{10} \\ \frac{1}{9} & 0 & 0 & \frac{1}{9} & \frac{7}{9} \end{pmatrix}. \tag{44}$$

4.1.2 ii

Here we find the equilibrial state of the matrix using the following:

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} & 0 & 0 \\ \frac{1}{7} & 0 & \frac{5}{7} & \frac{1}{7} & 0 \\ \frac{1}{5} & 0 & 0 & \frac{7}{10} & \frac{1}{10} \\ \frac{1}{9} & 0 & 0 & \frac{1}{9} & \frac{7}{9} \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} \tag{45}$$

From this, we get the following system of equations

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{8}\pi_2 + \frac{1}{7}\pi_3 + \frac{1}{5}\pi_4 + \frac{1}{9}\pi_5, \tag{46}$$
$$\pi_2 = \frac{1}{2}\pi_1 + \frac{5}{8}\pi_2, \tag{47}$$
$$\pi_3 = \frac{1}{4}\pi_2 + \frac{5}{7}\pi_3, \tag{48}$$
$$\pi_4 = \frac{1}{7}\pi_3 + \frac{7}{10}\pi_4 + \frac{1}{9}\pi_5, \tag{49}$$
$$\pi_5 = \frac{1}{10}\pi_4 + \frac{7}{9}\pi_5. \tag{50}$$

We can then use eq. (47)-(50) to obtain

$$\pi_5 \begin{pmatrix} \frac{10}{3} & \frac{40}{9} & \frac{35}{9} & \frac{20}{0} & 1 \end{pmatrix} = 1 \tag{51}$$

to then solve for $\pi_5 = \frac{9}{134}$. We then sub this back in to find our stable state of

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} = \begin{pmatrix} \frac{15}{76} & \frac{20}{67} & \frac{35}{134} & \frac{10}{67} & \frac{9}{134} \end{pmatrix}. \tag{52}$$

This gives us the proportion of employees in each class after many years.

4.1.3 iii

Assigning parameters: let $u = £10,000$. Class $1 = u$, class $2 = 4u$, class $3 = 6u$, class $4 = 10u$ and class $5 = 9u$. We multiply each salary by the probability of a worker being of that class after a long time (our stable state). This is expressed as

$$\begin{pmatrix} u & 4u & 6u & 10u & 9u \end{pmatrix} \begin{pmatrix} \frac{15}{76} \\ \frac{20}{67} \\ \frac{35}{134} \\ \frac{10}{67} \\ \frac{9}{134} \end{pmatrix} = \frac{681}{134} u \approx £50,821. \tag{53}$$

Now we have the average salary, we then multiply it by an the population $N = 10,000$ to obtain a total yearly salary bill of $£508,210,000$

4.1.4 iv

This is where we make use of the value given to us of the number of fired employees from the first class. We have to be aware that the new number of employees is equal to the ones fired and left. For the classes $2-4$, this is just the first column, but for class 1 we must make note that only $(1-u)/2$ have been replaced and the remaining are still class one and have already done the training. With this being the case, we multiply our stable state vector by our modified first column:

$$\begin{pmatrix} \frac{15}{76} & \frac{20}{67} & \frac{35}{134} & \frac{10}{67} & \frac{9}{134} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{7} \\ \frac{1}{5} \\ \frac{1}{9} \end{pmatrix} = \frac{45}{268}. \tag{54}$$

This is the portion of employees that undergo the training. Therefore, the average amount expended per employee is $\frac{45}{268}N \approx £3,358$. Assuming the total number of employees is the same as the previous question, this is a total of $\approx £33,582,089$.

4.2 b