MA3676 - 2018 Past Paper

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2 Question 2

2.1 a

Assigning parameters: $c = 9$.
We have the following difference equation, where the expected time to absorption $E[\mathcal{T}_n] = T_n$

$$T_n = \mathbb{E}[\mathcal{T}_n | +1]\mathbb{P}[+1] + \mathbb{E}[\mathcal{T}_n | -1]\mathbb{P}[-1]. \tag{1}$$

We then define the following expected values as

$$\mathbb{E}[\mathcal{T}_n | +1] = 1 + T_{n+1} \tag{2}$$
$$\mathbb{E}[\mathcal{T}_n | -1] = 1 + T_{n-1}. \tag{3}$$

We then get the following difference equation

$$T_n = \frac{1}{2}(1 + T_{n+1}) + \frac{1}{2}(1 + T_{n-1}) \tag{4}$$
$$-1 = \frac{1}{2}T_{n+1} - T_n + \frac{1}{2}T_{n-1}. \tag{5}$$

From this, we obtain the characteristic equation

$$\frac{1}{2}\lambda^2 - \lambda + \frac{1}{2} = 0 \tag{6}$$
$$\lambda^2 - 2\lambda + 1 = 0 \tag{7}$$
$$(\lambda - 1)^2 = 0. \tag{8}$$

With $p = q$, we then have the general solution to the homogeneous equation

$$T_n^{(g)} = A + Bn. \tag{9}$$

With a repeated root, we use the particular solution $T_n^{(p)}\alpha n^2$, remembering that T is a function of n, giving us

$$-1 = \frac{1}{2}\alpha(n+1)^2 - \alpha n^2 + \frac{1}{2}\alpha(n-1)^2. \tag{10}$$

Subbing in $n = 0$ to simplify our equation, we simplify to $\alpha = -1$ and therefore a particular solution of $-n^2$. Therefore, we have the following general solution

$$T_n = A + Bn - n^2. \tag{11}$$

We now look at our two boundary conditions. Firstly, when we reach the the position $9N$, we know that we have reached the end point end our expected number of steps must be zero. This gives us the boundary condition

$$T_{9N} = A + 9BN - (9N)^2 = 0 \tag{12}$$

Our other boundary condition exists at $n = 0$. We know that, when at position $n = 0$, we move $+1$ with probability 1. This can be expressed as

$$\mathbb{E}[\mathcal{T}_n] = \mathbb{E}[\mathcal{T}_0 | +1] \tag{13}$$
$$T_0 = T_1 + 1. \tag{14}$$

We plug our general solution into (14) to obtain

$$A = A + B - 1 + 1, \tag{15}$$

giving us $B + 0$ and therefore

$$T_n = A - n^2. \tag{16}$$

We now evaluate the first boundary condition where

$$T_{9N} = 0 = A - (9N)^2 \tag{17}$$
$$A = (9N)^2. \tag{18}$$

We then arrive at our solution

$$T_n = (9N)^2 - n^2. \tag{19}$$

Now we find the probability of absorption from our given starting position $S_0 = N$ with

$$T_N = (9N)^2 - N^2 \tag{20}$$
$$= 80N^2. \tag{21}$$

2.2 b

We know by definition that $\sum_{n=0}^{\infty} \mathbb{P}[Z = n] = 1$. Be aware that our function $C\alpha^n$ isn't valid when $n = 0$, as this is defined separately as $\mathbb{P}[Z = 0] = \frac{2}{3}$. Therefore, we have the following

$$1 = \sum_{n=0}^{\infty} \mathbb{P}[Z = n] \tag{22}$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} C\alpha^n. \tag{23}$$

Take note of the change from $n = 0$ to $n = 1$. From here, we then continue to solve for C in terms of α:

$$\frac{1}{3} = C\left(\sum_{n=1}^{\infty} \alpha^n\right) \tag{24}$$

$$= C\left(\sum_{n=0}^{\infty} \alpha^n - \sum_{n=0}^{0} \alpha^n\right) \tag{25}$$

$$= C\left(\sum_{n=10}^{\infty} \alpha^n - 1\right) \tag{26}$$

$$= C\left(\frac{1}{1-\alpha} - 1\right) \tag{27}$$

$$= C\left(\frac{\alpha}{1-\alpha}\right) \tag{28}$$

$$C = \frac{1-\alpha}{3\alpha}. \tag{29}$$

Subbing in the given $\alpha = \frac{2}{3}$ gives us the result of $C = \frac{1}{6}$ and therefore the distribution rule $\mathbb{P}[Z = n] = \frac{1}{6}\left(\frac{2}{3}\right)^n$, $n \geq 1$.

We know that the extinction probability ξ satisfies

$$G_{W_n}(\xi) = \xi \tag{30}$$

So our first job is to define our generating function for some parameter s. The formula for a generating function $G(s)$ is as follows

$$G(s) = s^n \sum_{n=0}^{\infty} \mathbb{P}[Z = n]. \tag{31}$$

We then use our values to proceed as

$$G(s) = s^n \left(\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{3}\right)^n + \frac{2}{3}\right) \tag{32}$$

$$= \frac{1}{6}\sum_{k=1}^{\infty} s^n \left(\frac{2}{3}\right)^n + s^0 \frac{2}{3} \tag{33}$$

$$= \frac{1}{6}\sum_{k=1}^{\infty} \left(\frac{2s}{3}\right)^n + \frac{2}{3} \tag{34}$$

$$= \frac{1}{6}\left[\sum_{k=0}^{\infty} \left(\frac{2s}{3}\right)^n - 1\right] + \frac{2}{3} \tag{35}$$

$$= \frac{1}{6}\left[\frac{1}{1-\frac{2s}{3}}\right] + \frac{2}{3} - \frac{1}{6} \tag{36}$$

$$= \frac{1}{6}\left[\frac{3}{3-2s}\right] + \frac{1}{2}. \tag{37}$$

$$\tag{38}$$

At this point we can sub in $s = \xi$ and solve.

$$G(\xi) = \frac{1}{6}\left[\frac{3}{3-2\xi}\right] + \frac{1}{2} = \xi \tag{39}$$

$$\frac{4-2\xi}{6-4\xi} = \xi \tag{40}$$

$$4 - 2\xi = \xi(6-4\xi) \tag{41}$$

$$4\xi^2 - 8\xi + 4 = 0 \tag{42}$$

$$(\xi - 1)(4\xi - 4) = 0 \tag{43}$$

giving us a repeated root at $\xi = +1$. This means that our probability of extinction is certain.

3 Question 3

3.1 a

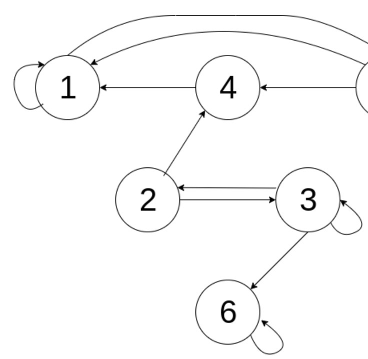
3.1.1 i

By definition, we only need to look at the most recent state value in order to determine the following step. Therefore, the only value we nee to be aware of here is that $X_7 = 3$. This means that at step 7, we are in state 3 with probability $= 1$ as this information is definite. In order to calculate our state probabilities at X_8, we calculate

$$\begin{aligned}
\pi(8) &= \pi(7)\mathbf{p} && (44)\\
&= \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3}\\ 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} && (45)\\
&= \begin{pmatrix}0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3}\end{pmatrix}. && (46)
\end{aligned}$$

By looking at the third element of our state vector, we see that $\mathbb{P}[X_8 = 3] = \frac{1}{3}$.

3.1.2 ii



We can see here that $\{1, 4, 5\}$ form a closed set of ergodic states and 6 is an absorbing state. 2 and 3 are both transient.
We then restructure our state-space from $(1\,2\,3\,4\,5\,6)$ to $(1\,4\,5\,6\,2\,3)$. This in turn gives us

$$\mathbf{p} = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}. \tag{47}$$

From here, it's easy enough to establish PQR, such that

$$\mathbf{P} = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2} & 0\\ 1 & 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{48}$$

$$\mathbf{Q} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}, \tag{49}$$

$$\mathbf{R} = \begin{pmatrix}0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{3}\end{pmatrix}. \tag{50}$$

3.1.3 iii

The probability is 0 as 2 is a transient state.

3.1.4 iv

The probability is 0 as 3 is a transient state.

3.1.5 v

As 5 and 1 are in the same closed set, we just need to find the equilibrial state of the relevant matrix for this closed set. This is found by

$$\begin{pmatrix}\pi_1 & \pi_4 & \pi_5\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_4 & \pi_5\end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{51}$$

From this, we obtain the following system of equations

$$\pi_1 = \frac{1}{2}\pi_1 + \pi_4 + \frac{1}{2}\pi_5, \tag{52}$$

$$\pi_4 = \frac{1}{2}\pi_5, \tag{53}$$

$$\pi_5 = \frac{1}{2}\pi_1. \tag{54}$$

We use the latter two to get our stable vector in the form

$$\boldsymbol{\pi} = \begin{pmatrix}2\pi_5 & \frac{1}{2}\pi_5 & \pi_5\end{pmatrix}. \tag{55}$$

We know that these must sum to one so

$$2\pi_5 + \frac{1}{2}\pi_5 + \pi_5 = 1 \tag{56}$$

$$\pi_5 = \frac{2}{7}. \tag{57}$$

We can then finalise our steady state as

$$\boldsymbol{\pi} = \begin{pmatrix}\frac{4}{7} & \frac{1}{7} & \frac{2}{7}\end{pmatrix}. \tag{58}$$

Therefore, our final probability is the steady state probability of $X = 1$ which is $\frac{4}{7}$ given that we start from within this closed set (which we did, state 5).

3.1.6 vi

3.2 b

We know one λ value is $\frac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$, and the general formula for a periodic matrix is given as

$$\lambda_k = e^{2i\pi k/d} \tag{59}$$

where d is the periodicity and $k = 0, 1, \dots, d-1$. Therefore we can see that $k = 1$ and $d = 3$. This gives the other two values for this periodic set of states

$$\lambda_2 = e^{4i\pi/3} = \frac{-1 - i\sqrt{3}}{2}, \tag{60}$$

$$\lambda_0 = e^0 = 1. \tag{61}$$

Looking at the matrix $\mathbf{p}$ we can see that state 2 is absorbing, so we can fill in the rest of this line with 0s and also add another $\lambda = 1$ to our list. As state 3 is clearly not one of the periodic state, we deduce that the three periodic states must be $1, 4$ and 5. We cannot tell what order these go in, so any order should suffice.

Finally, state 3 is transient and so other than filling in the row by adding the only possible value of $\frac{1}{2}$ to the line, all we know is it's eigenvalue has a magnitude strictly less than 1. This leaves us with

$$\mathbf{p} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{62}$$

and the eigenvalues $\{\frac{-1+i\sqrt{3}}{2}, 1, r, \frac{-1-i\sqrt{3}}{2}, 1\}$ where $|r| < 1$.

4 Question 4

4.1 a

Assigning parameters: $a = 13$, $b = 1$, $c = 2$, $d = 9$. Therefore

$$r_A = 1\% \tag{63}$$
$$r_B = 1\% \tag{64}$$
$$r_C = 2\% \tag{65}$$
$$r_D = 9\%. \tag{66}$$

4.1.1 i

Our transition matrix looks like the following:

$$\mathbf{p} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{67}$$

4.1.2 ii

First, we must find the steady-state vector for the transition matrix using $\boldsymbol{\pi}\mathbf{p} = \boldsymbol{\pi}$.

$$\begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D \end{pmatrix} = \begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{68}$$

This in turn gives us the following system of equations (ignoring $\pi_A = \dots$ as this is instead replaced by the summing to 1, as the previous question did.)

$$\pi_B = \frac{1}{4}\pi_A + \frac{1}{4}\pi_B, \tag{69}$$
$$\pi_C = \frac{1}{4}\pi_B + \frac{1}{4}\pi_C, \tag{70}$$
$$\pi_D = \frac{1}{2}\pi_C + \frac{1}{4}\pi_D. \tag{71}$$

Solving these in terms of π_A gives us the following steady-state vector

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_A & \frac{1}{3}\pi_a & \frac{1}{9}\pi_A & \frac{2}{27}\pi_A \end{pmatrix}. \tag{72}$$

Remembering that these must sum to 1, we can solve for π_A and get our final steady state.

$$\pi_A \left(\frac{27 + 9 + 3 + 2}{27} \right) = 1 \tag{73}$$
$$\pi_A = \frac{27}{41}. \tag{74}$$

This leaves us with

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{27}{41} & \frac{9}{41} & \frac{3}{41} & \frac{2}{41} \end{pmatrix}. \tag{75}$$

Remaining to be finished later.

4.2 b