MA3676 - 2018 Past Paper

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Contents

1 Question 1

1.1 a

1.1.1 i

$$\mathbb{P}[X_{j+1} = \dots | X_j = \dots, X_{j-1} = \dots, \dots] = \mathbb{P}[X_{j+1} = \dots | X_j = \dots] = \frac{1}{6} \tag{1}$$

As this does not depend on any value other than previous (which even this doesn't depend on), it is a Markov chain.

1.1.2 ii

We have $\mathbb{P}[X = x] = \frac{1}{6}$ for all values of $x = 1, 2, \dots, 6$. Therefore,

$$\begin{aligned} \mathbb{E}[X_n] &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots + \frac{1}{6} \cdot 6 && (2) \\ &= \frac{1}{6}(21) && (3) \\ &= \frac{7}{2} && (4) \end{aligned}$$

hence

$$\mathbb{E}[S_n] = \frac{7}{2}n. \tag{5}$$

1.1.3 iii

$$\begin{aligned} \mathbb{E}[X_{j+1} = \dots | X_j = \dots, X_{j-1} = \dots, \dots] &= \mathbb{E}[X_{j+1}] && (6) \\ &\neq X_j. && (7) \end{aligned}$$

As this doesn't hold true, X_n is not a martingale.

1.1.4 iv

1.1.4.1 A

1.1.4.2 B

$$\begin{aligned} \mathbb{E}[D_{n+1}|\mathcal{F}_n] &= \mathbb{E}[D_n + Y_{n+1}|\mathcal{F}_n] && (8) \\ &= D_n + \mathbb{E}[Y] && (9) \end{aligned}$$

Y_k looks at the 'score' obtained at each jump, so

$$\mathbb{E}[Y] = \frac{1}{6}(+1) + \frac{1}{6}(-1) + \frac{2}{3}(0) = 0. \tag{10}$$

Therefore

$$\mathbb{E}[D_{n+1}|\mathcal{F}] = D_n \tag{11}$$

showing that D_n is a martingale.

1.1.4.3 C

1.2 b

First, we test

$$\begin{aligned} G(1) &= (2 - 1)^{\frac{1}{2}} && (12) \\ &= 1. && (13) \end{aligned}$$

The first requirement is satisfied. Now, we expand $G(s)$ as follows

$$\begin{aligned}
G(s) &= (2-s)^{\frac{1}{2}} && (14)\\
&= \sqrt{2}\left(1-\frac{1}{2}s\right)^{\frac{1}{2}} && (15)\\
&= \sqrt{2}\left[1-\frac{1}{2}\frac{1}{2}s-\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{4}s^2+\cdots\right] && (16)\\
&= \sqrt{2}\left[1-\frac{1}{4}s-\frac{1}{32}s^2+\cdots\right]. && (17)
\end{aligned}$$

This is enough to see that, not only does the first probability take value greater than one, we also have negative probabilities. This means that $G(s)$ **is not** a valid generator.
The binomial expansion was used for the equations above.

2 Question 2

2.1 a

Assigning parameters: $c = 9$.
We have the following difference equation, where the expected time to absorption $E[\mathcal{T}_n] = T_n$

$$T_n = \mathbb{E}[\mathcal{T}_n| + 1]\mathbb{P}[+1] + \mathbb{E}[\mathcal{T}_n| - 1]\mathbb{P}[-1]. \tag{18}$$

We then define the following expected values as

$$\mathbb{E}[\mathcal{T}_n| + 1] = 1 + T_{n+1} \tag{19}$$
$$\mathbb{E}[\mathcal{T}_n| - 1] = 1 + T_{n-1}. \tag{20}$$

We then get the following difference equation

$$T_n = \frac{1}{2}(1 + T_{n+1}) + \frac{1}{2}(1 + T_{n-1}) \tag{21}$$
$$-1 = \frac{1}{2}T_{n+1} - T_n + \frac{1}{2}T_{n-1}. \tag{22}$$

From this, we obtain the characteristic equation

$$\frac{1}{2}\lambda^2 - \lambda + \frac{1}{2} = 0 \tag{23}$$
$$\lambda^2 - 2\lambda + 1 = 0 \tag{24}$$
$$(\lambda - 1)^2 = 0. \tag{25}$$

With $p = q$, we then have the general solution to the homogeneous equation

$$T_n^{(g)} = A + Bn. \tag{26}$$

With a repeated root, we use the particular solution $T_n^{(p)}\alpha n^2$, remembering that T is a function of n, giving us

$$-1 = \frac{1}{2}\alpha(n+1)^2 - \alpha n^2 + \frac{1}{2}\alpha(n-1)^2. \tag{27}$$

Subbing in $n = 0$ to simplify our equation, we simplify to $\alpha = -1$ and therefore a particular solution of $-n^2$. Therefore, we have the following general solution

$$T_n = A + Bn - n^2. \tag{28}$$

We now look at our two boundary conditions. Firstly, when we reach the the position $9N$, we know that we have reached the end point end our expected number of steps must be zero. This gives us the boundary condition

$$T_{9N} = A + 9BN - (9N)^2 = 0 \tag{29}$$

Our other boundary condition exists at $n = 0$. We know that, when at position $n = 0$, we move $+1$ with probability 1. This can be expressed as

$$\mathbb{E}[\mathcal{T}_n] = \mathbb{E}[\mathcal{T}_0| + 1] \tag{30}$$
$$T_0 = T_1 + 1. \tag{31}$$

We plug our general solution into (31) to obtain

$$A = A + B - 1 + 1, \tag{32}$$

giving us $B + 0$ and therefore

$$T_n = A - n^2. \tag{33}$$

We now evaluate the first boundary condition where

$$T_{9N} = 0 = A - (9N)^2 \tag{34}$$
$$A = (9N)^2. \tag{35}$$

We then arrive at our solution

$$T_n = (9N)^2 - n^2. \tag{36}$$

Now we find the probability of absorption from our given starting position $S_0 = N$ with

$$T_N = (9N)^2 - N^2 \tag{37}$$
$$= 80N^2. \tag{38}$$

2.2 b

We know by definition that $\sum_{n=0}^{\infty} \mathbb{P}[Z=n]=1$. Be aware that our function $C\alpha^n$ isn't valid when $n=0$, as this is defined separately as $\mathbb{P}[Z=0]=\frac{2}{3}$. Therefore, we have the following

$$1=\sum_{n=0}^{\infty} \mathbb{P}[Z=n] \tag{39}$$

$$=\frac{2}{3}+\sum_{n=1}^{\infty} C\alpha^n. \tag{40}$$

Take note of the change from $n=0$ to $n=1$. From here, we then continue to solve for C in terms of α:

$$\frac{1}{3}=C\left(\sum_{n=1}^{\infty}\alpha^n\right) \tag{41}$$

$$=C\left(\sum_{n=0}^{\infty}\alpha^n-\sum_{n=0}^{0}\alpha^n\right) \tag{42}$$

$$=C\left(\sum_{n=10}^{\infty}\alpha^n-1\right) \tag{43}$$

$$=C\left(\frac{1}{1-\alpha}-1\right) \tag{44}$$

$$=C\left(\frac{\alpha}{1-\alpha}\right) \tag{45}$$

$$C=\frac{1-\alpha}{3\alpha}. \tag{46}$$

Subbing in the given $\alpha=\frac{2}{3}$ gives us the result of $C=\frac{1}{6}$ and therefore the distribution rule $\mathbb{P}[Z=n]=\frac{1}{6}\left(\frac{2}{3}\right)^n$, $n\geq 1$.
We know that the extinction probability ξ satisfies

$$G_{W_n}(\xi)=\xi \tag{47}$$

So our first job is to define our generating function for some parameter s. The formula for a generating function $G(s)$ is as follows

$$G(s)=s^n\sum_{n=0}^{\infty}\mathbb{P}[Z=n]. \tag{48}$$

We then use our values to proceed as

$$G(s)=s^n\left(\sum_{n=1}^{\infty}\frac{1}{6}\left(\frac{2}{3}\right)^n+\frac{2}{3}\right) \tag{49}$$

$$=\frac{1}{6}\sum_{k=1}^{\infty}s^n\left(\frac{2}{3}\right)^n+s^0\frac{2}{3} \tag{50}$$

$$=\frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{2s}{3}\right)^n+\frac{2}{3} \tag{51}$$

$$=\frac{1}{6}\left[\sum_{k=0}^{\infty}\left(\frac{2s}{3}\right)^n-1\right]+\frac{2}{3} \tag{52}$$

$$=\frac{1}{6}\left[\frac{1}{1-\frac{2s}{3}}\right]+\frac{2}{3}-\frac{1}{6} \tag{53}$$

$$=\frac{1}{6}\left[\frac{3}{3-2s}\right]+\frac{1}{2}. \tag{54}$$

$$\tag{55}$$

At this point we can sub in $s = \xi$ and solve.

$$G(\xi) = \frac{1}{6}\left[\frac{3}{3-2\xi}\right] + \frac{1}{2} = \xi \tag{56}$$

$$\frac{4-2\xi}{6-4\xi} = \xi \tag{57}$$

$$4 - 2\xi = \xi(6-4\xi) \tag{58}$$

$$4\xi^2 - 8\xi + 4 = 0 \tag{59}$$

$$(\xi - 1)(4\xi - 4) = 0 \tag{60}$$

giving us a repeated root at $\xi = +1$. This means that our probability of extinction is certain.

3 Question 3

3.1 a

3.1.1 i

By definition, we only need to look at the most recent state value in order to determine the following step. Therefore, the only value we nee to be aware of here is that $X_7 = 3$. This means that at step 7, we are in state 3 with probability $= 1$ as this information is definite. In order to calculate our state probabilities at X_8, we calculate

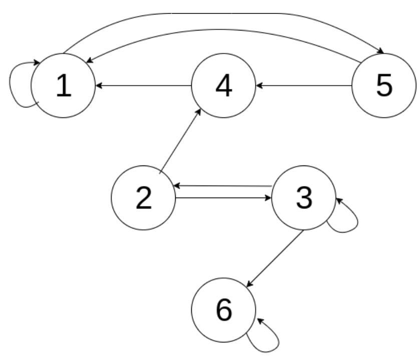
$$\pi(8) = \pi(7)\mathbf{p} \tag{61}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{62}$$

$$= \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{63}$$

By looking at the third element of our state vector, we see that $\mathbb{P}[X_8 = 3] = \frac{1}{3}$.

3.1.2 ii



We can see here that $\{1, 4, 5\}$ form a closed set of ergodic states and 6 is an absorbing state. 2 and 3 are both transient.

We then restructure our state-space from $(1\,2\,3\,4\,5\,6)$ to $(1\,4\,5\,6\,2\,3)$. This in turn gives us

$$\mathbf{p} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{64}$$

From here, it's easy enough to establish PQR, such that

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{65}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \tag{66}$$

$$\mathbf{R} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{67}$$

3.1.3 iii

The probability is 0 as 2 is a transient state.

3.1.4 iv

The probability is 0 as 3 is a transient state.

3.1.5 v

As 5 and 1 are in the same closed set, we just need to find the equilibrial state of the relevant matrix for this closed set. This is found by

$$\begin{pmatrix}\pi_1 & \pi_4 & \pi_5\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_4 & \pi_5\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 1 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}. \tag{68}$$

From this, we obtain the following system of equations

$$\pi_1 = \frac{1}{2}\pi_1 + \pi_4 + \frac{1}{2}\pi_5, \tag{69}$$

$$\pi_4 = \frac{1}{2}\pi_5, \tag{70}$$

$$\pi_5 = \frac{1}{2}\pi_1. \tag{71}$$

We use the latter two to get our stable vector in the form

$$\boldsymbol{\pi} = \begin{pmatrix}2\pi_5 & \frac{1}{2}\pi_5 & \pi_5\end{pmatrix}. \tag{72}$$

We know that these must sum to one so

$$2\pi_5 + \frac{1}{2}\pi_5 + \pi_5 = 1 \tag{73}$$

$$\pi_5 = \frac{2}{7}. \tag{74}$$

We can then finalise our steady state as

$$\boldsymbol{\pi} = \begin{pmatrix}\frac{4}{7} & \frac{1}{7} & \frac{2}{7}\end{pmatrix}. \tag{75}$$

Therefore, our final probability is the steady state probability of $X = 1$ which is $\frac{4}{7}$ given that we start from within this closed set (which we did, state 5).

3.1.6 vi

The probability to get absorbed in each of the closed sets is given by the elements of

$$\tilde{\mathbf{V}} = (\mathbb{I} - \mathbf{Q})^{-1}\tilde{\mathbf{R}} \tag{76}$$

where

$$\tilde{\mathbf{R}} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}. \tag{77}$$

This gives us the following

$$\tilde{\mathbf{R}} = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}\right)^{-1}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}. \tag{78}$$

$$= \begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}. \tag{79}$$

$$= 2\begin{pmatrix}\frac{2}{3} & \frac{1}{2}\\ \frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}. \tag{80}$$

$$= \begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}. \tag{81}$$

From this, we can determine that the probability of transition from state 3 to the closed set containing state 4 is $\frac{1}{3}$. We then multiply this by the stable state distribution value for state 4 found in the last question to give us $\frac{1}{3} \times \frac{1}{7} = \frac{1}{21}$.

3.2 b

We know one λ value is $\frac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$, and the general formula for a periodic matrix is given as

$$\lambda_k = e^{2i\pi k/d} \tag{82}$$

where d is the periodicity and $k = 0, 1, \ldots, d-1$. Therefore we can see that $k = 1$ and $d = 3$. This gives the other two values for this periodic set of states

$$\lambda_2 = e^{4i\pi/3} = \frac{-1-i\sqrt{3}}{2}, \tag{83}$$

$$\lambda_0 = e^0 = 1. \tag{84}$$

Looking at the matrix $\mathbf{p}$ we can see that state 2 is absorbing, so we can fill in the rest of this line with 0s and also add another $\lambda = 1$ to our list. As state 3 is clearly not one of the periodic state, we deduce that the three periodic states must be $1, 4$ and 5. We cannot tell what order these go in, so any order should suffice. Finally, state 3 is transient and so other than filling in the row by adding the only possible value of $\frac{1}{2}$ to the line, all we know is it's eigenvalue has a magnitude strictly less than 1. This leaves us with

$$\mathbf{p} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{85}$$

and the eigenvalues $\{\frac{-1+i\sqrt{3}}{2}, 1, r, \frac{-1-i\sqrt{3}}{2}, 1\}$ where $|r| < 1$.

4 Question 4

4.1 a

Assigning parameters: $a = 13$, $b = 1$, $c = 2$, $d = 9$. Therefore

$$r_A = 1\% \tag{86}$$
$$r_B = 1\% \tag{87}$$
$$r_C = 2\% \tag{88}$$
$$r_D = 9\%. \tag{89}$$

4.1.1 i

Our transition matrix looks like the following:

$$\mathbf{p} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{90}$$

4.1.2 ii

First, we must find the steady-state vector for the transition matrix using $\boldsymbol{\pi}\mathbf{p} = \boldsymbol{\pi}$.

$$\begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D \end{pmatrix} = \begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{91}$$

This in turn gives us the following system of equations (ignoring $\pi_A = \dots$ as this is instead replaced by the summing to 1, as the previous question did.)

$$\pi_B = \frac{1}{4}\pi_A + \frac{1}{4}\pi_B, \tag{92}$$
$$\pi_C = \frac{1}{4}\pi_B + \frac{1}{4}\pi_C, \tag{93}$$
$$\pi_D = \frac{1}{2}\pi_C + \frac{1}{4}\pi_D. \tag{94}$$

Solving these in terms of π_A gives us the following steady-state vector

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_A & \frac{1}{3}\pi_a & \frac{1}{9}\pi_A & \frac{2}{27}\pi_A \end{pmatrix}. \tag{95}$$

Remembering that these must sum to 1, we can solve for π_A and get our final steady state.

$$\pi_A \left(\frac{27 + 9 + 3 + 2}{27} \right) = 1 \tag{96}$$
$$\pi_A = \frac{27}{41}. \tag{97}$$

This leaves us with

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{27}{41} & \frac{9}{41} & \frac{3}{41} & \frac{2}{41} \end{pmatrix}. \tag{98}$$

Remaining to be finished later.

4.2 b