MA3614 - Complex Variable Methods and Applications

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Contents

1 Week 1

1.1 Fundamentals

1.1.1 Representations of z and $\overline{z}$

A complex number z can be defined in both cartesian or polar form:

$$z = x + iy = re^{i\theta}, \tag{1}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. Here, r is the modulus and θ is the argument.

Definition 1 *The principal argument of z is*

$$\arg z \in (-\pi, \pi], \tag{2}$$

where $\arg z$ *is multi-valued.*

Note, $|z|^2 = z\overline{z}$. $|z| =$ absolute value of z.

1.1.2 Multiplication, powers and roots of unity

Suppose $z = re^{i\theta}$, $z_1 = r_1e^{i\theta_1}$, $r_2e^{i\theta_2}$.

- **Multiplication:** $z_1z_2 = r_1r_2e^{i(\theta_1\theta_2)}$.
- **Powers:** $z^n = r^ne^{in\theta}$, $n = 0, \pm1, \pm2, \ldots$.
- Observe that $e^{2\pi i} = \exp(2\pi i) = 1$.
- **Roots of unity:** Let $\omega = \exp(2\pi i/n)$. $1, \omega, \omega^2, \ldots, \omega^{n-1}$ all satisfy $z^n - 1 = 0$ and are uniformly spaced on the unit circle.

1.1.3 Triangle inequality in $\mathbb{C}$

Definition 2

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \tag{3}$$

1.1.4 Convergence of a sequence in $\mathbb{C}$

Definition 3 *A sequence $z_0, z_1, z_2, \ldots$ converges to z if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that*

$$|z_n - z| < \epsilon, \quad \forall n \geq N. \tag{4}$$

From here on, $|\ |$ now means the absolute value of a complex number.

2 Week 2

2.1 Foundations of complex numbers

Theorem 1 *A polynomial of degree n can always be factorised in the form*

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{5}$$
$$= a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n). \tag{6}$$

where $a_0, \ldots, a_n, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $a_n \neq 0$.

2.1.1 Roots of the unity polynomial

Let $\omega = \exp(2\pi i/n)$. Let $\xi = \rho \exp(i\alpha)$ and let $z_0 = \sqrt[n]{\rho}\exp(i\alpha/n)$ be one solution. The n roots of ξ are $z_0, z_0\omega, \ldots, z_0\omega^{n-1}$.

2.1.2 Some definitions

Let $A \subset \mathbb{C}$. We write

$$f : A \to \mathbb{C}$$

with A denoting the domain of definition of f.

- **Open disk**: A set of the form
$$\{z \in \mathbb{C} : |z - z_0| < \rho\}, \quad \rho > 0. \tag{7}$$
The boundary is the unit circle $|z - z_0| = \rho$ which is *not* part of the set.
- **Unit disk**: This is the set
$$\{z \in \mathbb{C} : |z| < 1\}. \tag{8}$$
- **Neighbourhood**: A neighbourhood of a point z_0 means a disk of the form $\{z \in \mathbb{C} : |z - z_0| < \rho\}$ for some $\rho > 0$.
- **Interior point**: The interior point of A is a point $z_0 \in A$ such that a neighbourhood of z_0 is also in A.
- **Open set**: A set such that every point is an interior point.
- **Boundary point**: A boundary point of A is a point z_0 such that every neighbourhood of z_0 contains points which are in A and also contain points which are not in A.
- **Boundary**: The boundary of A is the set of all it's boundary points.
- **Polygonal path**: Let $w_1, w_2, \ldots, w_{n+1}$ be points in $\mathbb{C}$ and let l_k be the straight line segment joining w_k to w_{k+1}. The successive line segments $l_1, l_2, \ldots, l_{n+1}$ is a polygonal path joining w_1 to w_{n+1}.
- **Connected**: A set A is connected if every pair of points z_1 and z_2 in A can be joined by a polygonal path which is contained in A.
- **Domain**: An open connected set.
- **Region**: A domain or a domain together with some or all of the boundary points.
- **Bounded**: A set A is bounded if there exists $R > 0$ such that the set is contained in the disk $\{z : |z| < R\}$.
- **Unbounded**: A set is unbounded if it's not bounded.
- A domain (which is thus connected) and does not have holes.

2.1.3 Limits

Definition 4 *Let f be defined in a neighbourhood of z_0 and let $f_0 \in \mathbb{C}$. If for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that*

$$|f(z)| < \epsilon \text{ for all } z \text{ satisfying } 0 < |z - z_0| < \delta,$$

then we say that

$$\lim_{z \to z_0} f(z) = f_0. \tag{9}$$

2.1.4 Continuity

Definition 5 *A function $w = f(z)$ is continuous at $z = z_0$ provided $f(z_0)$ is defined and*

$$\lim_{z \to z_0} f(z) = f(0). \tag{10}$$

Suppose that $f(z)$ and $g(z)$ are continuous at z_0.

- $f(z) \pm g(z)$ and $f(z)g(z)$ are continuous at z_0.
- $f(z)/g(z)$ is continuous at z_0 provided $g(z) \neq 0$.

Suppose that $f(z)$ is continuous at z_0 and $g(z)$ is continuous at $f(z_0)$ then $g(f(z))$ is continuous at z_0. Let $f(z) = u(x, y) + iv(x, y)$. If f is continuous at $z_0 = x_0 + iy_0$ then u and v are both continuous as functions on $\mathbb{R}^2$ at (x_0, y_0). Conversely, if u and v are both continuous at (x_0, y_0) then f is continuous at $z_0 = x_0 + iy_0$.

3 Week 3

3.1 Functions and the Cauchy-Riemann equations

3.1.1 Analytic functions

Theorem 2 *Let f be a complex valued function defined in a neighbourhood of z_0. The derivative of f at z_0 is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{11}$$

provided the limit exists. Note that here $h \in \mathbb{C}$.

- A function f is analytic at z_0 if f is differentiable at all points in some neighbourhood of z_0.
- A function f is analytic in a domain if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an entire function if it is analytic on the whole complex plane $\mathbb{C}$.

3.1.2 Combining differentiable functions

Let f and g be differentiable at z_0. We have the following:

1. $$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0).$$
2. $$(cf)'(z_0) = cf'(z_0)$$
   for all constants $c \in \mathbb{C}$.
3. $$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0). \tag{12}$$
   This is the product rule.
4. $$(\frac{f}{g})'(z_0) = \frac{g(z_0)g'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \quad \text{if } g(z_0) \neq 0. \tag{13}$$
   This is the quotient rule.
5. Let now f be a function which is differentiable at $g(z_0)$. Then
   $$\frac{d}{dz} f(g(z)) \Big|_{z=z_0} = f'(g(z_0))g'(z_0). \tag{14}$$
   This is the chain rule.

3.1.3 The Cauchy-Riemann equations

Let $f(z) = u(x, y) + iv(x, y)$. When f is analytic at z_0 the following limit exists:

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{15}$$

By considering the case when h is real and then purely imaginary we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{16}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{17}$$

Equating the real and imaginary parts gives the Cauchy-Riemann equations.

Theorem 3 *The Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{18}$$

The Cauchy-Riemann equations in polar coordinates are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}. \tag{19}$$

4 Week 4

4.1 Analytic functions

4.1.1 Gradient

Theorem 4 *Te gradient of ϕ is*

$$\nabla\phi = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k}. \tag{20}$$

4.1.2 Directional derivative

Theorem 5 *The directional derivative of ϕ in the direction of a unit vector $\underline{n}$ is*

$$\frac{\partial \phi}{\partial n}(\underline{r}) = \left.\frac{\partial}{\partial s}\phi(\underline{r} + s\underline{n})\right|_{s=0} \tag{21}$$

$$= \left(n_1\frac{\partial \phi}{\partial x_1} + n_2\frac{\partial \phi}{\partial x_2} + n_3\frac{\partial \phi}{\partial x_3}\right)(\underline{r}) = \underline{n}\cdot\nabla\phi(\underline{r}). \tag{22}$$

When s is small

$$\phi(\underline{r} + s\underline{n}) - \phi(\underline{r}) \approx s\frac{\partial \phi}{\partial n}(\underline{r}) = (s\underline{n})\cdot\nabla\phi(\underline{r}). \tag{23}$$

4.1.3 Analytic function definition

Definition 6 *A function that is analytic holds the Cauchy-Riemann equations true.*

5 Week 5

5.1 Analytic functions

5.1.1 Harmonic functions

Theorem 6 *$\phi(x, y)$ is harmonic if*

$$\nabla^2\phi = \frac{\partial \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{24}$$

5.1.2 Harmonic Conjugate

Theorem 7 *If $f = u + iv$ is analytic then u and v are harmonic functions. v is said to be the harmonic conjugate of u.*

6 Week 6

6.1 Elementary functions of z

6.1.1 Representation of polynomials and zeros

Polynomials are entire functions and can be represented in several ways.

$$\begin{aligned}
p_n(z) &= \sum_{k=0}^{n} a_k z^k \\
&= \sum_{k=0}^{n} \frac{p_n^{(k)}(0)}{k!} z^k, \text{ finite Maclaurin series,} \\
&= \sum_{k=0}^{n} \frac{p_n^{(k)}(z_0)}{k!} (z - z_0)^k, \text{ Taylor polynomial ,} \\
&= a_n(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n), \text{ in terms of the zeros,} \\
&= a_n(z - \zeta_1)^{r_1}(z - \zeta_2)^{r_2}\cdots(z - \zeta_m)^{r_m},
\end{aligned}$$

where $\zeta_1, \ldots, \zeta_m$ are the distinct zeros and $r_1 + \cdots + r_m = n$.
At the zero ζ_k of multiplicity r_k we have

$$p_n(\zeta_k) = p'(\zeta_k) = \cdots = p_n^{(r_k-1)}(\zeta_k) = 0,\ p_n^{(r_k)}(\zeta_k) \neq 0. \tag{25}$$

6.1.2 Rational functions

Theorem 8 *A ration function is the ratio of two polynomials, p, q, such that*

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_n). \tag{26}$$

where $\zeta_1, \ldots, \zeta_n$ are singular points.

If the limits exists as $z \to \zeta_k$ then ζ_k is a removable singularity.
Otherwise $R(z)$ has a pole singularity at ζ_k.
A simple pole is the case when $1/R(z)$ has a simple zero at ζ_k.
The order of the pole of $R(z)$ is the multiplicity of the zero of $1/R(z)$.

6.1.3 Partial fractions representation

From eq. (26), when $\deg p(z) < \deg q(z)$ and the zeros of $q(z)$ are simple we have the partial fraction representation of the form

$$R(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - \zeta_k}. \tag{27}$$

When $\deg p(z) \geq \deg q(z)$ and the zeros of $q(z)$ are simple we have a representation of the form

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - \zeta_k}. \tag{28}$$

In either case, $A_k$is the residue at ζ_k.

6.1.4 Residues

When $R(z)$ is in the form of eq. (28), to get A_k we have

$$A_k = \lim_{z \to \zeta_k} (z - \zeta_k) R(z) = \lim_{z \to \zeta_k} \frac{(z - \zeta_k) p(z)}{q(z)},$$
$$= p(\zeta_k) \lim_{z \to \zeta_k} \frac{(z - \zeta_k)}{q(z)} = \frac{p(\zeta_k)}{q'(\zeta_k)}. \tag{29}$$

When $q(z)$ has a zero at ζ of multiplicity $r \geq 1$ we need terms involving

$$\frac{1}{z - \zeta}, \frac{1}{(z - \zeta)^2}, \ldots, \frac{1}{(z - \zeta)^r}.$$

The general case is as follows.
Let

$$R(z) = \frac{p(z)}{q(z)}. \tag{30}$$

We re-label to concentrate on on of the zeros of $q(z)$ at ζ and write

$$R(z) = \cdots + \frac{B_1}{z - \zeta} + \cdots + \frac{B_r}{(z - \zeta)^r} + \cdots. \tag{31}$$

Then

$$(z - \zeta)^r R(z) = B_r + B_{r-1}(z - \zeta) + \cdots + B_1 (z - \zeta)^{r-1} + (z - \zeta)^r (\text{a function at } \zeta). \tag{32}$$

To get the residue B_1 we have

$$B_1 = \frac{1}{(r-1)!} \lim_{z \to \zeta} \left(\frac{d^{r-1}}{dz^{r-1}} (z - \zeta)^r R(z) \right). \tag{33}$$

A similar type of calculation is done for the other coefficients.

7 Week 7

7.1 Elementary functions of z

7.1.1 General case of the residues

Let

$$R(z) = \frac{p(z)}{(z - \zeta_1)^{r_1} (z - \zeta)^{r_2} \cdots (z - \zeta_n)^{r_n}}. \tag{34}$$

With the procedures above we can get the coefficients in the following candidate representation of $R(z)$.

$$\left(\frac{A_{1,1}}{z - \zeta_1} + \cdots + \frac{A_{r_1,1}}{(z - \zeta_1)^{r_1}} \right) + \cdots + \left(\frac{A_{1,n}}{z - \zeta_n} + \cdots + \frac{A_{r_n,n}}{(z - \zeta_n)^{r_n}} \right). \tag{35}$$

The coefficients are

$$A_{i,j} = \frac{1}{(r_j - i)!} \lim_{z \to \zeta_j} \left(\frac{d^{r_j - i}}{dz^{r_j - i}} (z - \zeta_j)^{r_j} R(z) \right), \quad i = 1, 2, \ldots, r_j. \tag{36}$$

7.1.2 Complex trig functions

We define

$$\cosh z = \frac{1}{1}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{1}(e^z - e^{-z}),$$
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

As in the real case

$$\frac{d}{dz} \cosh z = \sinh z, \qquad \frac{d}{dz} \sinh z = \cosh z,$$
$$\frac{d}{dz} \cos z = -\sin z, \qquad \frac{d}{dz} \sin z = \cos z.$$

We also have the identities

$$\cos^2 z + \sin^2 z = \cosh^2 z - sinh^2 z = 1. \tag{37}$$

For all $z_1,\, z_2 \in \mathbb{C}$ we have the addition formulas

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \tag{38}$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2. \tag{39}$$

7.1.3 The real and imaginary parts of $\sin(z)$ and $\cos(z)$

With $z = x + iy$, $x, y \in \mathbb{R}$ we have

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{40}$$

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y. \tag{41}$$

The real and imaginary parts of these functions are hence harmonic functions.

8 Week 8

8.0.1 Exponential function

$$e^z \equiv \exp(z) := e^x e^{iy} = e^x(\cos y + i \sin y). \tag{42}$$

As in the real case we have for all $z,\, z_1,\, z_2 \in \mathbb{C}$,

$$\frac{d}{dz} e^z = e^z, \quad e^{-z} = 1/e^z, \quad e^{z_1+z_2} = e^{z_1} e^{z_2}. \tag{43}$$

The function $w = \exp(z)$ is periodic with period $2\pi i$ and is one-to-one on

$$G = \{z = x + iy :\, -\pi < y \leq \pi\} \tag{44}$$

with inverse

$$\text{Log } w = \ln|w| + i\text{Arg } w \tag{45}$$

which is the principal valued logarithm.

8.0.2 $\cot$ and $\tanh$

$$\cot z = \frac{\cos z}{\sin z}, \quad \tan z = \frac{\sin z}{\cos z} = -\cot(z + \pi/2) = \frac{1}{\tan(\pi/2 - z)}. \tag{46}$$

$\cot z$ has simple poles at $k\pi$ and $\tan z$ has simple poles at $\pi/2 + k\pi$ where $k \in \mathbb{Z}$.

8.0.3 Log z and the multi-valued $\log z$

Definition 7 *The principal valued logarithm is*

$$Log\ z = \ln|z| + iArg\ z. \tag{47}$$

The multi-valued version $w = \log z$ means all complex numbers such that

$$e^w = z$$

and the set of values is

$$\{\text{Log } z + 2k\pi i :\, k \in \mathbb{Z}\}.$$

In both cases

$$e^{\text{Log } z} = e^{\log z} = z.$$

8.0.4 Complex powers z^α

Definition 8 *The principal value of z^α is defined as*

$$e^{\alpha Log\ z}. \tag{48}$$

The possibly multi-valued version of this is

$$e^{\alpha \log z} \tag{49}$$

9 Week 9

9.1 Integrals, arcs and contours

9.1.1 Series and the residue more generally

Taylor series: If $f(z)$ is analytic in the disk $|z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{50}$$

and the series converges uniformly in $|z - z_0| \leq R' < R$.
Laurent series: If $f(z)$ is analytic in $0 < r < |z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}, \tag{51}$$

there is uniform convergence in $0 \leq r < r_1 \leq |z - z_0| \leq R_1 < R$.
Both series are unique once z_0 is specified.
All the coefficients can be written as loop integrals.
The coefficients a_{-1} is the residue at z_0 when $r = 0$.

9.2 Cauchy theorems

Let f be a function which is analytic in a domain D and let Γ be a positively orientated loop in D and let z be a point inside D.

Theorem 9 ***The Cauchy-Goursat theorem:***

$$\oint_{\Gamma} f(\zeta) d\theta = 0. \tag{52}$$

Theorem 10 ***The Cauchy integral formula:***

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{53}$$

Theorem 11 ***The generalised Cauchy integral formula:***

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n = 0, 1, 2, \ldots \tag{54}$$

9.2.1 Integral of a complex valued function

If $f : [a, b] \to \mathbb{C}$ with $f = u + iv$, u, $v \in \mathbb{R}$ then

$$\int_a^b f(x)\, dx = \int_a^b u(x)\, dx + i \int_a^b v(x)\, dx. \tag{55}$$

9.2.2 A smooth arc

Definition 9 *A set $\gamma \subset \mathbb{C}$ is a smooth arc if the set can be described in the form*

$$\{z(t) :\ a \leq t \leq b\} \tag{56}$$

where $z'(t)$ is continuous on $[a, b]$ and $z'(t) \neq 0$ on $[a, b]$.

The arc is said to be closed if the starting point $z(a)$ and the end point $z(b)$ are the same.
If the arc is not closed then we also require that $z(t)$ is one-to-one on $[a, b]$, i.e. it does not intersect itself.
If the arc is closed then we require that $z(t)$ is $one - to - one$ on $[a, b)$ with

$$z(b) = z(a), \quad z'(b) = z'(a).$$

Definition 10 *A smooth arc with a specific ordering of points is known as a directed smooth arc,*

9.2.3 A contour

Definition 11 *A contour is one point or a finite sequence of directed smooth arcs γ_k with the end of γ_k being the start of arc γ_{k+1}.*

Theorem 12 *Let $\gamma = \{z(t) : a \le t \le b\}$ and let $a = t_0 < t_1 < \cdots < t_m = b$. The length of the arc is approximately*

$$\sum_{i=1}^{m} |z(t_i) - z(t_{i-1})|. \tag{57}$$

When $t - t_{i-1}$ is small, we have

$$l(\gamma) = \textit{length of } \gamma = \int_a^b |z'(t)|\, dt. \tag{58}$$

Definition 12 *Let $a = t_0 < t_1 < \cdots < t_m = b$ and let*

$$A_m = \sum_{i=1}^{m} h_i f(z(t_{i-1/2})), \quad h_i = z(t_i) - z(t_{i-1}).$$

$$\begin{aligned} h_i f(z(t_{i-1/2})) &= (z(t_i) - z(t_{i=1}))f(z(t_{i-1/2})) \\ &\approx f(z(t_{i-1/2}))z'(t_{i-1/2})(t_i - t_{i-1}). \end{aligned}$$

$$\int_\gamma f(z)\, dz = \lim_{\substack{m\to\infty \\ max_i|h_i|\to 0}} A_m = \int_a^b f(z(t))z'(t)\, dt. \tag{59}$$

The value here does not depend on which particular valid parameterisation $z(t)$ that we use to describe γ.

9.2.4 The ML inequality

Lemma 1 *Let M and L be defined by*

$$M = \max_{z\in\Gamma} |f(z)| \quad \textit{and} \quad L = \textit{length of } \Gamma. \tag{60}$$

From the bound on $|f(z)|$ and the triangle inequality we have

$$\left|\sum_{i=1}^{m} h_i f(z(t_{i-1/2}))\right| \le \sum_{i=1}^{m} |h_i||f(z(t_{i-1/2}))| \le M \sum_{i=1}^{m} |h_i| \le ML.$$

As the bound above is independent of m and as the integral is an appropriate limit of such a sum we have

$$\left|\int_\gamma f(z)\, dz\right| \le ML. \tag{61}$$

9.2.5 Independence of path when $f = F'$

Theorem 13 *If there exists an anti-derivative F along the path then*

$$\frac{d}{dt} F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t). \tag{62}$$

This is the integrand in the expression for the contour integral.

Theorem 14 *Suppose that the function f(z) is continuous in a domain D and has an anti-derivative f(z) throughout D. Then for any contour Γ contained in D with initial point z_I and an end point z_E we have*

$$\int_\Gamma f(z)\, dz = F(z_E) - f(z_I). \tag{63}$$

10 Week 10

10.1 Loop integrals

10.1.1 Closed loops and powers of z

Let Γ denote a closed loop.
Let $n \in \mathbb{Z}$ and $z_0 \in \mathbb{C}$.
When $n \neq -1$ the anti-derivative of $(z - z_0)^n$ is $(z - z_0)^{n+1}/(n + 1)$ and as a consequence

$$\oint_\Gamma (z - z_0)^n \, dz = 0. \tag{64}$$

When $n = -1$ the function $1/(z - z_0)$ has an anti-derivative $\mathrm{Log}(z - z_0)$ but this function is discontinuous on a branch cut starting from z_0. The value of the integral depends on whether z_0 is inside or outside the loop.

$$\oint_\Gamma \frac{dz}{z - z_0} = \begin{cases} 2\pi i, & \text{if } z_0 \text{ is inside } \Gamma, \\ 0, & \text{if } z_0 \text{ is outside } \Gamma. \end{cases}$$

The integral does not exist in the usual sense when z_0 is on *Gamma*.

10.1.2 Path independence, loop integrals and anti-derivatives

The following are equivalent statements involving the integral of f:

1. All loop integrals of f are 0.
2. The value of the integral of f only depends on the end points.
3. There exists an anti-derivative F, i.e. $F' = f$.

10.1.3 Loop integrals and rational functions

Theorem 15 *If $z_1, \ldots, z_m$ are points inside Γ at which $R(z)$ has poles then*

$$\begin{aligned} \oint_\Gamma R(z) \, dz &= \sum_{k=1}^m A_k \oint_\Gamma \frac{dz}{z - z_k}, \\ &= 2\pi i \sum_{k=1}^m A_k. \end{aligned} \tag{65}$$

11 Week 17

11.1 Harmonic functions - further results

Suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D with u, $v \in \mathbb{R}$. We have the following:

- As all derivatives of f exist and are analytic it follows that all partial derivatives of u and v exist and are continuous.
- Both u and v are harmonic, i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$, with v being the harmonic conjugate of u.
- $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{66}$$

  As f' is analytic the first partial derivatives of u and v are harmonic functions. All the partial derivatives of u and v are harmonic.

11.1.1 Creating an analytic function from a harmonic function

Suppose we have a function ϕ which is harmonic in a domain D. Let

$$g(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}.$$

Define

$$u_1 = \frac{\partial \phi}{\partial x} \quad \textit{and} \quad v_1 = -\frac{\partial \phi}{\partial y}. \tag{67}$$

Check that the Cauchy-Riemann equations are satisfied.
If we now restrict D to be a simply connected domain then the property that g is analytic in D implies that all loop integrals of g are 0 and this in turn implies that g has an anti-derivative G in D, i.e. $G' = g$. If we represent G as $G = u + iv$, $u, v, \in \mathbb{R}$ then as G is analytic we now have

$$g = G' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}. \tag{68}$$

This implies that $\phi = u + C$, where C is a constant, and we know that a function v exists which is a harmonic conjugate of ϕ.

Theorem 16 *When a domain D is simply connected and u is a harmonic in D there exists a harmonic conjugate v such that $u + iv$ is analytic in D.*

This result is **not** true when D is an annulus.

12 Week 18

12.1 Functions defined by loop integrals

Let Γ denote a closed loop traversed once in the anti-clockwise direction and let $g(\zeta)$ denote any continuous function defined on Γ. If we define

$$G(z) = \oint_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta \tag{69}$$

then this defines an analytic function for z inside Γ and it also defines an analytic function for z outside Γ. As an example:

$$g(\zeta) = \frac{1}{2\pi i}\left(\frac{f(\zeta)}{\zeta - z_0}\right) \quad \text{gives } G(z) = \begin{cases} f'(z_0), & \text{if } z = z_0, \\ \frac{f(z)-f(z_0)}{z-z_0}, & \text{if } z \neq z_0. \end{cases} \tag{70}$$

In this case $g(\zeta)$ is defined inside the loop and has a singularity at $\zeta = z_0$.

12.2 Loop integrals of $f(z)/q(z)$ where q =polynomial

Suppose $f(z)$ is analytic inside a loop and

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \dots (z - z_n)^{r_n},$$

with $r_k \geq 1$ for $k = 1, \dots, n$. Using partial fractions we get

$$\frac{f(z)}{q(z)} = f(z)\left(\cdots + \frac{A_{1,k}}{z - z_k} + \cdots + \frac{A_{r_k,k}}{(z - z_k)^{r_k}} + \cdots\right). \tag{71}$$

By using the generalised Cauchy integral formula on each term for each point z_k inside Γ we can determine

$$\oint_\Gamma \frac{f(z)}{q(z)}\, dz. \tag{72}$$

12.3 Other versions of the formulae and entire functions

The Cauchy integral formula when Γ = circle of radius R:

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \in_0^{2\pi} f(\_0 + Re^{it})e^{-int}\, dt. \tag{73}$$

This is used to show that a bounded entire function is a constant (Liouville's theorem) by considering what happens when $n = 1$ and $R \to \infty$.

12.3.1 The fundamental theorem of algebra

Theorem 17 *Every non-constant polynomial with complex coefficients as at least one zero.*

12.3.2 Further Results

Lemma 2 *If $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D then f(z) is a constant.*

12.3.3 The mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt, \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{it})|\,dt. \tag{74}$$

Lemma 3 *Suppose $f(z)$ is analytic in a disk centered at z_0. If the maximum value of $|f(z)|$ over the disk is the value at the centre, i.e. the value $|f(z_0)|$, then $f(z)$ is constant on the disk.*

12.3.4 The maximum modulus theorem

Theorem 18 *If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point $z_0 \in D$ then f is a constant in D.*

12.4 Definitions: sequences in $\mathbb{C}$

- A sequence z_0, z_1, $z_2, \ldots$ converges to z if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$|z_n - z| < \epsilon \quad \text{for all } n \ge N. \tag{75}$$

- A sequence z_0, z_1, z_2, $\ldots$ is a Cauchy sequence if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$|z_n - z_m| < \epsilon \quad \text{for all } n \ge N \text{ and } m \ge M. \tag{76}$$

12.4.1 Result about convergence

A sequence in $\mathbb{C}$ converges if and only if it is a Cauchy sequence.

12.5 Definitions: series in $\mathbb{C}$

- Let c_0, c_1, $c_2, \ldots$ denote a sequence. A series is an expression of the form

$$c_0 + c_1 + c_2 + \cdots \quad \text{and we write as} \quad \sum_{k=0}^{\infty} c_k. \tag{77}$$

  The sequence of partial sums are given by

$$s_n = \sum_{k=0}^{n} c_k, \quad n = 0,\, 1,\, 2, \ldots \tag{78}$$

- The series converges if the sequence of partial sums converges and it diverges if the sequence of partial sums diverges. When we have convergence we say that

$$s = \sum_{k=0}^{\infty} c_k \tag{79}$$

  is the sum of the series.

- If $\sum |c_k|$ converges then $\sum c_k$ is absolutely convergent.

12.5.1 Results about series in $\mathbb{C}$

- If a series $\sum c_k$ converges then $c_n \to 0$ as $n \to \infty$.
- If the series $\sum |c_k|$ converges then $\sum c_k$ converges.
- **Comparison test**: If there exists K such that $|c_k| \leq M_k$ for all $k \geq K$ and $\sum M_k$ converges then $\sum c_k$converges.
- From the identity

$$(1-c)(1+c+c^2+\cdots+c^n) = 1-c^{n+1} \tag{80}$$

  we have that the geometric series

$$\sum_{k=0}^{\infty} c^k = \frac{1}{1-c}, \quad \text{when}|c| < 1. \tag{81}$$

- **Ratio test**: If $|c_{k+1}/c_k| \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.
- **Root test**: If $|c_k|^{1/k} \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.

12.6 Series of functions

Suppose that $f_0(z)$, $f_1(z), \ldots$ are all defined on D and let

$$F_n(z) = \sum_{k=0}^{n} f_k(z), \quad n = 0, 1, 2, \ldots \tag{82}$$

$\sum f_k(z)$ converges pointwise on D if $(F_n(z))$ converges $\forall z \in D$. The sequence converges uniformly to $F(z)$ on D if

$$\sup_{z \in D} |F_n(z) - F(z)| \to 0 \quad \text{as } n \to \infty. \tag{83}$$

A sufficient condition for a series to converges uniformly is the Weierstrass M-test: If $|f_k(z)| \leq M_k$ for all $z \in D$ and $\sum M_k$ converges then the series converges uniformly in D.
Uniform convergence preserves continuity: If $F_n(z)$, $n = 0, 1, 2, \ldots$ are continuous in D and $F_n \to F$ uniformly on D then the limit function $F(z)$ is also continuous in D.

12.7 Uniform convergence and analytic functions

Theorem 19 *Let $F_n(z)$ be a sequence of analytic functions in a simply connected domain D and converging uniformly to $F(z)$ in D. Then $F(z)$ is analytic in D.*

13 Week 19

13.1 Taylor series for analytic functions

If $f(z)$ is analytic at z_0 then the series

$$f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \tag{84}$$

is called the Taylor series for $f(z)$ around z_0.

Theorem 20 *If $f(z)$ is analytic in the disk $|z-z_0| < R$ then the Taylor series converges to $f(z)$ for all z in this disk and in any closed disk $|z-z_0| \leq R' < R$ the convergence is uniform.*

13.1.1 Key formula in the proof of the Taylor series

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + T_n(z) \tag{85}$$

where

$$T_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta-z)}\, d\zeta. \tag{86}$$

It can be shown that $\max\{|T_n(z)| : |z-z_0| \leq R'\} \to 0$ as $n \to \infty$.

13.1.2 Taylor's series, comments about R

If $f(z)$ is analytic at z_0 then the Taylor series is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \tag{87}$$

If $f(z)$ is analytic in $|z - z_0| < R$ then the series converges to $f(z)$ in this disk with uniform convergence in $|z - z_0| \leq R' < R$ for all $R' < R$.
If $f(z)$ is not an entire function then the largest R is such that $f(z)$ has a non-analytic point on $|z - z_0| = R$.

13.2 Maclaurin series case

Maclaurin series is the case of Taylor series when $z_0 = 0$.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k. \tag{88}$$

If $f(z)$ is analytic in $|z| < R$ then the series converges to $f(z)$ in this disk with uniform convergence in $|z| \leq R' < R$ for all $R' < R$.

13.3 Real coefficients, even functions, odd functions, etc.

If $f(z) = u(x, y) + iv(x, y)$ is real when z is real then

$$v(x, 0) = 0 \quad \text{and} \quad f^{(n)}(0) = \left.\frac{\partial^n u(x, 0)}{\partial x^n}\right|_{x=0} \quad \text{is real.} \tag{89}$$

If $R =$ radius of convergence and $0 < r < R$ then we have

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} f(re^{it}) e^{-int}\, dt \tag{90}$$

$$= \frac{1}{2\pi r^n} \int_{0}^{\pi} (f(re^{it}) + (-1)^n f(-re^{it})) e^{-int}\, dt. \tag{91}$$

If $f(-z) = f(z)$ then the Maclaurin series only has even powers.
If $f(-z) = -f(z)$ then the Maclaurin series only has odd powers.

13.4 Series you are expected to know

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots, \quad \text{valid for } |z| < 1 \tag{92}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \tag{93}$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} + \cdots + \frac{-z^n}{n!} + \cdots \tag{94}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \tag{95}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \tag{96}$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \tag{97}$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \tag{98}$$

$$\tag{99}$$

14 Week 20

14.1 The Koebe function, de Branges' theorem and a conjecture

The Koebe function says

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots . \tag{100}$$

Suppose that you consider all functions $g(z)$ which are analytic in the unit disk, are one-to-one and satisfy $g(0) = 0$ and $g'(0) = 1$. Such functions have Maclaurin series of the form

$$g(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \tag{101}$$

de Branges proved that $|a_n| \leq n$.
Bierberbach proved that $|a_2| \leq 2$.

14.2 Multiplying series - the Cauchy product

If $f(z)$ and $g(z)$ are both analytic in $|z - z_0| < R$ then $h(z) = f(z)g(z)$ is also analytic in $|z - z_0| < R$.
Let $z_0 = 0$.

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots , \tag{102}$$

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots , \tag{103}$$

$$h(z) = c_0 + c_1 z + c_2 z^2 + \cdots . \tag{104}$$

We get

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0. \tag{105}$$

This expression for c_n is known as the Cauchy product.

14.3 Leibnitz's formula for the nth derivative of a product

Repeatedly using the product rule gets

$$h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} f^{(n-k)}. \tag{106}$$

This is known as Leibnitz's rule for the nth derivative of a product.

14.4 The generalised L'Hopital's rule

If

$$g(z_0) = g'(z_0) = \cdots = g^{(m-1)}(z_0) = 0 \quad \text{and} \quad g^{(m)}(z_0) \neq 0 \tag{107}$$

and if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \tag{108}$$

then for z near z_0 we have

$$f(z) = a_m (z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots , \tag{109}$$

$$g(z) = b_m (z - z_0)^m + b_{m+1}(z - z_0)^{m+1} + \cdots , \tag{110}$$

$$\frac{f(z)}{g(z)} \to \frac{a_m}{b_m} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \quad \text{as } z \to z_0. \tag{111}$$

If the multiplicity of the zero of $g(z)$ at z_0 is greater than the multiplicity of the zero of $f(z)$ then there is no limit and $f(z)/g(z)$ has a singularity at z_0.

14.5 Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n. \tag{112}$$

The series always converges at $z = z_0$. When it converges at other points the region where it converges is a disk $\{z : |z-z_0| < R\}$ and it is analytic in the disk.
The largest R is the radius of convergence. When $R < \infty$, $\{z : |z-z_0| = R\}$ is the circle of convergence. In all cases

$$R = \frac{1}{\limsup |a_n|^{1/n}}. \tag{113}$$

$R = 0$when we only have convergence at $z = z_0$.
$R = \infty$ when we have convergence for all z.

15 Week 21

15.1 Cauchy-Hadamard theorem

Theorem 21

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{114}$$

has radius of convergence $R = \frac{1}{\alpha}$, *where* α *is the limit of*

$$b_n = \sup\{|a_m|^{1/m} : m \geq n\} \geq 0 \tag{115}$$

for the sequence $|a_n|^{1/n}$.

15.2 Properties of a function defined by a power series

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad R = \frac{1}{\limsup |a_n|^{1/n}}. \tag{116}$$

When $R > 0$ this defines an analytic function in $|z-z_0| < R$.
One way to relate the coefficients a_n to the derivatives of $f(z)$ is to use the generalized Cauchy integral formula. We take a loop Γ in the disk with z_0 inside the loop.

$$\frac{f^{(m)}(z_0)}{m!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{m+1}}\, dz \tag{117}$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n-(m+1)}}\, dz. \tag{118}$$

The only integral in the last line which is non-zero is when $n-(m+1) = -1$, i.e when $n = m$ we get

$$\frac{f^{(m)}(z_0)}{m!} = a_m. \tag{119}$$

15.3 Laurent series

A Laurent series is of the form

$$\sum_{-\infty}^{\infty} a_n (z-z_0)^n. \tag{120}$$

When it converges the region is an annulus $\{z : r < |z-z_0| < R\}$.

$$\sum_{n=-\infty}^{-1} a_n (z-z_0)^n, \quad \text{converges in } |z-z_0| > r. \tag{121}$$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \text{converges in } |z-z_0| < R. \tag{122}$$

To have a function defined at some points we need the coefficients a_n to be such that $r < R$.

15.3.1 Classifying zeros and poles

When $f(z)$ has a zero of multiplicity $m \geq 1$ at z_0 we have

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots = (z-z_0)^m g(z) \tag{123}$$

with $g(z)$ being analytic at z_0 and $g(z) = a_m \neq 0$.
If $f(z)$ has a removable singularity at z_0 then it has a Laurent series with no negative powers valid in $0 < |z-z_0| < R$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \lim_{z \to z_0} f(z) = a_0. \tag{124}$$

If $f(z)$ has a pole of order m then in $0 < |z-z_0| < R$ we have

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = \frac{\phi(z)}{(z-z_0)^m} \tag{125}$$

with $\phi(z)$ being analytic at z_0 and $\phi(z_0) = a_{-m} \neq 0$.
An essential singularity at z_0 has infinitely many negative powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \quad 0 < |z-z_0| < R. \tag{126}$$

16 Week 22

16.1 Integrating a Laurent series - the residue

If we have the Laurent series valid for $0 < |z-z_0| < r$ of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \tag{127}$$

and Γ is any loop in this region with z_0 as an interior point then

$$\oint_\Gamma f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \oint_\Gamma (z-z_0)^n\,dz = 2\pi i a_{-1}, \tag{128}$$

$$\mathrm{Res}(f,\, z_0) = a_{-1} = \text{Residue of } f(z) \text{ at } z = z_0. \tag{129}$$

16.2 The Residue theorem

If $z_1,\, z_2, \ldots,\, z_n$ are isolated singularities inside Γ and $C_1,\, C_2, \ldots,\, C_n$ are non-intersecting circles traversed once in the anti-clockwise direction then $\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)$ is the boundary of a region in which $f(z)$ is analytic and

$$\oint_\Gamma f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz \tag{130}$$

$$= 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k). \tag{131}$$

16.2.1 Techniques to calculate the residue

In the case of a simple pole pf $f(z)$ at z_0 most examples for calculating the residue have involved calculating the limit

$$\mathrm{Res}(f,\, z_0) = \lim_{z \to z_0} (z-z_0) f(z). \tag{132}$$

More generally, when we have a pole of order $m \geq 1$ we can calculate the residue by using

$$\mathrm{Res}(f,\, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)). \tag{133}$$

17 Week 23

17.1 The integrals on C_R^+ when we have a $a^{imz}term$

With $z = x + iy$, $miz = -my + imx$, $e^{imz} = e^{-my}e^{imx}$. When $m > 0$, $|e^{imz}| = e^{-my} <\leq 1$ when $y \geq 0$. When $\deg(Q) \geq \deg(P) + 2$ we have

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\, dz \to 0 \quad \text{and} \quad \int_{C_R^+} \frac{P(z)}{Q(z)} e^{imz}\, dz \to 0 \tag{134}$$

as $R \to \infty$ by using the ML inequality.
When $\deg(Q) = \deg(P) + 1$ Jordan's lemme also gives

$$\int_{C_R^+} \frac{P(z)}{Q(z)} e^{imz}\, dz \to 0 \tag{135}$$

as $R \to \infty$.
When $\deg(Q) = \deg(P) + 1$ there is also a constant $A \geq 0$ such that

$$\left| \frac{Q(Re^{i\theta})iRe^{i\theta}}{Q(Re^{i\theta})} \right| \leq A, \quad \text{for sufficiently large } R. \tag{136}$$

17.2 Counting zeros and poles

Suppose that $f(z)$ is analytic in a domain except for a finite number of poles. Let

$$G(z) = \frac{f'(z)}{f(z)}. \tag{137}$$

Let z_0 be a zero of multiplicity m and let z_p be a pole of $f(z)$ of order n.

$$\operatorname{Res}(G, z_0) = m, \quad \text{and} \quad \operatorname{Res}(G, z_p) = -n. \tag{138}$$

Let $f(z)$ be analytic inside a simple loop C and let $N_0(f)$ be the number of zeros of $f(z)$ inside C.

$$N_0(f) = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz. \tag{139}$$

If $g(z)$ is also analytic inside C and $|g(z)| < |f(z)|$ on C then

$$N_0(f + g) = N_0(f). \tag{140}$$

This is Rouche's called theorem.