7CCMCS00 - Foundations of Complex Systems Modelling

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1 Week 1

1.1 Linear Algebra

1.1.1 Fields

A field is a set $\mathbb{F}$ endowed with two binary operations, a multiplication $\bullet$ and an addition $+$. Both these operations map an ordered pair of elements of $\mathbb{F}$ to another element of $\mathbb{F}$, $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$. For each triplet a, b, $c \in \mathbb{F}$:

- **Associativity:** For both $\circ = +$ and $\circ = \bullet$, then $a \circ (b \circ c) = (a \circ b) \circ c$.
- **Commutativity:** For both $\circ = +$ and $\circ = \bullet$, then $a \circ b = b \circ a$.
- **Identities:** There exists two special elements in $\mathbb{F}$, zero (0) and one (1), such that $a + 0 = a$ and $a \bullet 1 = a$.
- **Inverse:** For each $a \in \mathbb{F}$ there is an element, denoted by $-a \in \mathbb{F}$, such that $a + (-a) = 0$. Similarly for each $a \in \mathbb{F} \setminus 0$, there is an element $\frac{1}{a} \in \mathbb{F}$ such that $a \bullet \frac{1}{a} = 1$.
- **Distributivity:** $a \bullet (b + c) = a \bullet b + a \bullet c$ shows distributivity of $\bullet$ over $+$.

1.1.2 Matrices: definitions

A matrix $\mathbf{A}$ is an $m \times n$-array of scalars $A_{ij} \equiv [\mathbf{A}]_{ij}$ from a given field $\mathbb{F}$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

$$\mathbf{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \in \mathcal{M}_{n \times m}(\mathbb{F}). \tag{1}$$

- Given $\mathbf{A}$, its transpose matrix $\mathbf{A}^T$ is such that $[\mathbf{A}^T]_{ij} = A_{ji}$.
- If $n = m$ then $\mathbf{A}$ is a square matrix.
- If $\mathbb{F} = \mathbb{R}$ and $\mathbf{A}^T = \mathbf{A}$, the matrix $\mathbf{A}$ is real symmetric.

- If $\mathbb{F} = \mathbb{C}$, its Hermitian transpose $\mathbf{A}^\dagger$ has $[\mathbf{A}^\dagger]_{ij} = \overline{A}_{ji}$.
- If $\mathbb{F} = \mathbb{C}$ and $\mathbf{A}^\dagger = \mathbf{A}$, the matrix $\mathbf{A}$ is Hermitian.
- We call vector matrices $\mathbf{v} \in \mathcal{M}_{n\times 1}(\mathbb{F}) \equiv \mathbb{F}^n$.

As an example:

$$\mathbf{A} = \begin{pmatrix} 1 & -i & 2 \\ 3i & 5 & 8i \end{pmatrix} \to \mathbf{A}^T = \begin{pmatrix} 1 & 3i \\ -i & 5 \\ 2 & 8i \end{pmatrix} \to \mathbf{A}^\dagger = \begin{pmatrix} 1 & -3i \\ i & 5 \\ 2 & -8i \end{pmatrix}. \tag{2}$$

A square matrix $\mathbf{A}$ is diagonal if $A_{ij} = 0$, $i \neq j$.

$$\mathbf{A} = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix} = \mathrm{diag}(A_{11},\, A_{22}, \ldots,\, A_{nn}). \tag{3}$$

The matrix $\mathbf{I}_n = \mathrm{diag}(A_{11},\, A_{22}, \ldots,\, A_{nn})$ is special and called the identity matrix.

1.1.3 Matrix operations

If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{n\times m}(\mathbb{F})$, their sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is an element-wise operation. For example:

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} + \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nm} \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} A_{11} + B_{11} & \cdots & A_{1m} + B_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} + B_{n1} & \cdots & A_{nm} + B_{nm} \end{pmatrix} \equiv \mathbf{C} \in \mathcal{M}_{n\times m}(\mathbb{F}). \tag{5}$$

The product $\mathbf{C} = \mathbf{A}\mathbf{B}$ is instead much more involved. If $\mathbf{A} \in \mathcal{M}_{n\times m}(\mathbb{F})$, $\mathbf{B} \in \mathcal{M}_{m\times n'}(\mathbb{F})$,

$$\mathbf{A}\mathbf{B} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nm} \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \sum_k A_{1k}B_{k1} & \cdots & \sum_k A_{1k}B_{kn'} \\ \vdots & \ddots & \vdots \\ \sum_k A_{nk}B_{k1} & \cdots & \sum_k A_{nk}B_{kn'} \end{pmatrix} \equiv \mathbf{C}. \tag{7}$$

Observe that if $\mathbf{C} = \mathbf{A}\mathbf{B}$, the matrix $\mathbf{B}\mathbf{A}$ is not defined unless $\mathbf{A}$ and $\mathbf{B}$ are both squared of the same dimension. Even in this case, however, $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$ in general.
If $\mathbf{A} \in \mathcal{M}_{n\times m}(\mathbb{F})$, we have that $\mathbf{I}_n\mathbf{A} = \mathbf{A}\mathbf{I}_m = \mathbf{A}$.

1.1.4 Linear Systems

Suppose we have to solve:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n = & b_1 \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n = & b_2 \\ \vdots & \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{nn}x_n = & b_n \end{cases}. \tag{8}$$

This can be written as:

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ v_n \end{pmatrix}. \tag{9}$$

1.1.5 The determinant

Definition 1 *Given a matrix* $\mathbf{A} \in \mathcal{M}_{\backslash\times\backslash}(\mathbb{F})$*, its determinant is given by the so-called Laplace expansion along any row* i*:*

$$det\mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} A_{ij}\, det\mathbf{A}^{(\mathbf{i'j'})}. \tag{10}$$

Here, the matrix $\mathbf{A}^{(i'j')}$ *is the* $(n-1)\times(n-1)$*-matrix obtained by removing the* i*th row and* j*th column from* $\mathbf{A}$*, and* $det\mathbf{A}^{(i'j')}$*-minor of* $\mathbf{A}$*.*

If $\det\mathbf{A} = 0$ then $\mathbf{A}$ is singular.

1.1.6 Geometric interpretation of the determinant

It is easily seen that the absolute value of the determinant of $\mathbf{A}$ is the volume of an n-dimensional parallelepid built from the column vectors of the matrix $\mathbf{A}$.

1.1.7 Properties of the determinant

1. If $\mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then $\det\mathbf{A} = \prod_{i=1}^{n} \lambda_i$.
2. $\det\mathbf{A} = \det\mathbf{A}^T$.
3. $\det c\mathbf{A} = c^n \det\mathbf{A}$.
4. If $A_{ik} = \lambda U_k + V_k$;

$$\det\mathbf{A} = \det\begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nk} & \cdots & A_{nn} \end{pmatrix} \tag{11}$$

$$= \lambda \det\begin{pmatrix} A_{11} & \cdots & U_1 & \cdots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \cdots & U_n & \cdots & A_{nn} \end{pmatrix} + \det\begin{pmatrix} A_{11} & \cdots & V_k & \cdots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \cdots & V_n & \cdots & A_{nn} \end{pmatrix}. \tag{12}$$

   It is said that the determinant is multilinear.

5. $\det\mathbf{AB} = \det\mathbf{A}\det\mathbf{B}$.
6. If we swap two columns (or rows) the determinant gets a minus sign: the determinant is alternating. This also means if $\det\mathbf{A}$ has two identical columns then $\det\mathbf{A} = 0$.

1.1.8 Inverse of a square matrix

A square matrix $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{C})$ is invertible if there exists a matrix $\hat{\mathbf{A}}$ such that

$$\hat{\mathbf{A}}\mathbf{A} = \mathbf{A}\hat{\mathbf{A}} = \mathbf{I}_n. \tag{13}$$

We denote $\mathbf{B} = \mathbf{A}^{-1}$. If it exists, this inverse is unique. Moreover,

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{14}$$

Theorem 1 ***Inverse of a square matrix:*** *Let* $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{F})$ *and let* $\mathbf{C}$ *its matrix of cofactors. Then, if* $det(\mathbf{A}) \neq 0$*, the inverse of* $\mathbf{A}$ *exists and is given by*

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}}\mathbf{C}^T. \tag{15}$$

Use $n = 2$ *as a simple example.*

1.1.9 Trace of a square matrix

For a square matrix we define its trace by

$$\mathrm{tr}\mathbf{A} = \sum_{i=1}^{n} A_{ii},\ \mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{F}). \tag{16}$$

Theorem 2 ***Trace and determinant:*** *Let* $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{C})$*. Then*

$$det\left(e^{\mathbf{A}}\right) = e^{tr\mathbf{A}}. \tag{17}$$

In this expression,

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^{k}. \tag{18}$$

1.1.10 Vector spaces

Let us assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let us focus on vectors $\mathbf{v} \in \mathbb{F}^n \equiv V$.
The set V is a vector space over $\mathbb{F}$ i.e., is a set of elements such that, given $\mathbf{u}$, $\mathbf{v} \in \mathbb{F}^n$, then $\alpha\mathbf{v} + \beta\mathbf{u} \in \mathbb{F}$ for any α, $\beta \in \mathbb{F}$. In particular, V contains the zero vector.
Given a set of K vectors $\{\mathbf{v}_k\}_{k=1,\ldots,K}$, its span is the set

$$\mathrm{Span}[\{\mathbf{v}_k\}_k] = \left\{\sum_{k=1}^{K} \alpha_k \mathbf{v}_k \,:\, \alpha_k \in \mathbb{F}\,\forall k\right\}. \tag{19}$$

The set of K vectors $\{\mathbf{v}_k\}_{k=1}^{K}$ is said to be linearly independent if the equation

$$\sum_k \alpha_k \mathbf{v}_k = \mathbf{0} \implies \alpha_k = 0\,\forall k \tag{20}$$

holds true.
A basis is a set $\{\mathbf{e}_k\}_k$ of linearly independent vectors which is maximal. This also means that

$$V = \mathrm{Span}[\{\mathbf{e}_k\}_k]. \tag{21}$$

The cardinalities of a basis V is the dimension of V, and all basis have the same cardinality (if finite).