Luke Dando
21131620

Homework 1

Question 1

Graph 1

Simple & directed.
$V = \{2, 3, 4, 5, 6, 7, 9, 10, 11\}$.
$E = \{(2, 3), (3, 4), (3, 5), (3, 7), (3, 9), (4, 7), (6, 3), (10, 3), (11, 3)\}$.

Graph 2

Simple & non-directed.
$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
$E = \{(1, 2), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2), (8, 2), (9, 2)\}$.

Graph 3

Simple & non-directed.
$V = \{1, 2, 3, 4, 5\}$.
$E = \{(1, 2), (1, 4), (2, 1), (2, 5), (3, 4), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4)\}$.

Question 2 (submitted for feedback)

Graph 1

Relabel $V = \{2, 3, 4, 5, 6, 7, 9, 10, 11\} \to \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Graph 2

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Graph 3

$$\mathbf{A_3} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Question 3

Let $\mathbf{u}_0$ be a (1×9)-vector of 1s. We then use our adjacency matrix $\mathbf{A}_1 \equiv \mathbf{A}$ to help calculate the total number of paths of a given length n. This is expressed as follows

$$p_n = |\mathbf{u}_n|_1 = |\mathbf{u}_0(\mathbf{A})^n|_1, \quad n \geq 1. \tag{4}$$

Here, $|\mathbf{u}|_1$ is the 1-norm of the vector u, summing the modulus of it's elements. This gives u the number of paths of length n available, evaluated as p_n. We can then work through each value of n up to 4 (although it isn't necessary as it isn't an iterative process).

$$\mathbf{u}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

$$= \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{6}$$

$$\rightarrow |\mathbf{u}_1|_1 = p_1 = 9. \tag{7}$$

This indicates there are a total of 9 different ways of traversing a path of length 1 exactly. This can be drawn trivially by hand to consolidate.
Next, $n = 2$;

$$\mathbf{u}_2 = \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$$= \begin{pmatrix} 4 & 1 & 0 & 0 & 4 & 0 & 0 & 4 & 4 \end{pmatrix}, \tag{9}$$

$$\rightarrow |\mathbf{u}_2|_1 = p_2 = 17. \tag{10}$$

This indicates there are a total of 17 different ways of traversing a path of length 2 exactly.
Next, $n = 3$;

$$\mathbf{u}_3 = \begin{pmatrix} 4 & 1 & 0 & 0 & 4 & 0 & 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{12}$$

$$\rightarrow |\mathbf{u}_3|_1 = p_3 = 4. \tag{13}$$

This indicates there are a total of 17 different ways of traversing a path of length 2 exactly. This is also the solution to the first part of this question. More specifically, the 4 paths of length three are;

$$(1, 5, 8, 9) \rightarrow (2) \rightarrow (3) \rightarrow (6). \tag{14}$$

Finally, we need to show there are 0 paths of length $n = 4$.

$$\mathbf{u}_4 = \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1\end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

$$= \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} = \mathbf{0}, \tag{16}$$

$$\to |\mathbf{u}_4|_1 = p_4 = 0. \tag{17}$$

We can see that there are no paths for length $n = 4$. Moreover, as we now have the zero-vector, all subsequent path lengths are forever 0 in number.

Question 4

Graph 1

$$i = (1, 2, 3, 4, 5, 6, 7, 8, 9), \tag{18}$$

$$k_i^{in}(\mathbf{A}_1) = (0, 4, 1, 1, 0, 2, 1, 0, 0), \tag{19}$$

$$k_i^{out}(\mathbf{A}_1) = (1, 4, 1, 0, 1, 0, 0, 1, 1). \tag{20}$$

Graph 2

$$i = (1, 2, 3, 4, 5, 6, 7, 8, 9), \tag{21}$$

$$k_i(\mathbf{A}_2) = (1, 8, 1, 1, 1, 1, 1, 1, 1). \tag{22}$$

Graph 3

$$i = (1, 2, 3, 4, 5), \tag{23}$$

$$k_i(\mathbf{A}_3) = (2, 2, 1, 3, 2). \tag{24}$$

Question 5

For a non-directed graph, as for any edge (i, j), there exists the reverse edge (j, i) such that

$$A_{ij} = A_{ji} \in \{1, 0\}. \tag{25}$$

Therefore

$$k_i^{in}(\mathbf{A}) = \sum_{n=0}^{N} A_{ij} = \sum_{n=0}^{N} A_{ji} = k_1^{out}(\mathbf{A}) = k_i(\mathbf{A}). \tag{26}$$

Let a non-directed $(n \times n)$-graph $\mathbf{G}$ be represented as

$$\mathbf{G} = \begin{pmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nn} \end{pmatrix}. \tag{27}$$

We then want to look at the diagonal of $\mathbf{G}^2$, which is represented as

$$\mathbf{G}^2 = \begin{pmatrix} \sum_{k=1}^{n} G_{1k} G_{k1} & & & \\ & \sum_{k=1}^{n} G_{2k} G_{k2} & & \\ & & \ddots & \\ & & & \sum_{k=1}^{n} G_{nk} G_{kn} \end{pmatrix}. \tag{28}$$

Therefore, we can conclude

$$(\mathbf{G}^2)_{ii} = \sum_{k=1}^{n} G_{ik} G_{ki}. \tag{29}$$

As it is a non-directed graph, we us eq. (25) to therefore conclude

$$(\mathbf{G}^2)_{ii} = \sum_{k=1}^{n} G_{ik} = \sum_{k=1}^{n} G_{ki} = k_i(\mathbf{G}). \tag{30}$$

Question 6

We know that to evaluate the in-degree of a vertex i in $\mathbf{A}$, we find

$$k_i^{in}(\mathbf{A}) = \sum_{n=1}^{N} A_{i,n}. \tag{31}$$

Therefore, for the sum of all in-degrees we have

$$\sum_{i=1}^{N} k_i^{in}(\mathbf{A}) = \sum_{i,n=1}^{N} A_{i,n}. \tag{32}$$

For the out-degree we then have

$$\sum_{i=1}^{N} k_i^{out}(\mathbf{A}) = \sum_{i,n=1}^{N} A_{n,i}. \tag{33}$$

As both i and n span the same set of values from set V;

$$\sum_{i,n=1}^{N} A_{i,n} = \underbrace{\sum_{i=1}^{N} k_i^{in}(\mathbf{A}) = \sum_{i=1}^{N} k_i^{out}(\mathbf{A})}_{} = \sum_{i,n=1}^{N} A_{n,i}. \tag{34}$$