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Global exponential stability of periodic neural networks with time-varying delays

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Abstract

In this paper, we consider the periodic neural networks with variable coefficients and time-varying delays, and investigate the existence of periodic solution and its global exponential stability. Various sufficient conditions ensuring the existence of periodic and its global exponential stability are given. The results obtained in this paper extend and generalize those given in previous literature.

Keywords: Neural networks; Variable coefficients; Time-varying delays; Global exponential stability; Periodic solution

1. Introduction

Recurrently connected neural networks, sometimes called Grossberg–Hopfield neural networks, have attracted increasing interest in both theoretical studies and engineering applications. Many important results on the existence and uniqueness of equilibrium point, global asymptotic stability and global exponential stability for autonomous neural networks have been established and successfully applied to signal processing system, especially in static image treatment, and to solve nonlinear algebraic equations, such application rely on the qualitative properties of stability.

Among the most popular models in the literature of autonomous neural networks is the following continuous-time delayed neural networks.

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(u_j(t-\tau_{ij})) + I_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where activation functions g_j and f_j satisfy certain defining conditions, and d_i, a_{ij}, b_{ij}, I_i $(i,j = 1, 2, \ldots, n)$ are constants.

All these neural networks have been extensively studied. In [1,2,4–6,8–11,14,17–21,23,26,27,29], the authors have obtained many a criterion for checking the existence, uniqueness, global asymptotical stability, global exponential stability and robust stability of neural networks. Several techniques have been introduced, and the combination of Lyapunov function (or functional) method with inequality technique and the combination of Lyapunov function (or functional) method with LMI technique have been widely adopted. But all results are focused upon autonomous neural networks.

However, as we well know, the nonautonomous phenomenon often occurs in many realistic systems. Particularly, when we consider a long-time dynamical behavior of a system, the parameters of the system usually will raise change along with time. In addition, in many

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applications, the property of periodic oscillatory solutions of a neural networks also is of great interest. Therefore, the research on the nonautonomous neural networks with delays is very important in like manner.

In this paper, we will consider the following dynamical systems and their periodic limits:

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t-\tau_{ij}(t))) + I_i(t), \tag{2}$$

where $i = 1, 2, \ldots, n, a_{ij}(t), b_{ij}(t), I_i(t) : R^+ \to R$ are continuously periodic functions with period $\omega > 0$, i.e., $d_i(t) = d_i(t+\omega), a_{ij}(t) = a_{ij}(t+\omega), b_{ij}(t) = b_{ij}(t+\omega), I_i(t) = I_i(t+\omega)$, $\tau_{ij}(t) = \tau_{ij}(t+\omega)$ for all $t > 0$ and $i, j = 1, 2, \ldots, n$.

From the mathematical point of view, system (2) is completely different from system (1), and a known mathematical method do not directly apply. Here, we develop some methods from functional differential equations (see [3,13,25]) to study the existence of periodic solution and its global exponential stability of system (2).

In fact, there have been considerable research on the nonautonomous neural networks (see [7,12,15,16,22,24,28,30]). In [22], when $n = 1$, the authors investigate the periodic solutions and almost periodic solution of the following first-order nonautonomous differential equation

$$\dot{x}(t) = -d(t)u(t) + a(t)f(u(t)) + I(t), \quad t > 0 \tag{3}$$

when $f(u) = \tanh(x)$ is a sigmoid-type function, $d(t) > 0, a(t)$ and $I(t)$ are continuous ω-periodic function with respect to the time variable t. Under assumptions $d(t) \geqslant a(t)$ and $1/\omega \int_0^{\omega} [d(s) - |a(s)|] \, \mathrm{d}s = \mu > 0$, the authors obtain a sufficient condition to ensure the existence, uniqueness and globally exponential stability of periodic solution for periodic system (3). In [7], by combining the theory of the exponential dichotomy and Lyapunov function, the authors study the existence and attractivity of almost periodic solutions for cellular neural networks with distributed delays and variable coefficients. In [15,16,24], the cellular neural networks with variable coefficients and time-varying delays are studied, by using the Lyapunov functional method, the techniques of matrix analysis and inequality analysis, the authors established a series of the criteria on the boundedness, global exponential stability and the existence of periodic solutions. In [12,30], the recurrent neural networks and BAM neural networks with periodic coefficients and delays are studied. Some new sufficient conditions ensuring the existence, uniqueness and global exponential stability of the periodic solution were obtained by using the continuation theorem based on coincidence degree and the Lyapunov functional method. In [28], the authors discuss the dynamical system with variable coefficients and constant delays, obtained the existence of periodic solution and its globally exponential stability.

In this paper, we will discuss the existence of periodic solution and its global exponential stability for nonautonomous system (2). We do not assume that the activations functions are bounded. We also do not use existing complicated theory to prove the existence of periodic solution. Utilizing the method given in [28], we obtain the existence of periodic solution and its globally exponential stability for dynamical system (2) by constructing a new Lyapunov function. We will see that the results obtained in this paper will extend and generalize the corresponding results existing in [12,28,30].

This paper is organized as follows. In Section 2, we will give some preliminaries. In Section 3, we will establish new criteria on the existence of periodic solution and its global exponential stability for periodic nonautonomous system (2). In Section 4, an example is given to illustrate the results obtained in this paper. In Section 5, we will give some concluding remarks of the results.

2. Some preliminaries

Definition 1. Class $H\{G_1, G_2, \ldots, G_n\}$ of functions: let $G = \mathrm{diag}[G_1, G_2, \ldots, G_n]$, where $G_i > 0$ $(i = 1, 2, \ldots, n)$. $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))^{\mathrm{T}}$ is said to belong to $H\{G_1, G_2, \ldots, G_n\}$, if the functions $g_i(x)$ $(i = 1, 2, \ldots, n)$ satisfy

$$\frac{|g_i(x+u) - g_i(x)|}{|u|} \leqslant G_i.$$

Let $\tau = \sup\{\tau_{ij}(t) : t \in [0, +\infty),\ i, j = 1, 2, \ldots, n\}$. We denote by $C[-\tau, 0]$ the Banach space of continuous functions $u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots, u_n(\cdot))^{\mathrm{T}} : [-\tau, 0] \to R^n$ with the following norm:

$$\|u(\cdot)\|_{(\xi,p)} = \left[\frac{1}{n}\sum_{i=1}^{n} \xi_i |u_i(\cdot)|^p\right]^{1/p}.$$

In this paper we always assume that all solutions of system (2) satisfy the following initial conditions:

$$u_i(\theta) = \phi_i(\theta), \quad \theta \in [-\tau, 0], \ i = 1, 2, \ldots, n, \tag{4}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in C[-\tau, 0]$. It is well known that by the fundamental theory of functional differential equations (see [3,13,25]), system (2) has a unique solution $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ satisfying the initial condition (4).

In the following, the Young inequality and Dini derivative will be used to prove the main results of this paper. On the Young inequality we have the following lemma.

Lemma 1 (*Cao [4] and Zheng and Chen [28]*). *Let $a > 0, b > 0$, $p > 1, q > 1$, and $1/p + 1/q = 1$. Then we have the inequality*

$$ab \leqslant \frac{1}{p}(a\varepsilon)^p + \frac{1}{q}(b\varepsilon^{-1})^q,$$

where $\varepsilon>0$. *The equality holds if and only if*

$(a\varepsilon)^p = (b\varepsilon^{-1})^q$.

On the Dini derivative we have the following lemma.

Lemma 2 (*Chen et al. [8]*). *Assume that* $u(t)$ *is a differentiable function defined on* R_+. *Then for any* $t \in R_+$ *the Dini right derivative* $D^+|u(t)|$ *of function* $|u(t)|$ *exists and has the expression as follows*:

$$D^+|u(t)| = \lim_{h\to 0^+}\sup \frac{1}{h}[|u(t+h)| - |u(t)|] = \sigma(u(t))\dot{u}(t),$$

where

$$\sigma(u(t))\dot{u}(t) = \begin{cases} \dot{u}(t) & \text{if } u(t)>0 \text{ or } u(t)=0 \text{ and } \dot{u}(t)>0,\\ -\dot{u}(t) & \text{if } u(t)<0 \text{ or } u(t)=0 \text{ and } \dot{u}(t)<0,\\ 0 & \text{if } u(t)=0 \text{ and } \dot{u}(t)=0 \end{cases}$$

and $\dot{u}(t) = \mathrm{d}u(t)/\mathrm{d}t$.

In this paper, for system (2) we introduce the following assumptions.

(H$_1$) Functions $d_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ $(i,j = 1,2,\ldots,n)$ are bounded and continuous functions defined on $t \in R_+ = [0,\infty)$.

(H$_2$) Functions $f(x) = (f_1(u), f_2(u), \ldots, f_n(u))^{\mathrm{T}} \in H\{F_1, F_2, \ldots, F_n\}$ and $g(u) = (g_1(u), g_2(u), \ldots, g_n(u))^{\mathrm{T}} \in H\{G_1, G_2, \ldots, G_n\}$.

(H$_3$) $\inf_{t\in R_+}\{1-\dot{\tau}_{ij}(t)\}>0$, in which $\dot{\tau}_{ij}(t)$ $(i,j = 1,2,\ldots,n)$ denotes the derivative of $\tau_{ij}(t)$.

3. Main results

In this section, we discuss the existence of periodic solution and its global stability. We propose a general and concise approach and give theorems on the existence of periodic solution and its global exponential stability.

Theorem 1. *Suppose that* (H$_1$) – (H$_3$) *hold and there are constants* $\xi_i>0, \alpha_{ij}, \beta_{ij} \in R$ $(i,j = 1,2,\ldots,n)$ *such that*

$$d_i(t)\xi_i - \sum_{j=1}^n \frac{1}{p} G_i\xi_j|a_{ji}(t)|^{\alpha_{ji}p} - \sum_{j=1}^n \frac{1}{q} G_j\xi_i|a_{ij}(t)|^{(1-\alpha_{ij})q} - \sum_{j=1}^n \frac{1}{q} F_j\xi_i|b_{ij}(t)|^{(1-\beta_{ij})p} - \sum_{j=1}^n \frac{1}{p} F_i\xi_j \frac{|b_{ji}(\psi_{ji}^{-1}(t))|^{\beta_{ji}p}}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} > 0 \tag{5}$$

for all $t\geqslant 0$ *and* $i = 1,2,\ldots,n$, *where* $\psi_{ij}^{-1}(t)$ *is the inverse function of* $\psi_{ij}(t) = t-\tau_{ij}(t)$ $(i,j = 1,2,\ldots,n)$ *and* $1\leqslant p<\infty$ *is a constant. Then the dynamical system* (2) *has a unique periodic solution* $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^{\mathrm{T}}$ *and, for any solution* $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$ *of system* (2), *there exists a scalar* $\varepsilon>0$ *such that*

$$|u_i(t+j\omega) - v_i(t)| = O(\mathrm{e}^{-\mathrm{j}\varepsilon\omega}) \quad i = 1,2,\ldots,n.$$

Proof. Define a function δ:

$$\delta(\varepsilon) = \min_{0\leqslant i\leqslant n}\left\{ d_i(t)\xi_i - \sum_{j=1}^n \frac{1}{p} G_i\xi_j|a_{ji}(t)|^{\alpha_{ji}p} - \sum_{j=1}^n \frac{1}{q} G_j\xi_i|a_{ij}(t)|^{(1-\alpha_{ij})q} - \sum_{j=1}^n \frac{1}{q} F_j\xi_i\,\mathrm{e}^{\varepsilon\tau}|b_{ij}(t)|^{(1-\beta_{ij})p} - \sum_{j=1}^n \frac{1}{p} F_i\xi_j\,\mathrm{e}^{\varepsilon\tau} \frac{|b_{ji}(\psi_{ji}^{-1}(t))|^{\beta_{ji}p}}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} - \varepsilon\xi_i \right\}.$$

Obviously, δ is continuous on R with respect to ε. From the inequality (5), we can see that $\delta(0)>0$, so there exists a positive number $\varepsilon>0$ such that $\delta(\varepsilon)>0$.

Further, let $\bar{u}_i(t) = u_i(t+\omega) - u_i(t), \bar{g}_i(u_i(t)) = g_i(u_i(t+\omega)) - g_i(u_i(t)), \bar{f}_i(u_i(t)) = f_i(u_i(t+\omega)) - f_i(u_i(t))$ and $w_i(t) = \mathrm{e}^{\varepsilon t}\bar{u}_i(t)$, $i = 1,2,\ldots,n$. Defining a Lyapunov function as follows:

$$L(t) = \sum_{i=1}^n \xi_i|w_i(t)|^p + \sum_{i=1}^n\sum_{j=1}^n \xi_i F_j\,\mathrm{e}^{\varepsilon\tau} \times \int_{t-\tau_{ij}(t)}^t \frac{|b_{ij}(\psi_{ij}^{-1}(s))|^{\beta_{ij}p}}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(s))}|w_j(s)|^p\,\mathrm{d}s.$$

Calculating the Dini upper right derivative of $L(t)$ with respect to t, we have

$$\begin{aligned} D^+L(t) = {} & \sum_{i=1}^n \xi_i p|w_i(t)|^{p-1}\,\mathrm{e}^{\varepsilon t}\mathrm{sign}(\bar{u}_i(t))\Bigg[-d_i(t)\bar{u}_i(t) + \varepsilon\bar{u}_i(t) \\ & + \sum_{j=1}^n a_{ij}(t)\bar{g}_j(u_j(t)) + \sum_{j=1}^n b_{ij}(t)\bar{f}_j(u_j(t-\tau_{ij}(t)))\Bigg] \\ & + \sum_{i=1}^n\sum_{j=1}^n \mathrm{e}^{\varepsilon\tau}\xi_i F_j \frac{|b_{ij}(\psi_{ij}^{-1}(t))|^{\beta_{ij}p}}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}|w_j(t)|^p \\ & - \sum_{i=1}^n\sum_{j=1}^n \mathrm{e}^{\varepsilon\tau}\xi_i F_j|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p \\ \leqslant {} & p\sum_{i=1}^n \xi_i\Bigg[-(d_i(t)-\varepsilon)|w_i(t)|^p \\ & + \sum_{j=1}^n |a_{ij}(t)|\mathrm{e}^{\varepsilon t}|\bar{g}_j(u_j(t))||w_i(t)|^{p-1} \\ & \times\sum_{j=1}^n |b_{ij}(t)|\mathrm{e}^{\varepsilon t}|\bar{f}_j(u_j(t-\tau_{ij}(t)))||w_i(t)|^{p-1}\Bigg] \\ & + \sum_{i=1}^n\sum_{j=1}^n \mathrm{e}^{\varepsilon\tau}\xi_i F_j \frac{|b_{ij}(\psi_{ij}^{-1}(t))|^{\beta_{ij}p}}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}|w_j(t)|^p \\ & - \sum_{i=1}^n\sum_{j=1}^n \mathrm{e}^{\varepsilon\tau}\xi_i F_j|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p. \end{aligned} \tag{6}$$

Since

$$\mathrm{e}^{\varepsilon t}|\bar{f}_j(u_j(t-\tau_{ij}(t)))| \leqslant F_j|w_j(t-\tau_{ij}(t))|\mathrm{e}^{\varepsilon\tau}, \tag{7}$$

$$\mathrm{e}^{\varepsilon t}|\bar{g}_j(u_j(t))| \leqslant G_j|w_j(t)|. \tag{8}$$

Further, from (6)–(8), we have

$$
\begin{aligned}
D^+L(t) \leqslant p\sum_{i=1}^{n}\xi_i\Bigg[&-(d_i(t)-\varepsilon)|w_i(t)|^p \\
&+\sum_{j=1}^{n}|a_{ij}(t)|G_j|w_j(t)||w_i(t)|^{p-1} \\
&+\sum_{j=1}^{n}|b_{ij}(t)|e^{\varepsilon\tau}F_j|w_j(t-\tau_{ij}(t))||w_i(t)|^{p-1}\Bigg] \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon\tau}\xi_iF_j\frac{|b_{ij}(\psi_{ij}^{-1}(t))|^{\beta_{ij}p}}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}|w_j(t)|^p \\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\varepsilon\tau}\xi_iF_j|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p. \qquad (9)
\end{aligned}
$$

Estimating the right-hand side of (9) by using Young inequality, we have that

$$
\begin{aligned}
&\sum_{j=1}^{n}|a_{ij}(t)|G_j|w_j(t)||w_i(t)|^{p-1} \\
&\quad=\sum_{j=1}^{n}G_j\{|a_{ij}(t)|^{\alpha_{ij}p}|w_j(t)|^p\}^{1/p} \\
&\qquad\times\{|a_{ij}(t)|^{(1-\alpha_{ij})q}|w_i(t)|^p\}^{1/q} \\
&\quad\leqslant\sum_{j=1}^{n}G_j\Bigg[\frac{1}{p}|a_{ij}(t)|^{\alpha_{ij}p}|w_j(t)|^p \\
&\qquad+\frac{1}{q}|a_{ij}(t)|^{(1-\alpha_{ij})q}|w_i(t)|^p\Bigg] \qquad (10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n}|b_{ij}(t)|e^{\varepsilon\tau}F_j|w_j(t-\tau_{ij}(t))||w_i(t)|^{p-1} \\
&\quad=\sum_{j=1}^{n}F_j\,e^{\varepsilon\tau}\{|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p\}^{1/p} \\
&\qquad\times\{|b_{ij}(t)|^{(1-\beta_{ij})q}|w_i(t)|^p\}^{1/q} \\
&\quad\leqslant\sum_{j=1}^{n}F_j\,e^{\varepsilon\tau}\Bigg[\frac{1}{p}|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p \\
&\qquad+\frac{1}{q}|b_{ij}(t)|^{(1-\beta_{ij})q}|w_i(t)|^p\Bigg]. \qquad (11)
\end{aligned}
$$

From (9)–(11), we finally have

$$
\begin{aligned}
D^+L(t) \leqslant p\sum_{i=1}^{n}\xi_i\Bigg\{&-(d_i(t)-\varepsilon)|w_i(t)|^p \\
&+\sum_{j=1}^{n}G_j\Bigg[\frac{1}{p}|a_{ij}(t)|^{\alpha_{ij}p}|w_j(t)|^p \\
&+\frac{1}{q}|a_{ij}(t)|^{(1-\beta_{ij})q}|w_i(t)|^q\Bigg] \\
&+\sum_{j=1}^{n}F_j\,e^{\varepsilon\tau}\Bigg[\frac{1}{p}|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p \\
&+\frac{1}{q}|b_{ij}(t)|^{(1-\beta_{ij})p}|w_i(t)^p|\Bigg]\Bigg\} \\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_iF_j\frac{|b_{ij}(\psi_{ij}^{-1}(t))|^{\beta_{ij}p}}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}e^{\varepsilon\tau}|w_j(t)|^p \\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_iF_j\,e^{\varepsilon\tau}|b_{ij}(t)|^{\beta_{ij}p}|w_j(t-\tau_{ij}(t))|^p \\
\leqslant p\sum_{i=1}^{n}\{&-(d_i(t)-\varepsilon)\xi_i+\sum_{j=1}^{n}\frac{1}{p}G_i\xi_j|a_{ji}(t)|^{\alpha_{ji}p} \\
&+\sum_{j=1}^{n}\frac{1}{q}G_j\xi_i|a_{ij}(t)|^{(1-\alpha_{ij})q} \\
&+\sum_{j=1}^{n}\frac{1}{q}F_j\xi_i\,e^{\varepsilon\tau}|b_{ij}(t)|^{(1-\beta_{ij})p} \\
&+\sum_{j=1}^{n}\frac{1}{p}F_i\xi_j\,e^{\varepsilon\tau}\frac{|b_{ji}(\psi_{ji}^{-1}(t))|^{\beta_{ji}p}}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t))}\}|w_i(t)|^p \\
=-p\sum_{i=1}^{n}&\delta|w_i(t)|^p\leqslant 0.
\end{aligned}
$$

Therefore, $L(t)$ is bounded, which implies

$$\sum_{i=1}^{n}\xi_i|e^{\varepsilon t}\bar{u}_i(t)|^p \qquad (12)$$

is bounded and

$$|u_i(t+\omega)-u_i(t)| = O(e^{-\varepsilon t}), \quad i=1,2,\ldots,n. \qquad (13)$$

Now, define a function $v(t)=[v_1(t),v_2(t),\ldots,v_n(t)]^T$ by

$$v_i(t)=\lim_{j\to\infty}u_i(t+j\omega).$$

Because of

$$u_i(t+j\omega)=u_i(t)+\sum_{k=1}^{n}\{u_i(t+k\omega)-u_i(t+(k-1)\omega)\}$$

and (13), $v(t)$ is well defined and is a periodic solution with period ω. Moreover,

$$|u_i(t+j\omega)-v_i(t)| = O(e^{-j\varepsilon\omega}) \quad \text{when } j\to\infty. \qquad (14)$$

Next, we prove the uniqueness of periodic solution of system (2). If $u(t), v(t)$ are two solutions with period ω of system (2). By similar method used before, it is easy to prove

$$|u_i(t+j\omega)-v_i(t+j\omega)| = O(e^{-j\varepsilon\omega}) \quad \text{when } j\to\infty. \qquad (15)$$

which means that the limit solution is unique. Theorem 1 is proved completely. □

As direct consequences of Theorem 1, we have the following corollaries.

Corollary 1. *Suppose that* (H_1)–(H_3) *hold and there are* $\xi_i > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$d_i(t)\xi_i - \sum_{j=1}^{n} G_i\xi_j|a_{ji}(t)| - \sum_{j=1}^{n} F_i\xi_j \frac{|b_{ji}(\psi_{ji}^{-1}(t))|}{1 - \dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} > 0 \tag{16}$$

for all $t \geqslant 0$. *Then the dynamical system* (2) *has a unique periodic solution* $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^{\mathrm{T}}$ *and, for any solution* $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$ *of system* (2), *there exists a scalar* $\varepsilon > 0$ *such that*

$$|u_i(t + j\omega) - v_i(t)| = O(\mathrm{e}^{-j\varepsilon\omega}) \quad i = 1, 2, \ldots, n.$$

In fact, in condition (5) taking $p = 1$ and $\alpha_{ij} = \beta_{ij} = 1$ $(i,j = 1, 2, \ldots, n)$, by Theorem 1, we easily obtain the result of Corollary 1.

Remark 1. In [30], the authors obtained the existence and globally exponential stability of periodic solutions for periodic delayed neural networks by utilizing the Mawhin coincidence degree theory and the Lyapunov functional methods. In [15], the authors obtained the existence and global exponential stability of periodic solutions for periodic neural networks with variable coefficients and time-varying delays by using the existence theory of periodic solutions for general functional differential equations. However, in this paper, we obtain the existence and global exponential stability by utilizing the general and very concise approach. Furthermore, the criteria obtained in this paper are different from ones of [12,16,24].

When $\tau_{ij}(t) = \tau_{ij}$, where τ_{ij} $(i,j = 1, 2, \ldots, n)$ are constants, then system (2) degenerates into the following form:

$$\begin{aligned}\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = &- d_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(u_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t - \tau_{ij})) + I_i(t),\end{aligned} \tag{17}$$

where $i = 1, 2, \ldots, n$. For system (17), we have the following result.

Theorem 2. *Suppose that* (H_1), (H_2) *hold and there are constants* $\xi_i > 0$, $\alpha_{ij}, \beta_{ij} \in R$ $(i,j = 1, 2, \ldots, n)$ *such that*

$$\begin{aligned}&d_i(t)\xi_i - \sum_{j=1}^{n} \frac{1}{p} G_i\xi_j|a_{ji}(t)|^{\alpha_{ji}p} - \sum_{j=1}^{n} \frac{1}{q} G_j\xi_i|a_{ij}(t)|^{(1-\alpha_{ij})q} \\ &\quad - \sum_{j=1}^{n} \frac{1}{q} F_j\xi_i|b_{ij}(t)|^{(1-\beta_{ij})p} - \sum_{j=1}^{n} \frac{1}{p} F_i\xi_j|b_{ji}(t)|^{\beta_{ji}p} > 0\end{aligned} \tag{18}$$

for all $t \geqslant 0$ *and* $i = 1, 2, \ldots, n$. *Then the dynamical system* (17) *has a unique periodic solution* $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^{\mathrm{T}}$ *and, for any solution* $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$ *of system* (17), *there exists a scalar* $\varepsilon > 0$ *such that*

$$|u_i(t + j\omega) - v_i(t)| = O(\mathrm{e}^{-j\varepsilon\omega}) \quad i = 1, 2, \ldots, n.$$

In order to complete the proof of Theorem 2, we only need define the following Lyapunov function similar to the proof of Theorem 1.

$$\begin{aligned}L_1(t) = &\sum_{i=1}^{n} \xi_i|w_i(t)|^p + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i F_j\, \mathrm{e}^{\varepsilon\tau} \\ &\times \int_{t-\tau_{ij}}^{t} |b_{ij}(t)||w_j(s)|^p\, \mathrm{d}s.\end{aligned}$$

Further, as consequences of Theorem 2, we also have the results similar to Corollary 1. Here, we omit it.

Remark 2. From Theorem 2, we can see that the results obtained in this paper improve and extend the main results given in [28].

For autonomous system (1), as consequences of Theorems 1 and 2, we have the following result.

Corollary 2. *Suppose that* (H_2) *and* (H_3) *hold and there are constants* $\xi_i > 0$, $\alpha_{ij}, \beta_{ij} \in R$ $(i,j = 1, 2, \ldots, n)$ *such that*

$$\begin{aligned}&d_i\xi_i - \sum_{j=1}^{n} \frac{1}{p} G_i\xi_j|a_{ji}|^{\alpha_{ji}p} - \sum_{j=1}^{n} \frac{1}{q} G_j\xi_i|a_{ij}|^{(1-\alpha_{ij})q} \\ &\quad - \sum_{j=1}^{n} \frac{1}{q} F_j\xi_i|b_{ij}|^{(1-\beta_{ij})p} - \sum_{j=1}^{n} \frac{1}{p} F_i\xi_j|b_{ji}| > 0\end{aligned} \tag{19}$$

for $i = 1, 2, \ldots, n$ *and* $1 \leqslant p < \infty$ *is a constant. Then the dynamical system* (1) *has a unique periodic equilibrium* $v^* = [v_1^*, v_2^*, \ldots, v_n^*]^{\mathrm{T}}$ *and, for any solution* $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$ *of system* (1), *there exists a scalar* $\varepsilon > 0$ *such that*

$$|u_i(t) - v_i^*| = O(\mathrm{e}^{-\varepsilon t}) \quad i = 1, 2, \ldots, n.$$

In order to complete the proof of Corollary 2, we only need construct the following Lyapunov function:

$$\begin{aligned}L_2(t) = &\sum_{i=1}^{n} \xi_i|w_i(t)|^p + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i F_j\, \mathrm{e}^{\varepsilon\tau} \\ &\times \int_{t-\tau_{ij}}^{t} |b_{ij}||w_j(s)|^p\, \mathrm{d}s.\end{aligned}$$

Similar to the proof of Theorem 1, we easily obtain the result of Corollary 2. Here, we omit it.

For autonomous system (1), we also have the following result.

Corollary 3. *Suppose that* (H_2) *holds and there are constants* $\xi_i > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$d_i\xi_i - \sum_{j=1}^{n} G_i\xi_j|a_{ji}| - \sum_{j=1}^{n} F_i\xi_j|b_{ji}| > 0. \tag{20}$$

Then the dynamical system (1) *has a unique periodic solution* $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^{\mathrm{T}}$ *and, for any solution* $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$ *of system* (1), *there exists a scalar* $\varepsilon > 0$ *such that*

$$|u_i(t + j\omega) - v_i(t)| = O(\mathrm{e}^{-j\varepsilon\omega}) \quad i = 1, 2, \ldots, n.$$

Remark 3. Let the matrices $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $F = \mathrm{diag}(F_1, F_2, \ldots, F_n)$, $G = \mathrm{diag}(G_1, G_2, \ldots, G_n)$, $A = (a_{ij})_{n\times n}$, $|A| = (|a_{ij}|)_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $|B| = (|b_{ij}|)_{n\times n}$.

We easily find that the condition (20) is equivalent to the matrix $D-(|A|^{\mathrm{T}}G+|B|^{\mathrm{T}}F)$ be a non-singular M-matrix. Thus, under this condition, we can obtain the result similar to Corollary 3.

Remark 4. It should be emphasized that parameters α_{ij}, β_{ij} play key role in Theorems 1–2 and Corollary 2. By appropriately choosing the parameters α_{ij}, β_{ij} (see Remark 3), we can see that it include many known assumptions as its special cases.

4. Illustrative example

Example. Let $n=2$, $f_j(u)=g_j(u)=\frac{1}{2}(|x+1|-|x-1|)$, $c_i(t)=5+\sin t, a_{ij}(t)=\sin t, b_{ij}(t)=\cos t, \tau_{ij}(t)=\frac{1}{2}\sin t+1$ $(i,j=1,2)$. We consider the following 2-dimensional neural networks:

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}=&-(5+\sin t)u_1(t)+\sum_{j=1}^{2}\sin t g_j(u_j(t))\\&+\sum_{j=1}^{2}\cos t f_j(u_j(t-\tau_{1j}(t)))+\sin t,\\\frac{\mathrm{d}u_2(t)}{\mathrm{d}t}=&-(5+\sin t)u_2(t)+\sum_{j=1}^{2}\sin t g_j(u_j(t))\\&+\sum_{j=1}^{2}\cos t f_j(u_j(t-\tau_{2j}(t)))+\cos t.\end{aligned}\tag{21}$$

Obviously, $\psi_{ij}(t)=t-\frac{1}{2}\sin t-1, \dot{\psi}_{ij}(t)=1-\frac{1}{2}\cos t>0$. Thus the inverse of $\psi_{ij}(t)$ $(i,j=1,2)$ exist, we denote it by $\bar{\psi}(t)$. By taking $\xi_i=1, F=(1,1), G=(1,1), p=1, \alpha_{ij}=\beta_{ij}=1$ $(i,j=1,2)$ then the condition (5) transform into

$$\begin{aligned}&d_i(t)\xi_i-\sum_{j=1}^{n}G_i\xi_j|a_{ji}(t)|-\sum_{j=1}^{n}F_i\xi_j\frac{|b_{ji}(\psi_{ji}^{-1}(t))|}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t))}\\&\quad=(5+\sin t)-2\times\sin t-2\times\frac{\cos(\bar{\psi}(t))}{1-\frac{1}{2}\cos(\bar{\psi}(t))}>0\end{aligned}$$

for all $t\geqslant 0$ and $i=1,2$. Hence system (21) has a unique globally exponentially stable 2π-periodic solution.

For numerical simulation, the following four cases are given: case 1 with the initial state $(\phi_1(t),\phi_2(t))=(-0.01\sin t+0.1, 0.01\cos t-0.1)$ for $t\in[-2.5,0]$; case 2 with the initial state $(\phi_1(t),\phi_2(t))=(-0.025\sin t+0.1, 0.025\cos t-0.1)$ for $t\in[-2.5,0]$; case 3 with the initial state $(\phi_1(t),\phi_2(t))=(-0.04\sin t+0.1, 0.04\cos t-0.1)$ for $t\in[-2.5,0]$; case 4 with the initial state $(\phi_1(t),\phi_2(t))=(-0.0625\sin t+0.1, 0.0625\cos t-0.1)$ for $t\in[-2.5,0]$; The following Fig. 1 depicts the time responses of state variables of $x_1(t)$ and $x_2(t)$ with step $h=0.01$, and Fig. 2 depicts the phase plots of state variable $x_1(t)$ and $x_2(t)$. It confirms that the proposed condition leads to the unique and globally exponentially stable 2π-periodic solution for this model.

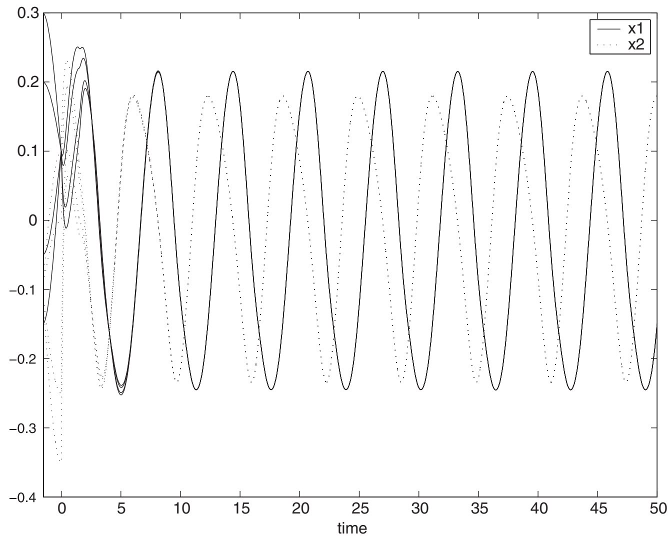

Fig. 1. Transient response of state variables $x_1(t)$ and $x_2(t)$ for Example 1.

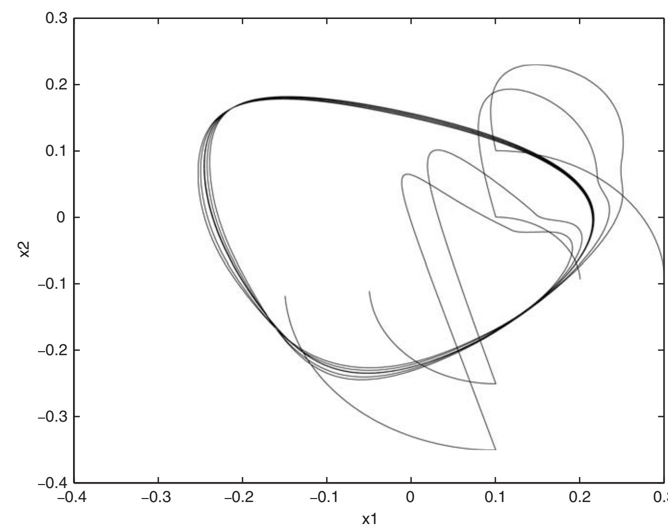

Fig. 2. Phase plots of state variables $x_1(t)$ and $x_2(t)$ for Example 1.

5. Conclusions

In this paper, different from [12,15,16,24,30], we introduce the new research method, that is, by introducing ingeniously many real parameters α_{ij}, β_{ij} and ω_i, applying Young inequality technique and Dini derivative. We obtain new criteria on the existence, uniqueness and global exponential stability of the periodic solutions for the general periodic nonautonomous neural networks with variable coefficients and time-varying delays. The results obtained in this paper improve and extend many previous works, and they are easy to check and apply in practice.

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