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On global exponential stability of generalized stochastic neural networks with mixed time-delays ☆

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Abstract

This paper is concerned with the global exponential stability analysis problem for a general class of stochastic neural networks with mixed time-delays. The mixed time-delays under consideration comprise both the discrete time-varying delays and the distributed time-delays. The main purpose of this paper is to establish easily verifiable conditions under which the delayed stochastic neural network is exponentially stable in the mean square in the presence of both the discrete and distributed delays. By employing a new Lyapunov–Krasovskii functional and conducting stochastic analysis, a linear matrix inequality (LMI) approach is developed to derive the criteria of the exponential stability. Furthermore, the main results are specialized to deal with the analysis problem for the global asymptotic stability within the same LMI framework. The proposed criteria can be readily checked by using some standard numerical packages such as the Matlab LMI toolbox. A simple example is provided to demonstrate the effectiveness and applicability of the proposed testing criteria.

Keywords: Stochastic neural networks; Discrete and distributed delays; Lyapunov–Krasovskii functional; Global exponential stability; Global asymptotic stability; Linear matrix inequality

1. Introduction

The well-known Hopfield neural networks were firstly introduced by Hopfield [14,15] in early 1980s. Since then, both the mathematical analysis and practical applications of Hopfield neural networks have gained considerable research attention. Hopfield neural networks have already been successfully applied in many different areas such as combinatorial optimization, knowledge acquisition and pattern recognition, see e.g. [20,21,28]. It should be pointed out that, these applications are largely dependent on the stability of the equilibrium point of neural networks. Stability, as one of the most important properties for neural networks, is crucially required when designing neural networks. It is often the case in practice that, the neural network is designed with only one equilibrium point, and this equilibrium point is expected to be globally stable. For example, the neural network that is applied to solve the optimization problem must have one unique equilibrium point and be globally stable.

In both the biological and artificial neural networks, the interactions between neurons are generally asynchronous, which give rise to the inevitable signal transmission delays. Also, in electronic implementation of analog neural networks,

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time-delay is usually time-varying due to the finite switching speed of amplifiers. It is known that time-delays may cause undesirable dynamic network behaviors such as oscillation and instability. Consequently, the stability analysis problems for delayed neural networks have received considerable research attention. So far, a large amount of results have appeared in the literature, see e.g. [1,2,7–10,19,26,27,29] and references therein, where the delay type can be constant, time-varying, or distributed, and the stability criteria can be delay-dependent or delay-independent. Note that continuously distributed delays have recently gained particular attention, since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths.

On the other hand, in real nervous systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. It has also been known that a neural network could be stabilized or destabilized by certain stochastic inputs [4]. Hence, the stability analysis problem for stochastic Hopfield neural networks has begun to attract research interests, and some initial results have been obtained, see e.g. [4,16,17,24]. It should be mentioned that, in most existing literature tackling stochastic neural networks, the time-delays have been assumed to be either discrete or distributed, and the stability criteria have been derived mainly based on the computation of matrix norms. To the best of the authors' knowledge, the exponential stability analysis problem for *stochastic* Hopfield neural networks with *both* the discrete and distributed time-delays has not been fully investigated, and remains important and challenging.

In this paper, we deal with the global exponential stability analysis problem for a class of stochastic Hopfield neural networks with simultaneous presence of both the discrete and distributed time-delays. By utilizing a novel Lyapunov–Krasovskii functional and using stochastic analysis tools, we show that the addressed stability analysis problem is solvable if two linear matrix inequalities are feasible. Hence, different from the commonly used matrix norm theories (such as the M-matrix method), a unified *linear matrix inequality* (LMI) approach is developed to establish sufficient conditions for the neural networks to be globally exponential stable in the mean square. Note that LMIs can be easily solved by using the Matlab LMI toolbox, and no tuning of parameters is required [5]. A numerical example is provided to show the usefulness of the proposed global stability condition.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X\geqslant Y$ (respectively, $X>Y$) where X and Y are symmetric matrices, means that $X-Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h>0$ and $C([-h,0];\mathbb{R}^n)$ denote the family of continuous functions φ from $[-h,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\|=\sup_{-h\leqslant\theta\leqslant 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\|=\sup\{|Ax|:|x|=1\}=\sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathscr{F}_0}([-h,0];\mathbb{R}^n)$ the family of all $\mathscr{F}_0$-measurable $C([-h,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(\theta):-h\leqslant\theta\leqslant 0\}$ such that $\sup_{-h\leqslant\theta\leqslant 0}\mathbb{E}|\xi(\theta)|^p<\infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2. Problem formulation

Consider, on a probability space $(\Omega,\mathscr{F},\mathscr{P})$, the following stochastic neural network with discrete and distributed time-delays of the form:

$$\begin{aligned} \mathrm{d}u_i(t) = {} & \left[-d_iu_i(t)+\sum_{j=1}^{n}a_{ij}f_j(u_j(t))+\sum_{j=1}^{n}b_{ij}g_j(u_j(t-\tau_1(t)))+\int_{t-\tau_2}^{t}\sum_{j=1}^{n}c_{ij}h_j(u_j(s))\,\mathrm{d}s+J_i\right]\mathrm{d}t \\ & +\sigma_i(t,u_1(t),\ldots,u_n(t),u_1(t-\tau_1(t)),\ldots,u_n(t-\tau_1(t)))\,\mathrm{d}w(t),\quad i=1,\ldots,n, \end{aligned} \tag{2.1}$$

where n is the number of the neurons in the neural network, $u_i(t)$ denotes the state of the ith neural neuron at time t, $f_j(u_j(t))$, $g_j(u_j(t))$ and $h_j(u_j(t))$ are the activation functions of the jth neuron at time t. The constants a_{ij}, b_{ij} and c_{ij} denote, respectively, the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the jth neuron on the i neuron. J_i is the external bias on the ith neuron, d_i denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $\tau_1(t)$ is the time-varying discrete time-delay with bound τ_1^*, i.e.,

$$0\leqslant\tau_1(t)\leqslant\tau_1^*, \tag{2.2}$$

while τ_2 describes the distributed time-delay. $w(t)$ is a zero-mean scalar Wiener process (Brownian motion) on $(\Omega, \mathscr{F}, \mathscr{P})$ with

$$\mathbb{E}[w(t)] = 0, \quad \mathbb{E}[w(t)^2] = t. \tag{2.3}$$

The stochastic neural network (2.1) can be rewritten in the following matrix–vector form:

$$\mathrm{d}u(t) = \left[-Du(t) + AF(u(t)) + BG(u(t-\tau_1(t))) + C\int_{t-\tau_2}^{t} H(u(s))\,\mathrm{d}s + J\right]\mathrm{d}t + \sigma(t, u(t), u(t-\tau_1(t)))\,\mathrm{d}w(t), \tag{2.4}$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}$, $D = \mathrm{diag}(d_1, \ldots, d_n)$, $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$, $J = [I_1, \ldots, I_n]^{\mathrm{T}}$, and $F(u(t)) = [f_1(u_1(t)), \ldots, f_n(u_n(t))]^{\mathrm{T}}$, $G(u(t-\tau_1)) = [g_1(u_1(t-\tau_1)), \ldots, g_n(u_n(t-\tau_1))]^{\mathrm{T}}$, $H(u(s)) = [h_1(u_1(s)), \ldots, h_n(u_n(s))]^{\mathrm{T}}$, $\sigma(t, u(t), u(t-\tau_1(t))) = [\sigma_1(t, u_1(t), \ldots, u_n(t), u_1(t-\tau_1(t)), \ldots, u_n(t-\tau_1(t))), \ldots, \sigma_n(t, u_1(t)), \ldots, u_n(t), u_1(t-\tau_1(t)), \ldots, u_n(t-\tau_1(t))]^{\mathrm{T}}$.

Remark 1. Note that the neural network model described in (2.4) comprises the discrete delay τ_1, the distributed delay τ_2 and the stochastic perturbation $w(t)$, which gives a better approximation of the real neurobiological model. To be specific, the interactions between neurons are typically asynchronous, which leads to unavoidable signal transmission delays. Time-delay also occurs in the communication and response of neurons owing to the unavoidable finite switching speed of amplifiers in the electronic implementation of analog neural networks. The distributed nature of the time-delays exhibits because neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Furthermore, neural networks may operate in an environment where all the signals are undoubtedly stochastic, either because a noise could be superimposed on a signal (non-stochastic) or, more generally, because the signal could be inherently stochastic in that a predictable part of the signal cannot be identified [3]. To this end, we can conclude that the model (2.4) could represent a broad class of neural networks.

Traditionally, the activation functions are assumed to be continuous, differentiable and monotonically increasing, such as the sigmoid-type of function. However, in many electronic circuits, the input–output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network. In this paper, we make following assumptions for the neuron activation functions.

Assumption 1. For $i \in \{1, 2, \ldots, n\}$, the neuron activation functions in (2.4) satisfy

$$l_i^- \leqslant \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leqslant l_i^+, \tag{2.5}$$

$$m_i^- \leqslant \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leqslant m_i^+, \tag{2.6}$$

$$\upsilon_i^- \leqslant \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leqslant \upsilon_i^+, \tag{2.7}$$

where l_i^-, l_i^+, m_i^-, m_i^+, υ_i^-, υ_i^+ are some constants.

Assumption 2. The neuron activation functions in (2.4) are bounded.

Remark 2. The constants l_i^-, l_i^+, m_i^-, m_i^+, υ_i^-, υ_i^+ in Assumption 1 are allowed to be positive, negative or zero. Hence, the description given in (2.5)–(2.7) is less restrictive than the traditional Lipschitz condition. Also, the resulting activation functions could be non-monotonic, and more general than the usual sigmoid functions.

Notice that when $\sigma(t, x, y) \equiv 0$, the stochastic system (2.4) becomes the following deterministic neural network system:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -Du(t) + AF(u(t)) + BG(u(t-\tau_1(t))) + C\int_{t-\tau_2}^{t} H(u(s))\,\mathrm{d}s + J, \tag{2.8}$$

for which the stability analysis has been conducted in [22,26]. Under Assumption 2, it is not difficult to prove the existence of equilibrium point of the system (2.8) by using Brouwer's fixed point theorem. Obviously, the equilibrium point of the system (2.8) is also the equilibrium point of the system (2.4). Denote by u^* the equilibrium point of the system (2.4). For convenience, we shift the equilibrium point u^* of (2.4) to the origin by letting $x(t) = u(t) - u^*$, and then the system (2.4) can be transformed into:

$$\mathrm{d}x(t) = \left[-Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^{t} \hat{H}(x(s))\,\mathrm{d}s\right]\mathrm{d}t + \hat{\sigma}(t, x(t), x(t-\tau_1(t)))\,\mathrm{d}w(t), \tag{2.9}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ is the state vector of the transformed system, $\hat{\sigma}(t, x(t), x(t-\tau_1(t))) := \sigma(t, x(t) + u^*, x(t-\tau_1(t)) + u^*)$ and the transformed neuron activation functions are

$$\hat{F}(x(t)) := [\hat{f}_1(x_1(t)), \ldots, \hat{f}_n(x_n(t))]^{\mathrm{T}} = F(u(t)) - F(u^*), \tag{2.10}$$

$$\hat{G}(x(t)) := [\hat{g}_1(x_1(t)), \ldots, \hat{g}_n(x_n(t))]^{\mathrm{T}} = G(u(t)) - G(u^*), \tag{2.11}$$

$$\hat{H}(x(t)) := [\hat{h}_1(x_1(t)), \ldots, \hat{h}_n(x_n(t))]^{\mathrm{T}} = H(u(t)) - H(u^*). \tag{2.12}$$

According to (2.5)–(2.7), it can be easily checked that the transformed neuron activation functions satisfy

$$l_i^- \leqslant \frac{\hat{f}_i(s_1) - \hat{f}_i(s_2)}{s_1 - s_2} \leqslant l_i^+ \quad (i = 1, \ldots, n), \tag{2.13}$$

$$m_i^- \leqslant \frac{\hat{g}_i(s_1) - \hat{g}_i(s_2)}{s_1 - s_2} \leqslant m_i^+ \quad (i = 1, \ldots, n), \tag{2.14}$$

$$\upsilon_i^- \leqslant \frac{\hat{h}_i(s_1) - \hat{h}_i(s_2)}{s_1 - s_2} \leqslant \upsilon_i^+ \quad (i = 1, \ldots, n). \tag{2.15}$$

Assumption 3. For the function $\hat{\sigma} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, there are constants ρ_1, ρ_2 such that

$$\hat{\sigma}^{\mathrm{T}}(t, x, y)\hat{\sigma}(t, x, y) \leqslant \rho_1 x^{\mathrm{T}} x + \rho_2 y^{\mathrm{T}} y. \tag{2.16}$$

Under the Assumptions 1 and 3, it is easy to check that functions $\hat{F}, \hat{G}, \hat{H}$ and $\hat{\sigma}$ satisfy the linear growth condition (cf. [18,23]). Thus we have the following proposition.

Proposition 1 (*Khasminskii [18]*). *For any initial data $\xi \in L^2_{\mathscr{F}_0}([-h, 0]; \mathbb{R}^n)$, Eq. (2.9) has a unique solution denoted by $x(t; \xi)$, or $x(t)$.*

It can be concluded that, for the stability analysis of system (2.4), it is sufficient to discuss the stability of the transformed system (2.9). Let $\tau = \max\{\tau_1^*, \tau_2\}$. We are now in a position to introduce the following concepts of stability.

Definition 1. System (2.9) is said to be stable in mean square if for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$\mathbb{E}|x(t;\xi)|^2 < \varepsilon, \; t > 0 \quad \text{when} \quad \sup_{-\tau \leqslant s \leqslant 0} \mathbb{E}|\xi(s)|^2 < \delta. \tag{2.17}$$

If, in addition to (2.17), the relation $\lim_{t\to 0} \mathbb{E}|x(t;\xi)|^2 = 0$ holds, then system (2.9) is said to be asymptotically stable in mean square.

Definition 2. System (2.9) is said to be exponentially stable in mean square if there exist positive constants $\alpha > 0$ and $\mu > 0$ such that every solution $x(t; \xi)$ of (2.9) satisfies

$$\mathbb{E}|x(t;\xi)|^2 \leqslant \mu \mathrm{e}^{-\alpha t} \sup_{-\tau \leqslant s \leqslant 0} \mathbb{E}|\xi(s)|^2, \quad \forall t > 0.$$

The main purpose of this paper is to establish LMI-based sufficient conditions under which the global exponential stability in mean square is guaranteed for the neural network (2.4) with both discrete and distributed time-delays.

3. Main results and proofs

The following lemmas will be used in establishing our main results.

Lemma 1. *Let x, y be any n-dimensional real vectors, and let P be a $n \times n$ positive semi-definite matrix. Then, the following matrix inequality holds*:

$$2x^{\mathrm{T}} P y \leqslant x^{\mathrm{T}} P x + y^{\mathrm{T}} P y.$$

Proof. The proof follows from the following simple inequality

$$(x^{\mathrm{T}} P^{1/2} - y^{\mathrm{T}} P^{1/2})(x^{\mathrm{T}} P^{1/2} - y^{\mathrm{T}} P^{1/2})^{\mathrm{T}} \geqslant 0.$$

immediately. □

Lemma 2 (*Gu [12]*). *For any positive definite matrix $M>0$, scalar $\gamma>0$, vector function $\omega:[0,\gamma]\to\mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds*:

$$\left(\int_0^{\gamma}\omega(s)\,\mathrm{d}s\right)^{\mathrm{T}}M\left(\int_0^{\gamma}\omega(s)\,\mathrm{d}s\right)\leqslant\gamma\left(\int_0^{\gamma}\omega^{\mathrm{T}}(s)M\omega(s)\,\mathrm{d}s\right). \tag{3.1}$$

For presentation convenience, in the following, we denote

$$L_1=\mathrm{diag}(l_1^+l_1^-,\ldots,l_n^+l_n^-),\quad L_2=\mathrm{diag}\left(\frac{l_1^++l_1^-}{2},\ldots,\frac{l_n^++l_n^-}{2}\right), \tag{3.2}$$

$$M_1=\mathrm{diag}(m_1^+m_1^-,\ldots,m_n^+m_n^-),\quad M_2=\mathrm{diag}\left(\frac{m_1^++m_1^-}{2},\ldots,\frac{m_n^++m_n^-}{2}\right), \tag{3.3}$$

$$\Upsilon_1=\mathrm{diag}(\upsilon_1^+\upsilon_1^-,\ldots,\upsilon_n^+\upsilon_n^-),\quad \Upsilon_2=\mathrm{diag}\left(\frac{\upsilon_1^++\upsilon_1^-}{2},\ldots,\frac{\upsilon_n^++\upsilon_n^-}{2}\right). \tag{3.4}$$

The main results of this paper are given in the following theorem.

Theorem 1. *Suppose that $\tau_1(t)$ is the discrete time-varying delay satisfying* (2.2) *and $\dot\tau_1(t)\leqslant\tau_0$ (here τ_0 is a constant), τ_2 is a constant describing the distributed delay, and ε_0 $(0<\varepsilon_0<1)$ is a fixed constant.*

Let Assumptions 1 and 3 hold. Then, the stochastic delayed neural network (2.9) is globally exponentially stable in mean square if there exist three positive definite matrices $P_1>0$, $P_2>0$, $P_3>0$, three diagonal matrices $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)>0$, $\Gamma=\mathrm{diag}(\gamma_1,\ldots,\gamma_n)>0$ and $\Delta=\mathrm{diag}(\delta_1,\ldots,\delta_n)>0$ and a scalar λ^* such that the following two LMIs hold:

$$P_1<\lambda^* I, \tag{3.5}$$

$$\Psi:=\begin{bmatrix}\Sigma & P_1A+\Lambda L_2 & \Gamma M_2 & P_1B & \Delta\Upsilon_2 & P_1C\\ A^{\mathrm{T}}P_1+\Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0\\ \Gamma M_2 & 0 & (1+\varepsilon_0\tau_1^*)P_2-\Gamma & 0 & 0 & 0\\ B^{\mathrm{T}}P_1 & 0 & 0 & -P_2 & 0 & 0\\ \Delta\Upsilon_2 & 0 & 0 & 0 & \tau_2P_3-\Delta & 0\\ C^{\mathrm{T}}P_1 & 0 & 0 & 0 & 0 & -\dfrac{1-\varepsilon_0}{\tau_2}P_3\end{bmatrix}<0, \tag{3.6}$$

where

$$\Sigma:=-P_1D-DP_1-\Lambda L_1-\Gamma M_1-\Delta\Upsilon_1+\varepsilon_0\tau_1^*I+\left(\rho_1+\frac{\rho_2}{1-\tau_0}\right)\lambda^*I. \tag{3.7}$$

Proof. In order to establish the stability conditions [6,13], we introduce the following Lyapunov–Krasovskii functional candidate $V(t)=V(x(t),t)\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}^+)$:

$$\begin{aligned}V(t):=&\,x^{\mathrm{T}}(t)P_1x(t)+\frac{\rho_2\lambda^*}{1-\tau_0}\int_{t-\tau_1(t)}^{t}x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s+\varepsilon_0\int_0^{\tau_1(t)}\int_{t-s}^{t}x^{\mathrm{T}}(\eta)x(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\\ &+\int_{t-\tau_1(t)}^{t}\hat G^{\mathrm{T}}(x(s))P_2\hat G(x(s))\,\mathrm{d}s+\varepsilon_0\int_0^{\tau_1(t)}\int_{t-s}^{t}\hat G^{\mathrm{T}}(x(\eta))P_2\hat G(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s\\ &+\int_0^{\tau_2}\int_{t-s}^{t}\hat H^{\mathrm{T}}(x(\eta))P_3\hat H(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s,\end{aligned} \tag{3.8}$$

By Itô's differential formula [18,23], the stochastic differential of $V(t)$ along (2.9) can be obtained as

$$\mathrm{d}V(t)=\mathscr{L}V(t)\,\mathrm{d}t+2x^{\mathrm{T}}(t)P_1[\hat\sigma(t,x(t),x(t-\tau_1(t)))]\,\mathrm{d}w(t), \tag{3.9}$$

where

$$\begin{aligned}\mathscr{L}V(t) = {} & \frac{\rho_2\lambda^*}{1-\tau_0}x^{\mathrm{T}}(t)x(t) - \frac{(1-\dot{\tau}_1(t))\rho_2\lambda^*}{1-\tau_0}x^{\mathrm{T}}(t-\tau_1(t))x(t-\tau_1(t)) \\ & + \hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)) - (1-\dot{\tau}_1(t))\hat{G}^{\mathrm{T}}(x(t-\tau_1(t)))P_2\hat{G}(x(t-\tau_1(t))) \\ & + \varepsilon_0\tau_1(t)x^{\mathrm{T}}(t)x(t) - \varepsilon_0\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\ & + \varepsilon_0\dot{\tau}_1(t)\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\ & + \varepsilon_0\tau_1(t)\hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)) - \varepsilon_0\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\ & + \varepsilon_0\dot{\tau}_1(t)\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\ & + \tau_2\hat{H}^{\mathrm{T}}(x(t))P_3\hat{H}(x(t)) - \int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s \\ & + 2x^{\mathrm{T}}(t)P_1\left(-Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right) \\ & + \hat{\sigma}^{\mathrm{T}}(t,x(t),x(t-\tau_1(t)))P_1\hat{\sigma}(t,x(t),x(t-\tau_1(t))). \end{aligned} \tag{3.10}$$

In terms of Assumptions 1, 3 and Lemma 2, we have

$$-\frac{(1-\dot{\tau}_1(t))\rho_2\lambda^*}{1-\tau_0}x^{\mathrm{T}}(t-\tau_1(t))x(t-\tau_1(t)) \leqslant -\rho_2\lambda^* x^{\mathrm{T}}(t-\tau_1(t))x(t-\tau_1(t)); \tag{3.11}$$

$$-(1-\dot{\tau}_1(t))\hat{G}^{\mathrm{T}}(x(t-\tau_1(t)))P_2\hat{G}(x(t-\tau_1(t))) \leqslant -(1-\tau_0)\hat{G}^{\mathrm{T}}(x(t-\tau_1(t)))P_2\hat{G}(x(t-\tau_1(t))); \tag{3.12}$$

$$\varepsilon_0\tau_1(t)x^{\mathrm{T}}(t)x(t) \leqslant \varepsilon_0\tau_1^* x^{\mathrm{T}}(t)x(t); \tag{3.13}$$

$$-\varepsilon_0\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s + \varepsilon_0\dot{\tau}_1(t)\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \leqslant -(1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s; \tag{3.14}$$

$$\varepsilon_0\tau_1(t)\hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)) \leqslant \varepsilon_0\tau_1^*\hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)); \tag{3.15}$$

$$-\varepsilon_0\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s + \varepsilon_0\dot{\tau}_1(t)\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \leqslant -(1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s; \tag{3.16}$$

$$\begin{aligned} & -\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s \\ & \quad = -(1-\varepsilon_0)\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s - \varepsilon_0\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s \\ & \quad \leqslant -\frac{1-\varepsilon_0}{\tau_2}\left(\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right)^{\mathrm{T}}P_3\left(\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right) - \varepsilon_0\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s; \ \text{(by Lemma 2)} \end{aligned} \tag{3.17}$$

$$\begin{aligned} & \hat{\sigma}^{\mathrm{T}}(t,x(t),x(t-\tau_1(t)))P_1\hat{\sigma}(t,x(t),x(t-\tau_1(t))) \\ & \quad \leqslant \lambda_{\max}(P_1)\hat{\sigma}^{\mathrm{T}}(t,x(t),x(t-\tau_1(t)))\hat{\sigma}(t,x(t),x(t-\tau_1(t))) \\ & \quad \leqslant \lambda_{\max}(P_1)\left[\rho_1 x^{\mathrm{T}}(t)x(t) + \rho_2 x(t-\tau_1(t))x^{\mathrm{T}}(t-\tau_1(t))\right] \quad \text{(by (2.16))} \\ & \quad \leqslant \lambda^*\left[\rho_1 x^{\mathrm{T}}(t)x(t) + \rho_2 x^{\mathrm{T}}(t-\tau_1(t))x(t-\tau_1(t))\right] \quad \text{(by (3.5)).} \end{aligned} \tag{3.18}$$

Substituting (3.11)–(3.18) into (3.10) leads to

$$\begin{aligned}
\mathscr{L}V(t) \leqslant\ & \frac{\rho_2\lambda^*}{1-\tau_0}x^{\mathrm{T}}(t)x(t) + \hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)) - (1-\tau_0)\hat{G}^{\mathrm{T}}(x(t-\tau_1(t)))P_2\hat{G}(x(t-\tau_1(t))) \\
& + \varepsilon_0\tau_1^* x^{\mathrm{T}}(t)x(t) - (1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\
& + \varepsilon_0\tau_1^*\hat{G}^{\mathrm{T}}(x(t))P_2\hat{G}(x(t)) - (1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\
& + \tau_2\hat{H}^{\mathrm{T}}(x(t))P_3\hat{H}(x(t)) - \frac{1-\varepsilon_0}{\tau_2}\left(\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right)^{\mathrm{T}}P_3\left(\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right) \\
& - \varepsilon_0\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s \\
& + 2x^{\mathrm{T}}(t)P_1\left(-Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^{t}\hat{H}(x(s))\,\mathrm{d}s\right) \\
& + \rho_1\lambda^* x^{\mathrm{T}}(t)x^{\mathrm{T}}(t) \\
=\ & \eta^{\mathrm{T}}(t)\Psi_1\eta(t) - (1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t}x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s - (1-\tau_0)\varepsilon_0\int_{t-\tau_1(t)}^{t}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\
& - \varepsilon_0\int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s,
\end{aligned} \tag{3.19}$$

where

$$\eta(t) = \left[x^{\mathrm{T}}(t) \quad \hat{F}^{\mathrm{T}}(x(t)) \quad \hat{G}^{\mathrm{T}}(x(t)) \quad \hat{G}^{\mathrm{T}}(x(t-\tau_1)) \quad \hat{H}^{\mathrm{T}}(x(t)) \quad \int_{t-\tau_2}^{t}\hat{H}^{\mathrm{T}}(x(s))\,\mathrm{d}s\right]^{\mathrm{T}}, \tag{3.20}$$

$$\Psi_1 = \begin{bmatrix}
-P_1D - DP_1 + \varepsilon_0\tau_1^*I + \left(\rho_1 + \frac{\rho_2}{1-\tau_0}\right)\lambda^*I & P_1A & 0 & P_1B & 0 & P_1C \\
A^{\mathrm{T}}P_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (1+\varepsilon_0\tau_1^*)P_2 & 0 & 0 & 0 \\
B^{\mathrm{T}}P_1 & 0 & 0 & -(1-\tau_0)P_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau_2P_3 & 0 \\
C^{\mathrm{T}}P_1 & 0 & 0 & 0 & 0 & -\frac{1-\varepsilon_0}{\tau_2}P_3
\end{bmatrix}. \tag{3.21}$$

By (2.13)–(2.15), we have

$$(\hat{f}_i(x_i(t)) - l_i^+x_i(t))(\hat{f}_i(x_i(t)) - l_i^-x_i(t)) \leqslant 0, \quad i = 1,\dots,n, \tag{3.22}$$

$$(\hat{g}_i(x_i(t)) - m_i^+x_i(t))(\hat{g}_i(x_i(t)) - m_i^-x_i(t)) \leqslant 0, \quad i = 1,\dots,n, \tag{3.23}$$

$$(\hat{h}_i(x_i(t)) - \upsilon_i^+x_i(t))(\hat{h}_i(x_i(t)) - \upsilon_i^-x_i(t)) \leqslant 0, \quad i = 1,\dots,n, \tag{3.24}$$

which are equivalent to

$$\begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} l_i^+l_i^-e_ie_i^{\mathrm{T}} & -\frac{l_i^+ + l_i^-}{2}e_ie_i^{\mathrm{T}} \\ -\frac{l_i^+ + l_i^-}{2}e_ie_i^{\mathrm{T}} & e_ie_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix} \leqslant 0, \quad i = 1,\dots,n, \tag{3.25}$$

$$\begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m_i^+m_i^-e_ie_i^{\mathrm{T}} & -\frac{m_i^+ + m_i^-}{2}e_ie_i^{\mathrm{T}} \\ -\frac{m_i^+ + m_i^-}{2}e_ie_i^{\mathrm{T}} & e_ie_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix} \leqslant 0, \quad i = 1,\dots,n, \tag{3.26}$$

$$\begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \upsilon_i^+\upsilon_i^- e_i e_i^{\mathrm{T}} & -\dfrac{\upsilon_i^+ + \upsilon_i^-}{2} e_i e_i^{\mathrm{T}} \\ -\dfrac{\upsilon_i^+ + \upsilon_i^-}{2} e_i e_i^{\mathrm{T}} & e_i e_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix} \leqslant 0, \quad i = 1, \ldots, n, \tag{3.27}$$

where e_i denotes the unit column vector having "1" element on its ith row and zeros elsewhere.

Thus we have

$$\begin{aligned}
&\eta^{\mathrm{T}}(t)\Psi_1\eta(t) - \sum_{i=1}^{n}\lambda_i \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} l_i^+ l_i^- e_i e_i^{\mathrm{T}} & -\dfrac{l_i^+ + l_i^-}{2} e_i e_i^{\mathrm{T}} \\ -\dfrac{l_i^+ + l_i^-}{2} e_i e_i^{\mathrm{T}} & e_i e_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix} \\
&\quad - \sum_{i=1}^{n}\gamma_i \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m_i^+ m_i^- e_i e_i^{\mathrm{T}} & -\dfrac{m_i^+ + m_i^-}{2} e_i e_i^{\mathrm{T}} \\ -\dfrac{m_i^+ + m_i^-}{2} e_i e_i^{\mathrm{T}} & e_i e_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix} \\
&\quad - \sum_{i=1}^{n}\delta_i \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \upsilon_i^+ \upsilon_i^- e_i e_i^{\mathrm{T}} & -\dfrac{\upsilon_i^+ + \upsilon_i^-}{2} e_i e_i^{\mathrm{T}} \\ -\dfrac{\upsilon_i^+ + \upsilon_i^-}{2} e_i e_i^{\mathrm{T}} & e_i e_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix} \\
&= \eta^{\mathrm{T}}(t)\Psi_1\eta(t) + \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\Lambda L_1 & \Lambda L_2 \\ \Lambda L_2 & -\Lambda \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\Gamma M_1 & \Gamma M_2 \\ \Gamma M_2 & -\Gamma \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix} + \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\Delta \Upsilon_1 & \Delta \Upsilon_2 \\ \Delta \Upsilon_2 & -\Delta \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix} \\
&= \eta^{\mathrm{T}}(t) \begin{bmatrix} \Sigma & P_1 A + \Lambda L_2 & \Gamma M_2 & P_1 B & \Delta \Upsilon_2 & P_1 C \\ A^{\mathrm{T}} P_1 + \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ \Gamma M_2 & 0 & (1+\varepsilon_0 \tau_1^*) P_2 - \Gamma & 0 & 0 & 0 \\ B^{\mathrm{T}} P_1 & 0 & 0 & -P_2 & 0 & 0 \\ \Delta \Upsilon_2 & 0 & 0 & 0 & \tau_2 P_3 - \Delta & 0 \\ C^{\mathrm{T}} P_1 & 0 & 0 & 0 & 0 & -\dfrac{1-\varepsilon_0}{\tau_2} P_3 \end{bmatrix} \eta(t) \\
&= \eta^{\mathrm{T}}(t)\Psi\eta(t) \\
&\leqslant \lambda_{\max}(\Psi)|\eta(t)|^2,
\end{aligned} \tag{3.28}$$

where $\lambda_{\max}(\Psi) < 0$ by (3.6), and Ψ, Ω and Ψ_1 are defined in (3.6), (3.7) and (3.21), respectively.

It follows from (3.25)–(3.28) that

$$\eta^{\mathrm{T}}(t)\Psi_1\eta(t) \leqslant \eta^{\mathrm{T}}(t)\Psi\eta(t) \leqslant \lambda_{\max}(\Psi)|\eta(t)|^2. \tag{3.29}$$

Therefore, from (3.19) and (3.29), we obtain

$$\begin{aligned}
\mathscr{L}V(t) &\leqslant \lambda_{\max}(\Psi)|\eta(t)|^2 - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\
&\quad - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s - \varepsilon_0 \int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s \\
&\leqslant \lambda_{\max}(\Psi)|x(t)|^2 - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\
&\quad - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s - \varepsilon_0 \int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s.
\end{aligned} \tag{3.30}$$

Eqs. (3.9) and (3.30) imply that

$$\begin{aligned}\mathrm{d}V(t) \leqslant &\bigg[\lambda_{\max}(\Psi)|x(t)|^2 - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s - (1-\tau_0)\varepsilon_0 \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\ &-\varepsilon_0 \int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s\bigg]\mathrm{d}t + 2x^{\mathrm{T}}(t)P_1[\hat{\sigma}(t,x(t),x(t-\tau_1(t)))]\,\mathrm{d}w(t).\end{aligned} \tag{3.31}$$

Let α be a constant to be determined. In order to deal with the exponential stability of (2.4), we proceed to calculate the stochastic differential of $\mathrm{e}^{\alpha t}V(t)$ along (2.9) as follows:

$$\mathrm{d}[\mathrm{e}^{\alpha t}V(t)] = \mathrm{e}^{\alpha t}[\alpha V(t)\,\mathrm{d}t + \mathrm{d}V(t)]. \tag{3.32}$$

Notice that

$$\begin{aligned}V(t) =\; & x^{\mathrm{T}}(t)P_1x(t) + \frac{\rho_2\lambda^*}{1-\tau_0}\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s + \varepsilon_0\int_0^{\tau_1(t)}\int_{t-s}^{t} x^{\mathrm{T}}(\eta)x(\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s + \varepsilon_0\int_0^{\tau_1(t)}\int_{t-s}^{t} \hat{G}^{\mathrm{T}}(x(\eta))P_2\hat{G}(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \int_0^{\tau_2}\int_{t-s}^{t} \hat{H}^{\mathrm{T}}(x(\eta))P_3\hat{H}(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s \\ \leqslant\; & \lambda_{\max}(P_1)x^{\mathrm{T}}(t)x(t) + \frac{\rho_2\lambda^*}{1-\tau_0}\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s + \varepsilon_0\int_0^{\tau_1(t)}\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(\eta)x(\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s + \varepsilon_0\int_0^{\tau_1(t)}\int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(\eta))P_2\hat{G}(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \int_0^{\tau_2}\int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(\eta))P_3\hat{H}(x(\eta))\,\mathrm{d}\eta\,\mathrm{d}s \\ \leqslant\; & \lambda_{\max}(P_1)x^{\mathrm{T}}(t)x(t) + \left(\frac{\rho_2\lambda^*}{1-\tau_0} + \varepsilon_0\tau_1^*\right)\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\ &+ (1+\varepsilon_0\tau_1^*)\int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s + \varepsilon_0\tau_2\int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s.\end{aligned} \tag{3.33}$$

Substituting (3.31) and (3.33) into (3.32), we obtain that

$$\begin{aligned}\mathrm{d}[\mathrm{e}^{\alpha t}V(t)] \leqslant \mathrm{e}^{\alpha t}&\left[(\lambda_{\max}(\Psi) + \alpha\lambda_{\max}(P_1))|x(t)|^2 - \left((1-\tau_0)\varepsilon_0 - \alpha\left(\frac{\rho_2\lambda^*}{1-\tau_0} + \varepsilon_0\tau_1^*\right)\right)\int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s\right. \\ &- ((1-\tau_0)\varepsilon_0 - \alpha(1+\varepsilon_0\tau_1^*))\int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s \\ &\left.-\varepsilon_0(1-\alpha\tau_2)\int_{t-\tau_2}^{t} \hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s\right]\mathrm{d}t \\ &+ 2\mathrm{e}^{\alpha t}x^{\mathrm{T}}(t)P_1[\hat{\sigma}(t,x(t),x(t-\tau_1(t)))]\,\mathrm{d}w(t).\end{aligned} \tag{3.34}$$

From now on, we take α to be a constant sufficiently small such that the following inequalities hold:

$$\begin{cases}\lambda_{\max}(\Psi) + \alpha\lambda_{\max}(P_1) < 0, \\ -\left((1-\tau_0)\varepsilon_0 - \alpha\left(\dfrac{\rho_2\lambda^*}{1-\tau_0} + \varepsilon_0\tau_1^*\right)\right) < 0, \\ -((1-\tau_0)\varepsilon_0 - \alpha(1+\varepsilon_0\tau_1^*)) < 0, \\ -\varepsilon_0(1-\alpha\tau_2) < 0.\end{cases} \tag{3.35}$$

Integrating both sides of (3.34) from 0 to t ($t>0$) and taking the mathematical expectation we have

$$\mathrm{e}^{\alpha t}\mathbb{E}V(t) \leqslant \mathbb{E}V(0). \tag{3.36}$$

Setting $k^* = \max_{1 \leqslant i \leqslant n}\{|m_i^-|, |m_i^+|, |\upsilon_i^-|, |\upsilon_i^+|\}$ and using (3.33), we can get

$$\begin{aligned}\mathbb{E}V(0) &\leqslant \mathbb{E}\Bigg(\lambda_{\max}(P_1)x^{\mathrm{T}}(0)x(0) + \left(\frac{\rho_2\lambda^*}{1-\tau_0} + \varepsilon_0\tau_1^*\right)\int_{-\tau_1(0)}^{0} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\ &\quad +(1+\varepsilon_0\tau_1^*)\int_{-\tau_1(0)}^{0}\hat{G}^{\mathrm{T}}(x(s))P_2\hat{G}(x(s))\,\mathrm{d}s + \varepsilon_0\tau_2\int_{-\tau_2}^{0}\hat{H}^{\mathrm{T}}(x(s))P_3\hat{H}(x(s))\,\mathrm{d}s\Bigg) \\ &\leqslant \left(\lambda_{\max}(P_1) + \left(\frac{\rho_2\lambda^*}{1-\tau_0} + \varepsilon_0\tau_1^*\right)\tau_1^* + (1+\varepsilon_0\tau_1^*)\tau_1^*k^* + \varepsilon_0\tau_2^2k^*\right)\sup_{-\tau\leqslant s\leqslant 0}\mathbb{E}|x(s)|^2.\end{aligned} \tag{3.37}$$

Obviously, we have

$$\mathbb{E}V(t) \geqslant \lambda_{\max}(P_1)\mathbb{E}|x(t)|^2. \tag{3.38}$$

Thus, it follows from (3.36), (3.37) and (3.38) that

$$\mathbb{E}|x(t;\xi)|^2 \leqslant \mu \mathrm{e}^{-\alpha t}\sup_{-\tau\leqslant s\leqslant 0}\mathbb{E}|\xi(s)|^2, \tag{3.39}$$

where $\mu = (\lambda_{\max}(P_1) + (\rho_2\lambda^*/(1-\tau_0) + \varepsilon_0\tau_1^*)\tau_1^* + (1+\varepsilon_0\tau_1^*)\tau_1^*k^* + \varepsilon_0\tau_2^2k^*)/\lambda_{\max}(P_1)$. This completes the proof of Theorem 1. □

Remark 3. In Theorem 1, sufficient conditions are provided for system (2.4) to be globally exponentially stable in mean square. It should be pointed out that, such conditions are expressed in the form of LMIs, where the variables of the LMIs are essentially the parameters of the addressed neural networks. Therefore, once an adequate neural network is established and the corresponding parameters are identified, we can analyze the exponential stability of the neural network by simply checking the feasibility of the LMIs. Note that the verification of the solvability of LMIs can be readily done by utilizing the numerically efficient Matlab LMI toolbox, and no turning of parameters will be needed [11]. In the past decade, LMIs have gained much attention for their computational tractability and usefulness in system engineering (see e.g. [5]) as the so-called interior point method has been proved to be numerically very efficient for solving the LMIs. The number of analysis and design problems that can be formulated as LMI problems is large and continues to grow.

Remark 4. It should be pointed out that, the main results given in Theorem 1 are general enough to cover many previous results in the literature. For example, if we do not consider the distributed delays, the stability analysis problem for stochastic neural networks with discrete delays has been studied [4,16,17,24]. If we further drop out the stochastic perturbations, the stability analysis problem for deterministic neural networks have been extensively investigated in [1,7–10,19,26] with or without distributed delays. We like to mention that, the present results include those in [26,25] as special cases.

In what follows, a corollary is obtained for the case of *constant* discrete time-delay.

Corollary 1. *Suppose that* $\tau_1(t) \equiv \tau_1^*$ *is a constant discrete time-delay, while the constant scalar* τ_2 *describes the distributed delay, and* ε_0 $(0<\varepsilon_0<1)$ *is a fixed constant. Furthermore, assume that Assumptions* 1 *and* 3 *hold. Then, the stochastic system* (2.9) *is globally exponentially stable in mean square if there exist three positive definite matrices* $P_1>0$, $P_2>0$, $P_3>0$, *three diagonal matrices* $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)>0$, $\Gamma = \mathrm{diag}(\gamma_1,\ldots,\gamma_n)>0$ *and* $\Delta = \mathrm{diag}(\delta_1,\ldots,\delta_n)>0$ *and a scalar* λ^* *such that the following LMIs hold*:

$$P_1 < \lambda^* I \tag{3.40}$$

$$\Psi = \begin{bmatrix} \Sigma & P_1A+\Lambda L_2 & \Gamma M_2 & P_1B & \Delta\Upsilon_2 & P_1C \\ A^{\mathrm{T}}P_1+\Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ \Gamma M_2 & 0 & (1+\varepsilon_0\tau_1^*)P_2-\Gamma & 0 & 0 & 0 \\ B^{\mathrm{T}}P_1 & 0 & 0 & -P_2 & 0 & 0 \\ \Delta\Upsilon_2 & 0 & 0 & 0 & \tau_2P_3-\Delta & 0 \\ C^{\mathrm{T}}P_1 & 0 & 0 & 0 & 0 & -\dfrac{1-\varepsilon_0}{\tau_2}P_3 \end{bmatrix} < 0, \tag{3.41}$$

where

$$\Sigma = -P_1D - DP_1 - \Lambda L_1 - \Gamma M_1 - \Delta\Upsilon_1 + \varepsilon_0\tau_1^*I + (\rho_1+\rho_2)\lambda^*I. \tag{3.42}$$

Proof. The proof of Corollary 1 follows immediately from that of Theorem 1 by noticing that $\dot{\tau}_1^* = 0$ for constant discrete time-delay, and is, therefore, omitted. □

If we are only interested in the global *asymptotic* stability for the stochastic system (2.9), a weaker LMI condition can be obtained as follows.

Corollary 2. *Suppose that $\tau_1(t)$ is the discrete time-varying delay satisfying* (2.2) *and $\dot{\tau}_1(t) \leqslant \tau_0$ (here τ_0 is a constant), and τ_2 is a constant describing the distributed delay. Furthermore, let Assumptions* 1 *and* 3 *hold. Then, the stochastic system* (2.9) *is globally asymptotically stable in mean square if there exist three positive definite matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, three diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) > 0$, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) > 0$ and $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n) > 0$ and a scalar λ^* such that the following LMIs hold*:

$$P_1 < \lambda^* I \tag{3.43}$$

$$\Phi = \begin{bmatrix} \Sigma & P_1 A + \Lambda L_2 & \Gamma M_2 & P_1 B & \Delta \Upsilon_2 & P_1 C \\ A^{\mathrm{T}} P_1 + \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ \Gamma M_2 & 0 & P_2 - \Gamma & 0 & 0 & 0 \\ B^{\mathrm{T}} P_1 & 0 & 0 & -P_2 & 0 & 0 \\ \Delta \Upsilon_2 & 0 & 0 & 0 & \tau_2 P_3 - \Delta & 0 \\ C^{\mathrm{T}} P_1 & 0 & 0 & 0 & 0 & -\dfrac{1}{\tau_2} P_3 \end{bmatrix} < 0, \tag{3.44}$$

where

$$\Sigma = -P_1 D - D P_1 - \Lambda L_1 - \Gamma M_1 - \Delta \Upsilon_1 + \left(\rho_1 + \frac{\rho_2}{1 - \tau_0} \right) \lambda^* I. \tag{3.45}$$

Proof. To avoid unnecessary duplication, we only give the sketch of the proof here, and omit the details. Construct the following Lyapunov–Krasovskii functional:

$$\begin{aligned} V(t) &:= x^{\mathrm{T}}(t) P_1 x(t) + \frac{\rho_2 \lambda^*}{1 - \tau_0} \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(s) x(s) \, \mathrm{d}s \\ &\quad + \int_{t-\tau_1(t)}^{t} \hat{G}^{\mathrm{T}}(x(s)) P_2 \hat{G}(x(s)) \, \mathrm{d}s + \int_0^{\tau_2} \int_{t-s}^{t} \hat{H}^{\mathrm{T}}(x(\eta)) P_3 \hat{H}(x(\eta)) \, \mathrm{d}\eta \, \mathrm{d}s, \end{aligned} \tag{3.46}$$

Then, following the similar line in calculating $\mathrm{d}V(t)$ (from (3.8) to (3.31)), we can obtain

$$\mathrm{d}V(t) \leqslant \lambda_{\max}(\Phi) |x(t)|^2 \, \mathrm{d}t + 2x^{\mathrm{T}}(t) P_1 [\hat{\sigma}(t, x(t), x(t - \tau_1(t)))] \, \mathrm{d}w(t), \tag{3.47}$$

from which we have

$$\frac{\mathrm{d}\mathbb{E}V(t)}{\mathrm{d}t} \leqslant \lambda_{\max}(\Phi) \mathbb{E}|x(t)|^2. \tag{3.48}$$

It can now be concluded from Lyapunov stability theory that the stochastic neural network is globally asymptotically stable in mean square. □

4. A numerical example

In this section, a simple example is presented here to illustrate the usefulness of our main results. Our aim is to examine the global exponential stability of the delayed stochastic system (2.9) with network parameters given as follows:

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & -1.5 \\ 1 & -1.5 & 0.7 \\ -0.8 & -1.8 & -1.6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1.2 & 0.8 \\ -0.9 & 0.8 & 1.2 \\ 0.7 & -1.3 & 1.4 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.8 & 0.9 & -0.7 \\ 0.5 & -1.5 & 0.6 \\ -0.6 & -0.8 & 1 \end{bmatrix}, \quad \tau_1^* = 0.4, \quad \tau_0 = 0.2, \quad \tau_2 = 0.3, \quad \rho_1 = 0.5, \quad \rho_2 = 0.5.$$

Fix $\varepsilon_0 = 0.01$ and take the activation function as follows:

$$f_1(x) = g_1(x) = h_1(x) = \tanh(-2x),$$
$$f_2(x) = g_2(x) = h_2(x) = \tanh(3x),$$
$$f_3(x) = g_3(x) = h_3(x) = \tanh(2x).$$

From the facts

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) = \frac{4\mathrm{e}^{2x}}{(\mathrm{e}^{2x}+1)^2}, \quad 0 < \frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) \leqslant 1,$$

one has

$$L_1 = M_1 = \Upsilon_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = M_2 = \Upsilon_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By using the Matlab LMI toolbox, we solve the LMI (3.5) and (3.6), and obtain

$$P_1 = \begin{bmatrix} 83.5481 & 4.6915 & 10.5867 \\ 4.6915 & 152.8165 & 0.7311 \\ 10.5867 & 0.7311 & 88.6308 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 113.2663 & -16.2461 & -33.1369 \\ -16.2461 & 68.7641 & 10.9967 \\ -33.1369 & 10.9967 & 128.8603 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 115.7898 & -1.6115 & -12.6454 \\ -1.6115 & 77.1325 & -13.8423 \\ -12.6454 & -13.8423 & 85.6061 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 351.5517 & 0 & 0 \\ 0 & 198.8177 & 0 \\ 0 & 0 & 174.2568 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 296.6493 & 0 & 0 \\ 0 & 155.7459 & 0 \\ 0 & 0 & 283.3358 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 126.5884 & 0 & 0 \\ 0 & 54.8625 & 0 \\ 0 & 0 & 97.7004 \end{bmatrix},$$

and $\lambda^* = 337.9705$. Therefore, it follows from Theorem 1 that the stochastic system (2.9) with given parameters is globally exponentially stable in mean square.

5. Conclusions

In this paper, we have dealt with the problem of global exponential stability analysis for a class of stochastic neural networks, which involve both the discrete and distributed time-delays. We have removed the traditional monotonicity and smoothness assumptions on the activation function. A linear matrix inequality (LMI) approach has been developed to solve the problem addressed. The conditions for the global exponential stability in mean square have been derived in terms of the positive definite solution to the LMIs, and a numerical example has been used to demonstrate the usefulness of the main results.

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