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Linear stability and Hopf bifurcation in a three-unit neural network with two delays ☆

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Abstract

Considered is a system of delay differential equations modeling a time-delayed connecting network of three neurons without self-feedback. We investigate the linear stability of the trivial solution and Hopf bifurcation of this system. The general formula for the direction, the estimation formula of period and stability of Hopf bifurcating periodic solution are also given.

keywords: Neural networks; Delay differential equations; Linear stability; Hopf bifurcation; Bifurcation direction

1. Introduction

In some artificial neural network applications, such as content-addressable memories, information is stored as stable equilibrium points of the system. In 1984, Hopfield [18] considered a simplified neural network model in which each neuron is represented by a linear circuit consisting of a resistor and a capacitor and is connected to the other neurons via nonlinear sigmoidal activation functions. Since then, dynamical characteristics of neural networks has become a subject of intensive research activity. We also mention that delays always arise in neural networks due to the processing of information. Hence, most of the models of neural networks are described by systems of delay differential equations (see, for example, [2,8,10,19,20,29,37]). For the general theory of delay differential equations (DDEs) we refer to [13–15]. Due to the complexity of the analysis, most work has focused on the situation where all connection terms in the network have the same time-delay. For neural networks with multiple time-delays, however, the analysis is usually simplified by considering networks with small number of neurons or with simple architectures.

In [33], Shayer and Campbell considered the following system of DDEs with three delays:

$$\dot{x}_1(t) = -\kappa x_1(t) + \beta \tanh(x_1(t-\tau_s)) + \alpha_{12} \tanh(x_2(t-\tau_2)),$$
$$\dot{x}_2(t) = -\kappa x_2(t) + \beta \tanh(x_2(t-\tau_s)) + \alpha_{21} \tanh(x_1(t-\tau_1)).$$

They gave some sufficient conditions on the linear stability of the trivial solution. Later, Lin et al. [25] studied the following three-unit neural network with only one delay,

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{3} a_{ij}\beta \tanh(x_j(t-\tau)), \quad i = 1, 2, 3,$$

where $a_{ii} = 0$, $i = 1, 2, 3$. They, under the assumption that Hopf bifurcation occurs, only gave the formula for the direction of Hopf bifurcation and the estimation formula of the period of the bifurcating periodic solution but did not consider the linear stability of the trivial solution. More results for Hopf bifurcation of some delayed neural networks with two or three neurons can be found in [24,23,4–6,9,11,12,21,31,36] and the references therein.

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In this paper, we consider a three-unit ring network modeled by the following system of DDEs in a parameter space consisting of the two delays, the internal decay rate and the connection strength. More precisely, we shall give conditions on the linear stability of the trivial solution of the system

$$\begin{aligned}\dot{x}_1(t) = &-\kappa x_1(t) + a\tanh(x_2(t-\tau_2)) + a\tanh(x_3(t-\tau_1)),\\ \dot{x}_2(t) = &-\kappa x_2(t) + a\tanh(x_3(t-\tau_2)) + a\tanh(x_1(t-\tau_1)),\\ \dot{x}_3(t) = &-\kappa x_3(t) + a\tanh(x_1(t-\tau_2))\\ &+ a\tanh(x_2(t-\tau_1)),\end{aligned} \tag{1.1}$$

where $\kappa\,(\geqslant 0)$ is the internal decay rate, $\tau_1\,(\geqslant 0)$ and $\tau_2(\geqslant 0)$ are the connection delays, and $a\neq 0$ is the connection strength. Without loss of generality, we assume in the sequel that $\tau_2 \geqslant \tau_1$. Moreover, using the normal form method and the center manifold theory introduced by Hassard et al. [16], we will regard the connection strength a as the parameter to study the direction, stability and the estimation of period of the Hopf bifurcating periodic solution.

This system can model the evolution of a three-unit ring network with delayed feedback. System (1.1) is bidirectional with two loops in the sense that one direction is with time-delay τ_1 and the other direction is with different time-delay τ_2. Ring networks have been found in a variety of neural structures such as neocortex [35], and even in chemistry and electrical engineering. In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks [17,28]. Systems with delays are studied in many biological research topics, as well as in several branches of engineering. In engineering examples [30], the delay is usually discrete because the present state of the system changes at time t in a manner affected by the value of some variables at time $t-\tau$ for some $\tau>0$. Epidemic models require incubation times, which can be modeled with two discrete delays [7,27]. Two delays enter physiological models of development of disease such as erythropoiesis [1] and cyclic granulopoiesis [26]. Moreover, neurological diseases have been studied with two-delay models [3]. We also know time-delays always occur in the signal transmission between neurons in biological neural networks. Delays are omnipresent in artificial neural networks and, in particular, in the electronic hardware implementation, due to the finite propagation velocity of neural signals and due to the finite switching speed of neurons. Neural networks are complex and large-scale nonlinear dynamical systems. But, for simplicity, many researchers have directed their attention to the study of simple systems and it is useful since the complexity found may be carried over to large networks. So, considering this network, without self-feedback described by system (1.1) and with the minimal size among all possible ring networks, is very helpful in studying large ring networks. In this system, each neuron is connected to two others via nonlinear sigmoidal activation function, $\tanh(x)$. The activation function $\tanh(x)$, which is one of the typical transmitting function among neurons, is widely used in application of neural networks [32,2,25]. So, we are particularly interested in studying system (1.1) and we think this work should be significant.

The rest of the paper is organized as follows: In Section 2, we discuss the linear stability of the trivial solution. The conditions on the existence of Hopf bifurcation are given in Section 3. Section 4 is devoted to the direction and stability analysis of Hopf bifurcation and to the estimation of the period of the Hopf bifurcating periodic solution. Finally, an illustrating example is given in Section 5 to conclude the paper.

2. Linear stability of the trivial solution

In this section, we focus on the linear stability of the trivial fixed point $(x_1,x_2,x_3)=(0,0,0)$ of the nonlinear DDE (1.1). Linearizing (1.1) about it produces

$$\begin{aligned}\dot{x}_1(t) &= -\kappa x_1(t) + ax_2(t-\tau_2) + ax_3(t-\tau_1),\\ \dot{x}_2(t) &= -\kappa x_2(t) + ax_3(t-\tau_2) + ax_1(t-\tau_1),\\ \dot{x}_3(t) &= -\kappa x_3(t) + ax_1(t-\tau_2) + ax_2(t-\tau_1).\end{aligned} \tag{2.1}$$

The characteristic equation of (2.1) is

$$\begin{aligned}0 = &\det\begin{pmatrix} \lambda+\kappa & -a\mathrm{e}^{-\lambda\tau_2} & -a\mathrm{e}^{-\lambda\tau_1}\\ -a\mathrm{e}^{-\lambda\tau_1} & \lambda+\kappa & -a\mathrm{e}^{-\lambda\tau_2}\\ -a\mathrm{e}^{-\lambda\tau_2} & -a\mathrm{e}^{-\lambda\tau_1} & \lambda+\kappa\end{pmatrix}\\ &= (\lambda+\kappa)^3 - 3a^2(\lambda+\kappa)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} - a^3(\mathrm{e}^{-3\lambda\tau_1}+\mathrm{e}^{-3\lambda\tau_2})\\ &= \chi_1(\lambda)\chi_2(\lambda),\end{aligned} \tag{2.2}$$

where

$$\chi_1(\lambda) = \lambda+\kappa - a(\mathrm{e}^{-\lambda\tau_1}+\mathrm{e}^{-\lambda\tau_2})$$

and

$$\chi_2(\lambda) = (\lambda+\kappa+\tfrac{1}{2}a(\mathrm{e}^{-\lambda\tau_1}+\mathrm{e}^{-\lambda\tau_2}))^2 + \tfrac{3}{4}a^2(\mathrm{e}^{-\lambda\tau_1}-\mathrm{e}^{-\lambda\tau_2})^2.$$

So either

$$\lambda+\kappa - a(\mathrm{e}^{-\lambda\tau_1}+\mathrm{e}^{-\lambda\tau_2}) = 0 \tag{2.3}$$

or

$$\lambda+\kappa+\tfrac{1}{2}a(\mathrm{e}^{-\lambda\tau_1}+\mathrm{e}^{-\lambda\tau_2}) = \pm\mathrm{i}\tfrac{\sqrt{3}}{2}a(\mathrm{e}^{-\lambda\tau_1}-\mathrm{e}^{-\lambda\tau_2}). \tag{2.4}$$

It is well known that the trivial fixed point of the nonlinear DDE (1.1) is locally asymptotically stable if all the roots, λ, of the characteristic equation (2.2) satisfy $\mathrm{Re}(\lambda)<0$ (see, for example, [22,34]). Our goal in this section is to give the largest subset of the parameters κ, a, τ_1, and τ_2, in which all the roots of the characteristic equation (2.2) have negative real parts. We shall refer this subset as to the stability region of the trivial fixed point.

Substituting $\lambda=\mu+\mathrm{i}\omega$ into the left sides of both (2.3) and (2.4) and separating the real and imaginary parts, we obtain

$$\begin{aligned}\mathrm{R}_{(2.3)}(\mu,\omega) &= \mu+\kappa - a\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1) - a\mathrm{e}^{-\mu\tau_2}\cos(\omega\tau_2),\\ \mathrm{I}_{(2.3)}(\mu,\omega) &= \omega + a\mathrm{e}^{-\mu\tau_1}\sin(\omega\tau_1) + a\mathrm{e}^{-\mu\tau_2}\sin(\omega\tau_2),\end{aligned}$$

$$\mathrm{R}_{(2.4)}(\mu,\omega) = \mu + \kappa + a\mathrm{e}^{-\mu\tau_1}\cos\left(\omega\tau_1 \pm \frac{\pi}{3}\right) + a\mathrm{e}^{-\mu\tau_2}\cos\left(\omega\tau_2 \mp \frac{\pi}{3}\right),$$

$$\mathrm{I}_{(2.4)}(\mu,\omega) = \omega - a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 \pm \frac{\pi}{3}\right) - a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 \mp \frac{\pi}{3}\right).$$

It follows that if $\lambda = \mu + \mathrm{i}\omega$ is a solution to (2.3) then

$$\mu = -\kappa + a\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1) + a\mathrm{e}^{-\mu\tau_2}\cos(\omega\tau_2), \tag{2.5}$$

$$\omega = -a\mathrm{e}^{-\mu\tau_1}\sin(\omega\tau_1) - a\mathrm{e}^{-\mu\tau_2}\sin(\omega\tau_2), \tag{2.6}$$

and if $\lambda = \mu + \mathrm{i}\omega$ is a solution to (2.4) then

$$\mu = -\kappa - a\mathrm{e}^{-\mu\tau_1}\cos\left(\omega\tau_1 \pm \frac{\pi}{3}\right) - a\mathrm{e}^{-\mu\tau_2}\cos\left(\omega\tau_2 \mp \frac{\pi}{3}\right), \tag{2.7}$$

$$\omega = a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 \pm \frac{\pi}{3}\right) + a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 \mp \frac{\pi}{3}\right). \tag{2.8}$$

Theorem 2.1. *If $\kappa > 2|a|$ or $\kappa = -2a$, then all the roots of the characteristic equation* (2.2) *have negative real parts.*

Proof. Let $\mathrm{R}(\mu) = \mu + \kappa - |a|\mathrm{e}^{-\mu\tau_1} - |a|\mathrm{e}^{-\mu\tau_2}$. Obviously,

$$\mathrm{R}_{(2.3)}(\mu,\omega) \geqslant \mathrm{R}(\mu) \quad \text{and} \quad \mathrm{R}_{(2.4)}(\mu,\omega) \geqslant \mathrm{R}(\mu). \tag{2.9}$$

Since $\kappa > 2|a|$ or $\kappa = -2a$, we have

$$\mathrm{R}(0) = \kappa - 2|a| \geqslant 0.$$

This, combined with $\mathrm{dR}(\mu)/\mathrm{d}\mu = 1 + |a|\tau_1\mathrm{e}^{-\mu\tau_1} + |a|\tau_2\mathrm{e}^{-\mu\tau_2} > 0$, implies that $\mathrm{R}(\mu) > 0$ for $\mu > 0$. Therefore, it follows from (2.9) that

$$\mathrm{R}_{(2.3)}(\mu,\omega) > 0 \quad \text{and} \quad \mathrm{R}_{(2.4)} > 0 \quad \text{for } \mu > 0 \text{ and } \omega \in \mathbb{R}. \tag{2.10}$$

Assume that $\lambda = \mu + \mathrm{i}\omega$ is a solution to (2.2). By (2.10), we only need to show that $\lambda \neq \mathrm{i}\omega$. If $\kappa > 2|a|$, this is true from $\mathrm{R}(0) = \kappa - 2|a| > 0$ and (2.9). Now, assume $\kappa = -2a$. First, we show that $\mathrm{R}_{(2.3)}(0,\omega)$ and $\mathrm{I}_{(2.3)}(0,\omega)$ cannot be zero simultaneously. Otherwise, it follows from $\mathrm{R}_{(2.3)}(0,\omega) = 0$ that there exists $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $\omega\tau_1 = (2m+1)\pi$ and $\omega\tau_2 = (2n+1)\pi$. Then $\mathrm{I}_{(2.3)}(0,\omega) = \omega \neq 0$. Similarly, we can show that $\mathrm{R}_{(2.4)}(0,\omega)$ and $\mathrm{I}_{(2.4)}(0,\omega)$ cannot be zero simultaneously. This completes the proof. □

Theorem 2.1 presents a delay-independent sufficient condition for the linear stability of the trivial solution. In other words, under the condition that $\kappa > 2|a|$ or $\kappa = -2a$, the delays τ_1 and τ_2 are harmless to (1.1). In addition, when $\kappa = 2a$, by the way of contradiction, we can easily obtain all the roots of $\chi_1(\lambda)$, namely, (2.5) and (2.6), have negative real parts except $\lambda = 0$ and all roots of $\chi_2(\lambda)$, namely, (2.7) and (2.8), have negative real parts. In the following, we give some delay-dependent conditions on the linear stability of the trivial solution to (1.1).

Theorem 2.2. *Assume $-a < \kappa < -2a$, $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant -1/2a$. Then all the roots of the characteristic equation* (2.2) *have negative real parts.*

Proof. Let $\lambda = \mu + \mathrm{i}\omega$ be a root of (2.2). Since roots of (2.2) appear in complex conjugate pairs, without loss of generality, we can assume that $\omega \geqslant 0$. By way of contradiction, we assume that $\mu \geqslant 0$. We distinguish two cases to finish the proof.

First, we assume that μ and ω satisfy (2.5) and (2.6) simultaneously. It follows from (2.6) that $\omega \leqslant -2a$ and hence $0 \leqslant \omega\tau_1 \leqslant \omega\tau_2 \leqslant 1 < \pi/3$. It also follows from (2.5) and (2.6) that

$$M_1(\mu,\omega) = 0, \tag{2.11}$$

where

$$M_1(\mu,\omega) = (\mu+\kappa)^2 + \omega^2 - 2a(\mu+\kappa)\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1) + 2a\omega\mathrm{e}^{-\mu\tau_1}\sin(\omega\tau_1) + a^2\mathrm{e}^{-2\mu\tau_1} - a^2\mathrm{e}^{-2\mu\tau_2}.$$

For fixed ω, we have

$$\begin{aligned} M_1(0,\omega) &= \kappa^2 + \omega^2 - 2a\kappa\cos(\omega\tau_1) + 2a\omega\sin(\omega\tau_1) \\ &\geqslant \kappa^2 - 2a\kappa\cos(\omega\tau_1) + \omega^2 + 2a\omega(\omega\tau_1) \\ &\quad (\text{because } \sin(\omega\tau_1) \leqslant \omega\tau_1) \\ &= \kappa^2 - 2a\kappa\cos(\omega\tau_1) + \omega^2(1 + 2a\tau_1) \\ &> 0 \quad (\text{because } -1 \leqslant 2a\tau_1 \leqslant 0). \end{aligned}$$

Taking the partial derivative of $M_1(\mu,\omega)$ with respect to μ, we have

$$\begin{aligned} \frac{\partial M_1(\mu,\omega)}{\partial\mu} &= 2(\mu+\kappa) - 2a\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1) \\ &\quad + 2a\tau_1(\mu+\kappa)\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1) \\ &\quad - 2a\omega\tau_1\mathrm{e}^{-\mu\tau_1}\sin(\omega\tau_1) - 2a^2\tau_1\mathrm{e}^{-2\mu\tau_1} \\ &\quad + 2a^2\tau_2\mathrm{e}^{-2\mu\tau_2} \\ &= 2\{(\mu+\kappa)[1 + a\tau_1\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1)] \\ &\quad - a\mathrm{e}^{-\mu\tau_1}[\cos(\omega\tau_1) + a\tau_1\mathrm{e}^{-\mu\tau_1}] \\ &\quad - 2a\omega\tau_1\sin(\omega\tau_1) + 2a^2\tau_2\mathrm{e}^{-2\mu\tau_2}\}. \end{aligned}$$

Note that $a < 0$ since $\kappa < -2a$ and $\kappa \geqslant 0$. Then the last two terms $-2a\omega\tau_1\sin(\omega\tau_1)$ and $2a^2\tau_2\mathrm{e}^{-2\mu\tau_2}$ are nonnegative. On the other hand, since $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant -1/2a$ and $\frac{1}{2} = \cos(\pi/3) < \cos 1 \leqslant \cos(\omega\tau_1) \leqslant 1$, we have

$$(\mu+k)[1 + a\tau_1\mathrm{e}^{-\mu\tau_1}\cos(\omega\tau_1)] \geqslant (\mu+k)(1-\tfrac{1}{2}) \geqslant 0$$

and

$$-a\mathrm{e}^{-\mu\tau_1}[\cos(\omega\tau_1) + a\tau_1\mathrm{e}^{-\mu\tau_1}] \geqslant -a\mathrm{e}^{-\mu\tau_1}(\cos 1 - \tfrac{1}{2}) > 0.$$

Therefore, $(\partial M_1(\mu,\omega))/\partial\mu > 0$. This, combined with $M_1(0,\omega) > 0$, implies that $M_1(\mu,\omega) > 0$ for $\mu \geqslant 0$, which contradicts with (2.11).

Now, we assume that μ and ω satisfy (2.7) and (2.8) simultaneously. We first assume that

$$\mu = -\kappa - a\mathrm{e}^{-\mu\tau_1}\cos\left(\omega + \frac{\pi}{3}\right) - a\mathrm{e}^{-\mu\tau_2}\cos\left(\omega\tau_2 - \frac{\pi}{3}\right), \tag{2.12}$$

$$\omega = a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 + \frac{\pi}{3}\right) + a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 - \frac{\pi}{3}\right). \tag{2.13}$$

It follows from (2.13) easily that $\omega \leqslant -2a$ and hence

$$0 \leqslant \omega\tau_1 \leqslant \omega\tau_2 \leqslant 1 < \frac{\pi}{3}.$$

Then

$$\frac{\pi}{3} \leqslant \omega\tau_1 + \frac{\pi}{3} \leqslant \omega\tau_2 + \frac{\pi}{3} < \frac{2\pi}{3} \quad \text{and}$$
$$-\frac{\pi}{3} \leqslant \omega\tau_1 - \frac{\pi}{3} \leqslant \omega\tau_2 - \frac{\pi}{3} < 0.$$

Thus,

$$\begin{aligned}\omega &= a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 + \frac{\pi}{3}\right) + a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 - \frac{\pi}{3}\right)\\ &< a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 - \frac{\pi}{3}\right)\\ &< -a.\end{aligned}$$

This, combined with the assumptions $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant -1/2a$, gives us

$$0 \leqslant \omega\tau_1 \leqslant \omega\tau_2 < \frac{1}{2} < \frac{\pi}{6}$$

and hence

$$\frac{\pi}{3} \leqslant \omega\tau_1 + \frac{\pi}{3} < \frac{\pi}{2}, \quad -\frac{\pi}{3} \leqslant \omega\tau_2 - \frac{\pi}{3} < -\frac{\pi}{6}.$$

Then

$$\begin{aligned}&a\mathrm{e}^{-\omega\tau_1}\sin\left(\omega\tau_1 + \frac{\pi}{3}\right) + a\mathrm{e}^{-\omega\tau_2}\sin\left(\omega\tau_2 - \frac{\pi}{3}\right)\\ &\quad < a\mathrm{e}^{-\omega\tau_1}\sin\frac{\pi}{3} + a\mathrm{e}^{-\omega\tau_2}\sin\left(-\frac{\pi}{3}\right)\\ &\quad = -\frac{\sqrt{3}}{2}a(\mathrm{e}^{-\omega\tau_2} - \mathrm{e}^{-\omega\tau_1})\\ &\quad \leqslant 0.\end{aligned}$$

This contradicts with the fact that $\omega \geqslant 0$. Finally, we assume that

$$\mu = -\kappa - a\mathrm{e}^{-\mu\tau_1}\cos\left(\omega\tau_1 - \frac{\pi}{3}\right) - a\mathrm{e}^{-\mu\tau_2}\cos\left(\omega\tau_2 + \frac{\pi}{3}\right), \tag{2.14}$$

$$\omega = a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 - \frac{\pi}{3}\right) + a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 + \frac{\pi}{3}\right). \tag{2.15}$$

Then it follows from (2.14) and (2.15) that

$$M_2(\mu, \omega) = 0, \tag{2.16}$$

where

$$\begin{aligned}M_2(\mu, \omega) &= (\mu + \kappa)^2 + \omega^2 - a^2\mathrm{e}^{-2\mu\tau_1} - a^2\mathrm{e}^{-2\mu\tau_2}\\ &\quad - 2a^2\mathrm{e}^{-\mu(\tau_1+\tau_2)}\cos\left(\omega\tau_1 - \omega\tau_2 - \frac{2\pi}{3}\right).\end{aligned}$$

Note that as before we still have $0 \leqslant \omega\tau_1 \leqslant \omega\tau_2 \leqslant \frac{1}{2} < \pi/6$. Then

$$-\frac{\pi}{2} < -\frac{\pi}{12} - \frac{\pi}{3} < \frac{1}{2}(\omega\tau_1 - \omega\tau_2) - \frac{\pi}{3} \leqslant -\frac{\pi}{3}$$

and

$$\begin{aligned}M_2(0, \omega) &= \kappa^2 + \omega^2 - 2a^2\\ &\quad - 2a^2\left[2\cos^2\left(\frac{1}{2}(\omega\tau_1 - \omega\tau_2) - \frac{\pi}{3}\right) - 1\right]\\ &= \kappa^2 + \omega^2 - 4a^2\cos^2\left(\frac{1}{2}(\omega\tau_1 - \omega\tau_2) - \frac{\pi}{3}\right)\\ &\geqslant \kappa^2 + \omega^2 - 4a^2(\tfrac{1}{2})^2\\ &= \kappa^2 + \omega^2 - a^2\\ &> 0.\end{aligned}$$

Here, we used the condition that $\kappa > -a$. Again, taking the partial derivative of $M_2(\mu, \omega)$ with respect to μ, we get

$$\begin{aligned}\frac{\partial M_2(\mu, \omega)}{\partial \mu} &= 2(\mu + \kappa) + 2a^2\tau_1\mathrm{e}^{-2\mu\tau_1} + 2a^2\tau_2\mathrm{e}^{-2\mu\tau_2}\\ &\quad + 2a^2(\tau_1 + \tau_2)\mathrm{e}^{-\mu(\tau_1+\tau_2)}\\ &\quad \times\left[2\cos^2\left(\frac{1}{2}(\omega\tau_1 - \omega\tau_2) - \frac{\pi}{3}\right) - 1\right]\\ &= 2\Big[(\mu + \kappa) + a^2\tau_1\mathrm{e}^{-2\mu\tau_1} + a^2\tau_2\mathrm{e}^{-2\mu\tau_2}\\ &\quad - a^2(\tau_1 + \tau_2)\mathrm{e}^{-\mu(\tau_1+\tau_2)}\\ &\quad + 2a^2(\tau_1 + \tau_2)\mathrm{e}^{-\mu(\tau_1+\tau_2)}\\ &\quad \times\cos^2\left(\frac{1}{2}(\omega\tau_1 - \omega\tau_2) - \frac{\pi}{3}\right)\Big].\end{aligned}$$

Noting,

$$\kappa - a^2(\tau_1 + \tau_2)\mathrm{e}^{-\mu(\tau_1+\tau_2)} > \kappa - a^2\left(-\frac{1}{a}\right) = \kappa + a > 0$$

since $\kappa > -a$, we have

$$\frac{\partial M_2(\mu, \omega)}{\partial \mu} > 0.$$

This, combined with $M_2(0, \omega) > 0$, implies that $M_2(\mu, \omega) > 0$ for $\mu \geqslant 0$, a contradiction to (2.16). This completes the proof. $\square$

In the remaining of this section, we will give two results on the unstability of the trivial solution. One is delay-independent and the other is delay-dependent.

Theorem 2.3. *Assume that* $0 \leqslant \kappa < 2a$. *Then, for any* τ_1 *and* τ_2, *the characteristic equation* (2.2) *has a root with positive real part.*

Proof. Under the assumption, we see that

$$\chi_1(0) = \kappa - 2a < 0.$$

Also note that

$$\lim_{\lambda\to\infty}\chi_1(\lambda) = \lim_{\lambda\to\infty}[\lambda + \kappa - a(\mathrm{e}^{-\lambda\tau_1} + \mathrm{e}^{-\lambda\tau_2})] = \infty.$$

It follows from the mean value theorem that there exists a $\lambda^* > 0$ such that $\chi_1(\lambda^*) = 0$. That is, (2.3) and hence (2.2) has a positive real root. This completes the proof. $\square$

Theorem 2.4. *Assume that $0 \leqslant \kappa < -a$. Then the characteristic equation* (2.2) *has a root with positive real part if one of the following two conditions holds.*

(i) $\tau_1 + \tau_2 < -3\pi/2a$ *and* $\tau_2 - \tau_1 < -\pi/6a$.
(ii) $\tau_1 + \tau_2 \leqslant -\pi/2a$.

Proof. Note that if λ is a solution to

$$\lambda + \kappa + \tfrac{1}{2}a(\mathrm{e}^{-\lambda\tau_1} + \mathrm{e}^{-\lambda\tau_2}) = -\mathrm{i}\tfrac{\sqrt{3}}{2}a(\mathrm{e}^{-\lambda\tau_1} - \mathrm{e}^{-\lambda\tau_2}), \tag{2.17}$$

then $\overline{\lambda}$ is a solution to

$$\lambda + \kappa + \tfrac{1}{2}a(\mathrm{e}^{-\lambda\tau_1} + \mathrm{e}^{-\lambda\tau_2}) = \mathrm{i}\tfrac{\sqrt{3}}{2}a(\mathrm{e}^{-\lambda\tau_1} - \mathrm{e}^{-\lambda\tau_2})$$

and vice versa. Thus, it suffices to show that (2.17) has a solution with positive real part. In the following, we will fix τ_1 and regard $\tau_2 \geqslant \tau_1$ as a parameter. Denote

$$\tilde{\Delta}(\lambda, \tau_2) = \lambda + \kappa + \tfrac{1}{2}a(\mathrm{e}^{-\lambda\tau_1} + \mathrm{e}^{-\lambda\tau_2}) + \mathrm{i}\tfrac{\sqrt{3}}{2}a(\mathrm{e}^{-\lambda\tau_1} - \mathrm{e}^{-\lambda\tau_2}).$$

Let $\lambda = \mu + \mathrm{i}\omega$. Then

$$\begin{aligned}\tilde{\Delta}(\lambda, \tau_2) = {} & \mu + \kappa + a\mathrm{e}^{-\mu\tau_1}\cos\left(\omega\tau_1 - \frac{\pi}{3}\right) \\ & + a\mathrm{e}^{-\mu\tau_2}\cos\left(\omega\tau_2 + \frac{\pi}{3}\right) \\ & + \mathrm{i}\left(\omega - a\mathrm{e}^{-\mu\tau_1}\sin\left(\omega\tau_1 - \frac{\pi}{3}\right)\right. \\ & \left. - a\mathrm{e}^{-\mu\tau_2}\sin\left(\omega\tau_2 + \frac{\pi}{3}\right)\right).\end{aligned}$$

Note that

$$\tilde{\Delta}(0, \tau_1) = \kappa + a < 0$$

and

$$\lim_{\lambda\to\infty} \tilde{\Delta}(\lambda, \tau_1) = \infty.$$

By the mean value theorem, there exists a $\lambda(\tau_1) > 0$ such that $\tilde{\Delta}(\lambda(\tau_1), \tau_1) = 0$. Since solutions continuously depend on the parameter τ_2, we assume that $\lambda(\tau_2) = \mu(\tau_2) + \mathrm{i}\omega(\tau_2)$ with $\omega(\tau_2) \geqslant 0$ is a solution to (2.17) passing through $\lambda(\tau_1)$. Then $\mu(\tau_2)$ and $\omega(\tau_2)$ satisfy (2.14) and (2.15) simultaneously, that is,

$$\begin{aligned}\mu(\tau_2) = {} & -\kappa - a\mathrm{e}^{-\mu(\tau_2)\tau_1}\cos\left(\omega(\tau_2)\tau_1 - \frac{\pi}{3}\right) \\ & - a\mathrm{e}^{-\mu(\tau_2)\tau_2}\cos\left(\omega(\tau_2)\tau_2 + \frac{\pi}{3}\right),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\omega(\tau_2) = {} & a\mathrm{e}^{-\mu(\tau_2)\tau_1}\sin\left(\omega(\tau_2)\tau_1 - \frac{\pi}{3}\right) \\ & + a\mathrm{e}^{-\mu(\tau_2)\tau_2}\sin\left(\omega(\tau_2)\tau_2 + \frac{\pi}{3}\right).\end{aligned} \tag{2.19}$$

First, suppose (i) holds. We show that there is no such τ_2 such that $\mu(\tau_2) = 0$ by way of contradiction. Assume there is such a τ_{20} such that $\mu(\tau_{20}) = 0$. Then

$$\kappa = -a\cos\left(\omega(\tau_{20})\tau_1 - \frac{\pi}{3}\right) - a\cos\left(\omega(\tau_{20})\tau_{20} + \frac{\pi}{3}\right),$$

$$\omega(\tau_{20}) = a\sin\left(\omega(\tau_{20})\tau_1 - \frac{\pi}{3}\right) + a\sin\left(\omega(\tau_{20})\tau_{20} + \frac{\pi}{3}\right),$$

or, equivalently,

$$\begin{aligned}\kappa = {} & -2a\cos\left(\frac{\omega(\tau_{20})(\tau_1 + \tau_{20})}{2}\right) \\ & \times\cos\left(\frac{\omega(\tau_{20})(\tau_{20} - \tau_1)}{2} + \frac{\pi}{3}\right),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\omega(\tau_{20}) = {} & 2a\sin\left(\frac{\omega(\tau_{20})(\tau_1 + \tau_{20})}{2}\right) \\ & \times\cos\left(\frac{\omega(\tau_{20})(\tau_{20} - \tau_1)}{2} + \frac{\pi}{3}\right).\end{aligned} \tag{2.21}$$

Note that $\lambda = 0$ is not a solution to (2.17). It follows from (2.21) that $0 < \omega(\tau_{20}) \leqslant -2a$. Then

$$\frac{\pi}{3} < \frac{\omega(\tau_{20})(\tau_{20} - \tau_1)}{2} + \frac{\pi}{3} < -a\left(-\frac{\pi}{6a}\right) + \frac{\pi}{3} = \frac{\pi}{2}$$

and

$$0 < \frac{\omega(\tau_{20})(\tau_{20} + \tau_1)}{2} < -a\left(-\frac{3\pi}{2a}\right) = \frac{3\pi}{2}.$$

Noting $0 \leqslant \kappa$ and $\omega(\tau_{20}) > 0$, we can easily see that (2.20) and (2.21) cannot hold simultaneously. This contradiction, combined with the continuity of $\lambda(\tau_2)$, implies that if (i) holds then $\mu(\tau_2) > 0$.

Now, we suppose (ii) holds. Again, we show $\mu(\tau_2) > 0$ by way of contradiction. Assume there is a τ_{20} such that $\mu(\tau_{20}) = 0$. Then (2.20) and (2.21) hold simultaneously. Also, we have $0 < \omega(\tau_{20}) \leqslant -2a$ and

$$0 < \frac{\omega(\tau_{20})(\tau_{20} - \tau_1)}{2} \leqslant \frac{\omega(\tau_{20})(\tau_1 + \tau_{20})}{2} \leqslant \frac{(-2a)(-\pi/2a)}{2} = \frac{\pi}{2}.$$

We claim that $(\omega(\tau_{20})(\tau_1 + \tau_{20}))/2 \neq \pi/2$. Otherwise, it follows from (2.21) that $\omega(\tau_{20}) \leqslant -\sqrt{3}a$ and then

$$\frac{\omega(\tau_{20})(\tau_1 + \tau_{20})}{2} \leqslant \frac{(-\sqrt{3}a)(-\pi/2a)}{2} = \frac{\sqrt{3}\pi}{4} < \frac{\pi}{2}.$$

It is a contradiction. This proves the claim. Since $(\omega(\tau_{20})(\tau_1 + \tau_{20}))/2 < \pi/2$, it follows from (2.20) that $\cos((\omega(\tau_{20})(\tau_{20} - \tau_1))/2 + \pi/3) \geqslant 0$ while it follows from (2.21) that $\cos((\omega(\tau_{20})(\tau_{20} - \tau_1))/2 + \pi/3) < 0$, a contradiction. Then again we have $\mu(\tau_2) > 0$. This completes the proof. $\square$

3. Existence of Hopf bifurcation

From the discussion in Section 2, we know that the linear stability of the trivial solution of system (1.1) depends on the involved parameters. In this section, we study the Hopf bifurcation from the trivial solution by regarding a as the bifurcation parameter. Recall that the characteristic equation of system (2.1) is

$$(\lambda + \kappa)^3 - 3a^2(\lambda + \kappa)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} - a^3(\mathrm{e}^{-3\lambda\tau_1} + \mathrm{e}^{-3\lambda\tau_2}) = 0. \tag{3.1}$$

Assume that (3.1) has a pair of purely imaginary solutions $\lambda = \pm\mathrm{i}\omega_0$ when $a = a_0$. Let $\lambda(a)$ be the solution of (3.1) passing through $(a_0, \mathrm{i}\omega_0)$. Differentiate (3.1) with respect to a to obtain

$$\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(a_0)}{\mathrm{d}a} = \frac{1}{M^2+N^2}\{M[6a_0(\kappa\cos(\omega\tau_1+\omega\tau_2) + \omega\sin(\omega\tau_1+\omega\tau_2)) + 3a_0^2(\cos(3\omega\tau_1)+\cos(3\omega\tau_2))] + N[6a_0(\omega\cos(\omega\tau_1+\omega\tau_2) - \kappa\sin(\omega\tau_1+\omega\tau_2)) - 3a_0^2(\sin(3\omega\tau_1) + \sin(3\omega\tau_2))]\}, \tag{3.2}$$

$$\frac{\mathrm{d}\,\mathrm{Im}\,\lambda(a_0)}{\mathrm{d}a} = \frac{1}{M^2+N^2}\{M[6a_0(\omega\cos(\omega\tau_1+\omega\tau_2) - \kappa\sin(\omega\tau_1+\omega\tau_2)) - 3a_0^2(\sin(3\omega\tau_1)+\sin(3\omega\tau_2))] - N[6a_0(\kappa\cos(\omega\tau_1+\omega\tau_2) + \omega\sin(\omega\tau_1+\omega\tau_2)) + 3a_0^2(\cos(3\omega\tau_1) + \cos(3\omega\tau_2))]\}, \tag{3.3}$$

where

$$M = 3(\kappa^2-\omega^2) - 3a_0^2\cos(\omega\tau_1+\omega\tau_2) + 3a_0^2(\tau_1+\tau_2)[\kappa\cos(\omega\tau_1+\omega\tau_2) + \omega\sin(\omega\tau_1+\omega\tau_2)] + 3a_0^3\tau_1\cos(3\omega\tau_1) + 3a_0^3\tau_2\cos(3\omega\tau_2),$$

$$N = 6\kappa\omega + 3a_0^2\sin(\omega\tau_1+\omega\tau_2) + 3a_0^2(\tau_1+\tau_2) \times[\omega\cos(\omega\tau_1+\omega\tau_2) - \kappa\sin(\omega\tau_1+\omega\tau_2)] - 3a_0^3\tau_1\sin(3\omega\tau_1) - 3a_0^3\tau_2\sin(3\omega\tau_2).$$

Let

$$\alpha'(a_0) = \frac{\mathrm{d}\,\mathrm{Re}\,\lambda(a_0)}{\mathrm{d}a}.$$

Then, using the Hopf bifurcation theory, we have the following result.

Theorem 3.1. *For given τ_1, τ_2 and κ, assume all roots of* (3.1) *at $a=a_0$ have negative real parts except the pair of purely imaginary ones $\pm \mathrm{i}\omega_0$. If $\alpha'(a_0)\neq 0$ then system* (1.1) *has a periodic solution bifurcating from $a=a_0$.*

4. Direction, period and stability of Hopf bifurcating periodic solution

In this section, we discuss the direction of Hopf bifurcation and the stability of the bifurcating periodic solution. First, we rewrite the DDE (1.1) as an ODE by using the Riesz representation theorem. Then we use the method of Hassard et al. [16] to establish the results on Hopf bifurcation.

Write $a=a_0+\sigma$. With the Taylor expansion of $\tanh(x)$ about $x=0$, (1.1) can be expressed in the vector form

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\kappa X(t) + (a_0+\sigma)M_1X(t-\tau_1) + (a_0+\sigma)M_2X(t-\tau_2) - \frac{1}{3}(a_0+\sigma)M_1X^3(t-\tau_1) - \frac{1}{3}(a_0+\sigma)M_2X^3(t-\tau_2) + O(X^5(t-\tau_1), X^5(t-\tau_2)), \tag{4.1}$$

where $X^{\mathrm{i}}(t) = (x_1^{\mathrm{i}}(t), x_2^{\mathrm{i}}(t), x_3^{\mathrm{i}}(t))^{\mathrm{T}}$, $i=1,\ 3,\ 5$,

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $X_t = (x_{1t}, x_{2t}, x_{3t})^{\mathrm{T}}$, $x_{it}(\theta) = x_i(t+\theta)$, $-\tau_2\leqslant\theta\leqslant 0$, $i=1,2,3$. Then system (4.1) can also be rewritten as

$$\dot{X}(t) = L_\sigma X_t + G(X_t,\sigma), \tag{4.2}$$

where, for $\varphi = (\varphi_1,\varphi_2,\varphi_3)^{\mathrm{T}} \in C \stackrel{\Delta}{=} C([-\tau_2,0],\mathbb{R}^3)$,

$$L_\sigma\varphi = -\kappa\varphi(0) + (a_0+\sigma)M_1\varphi(-\tau_1) + (a_0+\sigma)M_2\varphi(-\tau_2) \tag{4.3}$$

and

$$G(\varphi,\sigma) = -\tfrac{1}{3}(a_0+\sigma)M_1\varphi^3(-\tau_1) - \tfrac{1}{3}(a_0+\sigma)M_2\varphi^3(-\tau_2) + O(\varphi^5(-\tau_1),\varphi^5(-\tau_2)). \tag{4.4}$$

By the Riesz representation theorem, there exists a 3×3 matrix-valued bounded variation function $\eta(.,\sigma): [-\tau_2,0]\to\mathbb{R}^{3^2}$ such that

$$L_\sigma\varphi = \int_{-\tau_2}^{0} \mathrm{d}\eta(\theta,\sigma)\varphi(\theta), \quad \varphi\in C([-\tau_2,0],\mathbb{R}^3). \tag{4.5}$$

In fact, we can choose

$$\eta(\theta,\sigma) = \begin{cases} -\kappa\,\mathrm{Id}_3 & \text{if } \theta=0, \\ (a_0+\sigma)M_1\delta(t+\tau_1) & \text{if } -\tau_1\leqslant\theta<0, \\ -(a_0+\sigma)M_2\delta(t+\tau_2) & \text{if } -\tau_2\leqslant\theta<-\tau_1, \end{cases}$$

where $\delta(\theta)$ is the Dirac function and Id_3 is the 3×3 identity matrix.

Now, for $\varphi\in C^1([-\tau_2,0],\mathbb{R}^3)$, we define

$$(A_\sigma\varphi)(\theta) = \begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\theta} & \text{if } \theta\in[-\tau_2,0), \\ \int_{-\tau_2}^{0}\mathrm{d}\eta(\xi,\sigma)\varphi(\xi) = L_\sigma(\varphi) & \text{if } \theta=0 \end{cases} \tag{4.6}$$

and

$$(R_\sigma\varphi)(\theta) = \begin{cases} (0,0,0)^{\mathrm{T}} & \text{if } \theta\in[-\tau_2,0), \\ G(\varphi,\sigma) & \text{if } \theta=0. \end{cases} \tag{4.7}$$

Then, since $\mathrm{d}x_t/\mathrm{d}t = \mathrm{d}x_t/\mathrm{d}\theta$, we can further rewrite (4.1) as

$$\dot{X}_t = A_\sigma X_t + R_\sigma X_t, \tag{4.8}$$

which is the desired form for our purpose. Note that (4.8) is just the trivial equation $\mathrm{d}x_t/\mathrm{d}t = \mathrm{d}x_t/\mathrm{d}\theta$ for $\theta \in [-\tau_2, 0)$ while it is (4.1) for $\theta = 0$.

Recall that when $\sigma = 0$ (namely, $a = a_0$), the characteristic equation (3.1) has a pair of purely imaginary roots $\lambda(a_0) = \pm \mathrm{i}\omega_0$ and $\mathrm{d}\,\mathrm{Re}\,\lambda(a_0)/\mathrm{d}a \neq 0$. Then there is a neighborhood of $\sigma = 0$ such that for any σ in it there is a two-dimensional local center manifold C_0 of (4.1) in C, which contains the zero element of C and the orbits of Hopf periodic solutions are also located in C_0. In order to construct local coordinates to describe the center manifold C_0 near the origin, we need an inner product and the adjoint operator A_0^* of A_0. A_0^* is defined as

$$(A_0^*\psi)(s) = \begin{cases} -\dfrac{\mathrm{d}\psi}{\mathrm{d}s} & \text{if } s \in (0, \tau_2], \\ \int_{-\tau_2}^{0} \mathrm{d}\eta^{\mathrm{T}}(t, 0)\psi(-t) & \text{if } s = 0, \end{cases} \tag{4.9}$$

where $\psi \in C([0, \tau_2], \mathbb{R}^3)$, η^{T} denotes the transpose of η. The inner product is defined by

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0) - \int_{\theta=-\tau_2}^{0} \int_{\xi=0}^{\theta} \overline{\psi}^{\mathrm{T}}(\xi - \theta)\,\mathrm{d}\eta(\theta, 0)\varphi(\xi)\,\mathrm{d}\xi, \tag{4.10}$$

for $\psi \in C([0, \tau_2], \mathbb{R}^3)$ and $\varphi \in C$. Then as usual, $\langle \psi, A_0\varphi \rangle = \langle A_0^*\psi, \varphi \rangle$ for $(\varphi, \psi) \in \mathrm{Dom}(A_0) \times \mathrm{Dom}(A_0^*)$.

Let $q(\theta)$ be the eigenvector of A_0 corresponding to the eigenvalue $\lambda(a_0) = \mathrm{i}\omega_0$, namely,

$$A_0 q(\theta) = \mathrm{i}\omega_0 q(\theta). \tag{4.11}$$

Note that $q(\theta)$ is an eigenvector of A_0 associated with the eigenvalue $\mathrm{i}\omega_0$ if and only if $q(\theta) = (q_1 \mathrm{e}^{\mathrm{i}\omega_0\theta}, q_2 \mathrm{e}^{\mathrm{i}\omega_0\theta}, \mathrm{e}^{\mathrm{i}\omega_0\theta})^{\mathrm{T}}$ for $\theta \in [-\tau_2, 0]$, where the complex-valued vector $(q_1, q_2, 1)^{\mathrm{T}}$ satisfies

$$\begin{pmatrix} \lambda + \kappa & -a\mathrm{e}^{-\lambda\tau_2} & -a\mathrm{e}^{-\lambda\tau_1} \\ -a\mathrm{e}^{-\lambda\tau_1} & \lambda + \kappa & -a\mathrm{e}^{-\lambda\tau_2} \\ -a\mathrm{e}^{-\lambda\tau_2} & -a\mathrm{e}^{-\lambda\tau_1} & \lambda + \kappa \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} = 0.$$

So,

$$q_1 = -\mathrm{e}^{\mathrm{i}\omega_0(\tau_2-\tau_1)} q_2 + \frac{\mathrm{i}\omega_0 + \kappa}{a_0} \mathrm{e}^{\mathrm{i}\omega_0\tau_2},$$

$$q_2 = \frac{a_0 + (\mathrm{i}\omega_0 + \kappa)\mathrm{e}^{\mathrm{i}\omega_0(2\tau_2-\tau_1)}}{a_0 \mathrm{e}^{2\mathrm{i}\omega_0(\tau_2-\tau_1)} + (\mathrm{i}\omega_0 + \kappa)\mathrm{e}^{\mathrm{i}\omega_0\tau_2}}.$$

The eigenvector of A_0^* corresponding to the eigenvalue $-\mathrm{i}\omega_0$ is

$$\widetilde{q}^*(\xi) = (\widetilde{q}_1^* \mathrm{e}^{\mathrm{i}\omega_0\xi}, \widetilde{q}_2^* \mathrm{e}^{\mathrm{i}\omega_0\xi}, \mathrm{e}^{\mathrm{i}\omega_0\xi})^{\mathrm{T}}$$

for $\xi \in [0, \tau_2]$, where

$$\widetilde{q}_1^* = -\frac{1}{a_0}[(\mathrm{i}\omega_0 - \kappa)\mathrm{e}^{-\mathrm{i}\omega_0\tau_2}\widetilde{q}_2^* + a_0 \mathrm{e}^{-\mathrm{i}\omega_0(\tau_2-\tau_1)}],$$

$$\widetilde{q}_2^* = \frac{a_0 \mathrm{e}^{-2\mathrm{i}\omega_0(\tau_2-\tau_1)} - (\mathrm{i}\omega_0 - \kappa)\mathrm{e}^{-\omega_0\tau_2}}{a_0 - (\mathrm{i}\omega_0 - \kappa)\mathrm{e}^{-\mathrm{i}\omega_0(2\tau_2-\tau_1)}}.$$

Let

$$\begin{aligned} q^* &= (D\widetilde{q}_1^* \mathrm{e}^{\mathrm{i}\omega_0\theta}, D\widetilde{q}_2^* \mathrm{e}^{\mathrm{i}\omega_0\theta}, D\mathrm{e}^{\mathrm{i}\omega_0\theta})^{\mathrm{T}} \\ &= (q_1^* \mathrm{e}^{\mathrm{i}\omega_0\theta}, q_2^* \mathrm{e}^{\mathrm{i}\omega_0\theta}, q_3^* \mathrm{e}^{\mathrm{i}\omega_0\theta})^{\mathrm{T}}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} D = [&\overline{\widetilde{q}_1^*}(q_1 + a_0\tau_1 \mathrm{e}^{-\mathrm{i}\omega_0\tau_1} + a_0\tau_2 \mathrm{e}^{-\mathrm{i}\omega_0\tau_2} q_2) \\ &+ \overline{\widetilde{q}_2^*}(q_2 + a_0\tau_1 \mathrm{e}^{-\mathrm{i}\omega_0\tau_1} q_1 + a_0\tau_2 \mathrm{e}^{-\mathrm{i}\omega_0\tau_2}) \\ &+ a_0\tau_1 \mathrm{e}^{-\mathrm{i}\omega_0\tau_1} q_2 + a_0\tau_2 \mathrm{e}^{-\mathrm{i}\omega_0\tau_2} q_1 + 1]^{-1}. \end{aligned}$$

Here $\overline{\widetilde{q}_i^*}$ is the conjugate of $\widetilde{q}_i^*$, $i = 1, 2$. Then $\langle q^*(\xi), q(\theta) \rangle = 1$ and $\langle q^*(\xi), \overline{q}(\theta) \rangle = 0$. Let X_t be a solution of (4.8) at $\sigma = 0$. Define

$$\dot{z}(t) = \langle q^*, X_t \rangle$$

and

$$\begin{aligned} W(t, \theta) &= W(z(t), \overline{z}(t), \theta) \\ &= X_t(\theta) - z(t)q(\theta) - \overline{z}(t)\overline{q}(\theta) \\ &= X_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} \\ &\quad + W_{30}(\theta)\frac{z^3}{6} + W_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \tag{4.13}$$

Then z and $\overline{z}$ are local coordinates for C_0 in the direction of q^* and $\overline{q}^*$.

Noting that W is real if X_t is, we shall deal with real solutions only. It is easy to see that $\langle q^*, W \rangle = 0$. Now, for any solution $X_t \in C_0$ of (4.8),

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{X}_t \rangle \\ &= \langle q^*, A_0 X_t \rangle + \langle q^*, R_0 X_t \rangle \\ &= \langle A_0^* q^*, X_t \rangle + \langle q^*, R_0 X_t \rangle \\ &= \mathrm{i}\omega_0 \langle q^*, X_t \rangle + \overline{q}^*(0)G(X_t, 0) \\ &= \mathrm{i}\omega_0 z(t) + \overline{q}^*(0)G(W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{zq(\theta)\}), 0) \\ &= \mathrm{i}\omega_0 z(t) + \overline{q}^*(0)[f^1(W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{zq(\theta)\}, 0), \\ &\qquad f^2(W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{zq(\theta)\}, 0), \\ &\qquad f^3(W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{zq(\theta)\}, 0)]^{\mathrm{T}} \\ &= \mathrm{i}\omega_0 z(t) + \overline{q}^*(0)[(f_0^1, f_0^2, f_0^3)]^{\mathrm{T}} \\ &= \mathrm{i}\omega_0 z(t) + g(z, \overline{z}) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \dot{W}(t, \theta) &= \dot{X}_t - \dot{z}(t)q - \dot{\overline{z}}(t)\overline{q} \\ &= \begin{cases} A_0 W - 2\,\mathrm{Re}\{\overline{q}^*(0)(f_0^1, f_0^2, f_0^3)^{\mathrm{T}} q(\theta)\} & \text{if } \theta \in [-\tau_2, 0) \\ A_0 W - 2\,\mathrm{Re}\{\overline{q}^*(0)(f_0^1, f_0^2, f_0^3)^{\mathrm{T}} q(\theta)\} & \\ \quad + (f_0^1, f_0^2, f_0^3)^{\mathrm{T}} & \text{if } \theta = 0 \end{cases} \\ &= A_0 W + H(z, \overline{z}, \theta), \end{aligned} \tag{4.15}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{30}(\theta)\frac{z^3}{2} + \cdots.$$

After some simple calculations, we can get

$$\begin{aligned} f_0^1 = & -\tfrac{1}{3}a_0[(\mathrm{e}^{-3\mathrm{i}\omega_0\tau_1} + q_2^3\mathrm{e}^{-3\mathrm{i}\omega_0\tau_2})z^3 \\ & + 3(\mathrm{e}^{-\mathrm{i}\omega_0\tau_1} + q_2^2\overline{q}_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_2})z^2\overline{z} \\ & + (\mathrm{e}^{3\mathrm{i}\omega_0\tau_1} + \overline{q}_2^3\mathrm{e}^{3\mathrm{i}\omega_0\tau_2})\overline{z}^3 + \tfrac{3}{2}(W_{20}^3(-\tau_1)\mathrm{e}^{-2\mathrm{i}\omega_0\tau_1} \\ & + W_{20}^2(-\tau_2)q_2^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau_2})z^4 + \cdots], \end{aligned}$$

$$\begin{aligned} f_0^2 = & -\tfrac{1}{3}a_0[(q_1^3\mathrm{e}^{-3\mathrm{i}\omega_0\tau_1} + \mathrm{e}^{-3\mathrm{i}\omega_0\tau_2})z^3 \\ & + 3(q_1^2\overline{q}_1\mathrm{e}^{-\mathrm{i}\omega_0\tau_1} + \mathrm{e}^{-\mathrm{i}\omega_0\tau_2})z^2\overline{z} \\ & + (\overline{q}_1^3\mathrm{e}^{3\mathrm{i}\omega_0\tau_1} + \mathrm{e}^{3\mathrm{i}\omega_0\tau_2})\overline{z}^3 + \tfrac{3}{2}(W_{20}^1(-\tau_1)q_1^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau_1} \\ & + W_{20}^3(-\tau_2)\mathrm{e}^{-2\mathrm{i}\omega_0\tau_2})z^4 + \cdots], \end{aligned}$$

$$\begin{aligned} f_0^3 = & -\tfrac{1}{3}a_0[(q_2^3\mathrm{e}^{-3\mathrm{i}\omega_0\tau_1} + q_1^3\mathrm{e}^{-3\mathrm{i}\omega_0\tau_2})z^3 \\ & + 3(q_2^2\overline{q}_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_1} + q_1^2\overline{q}_1\mathrm{e}^{-\mathrm{i}\omega_0\tau_2})z^2\overline{z} \\ & + (\overline{q}_2^3\mathrm{e}^{3\mathrm{i}\omega_0\tau_1} + \overline{q}_1^3\mathrm{e}^{3\mathrm{i}\omega_0\tau_2})\overline{z}^3 + \tfrac{3}{2}(W_{20}^2(-\tau_1)q_2^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau_1} \\ & + W_{20}^1(-\tau_2)q_1^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau_2})z^4 + \cdots]. \end{aligned}$$

Here W_{20}^{i} is the ith component of W_{20} in (4.13). From (4.15), we can obtain $H_{20} = H_{11} = H_{02} = 0$ and calculate $H_{30}, H_{21}, \ldots$. From (4.13) and (4.14) and the relation

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}}, \tag{4.16}$$

we can get

$$W_{20} = W_{11} = W_{02} = 0, \quad W_{30} = (3\mathrm{i}\omega_0 I - A_0)^{-1}H_{30},$$
$$W_{21} = (\mathrm{i}\omega_0 I - A_0)^{-1}H_{21}, \quad W_{21} = \overline{W}_{12}, \quad W_{03} = \overline{W}_{30}, \ldots.$$

Then $W(t,\theta) = W(z,\overline{z},\theta)$ in (4.13) can be determined. Here we only calculate what we need. In (4.14), let

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{30}\frac{z^3}{6} + g_{21}\frac{z^2\overline{z}}{2} + \cdots.$$

We can easily obtain

$$g_{20} = g_{11} = g_{02} = 0,$$

$$\begin{aligned} g_{21} = & -2a_0[\overline{q}_1^*(\mathrm{e}^{-\mathrm{i}w\tau_1} + q_2^2\overline{q}_2\mathrm{e}^{-\mathrm{i}w\tau_2}) + \overline{q}_2^*(q_1^2\overline{q}_1\mathrm{e}^{-\mathrm{i}w\tau_1} + \mathrm{e}^{-\mathrm{i}w\tau_2}) \\ & + \overline{q}_3^*(q_2^2\overline{q}_2\mathrm{e}^{-\mathrm{i}w\tau_1} + q_1^2\overline{q}_1\mathrm{e}^{-\mathrm{i}w\tau_2})]. \end{aligned} \tag{4.17}$$

Hence we have

$$\dot{z} = \mathrm{i}\omega_0 z(t) + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.18}$$

From the above discussion, we can calculate the following quantities:

$$c_1(0) = \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{1}{2}g_{21} = \frac{1}{2}g_{21},$$
$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda(a_0)\}},$$
$$t_2 = -\frac{1}{\omega_0}[\mathrm{Im}\{c_1(0)\} + \mu_2\,\mathrm{Im}\{\lambda'(a_0)\}],$$
$$b_2 = 2\,\mathrm{Re}\{c_1(0)\} = \mathrm{Re}\,g_{21}.$$

Using the method of [16], we obtain the following result.

Theorem 4.1. *If $g_{21} \neq 0$, the bifurcating periodic solution of system* (1.1) *at the origin can be described by the following formulas.*

(i) *The direction of the Hopf bifurcating periodic solution is determined by*

$$\mu(\sigma) = \mu_2\sigma^2 + \cdots.$$

If $\alpha'(a_0) > 0$, then when $\mu_2 > 0$ (<0) the Hopf bifurcation is supercritical (*subcritical*) *and the bifurcating periodic solution exists for $a > a_0 (< a_0)$.*

(ii) *The period of the bifurcating periodic solution can be estimated by*

$$T(\sigma) = \frac{2\pi}{\omega_0}(1 + t_2\sigma^2 + \cdots).$$

The period increases (*decreases*) *if $t_2 > 0$ (<0).*

(iii) *The stability of the bifurcating periodic solution is determined by*

$$B(\sigma) = b_2\sigma^2 + \cdots.$$

When $b_2 > 0$ (<0) the bifurcating periodic solution is unstable (*stable*).

5. An example

In this section, we use the formulas obtained in Section 4 to calculate the Hopf bifurcation value and the direction of the Hopf bifurcation of system (1.1) with $\tau_1 = \pi/4$, $\tau_2 = 5\pi/4$ and $\kappa = 1$. One can easily check that when $a_0 = -\sqrt{6}/3$ the characteristic equation (3.1) has a pair of pure imaginary solution $\pm\mathrm{i}$. Then, by (3.2) and (3.3), we have

$$\alpha'(a_0) = -0.4203 \neq 0,$$
$$\left.\frac{\mathrm{d}\,\mathrm{Im}\,\lambda(a)}{\mathrm{d}a}\right|_{a=a_0} = 0.0599.$$

For this case, if $\mu > 0$ then

$$\begin{aligned} \mathrm{R}_{2.3}(\mu,\omega_0) = \mathrm{R}_{2.3}(\mu,1) & = \mu + 1 + \frac{\sqrt{6}}{3}\mathrm{e}^{-(\pi/4)\mu}\cos\left(\frac{\pi}{4}\right) \\ & \quad + \frac{\sqrt{6}}{3}\mathrm{e}^{-(5\pi/4)\mu}\cos\left(\frac{5\pi}{4}\right) \\ & \neq 0 \end{aligned}$$

and

$$\mu+1-\frac{\sqrt{6}}{3}\mathrm{e}^{-(\pi/4)\mu}\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)-\frac{\sqrt{6}}{3}\mathrm{e}^{-(5\pi/4)\mu}\times\cos\left(\frac{5\pi}{4}-\frac{\pi}{3}\right)\neq 0$$

and

$$\begin{aligned}&\mu+1-\frac{\sqrt{6}}{3}\mathrm{e}^{-(\pi/4)\mu}\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)-\frac{\sqrt{6}}{3}\mathrm{e}^{-(5\pi/4)\mu}\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)\\ &>\mu+1-\frac{\sqrt{6}}{3}\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)-\frac{\sqrt{6}}{3}\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)\\ &>\mu+1-\frac{\sqrt{6}}{3}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)+\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)\right]\\ &=\mu+1-\frac{\sqrt{6}}{3}\times\frac{\sqrt{6}}{2}\\ &=\mu\\ &\neq 0.\end{aligned}$$

It follows that $\mathrm{R}_{2.4}(\mu,\omega_0)=\mathrm{R}_{2.4}(\mu,1)\neq 0$ for $\mu>0$ and hence all roots of (3.1) have negative real parts except a pair of purely imaginary roots. According to Section 4, we can get

$$\begin{aligned}&\operatorname{Re} g_{21}=-1.3278,\\ &\operatorname{Im} g_{21}=0.6807,\\ &\mu_2=-1.5796,\\ &t_2=-0.2459,\\ &b_2=\operatorname{Re} g_{21}=-1.3278.\end{aligned}$$

So the bifurcating periodic solution of the system (1.1) is stable and the period of the bifurcating periodic solution is approximately

$$T(\sigma)=2\pi(1-0.2459\sigma^2+\cdots).$$

A numerical simulation near $a=-\sqrt{6}/3$ is shown in Fig. 1.

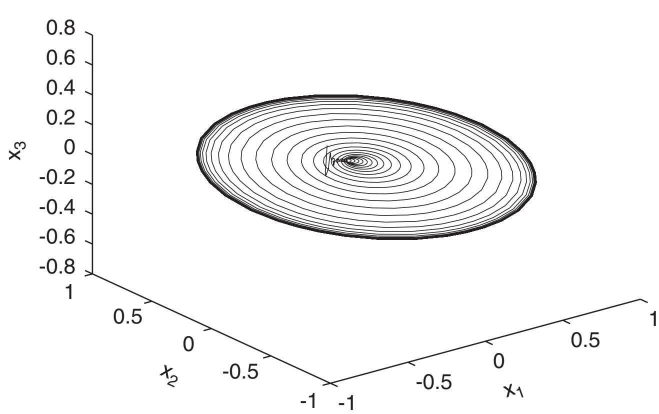

Fig. 1. Numerical solution $(x_1(t), x_2(t), x_3(t))$ of system (1.1), where $k=1$, $\tau_1=\pi/4$, $\tau_2=5\pi/4$, $a=-\sqrt{6}/3$.

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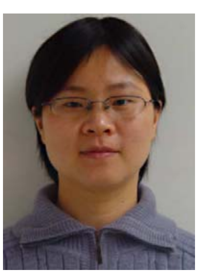
References

[1] J. Belair, S.A. Campbell, Stability and bifurcations of equilibria in a multiple-delayed differential equation, SIAM J. Appl. Math. 54 (1994) 1402–1424.
[2] J. Bélair, S.A. Campbell, P. van den Driessche, Frustration, stability, and delay-induced oscillations in a neural network model, SIAM J. Appl. Math. 56 (1996) 245–255.
[3] A. Beuter, J. Belair, C. Labrie, Feedback and delays in neurological diseases: a modeling study using dynamical systems, Bull. Math. Biol. 55 (1993) 525–541.
[4] S.A. Campbell, S. Ruan, G.S.K. Wolkowicz, J. Wu, Stability and bifurcation of a simple neural network with multiple time delays, Fields Institute Communications, vol. 21, American Mathematical Society, Providence, RI, 1998, pp. 65–79.
[5] Y. Chen, J. Wu, Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network, Physica D 134 (1999) 185–199.
[6] Y. Chen, J. Wu, T. Krisztin, Connection orbits from synchronous periodic solutions to phase-locked periodic solutions in a delay differential system, J. Differential Equations 163 (2000) 130–173.
[7] K.L. Cooke, J.A. Yorke, Some equations modeling growth processes and gonorrhea epidemics, Math. Biosci. 16 (1973) 75–101.
[8] P.V. Driessche, X. Zou, Global attractivity in delayed Hopfield neural network models, SIAM J. Appl. Math. 58 (1998) 1878–1890.
[9] T. Faria, On a planar system modelling a neuron network with memory, J. Differential Equations 168 (2000) 129–149.
[10] K. Gopalsamy, I. Leung, Delay induced periodicity in a neural network of excitation and inhibition, Physica D 89 (1996) 395–426.
[11] S. Guo, L. Huang, Hopf bifurcating periodic orbits in a ring of neurons with delays, Physica D 183 (2003) 19–44.
[12] S. Guo, L. Huang, Linear stability and Hopf bifurcation in a two-neuron network with three delays, Int. J. Bifurcation Chaos 8 (2004) 2799–2810.
[13] J. Hale, Theory of Functional Differential Equations, Springer, New York, 1997.
[14] J. Hale, H. Kocak, Dynamics and Bifurcations, Springer, New York, 1991.
[15] J. Hale, S.V. Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.
[16] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, 1981.
[17] M.W. Hirsch, Convergent activation dynamics in continuous-time networks, Neural Networks 2 (1989) 331–349.
[18] J.J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, Proc. Natl. Acad. Sci. USA 81 (1984) 3088–3092.
[19] L. Huang, J. Wu, Dynamics of inhibitory artificial neural networks with threshold nonlinearity, Fields Ins. Commun. 29 (2001) 235–243.
[20] L. Huang, J. Wu, The role of threshold in preventing delay-induced oscillations of frustrated neural networks with McCulloch–Pitts nonlinearity, Int. J. Math. Game Theory Algebra 11 (6) (2001) 71–100.
[21] L. Huang, J. Wu, Nonlinear waves in networks of neurons with delayed feedback: pattern formation and continuation, SIAM J. Math. Anal. 34 (2003) 836–860.
[22] V. Kolmanovskii, V. Nosov, Stability of Functional Differential Equation, Academic Press, Toronto, 1986.

[23] X.F. Liao, G. Chen, Local stability, Hopf and resonant codimension-two bifurcation in a Harmonic oscillator with two time delays, Int. J. Bifurcation Chaos 11 (2001) 2105–2121.
[24] X.F. Liao, Kwok-wo Wong, Z. Wu, Bifurcation analysis in a two-neuron system with continuously distributed delays, Physica D 149 (2001) 123–141.
[25] Y. Lin, R. Lemmert, P. Volkmann, Bifurcation of periodic solution in a three-unit neural network with delay, Acta. Math. Appl. Sin. 17 (2001) 375–382.
[26] N. MacDonald, An activation–inhibition model of cyclic granulopoiesis in chronic granulocytic leukemia, Math. Biosci. 54 (1980) 61–70.
[27] N. MacDonald, Biological Delay Systems: Linear Stability Theory, Cambridge University Press, New York, 1989.
[28] C.M. Marcus, R.M. Westervelt, Stability of analog neural networks with delay, Phys. Rev. A. 39 (1989) 347–359.
[29] C.M. Marcus, R.M. Westervelt, Stability of analog neural networks with delay, Phys. Rev. A 39 (1989) 347–359.
[30] K. Murakaimi, Bifurcated periodic solutions for delayed van der Pol equation, Neural Parallel Sci. Comput. 7 (1999) 1–16.
[31] L. Olien, J. Bélair, Bifurcations, stability, and monotonicity properties of a delayed neural network model, Physica D 102 (1997) 349–363.
[32] A. Potapov, M.K. Ali, Robust chaos in neural networks, Phys. Lett. A 227 (2000) 310–322.
[33] L.P. Shayer, S.A. Campbell, Stability, bifurcation and multistability in a system of two coupled neurons with multiple time delays, SIAM J. Appl. Math. 61 (2000) 673–700.
[34] G. Stépán, Retarded Dynamical Systems, Pitman Research Notes Mathematical Series, vol. 210, Longman Group, Essex, UK, 1989.
[35] J. Szenfagothai, The module-concept in cerebral cortex architecture, Brain Res. 95 (1967) 475–496.
[36] J. Wei, S. Ruan, Stability and bifurcation in a neural network model with two delays, Physica D 130 (1999) 255–272.
[37] J. Wu, Introduction to Neural Dynamics and Signal Transmission Delay, Walter de Gruyter, Berlin, 2001.



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