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Dynamical behaviors of Cohen–Grossberg neural networks with delays and reaction–diffusion terms ☆

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Abstract

In this paper, we study Cohen–Grossberg neural networks with delays and reaction–diffusion terms. By employing homotopic mapping theory and constructing suitable Lyapunov functional method we present some sufficient conditions ensuring the existence, uniqueness and globally exponential stability (GES) of the equilibrium point. These conditions obtained have important leading significance in the designs and applications of GES for reaction–diffusion neural circuit systems with delays. Finally, we show a numerical example to verify the theoretical analysis.

Keywords: Homotopic mapping theory; Cohen–Grossberg neural networks; Reaction–diffusion terms; Globally exponential stability; Equilibrium point; Lyapunov functional

1. Introduction

Cohen–Grossberg neural networks (CGNNs) were first introduced by Cohen–Grossberg [4]. CGNNs include Hopfield neural networks as special cases [1,15,17,18, 3,20,21]. The class of networks has good application in parallel computation, associative memory and optimization problems, which has been one of the most active areas of research and has received much attention. One can refer to the articles [19,2,16,14,9] for detailed discussion on these aspects. Thus, from the viewpoint of application, the dynamical study for CGNNs is quite important and significant, and cannot be replaced with the dynamical study for the traditional neural networks such as Hopfield neural networks and cellular neural networks. Furthermore, in the analysis of dynamical CGNNs for parallel computation and optimization, to increase the rate of convergence to the equilibrium point of the networks, it is necessary to ensure a desired exponential convergence rate of the networks trajectories, starting from arbitrary initial states to the equilibrium point which corresponds to the optimal solution. From the viewpoint of mathematics and engineering, it is required that the equilibrium point of CGNNs is globally exponential stability (GES). As a consequence, many authors have paid much effort to the research on GES of equilibrium point for CGNNs in Refs. [19,2,16,14,9]. However, strictly speaking, diffusion effects cannot be avoided in neural network models when electrons are moving in asymmetric electromagnetic field, thus we must consider the space is varying with the time. Refs. [11–13,7] have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations. It is also common to consider the diffusion in biological systems (such as immigration) [8,6,10]. To the best our knowledge, few authors study GES of Cohen–Grossberg neural networks with delays and reaction–diffusion terms.

Motivated by the above discussions, in this paper we analyze further problem of GES of Cohen–Grossberg

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neural networks with delays and reaction–diffusion terms, and give a set of sufficient conditions ensuring the existence, uniqueness and GES of the equilibrium point by using homotopic mapping theory and employing Lyapunov functional method.

2. Preliminary

Let $C \triangleq C([-\tau,0]\times R^m, R^n)$ be the Banach space of continuous functions which map $[-\tau,0]\times R^m$ into R^n with the topology of uniform convergence. Ω be an open bounded domain in R^m with smooth boundary $\partial\Omega$, and $mes\Omega>0$ denotes the measure of Ω. $L^2(\Omega)$ is the space of real functions on Ω which are L^2 for the Lebesgue measure. It is a Banach space for the norm

$$\|u(t)\| = \sum_{i=1}^{n} \|u_i(t)\|_2,$$

where

$$u(t) = (u_1(t),\ldots,u_n(t))^{\mathrm{T}}, \quad \|u_i(t)\|_2 = \left(\int_\Omega |u_i(t,x)|^2\,\mathrm{d}x\right)^{1/2}.$$

For any $\varphi(t,x)\in C[[-\tau,0]\times\Omega, R^n]$, we define

$$\|\varphi\| = \sum_{i=1}^{n} \|\varphi_i\|_2,$$

where

$$\varphi(t,x) = (\varphi_1(t,x),\ldots,\varphi_n(t,x))^{\mathrm{T}},$$
$$\|\varphi_i\|_2 = \left(\int_\Omega |\varphi_i(x)|_\tau^2\,\mathrm{d}x\right)^{1/2},$$
$$|\varphi_i(x)|_\tau = \sup_{-\tau\leqslant s\leqslant 0} |\varphi_i(s,x)|.$$

Consider the following delayed Cohen–Grossberg neural networks with reaction–diffusion terms

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} = \sum\limits_{l=1}^{m} \dfrac{\partial}{\partial x_l}\left(D_{il}\dfrac{\partial u_i(t,x)}{\partial x_l}\right) - a_i(u_i(t,x)) \\ \qquad \times\left[b_i(u_i(t,x)) - \sum\limits_{j=1}^{n} a_{ij}s_j(u_j(t,x))\right. \\ \qquad \left. - \sum\limits_{k=0}^{K}\sum\limits_{j=1}^{n} t_{ij}^{(k)} s_j(u_j(t-\tau_k,x)) + I_i\right], \quad t\geqslant 0, \quad x\in\Omega \\ u_i(t,x) = \varphi_i(t,x), \quad -\tau\leqslant t\leqslant 0, \quad x\in\Omega \\ \dfrac{\partial u_i(t,x)}{\partial n} = 0, \quad x\in\partial\Omega, \end{cases} \tag{1}$$

where $i=1,\ldots,n$; n is the number of neurons in the network; $u_i(t,x)$ denotes the state variable associated with the ith neurons at time t and in space x; $a_i(t,x)$ represents an amplification function; $b_i(t,x)$ represents an appropriately behaved function; a_{ij} represents the strength of the neuron interconnections within the network; $s_j(t,x)$ shows how the jth neuron reacts to the input; $t_{ij}^{(k)}$ represents the interconnection with delay τ_k; τ_k is delay and satisfies $0\leqslant\tau_k\leqslant\tau$; I_i is the constant input from outside the system; $D_{ik}\geqslant 0$ corresponds to the transmission diffusion operator along the ith neuron. $u(t,x) = (u_1(t,x),\ldots,u_n(t,x))^{\mathrm{T}}$, $x=(x_1,\ldots,x_m)^{\mathrm{T}}$.

Throughout the paper, we always assume that system (1) has a continuous solution denoted by $u(t,0,\varphi;x)$ or simply $u(t,x)$ if no confusion should occur.

Definition 1. $u(t,x)\equiv u^*\in R^n$ is said to be an equilibrium point of system (1), if the constant vector $u^*=(u_1^*,\ldots,u_n^*)^{\mathrm{T}}$ satisfies

$$b_i(u_i^*) - \sum_{j=1}^{n} a_{ij}s_j(u_j^*) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(u_j^*) + I_i = 0,$$
$$i=1,\ldots,n. \tag{2}$$

Lemma 1 (Cronin [5] *Kronecker's theorem*). *Assume that* $f:\overline{\Omega}\to R^n$ *is a continuous function*, $\deg(f,\Omega,p)\neq 0$, *then there exists* $x_0\in\Omega$, *such that* $f(x_0)=p$.

Lemma 2 (Cronin [5] *Homotopy invariance theorem*). *Assume that* $H(x,\lambda):\overline{\Omega}\times[0,1]\to R^n$ *is a continuous function, denote* $h_\lambda(x)=H(x,\lambda)$. *When* $\lambda\in[0,1]$ *and* $p\notin h_\lambda(\partial\Omega)$, $\deg(h_\lambda,\Omega,p)$ *is independent of* λ.

3. Existence and GES of the equilibrium point

In the paper, we always assume that

(H1) There exist m_i and M_i, such that

$$0<m_i\leqslant a_i(u_i)\leqslant M_i, \quad i=1,\ldots,n.$$

(H2) $b_i(\cdot)$ is differentiable, $\alpha_i\triangleq\inf_{u_i\in R}\{\dot{b}_i(u_i)\}>0$, where $\dot{b}_i(\cdot)$ is the derivative of $b_i(\cdot)$, $b_i(0)=0$, $i=1,\ldots,n$.

(H3) There exist constants β_j $(j=1,\ldots,n)$, such that

$$|s_j(x)-s_j(y)|\leqslant\beta_j|x-y|, \quad \forall x,y\in R.$$

(H4) There exist constants p_{ij}, q_{ij}, p_{ij}^*, q_{ij}^*, such that

$$\begin{aligned} &-2m_i\alpha_i + \sum_{j=1}^{n} M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + \sum_{j=1}^{n} M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} \\ &\quad + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} \\ &\quad + \sum_{k=0}^{K}\sum_{j=1}^{n} M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}<0. \end{aligned}$$

In order to study the existence and uniqueness of the equilibrium point, we consider the following algebraic equations associated with system (1)

$$b_i(u_i(t,x)) - \sum_{j=1}^{n} a_{ij}s_j(u_j(t,x)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(u_j(t,x)) + I_i = 0, \quad i=1,\ldots,n. \tag{3}$$

System (3) can be rewritten in the following vector form:

$$B(u) - AS(u) - TS(u) + I = 0, \tag{4}$$

where $u=(u_1,\ldots,u_n)^{\mathrm{T}}$, $B(u)=(b_1(u_1),\ldots,b_n(u_n))^{\mathrm{T}}$, $A=(a_{ij})_{n\times n}$, $T=(\sum_{k=0}^{K} t_{ij}^{(k)})_{n\times n}$, $S(u)=(s_1(u_1),\ldots,s_n(u_n))^{\mathrm{T}}$, $I=(I_1,\ldots,I_n)^{\mathrm{T}}$.

Theorem 1. *If* (H1)–(H4) *holds, then system* (1) *has a unique equilibrium point* u^*.

Proof. Let $h(u)=B(u)-AS(u)-TS(u)+I$. Define the homotopic mapping $H(u,\lambda):\bar{\Omega}\times[0,1]\to R^n$ as follows:

$$H(u,\lambda) \triangleq (H_1(u,\lambda),\ldots,H_n(u,\lambda))^{\mathrm{T}} = \lambda h(u) + (1-\lambda)B(u), \quad \lambda\in[0,1],$$

in which $\alpha = \mathrm{diag}(\alpha_i)$.

It follows from (H2) and (H4) that

$$|b_i(u_i)| \geqslant \alpha_i|u_i|, \quad i=1,\ldots,n, \tag{5}$$

$$|s_j(u_j)| \leqslant \beta_j|u_j| + |s_j(0)|, \quad j=1,\ldots,n. \tag{6}$$

Thus we have

$$\begin{aligned}|H_i(u,\lambda)| &= \left|\lambda\left(b_i(u_i) - \sum_{j=1}^{n} a_{ij}s_j(u_j) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(u_j) + I_i\right) + (1-\lambda)b_i(u_i)\right| \geqslant \alpha_i|u_i| - \sum_{j=1}^{n}|a_{ij}|\beta_j|u_j| \\ &\quad - \sum_{j=1}^{n}|a_{ij}||s_j(0)| - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|\beta_j|u_j| \\ &\quad - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)| - |I_i|.\end{aligned} \tag{7}$$

By (7), we obtain

$$\begin{aligned}&\sum_{i=1}^{n} M_i|u_i||H_i(u,\lambda)| \\ &\geqslant \sum_{i=1}^{n} M_i|u_i|\left(\alpha_i|u_i| - \sum_{j=1}^{n}|a_{ij}|\beta_j|u_j| - \sum_{j=1}^{n}|a_{ij}||s_j(0)| - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|\beta_j|u_j| - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)| - |I_i|\right) \\ &\geqslant \sum_{i=1}^{n} M_i\left(\alpha_i|u_i|^2 - \sum_{j=1}^{n}|a_{ij}|^{p_{ij}}\beta_j^{q_{ij}}|u_i||a_{ij}|^{1-p_{ij}}\beta_j^{1-q_{ij}}|u_j| - \sum_{j=1}^{n}|a_{ij}||s_j(0)||u_i| \right. \\ &\quad - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{p_{ij}^*}\beta_j^{q_{ij}^*}|u_i||t_{ij}^{(k)}|^{1-p_{ij}^*}\beta_j^{1-q_{ij}^*}|u_j| \\ &\quad \left. - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)||u_i| - |I_i||u_i|\right) \\ &\geqslant \sum_{i=1}^{n} M_i\left(\alpha_i|u_i|^2 - \frac{1}{2}\sum_{j=1}^{n}|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}}|u_i|^2 - \frac{1}{2}\sum_{j=1}^{n}|a_{ij}|^{2-2p_{ij}}\beta_j^{2-2q_{ij}}|u_j|^2 \right. \\ &\quad - \sum_{j=1}^{n}|a_{ij}||s_j(0)||u_i| - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*}|u_i|^2 \\ &\quad - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}|u_j|^2 \\ &\quad \left. - \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)||u_i| - |I_i||u_i|\right) \\ &= \sum_{i=1}^{n}\left(M_i\alpha_i - \frac{1}{2}\sum_{j=1}^{n}M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} - \frac{1}{2}\sum_{j=1}^{n}M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} \right. \\ &\quad \left. - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\right)|u_i|^2 \\ &\quad - \sum_{i=1}^{n}M_i\left(\sum_{j=1}^{n}|a_{ij}||s_j(0)| + \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)| + |I_i|\right)|u_i|.\end{aligned}$$

So, we obtain

$$\begin{aligned}&\int_{\Omega}\left(\sum_{i=1}^{n}M_i|u_i||H_i(u,\lambda)|\right)\mathrm{d}x \\ &\geqslant \int_{\Omega}\left[\sum_{i=1}^{n}\left(M_i\alpha_i - \frac{1}{2}\sum_{j=1}^{n}M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} - \frac{1}{2}\sum_{j=1}^{n}M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} \right.\right. \\ &\quad \left. - \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\right)|u_i|^2 \\ &\quad \left. - \sum_{i=1}^{n}M_i\left(\sum_{j=1}^{n}|a_{ij}||s_j(0)| + \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}||s_j(0)| + |I_i|\right)|u_i|\right]\mathrm{d}x \\ &\geqslant \delta_0\|u(t)\|_2^2 - L\|u(t)\|_2,\end{aligned}$$

where

$$\delta_0 = \min_i\left\{M_i\alpha_i - \frac{1}{2}\sum_{j=1}^n M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} - \frac{1}{2}\sum_{j=1}^n M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} - \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} - \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\right\} > 0,$$

$L = \max_i\{\int_\Omega M_i(\sum_{j=1}^n|a_{ij}||s_j(0)| + \sum_{k=0}^K\sum_{j=1}^n|t_{ij}^{(k)}||s_j(0)| + |I_i|)\,\mathrm{d}x\}$.

Take $U(R_0) = \{U \in R^n | \|u(t)\|_2 < R_0 = (L+1)/\delta_0\}$. For any $u \in \partial U(R_0)$, we have

$$\int_\Omega\left(\sum_{i=1}^n M_i|u_i||H_i(u,\lambda)|\right)\mathrm{d}x \geqslant \delta_0\|u(t)\|_2\left(\|u(t)\|_2 - \frac{L}{\delta_0}\right) > 0, \quad \forall\lambda \in [0,1],$$

i.e.,

$$H(u,\lambda) \neq 0, \quad \forall u \in \partial U(R_0), \quad \lambda \in [0,1].$$

Following from Lemma 2, we obtain

$$\begin{aligned}\deg(h(u), U(R_0), 0) &= \deg(H(u,1), U(R_0), 0)\\ &= \deg(H(u,0), U(R_0), 0)\\ &= 1.\end{aligned}$$

It follows from Lemma 1 that there exist at least an $u^* \in U(R^0)$, such that $h(u^*) = 0$, i.e., system (1) has at least an equilibrium point u^*.

Next, we prove the uniqueness of the equilibrium point. Suppose that $\bar{u}^*$ is also an equilibrium point of system (1), then

$$b_i(\bar{u}_i^*) - \sum_{j=1}^n a_{ij}s_j(\bar{u}_j^*) - \sum_{k=0}^K\sum_{j=1}^n t_{ij}^{(k)}s_j(\bar{u}_j^*) + I_i = 0, \quad i = 1,\ldots,n. \tag{8}$$

Thus

$$b_i(u_i^*) - b_i(\bar{u}_i^*) = \left(\sum_{j=1}^n a_{ij}s_j(u_j^*) - \sum_{j=1}^n a_{ij}s_j(\bar{u}_j^*)\right) + \left(\sum_{k=0}^K\sum_{j=1}^n t_{ij}^{(k)}s_j(u_j^*) - \sum_{k=0}^K\sum_{j=1}^n t_{ij}^{(k)}s_j(\bar{u}_j^*)\right). \tag{9}$$

From (H2) and (H3), we obtain

$$\sum_{i=1}^n \alpha_i M_i|u_i^* - \bar{u}_i^*|^2 \leqslant \sum_{i=1}^n\left(\frac{1}{2}\sum_{j=1}^n M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + \frac{1}{2}\sum_{j=1}^n M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} + \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} + \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\right)|u_i^* - \bar{u}_i^*|^2.$$

That is

$$\sum_{i=1}^n\left(-M_i\alpha_i + \frac{1}{2}\sum_{j=1}^n M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + \frac{1}{2}\sum_{j=1}^n M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} + \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} + \frac{1}{2}\sum_{k=0}^K\sum_{j=1}^n M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\right)|u_i^* - \bar{u}_i^*|^2 \leqslant 0,$$

which implies $u_i^* = \bar{u}_i^*$ $(i = 1,\ldots,n)$. Hence system (1) has a unique equilibrium point u^*. This completes the proof. $\square$

Denoting $y_i(t,x) = u_i(t,x) - u_i^*$, $i = 1,\ldots,n$, system (1) can be rewritten as follows:

$$\begin{cases}\dfrac{\partial y_i(t,x)}{\partial t} = \sum\limits_{l=1}^m \dfrac{\partial}{\partial x_l}\left(D_{il}\dfrac{\partial y_i(t,x)}{\partial x_l}\right) - a_i(y_i(t,x)) \\ \qquad \times\left[b_i(u_i(t,x)) - b_i(u_i^*) - \sum\limits_{j=1}^n a_{ij}s_j(u_j(t,x)) + \sum\limits_{j=1}^n a_{ij}s_j(u_j^*) - \sum\limits_{k=0}^K\sum\limits_{j=1}^n t_{ij}^{(k)}s_j(u_j(t-\tau_k,x)) + \sum\limits_{k=0}^K\sum\limits_{j=1}^n t_{ij}^{(k)}s_j(u_j^*)\right], \quad t \geqslant 0, \quad x \in \Omega \\ y_i(t,x) = \psi_i(t,x), \quad -\tau \leqslant t \leqslant 0, \quad x \in \Omega \\ \dfrac{\partial y_i(t,x)}{\partial n} = 0, \quad x \in \partial\Omega,\end{cases} \tag{10}$$

in which $i = 1,\ldots,n$, $\psi_i(t,x) = \varphi_i(t,x) - u_i^*$. $y(t,x) = (y_1(t,x),\ldots,y_n(t,x))^{\mathrm{T}}$, $\psi(t,x) = (\psi_1(t,x),\ldots,\psi_n(t,x))^{\mathrm{T}}$. Obviously, u^* for system (1) is GES if and only if the equilibrium point O of system (10) is GES. Thus in the following, we only consider GES of the equilibrium point O for system (10).

Theorem 2. *If* (H1)–(H4) *holds*, *then the equilibrium point O of system* (10) *is GES*.

Proof. From (H4), there exists a sufficiently small constant $0 < \lambda < \min_i\{m_i\alpha_i\}$, such that

$$2\lambda - 2m_i\alpha_i + \sum_{j=1}^n M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + \sum_{j=1}^n M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} + \sum_{k=0}^K\sum_{j=1}^n M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} + \sum_{k=0}^K\sum_{j=1}^n M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}\mathrm{e}^{2\lambda\tau} \geqslant 0.$$

Taking Lyapunov functional as follows:

$$\begin{aligned} V(t) = \int_\Omega \sum_{i=1}^{n} \Bigg(& |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} \\ & \times \int_{t-\tau_k}^{t} |y_j(s,x)|^2 \mathrm{e}^{2\lambda(s+\tau_k)}\,\mathrm{d}s \Bigg)\,\mathrm{d}x. \end{aligned} \tag{11}$$

Calculating the rate of change of $V(t)$ along (10), we have

$$\begin{aligned} D^+V(t) = & \int_\Omega \sum_{i=1}^{n} \Bigg(2\lambda |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} \\ & + 2|y_i(t,x)|\mathrm{e}^{2\lambda t} \mathrm{sign}(y_i(t,x))\dot{y}_i(t,x) \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t,x)|^2 \mathrm{e}^{2\lambda(t+\tau_k)} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} \\ & \times |y_j(t-\tau_k,x)|^2 \mathrm{e}^{2\lambda t} \Bigg)\,\mathrm{d}x \\ = & \int_\Omega \sum_{i=1}^{n} \Bigg[2\lambda |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} + 2|y_i(t,x)|\mathrm{e}^{2\lambda t}\mathrm{sign}(y_i) \\ & \times \left(\sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_{il} \frac{\partial y_i(t,x)}{\partial x_k} \right) - a_i(y_i(t,x)) \right) \\ & \times \Bigg[b_i(u_i(t,x)) - b_i(u_i^*) - a_{ij}s_j(u_j(t,x)) + a_{ij}s_j(u_j^*) \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} s_j(u_j(t-\tau_k,x)) + \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} s_j(u_j^*) \Bigg] \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t,x)|^2 \mathrm{e}^{2\lambda(t+\tau_k)} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} \\ & \times |y_j(t-\tau_k,x)|^2 \mathrm{e}^{2\lambda t} \Bigg]\,\mathrm{d}x. \end{aligned}$$

From the boundary condition, we have

$$\begin{aligned} & \int_\Omega \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial y_i(t,x)}{\partial x_l} \right) |y_i(t,x)|\,\mathrm{d}x \\ & = \sum_{l=1}^{m} \int_\Omega |y_i(t,x)| \frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial y_i(t,x)}{\partial x_l} \right)\mathrm{d}x \\ & = \sum_{l=1}^{m} \Bigg[\int_\Omega \frac{\partial}{\partial x_l}\left(|y_i(t,x)| D_{il}\frac{\partial y_i(t,x)}{\partial x_l} \right)\mathrm{d}x \\ & \quad - \int_\Omega \frac{\partial y_i(t,x)}{\partial x_l} D_{il} \frac{\partial y_i(t,x)}{\partial x_l}\,\mathrm{d}x \Bigg] \\ & = -\sum_{l=1}^{m} \int_\Omega D_{il}\left(\frac{\partial y_i(t,x)}{\partial x_l} \right)^2 |y_i(t,x)|\,\mathrm{d}x. \end{aligned}$$

Combining (H1)–(H4) with $2ab \leqslant a^2+b^2$, we obtain

$$\begin{aligned} D^+V(t) \leqslant & \int_\Omega \sum_{i=1}^{n} \Bigg[2\lambda |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} + 2\mathrm{e}^{2\lambda t}\Bigg(-m_i\alpha_i |y_i(t,x)|^2 \\ & + \sum_{j=1}^{n} |a_{ij}|\beta_j |y_i(t,x)||y_j(t,x)| \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|\beta_j |y_i(t,x)||y_j(t-\tau_k,x)| \Bigg) \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t,x)|^2 \mathrm{e}^{2\lambda(t+\tau_k)} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t-\tau_k,x)|^2 \mathrm{e}^{2\lambda t} \Bigg]\,\mathrm{d}x \\ \leqslant & \int_\Omega \sum_{i=1}^{n} \Bigg[2\lambda |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} + 2\mathrm{e}^{2\lambda t}\Bigg(-m_i\alpha_i |y_i(t,x)|^2 \\ & + \sum_{j=1}^{n} M_i |a_{ij}|^{p_{ij}} \beta_j^{q_{ij}} |y_i(t,x)||a_{ij}|^{1-p_{ij}} \beta_j^{1-q_{ij}} |y_j(t,x)| \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{p_{ij}^*} \beta_j^{q_{ij}^*} |y_i(t,x)||t_{ij}^{(k)}|^{1-p_{ij}^*} \\ & \times \beta_j^{1-q_{ij}^*} |y_j(t-\tau_k,x)| \Bigg) \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t,x)|^2 \mathrm{e}^{2\lambda(t+\tau_k)} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t-\tau_k,x)|^2 \mathrm{e}^{2\lambda t} \Bigg]\,\mathrm{d}x \\ \leqslant & \int_\Omega \sum_{i=1}^{n} \Bigg[2\lambda |y_i(t,x)|^2 \mathrm{e}^{2\lambda t} + \mathrm{e}^{2\lambda t}\Bigg(-2m_i\alpha_i |y_i(t,x)|^2 \\ & + \sum_{j=1}^{n} M_i |a_{ij}|^{2p_{ij}} \beta_j^{2q_{ij}} |y_i(t,x)|^2 \\ & + \sum_{j=1}^{n} M_i |a_{ij}|^{2-2p_{ij}} \beta_j^{2-2q_{ij}} |y_j(t,x)|^2 \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2p_{ij}^*} \beta_j^{2q_{ij}^*} |y_i(t,x)|^2 \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t-\tau_k),x| \Bigg) \\ & + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t,x)|^2 \mathrm{e}^{2\lambda(t+\tau_k)} \\ & - \sum_{k=0}^{K}\sum_{j=1}^{n} M_i |t_{ij}^{(k)}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} |y_j(t-\tau_k,x)|^2 \mathrm{e}^{2\lambda t} \Bigg]\,\mathrm{d}x \end{aligned}$$

$$\leqslant \sum_{i=1}^{n}\left[2\lambda - 2m_i\alpha_i + \sum_{j=1}^{n} M_i|a_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + \sum_{j=1}^{n} M_j|a_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*} + \sum_{k=0}^{K}\sum_{j=1}^{n} M_j|t_{ji}^{(k)}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}e^{2\lambda\tau}\right]\|y_i(t)\|_2^2 e^{2\lambda t} \leqslant 0, \tag{12}$$

which implies

$V(t) \leqslant V(0)$.

By Eq.(11) we have

$$V(0) = \int_\Omega \sum_{i=1}^{n}\left(|\psi_i(t,x)|^2 + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*} \times \int_{-\tau_k}^{0} |y_j(s,x)|^2 e^{2\lambda(s+\tau_k)}\,ds\right)dx$$
$$\leqslant \sum_{i=1}^{n}\left(\|\psi_i(t)\|_2^2 + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*} \times \int_{-\tau_k}^{0}\|y_j(s)\|_2^2 e^{2\lambda(s+\tau_k)}\,ds\right)$$
$$\leqslant \max_i\left\{1 + \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}(e^\tau - 1)\right\}\|\psi\|_2^2,$$

and

$V(t) \geqslant \|y(t)\|_2^2 e^{2\lambda t}$.

Hence

$$\|y(t)\|_2^2 e^{2\lambda t} \leqslant \max_i\left\{1 + \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}(e^\tau - 1)\right\}\|\psi\|_2^2,$$

which leads to

$\|y(t)\|_2 \leqslant \gamma\|\psi\|_2 e^{-\lambda t}$,

where

$$\gamma = \left(\max_i\left\{1 + \sum_{k=0}^{K}\sum_{j=1}^{n}|t_{ij}^{(k)}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}(e^\tau - 1)\right\}\right)^{1/2}.$$

This completes the proof. □

Corollary 1. *System* (1) *has a unique equilibrium point which is GES, if the conditions* (H_1)–(H3). *Assume furthermore that one of the following conditions hold*:

(H5)

$$-2m_i\alpha_i + \sum_{j=1}^{n} M_i|a_{ij}|^2\beta_j^2 + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|^2\beta_j^2 < 0.$$

(H6)

$$-2m_i\alpha_i + \sum_{j=1}^{n} M_i|a_{ij}|\beta_j + \sum_{j=1}^{n} M_j|a_{ji}|\beta_i + \sum_{k=0}^{K}\sum_{j=1}^{n} M_i|t_{ij}^{(k)}|\beta_j + \sum_{k=0}^{K}\sum_{j=1}^{n} M_j|t_{ji}^{(k)}|\beta_i < 0.$$

Proof. In (H4), let $p_{ij} = q_{ij} = p_{ij}^* = q_{ij}^* = 1$, then (H4) turns to (H5). Furthermore, we suppose that $p_{ij} = q_{ij} = p_{ij}^* = q_{ij}^* = \frac{1}{2}$, then (H4) turns to (H6). By Theorems 1 and 2, system (1) has a unique equilibrium point which is GES.

Remark 1. When $D_{ik} \equiv 0$, then system (1) becomes the system analyzed in [19,2,16,14,9]. It is worth noting that, in the paper, we did not need s_j is bounded. Thus, we improve the results in Refs. [19,2,16,14,9].

Remark 2. From Theorems 1 and 2, we see if reaction–-diffusion terms satisfy a weaker condition $D_{il} \geqslant 0$, then the effects for the existence and GES of the equilibrium point just come from the networks parameters, the stability is completely expressed by the relations of these parameters.

Remark 3. Although the assertions of exponential stability in Theorems 1 and 2 are independent of the delays, the convergence rate λ do depend on the delays τ_k.

4. Example

Example. Consider Cohen–Grossberg neural networks with delays and reaction–diffusion terms

$$\begin{cases} \frac{\partial u_1(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(D_{11}\frac{\partial u_1(t,x)}{\partial x}\right) - a_1(u_1(t,x)) \times \left[b_1(u_1(t,x)) - \sum_{j=1}^{2} a_{1j}s_j(u_j) - \sum_{j=1}^{2}\sum_{k=0}^{1} t_{1j}^{(k)} s_j(u_j(t-\tau_k,x))\right] \\ \frac{\partial u_2(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(D_{21}\frac{\partial u_2(t,x)}{\partial x}\right) - a_2(u_2(t,x)) \times \left[b_2(u_2(t,x)) - \sum_{j=1}^{2} a_{2j}s_j(u_j) - \sum_{j=1}^{2}\sum_{k=0}^{1} t_{2j}^{(k)} s_j(u_j(t-\tau_k,x))\right]. \end{cases} \tag{13}$$

Let $D_{11} > 0, D_{21} > 0$, $a_i = 4 + \sin u_i$, $b_i(u_i) = 4u_i$, $s_1(u_1) = \arctan u_1$, $s_2 = u_2$. Clearly, a_i satisfies (H1) with $m_i = 3$, $M_i = 5$, s_j satisfies (H2) with $\beta_j = 1$, b_i satisfies (H3) with $\alpha_i = 4$, $i,j = 1,2$. Moreover, we choose $a_{11} = \frac{1}{12}$, $a_{12} = \frac{1}{12}$, $a_{21} = \frac{1}{6}$, $a_{22} = \frac{1}{24}$, $t_{11}^{(0)} = \frac{1}{12}$, $t_{12}^{(0)} = \frac{1}{6}$, $t_{11}^{(1)} = \frac{1}{12}$, $t_{12}^{(1)} = \frac{1}{4}$, $t_{21}^{(0)} = \frac{1}{6}$, $t_{22}^{(0)} = \frac{1}{24}$, $t_{21}^{(1)} = \frac{1}{12}$, $t_{22}^{(1)} = \frac{1}{24}$. By simple calculation, we show that (H6) holds. It follows from Theorems 1 and 2 that system (13) has a unique equilibrium point $(0,0)^{\mathrm{T}}$ which is GES (Figs. 1–6).

5. Conclusions

In this paper, the dynamics of Cohen–Grossberg neural networks model with delays and reaction–diffusion is studied. By employing homotopic mapping theory and constructing Lyapunov functional method, some sufficient conditions have been obtained which guarantee the model to be GES. The given algebra conditions are useful in design and applications of reaction–diffusion Cohen–Grossberg neural networks. Moreover, our methods in the paper may be extended for more complex networks.

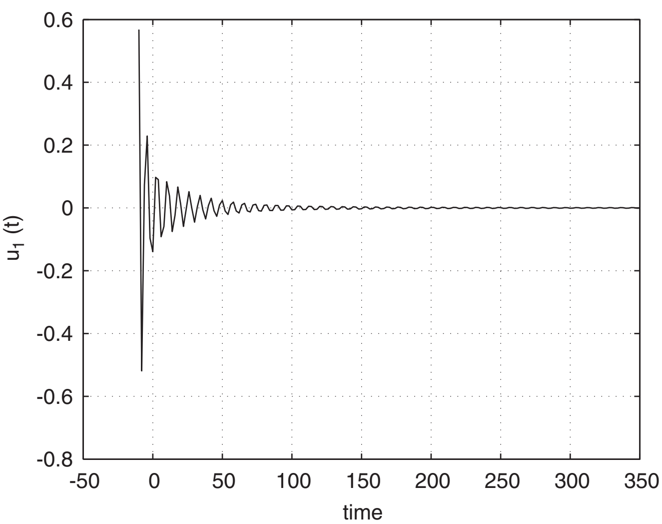

Fig. 1. The trajectory of u_1 versus time in the system (13) with $\tau_k = 5$.

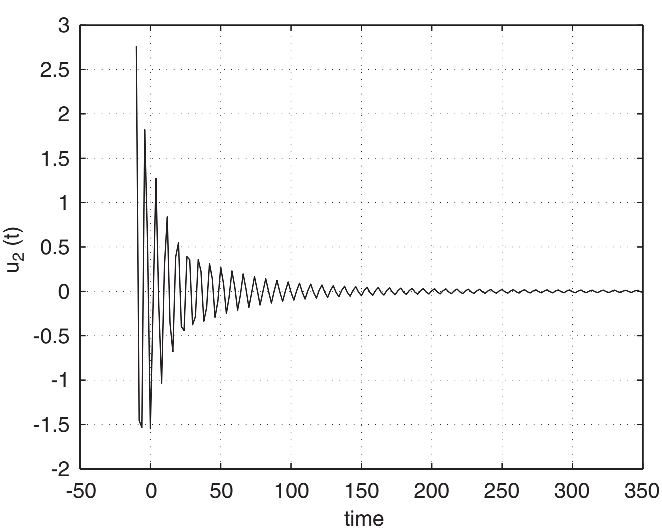

Fig. 2. The trajectory of u_2 versus time in the system (13) with $\tau_k = 5$.

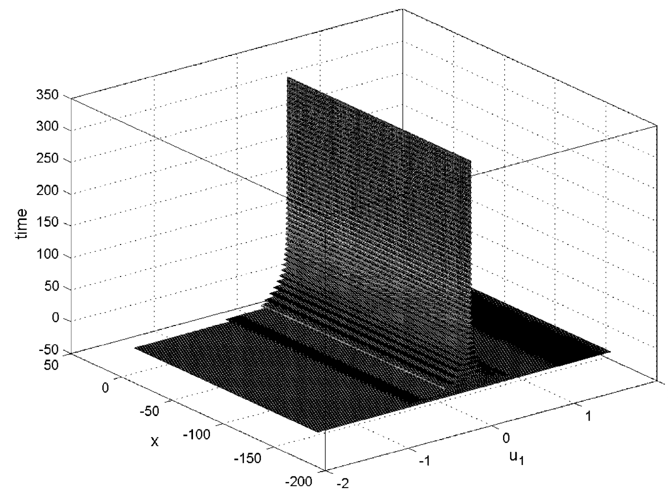

Fig. 3. The cube graph of u_1 and x about time in the system (13) with $\tau_k = 5$.

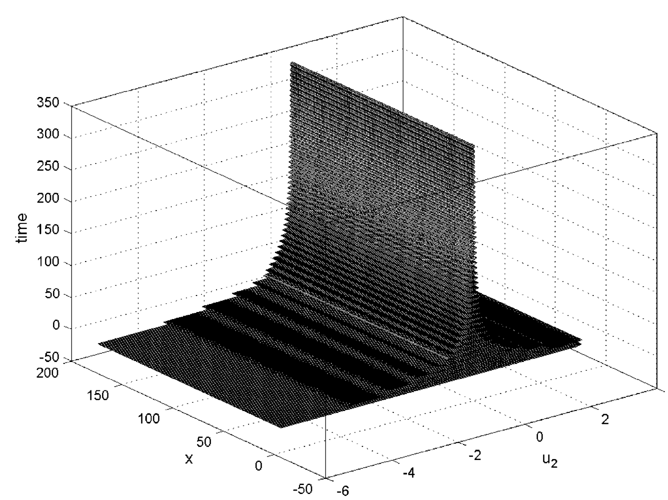

Fig. 4. The cube graph of u_2, x and time in the system (13) with $\tau_k = 5$.

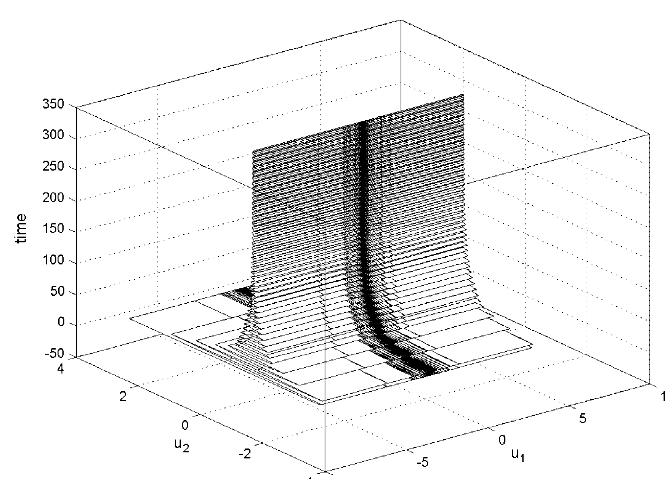

Fig. 5. The cube graph of u_1 and u_2 about time in the system (13) with $\tau_k = 5$.

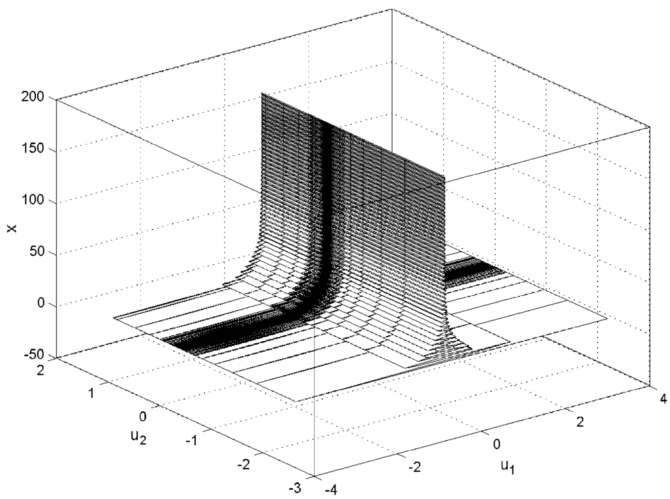

Fig. 6. The cube graph of u_1 and u_2 about x in the system (13) with $\tau_k = 5$.

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