Introduction to quaternions

Topics: Definition

Mathematics

Operations

Euler Angles (optional)

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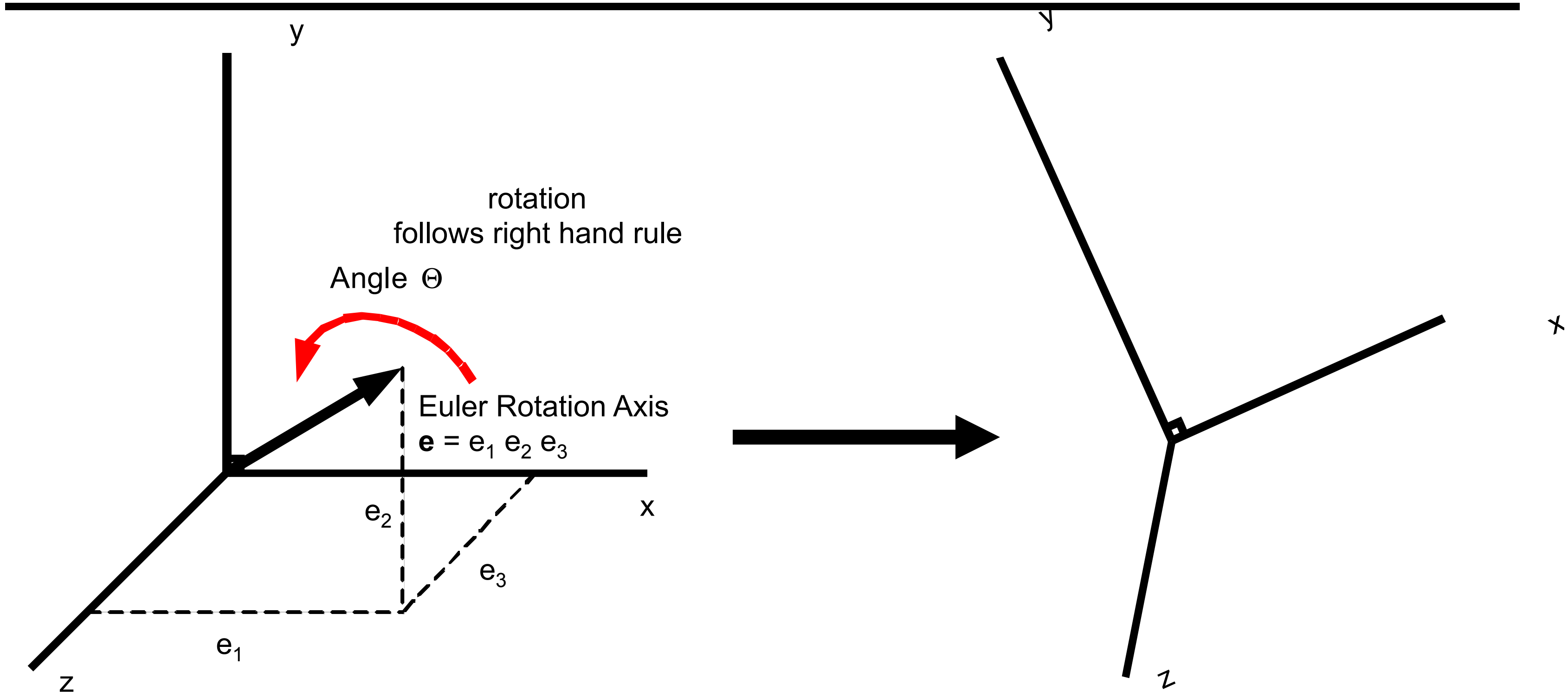
Euler's Theorem



Euler's Theorem: (paraphrased)

The rotational relationship between any two coordinate frames can be described by a unit vector about which rotation takes place and a total rotation angle.

Quaternion Definition

Discovered by William Rowan Hamilton in 1843 while walking with Lady Hamilton, crossing the Broom Bridge on the way to Dublin. He scratched the fundamental formulation

$$i^2 = j^2 = k^2 = ijk = -1$$

into a stone on the side of the bridge.

$$Q = q_1\ q_2\ q_3\ q_4 = \mathbf{e}\ \sin(\Theta/2)\ \cos(\Theta/2)$$

$$= (e_1\ e_2\ e_3\ )\sin(\Theta/2)\ \cos(\Theta/2)$$

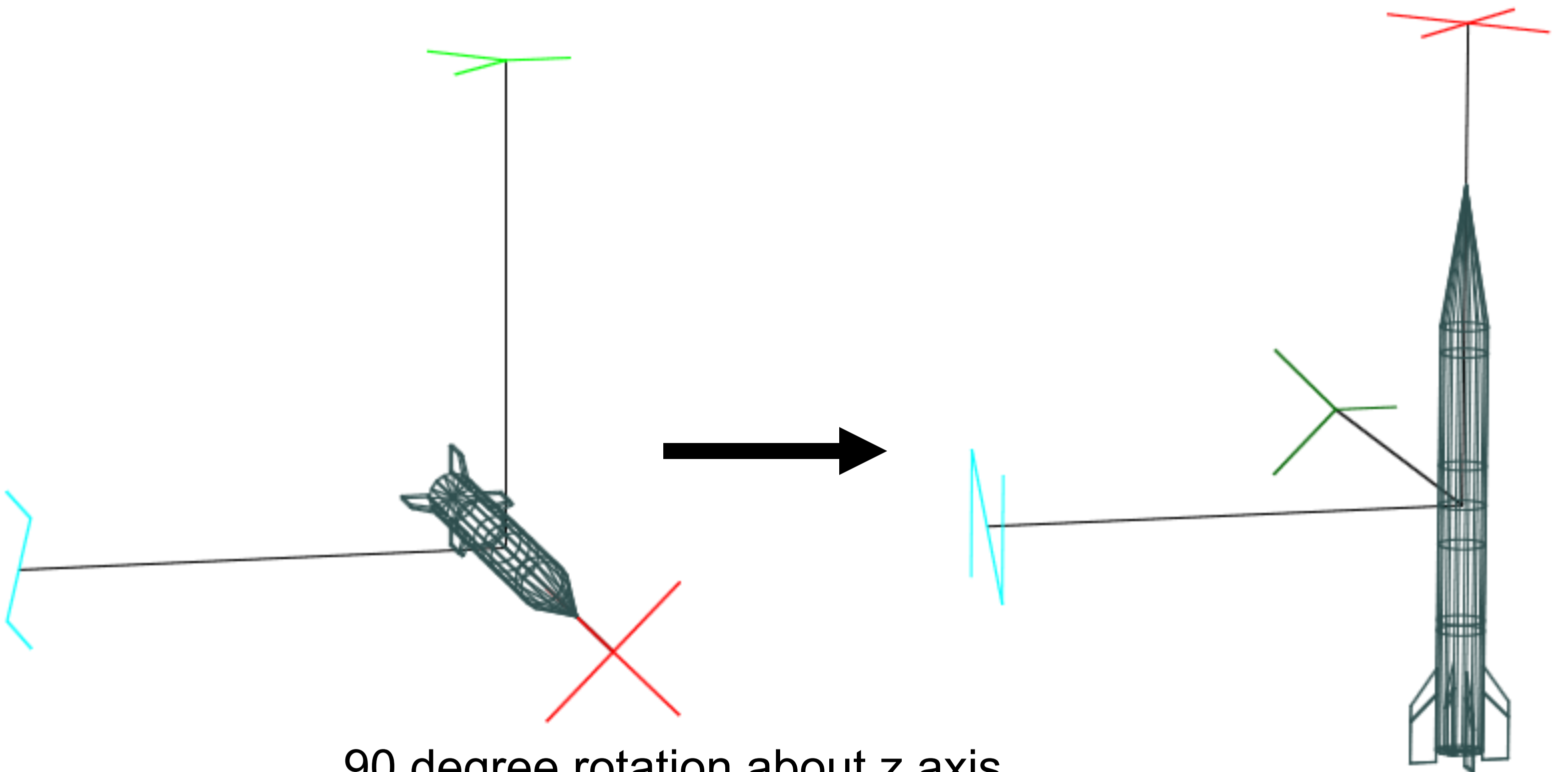
$$= \mathbf{q_1 i\ \ q_2 j\ \ q_3 k\ \ q_4}$$

$e_1\ e_2\ e_3$ **must** be unit vector

$q_1\ q_2\ q_3$ called the "vector elements"
q_4 called the "scalar element"

There is not an accepted standard on the order.
In some instances, the scalar element is first,
sometimes denoted q_0

Example



90 degree rotation about z axis

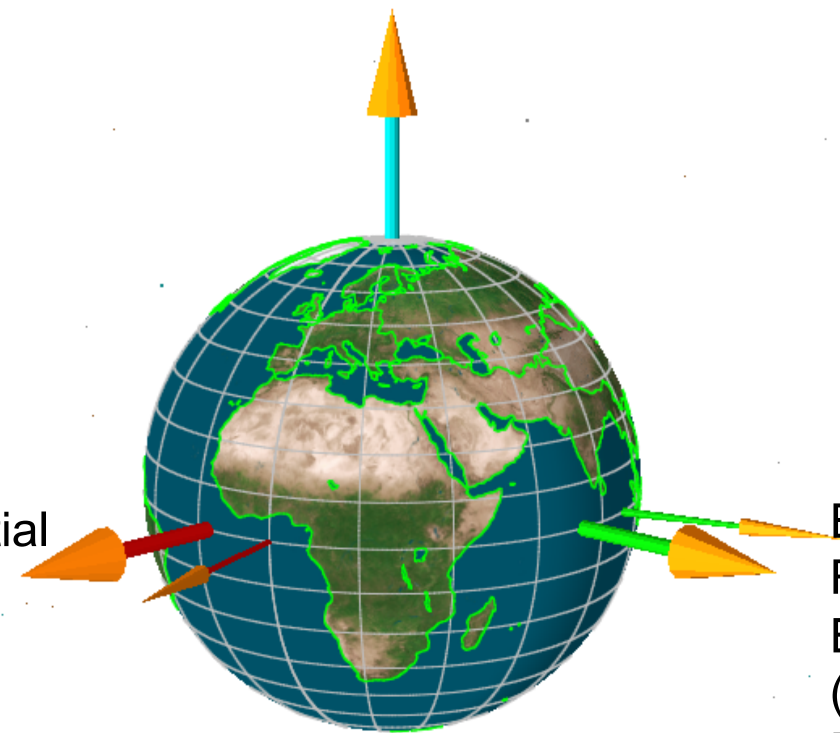
e = 0.0 0.0 1

Q = 0.0*sin(45), 0.0*sin(45), 1.0*sin(45), cos(45)

= 0.0 0.0 .707107 .707107

Coordinate Frames

Earth Centered Inertial
ECI



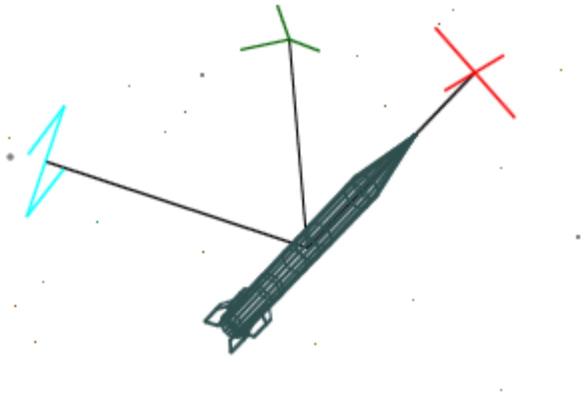
Earth Centered
Relative
ECR or ECEF
(Earth Centered,
Earth Fixed)
Mars Centered Inertial

ECI has:
Z axis along Earth rotation axis,
X axis along the Vernal equinox
(intersection of Equatorial and Ecliptic planes)
Fixed in inertial space.

ECR has:
Z axis along Earth rotation axis,
X axis through intersection of Prime
Meridian and Equator
Rotates with Earth.

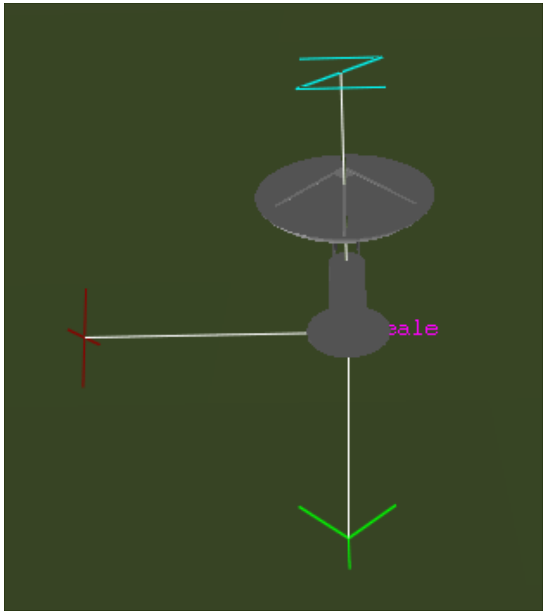
Sometimes ECI is defined as aligned with ECR at a particular time

More Coordinate Frames



Body Fixed

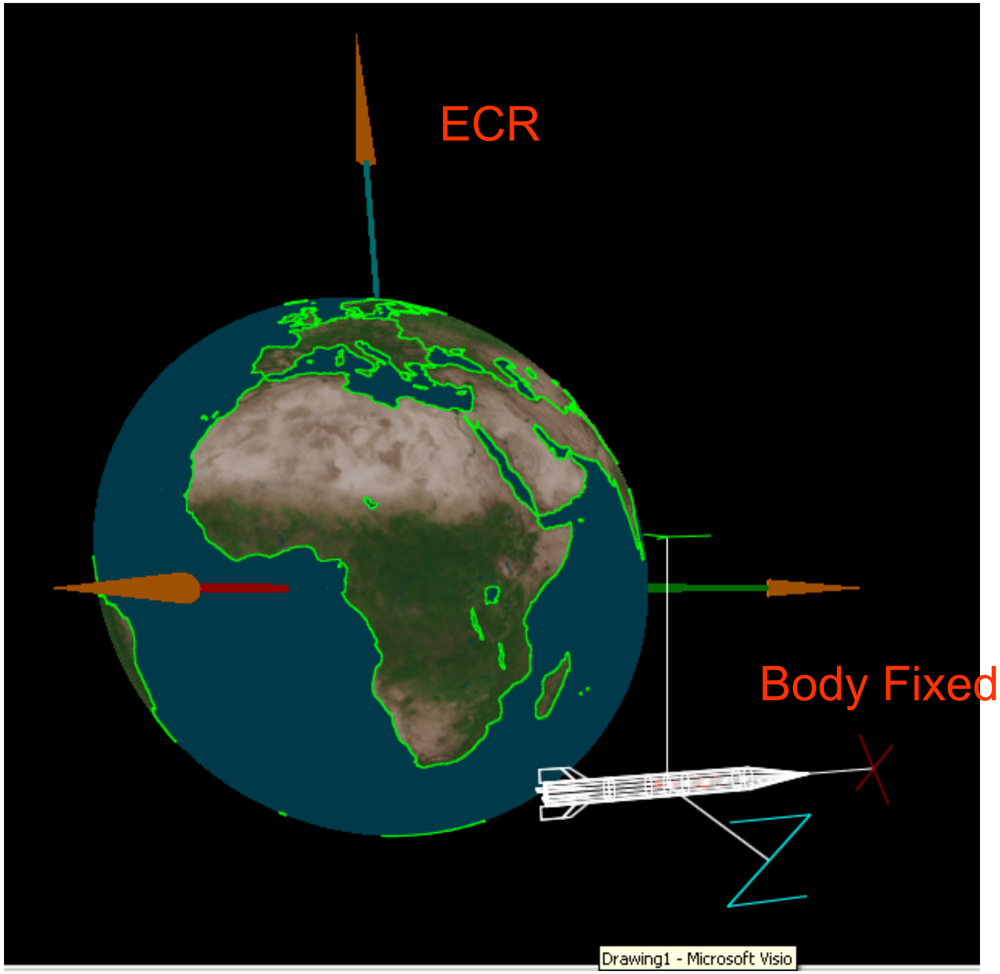
Sensor Boresight



An infinite number of coordinate frames can be defined

Bottom Line: Always know what coordinate frame(s) you are working in!

What Do We Do with Quaternions?



Describe vehicle orientation:
e.g. where, rotationally, is the vehicle relative to a given reference frame?

Transform vectors:
e.g. I know where a target is in ECR, where is it relative to the body (in the body fixed frame)?

Rotate vectors:
e.g. I know a vector in my vehicle coordinate frame, where is it pointing in ECR?

Quaternion Mathematics

Basic Functions:	Normalization Multiplication Inverse, Identity Equality Successive Rotations Two Coordinate frames Vector cross and dot products
Operations with Quaternions:	Vector Rotation, Transformation Attitude/Orientation Propagation Vehicle maneuver/re-orientation Attitude/orientation knowledge update Appendage Pointing

Quaternion Normalization

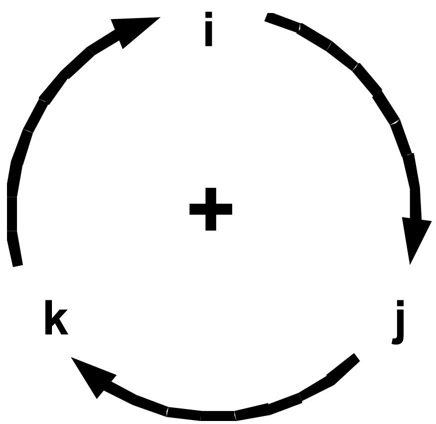
To normalize quaternion, make it's magnitude = 1.0

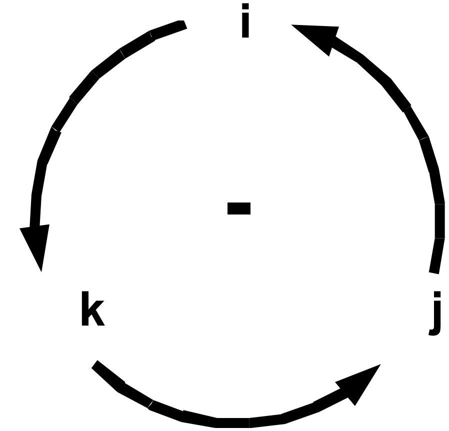
$$Q_n = \frac{q_1 \quad q_2 \quad q_3 \quad q_4}{\sqrt{q_1^2 \quad q_2^2 \quad q_3^2 \quad q_4^2}}$$

Any quaternion that represents a rotation MUST be normal.

Normalize before and after all operations.

Quaternion Multiplication





multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k \qquad jk = i \qquad ki = j$$

$$ji = -k \qquad ik = -j \qquad kj = -i$$

$$Q_A\, Q_B = (q_{a1}i \quad q_{a2}j \quad q_{a3}k \quad q_{a4})\,(q_{b1}i \quad q_{b2}j \quad q_{b3}k \quad q_{b4})$$

multiplication performed term by term

Commutivity, Associativity

$$Q_A\ Q_B \neq Q_B\ Q_A$$

Not Commutive

$$(Q_A\ Q_B)\ Q_C = Q_A\ (Q_B\ Q_C)$$

Associative

Quaternion Inverse, Identity

inverse: $Q^{-1} = Q^{*} = -q_1 i \;\; -q_2 j \;\; -q_3 k \;\; q_4$

(also called conjugate)

identity: $\mathbf{Q_I} = 0 \quad 0 \quad 0 \quad 1$

(a zero angle rotation)

Multiplication by conjugate

$$Q^{*} Q = Q Q^{*} = Q_I = 0 \quad 0 \quad 0 \quad 1$$

Multiplication by identity

$$Q Q_I = Q_I Q = Q$$

These are the only multiplicative operations that are commutative

$\mathbf{Q_A Q_B \neq Q_B Q_A}$ **in general**

Quaternion Equality

Reversing signs of all four elements of a quaternion
yields the same quaternion, mathematically

Interpretation:
Negative rotation about the opposite rotation axis

Warning:

Be very careful about software that may encounter an equivalent quaternion with reversed signs.

A 1^{o} error in one direction may be interpreted as a 359^{o} error in the opposite direction

IT HAS HAPPENED!!!!!!!!!!!!!!!

Successive Rotations

Given:

$$Q_1 \quad Q_2 \quad Q_3 \quad \ldots \quad Q_n = \text{ sequence of rotations}$$

$$Q_{total} = Q_1 \, Q_2 \, Q_3 \ldots Q_n$$

successive rotations are described by successive multiplications of the quaternions

Two Coordinate Frames

Given: Two coordinate frames, described by

$$\mathbf{Q_A} \quad \mathbf{Q_B}$$

both referenced to the same coordinate frame

To determine the quaternion going from A to B:

start with: $Q_B = Q_A \; Q_{AB}$

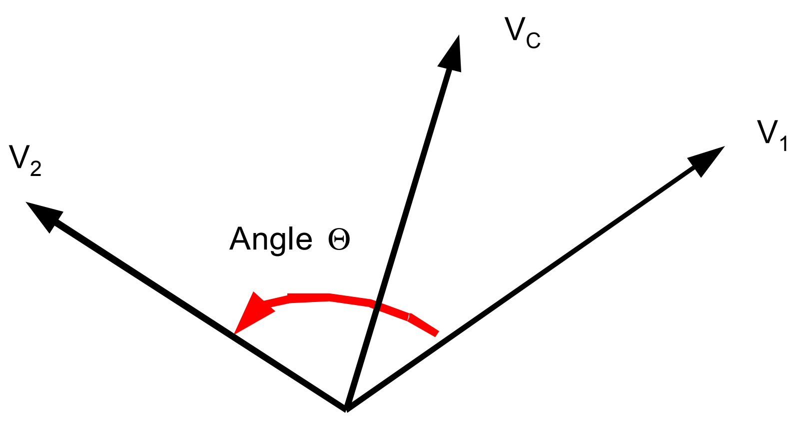
multiply both sides by inverse of Q_A

$$Q_A^* \, Q_B = Q_A^* \, Q_A \, Q_{AB}$$

or

$$\mathbf{Q_{AB} = Q_A^* \quad Q_B}$$

Vector Cross, Dot Products



Cross Product

$V_C = V_1 \mathbf{X} \ V_2$

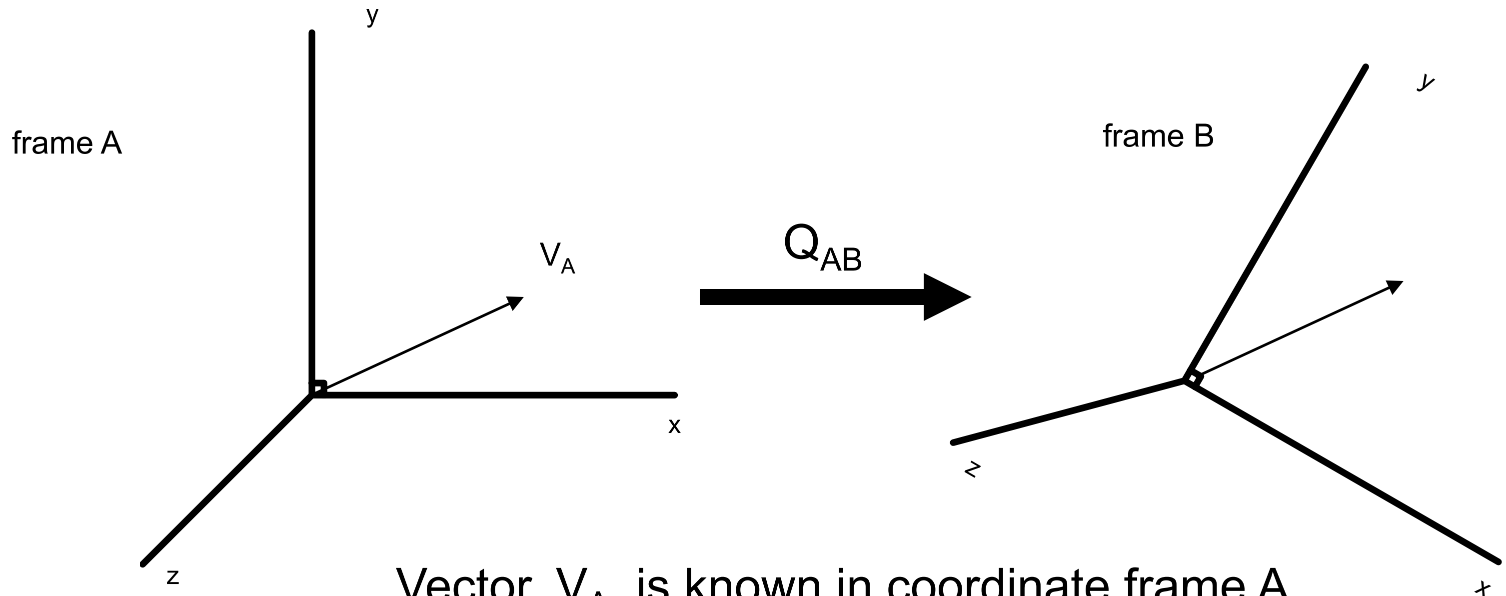
V_C perpendicular to both V_1 and V_2
$|V_C| = |V_1|\ |V_2| \sin(\Theta)$

Dot Product

$P = V_1 \bullet V_2$
$= |V_1|\ |V_2| \cos(\Theta)$

If V1 and V2 are unit vectors,
$\Theta = \cos^{-1}(P)$

Vector Transformation

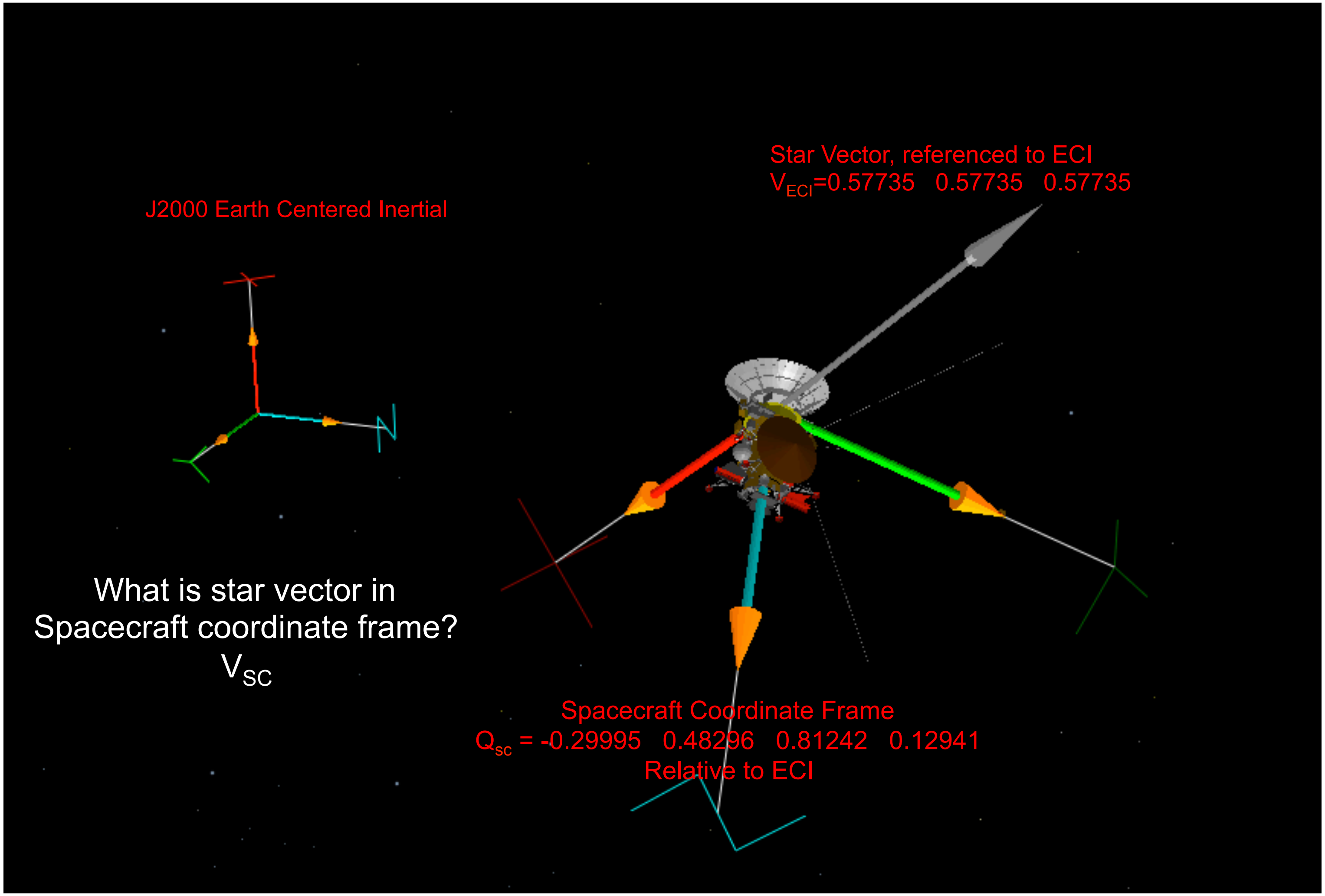


Vector, V_A, is known in coordinate frame A
What is it in frame B? - V_B

$$V_B = Q^*_{AB} \; V_A \; Q_{AB}$$

a zero (0) is appended to V to make it compatible with quaternion multiplication

Vector Transformation, Example: Vector from Star Catalog



Vector Transformation, Example: Vector from Star Catalog (in ECI) to Spacecraft Coordinate Frame

$$V_{SC} = Q^{*}_{SC} \; V_{ECI} \; Q_{SC}$$

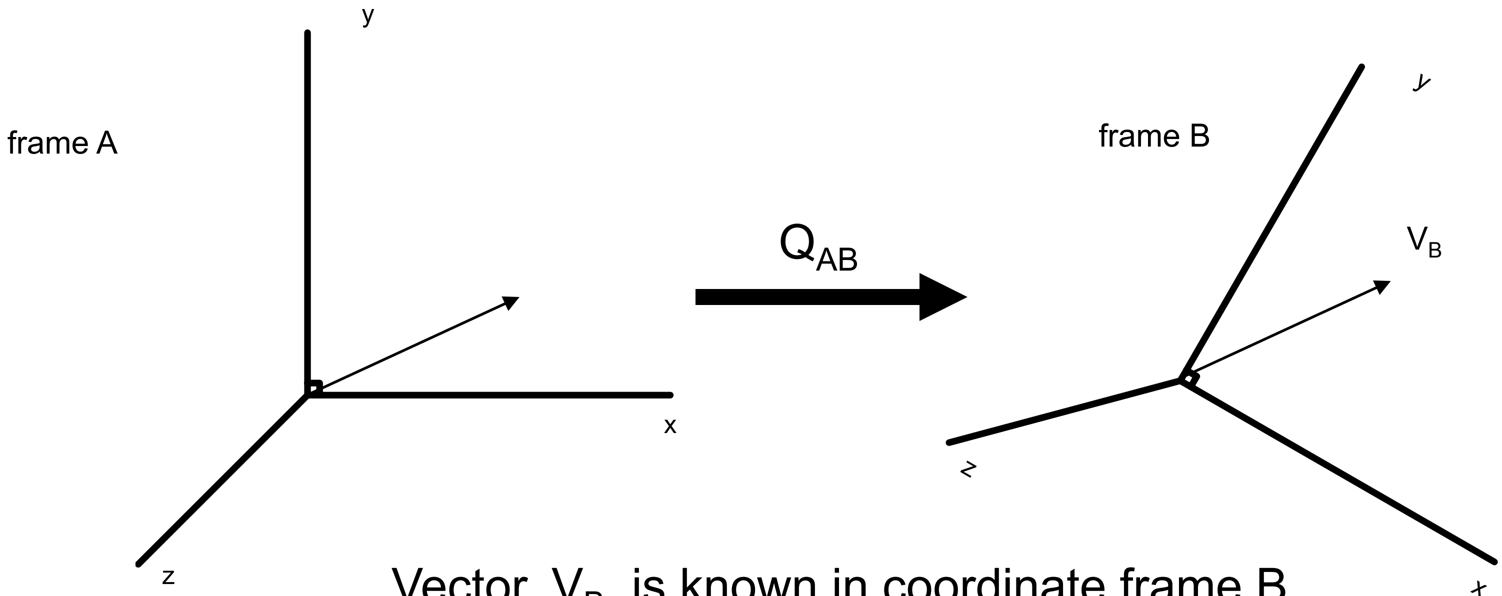
from two charts back

Q_{SC} = -0.29995   0.48296   0.81242   0.12941

V_{ECI} = 0.57735   0.57735   0.57735

V_{SC} = -0.85355 -0.16910   0.49280

Vector Rotation

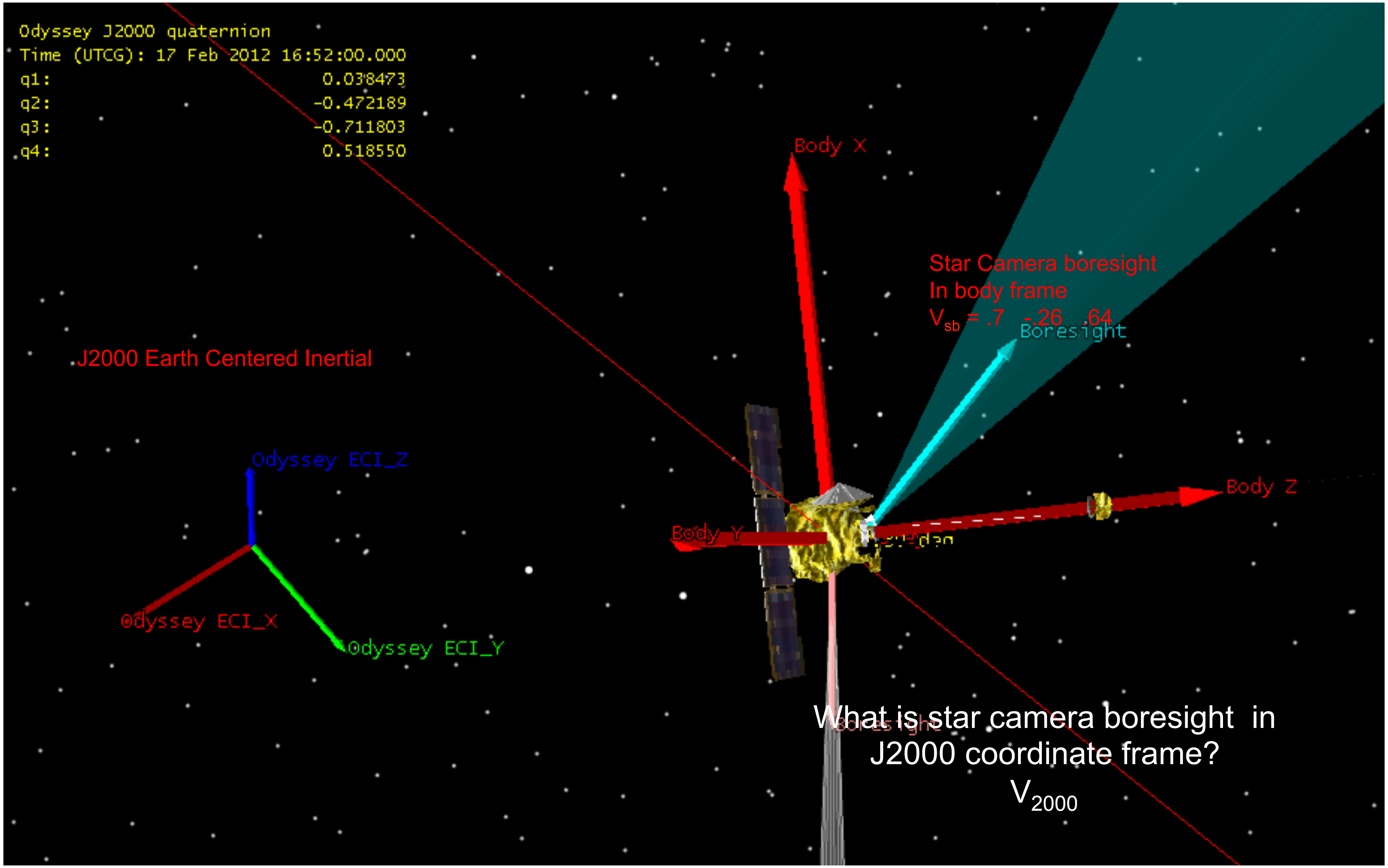


Vector, V_B, is known in coordinate frame B
What is it in frame A? - V_A

$$V_A = Q_{AB} \; V_B \; Q^*_{AB}$$

a zero (0) is appended to V to make it compatible with quaternion multiplication

Vector Rotation Example: Star Camera Boresight



Vector Rotation Example: Star Camera Boresight from SC Body Frame to ECI

$$V_A = Q_{AB} \; V_B \; Q^*_{AB}$$

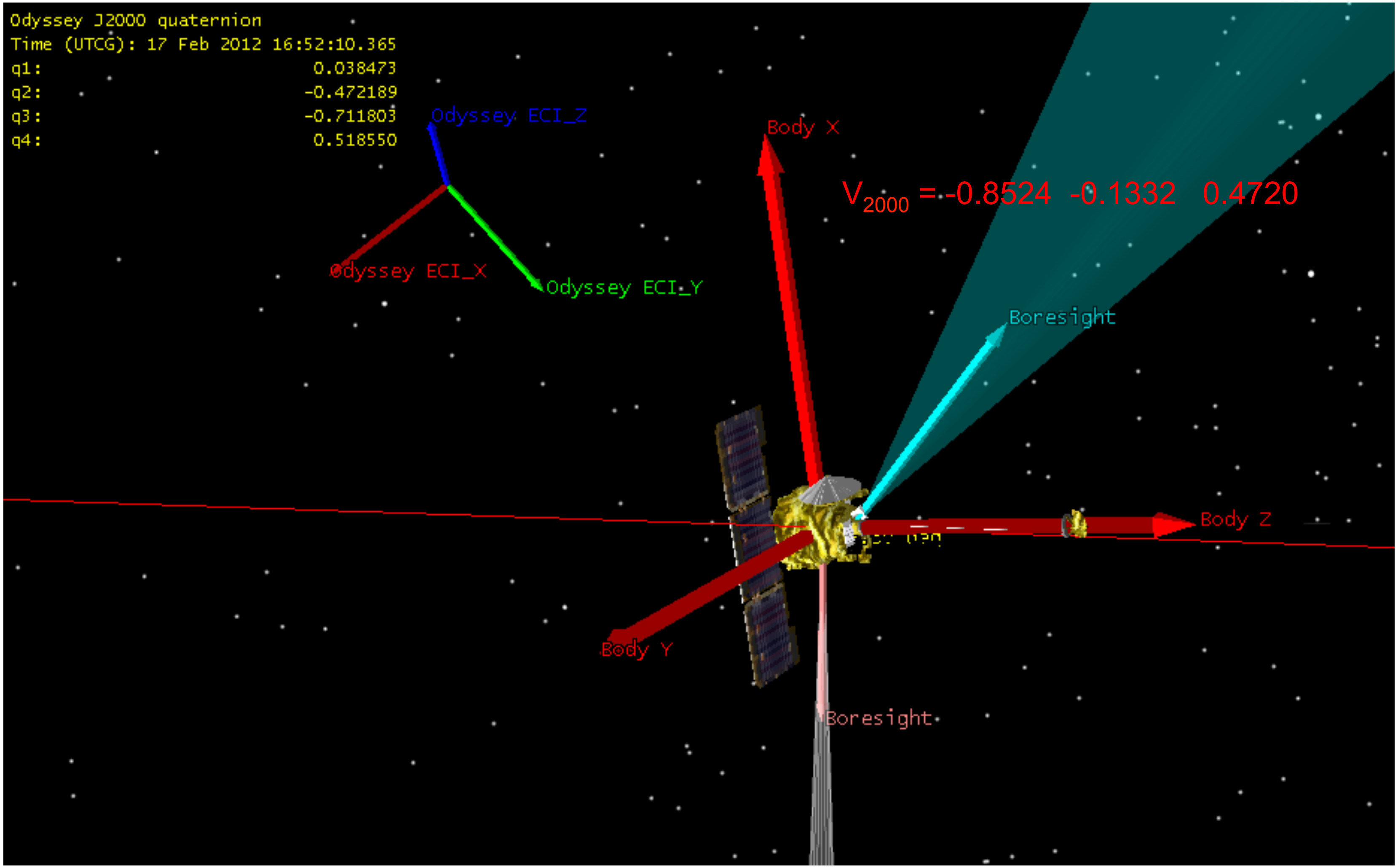
from two charts back

Q_{sc} = 0.038473 -0.472189 -0.711803 0.51855

V_{sb} = .7 -.26 .64

V_{2000} = -0.8524 -0.1332 0.4720

Another View



Attitude/Orientation Propagation

We know orientation at time t_1

$Q(t_1)$

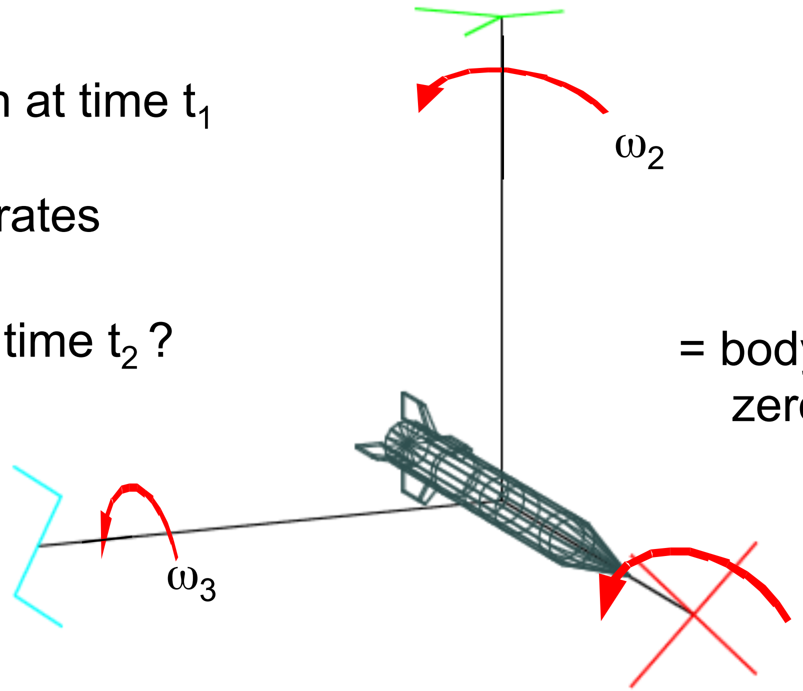
and rotational rates

ω

What is orientation at time t_2 ?

$t_2 = t_1 + \Delta t$

$Q(t_2)$



$\boldsymbol{\omega} = \omega_1 \;\; \omega_2 \;\; \omega_3 \;\; 0$

= body rotation vector with zero (0) appended

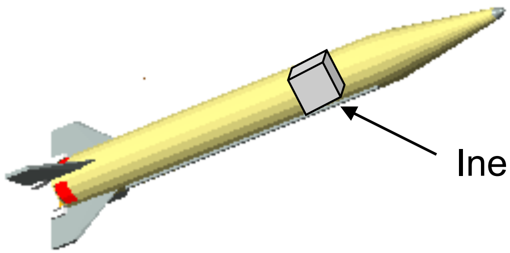
Calculate quaternion derivative

$$\dot{Q} = \tfrac{1}{2}\, Q\boldsymbol{\omega}$$ (do not normalize)

Integrate forward in time

$$Q(t_2) = \int_{t_1}^{t_2} \dot{Q}\, dt \quad + Q(t_1)$$

Attitude/Orientation Knowledge Update



Attitude/Orientation Propagation Process

Inertial Measurement Unit $\longrightarrow$ Body Rotational Rates – $[\omega\ 0.0]$

$\longrightarrow \int \frac{1}{2} Q\omega \longrightarrow Q_{est}$

Q_{est} is "estimated" attitude

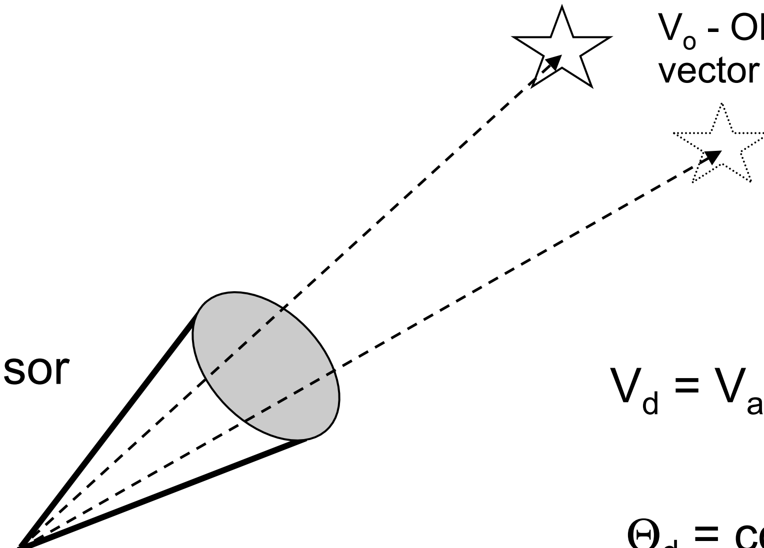
Errors accumulate over time, requires update from external reference

Star Camera, Sun sensor, planetary horizon sensor are common external reference

Updating using quaternions:

point sensor at known star
compare observed with anticipated star position
generate delta quaternion
update Q_{est}

Updating Attitude



$V_d = V_a \mathbf{X} \; V_o$ delta rotation axis (normalize)

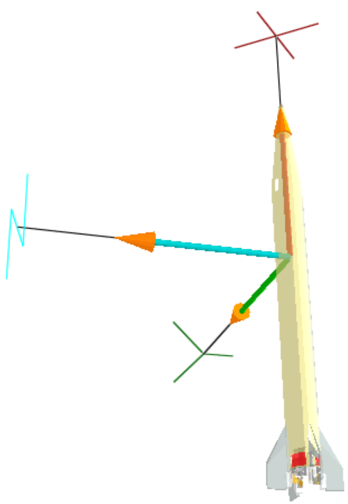
$\Theta_d = \cos^{-1} (V_a \bullet V_o)$ delta rotation angle

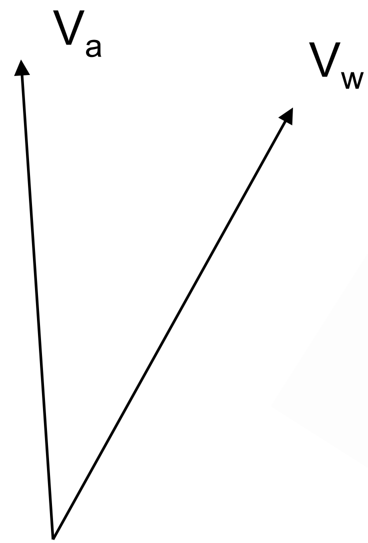
$Q_d = (V_d)\sin(\Theta_d/2) \quad \cos(\Theta_d/2)$
delta quaternion

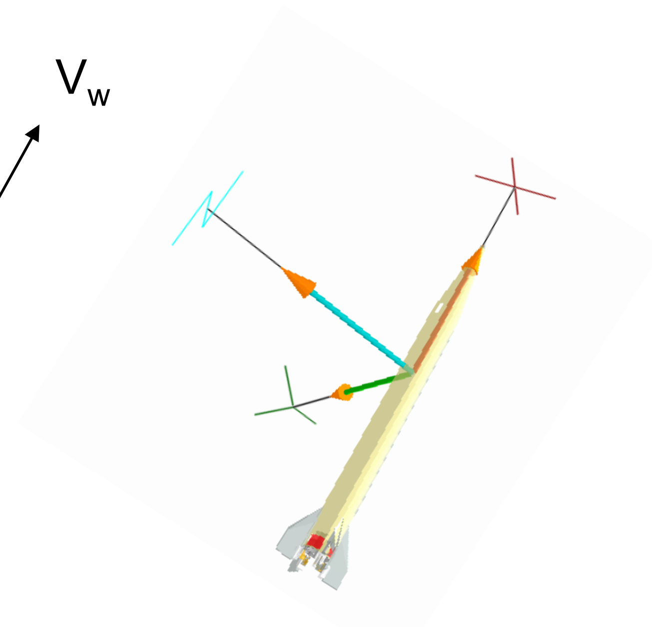
Update Q_{est}

$Q_{upd} = Q_{est}\, Q_d$ updated attitude estimate

Attitude Change/Reorientation







Where we are
commanded attitude = Q_a

Where we want to be
commanded attitude = Q_w

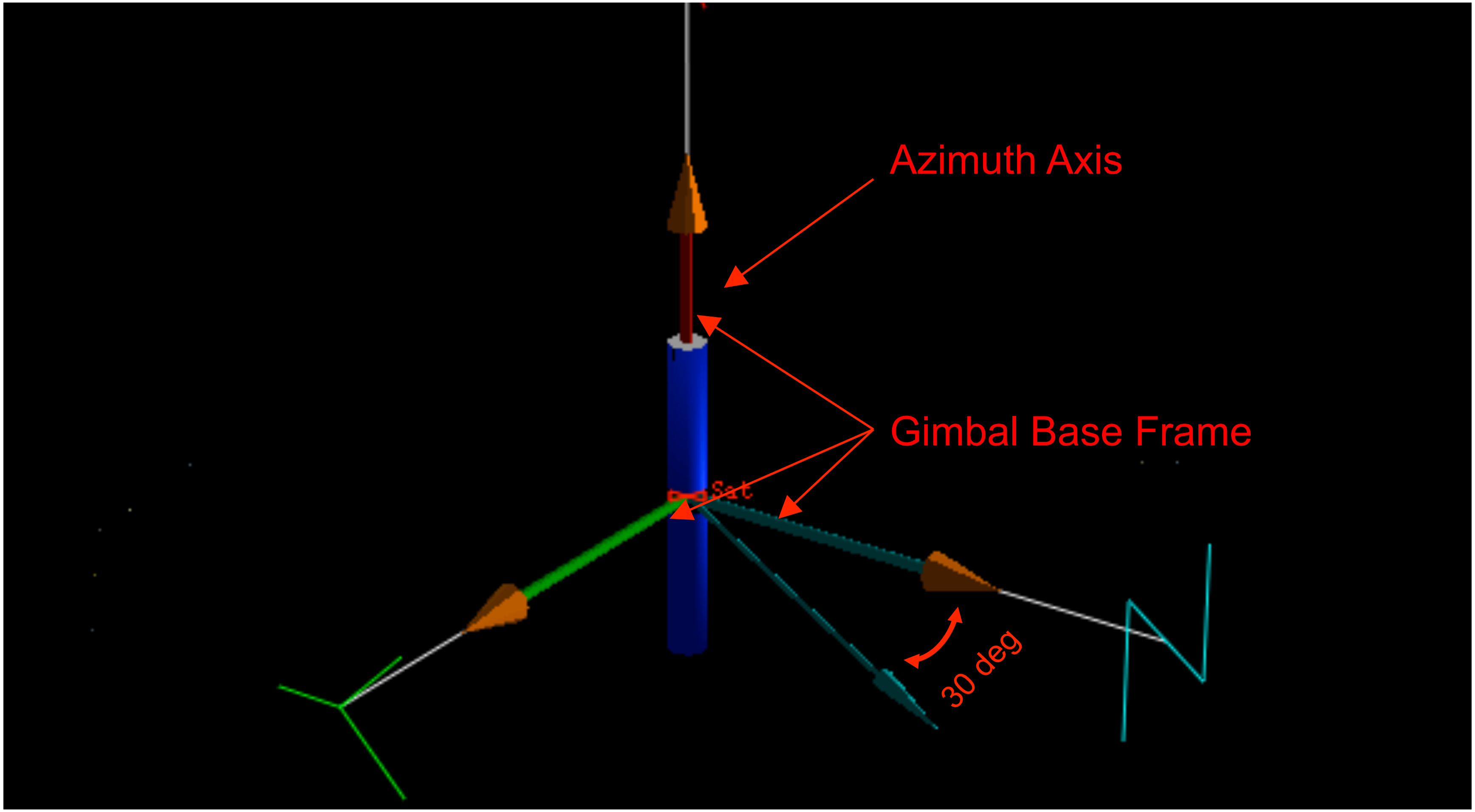
rotation axis $V_R = V_a \mathbf{X} V_w$ (normalize)

rotation angle $\Theta_R = \cos^{-1}(V_a \bullet V_w)$

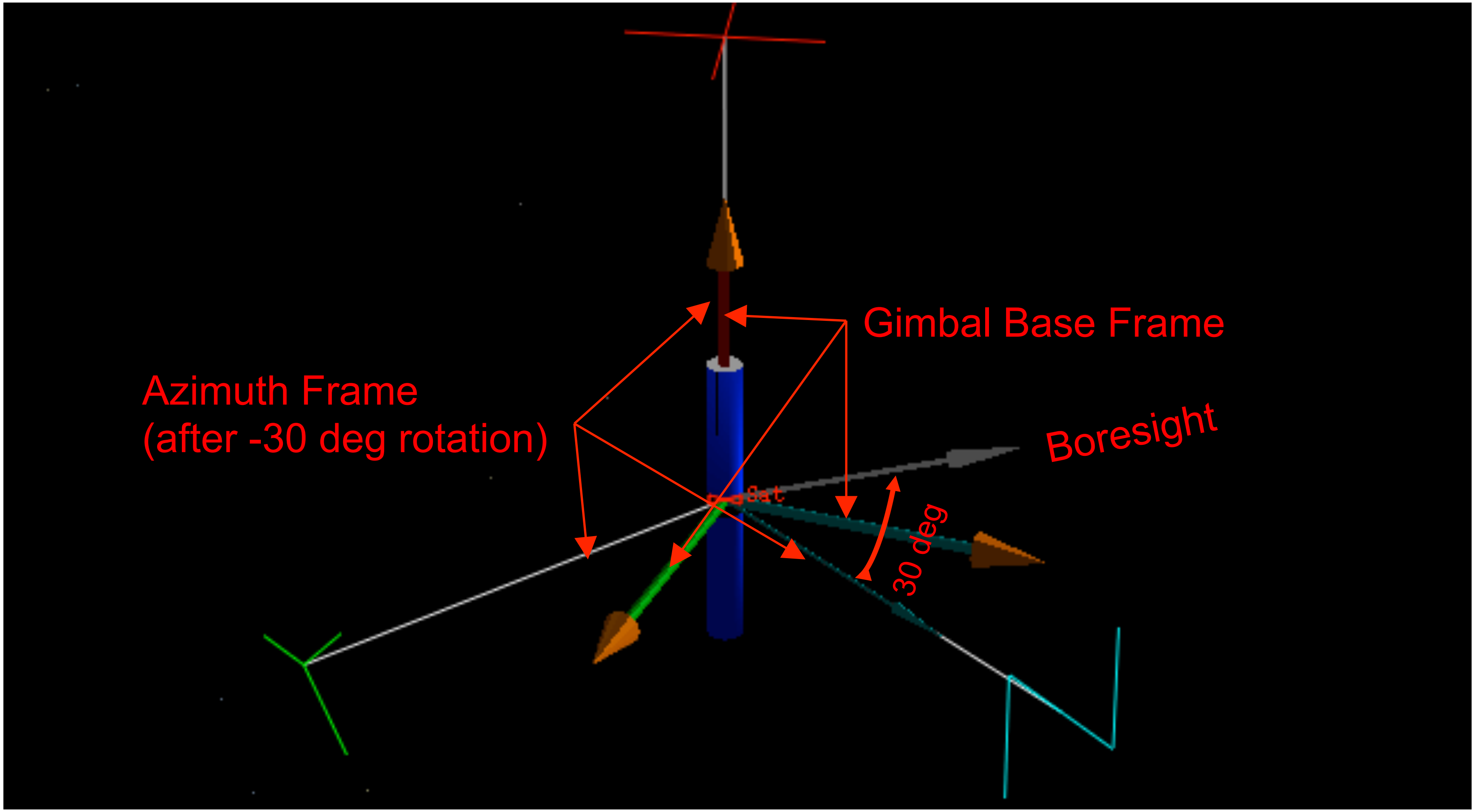
rotation quaternion $Q_R = V_R \sin(\Theta_R/2) \ \cos(\Theta_R/2)$

final attitude $\mathbf{Q_w = Q_a Q_R}$

Appendage Pointing
Azimuth

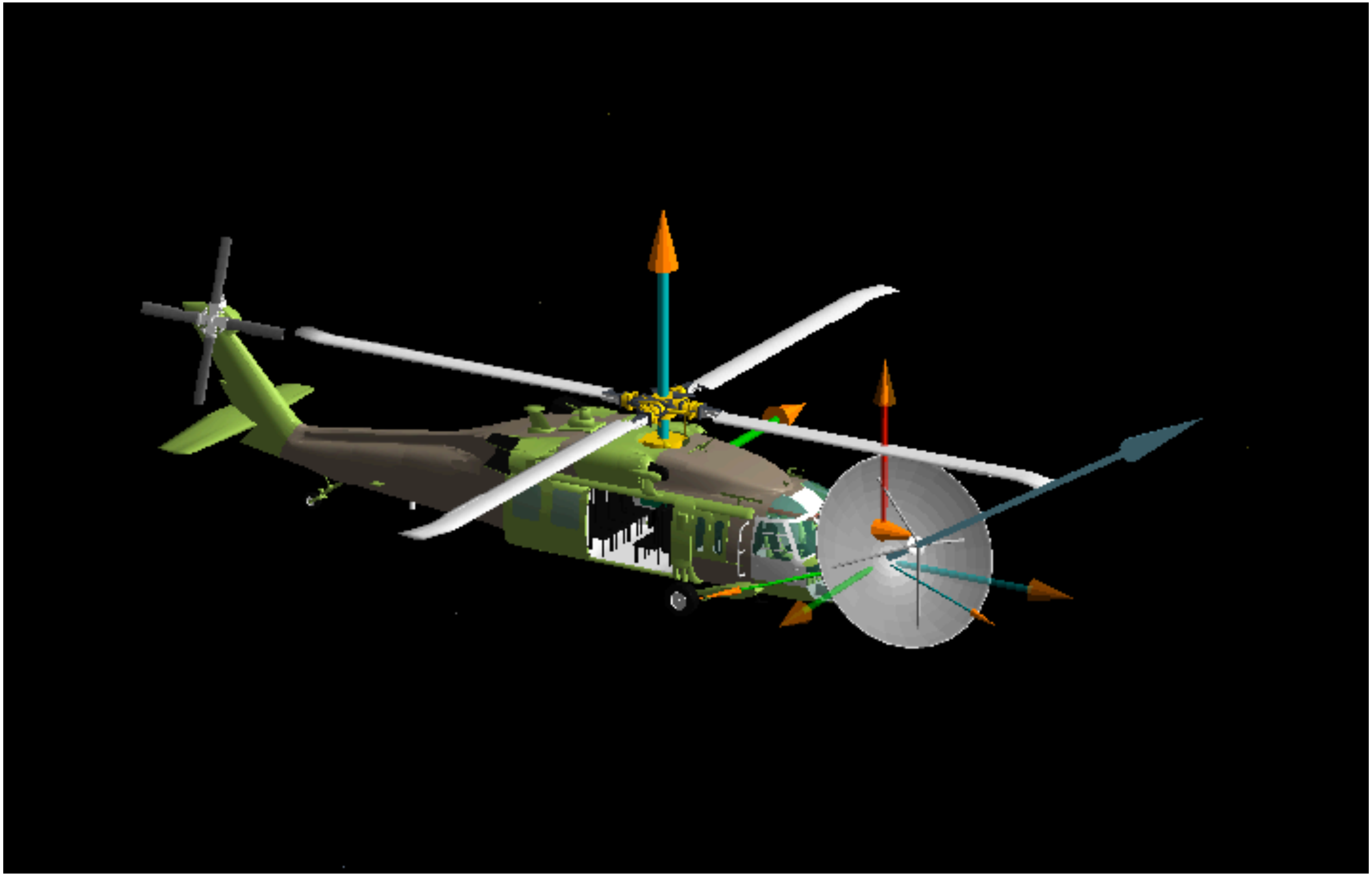


Appendage Pointing
Azimuth - Elevation

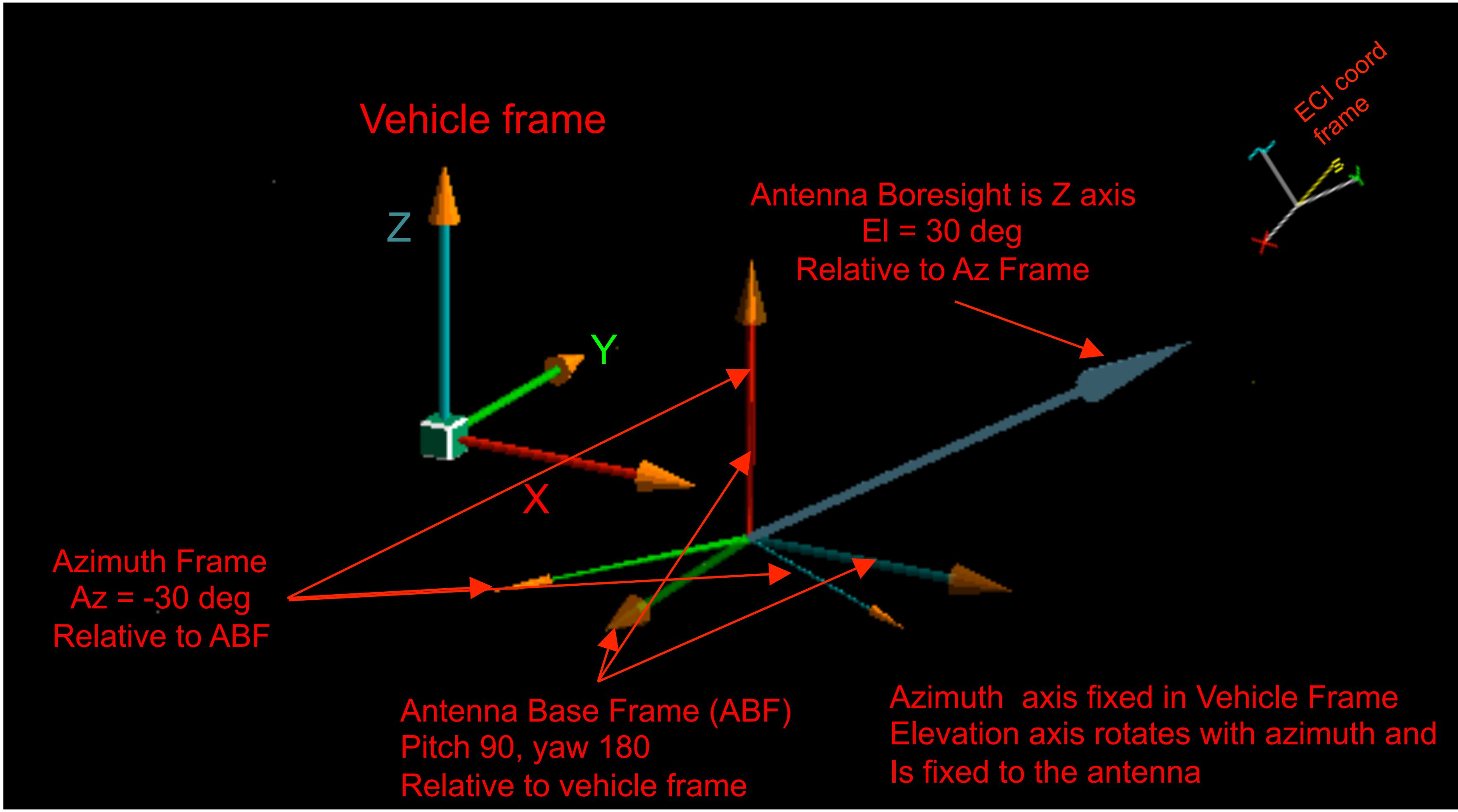


Appendage Pointing Example

Newest Top Secret Satellite



Appendage Pointing Example
Azimuth and Elevation Gimbal



Appendage Pointing Example

Quaternions

What is the orientation of the antenna relative to the universe (reference - ECI)?

Vehicle attitude	Q_v relative to ECI
Vehicle to ABF	Q_{P90} = 0.0 sin(45) 0.0 cos(45)
	Q_{Y180} = 0.0 0.0 1.0 0.0
	$Q_{VABF} = Q_{P90}\ Q_{Y180}$
Azimuth	Q_{az} = $\sin(\Theta_{AZ}/2)$ 0.0 0.0 $\cos(\Theta_{AZ}/2)$
Elevation	Q_{El} = 0.0 $\sin(\Theta_{EL}/2)$ 0.0 $\cos(\Theta_{EL}/2)$

Antenna boresight relative to ECI

$$Q_{ANT} = Q_v\ Q_{VABF}\ Q_{az}\ Q_{El}$$

Appendage Pointing:

The Problem

- We have:
  - Target Vector
    - Offboard Sensor
      - In ECI, ECR, MCI, etc.
    - Onboard sensor
      - In sensor frame, vehicle frame, etc
  - Gimbal frame
    - Relative to Vehicle
      - Many geometries

- What we need to find

Gimbal Angles To Point Payload At Target

Gimbal Geometries

Many

Azimuth, Elevation

Payload boresight may be offset Gimbal Axis

Roll, Pitch

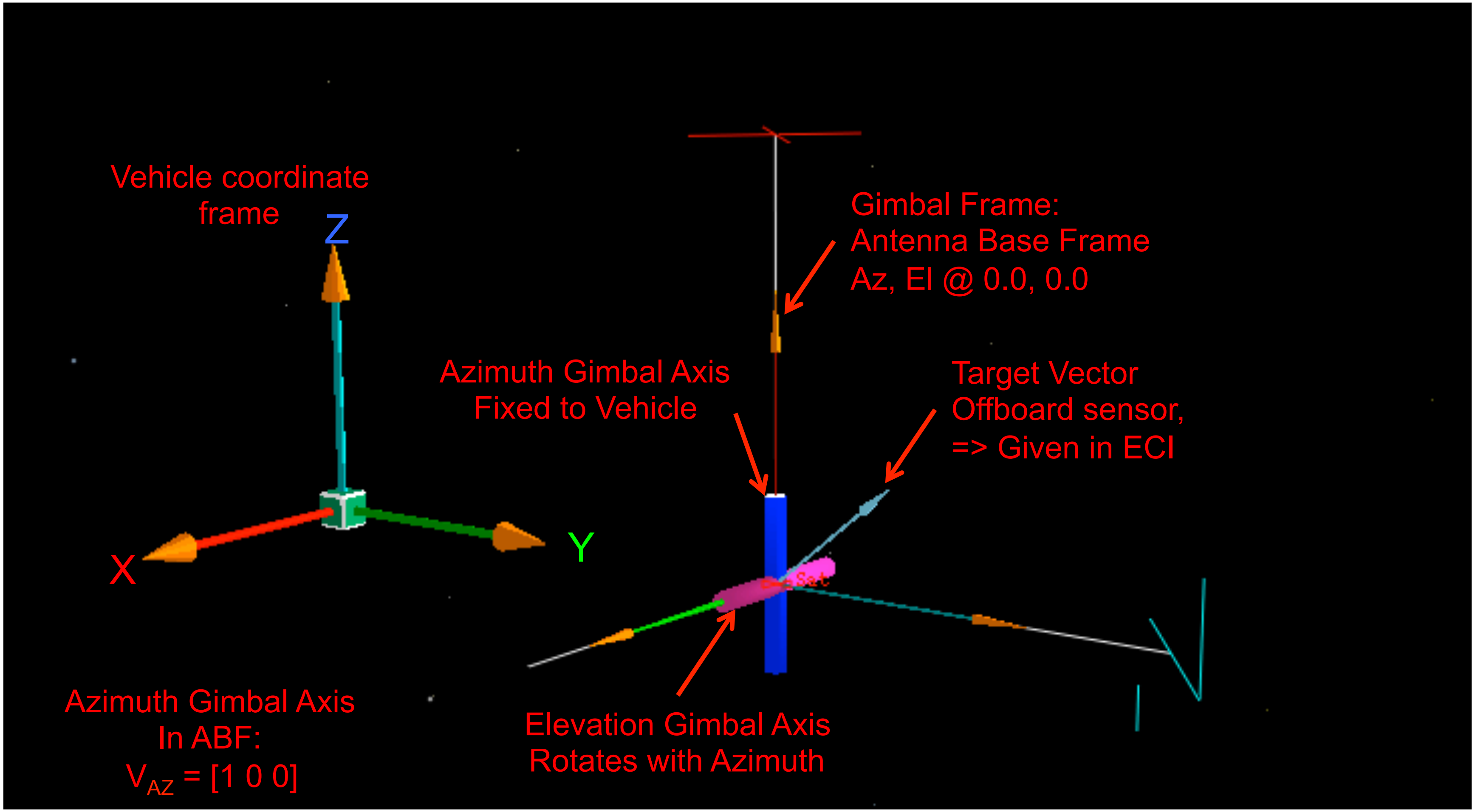
Pitch, Roll

Vehicle may be one or two axes of pointing
- Nadir, yaw for azimuth
- Spinner with spin axis orbit normal:
  - despun platform, into, out of spin for azimuth
  - elevation gimbal
- Single axis gimbal relative to nadir for push broom
- Single axis scanning gimbal

Each Has Unique Math to Point

Example:

Find Azimuth, Elevation Gimbal Angles to Point at Target Vector



Step 1:

Transform Target Vector into Gimbal Frame

(from earlier slide)

Vehicle attitude Q_v relative to reference (ECI)

Vehicle to ABF $Q_{P90} = 0.0 \ \sin(45) \ 0.0 \ \cos(45)$

$Q_{Y180} = 0.0 \ \ 0.0 \ \ 1.0 \ \ 0.0$

$Q_{VABF} = Q_{P90} \ Q_{Y180}$

ECI to ABF $Q_{E2ABF} = Q_v \ Q_{VABF}$

Target Vector in ECI $= V_{ECI}$

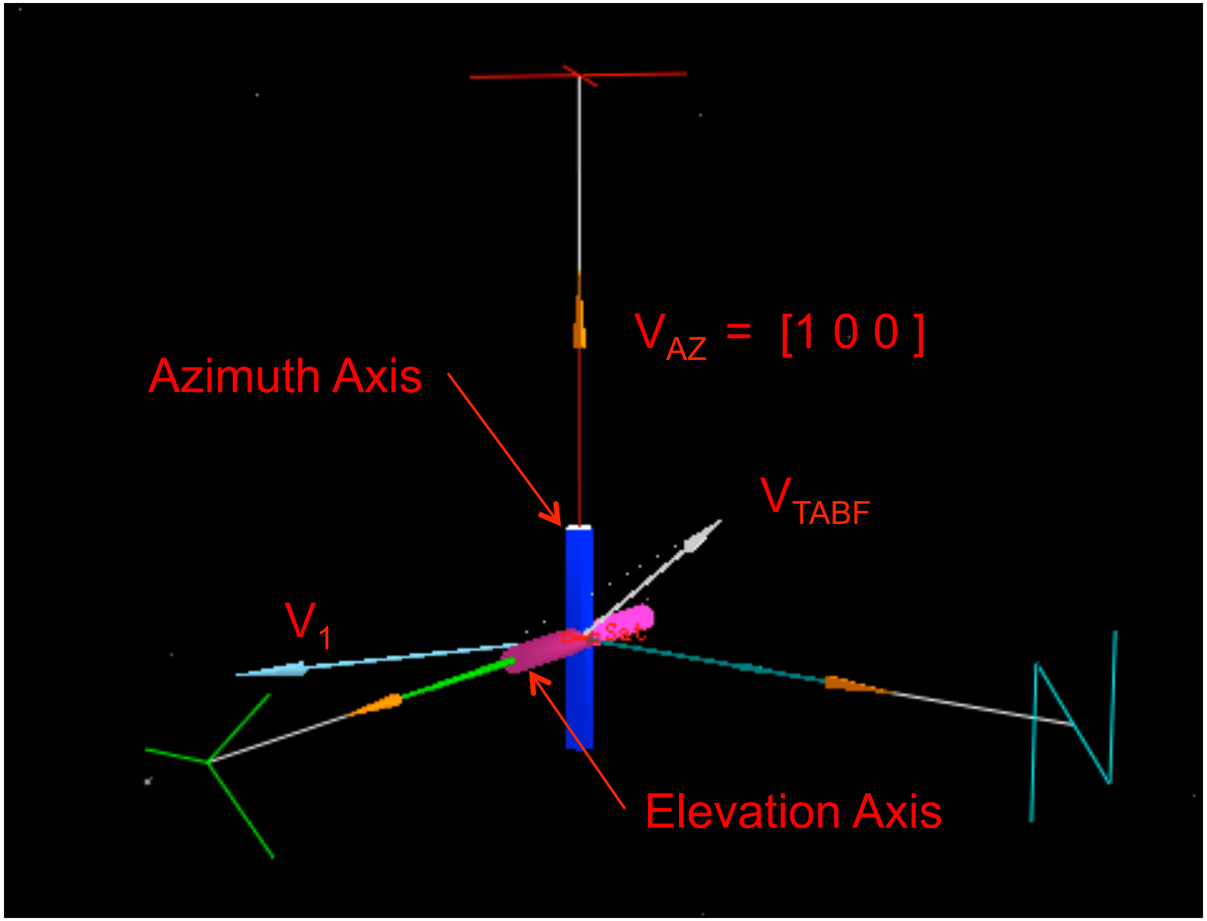
Target Vector in ABF $\mathbf{V_{TABF} = Q^*_{E2ABF} \ V_{ECI} \ \ Q_{E2ABF}}$

Step 2:

Find Azimuth Angle

Cross Product of Target Vector with Azimuth Axis

$$\mathbf{V_1 = V_{TABF} \times V_{AZ}} \text{ (normalize)}$$



V_1 is in ABF Y-Z Plane, Perpendicular to Target Vector

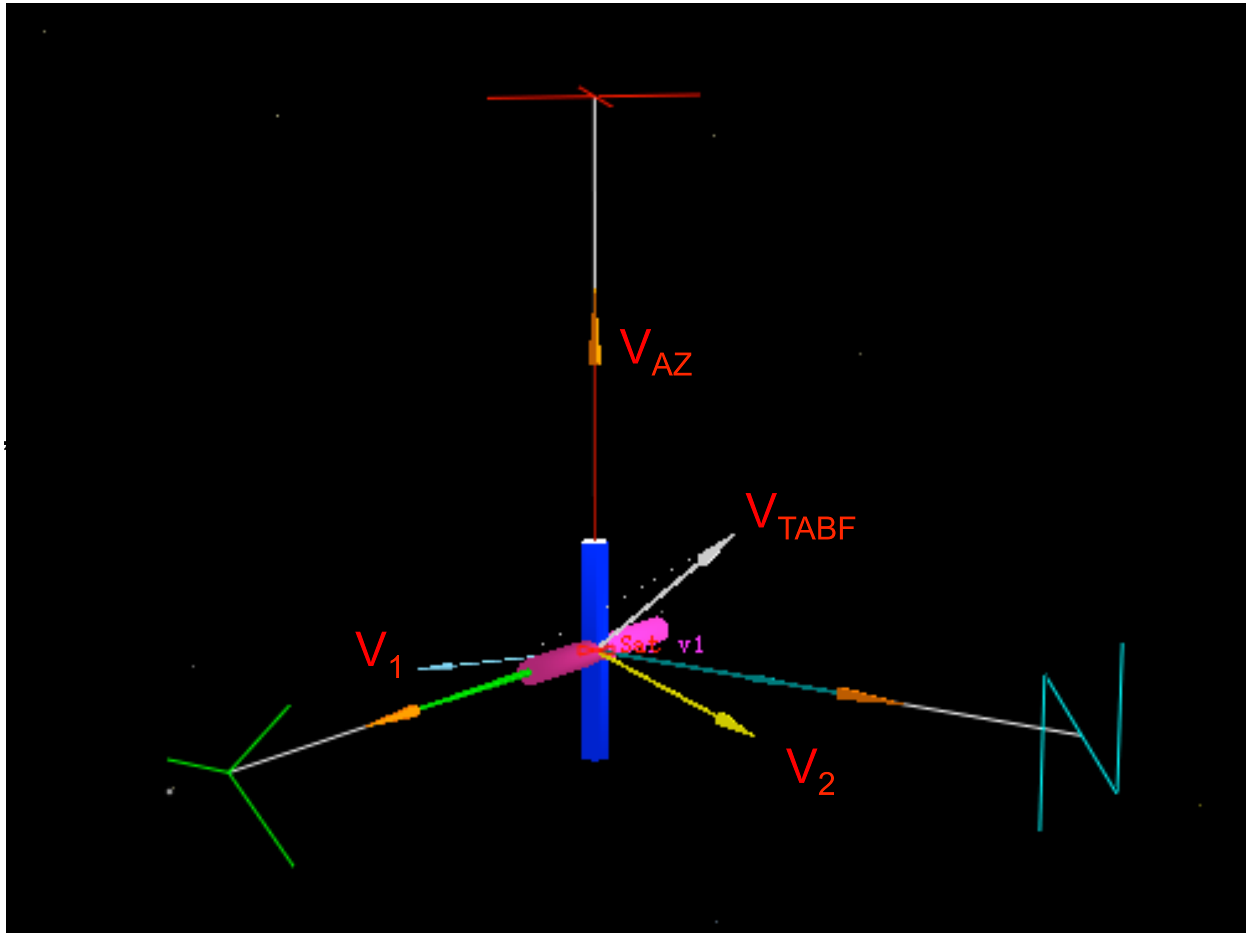
Step 2:

Find Azimuth Angle (cont'd)

Cross Product Az axis with V1

$$\mathbf{V_2 = V_{AZ} \ X \ V_1} \ \text{(normalize)}$$

V_1 is in ABF Y-Z Plane, Projection of V_{TABF} On Y-Z plane



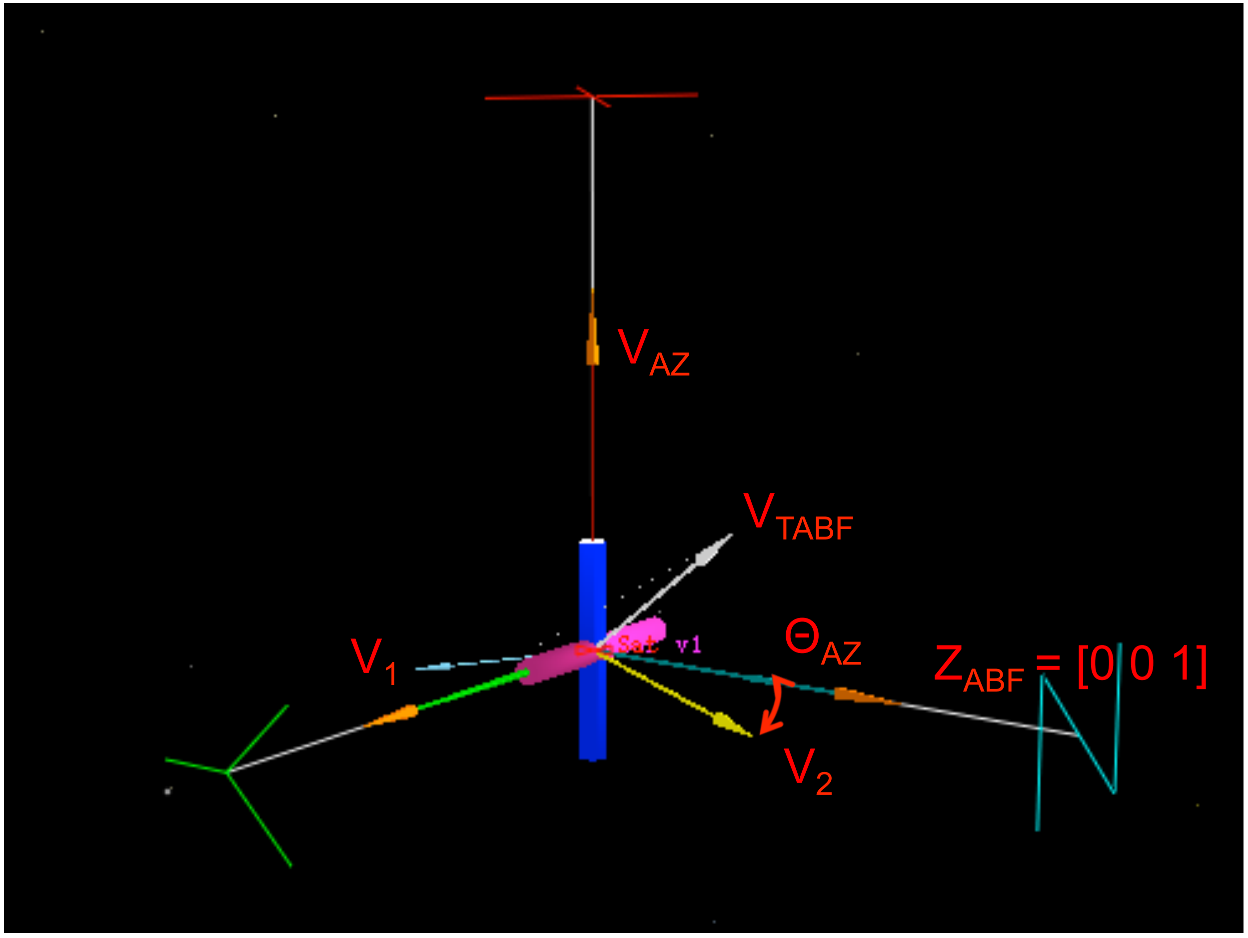
Step 2:

Find Azimuth Angle (more)

Azimuth Magnitude is Angle between ABF boresight vector (Z_{ABF}) and V_2

$|\Theta_{AZ}| = \cos^{-1}(V_2 \bullet Z_{ABF})$ (dot product)

Right Hand Rule
Determines Sign
Of Azimuth Angle



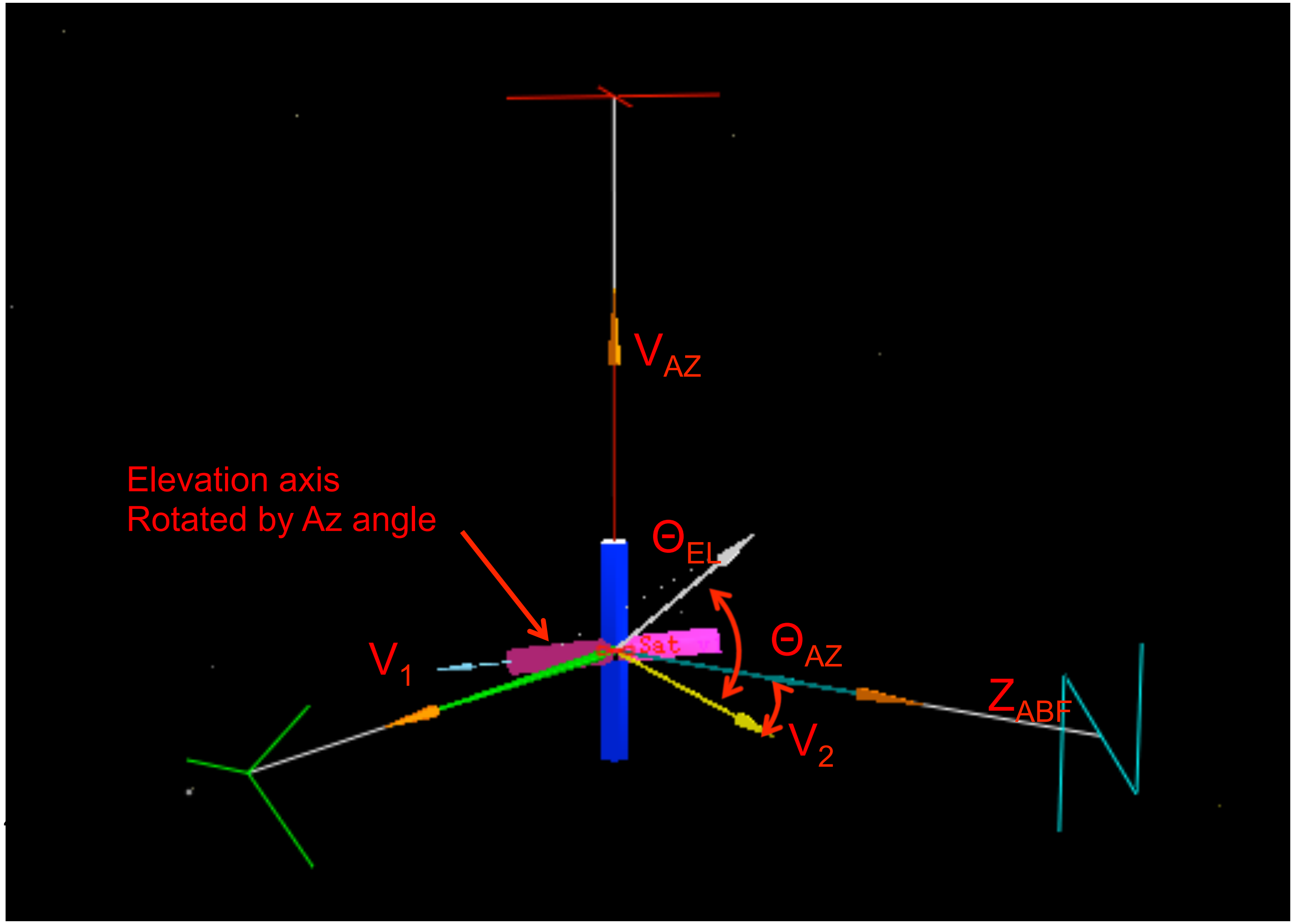
Step 3:

Find Elevation Angle

Elevation is Angle between V_{TABF} and V_2

$$\Theta_{EL} = \cos^{-1}(V_2 \bullet V_{TABF})$$

Right Hand Rule Determines Sign Of Elevation Angle



Conclusion

Quaternions simple, elegant to use
unambiguous
non-singular
straightforward operations

Questions?

Euler Angles

Three successive rotations about independent axes.
Sometimes called "yaw, pitch, roll" angles or
"heading, attitude and bank".

Describe rotational relationship between two coordinate frames

Quaternions and Euler angles contain the same information.

Conversion from one to the other:

one-to-one correspondence between a quaternion
and a set of Euler angles

Advantages, Disadvantages

Euler Angles

Advantages: Familiarity - nose up/down, left/right, roll left/right

Disadvantages: Ambiguous - 12 rotation sequences -
ypr (321), rpy (123), ypy (323), ...

Vector tranformations and rotations require calculating corresponding matrix or quaternion

Singularities - e.g. 90 pitch causes instantaneous 180 deg change in euler angles

Equations of motion are horrible

Advantages, Disadvantages (cont'd)

Quaternions:

Advantages:

No ambiguity

No singularities

Transformations: two quaternion multiplications

Equations of motion simple

Successive rotations: successive multiplications

Disadvantages:

Unfamiliar

let's look at some maneuvers

Transformation with Euler Angles

1) pick correct Euler sequence
2) calculate matrix
3) perform vector/matrix multiplication

Table E-1. The Attitude Matrix, A, for the 12 Possible Euler Angle Representations ($S \equiv$ sine, $C \equiv$ cosine, $1 \equiv x$ axis, $2 \equiv y$ axis, $3 \equiv z$ axis)

TYPE – 1 EULER ANGLE REPRESENTATION	MATRIX A	TYPE – 2 EULER ANGLE REPRESENTATION	MATRIX A
1 - 2 - 3	$\begin{bmatrix} C\psi C\theta & C\psi S\theta S\phi + S\psi C\phi & -C\psi S\theta C\phi + S\psi S\phi \\ -S\psi C\theta & -S\psi S\theta S\phi + C\psi C\phi & S\psi S\theta C\phi + C\psi S\phi \\ S\theta & -C\theta S\phi & C\theta C\phi \end{bmatrix}$	1 - 2 - 1	$\begin{bmatrix} C\theta & S\theta S\phi & -S\theta C\phi \\ S\psi S\theta & C\psi C\phi - S\psi C\theta S\phi & C\psi S\phi + S\psi C\theta C\phi \\ C\psi S\theta & -S\psi C\phi - C\psi C\theta S\phi & -S\psi S\phi + C\psi C\theta C\phi \end{bmatrix}$
1 - 3 - 2	$\begin{bmatrix} C\psi C\theta & C\psi S\theta C\phi + S\psi S\phi & C\psi S\theta S\phi - S\psi C\phi \\ -S\theta & C\theta C\phi & C\theta S\phi \\ S\psi C\theta & S\psi S\theta C\phi - C\psi S\phi & S\psi S\theta S\phi + C\psi C\phi \end{bmatrix}$	1 - 3 - 1	$\begin{bmatrix} C\theta & S\theta C\phi & S\theta S\phi \\ -C\psi S\theta & C\psi C\theta C\phi - S\psi S\phi & C\psi C\theta S\phi + S\psi C\phi \\ S\psi S\theta & -S\psi C\theta C\phi - C\psi S\phi & -S\psi C\theta S\phi + C\psi C\phi \end{bmatrix}$
2 - 3 - 1	$\begin{bmatrix} C\theta C\phi & S\theta & -C\theta S\phi \\ -C\psi S\theta C\phi + S\psi S\phi & C\psi C\theta & C\psi S\theta S\phi + S\psi C\phi \\ S\psi S\theta C\phi + C\psi S\phi & -S\psi C\theta & -S\psi S\theta S\phi + C\psi C\phi \end{bmatrix}$	2 - 1 - 2	$\begin{bmatrix} C\psi C\phi - S\psi C\theta S\phi & S\psi S\theta & -C\psi S\phi - S\psi C\theta C\phi \\ S\theta S\phi & C\theta & S\theta C\phi \\ S\psi C\phi + C\psi C\theta S\phi & -C\psi S\theta & -S\psi S\phi + C\psi C\theta C\phi \end{bmatrix}$
2 - 1 - 3	$\begin{bmatrix} C\psi C\phi + S\psi S\theta S\phi & S\psi C\theta & -C\psi S\phi + S\psi S\theta C\phi \\ -S\psi C\phi + C\psi S\theta S\phi & C\psi C\theta & S\psi S\phi + C\psi S\theta C\phi \\ C\theta S\phi & -S\theta & C\theta C\phi \end{bmatrix}$	2 - 3 - 2	$\begin{bmatrix} C\psi C\theta C\phi - S\psi S\phi & C\psi S\theta & -C\psi C\theta S\phi - S\psi C\phi \\ -S\theta C\phi & C\theta & S\theta S\phi \\ S\psi C\theta C\phi + C\psi S\phi & S\psi S\theta & -S\psi C\theta S\phi + C\psi C\phi \end{bmatrix}$
3 - 1 - 2	$\begin{bmatrix} C\psi C\phi - S\psi S\theta S\phi & C\psi S\phi + S\psi S\theta C\phi & -S\psi C\theta \\ -C\theta S\phi & C\theta C\phi & S\theta \\ S\psi C\phi + C\psi S\theta S\phi & S\psi S\phi - C\psi S\theta C\phi & C\psi C\theta \end{bmatrix}$	3 - 1 - 3	$\begin{bmatrix} C\psi C\phi - S\psi C\theta S\phi & C\psi S\phi + S\psi C\theta C\phi & S\psi S\theta \\ -S\psi C\phi - C\psi C\theta S\phi & -S\psi S\phi + C\psi C\theta C\phi & C\psi S\theta \\ S\theta S\phi & -S\theta C\phi & C\theta \end{bmatrix}$
3 - 2 - 1	$\begin{bmatrix} C\theta C\phi & C\theta S\phi & -S\theta \\ -C\psi S\phi + S\psi S\theta C\phi & C\psi C\phi + S\psi S\theta S\phi & S\psi C\theta \\ S\psi S\phi + C\psi S\theta C\phi & -S\psi C\phi + C\psi S\theta S\phi & C\psi C\theta \end{bmatrix}$	3 - 2 - 3	$\begin{bmatrix} C\psi C\theta C\phi - S\psi S\phi & C\psi C\theta S\phi + S\psi C\phi & -C\psi S\theta \\ -S\psi C\theta C\phi - C\psi S\phi & -S\psi C\theta S\phi + C\psi C\phi & S\psi S\theta \\ S\theta C\phi & S\theta S\phi & C\theta \end{bmatrix}$

Propagation with Euler Angles

Table E-2. Kinematic Equations of Motion for the 12 Possible Euler Angle Representations ($1 \equiv x$ axis, $2 \equiv y$ axis, $3 \equiv z$ axis; $\omega_1, \omega_2, \omega_3$ are components of the angular velocity along the body x, y, z axes.)

AXIS SEQUENCE		INDEX VALUES I	J	K	KINEMATIC EQUATIONS OF MOTION
TYPE 1	1 - 2 - 3	1	2	3	$\dot{\phi} = (\omega_I \cos\psi - \omega_J \sin\psi)\sec\theta$
	2 - 3 - 1	2	3	1	$\dot{\theta} = \omega_J \cos\psi + \omega_I \sin\psi$
	3 - 1 - 2	3	1	2	$\dot{\psi} = \omega_K - (\omega_I \cos\psi - \omega_J \sin\psi)\tan\theta$
	1 - 3 - 2	1	3	2	$\dot{\phi} = (\omega_I \cos\psi + \omega_J \sin\psi)\sec\theta$
	3 - 2 - 1	3	2	1	$\dot{\theta} = \omega_J \cos\psi - \omega_I \sin\psi$
	2 - 1 - 3	2	1	3	$\dot{\psi} = \omega_K + (\omega_I \cos\psi + \omega_J \sin\psi)\tan\theta$
TYPE 2	1 - 2 - 1	1	2	3	$\dot{\phi} = (\omega_K \cos\psi + \omega_J \sin\psi)\csc\theta$
	2 - 3 - 2	2	3	1	$\dot{\theta} = \omega_J \cos\psi - \omega_K \sin\psi$
	3 - 1 - 3	3	1	2	$\dot{\psi} = \omega_I - (\omega_K \cos\psi + \omega_J \sin\psi)\cot\theta$
	1 - 3 - 1	1	3	2	$\dot{\phi} = -(\omega_K \cos\psi - \omega_J \sin\psi)\csc\theta$
	3 - 2 - 3	3	2	1	$\dot{\theta} = \omega_J \cos\psi + \omega_K \sin\psi$
	2 - 1 - 2	2	1	3	$\dot{\psi} = \omega_I + (\omega_K \cos\psi - \omega_J \sin\psi)\cot\theta$