Project on Chandrasekhar Limit

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Preface

Present lecture note will try to cover....bla bla bla......

HR Diagram

Introduction

The Hertzsprung -Russell diagram is the one of the most important tool for the study of stellar evolution. Developed independently in 1911 by Ejnar Hertzsprung and by Henry Norris Russell in 1913, it plots the temperature of stars against their luminosity (theoritical HR diagram), or the color of the stars against their absolute magnitude (observational HR diagram).Depending on its initial mass, every star goes through specific evolutionary stages dictated by its internal structure and how it produces energy. Each of these stages corresponds to a change in the temperature and luminosity of the star, which can be seen to move to different regions on the HR diagram as it evolves. This reveals the true power of the HR diagram – astronomers can know a star's internal structure and evolutionary stage simply by determining its position in the diagram.

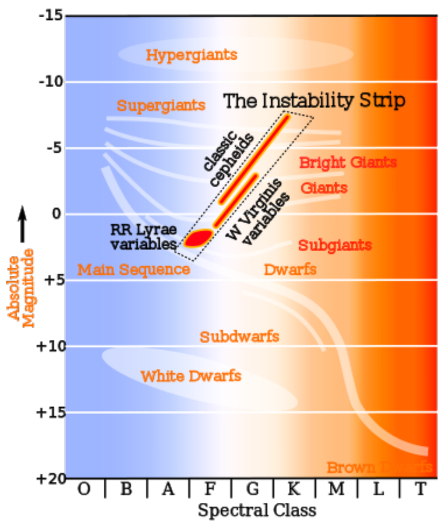

Figure 1: HR Diagram

Interpretation

Most of the stars occupy the region in the diagram along the line "Main Sequence".At the time when stars are found in the main sequence region are undergoing hydrogen fusion in their cores. The next concentration of the stars is on the horizontal branch (fusing helium in their cores and burning hydrogen in their shell surrounding the core.Another prominent feature is the Hertzsprung gap located in the region between A5 and G0 spectral type and between +1 and -3 absolute magnitudes (i.e., between the top of the main sequence and the giants in the horizontal branch). RR Lyrae variable stars can be found in the left of this gap on a section of the diagram called the instability strip. Cepheid variables also fall on the instability strip, at higher luminosities.The top right of the diagram is where the Blue Supergiants reside. The Red Giant stars are immediately down and to the right of the blue supergiants

HR diagrams can be used to roughly measure how far away a star cluster or galaxy by comparing the apparent magnitudes of the stars in the cluster to the absolute magnitudes of star with known distance.The observed group is then shifted in the vertical direction, until the two main sequences overlap. The difference in magnitude that was bridged in order to match the two groups is called the distance modulus and is a direct measure for the distance (ignoring extinction). This technique is known as main sequence fitting and is a type of spectroscopic parallax. Not only the turn-off in the main sequence can be used, but also the tip of the red giant branch stars.

1 Photon Gas

Photon gas is a gas like collection of photons which has many similar properties to conventional gases like hydrogen or neon. Photons are a part of family of Bosons , Particles that follow the Bose-Einstein statistics and with integer spin A gas of bosons with only one type of particle is uniquely described by three state functions such as the temperature, volume, and the number of particles. We can consider Light as a phton gas comprising of photons with energy-momentum relation as

$$\epsilon = pc$$

We know that the photon gas follow the Bose-Einstein statistics so the distribution function for the photons will be-

$$f = \frac{1}{e^{\beta(\epsilon)} - 1} \tag{1}$$

Now we can use the distribution function and energy momentum relation to calculate the macroscopic parameters like number of particles (N), total internal energy (U), and Pressure (P). Starting with the number of particles, it can be calculated as-

$$N = 2\int \frac{d^3x d^3p}{h^3} f(\epsilon)$$

Putting in the distribution function we get,

$$N = 2\int \frac{d^3x d^3p}{h^3} \frac{1}{e^{\beta(\epsilon)} - 1}$$

eqn. is multiplied by 2 to account the both (+1,-1) spins of the photons and we know that

$$\int d^3x = V$$

substituting the above relation in the eqn. we get

$$N = \frac{2V}{h^3} \int \frac{4\pi^2 pdp}{e^{\beta px} - 1}$$

$$Let \beta pc = x$$

$$\implies dp = \frac{kT}{c} dx$$

On substituting the values in the eqn. , we get-

$$N = \frac{8\pi(kT)^3}{h^3c^3} \int_0^\infty \frac{x^2}{e^x - 1} dx$$

The integral term in above equation is a special integral known as Reimann Zeta function , denoted by $\zeta(n)$. This function is defined as-

$$\zeta(n) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx$$

On comparing both the equations we can conclude -

$$\frac{N}{V} = \frac{8\pi(kT)^3}{h^3c^3} \zeta(3)\Gamma(3)$$

$$\implies \frac{N}{V} = 16\pi(\frac{kT}{hc})^3\zeta(3) \propto T^3$$

Hence from the above equation it can be concluded that the number density of a photon gas system is directly proportional to the **third power of Temperature(T)**.

Moving on to the calculation of Internal Energy of the photon gas. The energy of a single photon is the microscopic variable in this case. The total internal energy is defined as

$$U = 2\int \frac{d^3xd^3p}{h^3}\frac{\epsilon}{e^{\beta\epsilon}-1}$$
$$\implies U = \frac{2V}{h^3}\int 4\pi p^2 dp \frac{\epsilon}{e^{\beta\epsilon}-1}$$

we know,

$$\epsilon = pc$$

using this relation we get-

$$\frac{U}{V} = \frac{8\pi c}{h^3}\int \frac{p^3}{e^{\beta pc}-1}dp = \int u_V d\nu$$

Where

$$u_\nu d\nu = \frac{8\pi c}{h^3}\frac{p^3}{e^{\beta pc}-1}dp$$

substituting $p = \frac{h\nu}{c}$ and $dp = \frac{hd\nu}{c}$ in the above equation, we get

$$u_\nu d\nu = \frac{8\pi\nu^3 h}{c^3}\frac{d\nu}{e^{\beta h\nu}-1}$$

The term u_ν, in above equation represents the spectral distribution of energy in the black body radiation.

$$\begin{aligned} \frac{U}{V} &= \int u_\nu d\nu \\ &= \frac{8\pi c}{h^3}\int \frac{p^3dp}{e^{\beta pc}-1} \\ Let\beta pc &= x \\ \implies dp &= \frac{kT}{c}dx \end{aligned} \tag{2}$$

On substituting the above relation in the equation we get

$$\begin{aligned} \frac{U}{V} &= \frac{8\pi c}{h^3}(\frac{kT}{c})^4\int_0^\infty \frac{x^{4-1}}{e^x-1}dx \\ &= \frac{8\pi}{(hc)^3}(kT)^4\zeta(4) \propto T^4 \\ &= \frac{8\pi}{h^3c^3}(kT)^4\zeta(4) \end{aligned} \tag{3}$$

Here

$$\zeta(4) = \frac{\pi^4}{90}$$

Hence,

$$\frac{U}{V} = \frac{8\pi^5}{15h^3c^3}(kT)^4$$

Hence from the above equation it is conclusive that the energy density is directly proportional to forth power of temperature This relation goes a long way in deriving the Stefan-Boltzmann law of radiation.

2 Calculating the inside temperature of a star

Surface temperature of any star can be directly calculated by using either the Stefan-Boltzman law or Weins displacement law.In case of Sun it comes out to be around $(5x10^3K)$.However to calculate the temperature inside a star above mentioned laws can not be used as the temperature gradually increases as we move inside the star.So, here we are going to calculate the average temperature inside a star.

Let us take the average mass density :

$$\begin{aligned}\rho(r) \approx \langle\rho\rangle &= \frac{M}{\frac{4}{3}\pi R^3} \\ \langle\rho\rangle &= \frac{\int_0^R \rho(r)4\pi r^2 dr}{\int_0^R 4\pi r^2 dr} = \frac{M}{\frac{4}{3}\pi R^3}\end{aligned} \tag{4}$$

where ρ is the density, M is the mass of the object, R is its radius. Here we have considered a small sphere inside a big sphere,as its density is constant we can integrate over the radius from 0 to R to get the average mass density Now we have to calculate the Gravitational potential of the star Ω :

$$\begin{aligned}\Omega &= -\int_0^R \frac{GM(r)}{r}4\pi r^2\rho(r)dr \\ &\approx -\int_0^R G\frac{4}{3}\pi r^3\rho(r)4\pi r\rho(r) \\ &= -G\frac{4\pi}{3}4\pi\langle\rho\rangle^2[\frac{R^5}{5}] \\ &= -\frac{3}{5R}[\frac{4\pi}{3}R^3\langle\rho\rangle]^2\end{aligned} \tag{5}$$

Here G is the Gravitational constant

now using $M(r) = \frac{4}{3}\pi r^3\rho(r)$ and $\rho \approx \langle\rho\rangle$

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}$$

Now the internal energy U

$$U = \frac{3}{2}PV = \frac{3}{2}\int_0^R P(r)4\pi r^2 dr$$

here we have assumed the pressure to be the function of r

$$P = \frac{\langle\rho\rangle}{m}K_BT$$

where $m = mass of gas particles$

$$\begin{aligned}U &= \frac{3k_B}{2m}\int_0^R T(r)\langle\rho\rangle 4\pi r^2 dr \\ &= \frac{3}{2}\frac{k_B}{m}M\langle T\rangle\end{aligned} \tag{6}$$

$$\langle T\rangle = \frac{1}{M}\int_0^R 4\pi r^2 dr\langle\rho\rangle T(r) \tag{7}$$

Using the virial theorem we can relate the Total kinetic energy of a self gravitating body due to the motions of the constituent parts, U to the Gravitational Potential energy, Ω of the body.

Virial theorem

$$2U + \Omega = 0$$

$$\begin{aligned} -2(\frac{3}{2}k_B\langle T\rangle\frac{M}{m}) &= -\frac{3}{5}\frac{GM^2}{R} \\ \langle T\rangle &= \frac{1}{5}\frac{GMm}{k_B R} \end{aligned} \tag{8}$$

where $\langle\rho\rangle\frac{4}{3}\pi R^3 = M \implies R^3 = \frac{3M}{4\pi\langle\rho\rangle}$

$$\langle T\rangle = \frac{1}{5}\frac{Gm}{k}M\left(\frac{4\pi\langle\rho\rangle}{3M}\right)^{\frac{1}{3}} \tag{9}$$

Where $\langle T\rangle$ is directly proportional to M and $\langle\rho\rangle$ as follows-

$$\langle T\rangle \propto M^{\frac{2}{3}}\langle\rho\rangle^{\frac{1}{3}}$$

Calculation of the average temperature of the Sun :

$$m_H = 1.6\times 10^{-27}Kg$$
$$G = 6.67\times 10^{-11}m^3/kg/s^2, k_B = 1.38\times 10^-23m^2kgs^{-2}k^-1$$

values for the SUN :

$$M = 1.9\times 10^{30}kg, R = 6.9\times 10^8 m$$

$$\begin{aligned} \langle T\rangle &= \frac{1}{5}\frac{6.6\times 10^{-11}\times 1.9\times 10^{30}}{1.38\times 10^{-23}\times 6.9\times 10^8}\times 1.6\times 10^{-27} \\ &= \left(\frac{6.6\times 1.9\times 1.6}{5\times 1.38\times 10^{-23}}\right)\times\frac{10^{-8}}{10^{-15}} \end{aligned} \tag{10}$$

Hence the average temperature inside the Sun is

$$4\times 10^6 K$$

The Stellar Cycle

Introduction

In this section the main story of stellar cycle is covered.The stars are divided into two subtypes,small star(whose size <Mo(Mass of sun)). and Large Star.These large stars can be further divided into two subgroup of stars based on its size. One is stars with size (<20Mo) and other are stars with size(>20Mo).

Small Stars

In the late stage of Small Star all the hydrogen in the core of the stars gets convert into Helium.Thus due to absence of hydrogen nuclear fusion the gravitational force causes the core of the star to collapse and the outside of the core star expands. Thus as it expands it goes away from the core it decreases its temperature and thus it colour changes to Reddish.This stage of star is called Red Giant. At this steps the temperature of the core increases and thus it provides enough energy to star nuclear fusion of helium into carbon or oxygen. After that a stage comes when the helium also gets over. In this stage the outer surface of the star fades away and only the oxygen and carbon containing core is left behind. This core structure is similar to the size of the planet and this celestial body is called Planetary Nebulae. When this helium gets over the gravitational force further compress the core and due to the immense gravitational force the atoms inside the core breaks and the electrons are release. This electron produces degeneracy pressure in the counter direction of the Gravitational force. Thus it forms a stable structure called white dwarf.After that due to lack of source of energy the white dwarf shades away from our vision.

Big Stars

In the case of big stars when the hydrogen gets converted into helium then a red super giant is formed which makes the core of the star so hot and provide it with so much energy that it starts a chain of nuclear fusion reaction. The nuclear reaction stops at Iron which is a stable element. At this point the core of the star containing iron is very small and the outside of the star becomes so big that outside of the star explodes.The star in this stage is called Super Nova.In this stage of temperature and energy becomes so high that outside the core area the iron further gets nuclear fused to higher element like copper Nickel Zinc etc. Super Nova is one of the brightest things in the observable universe and it can also be seen with Naked Eyes.Now the iron fate of iron containing core is determined based on its mass.

2.1 Neutron Star

It occurs when the mass of the core star is between 1.4Mo and 2Mo. Due to the high mass of iron containing core the Gravitational force becomes so high that the electron degeneracy pressure cannot counter that strong Gravitational force.Thus it further collapse.At this point the nucleus breaks and the electron fuses with the proton to form neutron. So the core gets occupied by only neutron. Now this neutron produces degeneracy pressure in the counter direction of Gravitational force and it attains an equilibrium. Neutron star has huge pressure. Neutron Stars have huge magnetic field and it spins in a very random way. When two neutron star comes closer by spinning at one point the two neutron stars blasts each other which may result either in the spreading out of the various particle in Neutron Star and death of the neutron star or the formation of black hole. These huge collision produce ripples in the space time which forms Gravitational Waves and these

2.2 Black Hole

It happens when the mass of the core of the star is more than 2Mo. Here due to more high mass the Gravitational force is strong enough to supress even the neutron degeneracy pressure and it is believe that this high Gravitational force converts the Giant Star into a black hole with infinite energy and pressure. The gravitational pull of black distorts the space time curvature in such a way that even light cannot pass through it.

White Dwarf

Introduction

White Dwarf is a celestial body which is formed by the stars of the <Mass of the sun (Mo) at their last stage.It is mainly the core of the star .The White Dwarf remain in it equilibrium because of the electron degeneracy pressure which counter the gravitational force . As there are no source of Energy in White Dwarf the temperature inside White Dwarf cools down.The White Dwarf is made up of waste material of star formed during nuclear fusion reaction thus the core is made up of either carbon or oxygen. This elements causes the shinning of the White Dwarf at its initial phase. As the atom inside the White Dwarf cannot do further Nuclear Fusion. So this White Dwarf Gradually looses its temperature and it generally converts into Black Dwarf(It basically gets removed from our Vision).

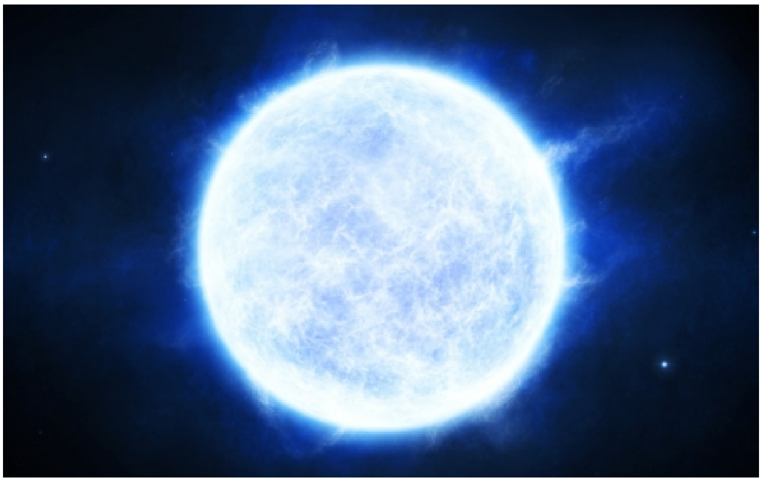

Figure 2: Figure: White Dwarf

Equilibrium of White Dwarf

White Dwarf maintains its equilibrium state due to the Electron Degeneracy Pressure. This pressure creates a force in the outward direction of the star core and it balances the Gravitational Force. Now using Statistical Mechanics we can find various properties of White Dwarfs like its number density, Internal Energy, Pressure. We must remember that Electrons are Fermions.So it obeys Fermi-Dirac Equation which $f(E) = 1/e^{(E-U)/KT} + 1$.

Where K= Boltzman Constant

T= Temperature

E= Energy

U= Fermi Energy

Number Density

Now using these relation we can say the number density of White Dwarf

We know $n = N/V$

Where $n = numberDensity$

$N = No.of particles$

$V = Volume$
To know the Number of Particles we should integrate the Fermions Distribution over the Phase Space

$$N = g/h^3 \int d^3p.d^3x 1/e^{(E-U)/KT} + 1$$

Now $d^3x = V$

$$N = gV/h^3 \int d^3p.1/e^{(E-U)/KT} + 1$$

Here g is the spin Degeneracy of Electron

$h = PlanksConstant$

$$g = 2s + 1$$

For electron

$$s = 1/2$$

So for electron

$$g = 2$$

so

$$N = 2V/h^3 \int d^3p.1/e^{(E-U)/KT} + 1$$

and

$$n = N/V$$

so

$$n = 2/h^3 \int d^3p.1/e^{(E-U)/KT} + 1$$

Now when T tends to 0K ,We have three cases

1. When E>U

$$\frac{1}{e^{(E-U)/KT} + 1} = 0$$

$n = 0$

2.When E<U

$$\frac{1}{e^{(E-U)/KT} + 1} = 1$$

.

So

$$n = 2/h^3 \int d^3p.$$

$$n = 2/h^3 \int 4\pi p^2 dp$$

$$n = 2/h^3 4\pi pf^3/3$$

When $E = U$

$$\frac{1}{e^{(E-U)/KT} + 1} = 1/2$$

$$n = 2/2h^3 \int 4\pi p^2 dp$$

$$n = 1/h^3 4\pi pf^3/3$$

Where pf is the fermi energy.

3 Internal Energy

Internal Energy(U) =

$$gV/h^3 \int d^3pE1/e^{1/(E-u)/KT} + 1$$

At $T = 0K$

1.When $E > U$

$$f(E) = 0$$

Thus

$$U = 0$$

2. When $E < U$

$$f(E) = 1$$

Thus

$$U = gV/h^3 \int Ed^3p$$

$$U = gV/h^3 \int 4\pi p^2 Edp$$

For Non Relativistic Case

$$E = p^2/2m$$

$$U = gV/h^3 \int 4\pi p^2 p^2/2mdp$$

$$U = gV/h^3 \int 4\pi p^4/2mdp$$

$$U = gV/h^3 4\pi p^5/10m$$

Now for Relativistic Case

$$E = \sqrt{p^2c^2 + m^2c^4}$$

$$U = gV/h^3 \int 4\pi p^2 \sqrt{p^2c^2 + m^2c^4}dp$$

For Ultra Relativistic Case

$$E = pc$$

$$U = gV/h^3 \int 4\pi p^2 pcdp$$

$$U = gV/h^3 \int 4\pi p^3 cdp$$

$$U = gVc/h^3 4\pi p^4/4$$

3. When $E = U$

$$f(E) = 1/2$$

$$U = gV/2h^3 \int 4\pi p^2 Edp$$

Now for Non Relativistic Case

$$U = gV/2h^3 \int 4\pi p^2 p^2/2mdp$$

$$U = gV/2h^3 4\pi p^5/10m$$

Now for Relativistic Case-

$$U = gV/2h^3 \int 4\pi p^2 \sqrt{p^2c^2 + m^2c^4}dp$$

Now for Ultra relativistic Case

$$U = gV/2h^3 \int 4\pi p^2 pcdp$$

$$U = \frac{gVc}{2h^3}\pi p^4$$

4 Pressure

$$P = g/h^3 \int (d^3p(pv/3)$$

$$P = g/h^3 \int (4\pi p^2(pv/3)dp$$

Now Non Relativistic Case -

$$velocity(v) = p/m$$

$$P = g/h^3 \int (4\pi p^4/3m)dp$$

$$P = g/h^3(4\pi p^5/15m)$$

Now Relativistic Case -

$$Velocity(v) = pc^2/\sqrt{p^2c^2 + m^2c^4}$$

$$P = g/h^3 \int (4\pi p^2(p^2c^2/\sqrt{p^2c^2 + m^2c^4}/3)dp$$

Now for Ultra Relativistic Case -

$$velocity(v) = c$$

$$P = g/h^3 \int 4\pi p^2 c/3dp$$

$$P = gc/h^3 4\pi p^3/9$$

5 Calculating Chandrasekhar Limit :

We will be calculating the Chandrasekhar limit using 2 different approaches -

N = Total no. of electrons in the collapsing stars
M = Total Mass of the Star

$$M = Nm_e + 2Nm_p \approx 2Nm_p$$
$$m_p >> m_e$$

$$\rho = \frac{M}{V} = \frac{Nm_p}{V} = nm_p \tag{11}$$
$$n : \text{number density of particles}$$

As the star is continuously compressing n is getting larger but there is a limit to the compression set by **Heisenberg Uncertainty Principle :**

$$\begin{aligned}\Delta x \Delta p &\approx \hbar \\ \Delta x m_e \Delta v &= \hbar \\ \Delta x m_e v &= \hbar \\ v &= \frac{\hbar}{m_e \Delta x}\end{aligned} \tag{12}$$

$$\text{Volume Available for each particle} = \tfrac{V}{N}$$

$$\begin{aligned}\Delta x &\approx \left(\frac{V}{N}\right)^{\frac{1}{3}} \\ v &= \frac{\hbar}{m_e}\left(\frac{N}{V}\right)^{\frac{1}{3}} \\ v &= \frac{\hbar}{m_e} n^{\frac{1}{3}}\end{aligned} \tag{13}$$

In a classical gas for **Non-Relativistic Case** -

$$\frac{1}{2}mv^2 \approx \frac{3}{2}k_B T \approx nk_B T$$

similarly

$$\begin{aligned}P &\approx nm_e v^2 \\ &\approx nm_e \left[\frac{\hbar}{m_e} n^{\frac{1}{3}}\right]^2 \\ &\approx \frac{\hbar^2}{m_e} n^{\frac{5}{3}} \\ &= \frac{\hbar^2}{m_e m_p^{\frac{5}{3}}} \rho^{\frac{5}{3}}\end{aligned} \tag{14}$$

Stellar Equations -

$$\begin{aligned}\frac{dM}{dr} &= 4\pi r^2 \rho \\ \frac{dP}{dr} &= -\frac{GM}{r^2}\rho\end{aligned} \tag{15}$$

Using above equations we can find the Gravitational pressure to be-

$$P_G = \alpha \frac{GM^2}{R^4} \tag{16}$$

Now equating the gas pressure and the gravitational pressure :

$$\begin{aligned} \alpha \frac{GM^2}{R^4} &= \frac{\hbar^2}{m_e m_p^{\frac{5}{3}}} \left[\frac{M}{\frac{4}{3}\pi R^3} \right]^{\frac{5}{3}} \\ R &= \frac{\hbar^2}{G m_e m_p^{\frac{5}{3}}} M^{-\frac{1}{3}} \end{aligned} \tag{17}$$

In a classical Gas for **Non-Relativistic Case** -

$$R \propto M^{-\frac{1}{3}}$$

In a classical Gas for **Ultra Relativistic Case** -

$$\begin{aligned} K.E &= pc \\ P &\approx nK.E \\ P &\approx npc \\ &\approx n m_e v c \\ &\approx n m_e \frac{\hbar}{m_e} n^{\frac{1}{3}} \\ n &= \frac{N}{V} = \frac{3M}{8 m_p \pi R^3} \\ P &= \hbar c \left(\frac{3M}{8 m_p \pi R^3} \right)^{\frac{4}{3}} \end{aligned} \tag{18}$$

We know from (16) that -

$$P_G = \alpha \frac{GM^2}{R^4}$$

now equating both the gravitational pressure and electronic degeneracy pressure we get-

$$\begin{aligned} \alpha \frac{GM^2}{R^4} &= \hbar c \left(\frac{3M}{8 m_p \pi R^3} \right)^{\frac{4}{3}} \\ M &= \left[\left(\frac{\hbar c}{\alpha G} \right)^3 \left(\frac{3}{8 m_p \pi} \right)^4 \right]^{\frac{1}{2}} \end{aligned} \tag{19}$$

6 For Relativistic Case

We will first find the value of Fermi Momentum

$$FermiMomentum(P_f) = (\frac{3n}{\pi})^{1/3}h \tag{20}$$

$$n = \frac{\rho}{2mp} \tag{21}$$

m_p = mass of proton

n is the Number Density

Now Electron Degenarcy Pressure of White Dwarf is

$$Pressure(P_e) = \frac{gV}{h^3}\int_0^{Pf} \frac{4\pi p^2 pv}{3}dp \tag{22}$$

For relativistic case

$$v = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} \tag{23}$$

On combining equation 19 and 20

$$P_e = \frac{8\pi V}{h^3}\int_0^{P_f} \frac{(p^4 dp)}{\sqrt{(p/mc)^2 + 1}} \tag{24}$$

Let

$$p/mc = sinh\theta = x$$

$$dp = mc(cosh\theta d\theta)$$

$$P_e = 8\pi V m^4 c^5/h^3 \int_0^{\theta_f} sinh^4\theta\theta d\theta \tag{25}$$

$$P_e = 8\pi V m^4 c^5/h^3 f(x) \tag{26}$$

$$f(x) = \int (sinh^4\theta) = x\sqrt{x^2+1}(2x^2-3) + 3arcsinhx$$

$$wherex = P_f/mc$$

By combining equation 18 and 19

$$x = (9M/(64\pi^2 mp)^{1/3} h/mcR \tag{27}$$

Also

Now this f(x) goes into two different forms in two extreme cases

$$f(x) = 8/5x^5 - 4/7x^7 + 1/3x^9 - 5/22x^11 + ...whenx <<<< 1$$

f(x)= $2x^4 - 2x^2 + 3(ln2x - 7/12) + 5/4x^{-2} + ...whenx >>>>> 1$(28)

In Relativitistic case x»»»1 so

$$f(x) = 2x^4 - 2x^2 + 3(ln2x - 7/12) + 5/4x^{-2} + ...$$

For simlplicicity we can also assume

$$f(x) = 2x^4 - 2x^2$$

$$P_e = 8\pi V m^4 c^5 2(x^4 - x^2)/h^3 \tag{29}$$

$$P_e = 8\pi V m^4 c^5 2x^2(x^2-1)/h^3 \tag{30}$$

Now Gravitational Prssure is

$$P_g = \alpha GM^2/R^4 (Equatiom 16)$$

Now equating Electron Degenarcy Pressure = Gravitational Pressure

$$P_e = P_g$$

$$\alpha GM^2/R^4 = 8\pi V m^4 c^5 2x^2(x^2-1)/h^3 \tag{31}$$

By combining equation 25 and 29 and simplifying we get

$$R \approx \frac{(9\pi)^{1/3}h}{2mc}\left(\frac{M}{m_p}\right)^{1/3}\sqrt{1-\left(\frac{M}{M_o}\right)^{2/3}} \tag{32}$$

$$M_o = (9/64)(3\pi/\alpha^3)^{1/2}\frac{(hc/G)^{3/2}}{m_p^2} \tag{33}$$

Now for M>Mo there are no real solutions so we can say that white dwarf does not exist for M>Mo.
Now on calculating equation Mo we get

$$M_o \approx 10^{33} g$$

But after detailed study by Dr Chandrasekhar and comparing with the mass of sun (M_{Sun})

$$M_o = \frac{5.75}{\mu_e^2} M_{Sun}$$

μ_e is the degree of ionisation of the fermi gas which is 2
Thus

$$M_o = \frac{5.75}{4} M_{Sun} \approx 1.44 M_{Sun} \tag{34}$$

Magnetic Field in White Dwarf-

White Dwarfs have a very high density and rotate at high angular velocity and are highly magnetised as well. Observations have shown that the surface magnetic field of the white dwarf can be as high as 10^9G, According to Virial Theorem arguments the magnetic field in the centre of the white dwarf can go upto 10^{13}G. If the central magnetic field is determined through analytical calculations in both Newtonian and General Relativity it comes out to be around 10^{12-16}G.

Effect of Magnetic field in the Chandrasekhar Limit

The Magnetic Field in the White Dwarfs can magnificiently effect the Chandrasekhar Limit. The Magnetic field can contribute in the Electron Degenerated gas pressure and hence can change the Chandrasekhar limit. The Magnetic field can change the momentum of the electrons which will impact the pressure against the Gravitational pressure.

There can be different ways to incorporate the magnetic field in the calculation of Chandrasekhar Limit. We will be using Quantum mechanical way to include it.We consider the magnetic field to be Quantized in different Landau Levels and magnetic field to in one direction only.

We will be doing the calculations for the LLL(Low Landau Levels) for now:
l = Landau Level

Magnetic field is considered to be in z direction,pressure in the x and y direction will be same due to symmetry while the pressure in the z direction will be different because of the different motion of particles by the virtue of magnetic field.

Non relativistic case

$$\begin{aligned} P_x &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_x}{m}\right)\left(\frac{p_x}{2\pi}\right) dp_z \\ &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \left(\frac{leB}{m}\right) \frac{p_{zmax.}}{2\pi} \\ &= \frac{(eB)^2}{2\pi^2 m} \sum_0^{l_{max.}} l p_{zmax.} \\ &= \frac{(eB)^2}{2\pi^2 m} \sum_0^{l_{max.}} l\sqrt{\mu^2 - 2leB - m^2} \end{aligned} \tag{35}$$

$$\begin{aligned} P_z &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_z}{m}\right)\left(\frac{p_z}{2\pi}\right) dp_z \\ &= \sum_0^{l_{max.}} \frac{eB}{6\pi^2 m} \left(p_{zmax.}\right)^3 \\ &= \sum_0^{l_{max.}} \frac{eB}{6\pi^2 m} \left(\mu^2 - 2leB - m^2\right)^{\frac{3}{2}} \end{aligned} \tag{36}$$

Ultra-relativistic case-

$$
\begin{aligned}
P_x &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_x}{m}\right) \left(\frac{p_x}{\sqrt{p_z^2 + 2leB}}\right) dp_z \\
p_x &= \sqrt{2leB} \\
P_x &= \sum_0^{l_{max.}} \frac{le^2 B^2}{\pi} \int_0^{p_{max}} \frac{dp_z}{\sqrt{p_z^2 + 2leB}} \\
&= \frac{e^2 B^2}{\pi} \sum_0^{l_{max.}} l \ln\left(\sqrt{p_{zmax.}^2 + 2leB} + p_{zmax.}\right) \\
&= \frac{e^2 B^2}{\pi} \sum_0^{l_{max.}} l \sinh^{-1}\left(\frac{p_{zmax.}}{2leB}\right)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
P_z &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_z}{m}\right) \left(\frac{p_z}{\sqrt{p_z^2 + 2leB}}\right) dp_z \\
&= \frac{eB}{\pi} \sum_0^{l_{max.}} \frac{p_z^2}{\sqrt{p_z^2 + 2leB}} \\
&= \frac{eB}{\pi} \sum_0^{l_{max}} p_z \int_0^{p_{zmax.}} p_z \left(1 + \frac{2leB}{p_z^2} + ...\right) dp_z \\
&= \frac{eB}{\pi} \sum_0^{l_{max}} p_z \left(\frac{p_{zmax.}^2}{2} + 2leB \ln\left(p_{zmax.}\right)\right) \\
&= \frac{eB}{\pi} \left[\frac{p_{zmax.} l_{max.}}{2} + eBl(l+1) \ln p_{zmax.}\right]
\end{aligned} \tag{38}
$$

Relativistic case-

$$
\begin{aligned}
P_x &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_x}{m}\right) \left(\frac{p_x}{\sqrt{p_z^2 + m^2 + 2leB}}\right) dp_z \\
p_x &= \sqrt{2leB} \\
P_x &= \sum_0^{l_{max.}} \frac{le^2 B^2}{\pi} \int_0^{p_{max}} \frac{dp_z}{\sqrt{p_z^2 + m^2 + 2leB}} \\
&= \frac{e^2 B^2}{\pi} \sum_0^{l_{max.}} l \ln\left(\sqrt{p_{zmax.}^2 + m^2 + 2leB} + p_{zmax.}\right) \\
&= \frac{e^2 B^2}{\pi} \sum_0^{l_{max.}} l \sinh^{-1}\left(\frac{p_{zmax.}}{2leB + m^2}\right)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
P_z &= \sum_0^{l_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{zmax}} \left(\frac{p_z}{m}\right) \left(\frac{p_z}{\sqrt{p_z^2 + m^2 + 2leB}}\right) dp_z \\
&= \frac{eB}{\pi} \sum_0^{l_{max.}} \frac{p_z^2}{\sqrt{p_z^2 + m^2 + 2leB}} \\
&= \frac{eB}{\pi} \sum_0^{l_{max}} p_z \int_0^{p_{zmax.}} p_z \left(1 + \frac{2leB + m^2}{p_z^2} + ...\right) dp_z \\
&= \frac{eB}{\pi} \sum_0^{l_{max}} p_z \left(\frac{p_{zmax.}^2}{2} + (m^2 + 2leB) \ln\left(p_{zmax.}\right)\right) \\
&= \frac{eB}{\pi} \left[\frac{p_{zmax.} l_{max.}}{2} + eBl(l+1) \ln p_{zmax.} + m^2 \ln p_{zmax.} l_{max.}\right]
\end{aligned} \tag{40}
$$

Neutron Star

Introduction

Neutron star is the result of the supernova (explosion) resulted by the gravitational collapse of a super-massive star after it exhausted its fuel in nuclear fusion.The pressure inside the neutron star generated by the radiation and degeneracy of the electron is less the gravitational pressure which results in the fusion of electron and proton in the core to form neutrons and which highly increases the density of the star and atomic mass as well.For a fact, a sugar cube of neutron star matter will weigh more than all the humanity!!.Another measure of the enormous gravity of the neutron star can be stated as the escape velocity of a rocket from a neutron star is more the half of the speed of light!!.Though most of the core of the neutron star is composed of neutron resulted by the fusion of electron and proton but still there are electrons and protons present in the core in company of neutrons.The matter present in the vary core of the neutron star is known as super fluid neutrons electrons and superconducting protons.

Pulsars

Radio astronomers were the first to discover celestial objects which correspond to the neutron stars hypothesized by Baade and Zwicky calculated to be possible by Landau and by Oppenheimer and Volkoff. In 1967 Bell and Hewish discovered regularly pulsing radio sources, ultimately to be called **Pulsar**.

Upon further investigation, pulsars were deduced by Gold to be magnetized spinning neutron stars. Such a star can apparently produce a rotating beam of radiation which leads to a series of regular spaced pulses when observed by anyone who happens to lie in the path of sweeping beam.

In 1969, Goldreich and Julian pointed out that the combination of rapid rotation plus strong magnetic field must, by Maxwell's equations, induce strong electric fields near the surface of the star. The electric fields should force electric charges to flow from the surface of the star. It is now thought that charged particles may flow out of the magnetic polar caps of the neutron star, essentially parallel to the magnetic field lines (figure 2). The acceleration of the charged particles as they try to follow the curved trajectories required by the magnetic field structure will cause them to radiate. The high energies and densities of the resulting radiation field may lead to the creation of a "**pair plasma**" in the space surrounding the star, where electrons and positrons are created and annihilated in great profusion. Gamma ray astronomers hope to detect the line photons resulting from this annihilation to test the validity of these models.

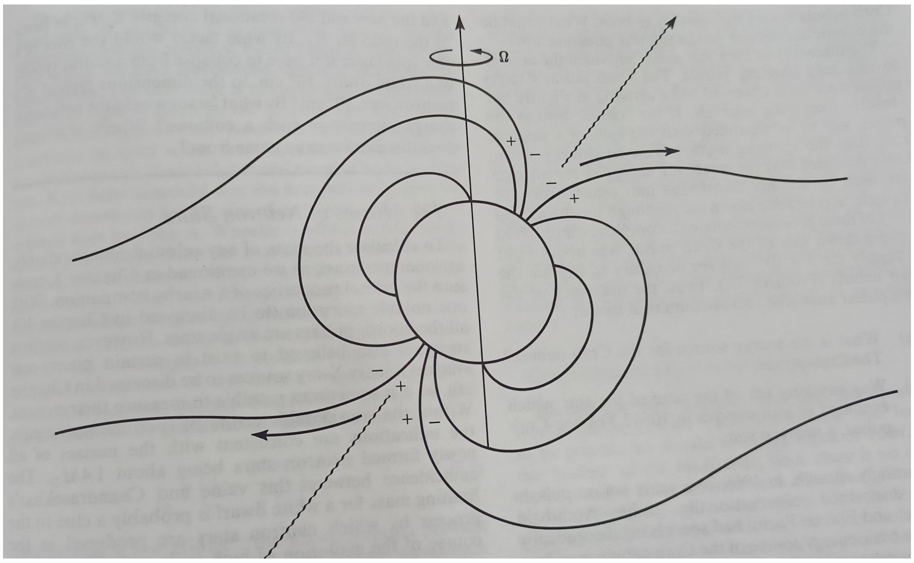

Figure 3: Magnetosphere structure for Pulsar

Inside Neutron Star

Neutron stars star have a solid crust over liquid core.The crust are extremely hard the outermost crust is made up of ions that are formed during Supernova Here the the gravity squeezes the nuclei into crystal lattice with sea of electron flowing through them .Now the deeper layers the gravity squeezes the nucleus closer together.It is composed of Super Fluid Neutrons. The concentration of proton decreases and neutron increases.At the base of the crust the nuclear are squeezed by Gravitational Pull so high that it touch to touch. This phase is called Gnochi Phase. Proton and Neutron makes long cylinder like structure. This is called Spaghetti Phase and Lasagna Phase. This phases are named so because millions of Protons and Neutron come together due to high gravitational Pull to make this type of Structure.This type of structure are called Nuclear Pasta. This structure are highly dense and almost unbreakable. Beneath the Pasta their is core. The outer core is made up of Superconducting Proton. The inner core is one of the mysterious part of Neutron Star. Scientist are still unaware about the components and it properties and composition. It is believe that the pressure inside the core is so high that the Neutrons breaks into Up Quarks and Down Quarks and they are hold by Gluons particle and it together forms a Plasma State named Quark Gluon Plasma. It is also believed that if one goes more in the core the pressure increases which results in the production of different types of Quarks known as Strange Quarks.But some Scientist believes that the pressure due to Gravity inside core of Neutron Star is not sufficient to form Quarks.So that is why the actual composition of the core of Neutron star is till a matter of debate.

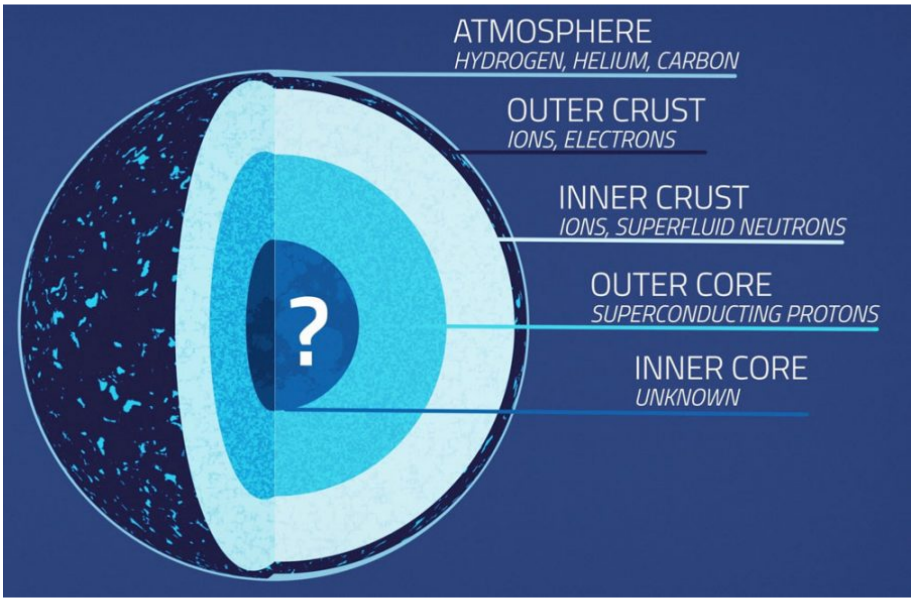

Figure 4: Layers of Neutron Star

7 Energy Released By Pulsar

At Initial Phases of Pulsar the spin is high but its spin reduces as time passes by. The loss of Spin is due to conversion of Rotational Kinetic Energy of Pulsar into various others forms of Energy. Let us Consider the Pulsar inside Crab Nebula

We Know Rotational Kinetic Energy (K.E)=$1/2I\omega^2$

Also $\omega = 2\pi/T$

K.E.= $1/2I(2\pi/T)^2$

On putting values Initial Kinetic Energy =$2 * 10^{42}$ Joule

I= Moment of Inertia ($I = 2/5MR^2$)

T = Time Period

Now it is observed that the Initial T of Pulsar of Crab Nebula = 0.33s

and also the Rate of Change of Time Period =(dT/dt)= $10^{-12.4}$

So d(K.E)/dt= $-1/2I(4\pi^2 dT/dt/T^3)$

On putting Values $d(K.E)/dt = 4 * 10^{31}$

Thus energy loss per second $= 2 * 10^{42}/4 * 10^{31} = 2 * 10^{-12}$

Therefore percentage loss $= 2 * 10^{-10}$

$$
\begin{aligned}
p_x^2 + p_y^2 &= 2neB \\
P_x &= \sum_{n=0}^{n_{max}} \frac{dp_x dp_y}{(2\pi)^2} \int_0^{p_{max}} \frac{p_x}{m}\frac{p_x}{2\pi} dp_z
\end{aligned} \tag{41}
$$

Now equating the Gravitational pressure in x direction with above pressure-

$$
\begin{aligned}
\frac{(eB)^2}{2\pi^2 me}\hbar\left(3\pi^2\frac{N}{V}\right)^{\frac{1}{3}} &= \alpha\frac{GM^2}{3R^4} \\
R &= \left(\frac{\alpha G}{3K}\right)^{\frac{1}{3}} M^{\frac{5}{9}}
\end{aligned} \tag{42}
$$

similarly for z direction

$$
R = \frac{16\alpha G\pi m^2}{3eB} M \tag{43}
$$

Considering Full Anisotropy of Stars

8 Lowest Landau Level

In lowest Landau Level the Pressure along x axis and y axis vanishes i.e, P_x and P_y is 0. and its P_z component only have its impact. Its shape can be thought of a cylinder in z axis having infinitely large height along z axis. So the equation of P_z gets modified

$$P_z = \frac{meB}{\pi} ln(\frac{3M}{8m_p ReB}) + \frac{1}{2m_e}(\frac{3M}{8m_p ReB})^2$$

Also due to 1-D The Gravitational Pressure becomes

Assumption : The Gravitational pressure is measured w.r.t to an object having 1 unit mass a unit far from the L.L.L star also the lengh of the 1D Star is also R

$$P_G = \alpha \frac{GM}{a(a+R)}$$

As

$$P = \frac{dE}{dV}$$

Now for 1 D

$$dV = dR$$

thus

$$P = \frac{dE}{dR}$$

Thus

$$P_G = \alpha \frac{GM}{a(a+R)}$$

Equating the two equation we get

$$\frac{meB}{\pi} ln(\frac{3M}{8m_p R^3 eB}) + \frac{1}{2m_e}(\frac{3M}{8m_p R^3 eB})^2 = \alpha \frac{GM_2}{3R^4}$$

The star in Lowest Landau Level will act as black Hole as its Gravitational force increases as we go towards it and when one reach to the LLL compact star the Gravitational force and pressure goes to infinity like that of a black hole.

9 For First Landau Level

For First Landau Level we have the equate the degeneracy pressure caused due to Px Py and Pz and equate it with the gravitational pressure . Now as we know due to Quantum Hall effect the pressure along Px and Py is same but Pz is different. If we assume the the white dwarf to be spherical.

So the Gravitational Pressure of White Dwarf is :

$$P_G = \alpha \frac{GM}{R^4}$$

Degenaracy Pressure

We know that

$$P_x = \frac{(eB)^2}{(m_e \pi)^2} (ln\sqrt{p_z^2 + m^2 + 2eB} + p_z)$$

According to Symmetry we know that P_x=P_y

Now

$$P_z = \frac{eB}{m_e \pi} (\frac{p_z^2}{2} + (m^2 + 2eB) ln(p_z)$$

Now total Degenaracy Pressure equals

$$P_x + P_y + P_z = 2\frac{(eB)^2}{(m_e \pi)^2} (ln\sqrt{p_z^2 + m^2 + 2eB} + p_z) + \frac{eB}{m_e \pi} (\frac{p_z^2}{2} + (m^2 + 2eB) ln(p_z)) = \alpha \frac{GM}{R^4}$$

$$p_z = \frac{(2\pi)^2 (\lambda_e)^3 m_e c^2}{2Bg_n} n_e$$

Also

n_e= Number Density of electron

$n_e = \frac{N}{V}$ (here N is constant for a particular Landau Level)