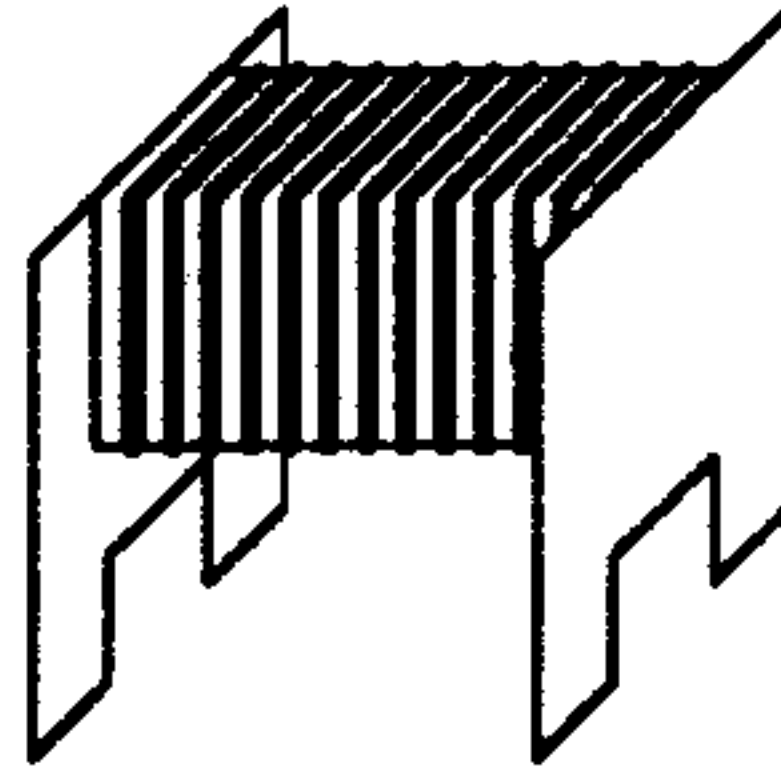


Figure 2: Scheme of electromagnet



## 10 Solution of two problems of 10<sup>th</sup> IYPT

### 10.1 Coin

Hynek Nemec - Zdenek Kluiber

#### Problem

From what height must a coin with heads up be dropped, so that the probability of landing with heads or tails up is equal?

#### 10.1.1 Introduction

The dropping or throwing of a coin is one way to decide some disputes. It is known, that if a coin is dropped from a great height, the probability of landing with heads up or tails up is equal. In the following article we will show you the lowest height to obtain such results.

If we drop the coin by hand it is very difficult to measure the height of center of mass. For this reason we used an electromagnet with an adapter from a thin metal sheet (see fig. 2). The coin we used in our experiments was the Czech 5 Kč; we let it fall to a board with syntethic surface. The parameters were the following: thickness of coin – 1.85 mm, radius of coin – 11.5 mm, mass of coin – 4.87 g, coefficient of friction between coin and board – 0.12 and coefficient of restitution between board and coin  $\approx 0.88$  for heights around 20 mm.

#### 10.1.2 Distribution of initial angular velocities and initial angles

It is impossible to drop the coin in the exact same way, – during every experiment the initial angular velocity is different. We determined

Figure 3: Apparatus for the measurement of angular velocities

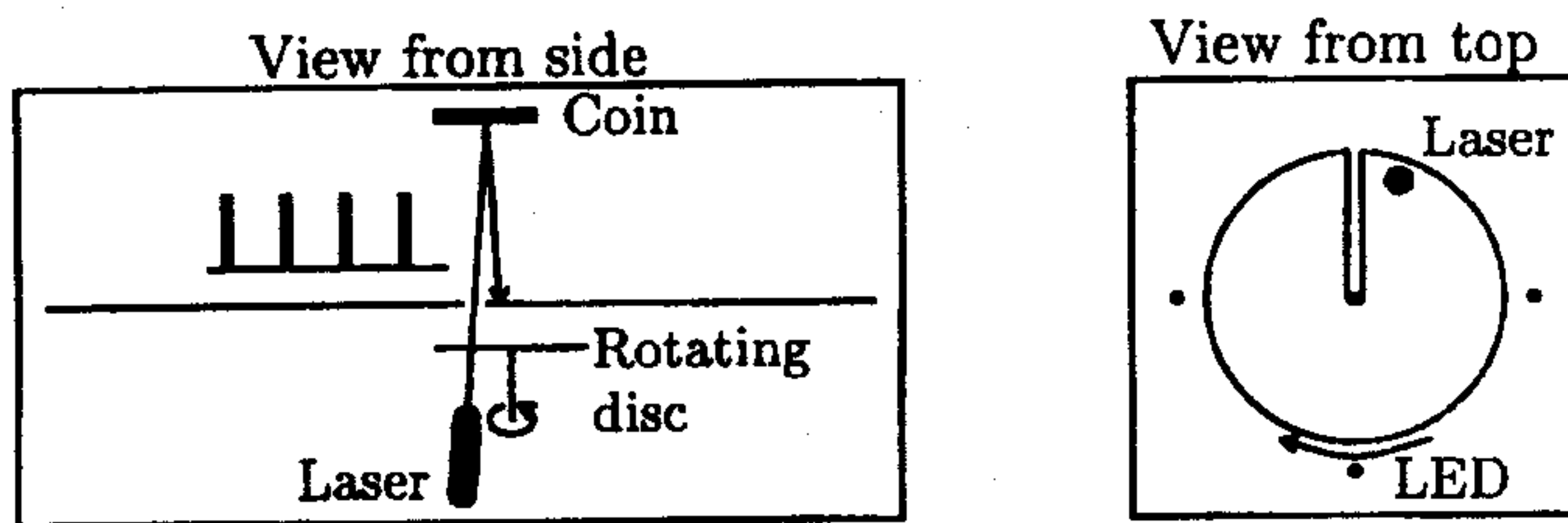


Figure 4: Example of the photo for measurement of distributions

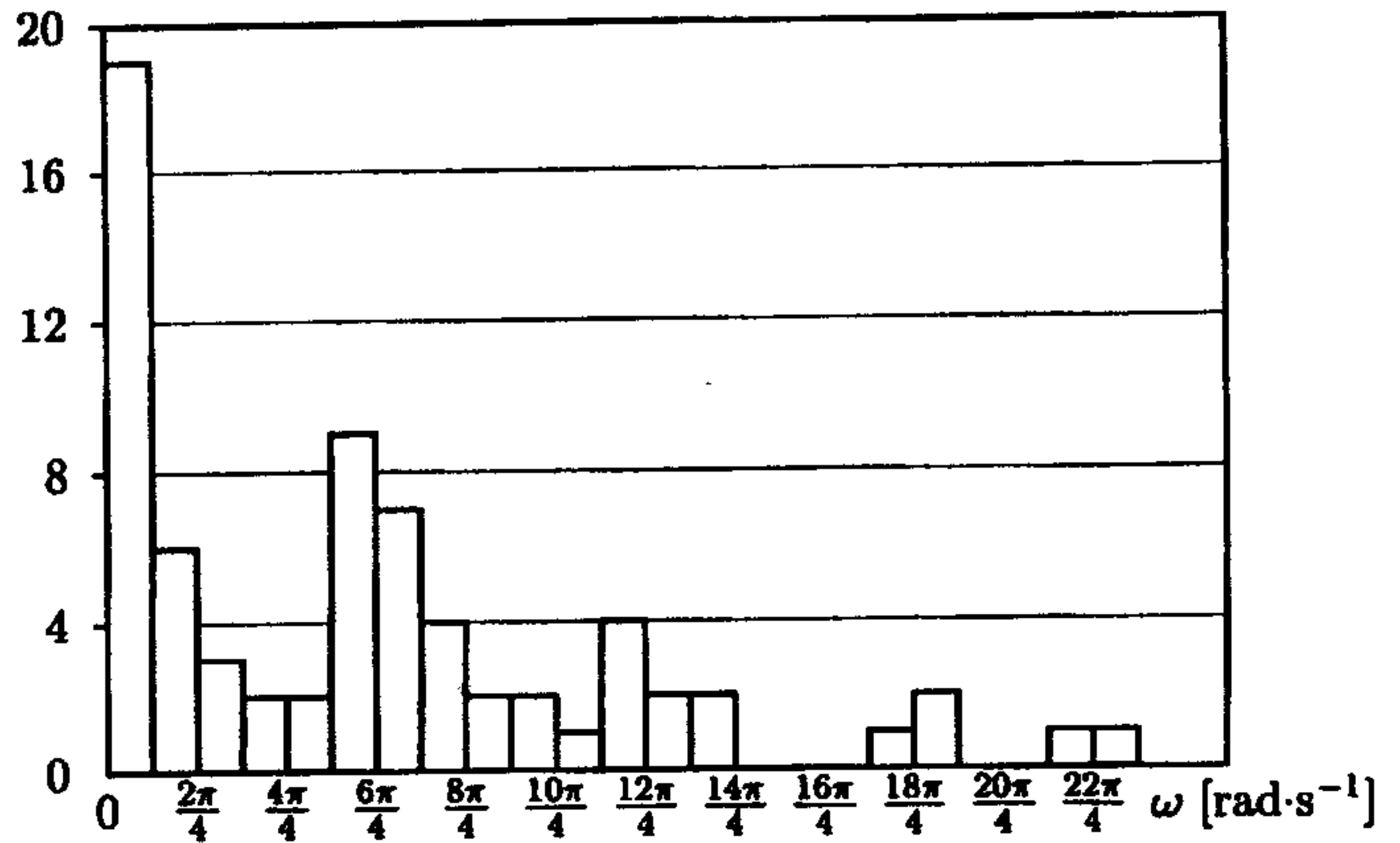


initial angular velocities experimentally. Scheme of apparatus is in fig. 3: the laser ray directed towards the coin was interrupted by rotating a disc (100 rt/s) with a slot – we got short pulses with a frequency of 100 Hz. We took pictures of tracks of rays reflected from the coin (there was a reflective foil on one side of the coin) to a board with a marked dimension (by light emitting diodes). By simple computation we obtained, from displacements of neighbouring tracks, an angular velocity of the coin.

Accuracy of this method depends mostly on the surface quality of the coin and also on the accuracy of light marks on the board and on resolution of photos. An example of a typical photo is in fig. 4. Distribution of initial angular velocities is represented in fig. 5.

It is evident from photos, that some first tracks lie evenly in line (10

Figure 5: Distribution of initial angular velocities



tracks on 100 Hz is equivalent with 5 cm of height, 15 with 11 cm) – it isn't possible to observe the influence of bypassing on photos. It is also evident, that at this time the coin is falling in one plane.

### 10.1.3 Impact of coin to board

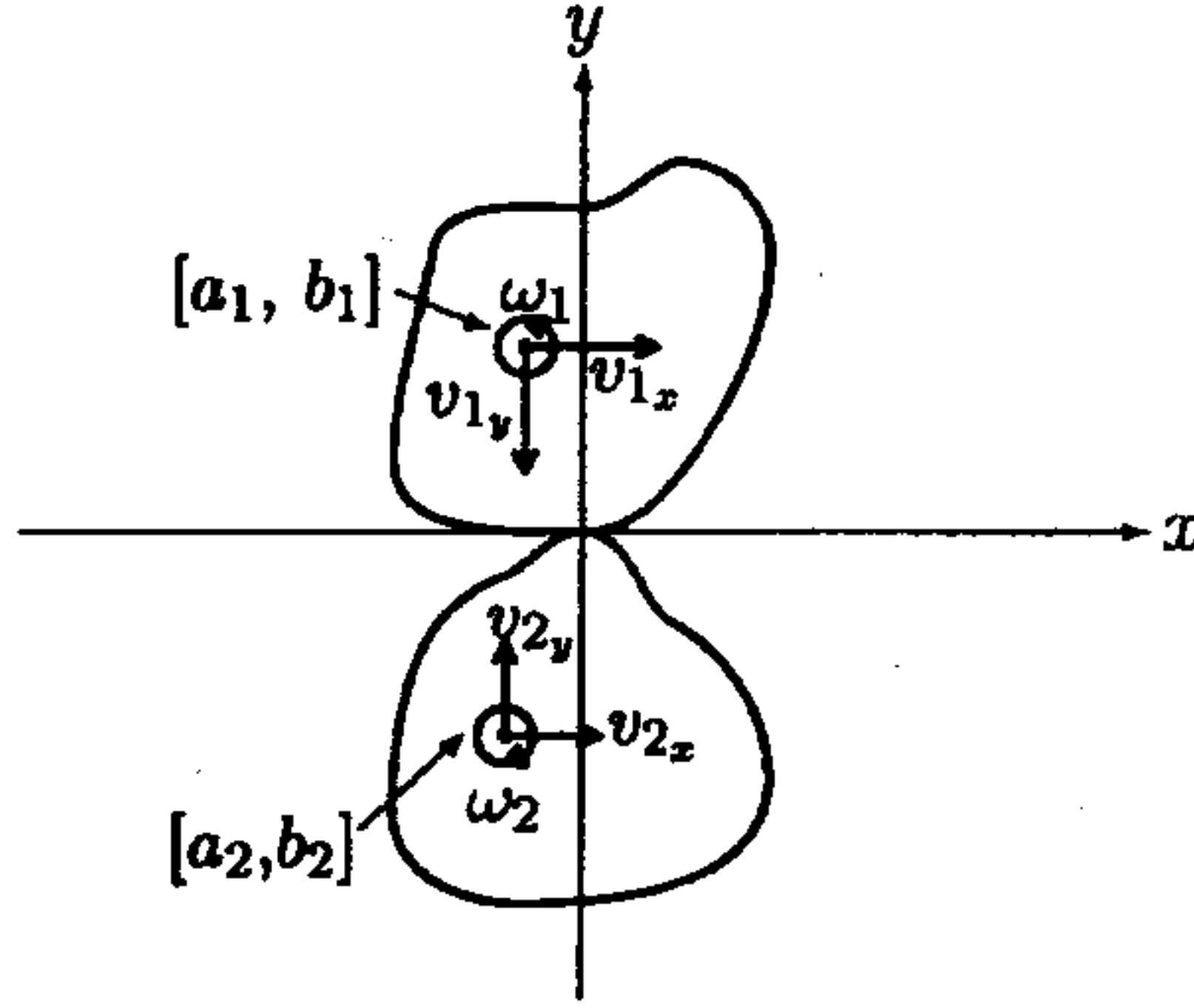
As is evident from the last experiment, we can only consider the impact of a coin in a plane for heights up to 10 cm. makes the solution easier.

In section 10.1.3 we solve a general impact of two bodies in the plane (due to laws of conservation of momentum and angular momentum of bodies) and in section 10.1.3 we adjust some expressions for the case of impact of a coin to board.

**General impact of two bodies in the plane** Let's have two bodies with masses  $m_1$  and  $m_2$ , moments of inertia  $J_1$  and  $J_2$ , moving with velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and angular velocities  $\omega_1$  and  $\omega_2$ . The coordinate system is defined with origin in the point of contact of bodies, where the  $x$ -axis is tangential to both bodies. We can resolve the velocities  $\mathbf{v}_i$  ( $i = 1, 2$ ) we can into directions  $x$  and  $y$  and the corresponding components we denote as  $v_{ix}$  and  $v_{iy}$  (see fig. 6). By index 0 we denote the state before the impact and by  $f$  the state after impact.

In every time of collision there are valid laws of conservation of mo-

Figure 6: Description of impact of two bodies



momentum and moment of momentum – we write them in the form

$$m_i (v_{ix} - v_{ix0}) = \pm P_t \quad (1)$$

$$m_i (v_{iy} - v_{iy0}) = \pm P_n \quad (2)$$

$$J_i (\omega_i - \omega_{i0}) = \pm b_i P_t \mp a_i P_n, \quad (3)$$

where  $a_i$  and  $b_i$  is horizontal and vertical component of radius vector of center of gravity and  $P_t$  and  $P_n$  is tangential and normal impulse of force (upper sign is for body 1, lower for body 2). Let's define the relative velocity of sliding and compression as

$$S = v_{1x} + b_1 \omega_1 - (v_{2x} + b_2 \omega_2) \quad (4)$$

$$C = v_{1y} - a_1 \omega_1 - (v_{2y} - a_2 \omega_2). \quad (5)$$

We substitute from equations (1–3) and get

$$S = S_0 + B_1 P_t - B_3 P_n \quad (6)$$

$$C = C_0 - B_3 P_t + B_2 P_n, \quad (7)$$

where  $S_0$  and  $C_0$  (initial velocity of sliding and compression) and  $B_1$ ,  $B_2$  and  $B_3$  are system constants defined as

$$B_1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{b_1^2}{J_1} + \frac{b_2^2}{J_2} \quad (8)$$

$$B_2 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{a_1^2}{J_1} + \frac{a_2^2}{J_2} \quad (9)$$

$$B_3 = \frac{a_1 b_1}{J_1} + \frac{a_2 b_2}{J_2} \quad (10)$$

$$S_0 = v_{1x0} + b_1 \omega_{10} - (v_{2x0} + b_2 \omega_{20}) \quad (11)$$

$$C_0 = v_{1y0} - a_1 \omega_{10} - (v_{2y0} - a_2 \omega_{20}). \quad (12)$$



Now we solve the collision by graphic method (see [1]). We transform the course of impact into coordinates  $[P_t, P_n]$  and we will follow the path of imaginary point  $Q = [P_t, P_n]$ . At the start,  $Q = 0$  is necessary and the normal impulse  $P_n$  will increase during the impact.

Let's introduce some useful points: Intersections of line of limiting friction with the lines  $C = 0$  of maximal compression ( $P_{nc}$ ) and the line ( $S = 0$ ) of no sliding ( $P_{ns}$ ) are defined by formulas

$$P_{ns} = \frac{S_0}{B_3 + B_1 f \operatorname{sgn} S_0} \quad (13)$$

$$P_{nc} = -\frac{C_0}{B_2 + B_3 f \operatorname{sgn} S_0}. \quad (14)$$

Intersection  $P_{nsc}$  of lines  $S = 0$  and  $C = 0$  we express as

$$P_{nsc} = \frac{S_0 B_3 + C_0 B_1}{B_3^2 - B_1 B_2}. \quad (15)$$

In the first phase of impact, the point  $Q$  follows the line of limiting friction  $P_t = -f P_n \operatorname{sgn} S_0$  (if possible) until it intersects one of the lines of maximal compression ( $C = 0$ ) or no sliding ( $S = 0$ ).

If the path of point  $Q$  intersects the line  $C = 0$ , then the final impulse  $P_{nf}$  is determined by the formula  $P_{nf} = P_{nc}(1 + \varepsilon)$ , where  $\varepsilon$  is coefficient of restitution.

Another situation will occur, if the path of point  $Q$  intersects line  $S = 0$ . According to friction and direction of sliding, there are three possibilities: point  $Q$  will follow the line of no sliding (enough friction is available to prevent sliding) or it will follow the line of limiting friction (if  $\operatorname{sgn} S$  won't change) or travel along the line of reversed limiting friction (defined as  $P_n = 2P_{ns} + f P_t \operatorname{sgn} S_0$ ), if the direction of sliding will change.

Using an absolute value and function  $\operatorname{sgn}$  we get only a few cases:

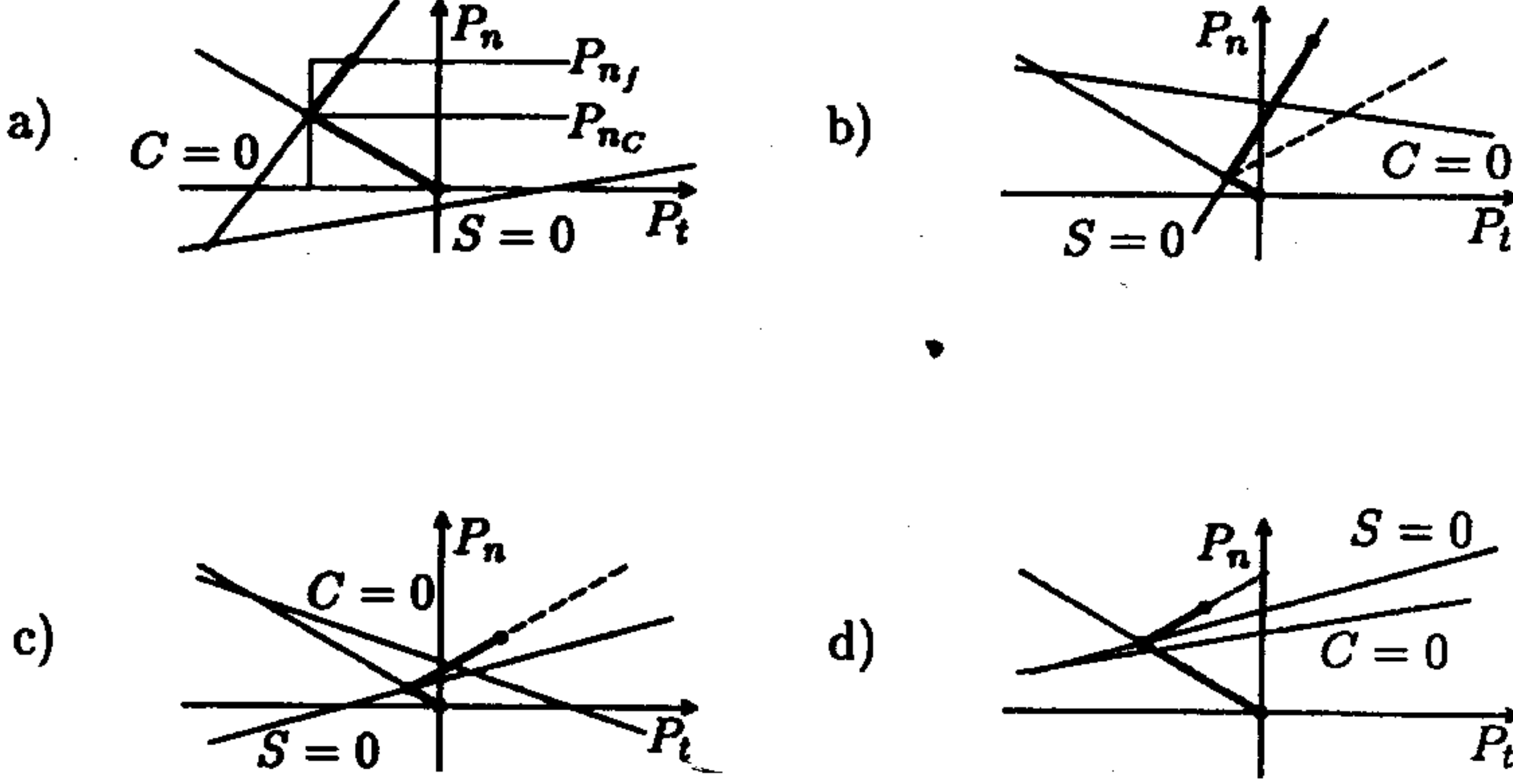
- a)  $P_{ns} \leq 0$  or  $P_{nc}(1 + \varepsilon) \leq P_{ns}$  (see fig. 7a): line  $S = 0$  can't be intersected, because its intersection with the initial phase lies under the  $P_t$ -axis.  $P_{nf}$  and  $P_{tf}$  are determined by expressions

$$P_{nf} = (1 + \varepsilon) P_{nc} \quad (16)$$

$$P_{tf} = -f P_{nf} \operatorname{sgn} S_0. \quad (17)$$

- b)  $P_{ns} > 0$  and  $P_{nc}(1 + \varepsilon) > P_{ns}$  and  $\left| \frac{B_1}{B_3} \right| \geq \frac{1}{f}$  (fig. 7b): line  $S = 0$  is intersected during impact and its slope is greater than slope of line of limiting friction. Point  $Q$  travels only along the line

Figure 7: Possible arrangement of lines  $S = 0$ ,  $C = 0$  and initial sliding. The thick line marks path of point  $Q$ , dashed line reversed limiting friction



$S = 0$  (enough friction is available to prevent sliding) and the impact is terminated at the point with coordinates

$$P_{nf} = (1 + \epsilon)P_{nsc} \quad (18)$$

$$P_{tf} = \frac{P_{nf}B_3 - S_0}{B_1} \quad (19)$$

- c)  $P_{ns} > 0$  and  $P_{nc}(1 + \epsilon) > P_{ns}$  and  $\left|\frac{B_1}{B_3}\right| < \frac{1}{f}$  and  $P_{ns} < P_{nc}$  (fig. 7c): the slope of the line  $S = 0$  is less than the slope of line of limiting friction; line  $S = 0$  is intersected before line  $C = 0$ . By geometric consideration, we obtain

$$P_{nf} = (1 + \epsilon) \frac{2P_{ns}fB_3 + C_0 \operatorname{sgn} S_0}{fB_3 - B_2 \operatorname{sgn} S_0} \quad (20)$$

$$P_{tf} = f(P_{nf} - 2P_{ns}) \operatorname{sgn} S_0 \quad (21)$$

- d)  $P_{ns} > 0$  and  $P_{nc}(1 + \epsilon) > P_{ns}$  and  $\left|\frac{B_1}{B_3}\right| < \frac{1}{f}$  and  $P_{ns} < P_{nc}$  (fig. 7d): as the previous case, only the line  $C = 0$  is intersected before  $S = 0$ . Evidently

$$P_{nf} = (1 + \epsilon)P_{nc} \quad (22)$$

$$P_{tf} = f(P_{nf} - 2P_{ns}) \operatorname{sgn} S_0 \quad (23)$$

The values after collision we determine from equations (1-3):

$$v_{1x} = v_{1x0} + \frac{P_{tf}}{m_1} \quad (24)$$

$$v_{1y} = v_{1y0} + \frac{P_{nf}}{m_2} \quad (25)$$

$$\omega_1 = \omega_{10} + \frac{b_1 P_{tf} - a_1 P_{nf}}{J_1} \quad (26)$$

**Application of general formulas to collision of a coin and board** Let's consider a board as body number 2; we assume  $m_1 \ll m_2$ ,  $J_1 \ll J_2$ ,  $v_2 \rightarrow 0$  and  $\omega_2 \rightarrow 0$  and we assume the coin as a cylinder with radius  $R$ , height  $h$  and density  $\rho$ . Then

$$m_1 = \rho \pi R^2 h \quad (27)$$

$$J_1 = \frac{m_1 (3R^2 + h^2)}{12} \quad (28)$$

$$B_1 = \frac{1}{m_1} + \frac{b_1^2}{J_1} \quad (29)$$

$$B_2 = \frac{1}{m_1} + \frac{a_1^2}{J_1} \quad (30)$$

$$B_3 = \frac{a_1 b_1}{J_1}, \quad (31)$$

Values of  $C_0$  and  $S_0$  are evident from (12) and (11). We define an angle of rotation  $\varphi$  as a displacement from vertical direction. We obtain coordinates of the center of gravity by geometrical consideration:

$$a_1 = R \cos \varphi \operatorname{sgn}(\sin \varphi) - \frac{\sin \varphi \operatorname{sgn}(\cos \varphi)}{2} \quad (32)$$

$$b_1 = |R \sin \varphi| + \left| \frac{h \cos \varphi}{2} \right| \quad (33)$$

$$d = y - |R \sin \varphi| + \left| \frac{h \cos \varphi}{2} \right|. \quad (34)$$

#### 10.1.4 Computer simulation, experimental results

**Description and simulation of fall of coin** Fall of the coin is described by a set of movement differential equations

$$\ddot{x} = 0 \quad (35)$$

$$\ddot{y} = g \doteq -9.81 \quad (36)$$

$$\ddot{\varphi} = 0 \quad (37)$$

until  $d > 0$ . If  $d = 0$ , a collision occurs<sup>1</sup>. If the total energy (potential+kinetic) is greater than the energy necessary to turn the coin, the computation continues. In the opposite case, we determine from  $\text{sgn}(\cos \varphi)$  if the coin is with heads or tails up.

It is evident from experiments and simulations that the coin turns only when it has high energies ( $\gg mg\sqrt{R^2 + h^2}$ ); if it has low energies, the impact will limit to vibrations.

The result of simulation is the diagram<sup>2</sup> of falls of coin (see example on the fig. 8) in dependence on initial height and on initial angular velocities. (Evidently, it is possible to compute dependence on other parameters, as initial angle, initial velocity, coefficients of restitution and friction etc. Simulation of fig.8 took 4 hours.)

By substituting a histogram (fig. 5) into this set of results we obtain a dependence of probability of landing with heads up at the height (fig. 9) from which the coin was dropped. From this graph we can deduct the height where the probability of landing with heads and tails up is the same: for parameters given in the introduction, it is between  $1.80R$  and  $1.85R$ .

**Experimental results** We measured the relative frequency of landing with heads up depending on the height – it was in the range  $1.5R$  and  $8R$  (see fig. 9). An error of height measuring should not exceed  $0.3 \text{ mm}$ .

From experimental results we can speculate that the same probability of landing with heads and tails up lie in the ranges  $1.79R$  and  $1.91R$ . Theoretical results are in agreement with experimental ones, although we neglected vibrational character of collision and bypassing of coin. For a more exact description, it would be necessary to solve vibrations of coin and board [1] and for greater heights describe bypassing of coin [2].

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<sup>1</sup>If modelling this process, it is usually  $d \neq 0$ . That's why it is good to compute as long as  $d$  reaches zero. For to find a zero value (or smallest value) it is good to use the Newton's iteration method.

<sup>2</sup>The unit of height is in multiple of radius of coin  $R$  to get an image of height compared with dimension of coin.



Figure 8: An example of diagrammed falls of coin. Parameters:  $\varepsilon = 0.88$ ,  $f = 0.12$ , initial angle  $\varphi = 0$  and initial velocity  $v = 0$ . On the horizontal axis there are initial angular velocities and on the vertical axis, there is the height, from which the coin falls. Black portions mark places where the coin landed tails up

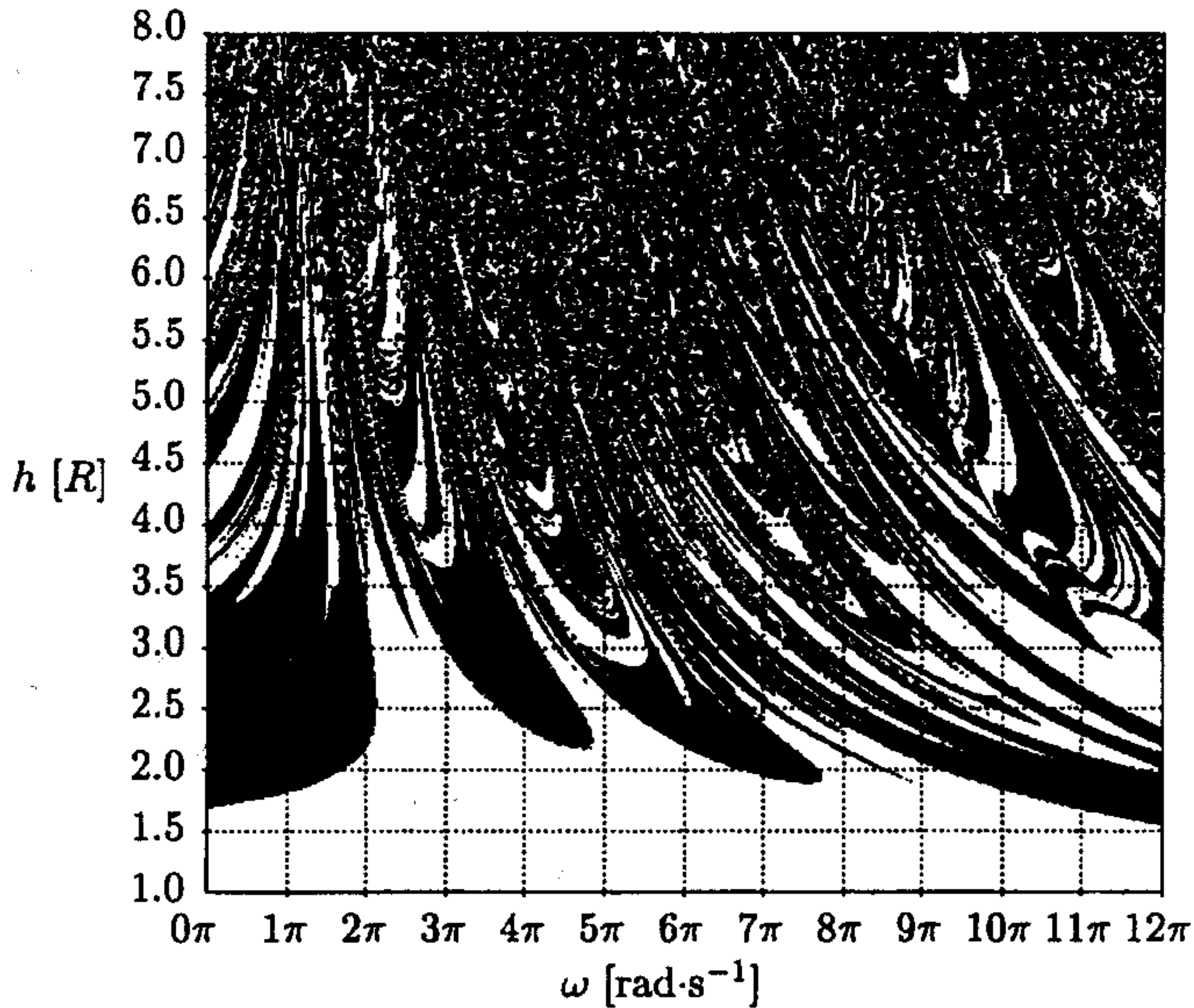
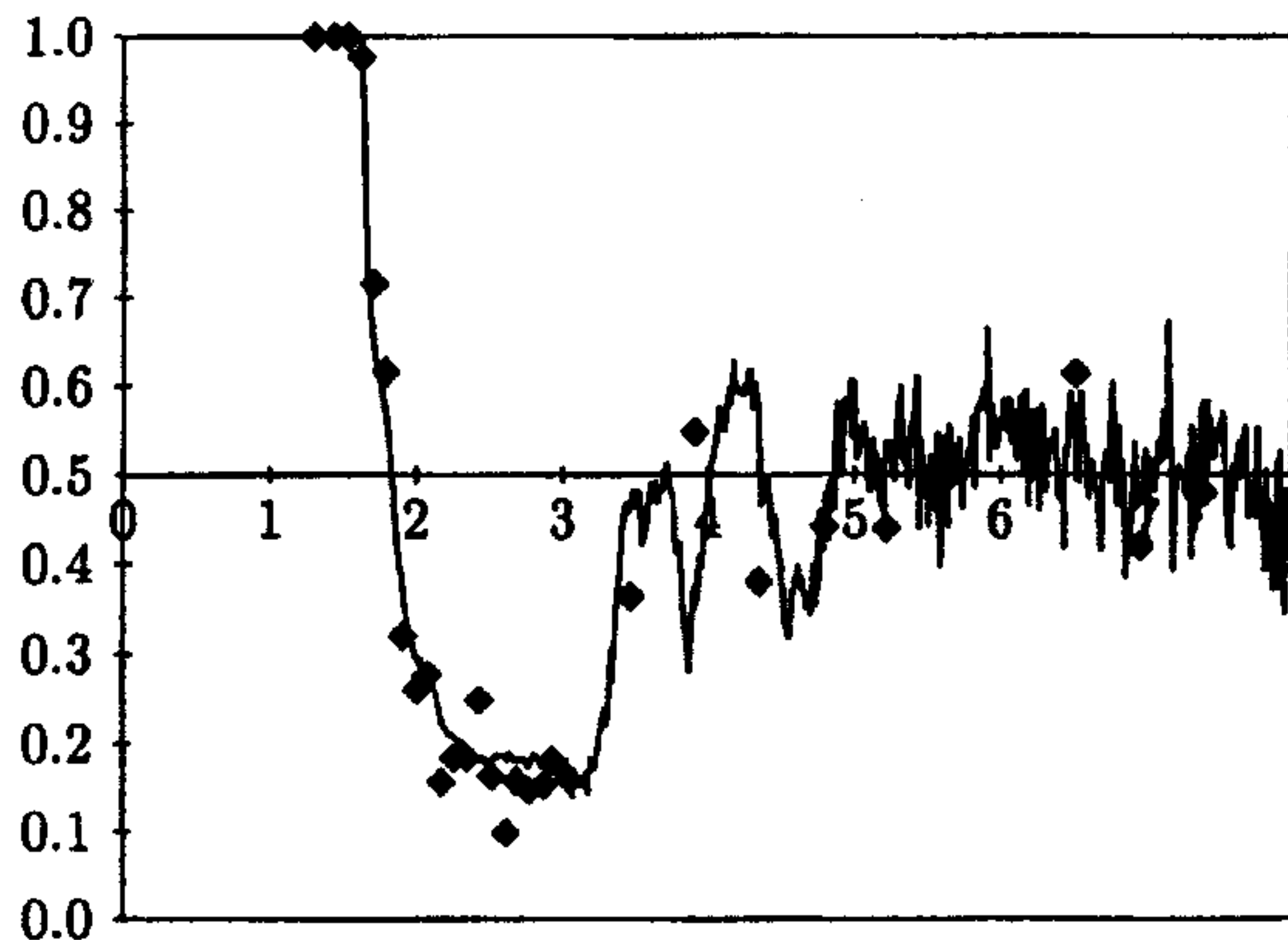


Figure 9: Dependence of probability of landing with heads up on height. The connected curve is a result of theory, the marks  $\blacklozenge$  expresses the results of experiments. The coin was dropped 500 times per height



## References

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- [3] Horák, Z. — Krupka, F. — Šindelář, V.: *Technická fysika*. Praha, Státní nakladatelství technické literatury, 1961.