Linear Optimal Control

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0.0/25.0 points (graded)

Consider the scalar equation

$$\dot{x} = -4x + 2u,$$

and the infinite horizon cost function

$$J = \int_0^\infty \left[32x^2 + u^2\right] dt.$$

(a) Assume that the optimal cost-to-go function is of the form $J^\star = px^2$. What value of p satisfies the Hamilton-Jacobi-Bellman conditions for optimality?

Answer: 2.0

Explanation

Given the equation of motion

$$\dot{x} = -4x + 2u,$$

the infinite horizon cost function, $J = \int_0^\infty \left[32x^2 + u^2\right] dt$ and the form of the cost-to-go ($J^*(x) = px^2$), we can substitute into the Hamilton-Jacobi-Bellman equation as follows. We know:

$$0 = \min_u \left\{ g(\mathbf{x}, \mathbf{u}) + \frac{\partial J^*}{\partial x} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right\}$$

We also know that

$$g(\mathbf{x}, \mathbf{u}) = 32x^2 + u^2$$
$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = -4x + 2u$$
$$\frac{\partial J^*}{\partial x} = 2px$$
$$\frac{\partial J^*}{\partial t} = 0$$

Substituting in we get:

$$0 = \min_u \left\{32x^2 + u^2 + (2px)(-4x + 2u)\right\} \quad (Eq. 1)$$

Now, let

$$\mathcal{H} = 32x^2 + u^2 + (2px)(-4x + 2u)$$

We know that for u to minimize $\mathcal{H}$, we must have $\frac{\partial \mathcal{H}}{\partial u} = 0$.

$$\frac{\partial \mathcal{H}}{\partial u} = 2u + (2px)(2)$$

We can now see that the optimal u, $u^\star$, is $-2px$. Plugging back in to Eq. 1, we get

$$0 = \left\{32x^2 + (-2px)^2 + (2px)(-4x + 2(-2px))\right\}$$

Which simplifies to:

$$0 = 32x^2 - 4p^2x^2 - 8px^2$$

Because this must be true for all x, this reduces to following quadratic form:

$$0 = -4p^2 - 8p + 32$$

From this we know that $p = \{2, -4\}$. Because J^* should be positive definite we choose $p = 2$.

(b) Given that the optimal feedback controller associated with $J^\star$ is $u^\star = -Kx$, what is the value of K?

Answer: 4.0

Explanation

Once we have solved for p as in part (a), we can easily find $u^\star$. From before we know that $u^\star = -2px$, so $u^\star = -Kx$ where $K = 4$.

(c) Suppose we change our cost to the following:

$$J = \int_0^\infty \left[96x^2 + 3u^2\right] dt.$$

Which of the following statements is true? (Select all that apply)

- [ ] The optimal controller (K) gets multiplied by 3
- [ ] The optimal controller (K) gets divided by 3
- [ ] The optimal cost-to-go gets multiplied by 3 ✔
- [ ] The optimal cost-to-go gets divided by 3
- [ ] None of the above

Explanation

The optimal cost-to-go gets multiplied by 3, but the controller does not change. This can be easily verified by going through the same calculations as in part (a) and (b). This is a feature of LQR (Linear Quadratic Regulator) problems in general. Scaling both the cost on state and action does not change the controller (but only changes the optimal cost-to-go).

Submit You have used 0 of 2 attempts

Answers are displayed within the problem