6.832 Midterm

Name: ______________________________

October 23, 2014

Please do not open the test packet until you are asked to do so.

- You will be given 85 minutes to complete the exam.
- Please write your name on this page, and on any additional pages that are in danger of getting separated.
- We have left workspace in this booklet. Scrap paper is available from the staff. Any scrap paper should be handed in with your exam.
- YOU MUST WRITE ALL OF YOUR ANSWERS IN THIS BOOKLET (not the scrap paper).
- The test is open notes.
- The test is out of 35 points.

Good luck!

Problem	Possible	Your Score
Problem 1	5	
Problem 2	10	
Problem 3	10	
Problem 4	10	
Total	35	

Problem 1 (5 pts) *The Linear Quadratic Regulator*

Suppose you are given a stabilizable linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

and the cost-to-go function

$$J(\mathbf{x}_0) = \int_0^\infty \left[\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right] dt, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

with $\mathbf{Q}^T = \mathbf{Q} \succ 0$, $\mathbf{R}^T = \mathbf{R} \succ 0$, *and* $eig(\mathbf{A}) \neq 0$. *I can plot the level-set of the optimal cost-to-go function* $J^*(\mathbf{x}) = 1$*; for this problem formulation the optimal cost-to-go is a positive-definite quadratic function,* $J^* = \mathbf{x}^T\mathbf{S}\mathbf{x}$, $\mathbf{S} = \mathbf{S}^T \succ 0$ *and the level-set is an ellipse. The optimal control is given by* $\mathbf{u}^* = -\mathbf{K}\mathbf{x}$.

a) *If I were to double the value of* $\mathbf{R}$ *(so* $\mathbf{R}_{new} = 2\mathbf{R}$*), what happens to the 1 level-set of the cost-to-go? (Circle one)*

   (i) *It gets bigger (the volume of the ellipse increases)*

   (ii) *It gets smaller (the volume of the ellipse decreases)*

   (iii) *The size does not change (the volume of the ellipse remains constant)*

   Give a mathematical justification for your answer:

b) *Now suppose that you have doubled both* $\mathbf{Q}$ *and* $\mathbf{R}$*, so*

$$\mathbf{Q}_{new} = 2\mathbf{Q}, \quad \mathbf{R}_{new} = 2\mathbf{R}.$$

Which of the following statements about the resulting optimal cost-to-go,

$$J^* = \mathbf{x}^T \mathbf{S}_{new} \mathbf{x},$$

are true (circle all that apply).

(i) $\mathbf{S}_{new} = \frac{1}{4}\mathbf{S}$

(ii) $\mathbf{S}_{new} = \frac{1}{2}\mathbf{S}$

(iii) $\mathbf{S}_{new} = \mathbf{S}$

(iv) $\mathbf{S}_{new} = 2\mathbf{S}$

(v) $\mathbf{S}_{new} = 4\mathbf{S}$

Give a mathematical justification for your answer:

c) *Continuing with the formulation in part (b), which of the following statements about the resulting optimal controller,*

$$\mathbf{u}^*_{new} = -\mathbf{K}_{new}\mathbf{x},$$

are true (circle all that apply):

(i) $\mathbf{K}_{new} > \mathbf{K}$

(ii) $\mathbf{K}_{new} = \mathbf{K}$

(iii) $\mathbf{K}_{new} < \mathbf{K}$

where the inequality is taken element-wise. Remember the $-\mathbf{K}$ *in the description above.*

Give a mathematical justification for your answer:

Problem 2 (10 pts) *Lyapunov analysis*

a) *Suspose you have a system $\dot{\mathbf{x}} = f(\mathbf{x})$ with $f(0) = 0$ and a positive-definite scalar function $V(\mathbf{x})$ where you have successfully verified that*

$$\dot{V}(0) = 0$$
$$\dot{V}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \textit{ with } 0 < \sum_i |x_i| \leq 1.$$

Describe the set of initial conditions for which you can guarantee that the system will arrive at the origin as $t \to \infty$. Explain your answer.

b) Consider an uncertain nonlinear system of the form

$$\dot{\mathbf{x}} = f_1(\mathbf{x}) + \alpha f_2(\mathbf{x}), \quad \alpha = \{0.8, 1.1\}.$$

In words, the uncertain gain α is known to take one of exactly two values – either 0.8 or 1.1, but we do not know apriori which one.

(i) Suppose that you know that the origin, $\mathbf{x} = 0$, is a fixed point for the system $\dot{\mathbf{x}} = f_1(\mathbf{x}) + f_2(\mathbf{x})$. Is the origin guaranteed to be a fixed point for the uncertain system? Circle yes or no.

YES *or* NO

Explain your answer:

(ii) Suppose that you are given a radially-unbounded, positive-definite Lyapunov function, $V(\mathbf{x})$, which satisfies the conditions

$$\forall \mathbf{x} \neq 0, \dot{V}(\mathbf{x}, 0.8) < 0, \quad \dot{V}(0, 0.8) = 0,$$
$$\forall \mathbf{x} \neq 0, \dot{V}(\mathbf{x}, 1.1) < 0, \quad \dot{V}(0, 1.1) = 0,$$

where I've used the notation $\dot{V}(\mathbf{x}, \alpha) = \frac{\partial V}{\partial \mathbf{x}} \left[f_1(\mathbf{x}) + \alpha f_2(\mathbf{x}) \right]$.
Which of the following can we conclude about the system

$$\dot{\mathbf{x}} = f_1(\mathbf{x}) + f_2(\mathbf{x})$$

i. The origin is globally stable in the sense of Lyapunov (i.s.L.).
ii. The origin is globally asymptotically stable.
iii. The origin is globally exponentially stable.
iv. None of the above.

Circle all that are true. Provide a mathematical justification for your answer:

(iii) Suppose that you are given a radially-unbounded, positive-definite Lyapunov function candidate, $V(\mathbf{x})$, which you know fails to satisfy the Lyapunov conditions globally, but you would like use sums-of-squares (SOS) optimization to verify the conditions:

$$\forall \mathbf{x} \in \{\mathbf{x} : V(\mathbf{x}) \leq 1\}, \dot{V}(\mathbf{x}, 0.8) \leq 0$$
$$\forall \mathbf{x} \in \{\mathbf{x} : V(\mathbf{x}) \leq 1\}, \dot{V}(\mathbf{x}, 1.1) \leq 0.$$

Write down a sums-of-squares program by listing the decision variables and all *of the required sums-of-squares constraints in the boxes below.*

find

(list decision variables)

subject to *is SOS*

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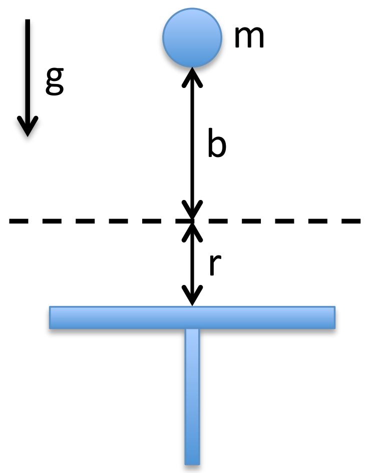
Problem 3 (10 pts) *Trajectory optimization*

Imagine the simple model of a juggling robot illustrated below, which consists of a paddle that can move vertically (with configuration described by the position r) and a ball (modeled as a point with mass m at vertical height b).



Our input is direct control of the velocity of the paddle, resulting in a hybrid state space model with

$$\mathbf{x} = \begin{bmatrix} r \\ b \\ \dot{b} \end{bmatrix}, \quad \dot{\mathbf{x}} = f(\mathbf{x}, u) = \begin{bmatrix} u \\ \dot{b} \\ -g \end{bmatrix},$$

and a simple elastic collision model,

$$\dot{b}^{+} = \dot{r} - .9(\dot{b}^{-} + \dot{r}).$$

We assume the paddle is sufficiently massive to be unaffected by the collision with the ball.

Let us formulate a direct transcription trajectory optimization problem with 11 knot points to find a periodic solution for this system with exactly one collision per period. We will use the decision variables

$$\mathbf{x}_0, ...\mathbf{x}_{10}, \quad u_0, ..., u_9, \quad h.$$

and add the dynamic constraints

$$\mathbf{x}_{n+1} = \mathbf{x}_n + f(\mathbf{x}_n, u_n)h,$$

where $h > 0.01$ is the timestep between knot points.

a) *Let us enforce that the paddle and the ball only come into contact at height 0. Write down all of the required constraints that you must add to the program relating to the hybrid guard and reset (aka collision). Your answers should be written in terms of differentiable functions of the decision variables named above.*

b) *To further constrain the solution, let us require that the apex of the ball occurs at a height of 1. Write any additional constraints required to enforce this. Your answers should be written in terms of differentiable functions of the decision variables named above.*

c) *Do you expect the problem you have formulated so far to have a unique solution? If not, explain why and provide a reasonable additive-cost objective function written in terms of the decision variables.*

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Problem 4 (10 pts) *Nonlinear Dynamics and the Hamilton-Jacobi-Bellman Equations*

a) *Suppose you are given the system*

$$\dot{x} = x + u,$$

where x and u are scalars, and the cost function,

$$J(x_0) = \int_0^\infty g(x(t), u(t))dt, \quad g(x,u) = x^2 + u^2, \quad x(0) = x_0,$$

which we would like to minimize. Suppose you are also given the candidate cost-to-go function

$$\hat{J}(x) = x^4 - x^2 + \frac{1}{4}.$$

Use the optimality conditions to derive the control law associated with this cost-to-go:

$$\hat{u} = \operatorname*{argmin}_u \left[g(x,u) + \frac{\partial \hat{J}}{\partial x} f(x,u) \right].$$

Show your work.

$\hat{u} =$ $\boxed{}$

b) Is the controller derived above optimal? Circle one of the following:

YES *or* NO *or* INCONCLUSIVE

Justify your answer and show your work.

c) Using the system and controller from part (a), what are the fixed points of the closed-loop system. For each fixed point, say if it is locally unstable, locally stable i.s.L, locally asymptotically stable, and/or locally exponentially stable.

d) *Now consider a similar problem, but we will remove the input term from the cost function and add input limits to the system, so that we have*

$$\dot{x} = x + u, \quad |u| \leq 1,$$

and the cost function,

$$J(x_0) = \int_0^\infty g(x(t), u(t))dt, \quad g(x,u) = x^2, \quad x(0) = x_0,$$

which we would like to minimize. Using the same candidate cost-to-go function

$$\hat{J}(x) = x^4 - x^2 + \frac{1}{4},$$

use the optimality conditions to derive the control law associated with the cost to go for this problem. Show your work.

$\hat{u} =$

e) Using the system and controller from part (d), what are the fixed points of the closed-loop system. For each fixed point, say if it is locally unstable, locally stable i.s.L, locally asymptotically stable, and/or locally exponentially stable.

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