# Problem: Discrete Display

## Discrete Systems

0.0/20.0 points (graded)

For a univariate dynamic system $\dot{x} = f(x)$ we have seen via graphical analysis that $x^*$ is a locally stable equilibrium if the following conditions hold

1. $f(x^*) = 0$
2. $\frac{\partial f}{\partial x}(x^*) < 0$

Otherwise stated, that $f(x)$ has a zero-crossing at $x^*$ with negative slope.

Now, consider a simple discretization of this continuous system, where for some fixed time step $h$ we have:

$$x[k+1] = x[k] + hf(x[k])$$

For arbitrary $h$, the two conditions above are *not* sufficient for stability of the discrete system. Provide a counterexample demonstrating this by giving values for $x\_star$, $f(x)$, and $h$ below.

```
syms x
x_star =
h =
f =

```

Unanswered

```
% One possible solution is described
syms x
% pick a favorite system which is stable in the continuous sense
% any linear system will do. We'll also take the origin to be
% the equilibrium, for simplicity.
x_star = 0
f = -x

% now, let's try some values for h and see what happens
h = .1;
% simulate 100 steps, from x_0 close to 0
x_sim(1) = .01;
for i=2:100,
x_sim(i) = x_sim(i-1) + h*-x_sim(i-1);  % our discrete update rule
end
% x_sim(10) = 2.9e-7, so it looks like the discrete system is also stable
% try again with a much larger h*-x_sim
h = 10;
x_sim(1) = .01;
for i=2:100,
x_sim(i) = x_sim(i-1) + h*-x_sim(i-1);  % our discrete update rule
end
% Now x_sim(100) = -2.9e92, a massive number, so it's safe to say that this is unstable
```

**Run Code**

Find the upper bound $h^*$ such that all $h < h^*$ results in a stable discrete system. Write your answer in terms of $G$, where $G = \left|\frac{\partial f}{\partial x}(x^*)\right| > 0$

**Answer:** 2/G

### Explanation

Using graphical analysis for one dimensional continuous systems, we said that for an equilibrium to be stable, the dynamic flow had to point toward the equilibrium. Put another way, that the dynamics had to lead the system *closer* to the equilibrium. We saw this graphically by drawing arrows along the axis. Part 1 of this problem illustrated that if $h$ is too large, the system can become unstable. What does it mean for the discrete system to get closer? It is, surprisingly, easier to formulate this criteria in the discrete case. Closer just means that $|x[k+1]-x^*|<|x[k]-x^*|$. For simplicity, we'll assume $x^*=0$. Substituting and simplifying, we get:

$$\begin{aligned} |x[k+1]| &< |x[k]| \\ |x[k]+hf(x[k]| &< |x[k]| \\ (x[k]+hf(x[k])^2 &< x[k]^2 \\ 2hx[k]f(x[k])+h^2f(x[k])^2 &< 0 \\ hf(x[k])^2 &< -2x[k]f(x[k]) \\ h &< -\frac{2x[k]}{f(x[k])} \end{aligned}$$

This must hold as $x[k]\to x^*$, so by linearization of $f$ or through Taylor expansion, the right hand side simplifies to $\frac{2}{G}$. The Taylor expansion argument is given below:

$$\begin{aligned} \lim_{x[k]\to 0} -\frac{2x[k]}{f(x[k])} &= \lim_{x[k]\to 0} -\frac{2x[k]}{f(0)+x[k]f'(0)+.5x[k]^2f''(0)+\ldots} \\ &= \lim_{x[k]\to 0} -\frac{2}{f'(0)+.5x[k]f''(0)+\ldots} \\ &= -\frac{2}{f'(0)} \\ &= \frac{2}{G} \end{aligned}$$

Submit You have used 0 of 3 attempts

Answers are displayed within the problem

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