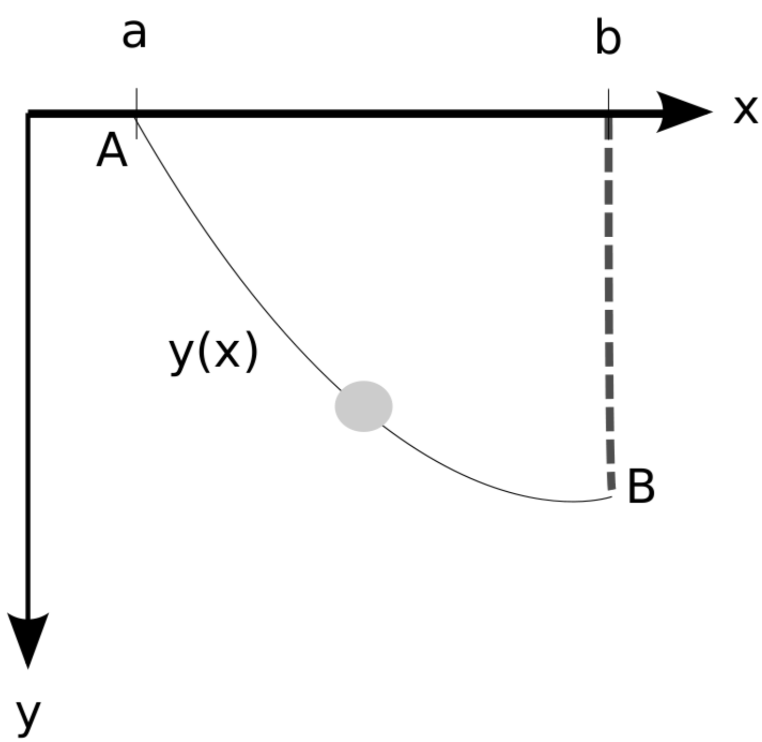
Brachistochrone

The Brachistochrone Problem, Part 1

0.0/10.0 points (ungraded)

In 1696, Johann Bernoulli challenged his fellow mathematicians to solve the "problem of the brachistochrone". The story goes that Newton stayed up all night and had the correct solution by the morning. Hopefully, you'll be quite a bit quicker than he was!

In this question, we will formulate this problem as a minimum-time optimal control problem.



Given two points A and B as shown in the figure above, the goal is to find a curve $y(x)$ joining A and B such that a body constrained to move along this curve under the influence of gravity goes from A to B in the least possible time. Think of the curve as a wire and the body as a bead that moves along the wire. We will make the following assumptions:

- There is no friction between the bead and the wire.

- The point A is higher than B
- The bead is released at rest from A
- The potential energy of the bead is 0 at the point A.

Let $u\left(x\right)$, which is our equivalent to "control" for this problem, be $\frac{dy}{dx}$. Then, the arclength of the curve from A to B is $\int_a^b \sqrt{1+u(x)^2}dx$

(a) Compute the speed of the bead $v\left(y\right)$ as a function of y. Your expression should be in terms of the gravitational acceleration g (in addition to y). Type in your expression below. (Hint: Use the fact that energy is conserved).

Explanation

Since energy is conserved, we know $0 = .5mv^2 - mgy$, so v = \sqrt\{2gy}.

Answer: sqrt(2*g*y)

(b) We can use the answer from the previous part to write down the brachistochrone problem as a minimum-time optimal control problem where the "control input" is $u\left(x\right)$, and the cost is of the form:

$$J\left(u\left(x\right), y\left(x\right)\right) = \int_a^b g\left(u\left(x\right), y\left(x\right)\right)dx.$$

Write down the expression for $g\left(u, y\right)$ below. (Again, please abbreviate $y\left(x\right)$ and $u\left(x\right)$ to y and u respectively. Thus, your answer should be in terms of the symbols u, y, and g).

Answer: sqrt((1+u^2)/(2*g*y))

Explanation

Total time here can be written as the integral of the differential length over the speed, integrated along the curve, so $T = \int_a^b \sqrt{\frac{1+u^2}{2gy}}$

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Answers are displayed within the problem

The Brachistochrone Problem, Part 2

0.0/15.0 points (ungraded)

For this final part, we will use Pontryagin's Minimum Principle to derive the differential equations that describe the solution.

Again, thinking of this as an optimal control problem, we introduce the costate variable $\lambda\left(x\right)$ and write the Hamiltonian

$$H\left(y\left(x\right), u\left(x\right), \lambda\left(x\right), x\right) = \lambda\left(x\right) u\left(x\right) + g\left(u\left(x\right), y\left(x\right)\right)$$

(c) Pontryagin's Minimum princple states that $u\left(x\right)$ minimizes the Hamiltonian for all $y\left(x\right), \lambda\left(x\right), x$. Use this to solve for λ in terms of the symbols u, x, g, and y.

λ=

Answer: -u/sqrt(2*g*y*(1+u^2))

Explanation

Taking the derivative $0 = \frac{\partial H}{\partial u} = \lambda + \frac{\partial g}{\partial u} = \lambda + \frac{u}{\sqrt{(1+u^2)2gy}}$, we can easily find λ as given in the answer.

(d)Again from Pontryagin, we know that the costate dynamics are $\lambda' = -\frac{\partial g(u,y)}{\partial y}$ Find the dynamics of the Hamiltonian, $H' = \frac{dH}{dx}$. Again, write your answer in terms of the symbols u, x, g, and y. HINT: Your answer will not depend on $u' = \frac{du}{dx}$!

Answer: 0

Explanation

Expand the derivative H' and substitute:

$$\begin{aligned} H' &= \frac{\partial H}{\partial y}\frac{dy}{dx} + \frac{\partial H}{\partial u}\frac{du}{dx} + \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dx} + \frac{\partial H}{\partial x} \\ &= \frac{\partial g}{\partial y}y' + 0 + u\lambda' + 0 \\ &= \frac{\partial g}{\partial y}u - u\frac{\partial g}{\partial y} \\ &= 0 \end{aligned}$$

(e) Let $k = H\left(y\left(a\right), u\left(a\right), \lambda\left(a\right), a\right)$ be the initial value of the Hamiltonian. Use the results above to find a governing equation of motion for the Brachistochrone. Your answer should be of the form

$$1 = \left(\alpha + u^2\right)\beta$$

Find α and β in terms of $x, g, y,$ and k.

$\alpha =$

Answer: 1

$\beta =$

Answer: 2*g*y*k^2

You should confirm that there is a parametric solution solution to this differential equation of the form (called a *cycloid*)

$$\begin{aligned} x\left(s\right) &= \frac{1}{4gk^2}\left(s - \sin s\right) \\ y\left(s\right) &= \frac{1}{4gk^2}\left(1 - \cos s\right) \end{aligned}$$

Explanation

From the previous part, we know that H is constant along the trajectory, so we can write

$$H\left(y\left(x\right),u\left(x\right),\lambda\left(x\right),x\right)=\lambda\left(x\right)u\left(x\right)+g\left(u\left(x\right),y\left(x\right)\right)$$

$$k=-\frac{u}{\sqrt{\left(1+u^2\right)2gy}}u+\sqrt{\frac{1+u^2}{2gy}}$$

$$k=-\frac{u^2}{\sqrt{\left(1+u^2\right)2gy}}+\frac{1+u^2}{\sqrt{\left(1+u^2\right)2gy}}$$

$$k=\frac{1}{\sqrt{\left(1+u^2\right)2gy}}$$

$$1=\left(1+u^2\right)2gyk^2$$

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Answers are displayed within the problem