

This document provides detailed derivation steps for (12, (13), as follows:

According to (10) and (11), it can be deduced as follows [31]:

$$\begin{aligned} \mathbf{H}_w \mathbf{g}_{wi} &= (\mathbf{G}_w^T \mathbf{G}_w + \mathbf{g}_{wi} \mathbf{g}_{wi}^T)^{-1} \mathbf{g}_{wi} \\ &= \mathbf{H}_{wi} \mathbf{g}_{wi} - \mathbf{H}_{wi} \mathbf{g}_{wi} (1 + \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi})^{-1} \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} \end{aligned} \quad (\text{A1})$$

where $\mathbf{S}_{wi,i} = (1 + \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi})^{-1} = 1 - \mathbf{g}_{wi}^T \mathbf{H}_w \mathbf{g}_{wi}$.

$$\begin{aligned} \hat{\mathbf{x}}_w &= \mathbf{H}_w \mathbf{G}_w^T \mathbf{y}_w \\ &= \mathbf{H}(\mathbf{G}_{wi}^T \mathbf{y}_{wi} + (\mathbf{y}_w)_i \mathbf{g}_{wi}) \\ &= (\mathbf{H}_{wi} - \mathbf{S}_{wi,i} \mathbf{H}_{wi} \mathbf{g}_{wi} \mathbf{g}_{wi}^T \mathbf{H}_{wi})(\mathbf{G}_{wi}^T \mathbf{y}_{wi} + (\mathbf{y}_w)_i \mathbf{g}_{wi}) \\ &= \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi} + (\mathbf{y}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi} - \mathbf{S}_{wi,i} \mathbf{H}_{wi} \mathbf{g}_{wi} \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi} \\ &\quad - \mathbf{S}_{wi,i} (\mathbf{y}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi} \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi} + (\mathbf{y}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi} (1 - \mathbf{S}_{wi,i} \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi}) \\ &\quad - \mathbf{S}_{wi,i} (\mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi}) \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi} + \mathbf{S}_{wi,i} (\mathbf{y}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi} - \mathbf{S}_{wi,i} (\mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi}) \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi} + \mathbf{S}_{wi,i} ((\mathbf{y}_w)_i - \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_{wi}^T \mathbf{y}_{wi}) \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \hat{\mathbf{x}}_{wi} + \mathbf{S}_{wi,i} (\mathbf{e}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi} \end{aligned} \quad (\text{A2})$$

The relationship between the residual \mathbf{e} and \mathbf{e}_i is as follow:

$$\begin{aligned} \mathbf{e}_w &= \mathbf{y}_w - \mathbf{G}_w \hat{\mathbf{x}}_w \\ &= \mathbf{y}_w - \mathbf{G}_w (\hat{\mathbf{x}}_{wi} + \mathbf{S}_{wi,i} (\mathbf{e}_w)_i \mathbf{H}_{wi} \mathbf{g}_{wi}) \\ &= \mathbf{e}_w - \mathbf{S}_{wi,i} (\mathbf{e}_w)_i \mathbf{G}_w \mathbf{H}_{wi} \mathbf{g}_{wi} \end{aligned} \quad (\text{A3})$$

It can be obtained that the relationship between $(\mathbf{e}_w)_i$ and $(\mathbf{e}_{wi})_i$ is as follow:

$$(\mathbf{e}_w)_i = (\mathbf{e}_{wi})_i - \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} = \mathbf{S}_{i,i} (\mathbf{e}_i)_i \quad (\text{A4})$$

The relationship between the residual $WSSE$ and $WSSE_i$ is as follow:

$$\begin{aligned} WSSE &= \mathbf{e}_w^T \mathbf{e}_w \\ &= \mathbf{e}_{wi}^T \mathbf{e}_{wi} - 2 \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i \mathbf{e}_{wi}^T \mathbf{G}_w \mathbf{H}_{wi} \mathbf{g}_{wi} + \mathbf{S}_{wi,i}^2 (\mathbf{e}_{wi})_i^2 \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_w^T \mathbf{G}_w \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \mathbf{e}_{wi}^T \mathbf{e}_{wi} - 2 \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i ((\mathbf{y}_{wi} - \mathbf{G}_w \mathbf{H}_{wi} \mathbf{y}_{wi})^T \mathbf{G}_{wi} \\ &\quad + (\mathbf{e}_{wi})_i \mathbf{g}_{wi}^T) \mathbf{H}_{wi} \mathbf{g}_{wi} + \mathbf{S}_{wi,i}^2 (\mathbf{e}_{wi})_i^2 \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{G}_w^T \mathbf{G}_w \frac{\mathbf{H}_w \mathbf{g}_{wi}}{\mathbf{S}_{wi,i}} \\ &= \mathbf{e}_{wi}^T \mathbf{e}_{wi} - 2 \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i^2 \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} + \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i^2 \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= \mathbf{e}_{wi}^T \mathbf{e}_{wi} + (\mathbf{e}_{wi})_i^2 - \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i^2 \mathbf{g}_{wi}^T \mathbf{H}_{wi} \mathbf{g}_{wi} \\ &= WSSE_i + \mathbf{S}_{wi,i} (\mathbf{e}_{wi})_i^2 \end{aligned} \quad (\text{A5})$$

Substituting (31) into (32), we obtain:

$$WSSE_i = WSSE - (\mathbf{e}_w)_i^2 / \mathbf{S}_{wi,i} \quad (\text{A6})$$

For two fault elimination:

$$WSSE_{i,j} = WSSE - (\mathbf{e}_w)_i^2 / \mathbf{S}_{wi,i} - (\mathbf{e}_{wi})_j^2 / \mathbf{S}_{wj,j}^i \quad (\text{A7})$$

where

$$\begin{aligned} \mathbf{S}_{i,j}^i &= \mathbf{w}_j (1 + \mathbf{w}_j \mathbf{g}_j^T \mathbf{H}_{i,j} \mathbf{g}_j)^{-1} \\ &= \mathbf{w}_j (1 + (\mathbf{w}_j)^2 \mathbf{g}_j^T \mathbf{H}_i \mathbf{g}_j / \mathbf{S}_{i,j}^i)^{-1} \end{aligned} \quad (\text{A8})$$

The relationship between the fault detection statistics after two satellites are removed and the fault detection statistics of all satellites is as follows:

$$WSSE_{i,j} = WSSE - \frac{(e_w)_i^2}{S_{wi,i}} - \frac{\left((e_w)_j - \frac{S_{wi,j}}{S_{i,i}} (e_w)_i \right)^2}{S_{wj,j} - \frac{S_{wi,j}^2}{S_{wi,i}}} \quad (A9)$$

$$= WSSE - \frac{S_{wi,i}(e_w)_j^2 - 2S_{wi,j}(e_w)_i(e_w)_j + S_{wj,j}(e_w)_i^2}{S_{wi,i}S_{wj,j} - S_{wi,j}^2}$$

$$T_{i,j} = \frac{S_{wi,i}(e_w)_j^2 - 2S_{wi,j}(e_w)_i(e_w)_j + S_{wj,j}(e_w)_i^2}{S_{wi,i}S_{wj,j} - S_{wi,j}^2} \quad (A10)$$

By mathematical induction, we can derive multiple satellite formulas:

$$T_{\{i,j,k,\dots,n\}} \leftarrow \frac{\sum_{I \in \{i,j,k,\dots,n\}} \sum_{J \in \{n,\dots,k,j,i\}} (e_w)_I (e_w)_J \text{cofactor}(S_{I,J})}{\det(S_{\{i,j,k,\dots,n\}})} \quad (A11)$$

$S_{\{i,j,k,\dots,n\}}$ represents the submatrix composed of S , \det represents the calculation of the matrix determinant. The expression of $\text{cofactor}(S_{I,J})$ is as follows:

$$\text{cofactor}(S_{I,J}) \begin{cases} S_{I,J}, & \text{if } I = J \\ -S_{I,J}, & \text{if } I \neq J \end{cases} \quad (A12)$$