This document provides detailed derivation steps for (12), (13), as follows:

According to (10) and (11), it can be deduced as follows [31]:

$$\begin{aligned}\boldsymbol{H}_w\boldsymbol{g}_{wi} &= (\boldsymbol{G}_w^T\boldsymbol{G}_w + \boldsymbol{g}_{wi}\boldsymbol{g}_{wi}^T)^{-1}\boldsymbol{g}_{wi}\\ &= \boldsymbol{H}_{wi}\boldsymbol{g}_{wi} - \boldsymbol{H}_{wi}\boldsymbol{g}_{wi}(1+\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi})^{-1}\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\end{aligned} \tag{A1}$$

where $\boldsymbol{S}_{wi,i} = (1+\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi})^{-1} = 1-\boldsymbol{g}_{wi}^T\boldsymbol{H}_w\boldsymbol{g}_{wi}$.

$$\begin{aligned}\hat{\boldsymbol{x}}_w &= \boldsymbol{H}_w\boldsymbol{G}_w^T\boldsymbol{y}_w\\ &= \boldsymbol{H}(\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + (\boldsymbol{y}_w)_i\boldsymbol{g}_{wi})\\ &= (\boldsymbol{H}_{wi} - \boldsymbol{S}_{wi,i}\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi})(\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + (\boldsymbol{y}_w)_i\boldsymbol{g}_{wi})\\ &= \boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + (\boldsymbol{y}_w)_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} - \boldsymbol{S}_{wi,i}\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi}\\ &\quad - \boldsymbol{S}_{wi,i}(\boldsymbol{y}_w)_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + (\boldsymbol{y}_w)_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}(1-\boldsymbol{S}_{wi,i}\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi})\\ &\quad - \boldsymbol{S}_{wi,i}(\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi})\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + \boldsymbol{S}_{wi,i}(\boldsymbol{y}_w)_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} - \boldsymbol{S}_{wi,i}(\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi})\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi} + \boldsymbol{S}_{i,i}((\boldsymbol{y}_w)_i - \boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_{wi}^T\boldsymbol{y}_{wi})\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \hat{\underline{\boldsymbol{x}}}_{wi} + \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\end{aligned} \tag{A2}$$

The relationship between the residual $\boldsymbol{e}$ and $\boldsymbol{e}_i$ is as follow:

$$\begin{aligned}\boldsymbol{e}_w &= \boldsymbol{y}_w - \boldsymbol{G}_w\hat{\boldsymbol{x}}_w\\ &= \boldsymbol{y}_w - \boldsymbol{G}_w(\hat{\underline{\boldsymbol{x}}}_{wi} + \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i\boldsymbol{H}_{wi}\boldsymbol{g}_{wi})\\ &= \boldsymbol{e}_{wi} - \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i\boldsymbol{G}_w\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\end{aligned} \tag{A3}$$

It can be obtained that the relationship between $(\boldsymbol{e}_w)_i$ and $(\boldsymbol{e}_{wi})_i$ is as follow:

$$(\boldsymbol{e}_w)_i = (\boldsymbol{e}_{wi})_i - \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} = \boldsymbol{S}_{i,i}(\boldsymbol{e}_i)_i \tag{A4}$$

The relationship between the residual *WSSE* and $WSSE_i$ is as follow:

$$\begin{aligned}WSSE &= \boldsymbol{e}_w^T\boldsymbol{e}_w\\ &= \boldsymbol{e}_{wi}^T\boldsymbol{e}_{wi} - 2\boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i\boldsymbol{e}_{wi}^T\boldsymbol{G}_w\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} + \boldsymbol{S}_{wi,i}^2(\boldsymbol{e}_{wi})_i^2\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_w^T\boldsymbol{G}_w\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \boldsymbol{e}_{wi}^T\boldsymbol{e}_{wi} - 2\boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i((\boldsymbol{y}_{wi} - \boldsymbol{G}_w\boldsymbol{H}_{wi}\boldsymbol{y}_{wi})^T\boldsymbol{G}_{wi}\\ &\quad + (\boldsymbol{e}_{wi})_i\boldsymbol{g}_{wi}^T)\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} + \boldsymbol{S}_{wi,i}^2(\boldsymbol{e}_{wi})_i^2\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{G}_w^T\boldsymbol{G}_w\frac{\boldsymbol{H}_w\boldsymbol{g}_{wi}}{\boldsymbol{S}_{wi,i}}\\ &= \boldsymbol{e}_{wi}^T\boldsymbol{e}_{wi} - 2\boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i^2\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi} + \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i^2\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= \boldsymbol{e}_{wi}^T\boldsymbol{e}_{wi} + (\boldsymbol{e}_{wi})_i^2 - \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i^2\boldsymbol{g}_{wi}^T\boldsymbol{H}_{wi}\boldsymbol{g}_{wi}\\ &= WSSE_i + \boldsymbol{S}_{wi,i}(\boldsymbol{e}_{wi})_i^2\end{aligned} \tag{A5}$$

Substituting (31) into (32), we obtain:

$$WSSE_i = WSSE - (\boldsymbol{e}_w)_i^2 / \boldsymbol{S}_{wi,i} \tag{A6}$$

For two fault elimination:

$$WSSE_{i,j} = WSSE - (\boldsymbol{e}_w)_i^2 / \boldsymbol{S}_{wi,i} - (\boldsymbol{e}_{wi})_j^2 / \boldsymbol{S}_{wj,j}^i \tag{A7}$$

where

$$\begin{aligned}\boldsymbol{S}_{j,j}^i &= w_j(1+w_j\boldsymbol{g}_j^T\boldsymbol{H}_{i,j}\boldsymbol{g}_j)^{-1}\\ &= w_j(1+(w_j)^2\boldsymbol{g}_j^T\boldsymbol{H}_i\boldsymbol{g}_j / \boldsymbol{S}_{j,j}^i)^{-1}\end{aligned} \tag{A8}$$

The relationship between the fault detection statistics after two satellites are removed and the fault detection statistics of all satellites is as follows:

$$WSSE_{i,j} = WSSE - \frac{(\boldsymbol{e}_w)_i^2}{\boldsymbol{S}_{wi,i}} - \frac{\left((\boldsymbol{e}_w)_j - \frac{\boldsymbol{S}_{wi,j}}{\boldsymbol{S}_{i,i}}(\boldsymbol{e}_w)_i\right)^2}{\boldsymbol{S}_{wj,j} - \frac{\boldsymbol{S}_{wi,j}^2}{\boldsymbol{S}_{wi,i}}} \tag{A9}$$

$$= WSSE - \frac{\boldsymbol{S}_{wi,i}(\boldsymbol{e}_w)_j^2 - 2\boldsymbol{S}_{wi,j}(\boldsymbol{e}_w)_i(\boldsymbol{e}_w)_j + \boldsymbol{S}_{wj,j}(\boldsymbol{e}_w)_i^2}{\boldsymbol{S}_{wi,i}\boldsymbol{S}_{wj,j} - \boldsymbol{S}_{wi,j}^2}$$

$$T_{i,j} = \frac{\boldsymbol{S}_{wi,i}(\boldsymbol{e}_w)_j^2 - 2\boldsymbol{S}_{wi,j}(\boldsymbol{e}_w)_i(\boldsymbol{e}_w)_j + \boldsymbol{S}_{wj,j}(\boldsymbol{e}_w)_i^2}{\boldsymbol{S}_{wi,i}\boldsymbol{S}_{wj,j} - \boldsymbol{S}_{wi,j}^2} \tag{A10}$$

By mathematical induction, we can derive multiple satellite formulas:

$$T_{\{i,j,k,\dots,n\}} \leftarrow \frac{\sum_{I\in\{i,j,k,\dots,n\}}\sum_{J\in\{n,\dots,k,j,i\}}(\boldsymbol{e}_w)_I(\boldsymbol{e}_w)_J\, cofactor(\boldsymbol{S}_{I,J})}{det(\boldsymbol{S}_{\{i,j,k,\dots.n\}})} \tag{A11}$$

$\boldsymbol{S}_{\{i,j,k,\dots.n\}}$ represents the submatrix composed of $\boldsymbol{S}$, *det* represents the calculation of the matrix determinant. The expression of $cofactor(\boldsymbol{S}_{I,J})$ is as follows:

$$cofactor(\boldsymbol{S}_{I,J})\begin{cases} \boldsymbol{S}_{I,J}, & if\ I = J \\ -\boldsymbol{S}_{I,J}, & if\ I \neq J \end{cases} \tag{A12}$$